Entirely Harmonic Domain Calculation of Multiphase Nonsinusoidal Steady State

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Abstract—This paper proposes an algorithm for obtaining the periodic steady-state solution of a multiphase network including nonlinear, switching, and frequency dependent elements. Unlike existing methods which deal with nonlinear and switching elements in the time domain, the approach presented is entirely in the harmonic domain. The method will be used for the harmonic analysis of power systems and for steady-state initialization in electromagnetic transient analysis. The algorithm takes rigorously into account the inter-harmonic couplings in the Jacobian matrix of the proposed Newton–Raphson iteration process so that a quadratic convergence rate is achieved. Linear, nonlinear, switching, and frequency dependent elements are modeled in a modular approach, and any network topology can be handled by extending the Modified Nodal Equations approach to the harmonic domain case. First the algorithm is described and then applied to a test case to demonstrate its computational performance.

Index Terms—Multiport circuits, nonlinear circuits, periodic functions, power system harmonics, switching circuits.

I. INTRODUCTION

POWER systems include nonlinear, switching, and frequency dependent elements. Iron cores of transformers, loads with rectifiers, and discharge lamps are nonlinear elements that generate a nonsinusoidal current even if a purely sinusoidal voltage is applied. HVdc converters and power-electronics devices are switching elements that control the currents by semiconductor switch operations. Overhead lines, underground cables, and transformer impedances are frequency dependent elements due to skin effect. To predict the harmonic content and propagation of a network with such elements, an algorithm for calculating the periodic steady-state solution is required. The algorithm can also be used for steady-state initialization in electromagnetic transient analysis.

This field of research was pioneered by Xia and Heydt [1], [2] using a decoupled positive-sequence circuit. Later, Denset *et al.* pointed out that the network must be modeled in phase coordinates since noncharacteristic harmonics cannot be properly evaluated by a positive-sequence circuit [3]. Subsequently proposed multiphase methods can be classified, in terms of their formulation methodologies, into three categories: harmonic domain methods [4]–[8], time domain methods [9], [10], and hybrid

methods [11], [12]. The harmonic domain methods deal with all network elements in the harmonic domain. Although the treatment of linear elements including frequency dependent elements is straightforward, that of nonlinear and switching elements which produce couplings between different harmonics is of concern. A practical solution used by most of the existing harmonic domain methods [5]–[8] is to use the values at the previous iteration step to eliminate the harmonic couplings in the formulation at the expense of slower convergence rate (this is basically equivalent to injecting the currents of nonlinear and switching elements obtained at the previous iteration step). On the other hand, the harmonic domain method presented in [4] takes rigorously into account the harmonic couplings based on the state-space representation of the network, but deriving the state equations can be difficult in particular network configurations. Another solution is to use the hybrid methods which deal with nonlinear and switching elements in the time domain while the remaining linear part is treated in the harmonic domain. The time domain methods seem to be at a development stage since the methods are applied only to simple networks.

This paper presents an algorithm to calculate the periodic steady-state solution of a multiphase network including nonlinear, switching, and frequency dependent elements. The approach presented formulates the network equations entirely in the harmonic domain by extending the Modified Nodal Equations approach [13] to the harmonic domain case so that the network equations can always be derived regardless of network topology. The network equations are then solved by a Newton–Raphson iteration process. In a modular approach, the harmonic domain branch constitutive relations and their linearized admittance matrices are derived for lumped linear elements, frequency dependent elements, switching elements, and nonlinear elements. The linearized admittance matrices, which correspond to the individual Jacobian matrices of the network elements, are derived based on the same principle as the method presented in [4]. Thus, the proposed algorithm takes rigorously into account the harmonic couplings and this achieves a quadratic convergence rate. Since the proposed formulation includes network equations for all harmonics, the Jacobian matrix of the entire network can be very large. It is shown that the large Jacobian matrix can be efficiently handled by taking advantage of its sparsity.

The paper first describes the proposed algorithm, and then the steady-state solution of a 500 kV double-circuit three-phase test network including nonlinear inductors, an SVC, and frequency dependent transmission lines is calculated. It is shown that the convergence rate of the proposed algorithm is quadratic.

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II. HARMONIC DOMAIN NOTATION

A. Operator of Time Derivative

The complex Fourier series of a time domain T -periodic scalar quantity $x(t)$ is given by

$$x(t) = \sum_{h=-\infty}^{\infty} x_h e^{jh\omega_0 t} \quad (1)$$

where x_h ($h = 0, \pm 1, \pm 2, \dots$) are the Fourier coefficients and $\omega_0 = 2\pi/T$ is the base angular frequency. Let $y(t)$ be the time derivative of $x(t)$:

$$y(t) = \frac{dx(t)}{dt} = \sum_{h=-\infty}^{\infty} jh\omega_0 x_h e^{jh\omega_0 t} \quad (2)$$

where $y(t)$ is also T -periodic. Considering harmonics only up to the $\pm K$ -th, the coefficients of the exponential terms can be expressed in the following matrix notation:

$$y = Sx \quad (3)$$

where $x, y \in \mathbf{C}^M$ ($M = 2K + 1$) are the harmonic domain vectors corresponding to $x(t)$ and $y(t)$, defined by

$$x = \begin{bmatrix} x_{-K} \\ \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \\ x_K \end{bmatrix}, \quad y = \begin{bmatrix} y_{-K} \\ \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \\ y_K \end{bmatrix} \quad (4)$$

and S is the diagonal $M \times M$ matrix defined by

$$S = \text{diag}(jk\omega_0) \quad \text{for } k = -K, \dots, K. \quad (5)$$

Thus S is the operator of time derivative in the harmonic domain [4], [14]. Note that S is singular because k includes zero.

B. Incremental Expression for Time Domain Nonlinearity

Next, we consider the nonlinear algebraic relation between two T -periodic scalar variables $x(t)$ and $y(t)$ in the time domain:

$$y(t) = f(x(t)). \quad (6)$$

For incremental changes of $x(t)$ and $y(t)$, (6) is linearized as

$$\Delta y(t) = f_x(t) \Delta x(t), \quad (7)$$

where $f_x(t)$ is the derivative of $f(\cdot)$ with respect to $x(t)$. Thus $f_x(t)$ is also T -periodic. Let Δx and Δy be the harmonic domain counterparts of $\Delta x(t)$ and $\Delta y(t)$ defined similarly to x and y in (4). The harmonic domain relation for (7) is

$$\Delta y = F \Delta x \quad (8)$$

where F is an $M \times M$ Hermitian Toeplitz matrix consisting of the Fourier coefficients c_k of $f_x(t)$ up to the $\pm K$ -th harmonic [4], [14]. Thus, for $K = 3$, F takes the form

$$F = \begin{bmatrix} c_0 & c_1^* & c_2^* & c_3^* & & & \\ c_1 & c_0 & c_1^* & c_2^* & c_3^* & & \\ c_2 & c_1 & c_0 & c_1^* & c_2^* & c_3^* & \\ c_3 & c_2 & c_1 & c_0 & c_1^* & c_2^* & c_3^* \\ & c_3 & c_2 & c_1 & c_0 & c_1^* & c_2^* \\ & & c_3 & c_2 & c_1 & c_0 & c_1^* \\ & & & c_3 & c_2 & c_1 & c_0 \end{bmatrix}. \quad (9)$$

Here the relation $c_{-k} = c_k^*$ (complex conjugate) is used.

C. Half-Wave Symmetry

If the driving force of a network is purely sinusoidal and if all nonlinear elements and periodically-operating switches result in only even order coefficients in matrix (9), all voltage and current waveforms in the network exhibit “half-wave symmetry”; there are no even harmonics in the voltage and current waveforms and no odd harmonics in the coefficient matrices. This is the usual case for ac power networks. If the network of interest exhibits this half-wave symmetry, the time domain derivative (3) and multiplication (8), expressed in the harmonic domain, can be compacted so that x, y, S , and F are given by (for $K = 3$)

$$x = \begin{bmatrix} x_{-3} \\ x_{-1} \\ x_1 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} y_{-3} \\ y_{-1} \\ y_1 \\ y_3 \end{bmatrix} \quad (10)$$

$$S = \text{diag}(-j3\omega_0, -j\omega_0, j\omega_0, j3\omega_0) \quad (11)$$

$$F = \begin{bmatrix} c_0 & c_2^* & & \\ c_2 & c_0 & c_2^* & \\ & c_2 & c_0 & c_2^* \\ & & c_2 & c_0 \end{bmatrix}. \quad (12)$$

In this case, S is not singular and $M = K + 1$.

D. Operator of Time Integral

The operator of time integral in the harmonic domain is given by S^{-1} in case of half-wave symmetry. Otherwise, the inverse of S cannot be taken, and therefore we use the Moore–Penrose pseudoinverse S^+ defined by

$$S^+ = \text{diag}\left(\dots, -\frac{1}{j2\omega_0}, -\frac{1}{j\omega_0}, 0, \frac{1}{j\omega_0}, \frac{1}{j2\omega_0}, \dots\right). \quad (13)$$

Note that S^+ ignores the dc component. Thus, if S^+ is used, a dc component would not be properly calculated.

E. HDAF and HDAM

In this paper, the harmonic domain nonlinear function $f(\cdot)$ between the vectors i and v of the branch current and voltage of a network element:

$$i = f(v) \quad (14)$$

is called HDAF (Harmonic Domain Admittance Function). The harmonic domain matrix Y that relates to each other the incremental vectors Δi and Δv of the branch current and voltage:

$$\Delta i = Y \Delta v \quad (15)$$

is called HDAM (Harmonic Domain Admittance Matrix). In the case of a single-phase network element, the harmonic domain vectors i , v , Δi , and Δv take the form (4), $f(\cdot) \in \mathbf{C}^M \rightarrow \mathbf{C}^M$, and $Y \in \mathbf{C}^{M \times M}$. In the case of a multiphase element (the three-phase case is shown throughout the paper for example of the multiphase case), the branch current and voltage vectors take the form

$$i = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}, \quad v = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \quad (16)$$

where the subvectors $[i_a]$, $[i_b]$, $[i_c]$, $[v_a]$, $[v_b]$, and $[v_c]$ are in the form (4). The HDAF is $f(\cdot) \in \mathbf{C}^{3M} \rightarrow \mathbf{C}^{3M}$, and the HDAM is expressed as

$$Y = \begin{bmatrix} [Y_{aa}] & [Y_{ab}] & [Y_{ac}] \\ [Y_{ba}] & [Y_{bb}] & [Y_{bc}] \\ [Y_{ca}] & [Y_{cb}] & [Y_{cc}] \end{bmatrix} \in \mathbf{C}^{3M \times 3M} \quad (17)$$

where $[Y_{ij}] \in \mathbf{C}^{M \times M}$ ($i, j = a, b, c$).

In the proposed method, we use a modular approach [8], [12] where the HDAFs and HDAMs of all network elements are individually calculated and then assembled to form the global HDAF and HDAM of the whole network to be used for a Newton–Raphson iteration.

III. HDAFs AND HDAMS OF NETWORK ELEMENTS

The proposed method is built on the types of passive network elements shown in Fig. 1 where the reference directions of the branch voltages and currents are defined. The network elements whose HDAFs and HDAMs are derived below are classified into (a) or (b) of Fig. 1.

Because in the case of a linear network element the incremental current-voltage relation $\Delta i = Y \Delta v$ is also satisfied by the full current-voltage relation $i = Yv$, the HDAF is readily obtained by the HDAM. Thus only HDAMs are derived for linear elements. For nonlinear elements both HDAFs and HDAMs are derived.

A. Single-Phase Linear R , L , C Elements

Using the operators of time derivative and integral, the HDAMs of the single-phase linear time-invariant resistor R , inductor L , and capacitor C are respectively given by

$$Y_R = \left(\frac{1}{R}\right) I, \quad Y_L = \left(\frac{1}{L}\right) S^{-1}, \quad Y_C = CS \quad (18)$$

where I is the $M \times M$ identity matrix. Because those network elements do not have inter-harmonic couplings, their HDAMs are diagonal matrices. The expressions for the inductor and the capacitor are valid in case of half-wave symmetry.

Fig. 1. The types of passive network elements, considered in the proposed algorithm, with their reference directions of the branch voltages and currents. The above elements are connected from the p -th to the q -th node where the letters p and q enclosed in circles are node indices. In (b) the three-phase case is shown. (a) Single-phase element; (b) multiphase element.

If the network does not exhibit half-wave symmetry, a small resistance r_0 which is effective only for the dc component is inserted in series to the inductor:

$$Y_L = \left(\frac{1}{L}\right) S^+ + G_{0L} \quad (19)$$

where the $(K+1, K+1)$ element of G_{0L} is $1/r_0$ and the remaining elements are all zeros. Similarly, a small conductance g_0 , effective only for dc, is connected in parallel to the capacitor:

$$Y_C = CS + G_{0C} \quad (20)$$

where the $(K+1, K+1)$ element of G_{0C} is g_0 and the remaining elements are zeros. Instead of using the small resistance, if the inductor is accompanied by the series resistor R , we consider the two elements as one:

$$Y_{LR} = (LS + RI)^{-1}. \quad (21)$$

If the capacitor is accompanied by the parallel conductance G , the two elements are again considered as one:

$$Y_{CG} = CS + GI. \quad (22)$$

B. Single-Phase Linear Frequency Dependent Elements

We assume that the characteristic of a single-phase linear frequency dependent element is given by the frequency-scan data Y_k of its admittance at discrete frequency points $\omega = k\omega_0$ ($k = 0, 1, 2, \dots$). The HDAM of the frequency dependent element is given by the following formulas. In case of no half-wave symmetry:

$$Y_{FD} = \text{diag}(Y_k) \quad \text{for } k = -K, \dots, K. \quad (23)$$

In case of half-wave symmetry:

$$Y_{FD} = \text{diag}(\dots, Y_{-3k}, Y_{-k}, Y_k, Y_{3k}, \dots) \quad (24)$$

where $Y_{-k} = Y_k^*$.

C. Multiphase Linear Frequency Dependent Elements

We assume that the characteristic of a multiphase linear frequency dependent element is also given by the frequency-scan

data of its admittance matrix at discrete frequency points $\omega = k\omega_0$ ($k = 0, 1, 2, \dots$):

$$\begin{bmatrix} Y_{aa,k} & Y_{ab,k} & Y_{ac,k} \\ Y_{ba,k} & Y_{bb,k} & Y_{bc,k} \\ Y_{ca,k} & Y_{cb,k} & Y_{cc,k} \end{bmatrix}. \quad (25)$$

If the submatrices $[Y_{ij}]$ ($i, j = a, b, c$) of (17) are obtained as shown below, (17) gives the HDAM of the multiphase frequency dependent element. In case of no half-wave symmetry:

$$[Y_{ij}] = \text{diag}(Y_{ij,k}) \quad \text{for } k = -K, \dots, K. \quad (26)$$

In case of half-wave symmetry:

$$[Y_{ij}] = \text{diag}(\dots, Y_{ij,-3}, Y_{ij,-1}, Y_{ij,1}, Y_{ij,3}, \dots). \quad (27)$$

In the above two equations, $Y_{ij,k}$ in the right-hand sides are given by the elements of (25).

D. Multiphase Linear R, L Elements

If the following admittance matrix is used for (25), (17) gives the HDAM of a multiphase linear series RL circuit:

$$\left(\begin{bmatrix} R_{aa} & R_{ab} & R_{ac} \\ R_{ba} & R_{bb} & R_{bc} \\ R_{ca} & R_{cb} & R_{cc} \end{bmatrix} + jk\omega_0 \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} \\ L_{ba} & L_{bb} & L_{bc} \\ L_{ca} & L_{cb} & L_{cc} \end{bmatrix} \right)^{-1}. \quad (28)$$

E. Periodically-Switching Elements

Power electronics controllers are periodically-switching elements synchronized with the power frequency. Let us consider a switching device which controls the current of a series inductor as shown in Fig. 2. In the time domain, the current-voltage relation of the element is expressed by

$$i(t) = p(t) \frac{1}{L} \int \{p(t)v(t)\} dt \quad (29)$$

where the T -periodic function $p(t)$ defines the switching instances by giving the value 1 when the switching device is on and 0 when it is off. We first focus on the integrand

$$v_L(t) = p(t)v(t). \quad (30)$$

Since this is a time-domain multiplication of two variables same as (7), the procedure from (7) to (8) can be used to obtain the corresponding harmonic domain expression

$$v_L = Pv \quad (31)$$

where P is an $M \times M$ Hermitian Toeplitz matrix consisting of the Fourier coefficients¹ of $p(t)$ taking the form of (9) (note that the Fourier coefficients can be derived analytically).

Applying the above to each multiplication in (29), we finally obtain the HDAM

$$Y_{sw} = P \left(\frac{1}{L} \right) S^{-1}P \quad (32)$$

¹The high order Fourier coefficients of $p(t)$ are generally large because of switch operations, and thus the truncation of the Fourier coefficients causes an error corresponding to the Gibbs phenomenon. The error can be reduced by the application of the Sigma Factor window [15] (See Appendix-A).

Fig. 2. Switching device which controls the current of the series inductor L . This element is connected from the p -th to the q -th node where the letters p and q enclosed in circles are node indices.

(this is for the half-wave symmetry case and S^{-1} is replaced by S^{+} in case of no half-wave symmetry). Finally, we generalize (32) to

$$Y_{sw} = PY_xP \quad (33)$$

where Y_x is the HDAM of the element whose current is controlled by the switching device. Since (33) is for full values of i and v unlike (7), the HDAF is readily obtained by $i = Y_{sw}v$. To calculate P , the switching instants have to be known. However, the on/off instants of a switching element depend on the current and voltage which are not known at this stage. Thus, we introduce the approximation that the on/off instants are determined using the solution at the previous iteration step. The approximation slightly derates the quadratic convergence of a Newton-Raphson iteration, since the on/off instants are determined like a fixed-point iteration procedure. With this approximation, we can sequentially determine the on/off instants using the information of the circuit condition at the previous iteration step and the logic of the control system associated with the switching device.

Instead of introducing the above approximation, it is possible to derive the HDAF and HDAM of a switching device considering the circuit condition. This achieves complete quadratic convergence, but the HDAF and the HDAM have to be derived for each switching device and the HDAF cannot be obtained readily by the HDAM. Reference [16] derives the HDAFs/HDAMs of SVCs and TCSCs.

Coarsely approximated values of the switching instants can be obtained by a formula which gives the behavior of the switching device at the power frequency assuming a balanced condition. At the first iteration step, these values can be used as the initial guess to calculate P . At each iteration step, the HADF and HDAM of a switching device are updated as described above.

F. Nonlinear R, L, C Elements

Because the time domain current-voltage relation of a nonlinear resistor can be written as $i(t) = f(v(t))$ as in (6), the derivation of the HDAM simply follows the procedure from (6) to (9). Thus the HDAM Y_{NR} of the nonlinear resistor is obtained in the form (9), where the Fourier coefficients of a T -periodic function, obtained by linearizing $f(\cdot)$ with respect to $v(t)$, are the entries of Y_{NR} . To get the HDAF expression of the nonlinear resistor, we symbolically denote the process for obtaining the harmonic domain vectors i and v of a T -periodic branch current $i(t)$ and voltage $v(t)$ by $i = \mathcal{F}\{i(t)\}$ and $v = \mathcal{F}\{v(t)\}$ where $\mathcal{F}$ stands for the coefficients of the exponential Fourier series. Using this we obtain the HDAF:

$$i = \mathcal{F}\{f(\mathcal{F}^{-1}\{v\})\}. \quad (34)$$

The dynamics of a nonlinear inductor is described by

$$v(t) = \frac{d\phi(t)}{dt}, \quad i(t) = f(\phi(t)) \quad (35)$$

where $\phi(t)$ is the flux. The procedure from (1) to (3) applies to the first equation to get $v = S\phi$, or for incremental values of v and ϕ ,

$$\Delta v = S\Delta\phi. \quad (36)$$

The procedure from (6) to (9) applies to the second equation of (35) to get

$$\Delta i = F\Delta\phi \quad (37)$$

where F takes the form (9). The entries of F are the Fourier coefficients of a T -periodic function obtained by linearizing $f(\cdot)$ with respect to $\phi(t)$. In case of half-wave symmetry, eliminating $\Delta\phi$ from (36) and (37) gives the HDAM

$$Y_{NL} = FS^{-1} \quad (38)$$

and the HDAF is

$$i = \mathcal{F}\{f(\mathcal{F}^{-1}\{S^{-1}v\})\}. \quad (39)$$

In case of no half-wave symmetry, S^{-1} is replaced with S^+ in (38) and (39) and a small resistance r_0 , which is effective only for the dc component, is inserted in series. The HDAM and HDAF are

$$Y_{NL} = FS^+ + G_{0L}, \quad (40)$$

$$i = \mathcal{F}\{f(\mathcal{F}^{-1}\{S^+v\})\} + G_{0L}v. \quad (41)$$

The dynamics of a nonlinear capacitor is described by

$$i(t) = \frac{dq(t)}{dt}, \quad q(t) = f(v(t)) \quad (42)$$

where $q(t)$ is the charge. Applying the procedures from (1) to (3) and from (6) to (9) to the above equations, we obtain $\Delta i = SF\Delta v$. Therefore, in case of half-wave symmetry, the HDAM and the HDAF of the nonlinear capacitor are

$$Y_{NC} = SF, \quad (43)$$

$$i = S\mathcal{F}\{f(\mathcal{F}^{-1}\{v\})\}. \quad (44)$$

In case of no half-wave symmetry, S^{-1} is replaced with S^+ and a small conductance g_0 , effective only for dc, is connected in parallel. The HDAM and HDAF are

$$Y_{NC} = SF + G_{0C}, \quad (45)$$

$$i = S\mathcal{F}\{f(\mathcal{F}^{-1}\{v\})\} + G_{0C}v. \quad (46)$$

IV. SOLUTION PROCESS

A. Modified Nodal Equations in the Harmonic Domain

We extend the Modified Nodal Equations approach proposed by Ho *et al.* [13] to the harmonic domain case and then the harmonic domain network equations are solved to obtain the periodic steady-state solution. Consider a network with N_n nodes. The network contains N_v independent voltage sources and some

independent current sources, whose reference directions for the voltages and currents are defined as in Fig. 3. First, the unknown vector z is defined as

$$z = \begin{bmatrix} [v_1] \\ [v_2] \\ \vdots \\ [v_{N_n}] \\ [i_1] \\ \vdots \\ [i_{N_v}] \end{bmatrix} \in \mathbf{C}^N \quad (47)$$

where $N = M(N_n + N_v)$. The subvectors v_n ($n = 1, 2, \dots, N_n$) are the harmonic domain vectors of the node voltages, and i_n ($n = 1, 2, \dots, N_v$) are those of the currents through the independent voltage sources. Each of the subvectors is a harmonic domain vector which takes the form (4).

Consider a node, in the network, to which passive network elements, independent voltage sources, and independent current sources are connected. The passive elements include all kinds of elements whose HDAFs and HDAMs are given in Section III. The harmonic domain KCL (Kirchhoff's Current Law) for each node is written as

$$\sum_n f_n(v_{b,n}) + \sum_n i_n = -\sum_n i_{s,n}. \quad (48)$$

The first term of the left-hand side is the sum of the currents leaving the node through the passive elements whose HDAFs $f_n(\cdot)$ are described as function of the branch voltages $v_{b,n}$ (in the case of a multiphase element, $f_n(\cdot)$ represents a part of the HDAF for the corresponding phase). The second term is the sum of the currents i_n leaving the node through the independent voltage sources. The right-hand side term is the sum of the currents $i_{s,n}$ leaving through the independent current sources, where the Fourier coefficients of the waveform $i_{s,n}(t)$ specified for the n -th independent current source give the elements of $i_{s,n}$ in the form (4). For the n -th independent voltage source connected from the p -th to the q -th node, we have the following additional branch equations:

$$v_p - v_q = v_{s,n} \quad (49)$$

where v_p and v_q are the node voltages at p and q respectively, and $v_{s,n}$ consists of the Fourier coefficients of the waveform $v_{s,n}(t)$ specified for the n -th independent voltage source.

Since (48) and (49) are harmonic domain expressions, each expression contains M equations. Thus, if we put (48) for all nodes and (49) for all independent voltage sources into a single vector equation, we obtain Modified Nodal Equations in the harmonic domain, namely a global HDAF, containing N equations altogether:

$$f(z) = -u \quad (50)$$

where $f(\cdot) \in \mathbf{C}^N \rightarrow \mathbf{C}^N$ and $u \in \mathbf{C}^N$. In the above equation, the first term of the left-hand side in (48) is considered as function of the node voltages because the branch voltages $v_{b,n}$ are the differences of two node voltages. The vector u in (50) is obtained by the right-hand side expressions of (48) and (49).

B. Newton–Raphson Iteration

Applying the Newton-Raphson algorithm to (50), we obtain the following iteration process:

$$z^{(m)} = z^{(m-1)} + \Delta z^{(m)}, \quad \text{for } m = 1, 2, 3, \dots \quad (51)$$

The correction term $\Delta z^{(m)}$ is given by the solution of

$$J(z^{(m-1)}) \Delta z^{(m)} = -\left\{f(z^{(m-1)}) + u\right\} \quad (52)$$

where

$$J(z^{(m)}) = \left. \frac{\partial f}{\partial z} \right|_{z=z^{(m)}} \quad (53)$$

is the Jacobian matrix, namely the global HDAM. If an appropriate initial guess is not known, $z^{(0)}$ obtained as the solution of $J(0)z^{(0)} = -u$, with $J(0)$ constructed assuming $z = 0$, can be used for the initial solution.

C. Jacobian Matrix

As explained in [13], the Jacobian matrix J in (53) can be formed by adding and subtracting the “stamps” of all network elements to and from appropriate portions of J .

For a passive single-phase element connected from the p -th to the q -th node [Fig. 1(a)], the HDAM is added to the following two diagonal submatrices of J , one submatrix corresponding to the rows of the p -th node and the columns of the node voltage v_p , and the other submatrix corresponding to the rows of the q -th node and the columns of the node voltage v_q . Hereafter, we denote the former submatrix by $J\{p, p\}$ and the latter by $J\{q, q\}$. Also, we subtract the HDAM from the two off-diagonal matrices $J\{p, q\}$ and $J\{q, p\}$. This procedure is illustrated by the stamps shown in Table I(a).

For a passive multiphase element connected from the p_a -th, p_b -th, and p_c -th nodes to the q_a -th, q_b -th, and q_c -th nodes [Fig. 1(b)], the submatrices Y_{ij} ($i, j = a, b, c$) of the HDAM in (17) are added to and subtracted from the submatrices of J as in Table I(b).

For an independent voltage source connected from the p -th to the q -th node [Fig. 3(a)], the $M \times M$ identity matrix I is added to the submatrices $J\{p, n\}$ and $J\{n, p\}$, and subtracted from $J\{q, n\}$ and $J\{n, q\}$ as in Table I(c). The rows denoted by n correspond to the additional branch equations (49) of the voltage source, and the columns denoted by n correspond to the branch current i_n through the voltage source [RHS: the right-hand side vector u in (50)].

Starting with the $N \times N$ zero matrix, we scan all passive network elements (single-phase and multiphase) and all independent voltage sources to include their contributions as described in the above, and we finally obtain the complete Jacobian matrix J . Also, note the following remarks.

- Because the voltages and currents of independent current sources are not included in the unknown vector z , they do not contribute to J .
- If one of the nodes p and q is grounded, corresponding rows and columns in Table I are eliminated.
- Controlled sources, which change their voltage or current in proportion to the voltage or current of another branch, can be dealt with as described in [17].

TABLE I
STAMPS FOR PASSIVE NETWORK ELEMENTS AND INDEPENDENT VOLTAGE SOURCES: (a) SINGLE-PHASE ELEMENT; (b) MULTIPHASE ELEMENT (THREE-PHASE CASE); (c) INDEPENDENT VOLTAGE SOURCE

	v_p	v_q
p	$+Y$	$-Y$
q	$-Y$	$+Y$

(a)

	v_{pa}	v_{pb}	v_{pc}	v_{qa}	v_{qb}	v_{qc}
p_a	$+Y_{aa} + Y_{ab} + Y_{ac}$	$-Y_{aa}$	$-Y_{ab}$	$-Y_{ac}$		
p_b	$+Y_{ba}$	$+Y_{bb} + Y_{bc}$	$-Y_{ba}$	$-Y_{bb}$	$-Y_{bc}$	
p_c	$+Y_{ca}$	$+Y_{cb}$	$+Y_{cc}$	$-Y_{ca}$	$-Y_{cb}$	$-Y_{cc}$
q_a	$-Y_{aa}$	$-Y_{ab}$	$-Y_{ac}$	$+Y_{aa}$	$+Y_{ab}$	$+Y_{ac}$
q_b	$-Y_{ba}$	$-Y_{bb}$	$-Y_{bc}$	$+Y_{ba}$	$+Y_{bb}$	$+Y_{bc}$
q_c	$-Y_{ca}$	$-Y_{cb}$	$-Y_{cc}$	$+Y_{ca}$	$+Y_{cb}$	$+Y_{cc}$

(b)

	v_p	v_q	i_n	RHS
p			$+I$	
q			$-I$	
n	$+I$	$-I$		$-v_{s,n}$

(c)

Fig. 3. The independent voltage and current sources, considered in the proposed algorithm, with their reference directions of the branch voltages and currents. The above elements are connected from the p -th to the q -th node where the letters p and q enclosed in circles are node indices. In (a) $v_s(t)$ is given, and in (b) $i_s(t)$ is given. (a) Independent voltage source; (b) independent current source.

D. Values of r_0 and g_0

When the network does not exhibit half-wave symmetry, we use a small series resistance r_0 and a small parallel conductance g_0 for modeling inductors and capacitors respectively. Smaller values of r_0 and g_0 give more accurate behavior of the inductors and the capacitors for the dc component. However, too small values will result in an ill-conditioned Jacobian matrix, and thus the values should be determined considering the machine epsilon of the computer being used.²

²If we decide to preserve 10 digits for obtaining the solution using a 16-digit floating-point processor, the largest impedance (in the network) multiplied by 10^{-6} is set to r_0 and the largest admittance multiplied by 10^{-6} to g_0 .

Fig. 4. 500 kV double-circuit three-phase test network. (a) Network diagram; (b) transformer; (c) load; (d) SVC.

V. TEST NETWORK

The periodic steady-state solution of the test network shown in Fig. 4 is calculated by the proposed algorithm. The test network is a 500 kV double-circuit transmission system. The parameter values used in the simulation are given in Table II. Al-

TABLE II
PARAMETER VALUES USED IN THE SIMULATION

Generators: the voltages of the generators (500 kV base).	
G1: $1.01 \angle 23.91^\circ$ [pu]	G2: $1.03 \angle 5.07^\circ$ [pu]
G3: $1.03 \angle 3.14^\circ$ [pu]	G4: $1.03 \angle 0.00^\circ$ [pu]
Transformers: the leakage inductances L_0 of the transformers.	
TR1: 10.6 [mH]	TR2: 15.3 [mH]
TR3: 15.3 [mH]	TR4: 5.17 [mH]
TR5: 6.63 [mH]	
Transmission Lines: the lengths of the transmission lines.	
TL1, TL4, TL5, TL6, TL7: 100 [km]	
TL2, TL3: 50 [km]	
Loads: the resistance R_p and inductance L_Q values of the loads.	
LD1: 58.0 [Ω], 57.4 [mH]	LD2: 86.6 [Ω], 86.6 [mH]
LD3: 85.1 [Ω], 89.4 [mH]	LD4: 14.8 [Ω], 6.71 [mH]
LD5: 130. [Ω], 3.45 [mH]	LD6: 129. [Ω], 3.41 [mH]
SVC: the switched inductance L , the fixed capacitance C , and the firing delay angle α of the SVC.	
$L = 995$ [mH], $C = 7.07$ [μF], $\alpha = 29.5^\circ$

Fig. 5. Current-flux ($i - \phi$) curve of the transformer magnetizing branches.

Fig. 6. Conductor configuration of the 500 kV double-circuit transmission lines.

though Fig. 4(a) is shown on a one-line diagram, all components are represented by three-phase models.

In this test case, the generators are represented by balanced three-phase sinusoidal voltage sources. The transformers are represented by the simplified equivalent shown in Fig. 4(b) with

Fig. 7. The calculated voltage waveforms at Bus A are shown in (a). The harmonic spectra of the voltages at Buses A-C are shown in (b) (the magnitudes are normalized by the corresponding fundamental components). (a) Voltage waveforms at Bus A; (b) harmonic spectra of voltages at Buses A-C.

the leakage inductances L_0 and the magnetizing branches L_N . All magnetizing branches have the same saturation characteristic defined by

$$i = 0.01\phi + 5 \times 10^{-68}\phi^{23} \quad (54)$$

where i is the current and ϕ the flux, and the current-flux ($i-\phi$) curve is shown in Fig. 5. The transmission lines are all untransposed and have the same conductor configuration shown in Fig. 6. At both ends of the transmission lines phases a and a', b and b', and c and c' are connected, and thus the admittance is given by the 6×6 matrix

$$\begin{bmatrix} Y_0(\omega) \coth(\Gamma(\omega)l) & -Y_0(\omega) \operatorname{cosech}(\Gamma(\omega)l) \\ -Y_0(\omega) \operatorname{cosech}(\Gamma(\omega)l) & Y_0(\omega) \coth(\Gamma(\omega)l) \end{bmatrix} \quad (55)$$

where $Y_0(\omega)$ is the 3×3 characteristic admittance matrix, $\Gamma(\omega)$ the 3×3 propagation constant matrix, and l the length. $Y_0(\omega)$

Fig. 8. Calculated current waveforms through the nonlinear inductors of TR1.

Fig. 9. Calculated current waveforms through the inductors of the SVC.

and $\Gamma(\omega)$ are calculated considering full frequency dependence [18], [19]. Substituting (55) into (25), the HDAMs of the transmission lines are obtained. The loads are represented by the $R-L$ series circuit shown in Fig. 4(c). The SVC shown in Fig. 4(d) is an FC-TCR type and the HDAM of each thyristor-inductor branch is given by (32).

With up to the ± 49 th harmonic component the solution is calculated considering half-wave symmetry. The initial solution is obtained by solving $J(0)z^{(0)} = -u$. Fig. 7(a) shows the voltage waveforms at Bus A which is distorted by the SVC, and Fig. 7(b) the harmonic spectra of the voltages at Buses A-C. It is observed that the harmonics generated by the SVC propagate along the transmission lines. Fig. 8 shows the magnetizing currents of TR1. The magnetizing currents of TR2-4 are similar to those of TR1. The imbalanced impedances of the untransposed transmission lines create the different peak values of the magnetizing currents phase by phase, and this imbalance can be observed also in Fig. 7(b). Fig. 9 shows the currents through the inductors of the SVC.

The Newton-Raphson iteration is terminated when the relative norm of the correction term $\Delta z^{(m)}$ becomes less than 10 times of machine epsilon ε :

$$\frac{\|\Delta z^{(m)}\|_2}{\|z^{(m)}\|_2} < 10\varepsilon. \quad (56)$$

The variation of the relative norm of the correction term during the iteration is shown in Fig. 10. ε of the computer used is 2.22×10^{-16} , and (56) is satisfied at the fifth step. The convergence rate is observed to be quadratic. Reference [20] compares the convergence rates of various iteration schemes and shows that a quadratic convergence is the fastest.

The size of the Jacobian matrix J is 2400×2400 , and its sparsity pattern is shown in Fig. 11. Sub-blocks corresponding

Fig. 10. Relative norm of the correction term $\Delta z^{(m)}$ with respect to iteration steps m .

Fig. 11. Sparsity pattern of Jacobian matrix.

to the HDAMs of linear elements have a diagonal pattern, because there are no couplings among harmonics of different orders. On the other hand, sub-blocks corresponding to nonlinear and switching elements show patterns of couplings among different orders of harmonics. To store J in memory and to solve (52), the sparsity of J was taken into account [21]. With the sparse matrix technique, the storage required was 1.2 MB and the computation time for this simulation was 19.5 seconds using a personal computer with Pentium 4 CPU (2.4 GHz). These requirements are acceptable for practical usage. The program is coded using MATLAB ver. 6.5.

VI. CONCLUSIONS

In this paper an entirely harmonic domain algorithm for obtaining the periodic steady-state solution of a multiphase network with nonlinear, switching, and frequency dependent elements is presented. The proposed Newton-Raphson iteration process takes rigorously into account the couplings among different harmonics and a quadratic convergence rate is achieved.

Fig. 12. Calculated results with and without the sigma factor window.

By extending the modified nodal equations approach to the harmonic domain case, the network equations can always be derived regardless of network topology, and the network elements are modeled in a modular approach. The algorithm was applied to obtain the periodic steady-state solution of a 500 kV double-circuit three-phase network with nonlinear inductors, an SVC, and frequency dependent transmission lines.

APPENDIX

A. Effectiveness of Sigma Factor Window

To show the effectiveness of the Sigma Factor window [15], the function

$$p(t) = \begin{cases} 1, & (-0.5 < t < 0.5) \\ 0, & \text{otherwise} \end{cases} \quad (57)$$

is reproduced with its Fourier coefficients

$$c_k = \frac{\sin(k\frac{\pi}{2})}{k\pi} \quad (k = 0, \pm 1, \pm 2, \dots) \quad (58)$$

only up to the ± 19 th with and without the Sigma Factor window. As shown in Fig. 12, the Sigma Factor eliminates the oscillation called the Gibbs phenomenon and gives a better approximation to the original function.

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