Comments on “Two-Dimensional Interpolation by Generalized Spline Filters Based on Partial Differential Equation Image Models”

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Abstract—The generalized spline formula corresponding to the separable semicausal Partial Differential Equation (PDE) image model is given in its correct form. In addition, the correct formula for the second-order B-spline is derived. Finally, it is shown that there is no reason to consider separable PDE image models only, at least when describing the concept of generalized splines.

In this correspondence we introduce corrections in some formulas of the above paper which, regardless of their origin, may cause problems in the understanding of the paper. For the convenience of future readers, we identify the errors first and subsequently provide the correct formulas.

The generalized spline formula for the semicausal PDE model, as given in the paper by (6), should be corrected. In this case

\[ L_z(D_1) L_z(D_z) s(z) = 0, \quad z = x, y \]

and \( L^* \) is the formal adjoint operator of \( L \).

The function \( s_1(\cdot) \) is correctly given by (18) of the paper. However, \( s_1(\cdot) \), as given in (18), is incorrect. The adjoint operator for \( L_1(\cdot) \) is

\[ L_1^*(D_1) = L_1(D_1) = D_1^2 + \alpha_1^2. \]

Therefore, \( s_1(\cdot) \) is the general solution of

\[ (D_1^2 + \alpha_1^2)^2 s_1(x) = 0. \]

That is,

\[ s_1(x) = k_1 \cos \alpha_1 x + k_2 \sin \alpha_1 x + k_3 x \cos \alpha_1 x + k_4 x \sin \alpha_1 x. \]

In addition, the second-order B-spline given in (31) of the paper is also incorrect. The first-order B-spline corresponding to the operator

\[ L(D_1) = D_1 + \alpha \]

is given by (30), i.e.,

\[ B_1(x, \alpha) = \begin{cases} \frac{1}{e^{2\alpha R} - 1} \left[ \exp \left[ \alpha(x + 2R) \right] - e^{-\alpha x} \right] \\ \frac{2e^{\alpha R}}{e^{2\alpha R} - 1} \sinh \left[ \frac{\alpha(x + R)}{2} \right] \end{cases} \]

and

\[ B_1^* (x, \alpha) = \frac{1}{e^{2\alpha R} - 1} \left[ \exp \left[ \alpha(2R - x) \right] - e^{\alpha x} \right] \]

\[ = \frac{2e^{\alpha R}}{e^{2\alpha R} - 1} \sinh \left[ \frac{\alpha(R - x)}{2} \right]. \]

The second-order B-spline corresponding to the operator

\[ L(D_1) = D_1^2 - \alpha^2 = (D_1 + \alpha) (D_1 - \alpha) \]

is obtained by convolving \( B_1(x, \alpha) \) with \( B_1(x, -\alpha) \). In the paper, the second-order B-spline is given by (31), i.e.,

\[ B_2(x, \alpha, -\alpha) = \begin{cases} B_1(x, \alpha, -\alpha) & -2R \leq x \leq -R \\ B_2^* (x, \alpha, -\alpha) & -R \leq x \leq 0 \end{cases} \]

where

\[ \begin{array}{c}
B_1^* (x, \alpha) = \frac{1}{(e^{2\alpha R} - 1)^2} \left[ (1/\alpha) \exp \left[ \alpha(4R + x) \right] \\
+ 2 \exp \left[ \alpha(2R + x) \right] \\
- 2 \exp \left[ \alpha(2R - x) \right] \\
+ \frac{\alpha}{2} \exp \left[ \alpha(2R + x) \right] \\
- \frac{\alpha}{2} \exp \left[ \alpha(2R - x) \right] \\
- x \exp \left[ \alpha(4R + x) \right] + e^{-\alpha x} \right] \end{array} \]

The second-order B-spline should be a continuous function of \( x \in [-2R, 2R] \) and, therefore, be continuous at point \( x = -R \). Hence, \( B_2(\cdot) \) should satisfy the equation

\[ B_2^* (x, \alpha, -\alpha) \big|_{x=-R} = B_2^* (x, \alpha, -\alpha) \big|_{x=-R} \]

It can be easily verified that (2) is not satisfied by (31) regardless of what constant \( h \), undefined in the paper, is.

In the sequel, the correct expression of \( B_2(\cdot) \) is obtained. For \( x \in [-2R, -R] \),

\[ B_2(x, \alpha, -\alpha) = B_2^* (x, \alpha, -\alpha) \]

After some algebra

\[ B_2(x, \alpha, -\alpha) = \frac{2e^{2\alpha R}}{(e^{2\alpha R} - 1)^2} \left[ (x + 2R) \cosh \left[ \alpha(x + 2R) \right] - \frac{1}{\alpha} \sinh \left[ \alpha(x + 2R) \right] \right]. \]
which is the same as the expression given in the paper.\footnote{For \( x \in [-R, 0] \),
\[
B_2(x, \alpha, -\alpha) = B_2^* (x, \alpha, -\alpha)
\]
\[
= \int_{-\infty}^{+\infty} B_1(y, \alpha) B_1(x - y, -\alpha) \, dy
\]
\[
= \int_{-R}^{x} B_1(y, \alpha) B_1^*(x - y, -\alpha) \, dy
\]
\[
+ \int_{x}^{+R} B_1(y, \alpha) B_1^*(x - y, -\alpha) \, dy
\]
After some algebra
\[
B_2^*(x, \alpha, -\alpha) = \frac{2e^{2\alpha R}}{(e^{2\alpha R} - 1)^3} \left[ \frac{(2/\alpha) \sinh (\alpha x) - 2(2R)}{\cosh (\alpha x) - x \cosh [\alpha(x + 2R)]} \right]
\]
Hence, the second-order \( B \)-spline is given by (1), where \( B_1^*(\cdot) \) and \( B_2^*(\cdot) \) are given by (3) and (4), respectively.

It can easily be verified that the \( B \)-spline, as derived, satisfies (2). On the other hand, \( B_2^*(\cdot) \) satisfies the properties which characterize the second-order \( B \)-spline. That is, the first and second derivatives are continuous at \( x = 0 \), and the first derivative vanishes at the endpoints. By coincidence, the above conditions are also satisfied by (31) of the paper.

Our final comment concerns a simplifying consideration made implicitly in the paper.\footnote{Our equation (18) agrees with our equation.} According to the title, image interpolation is based on PDE image models. However, the interpolation is based on separable PDE image models only, since the nonseparable PDE models proposed and used by Jain to represent images \cite{1}, are not considered by the authors. This seems to be reasonable for the part of the paper\footnote{A correct form is shown in the Appendix.} where the \( B \)-spline interpolator is developed, due to the analytical complexity of the problem.

In the case where the PDO is separable, that is, \( L(D_x, D_y) \) is the adjoint operator for \( L(\cdot) \). In the case where the PDO is separable, that is,
\[
L(D_x, D_y) = L_x(D_x) L_y(D_y),
\]
it is obvious that the condition
\[
L^* L_x = L_x^* L_x L_y = 0
\]
given in the paper,\footnote{Following directly as a special case of (6).} follows directly as a special case of (6).

\section*{References}
\footnote{We appreciate the efforts of Karayiannis and Venetsanopoulos in scrutinizing the contents of our paper. However, as we will explain, there were no errors in our derivation—just misprints.}


\section*{Authors’ Reply\footnote{Manuscript received June 10, 1987.}}

T. C. Chen and R. J. P. deFigueiredo

We appreciate the efforts of Karayiannis and Venetsanopoulos in scrutinizing the contents of our paper.\footnote{T. C. Chen is with Bell Communications Research, Box 7020, Redbank, NJ 07701. R. J. P. deFigueiredo is with the Department of Electrical and Computer Engineering and the Department of Mathematical Sciences, Rice University, Houston, TX 77251-1892. IEEE Log Number 8717267.} However, as we will explain, there were no errors in our derivation—just misprints.

Actually, the comments of Karayiannis and Venetsanopoulos may be grouped into two parts. (I) The authors state that our equations (18) and (31) are not correct, and they present alternate equations. (II) They propose a 2-D nonseparable partial differential operator (PDO) for 2-D splines as a more general operator than the separable PDO’s which we considered in our paper.\footnote{Our reply to each set of comments is formulated separately in the following.} Our reply to both sets of comments is formulated separately in the following.

\section*{Reply to Part I}

a) There was a printing error in equation (6) on p. 632 of our paper.\footnote{0096-3518/87/1200-1779$01.00 © 1987 IEEE} The first \textquotedblleft 4\textquotedblright\ sign should have been \textquotedblleft 1\textquotedblright\ Our equation (18) is a direct derivation of the correct form of (6). Actually, Table I on p. 638 of our paper shows the correct form of (6). Corrected forms of (6), (18), and Table I are shown in the Appendix.

b) There are also some misprints in our equation (31). The character \textquotedblleft s\textquotedblright\ should have been \textquotedblleft R\textquotedblright\ (defined by us earlier in the paper), and there are four parentheses missing. A correct form is shown in the Appendix. In fact, Karayiannis and Venetsanopoulos’ equation (4) agrees with our equation.

c) Our conclusion is that the above six typographical errors have no impact at all on the main concept of our paper\footnote{Manuscript received June 10, 1987.} and, more generally, of [1]. Also, how (18) and (31) are derived is clearly described in our paper.\footnote{T. C. Chen is with Bell Communications Research, Box 7020, Redbank, NJ 07701. R. J. P. deFigueiredo is with the Department of Electrical and Computer Engineering and the Department of Mathematical Sciences, Rice University, Houston, TX 77251-1892. IEEE Log Number 8717267.} In view of the above, we feel that the rather long derivation of Karayiannis and Venetsanopoulos is quite redundant and misleading with regard to our original work.