The state model properties relevant in this matter are detectability and stabilizability which are characteristic of every internally stable realization. Using Bézout's identities (7) and (13) the results presented in the paper for the single-input single-output case can be extended in a straightforward manner to the multi-input multi-output case.

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The Use of Strictly Causal Filters in the Approximation of Two-Dimensional Asymmetric Half-Plane Filters

TAKAO HINAMOTO AND A. N. VENETSANOPOULOS

Abstract—This paper considers the problem of approximating two-dimensional (2-D) asymmetric half-plane digital filters using strictly causal filters. A technique is developed by means of certain mapping techniques and singular value decomposition of two finite Hankel matrices. First, a given impulse response over an asymmetric half-plane is transferred into the open first quadrant via an invertible mapping. Second, the data in the transformed domain (open first-quadrant) are approximated by a strictly causal separable-denominator recursive filter using singular value decomposition. Finally, the resultant filter is transferred back to the original coordinates. Since the resulting filter is simpler than a causal filter being separable in the denominator, the implementation is advantageous in addition to having a very easy stability check. Two examples are presented to illustrate the utility of the proposed technique.

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I. INTRODUCTION

In recent years, a few techniques have been reported to design asymmetric half-plane weakly causal 2-D digital filters in the spatial-domain [1], [2]. The concept "weakly causal" is a generalization of the concept "causal." Any weakly causal 2-D digital filter can be expressed in terms of a recursive model and its impulse response is defined over an asymmetric half-plane including the closed first quadrant. It is known [3] that the inverse of a causal 2-D digital filter often becomes a weakly causal 2-D digital filter. The inverse filter is considered to be useful in image restoration. On the other hand, a mapping technique from the closed first quadrant to an asymmetric half-plane has been investigated [3], [4]. This mapping technique enables one to use an algorithm for designing quarter-plane causal 2-D digital filters for the purpose of designing asymmetric half-plane weakly causal 2-D digital filters [2]. Lashgari *et al.* [5] has presented a technique based on singular value decomposition to approximate a quarter-plane causal 2-D digital filter by one being separable in the denominator. This technique is used for the approximation in [2].

In this paper, the problem of approximating asymmetric half-plane weakly causal 2-D digital filters is considered in the spatial domain. An invertible mapping is proposed to move a quarter-plane causal impulse response into the open first quadrant, in such a way that its structure is kept unaffected. This mapping is combined with the invertible mapping reported in [3] to move an asymmetric half-plane weakly causal impulse response into the open first-quadrant. The data in the transferred domain (open first quadrant) are approximated by a strictly causal 2-D digital filter being separable in the denominator. This is done via singular value decomposition of two finite Hankel matrices as in [5], [11]. Finally, the resultant filter is transferred back to the original coordinates.

II. APPROXIMATION OF STRICTLY CAUSAL FILTERS

Let a 2-D digital filter be given in terms of the impulse response as

$$F(z_1, z_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} F_{ij} z_1^{-i} z_2^{-j} \quad (1)$$

where the F_{ij} 's denote $\lambda \times \gamma$ real matrices. The 2-D digital filter, (1), is said to be strictly causal since F_{ij} is assumed to be null for all $i, j \leq 0$.

Let the filter to approximate the impulse response (1) be specified as

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_0 & A_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(i, j)$$

$$y(i, j) = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}, \quad i, j \geq 0 \quad (2)$$

where $x^h(i, j)$ is the $n \times 1$ horizontal state vector, $x^v(i, j)$ is the $m \times 1$ vertical state vector, $u(i, j)$ is the $\gamma \times 1$ input vector, $y(i, j)$ is the $\lambda \times 1$ output vector, and A_1, A_2, A_0, B, C are real constant matrices of appropriate dimensions. The filter (2) is a special form of the Roesser state-space model [6] and is simpler than one used in [5]. This filter also includes the Attasi state-space model [7], [11].

The transfer function $H(z_1, z_2)$ of (2) is given as [7]

$$H(z_1, z_2) = [0 \quad C] \begin{bmatrix} z_1 I_n - A_1 & 0 \\ -A_0 & z_2 I_m - A_2 \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} H_{ij} z_1^{-i} z_2^{-j} \quad (3)$$

where

$$H_{ij} = CA_2^{i-1} A_0 A_1^{j-1} B$$

and I_n denotes the n -dimensional identity matrix. This implies that the filter specified by (2) is strictly causal and has a strictly proper 2-D transfer function [8] being separable in the denominator.

The problem being considered here can be stated formally as follows: given the first NM samples, $\{F_{ij}|i=1,2,\dots,N; j=1,2,\dots,M\}$, of the impulse response in (1), find the filter $\{A_1, A_2, A_0, B, C\}_{n,m}$ of (2), which gives a reasonable approximation in some sense to the finite bisequence $\{F_{ij}|i=1,2,\dots,N; j=1,2,\dots,M\}$.

For the finite bisequence, the corresponding Hankel matrices can be written as [5]

$$\bar{F}_{NN}^{(M)} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 & \cdots & \bar{F}_N \\ \bar{F}_2 & \bar{F}_3 & \cdots & \bar{F}_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_N & \bar{F}_{N+1} & \cdots & \bar{F}_{2N} \end{bmatrix} \\ \hat{F}_{MM}^{(N)} = \begin{bmatrix} \hat{F}_1 & \hat{F}_2 & \cdots & \hat{F}_M \\ \hat{F}_2 & \hat{F}_3 & \cdots & \hat{F}_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{F}_M & \hat{F}_{M+1} & \cdots & \hat{F}_{2M} \end{bmatrix} \quad (4)$$

where

$$\bar{F}_i = [F_{i1}^T, F_{i2}^T, \dots, F_{iM}^T]^T \text{ and } \hat{F}_j = [F_{1j}, F_{2j}, \dots, F_{Nj}]$$

and F^T stands for the transpose of any matrix F . Let these matrices be decomposed using singular value decomposition [9] as

$$\bar{F}_{NN}^{(M)} = [U_1 \quad U_3] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_3 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_3^T \end{bmatrix} \\ \hat{F}_{MM}^{(N)} = [U_2 \quad U_4] \begin{bmatrix} \Sigma_2 & 0 \\ 0 & -\Sigma_4 \end{bmatrix} \begin{bmatrix} V_2^T \\ V_4^T \end{bmatrix} \quad (5)$$

where

$$\Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\} \quad \Sigma_3 = \text{diag}\{\sigma_{n+1}, \sigma_{n+2}, \dots\} \\ \Sigma_2 = \text{diag}\{\delta_1, \delta_2, \dots, \delta_m\} \quad \Sigma_4 = \text{diag}\{\delta_{m+1}, \delta_{m+2}, \dots\} \\ \sigma_1 \geq \sigma_2 \geq \dots > 0, \quad \sigma_n \gg \sigma_{n+1} \\ \delta_1 \geq \delta_2 \geq \dots > 0, \quad \delta_m \gg \delta_{m+1}.$$

Furthermore, let [5], [9]

$$B = \text{First } \gamma\text{-columns of } \Sigma_1^{1/2} V_1^T \quad A_1 = \tilde{P} P^T [P P^T]^{-1} \\ C = \text{First } \lambda\text{-rows of } U_2 \Sigma_2^{1/2} \quad A_2 = [Q^T Q]^{-1} Q^T \tilde{Q} \quad (6)$$

where

$$P = \Sigma_1^{1/2} V_1^T \begin{bmatrix} I_{\gamma(N-1)} \\ 0 \end{bmatrix} \quad \tilde{P} = \Sigma_1^{1/2} V_1^T \begin{bmatrix} 0 \\ I_{\gamma(N-1)} \end{bmatrix} \\ Q = [I_{\lambda(M-1)} \quad 0] U_2 \Sigma_2^{1/2} \quad \tilde{Q} = [0 \quad I_{\lambda(M-1)}] U_2 \Sigma_2^{1/2} \\ \Sigma_1^{1/2} = \text{diag}\{\sqrt{\sigma_1}, \sqrt{\sigma_2}, \dots, \sqrt{\sigma_n}\} \\ \Sigma_2^{1/2} = \text{diag}\{\sqrt{\delta_1}, \sqrt{\delta_2}, \dots, \sqrt{\delta_m}\}.$$

Equation (6) is used to find the best A_0 as follows. The matrix A_0 minimizing

$$J = \|F - Q_M A_0 P_N\| \quad (7)$$

is given as [10]

$$A_0 = [Q_M^T Q_M]^{-1} Q_M^T F P_N^T [P_N P_N^T]^{-1} \quad (8)$$

where

$$F = \begin{bmatrix} F_{11} & F_{21} & \cdots & F_{N1} \\ F_{12} & F_{22} & \cdots & F_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ F_{1M} & F_{2M} & \cdots & F_{NM} \end{bmatrix} \quad Q_M = \begin{bmatrix} C \\ CA_2 \\ \vdots \\ CA_2^{M-1} \end{bmatrix}$$

$$P_N = [B, A_1 B, \dots, A_1^{N-1} B] \quad \|X\|^2 = \text{tr}(X^T X).$$

Unlike the method in [5], the matrix A_0 is determined, so that (7) is minimized. This idea can be easily extended to causal case [5].

The filter, (2), to approximate the finite bisequence $\{F_{ij}|i=1,2,\dots,N; j=1,2,\dots,M\}$ has the coefficient matrices given by (6) and (8).

III. MAPPING TECHNIQUES

Let $\bar{Q}_1$ and Q_1 be the closed and open first quadrants, respectively, of a 2-D real vector space, respectively ($\bar{Q}$ contains the coordinate axes). Consider an invertible mapping $\psi: \bar{Q}_1 \cap Z^2 \rightarrow Q_1 \cap Z^2$ given by

$$\psi(k, h) = (k+1, h+1), \quad (k, h) \in \bar{Q}_1 \quad (9)$$

where Z is the set of integers. This represents as right and up shifts of the impulse response in the power series expansion. Then, the inverse mapping $\psi^{-1}: Q_1 \cap Z^2 \rightarrow \bar{Q}_1 \cap Z^2$ can be described as

$$\psi^{-1}(i, j) = (i-1, j-1), \quad (i, j) \in Q_1. \quad (10)$$

This transformation can be interpreted in the frequency domain as follows. Let $G(\alpha_1, \alpha_2)$ be the 2-D z -transform of a quarter-plane causal filter having an impulse response $\{G_{kh}|k=0,1,\dots; h=0,1,\dots\}$, i.e.,

$$G(\alpha_1, \alpha_2) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} G_{kh} \alpha_1^{-k} \alpha_2^{-h} \quad (11)$$

and let $F(z_1, z_2)$ in (1) be the corresponding z -transform of the impulse response in the transformed domain. Then,

$$F(z_1, z_2) = z_1^{-1} z_2^{-1} G(\alpha_1, \alpha_2), \quad \text{for } \alpha_1 = z_1 \text{ and } \alpha_2 = z_2 \quad (12)$$

can be obtained. Inversely,

$$G(\alpha_1, \alpha_2) = z_1 z_2 F(z_1, z_2), \quad \text{for } z_1 = \alpha_1 \text{ and } z_2 = \alpha_2 \quad (13)$$

can be written.

Let the support of a 2-D impulse response $G_{kh}, (k, h) \in Z^2$ be defined as $S_G = \{(k, h) | (k, h) \in Z^2, G_{kh} \neq 0\}$, and a cone C^* be

defined as a subset of the 2-D real vector space such that if $(x, y) \in C^*$, then $(\mu x, \mu y) \in C^*$ for all $\mu \geq 0$. Let a causality cone $C_{pr,qt}$ be the intersection of two half planes $\rho_{pr} = \{(x, y) | px + ry \geq 0\}$ and $\rho_{qt} = \{(x, y) | qx + ty \geq 0\}$ [3] where p, r, q , and t are nonnegative integers satisfying $qr - pt = -1$, and (x, y) is an element in the 2-D real vector space. Moreover, let the 2-D impulse response $G_{kh}, (k, h) \in Z^2$ be called weakly causal with causality cone $C_{pr,qt}$ if there exists a cone C^* satisfying [2]

$$C^* \cap (-C^*) = \{(0,0)\}, \quad \bar{Q}_1 \subset C^*, \quad S_G \subset C^* \subset C_{pr,qt}.$$

Suppose that $C_{pr,qt}$ is the causality cone associated with a weakly causal impulse response $G_{kh}, (k, h) \in S_G$. Then, a possible invertible (one-to-one and onto) mapping $\phi: C_{pr,qt} \cap Z^2 \rightarrow \bar{Q}_1 \cap Z^2$, such that the origin is a fixed point $(\phi(0,0) = (0,0))$ is given as [3]

$$\phi(k, h) = (pk + rh, qk + th). \quad (14)$$

From the foregoing, an invertible mapping $\psi\phi: C_{pr,qt} \cap Z^2 \rightarrow Q_1 \cap Z^2$ which is given by

$$\psi\phi(k, h) = (pk + rh + 1, qk + th + 1) \quad (15)$$

can be considered. Then, the inverse mapping $(\psi\phi)^{-1}: Q_1 \cap Z^2 \rightarrow C_{pr,qt} \cap Z^2$ becomes

$$\begin{aligned} (\psi\phi)^{-1}(i, j) &= \phi^{-1}\psi^{-1}(i, j) \\ &= (t(i-1) - r(j-1), -q(i-1) + p(j-1)). \end{aligned} \quad (16)$$

This mapping $\psi\phi$ makes it possible to move the impulse response of a weakly causal filter into the open first quadrant, in such a way that its structure is kept unaffected. In the frequency domain, this transformation can be interpreted as follows. Let $G(s_1, s_2)$ be the 2-D z-transform of a weakly causal impulse response G_{kh} , i.e.,

$$G(s_1, s_2) = \sum_{(k,h) \in S_G} G_{kh} s_1^{-k} s_2^{-h} \quad (17)$$

and let $F(z_1, z_2)$ in (1) be the corresponding z-transform of the impulse response in the transformed domain. Then

$$F(z_1, z_2) = z_1^{-1} z_2^{-1} G(s_1, s_2), \quad \text{for } s_1 = z_1^p z_2^q \text{ and } s_2 = z_1^r z_2^t \quad (18)$$

is obtainable and inversely

$$G(s_1, s_2) = z_1 z_2 F(z_1, z_2), \quad \text{for } z_1 = s_1^t s_2^{-q} \text{ and } z_2 = s_1^{-r} s_2^p. \quad (19)$$

Letting $G_{kh}, (k, h) \in S_G$ be a 2-D weakly causal impulse response with causality cone $C_{pr,qt}$, an algorithm is summarized as follows.

Step 1: Transfer the data $G_{kh}, (k, h) \in C_{pr,qt}$ to the open first quadrant via the invertible transformation $\psi\phi$:

$$\psi\phi(k, h) = (pk + rh + 1, qk + th + 1)$$

namely, find a 2-D strictly causal impulse response F_{ij} , such that

$$F_{ij} = G_{(\psi\phi)^{-1}(i,j)}, \quad i=1,2,\dots,N; \quad j=1,2,\dots,M.$$

Step 2: Find the approximate filter $\{A_1, A_2, A_0, B, C\}_{n,m}$ specified by (2) for the finite bisequence $\{F_{ij} | i=1,2,\dots,N; j=1,2,\dots,M\}$ by making use of the approximation technique in Section II.

Step 3: The approximate weakly causal filter $\hat{G}_{kh}$ with causality cone $C_{pr,qt}$ is obtained from the $(\psi\phi)^{-1}$ transformation of the strictly causal approximate filter H_{ij} , i.e.,

$$\hat{G}_{kh} = H_{\psi\phi(k,h)}, \quad (k, h) \in C_{pr,qt}$$

where

$$H_{ij} = CA_2^{-1}A_0A_1^{-1}B, \quad i, j=1,2,\dots.$$

The approximate weakly causal filter can be written in the frequency domain as

$$\hat{G}(s_1, s_2) = s_1^{t-r} s_2^{p-q} C [s_1^{-r} s_2^p I_m - A_2]^{-1} A_0 [s_1^t s_2^{-q} I_n - A_1]^{-1} B. \quad (20)$$

This can be expressed in the spatial domain as

$$\begin{aligned} \begin{bmatrix} x^h(i+t, j-q) \\ x^v(i-r, j+p) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ A_0 & A_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(i, j) \\ y(i, j) &= [0 \quad C] \begin{bmatrix} x^h(i+t-r, j+p-q) \\ x^v(i+t-r, j+p-q) \end{bmatrix} \end{aligned} \quad (21)$$

where $pi + rj \geq 0$, $qi + tj \geq 0$, and $pt - qr = 1$. This state-space model can be rewritten as

$$\begin{aligned} \begin{bmatrix} \tilde{x}^h(i+t, j-q) \\ \tilde{x}^v(i-r, j+p) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ A_0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{x}^h(i, j) \\ \tilde{x}^v(i, j) \end{bmatrix} + \begin{bmatrix} A_1 B \\ A_0 B \end{bmatrix} u(i, j) \\ y(i, j) &= [CA_0 \quad CA_2] \begin{bmatrix} \tilde{x}^h(i, j) \\ \tilde{x}^v(i, j) \end{bmatrix} + CA_0 B u(i, j) \end{aligned} \quad (22)$$

where

$$\tilde{x}^h(i, j) = A_1 x^h(i, j) \quad \tilde{x}^v(i, j) = x^v(i+t, j-q).$$

Notice that the same arguments can be exactly applied to the filter:

$$\begin{aligned} \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= \begin{bmatrix} A_1 & A_0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(i, j) \\ y(i, j) &= [C \quad 0] \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}, \quad i, j \geq 0 \end{aligned} \quad (23)$$

instead of the filter specified by (2). This is due to duality.

IV. ILLUSTRATIVE EXAMPLES

Example 1: Quarter-Plane Gaussian Filter

In this example [1], [5] the problem is to approximate the impulse response of the "Gaussian Filter" given by

$$G_{kh} = 0.256322 \exp \left[-0.103203 \{ (k-4)^2 + (h-4)^2 \} \right]$$

where the support S_G is

$$S_G = \{ (k, h) | k=0,1,\dots,10; h=0,1,\dots,10 \}.$$

For this example, the causality cone is $C_{10,01}$, i.e., $p=t=1$ and $r=q=0$.

Moving the Gaussian filter into the open first-quadrant using the invertible mapping ψ , (9), gives

$$F_{ij} = 0.256322 \exp \left[-0.103203 \{ (i-5)^2 + (j-5)^2 \} \right]$$

where the support S_F becomes

$$S_F = \{ (i, j) | i=1,2,\dots,11; j=1,2,\dots,11 \}.$$

Fig. 1. The impulse response of the Gaussian filter defined over the closed first-quadrant where $(n, m) = (3, 3)$.

In this case, decomposing the corresponding Hankel matrices, (4), using singular value decomposition, the two sets of singular values σ_k and δ_k , $k=1, 2, \dots$, were given by

$$\sigma_1 = \delta_1 = 2.04444$$

$$\sigma_2 = \delta_2 = 0.86523$$

$$\sigma_3 = \delta_3 = 0.24604$$

$$\sigma_4 = \delta_4 = 0.05270$$

$$\sigma_5 = \delta_5 = 0.00992$$

$$\sigma_6 = \delta_6 = 0.00362$$

$$\sigma_7 = \delta_7 = 0.00017$$

where the two sets of singular values are identical since $H_{ij} = H_{ji}$. These values differ from those in [5], because of the different data structure of Hankel matrices. Then, the approximate model of order (3,3) was given by (21) and (22) with $p=t=1$, $r=q=0$ and

$$A_1 = A_2 = \begin{bmatrix} 0.8957902 & -0.2591658 & -0.0397439 \\ 0.2591732 & 0.6627084 & -0.3668608 \\ -0.0397601 & 0.3670085 & 0.4278232 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 2.1584389 & 2.0631170 & 0.9911916 \\ 3.2918956 & 3.1465175 & 1.5116940 \\ 2.3648684 & 2.2604300 & 1.0859875 \end{bmatrix}$$

$$B = C^T = \begin{bmatrix} -0.5876788 \\ 0.5609889 \\ -0.2702164 \end{bmatrix}$$

The filter impulse response in this case is shown in Fig. 1. The other results of the application of the algorithm to this example are summarized in comparison with [1], [5] in Table I where

$$\epsilon_2 = \frac{\left\{ \sum_{(i,j) \in S_F} (F_{ij} - H_{ij})^2 \right\}^{1/2}}{\left\{ \sum_{(i,j) \in S_F} F_{ij}^2 \right\}^{1/2}} \times 100$$

$$\epsilon_\infty = \frac{\max_{(i,j) \in S_F} |F_{ij} - H_{ij}|}{\max_{(i,j) \in S_F} |F_{ij}|} \times 100$$

and the number of parameters is given in terms of a rational transfer function.

Table I shows that the resulting filter does not have a better approximation than that in [5] in the case where the filter order is

TABLE I
ERROR ANALYSIS OF SOME DESIGNS

	Realization Order	ϵ_2	ϵ_∞	Maximum Negative Ripple in Support	Number of Parameters
Our Designs	(3,3)	5.52	6.07	always > 0	15
	(4,4)	1.04	1.10	always > 0	24
Design in [1]	(4,2)	10.78	9.19	-0.04479	29
Design in [5]	(3,3)	2.92	3.87	always > 0	22

the same as in [5]. This is due to more restrictive structure. However, if the number of parameters is nearly equal to that in [5], the resulting filter has better approximation than that in [5]. In general, the filter in [1] has many parameters because of its more general structure.

Example 2: Asymmetric Half-Plane Gaussian Filter

In this example [1], [2], the problem is to approximate the impulse response of the asymmetric half-plane Gaussian filter given by

$$G_{kh} = 0.256322 \exp \left[-0.103203 \{ (k-5)^2 + h^2 \} \right],$$

$$(k, h) \in S_G$$

where the support S_G is

$$S_G = \{ (k, h) | 0 \leq k \leq 12; -k \leq h \leq 16-k \}.$$

For this example, the causality cone is $C_{10,11}$, i.e., $p=q=t=1$ and $r=0$.

Equation (15) was used to move the data G_{kh} , $(k, h) \in C_{10,11}$ into the open first quadrant, and the resultant 2-D strictly causal impulse response $\{F_{ij} | (i, j) \in S_F\}$ was found by referring to (1) where the support S_F is

$$S_F = \{ (i, j) | 1 \leq i \leq 13; 1 \leq j \leq 17 \}.$$

Let the corresponding Hankel matrices, (4), be decomposed using singular value decomposition. Then, the two sets of singular values σ_k and δ_k , $k=1, 2, \dots$, were obtained as

$$\sigma_1 = 1.96867 \quad \delta_1 = 2.11738$$

$$\sigma_2 = 1.24200 \quad \delta_2 = 1.10604$$

$$\sigma_3 = 0.66248 \quad \delta_3 = 0.49593$$

$$\sigma_4 = 0.31901 \quad \delta_4 = 0.20446$$

$$\sigma_5 = 0.14227 \quad \delta_5 = 0.07954$$

$$\sigma_6 = 0.05950 \quad \delta_6 = 0.02956$$

$$\sigma_7 = 0.02347 \quad \delta_7 = 0.01056$$

$$\sigma_8 = 0.00874 \quad \delta_8 = 0.00364$$

$$\sigma_9 = 0.00310 \quad \delta_9 = 0.00122$$

$$\sigma_{10} = 0.00116 \quad \delta_{10} = 0.00044$$

$$\sigma_{11} = 0.00069 \quad \delta_{11} = 0.00026$$

$$\sigma_{12} = 0.00061 \quad \delta_{12} = 0.00022$$

$$\sigma_{13} = 0.00059 \quad \delta_{13} = 0.00021.$$

In this case, the approximate model of order (4,3) was given by

TABLE II
ERROR ANALYSIS OF SOME DESIGNS

	Realization Order	$\bar{\epsilon}_2$	$\bar{\epsilon}_\infty$	$\bar{\epsilon}_-$	Number of Parameters
Our Designs	(3,3)	28.17	24.03	-0.03454	15
	(4,3)	18.48	16.50	-0.02345	19
	(4,4)	13.49	13.82	-0.01597	24
Design in [1]	(4,2)	25.18	22.57	-0.04732	29
Design in [2]	(3,3)	21.14	20.63	-0.0304	22

(21) or (22) with $p = q = t = 1$, $r = 0$,

$$A_1 = \begin{bmatrix} 0.9159366 & -0.2674079 & 0.0213890 & -0.0192728 \\ 0.2887740 & 0.7287339 & -0.3675505 & 0.0693925 \\ -0.0358007 & 0.4423472 & 0.5330255 & -0.3927384 \\ 0.0524080 & -0.550136 & 0.5324059 & 0.3547614 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.9161460 & -0.2169250 & 0.0223083 \\ 0.2444246 & 0.7681495 & -0.2879143 \\ -0.0557063 & 0.3645970 & 0.6326041 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.4772919 \\ 0.5719763 \\ -0.4216764 \\ 0.2457270 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0.4371944 & 0.9474431 & 1.3082020 & 1.0360237 \\ 0.7880140 & 1.4413617 & 1.3462039 & 0.5629451 \\ 0.7548796 & 0.7345584 & -0.0125696 & -0.4788091 \end{bmatrix}$$

$$C = [-0.5365690 \quad 0.5340841 \quad -0.3504396]$$

and the impulse response is shown in Fig. 2. The other results of the application of the algorithm to this example are summarized in comparison with [1], [2] in Table II where the number of parameters is given in terms of a rational transfer function and

$$\bar{\epsilon}_2 = \frac{\left[\sum_{(k,h) \in S_G} w_{kh} (G_{kh} - \hat{G}_{kh})^2 \right]^{1/2}}{\left[\sum_{(k,h) \in S_G} w_{kh} G_{kh}^2 \right]^{1/2}} \times 100$$

$$\bar{\epsilon}_\infty = \frac{\max_{(k,h) \in S_G} w_{kh} |G_{kh} - \hat{G}_{kh}|}{\max_{(k,h) \in S_G} w_{kh} |G_{kh}|} \times 100$$

$$\bar{\epsilon}_- = \text{Max Negative Ripple of } \hat{G}_{kh} \text{ in Region } S_G$$

Fig. 2. The impulse response of the Gaussian filter defined over an asymmetric half-plane where $(n, m) = (4, 3)$.

with

$$w_{kh} = \begin{cases} 1.0, & \forall (k, h) \in \{(k, h) | G_{kh} \leq 0.03\} \cap S_G \\ 0.6, & \forall (k, h) \in \{(k, h) | G_{kh} > 0.03\} \cap S_G. \end{cases}$$

From Table II it is clear that if the number of parameters in the filter is nearly equal to that in [2], the resulting filter has better approximation than that in [2].

V. CONCLUSION

A technique has been presented for approximating asymmetric half-plane digital filters using a strictly causal filter with separable denominator. This is done via two mapping techniques and singular value decomposition of two finite Hankel matrices. The order of the approximating filter can be reasonably chosen by checking the singular values and since the rational transfer function has separable denominator in the transformed domain (open first quadrant), the stability test is trivial. Two illustrative examples have demonstrated that the approximation using a causal separable denominator filter [2], [5] is better than the one using a strictly causal separable denominator filter provided they have the same order. This is owing to the more restrictive structure, i.e., less number of independent parameters in a strictly causal filter. However, these examples have also showed that the reverse is true if both filters have almost the same number of independent parameters. This implies that the number of denominator parameters is more important in the approximation provided the total number of independent parameters is roughly fixed. As a result, a proposed technique is more effective and the implementation is often simpler. Unlike the causal case [5], the matrix CA_0B in (22) is not null in general even if the corresponding term is null in the original 2-D digital filter. This is due to the more restrictive structure of the filter. Note that the Attasi model can be imbedded in the filter treated here [7], [11].

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