### THE CRITICAL STATE LINE OF NON-PLASTIC TAILINGS

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THE CRITICAL STATE LINE OF NON-PLASTIC TAILINGS

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Abstract

The probability of failure of tailing dams and associated risks demand improvements in engineering practice. The critical state line provides a robust framework for the characterization of mine tailings. New experimental data for non-plastic platinum tailings and a large database for tailings and non-plastic soils (grain size between 2\(\mu\)m and 500\(\mu\)m) show that the critical state parameters for non-plastic tailings follow the same trends as non-plastic soils as a function of particle-scale characteristics and extreme void ratios. Critical state lines determined for extreme tailings gradations underestimate the range of critical state parameters that may be encountered in a tailings dam; in fact, mixtures with intermediate fines content exhibit the densest granular packing at critical state. The minimum void ratio \(e_{\text{min}}\) captures the underlying role of particle shape and grain size distribution on granular packing, and emerges as a valuable index property to inform sampling strategies for the assessment of spatial variability. Mineralogy does not significantly affect the intercept \(\Gamma_{100}\) but it does affect the slope \(\lambda\). The friction coefficients \(M\) of tailings are similar to those of other non-plastic soils; while mineralogy does not have a significant effect on friction, more angular grains lead to higher friction coefficients.

Key words: Tailings, critical state line, minimum void ratio, non-plastic soils.
Introduction

The historical failure rate of tailing dams and the associated risks demand improvements in material characterization, the design of the containment structure, construction practices and monitoring technology (Santamarina et al. 2019). The critical state line CSL provides a robust frame of reference for the assessment of mine tailings (Carrera et al. 2011; Castro et al. 1982; Fourie and Papageorgiou 2001; Jefferies and Been 2015; Li et al. 2018; Vermeulen 2001). The critical state line swings through the confinement-shear-volume space $p'-q-e$ and generates two projections. The $q-p'$ projection is the linear Coulomb strength model

$$q = Mp'$$

where the $M$ factor captures the frictional strength. The $e-p'$ projection (Figure 1) follows a semi-logarithmic trend

$$e = \Gamma_{p'} - \lambda \log_{10} \left( \frac{p'}{p^*} \right)$$

where $e$ is the void ratio, $\lambda$ is the slope, $p^*$ is an arbitrary reference stress, and the intercept $\Gamma_{p'}$ is the void ratio when $p' = p^*$. This study adopts $p^* = 100$ kPa to anchor the linear approximation in Equation 2 around stress levels that are relevant to field conditions. Although the reference stress $p^*$ does not affect the critical state line in itself, it does affect the strength of the correlations between $\Gamma_{p'}$ and other parameters (Torres-Cruz 2019). This manuscript adopts the classical definitions of the mean effective stress $p' = (\sigma'_1 + \sigma'_2 + \sigma'_3)/3$ and deviator stress $q = \sigma'_1 - \sigma'_3$. 

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There is strong lateral and vertical variability in grain size distribution within tailings dams (Carrera et al. 2011; Fourie and Papageorgiou 2001; Li 2017). Then, the proper selection of tailings samples is crucial to critical state characterization. One approach is to test the extreme, coarsest and finest, gradations in the deposit (Jefferies and Been 2015). However, this approach may be misleading as illustrated by multiple critical state studies of sand-silt mixtures (Papadopoulou and Tika 2008; Rahman and Lo 2008; Thevanayagam et al. 2002; Yang et al. 2006; Zlatović and Ishihara 1995): the critical state line CSL shifts downwards (a reduction in $\Gamma_{p*}$) as the silt content increases from 0% to approximately 30%, but it shifts upwards (an increase in $\Gamma_{p*}$) as the silt content exceeds 30%. Therefore, analyses based on the two extreme gradations underestimate the range of critical states present at a non-plastic tailings dam.

Previous studies report a correlation between the intercept $\Gamma_{100}$ and the minimum void ratio $e_{\text{min}}$ for non-plastic soils (Cho et al. 2006; Torres-Cruz 2019). In addition, there is some correlation (1) between the slope $\lambda$ and the gap between extreme void ratios ($e_{\text{max}}-e_{\text{min}}$), i.e., the volumetric compression potential (Cho et al. 2006; Cubrinovski and Ishihara 2000), and (2) between $\lambda$ and $e_{\text{min}}$, i.e., $e_{\text{min}}$ alone can provide some information on volumetric compressibility (Torres-Cruz 2019). Overall, these trends capture the underlying role of particle shape and grain size distribution on granular packing and extreme void ratios $e_{\text{max}}$ and $e_{\text{min}}$ (Cho et al. 2006; Torres-Cruz 2019). Traditionally, extreme void ratios are used to characterize coarse grained soils, i.e., retained on sieve #200. However, they can be equally useful to characterize the behavior of non-plastic silts (Lade et al. 1998; Carrera et al. 2011; Park and Santamarina 2017; Li and Coop 2018; Torres-Cruz 2019).

The scatter in the trends between critical state parameters and extreme void ratios may be too high for reliable predictions of $\Gamma_{100}$ and $\lambda$. However, these correlations suggest the possibility
that an easily measured index property could assist in assessing the potential spatial variability of critical state parameters in non-plastic tailings, such as those that result from the extraction of gold, platinum, copper, and iron (Bedin et al. 2012; Li et al. 2018; Li 2017; Li and Coop 2018; this work – Note: kimberlite and some types of iron tailings can exhibit considerable plasticity).

This study explores the determination of the critical state line of non-plastic platinum tailings, analyzes results in the context of a large database of CSLs compiled from the literature (25 non-plastic tailings and 132 non-plastic soils), and seeks to identify trends between critical state parameters ($
\Gamma_{100}$, $\lambda$ and $M$) and index properties that can be used to readily assess spatial variability in tailings dams.

**Triaxial testing program**

*Materials and methods: Platinum tailings*

Platinum tailings were sourced from the upper beach of an upstream spigotted tailings dam located in the North West Province of South Africa. The tailings are non-plastic, angular shaped (Figure 2) and have a complex mineralogy dominated by enstatite (27%), bytownite (27%), chromite (21%), and hornblende (8%) with traces (<5%) of diopside, epidote, talc, and biotite, among other minerals (Amponsah-Dacosta 2017).

The laboratory investigation centers around six mixtures prepared with pre-sieved tailings fractions. The mixtures have a fines content that ranges from FC = 10 to 98% (Figure 3 and Table 1 – Note: fines are < 75 μm). This range of fines content covers field observations (from 33% to 95% - Torres-Cruz 2016). This study follows two methods to measure $e_{\text{min}}$: the standard...
ASTM D1557 and a non-standard method that benefits from a small sample size (similar to that proposed by Lade et al. 1998, as described in Torres-Cruz 2016). Although not applicable to FC > 15%, this study involved the ASTM D4254 to measure $e_{\text{max}}$ due to a lack of a standardized alternative. Figure 4 presents $e_{\text{max}}$ and $e_{\text{min}}$ values for the six mixtures, and shows similar trends for $e_{\text{min}}$ values gathered with the two methods. Clearly, the extreme specimens (FC = 10% and 98%) exhibit the highest $e_{\text{max}}$ and $e_{\text{min}}$. Conversely, the lowest values correspond to mixtures with 30% ≤ FC ≤ 47%, in agreement with other studies of sand-silt mixtures (Papadopoulou and Tika 2008; Rahman and Lo 2008; Thevanayagam et al. 2002; Yang et al. 2006; Zlatović and Ishihara 1995; Park and Santamarina 2017).

Based on these results, critical state testing focused on three mixtures with distinctly different $e_{\text{min}}$ values: FC = 10%, FC = 30%, and FC = 81%. The preparation of triaxial test specimens (70 mm diameter, 141 mm height) involved moist tamping to achieve loose contractive specimens (Ishihara 1993) in order to minimize strain localization (Jefferies and Been 2015 – Note: ends were not lubricated as non-uniform radial strains may remain; refer to Rees 2010). Air-CO$_2$ replacement prior to water injection and back-pressure produced high degrees of saturation; in fact, all specimens exhibited a Skempton’s parameter $B \geq 0.96$. The stress path consisted of isotropic consolidation applied in a single stage followed by monotonic shearing, either under drained CD or undrained CU conditions. The displacement-controlled loading frame applied the deviatoric stress at constant cell pressure. Water content measurements at the end of the test enabled void ratio determinations (refer to Verdugo and Ishihara 1996). Table 2 presents a summary of the testing program (additional experimental details in Torres-Cruz 2016).
Results

Figure 5 illustrates typical response curves obtained for platinum tailings specimens subjected to drained and undrained loading in terms of deviatoric stress $q$, pore pressure $u$, and void ratio $e$ vs. vertical strain $\varepsilon_a$. The hyperbolic model fitted to the pre-peak portion of the deviatoric stress vs. axial strain curve $q$-$\varepsilon_a$ corrected early seating and misalignment effects and identified the true start of loading (Bishop and Henkel 1957).

On the other end, most specimens do not reach stable pore pressure $u$ (CU tests) or void ratio $e$ (CD tests) within the strain level attainable in triaxial tests (Figure 5). Several authors have suggested extrapolation procedures that allow for improved estimates of critical state conditions (Li 2017; Murthy et al. 2007). The following criteria were implemented in this study (Figure 6 - refer to Carrera et al. 2011): (a) extrapolate $q/p'$ to $\delta u/\delta \varepsilon_a = 0$ for undrained tests, (b) extrapolate $q-p'$ trends to asymptotic $M$, and (c) extrapolate void ratio to the critical state condition of zero dilatancy $\delta \varepsilon_v/\delta \varepsilon_a = 0$ for drained tests. Table 2 summarizes the critical state values determined for all tests.

Figure 7 shows the critical states for all CD and CU tests for the three selected mixtures. The CSLs on the $e$-$p'$ projection follow the semi-log linearization and are distinctly different for the three mixtures (Figure 7a). More importantly, the intercept $\Gamma_{100}$ is lowest for the mixture with fines content FC=30%, in agreement with extreme void ratio trends (Figure 4). Figure 7b shows the $q$-$p'$ projections of the CSLs; the computed $M$ values for the three mixtures fall within a narrow range of $M = 1.27 \pm 0.02$. This is consistent with previous results which show that $M$ is largely independent of grain size distribution (Bandini and Coop 2011; Carrera et al. 2011; Li et al. 2015). Table 1 lists the values of $\Gamma_{100}$, $\lambda$, and $M$ for each mixture. Distinctly different $e$-$p'$ projections but indistinguishable $q$-$p'$ projections imply that the dilative tendency of a specimen...
at an initial \( e_o \) and \( p_o \) will depend on its gradation, yet, gradation will not affect the stress ratio the specimen will reach at critical state.

**Insights from the database**

**Data sources**

The following analyses take into consideration the critical state lines of the three platinum mixtures described above, 25 tailings reported in the literature (Table 3), and 132 non-plastic soils collected from published studies (including natural soils and material from rock crushers - Torres-Cruz 2019). Figure 8 illustrates the range in particle size distributions –between 2\( \mu \) and 500\( \mu \)– for tailings in the database.

**Uncertainty in \( \Gamma_{100} \) and \( \lambda \).**

The least squares solution identifies model parameters by minimizing the sum of squared errors \( \text{SSE} = \sum (e_m - e_p)^2 \) between measured \( e_m \) and predicted \( e_p \) void ratios at critical state. Let’s consider critical state data for six tailings, and: (1) identify the optimal \( \Gamma_{100} \) and \( \lambda \) values, then (2) vary either \( \Gamma_{100} \) or \( \lambda \) to compute slices of the error surface across the optimum set (Santamarina and Fratta 2006). The horizontal axes in Figure 9 cover the range of possible \( \Gamma_{100} \) or \( \lambda \) values for non-plastic soils, as inferred from the database. The rate of convergence towards the minimum square error limits the accurate computation of the intercept \( \Gamma_{100} \) and slope \( \lambda \) for a given error. That is, the rate of convergence is a measure of invertibility; thus, Figures 9a and 9b show that the intercept \( \Gamma_{100} \) is better inverted than the slope \( \lambda \).

Alternatively, standard errors \( SE_{\lambda} \) and \( SE_{\Gamma_{100}} \) reflect the uncertainty in \( \lambda \) and \( \Gamma_{100} \) through
\[ SE_{\lambda} = \sqrt{\frac{\sum (e_p - e_m)^2}{N - 2}} \]

(3)

and

\[ SE_{\Gamma_{100}} = SE_{\lambda} \sqrt{\frac{\sum \left(\frac{p'_m}{p'_{avg}}\right)^2}{N}} \]

(4)

where \( N \) is the number of data points, and sub-indices stand for \( m = \text{measured} \) and \( \text{avg} = \text{average} \) (Navidi 2015). The normalizations \( NSE_{\lambda} = SE_{\lambda}/\text{range}(\lambda) \) and \( NSE_{\Gamma_{100}} = SE_{\Gamma_{100}}/\text{range}(\Gamma_{100}) \) facilitate the analysis. Figure 10 compares the normalized standard errors \( NSE_{\Gamma_{100}} \) and \( NSE_{\lambda} \) computed for 91 critical state lines with known \( e-p' \) data points. The normalized error of the slope \( NSE_{\lambda} \) is 3 to 12 times larger than the normalized error of the intercept \( NSE_{\Gamma_{100}} \) in both tailings and non-plastic soils.

**Critical state parameters and index properties**

Previous studies showed that particle-scale characteristics such as shape and grain size distribution affect index properties and critical state parameters (Torres-Cruz 2019; Cho et al. 2006; Cubrinovski and Ishihara 2000). This section explores the correlation between index properties and critical state parameters for tailings and non-plastic soils. The database is dominated by soils that do not exhibit cementation or crushing at effective stress levels of
interest, typically $\sigma' < 1$MPa (Note: see Jung et al. 2012 for the effect of cementation on critical state).

Figure 11 shows that tailings fall along the same $\Gamma_{100} - e_{\text{min}}$ and $\Gamma_{100} - e_{\text{max}}$ trends of other non-plastic soils. There is no clear clustering when the data is discriminated by mineralogy (not shown here). Furthermore the intercept $\Gamma_{100}$ is:

- higher than the minimum void ratio for 96% of the database; the overall trend is $\Gamma_{100} \approx 1.4 \cdot e_{\text{min}}$ thus, the spatial variability of $e_{\text{min}}$ is a good indicator of potential spatial variability of $\Gamma_{100}$ in non-plastic tailings
- lower than the maximum void ratio for 94% of the database, typically around $\Gamma_{100} \approx 0.8 \cdot e_{\text{max}}$.

Note that the spread in $e_{\text{max}}$ is much higher than in $e_{\text{min}}$ ($1.2 \leq e_{\text{max}} / e_{\text{min}} \leq 2.2$, for 90% of the database). This hinders correlations that involve $(e_{\text{max}} - e_{\text{min}})$.

Figure 12a shows that tailings populate the same area as other non-plastic soils in the $\lambda - e_{\text{min}}$ space. The range of possible slope values $\lambda$ increases with $e_{\text{min}}$, in other words, the potential contractibility at critical state decreases for soils with low $e_{\text{min}}$; in fact $\lambda < (e_{\text{min}} / 3)$ for 95% of the database. High hardness soils (e.g. quartz and silica) exhibit lower slopes $\lambda$, mostly in the range of $\lambda < 0.10$ (Figure 12b). Therefore, hardness affects contractiveness at critical state even though most of the data corresponds to intermediate stress levels where marked crushing is unlikely (all but one of the CSLs are defined for $p' < 4$ MPa). The scatter in Figure 12 reflects the complex interactions between grain size distribution, particle shape and mineralogy. The scatter also reflects the limited invertibility of the slope $\lambda$ (refer to Figures 9 and 10). There is no trend or clustering in plots of slope $\lambda$ versus void ratio gap $(e_{\text{max}} - e_{\text{min}})$; however, note that the contraction in one log cycle is $\lambda < (e_{\text{max}} - e_{\text{min}})$ for the entire database.
The relative contractiveness $R_c$ shows the position of the CSL between two extreme density conditions (Verdugo and Ishihara 1996):

\[
R_c = \frac{(e_{\text{max}})_{100} - \Gamma_{100}}{(e_{\text{max}})_{100} - (e_{\text{min}})_{100}}
\]

where $(e_{\text{max}})_{100}$ and $(e_{\text{min}})_{100}$ are the void ratios attained by isotropically compressing the soil to 100 kPa from its loosest $e_{\text{max}}$ and densest $e_{\text{min}}$ conditions. The values of $(e_{\text{max}})_{100}$ and $(e_{\text{min}})_{100}$ are unknown for most soils, therefore, the estimate of $R_c$ uses $e_{\text{max}}$ and $e_{\text{min}}$. The computed relative contractiveness values for the database spreads the full $R_c = 0$ to 1 range (not shown here). Tailings cannot be distinguished from other non-plastic soils. There is a slight tendency for rounded soils to exhibit lower $R_c$ than angular soils, and there is a weak inverse trend between $R_c$ and $D_{50}$.

The database allows us to explore correlations with particle size. In particular:

- the maximum void ratio $e_{\text{max}}$ is independent of particle size while $D_{50} > 100$ μm, but it increases for finer particles probably due to electrostatic interactions (Figure 13a)
- the minimum void ratio $e_{\text{min}}$ and the intercept $\Gamma_{100}$ are independent of particle size for all entries in the database (10 μm < $D_{50}$ < 2000 μm - Figures 13b and 13c); indeed, the effect of electrostatic interactions diminishes for the high energy conditions that prevail in $e_{\text{min}}$ testing and under $p' = 100$ kPa (Santamarina 2003).

Figure 13d presents the friction parameter $M$ vs the mean grain size $D_{50}$. The $M$-value varies widely for tailings from $M = 1.1$ to $M = 1.8$, in a similar range to non-plastic soils. The
associated variation in critical state friction angle is from \( \phi_{cs} = 27.7^\circ \) to 43.8\(^\circ\) (Note: for reference, \( M \approx 0.9 \) or \( \phi_{cs} \approx 23^\circ \) for glass beads). Previous studies have shown that mineralogy and particle angularity largely define the value of \( M \) (Cho et al. 2006; Sadrekarimi and Olson 2011; Li et al. 2015; Yang and Luo 2015). However, the database shows significant overlap: the \( M \)-value of angular soils is approximately 14\% larger than for rounded soils (Figure 13d), and the data do not reveal any distinct effect of mineralogy on \( M \).

Figures 11, 12 and 13 are based on published data produced by a large number of laboratories around the world. Therefore, there are potential differences in test protocols and data interpretation, including inconsistent assessment of particle shape and mineralogy. Then, apparent discrepancies between these results and previously reported observations from focused studies highlight the need for consistent assessment and reporting of soil properties.

Conclusions

This study explored critical state parameters for South African platinum tailings, identified critical state data for 25 other tailings, and considered additional data for 132 non-plastic soils previously reported in the literature. Inherent limitations in the strain levels attainable in triaxial tests hindered the determination of critical state; therefore, robust extrapolation strategies helped define consistent critical states for both drained and undrained tests.

The complete dataset allows the comparison of tailings to other non-plastic soils, and the identification of trends between critical state parameters and index properties. Salient conclusions follow:
1. Data of void ratio vs. confinement $e-p'$ at critical state extracted from triaxial test data allow the estimation of the intercept $\Gamma_{100}$ at $p'=100$ kPa with better accuracy than the slope $\lambda$.

2. The critical state parameters for non-plastic tailings fall on the same trends as data for a wide range of non-plastic soils. This suggests that inferences about soil behavior made from non-plastic soils (non tailings) can be reasonably adopted for non-plastic tailings provided that these inferences account for potential differences in particle shape, grain size distribution, and mineralogy.

3. Mineralogy does not significantly affect the intercept $\Gamma_{100}$ and related correlations, but it does affect the slope $\lambda$ even when tests are conducted at intermediate stress levels below crushing. In fact, almost all high hardness soils (quartz and silica) have $\lambda < 0.10$, while soils with other mineralogies and variable hardness span the entire range of $\lambda$.

4. Published data and new experimental results show that the intercept $\Gamma_{100}$ as well as the value of $e_{\text{min}}$ are lowest for mixtures with intermediate fines content, typically around FC$\approx$30%. Therefore, the assessment of field conditions based on extreme gradations underestimates the range of critical state parameters in the tailings dam.

5. The spatial variability of $e_{\text{min}}$ within a tailings deposit anticipates spatial variability in $\Gamma_{100}$. Additionally, the range of possible $\lambda$ values increases with $e_{\text{min}}$. These observations can be used to inform sampling strategies.

6. The range of the friction coefficient $M$ of tailings is similar to the range of values exhibited by other non-plastic soils. On average, angular soils exhibit greater friction coefficient $M$ than rounded soils, in agreement with previous studies. Mineralogy and mean particle size do not appear to significantly influence $M$. 
Acknowledgements

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ASTM. 2002. D1557-02: Standard test methods for laboratory compaction characteristics of soil using modified effort (56,000 lb-lbf/ft$^3$ (2,700 kN-m/m$^3$)). ASTM International, West Conshohocken, PA, USA.


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List of common symbols

\[\Gamma_p^*\] Critical state void ratio at a mean effective stress of \(p^*\)
\[\varepsilon_a\] Axial strain
\[\varepsilon_v\] Volumetric strain
\[\lambda\] Idealized slope of the CSL in \(e-\log_{10}(p'/p^*)\) space
\[\sigma_i'\] Effective principal stress in the \(i\) direction
\[\varphi_{cs}\] Critical state friction angle
CD Consolidated drained
CU Consolidated undrained
CSL Critical state line
\[D_{50}\] Mean particle size
\[e\] Void ratio
\[e_{\text{min}}\] Minimum void ratio
\[e_{\text{max}}\] Maximum void ratio
\((e_{\text{max}})_{100}\) Void ratio attained after isotropic consolidation to 100 kPa from \(e_{\text{max}}\)
\((e_{\text{min}})_{100}\) Void ratio attained after isotropic consolidation to 100 kPa from \(e_{\text{min}}\)
\[e_m\] Measured void ratio
\[e_p\] Predicted void ratio
FC Fines content (percentage passing 75 \(\mu\)m sieve)
\[M\] Critical state friction ratio
\[N\] Number of data points
\[\text{NSE}_{\lambda}\] Normalized standard error of the slope \(\lambda\)
\[\text{NSE}_{\Gamma_{100}}\] Normalized standard error of the intercept \(\Gamma_{100}\)
\[p^*\] Arbitrary reference stress used to model the CSL in \(e-\log_{10}(p'/p^*)\) space
\[p'\] Mean effective stress
\[p'_{m}\] Measured mean effective stress
\[p'_{\text{avg}}\] Average mean effective stress
\[q\] Deviator stress
\[R_c\] Relative contractiveness
SSE Sum of squared errors
\[\text{SE}_{\lambda}\] Standard error of the slope \(\lambda\)
\[\text{SE}_{\Gamma_{100}}\] Standard error of the intercept \(\Gamma_{100}\)
\[u\] Pore water pressure
Table 1. Properties of mixtures of platinum tailings.

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<th>FC (%)</th>
<th>D&lt;sub&gt;50&lt;/sub&gt; (μm)</th>
<th>C&lt;sub&gt;u&lt;/sub&gt;&lt;sup&gt;a&lt;/sup&gt;</th>
<th>G&lt;sub&gt;s&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</th>
<th>e&lt;sub&gt;min&lt;/sub&gt;&lt;sup&gt;c&lt;/sup&gt;</th>
<th>e&lt;sub&gt;max&lt;/sub&gt;&lt;sup&gt;d&lt;/sup&gt;</th>
<th>e&lt;sub&gt;max&lt;/sub&gt;&lt;sup&gt;e&lt;/sup&gt;</th>
<th>Γ&lt;sub&gt;100&lt;/sub&gt;</th>
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<td>0.70</td>
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<td>81</td>
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a) Coefficient of uniformity (D<sub>60</sub>/D<sub>10</sub>)
b) Specific gravity, ASTM D854
c) Non-standard procedure
d) ASTM D1557
e) ASTM D4254
Table 2. Critical state values and end of test conditions of triaxial tests on platinum tailings.

<table>
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<th>FC (e\text{\textsubscript{min}})</th>
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<td></td>
<td></td>
<td></td>
<td>e</td>
<td>p' (kPa)</td>
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<tr>
<td>Low 10% (0.70)</td>
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<td>CU</td>
<td>1.020</td>
<td>177</td>
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<td>CU</td>
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<td>0.813</td>
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</tr>
<tr>
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<td>CU</td>
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</tr>
<tr>
<td></td>
<td>30-6\textsuperscript{b}</td>
<td>CD</td>
<td>0.802</td>
<td>285</td>
</tr>
<tr>
<td></td>
<td>30-7\textsuperscript{b}</td>
<td>CD</td>
<td>0.773</td>
<td>502</td>
</tr>
<tr>
<td>High 81% (0.63)</td>
<td>81-1\textsuperscript{b}</td>
<td>CU</td>
<td>0.885</td>
<td>326</td>
</tr>
<tr>
<td></td>
<td>81-2\textsuperscript{b}</td>
<td>CU</td>
<td>0.889</td>
<td>335</td>
</tr>
<tr>
<td></td>
<td>81-3\textsuperscript{b}</td>
<td>CU</td>
<td>0.891</td>
<td>335</td>
</tr>
<tr>
<td></td>
<td>81-4</td>
<td>CU</td>
<td>0.916</td>
<td>119</td>
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<td></td>
<td>81-5</td>
<td>CD</td>
<td>0.879</td>
<td>448</td>
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<td></td>
<td>81-6\textsuperscript{b}</td>
<td>CU</td>
<td>0.914</td>
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<td>81-7</td>
<td>CD</td>
<td>0.892</td>
<td>239</td>
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\textsuperscript{a} Sign convention: positive for contraction and negative for dilation
\textsuperscript{b} Extrapolation scheme applied
\textsuperscript{c} Average M value of other tests done on the same mixture
Table 3. Properties of tailings compiled from published studies.

<table>
<thead>
<tr>
<th>Name (FC)</th>
<th>Symbol</th>
<th>$D_{50}$ (μm)</th>
<th>$C_u$</th>
<th>$e_{\text{min}}$</th>
<th>$e_{\text{max}}$</th>
<th>$\Gamma_{100}$</th>
<th>$\lambda$</th>
<th>$M$</th>
<th>$p'$ range (kPa)</th>
<th>$G_s$</th>
<th>Mineralogy</th>
<th>Particle shape</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazilian Gold (65)</td>
<td>BAU</td>
<td>65</td>
<td>6.9</td>
<td>0.90</td>
<td>2.00</td>
<td>0.89</td>
<td>0.206</td>
<td>1.13</td>
<td>100 - 780</td>
<td>3.10</td>
<td>Qtz=27%, Ab=25%, Chl=35%</td>
<td>A-SA</td>
<td>Bedin et al. (2012), Li et al. (2018)</td>
</tr>
<tr>
<td>Brazilian Gold (95)</td>
<td>10</td>
<td>7.7</td>
<td>0.68</td>
<td>1.20</td>
<td>0.89</td>
<td>0.176</td>
<td>1.41</td>
<td>1.43</td>
<td>30 - 900</td>
<td>2.89</td>
<td></td>
<td>A-SA</td>
<td></td>
</tr>
<tr>
<td>Deixing Copper (95)</td>
<td>DCU</td>
<td>30</td>
<td>5.1</td>
<td>0.62</td>
<td>1.28</td>
<td>0.84</td>
<td>0.126</td>
<td>1.43</td>
<td>25 - 3700</td>
<td>3.75</td>
<td>Fa=78%, Mag=22%</td>
<td>A-SA</td>
<td>Li (2017)</td>
</tr>
<tr>
<td>Hilton Mines (2.5)</td>
<td>HMS</td>
<td>200</td>
<td>–</td>
<td>0.62</td>
<td>1.05</td>
<td>0.98</td>
<td>1.17</td>
<td>1.42</td>
<td>50 - 1000</td>
<td>–</td>
<td></td>
<td>A-SA</td>
<td>Jefferies and Been (2015)</td>
</tr>
<tr>
<td>Lornex Copper (7)</td>
<td>HVC</td>
<td>260</td>
<td>2.7</td>
<td>0.68</td>
<td>1.08</td>
<td>1.00</td>
<td>0.264</td>
<td>1.40</td>
<td>70 - 1300</td>
<td>2.68</td>
<td>Qtz=100%, Mica=27%, Ill=15%, Fsp=11%</td>
<td>A-SA</td>
<td>Castro et al. (1982), Robertson et al. (2000), (Fear) Wride et al. (2000)</td>
</tr>
<tr>
<td>Highland Valley Copper (8)</td>
<td>200</td>
<td>2.8</td>
<td>0.54</td>
<td>1.06</td>
<td>0.84</td>
<td>0.068</td>
<td>–</td>
<td>1 - 220</td>
<td>2.66</td>
<td></td>
<td></td>
<td>A-SA</td>
<td></td>
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<tr>
<td>Merriespruit Gold (0)</td>
<td>MER</td>
<td>130</td>
<td>1.9</td>
<td>0.74</td>
<td>1.22</td>
<td>1.11</td>
<td>0.061</td>
<td>1.24</td>
<td>2 - 200</td>
<td>2.7</td>
<td></td>
<td>A-SA</td>
<td>Fourie and Papageorgiou (2001), Papageorgiou (2004), Tshabalala (2003)</td>
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<tr>
<td>Merriespruit Gold (20)</td>
<td>120</td>
<td>4.7</td>
<td>0.70</td>
<td>1.33</td>
<td>1.10</td>
<td>0.157</td>
<td>1.17</td>
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<tr>
<td>Merriespruit Gold (30)</td>
<td>100</td>
<td>30</td>
<td>0.58</td>
<td>1.33</td>
<td>0.90</td>
<td>0.073</td>
<td>1.12</td>
<td>2 - 130</td>
<td>2.7</td>
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<td>A-SA</td>
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</tr>
<tr>
<td>Merriespruit Gold (60)</td>
<td>60</td>
<td>25</td>
<td>0.66</td>
<td>1.83</td>
<td>0.76</td>
<td>0.024</td>
<td>1.19</td>
<td>10 - 125</td>
<td>2.7</td>
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<td>A-SA</td>
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<tr>
<td>Syncrude Oil Sand (3.5)</td>
<td>OIL</td>
<td>207</td>
<td>–</td>
<td>0.54</td>
<td>0.90</td>
<td>0.73</td>
<td>0.065</td>
<td>1.33</td>
<td>10 - 620</td>
<td>2.64</td>
<td>Qtz=90%, Fsp=5%, Kln=5%</td>
<td>SA-SR</td>
<td>Jefferies and Been (2015), Robertson et al. (2000), (Fear) Wride et al. (2000)</td>
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<tr>
<td>Syncrude Mildred Lake Oil Sand (10)</td>
<td>160</td>
<td>2.2</td>
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<td>0.85</td>
<td>0.035</td>
<td>–</td>
<td>1 - 370</td>
<td>2.66</td>
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<td>A-SA</td>
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<tr>
<td>Syncrude Oil Sand (12)</td>
<td>170</td>
<td>2.4</td>
<td>0.52</td>
<td>0.96</td>
<td>0.83</td>
<td>0.062</td>
<td>1.19</td>
<td>10 - 800</td>
<td>2.62</td>
<td>Qtz=95%</td>
<td>A-SA</td>
<td>Sladen and Hanford (1987)</td>
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<tr>
<td>Panzhihua Iron UB (19)</td>
<td>PFE</td>
<td>220</td>
<td>10</td>
<td>0.50</td>
<td>1.10</td>
<td>0.79</td>
<td>0.252</td>
<td>1.41</td>
<td>180 - 11000</td>
<td>3.37</td>
<td>Dii=35%, Lab=30%, Hbl=15%</td>
<td>A-SA</td>
<td>Li (2017), Li and Coop (2018)</td>
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<tr>
<td>Panzhihua Iron MB (68)</td>
<td>35</td>
<td>9.0</td>
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<td>Panzhihua Iron PO (93)</td>
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<td>0.70</td>
<td>1.22</td>
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<td>0.185</td>
<td>1.40</td>
<td>25 - 2000</td>
<td>3.11</td>
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<td>Mizpah Dam Gold (72)</td>
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<td>Vermeulen (2001)</td>
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<td>Pay Dam Gold (77)</td>
<td>25</td>
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<td>Mica=9%</td>
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<td>Witwatersrand Gold UB (41)</td>
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<td>23</td>
<td>–</td>
<td>1.24</td>
<td>–</td>
<td>–</td>
<td>1.56</td>
<td>50 - 600</td>
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<td>Qtz=80%, traces of Ms, Prl, Ill</td>
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<td>Chang et al. (2011)</td>
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<tr>
<td>Witwatersrand Gold MB (56)</td>
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<td>11</td>
<td>–</td>
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<td>–</td>
<td>–</td>
<td>1.78</td>
<td>40 - 210</td>
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<td>Witwatersrand Gold PO (99)</td>
<td>5</td>
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<td>–</td>
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<td>130 - 330</td>
<td>2.75</td>
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<td>Stava Fluorite (0)</td>
<td>STA</td>
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<td>60 - 1100</td>
<td>2.72</td>
<td>Qtz=78%, Cal=10%</td>
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<td>Carrera et al. (2011)</td>
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<td>0.69</td>
<td>0.85</td>
<td>0.63</td>
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<td>1.45</td>
<td>45 - 1350</td>
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<td>2.83</td>
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Figure captions:

Figure 1. Typical critical state line CSL and its idealization in $e$-$\log_{10}(p')$ space. The choice of $\Gamma_{100}$ avoids extrapolation effects.

Figure 2. Scanning electron microscope images of platinum tailings. (a) Coarse, (b) intermediate, and (c) fine particles.

Figure 3. Particle size distribution curves of platinum tailings.

Figure 4. Extreme void ratios vs fines content of platinum tailings.

Figure 5. Representative results of triaxial tests on platinum tailings. Drained tests: (a) deviatoric stress $q$ vs axial strain $e_a$, (b) void ratio $e$ vs $e_a$. Undrained tests: (c) $q$ vs $e_a$, (d) pore pressure $u$ vs $e_a$.

Figure 6. Estimation of critical state parameters: Extrapolation to critical state. Undrained test 81-1: (a) stress ratio $q/p'$ vs. rate of pore pressure change $\delta u/\delta e_a$ and (b) critical state in $q$-$p'$ plane. Drained test 10-3: (c) void ratio $e$ vs. dilatancy $\delta e_s/\delta e_a$.

Figure 7. Critical state lines for all mixtures of platinum tailings. (a) Projection on the void ratio vs. mean effective stress $e$-$p'$ plane and (b) projection on the deviatoric vs. mean effective stress $q$-$p'$ plane.

Figure 8. Particle size distribution. Ranges for the tailings listed in Table 3 and reported in this work. Data from references indicated in Table 3. Note: PLA refers to the platinum tailings reported herein. All other symbols as indicated in Table 3. The number next to each symbol indicates fines content.

Figure 9. Invertibility plots for (a) intercept $\Gamma_{100}$ and (b) slope $\lambda$. Note: PLA refers to the platinum tailings reported herein. All other symbols as indicated in Table 3. The number next to each specimen label indicates fines content.

Figure 10. Comparison between the normalized standard errors for the intercept $NSE_{\Gamma_{100}}$ and the slope $NSE_{\lambda}$. Note: PLA refers to the platinum tailings reported herein. All other symbols as indicated in Table 3.

Figure 11. Intercept $\Gamma_{100}$ vs (a) minimum void ratio $e_{\min}$ and (b) maximum void ratio $e_{\max}$. Note: PLA refers to the platinum tailings reported herein. All other symbols as indicated in Table 3.

Figure 12. Slope $\lambda$ vs minimum void ratio $e_{\min}$ with emphasis on: (a) tailings, and (b) mineralogy. Note: PLA refers to the platinum tailings reported herein. En/Byt/Chr refers to Enstatite/Bytownite/Chromite, and Sil to Silica. All other symbols as indicated in Table 3.

Figure 13. Effect of mean particle size $D_{50}$ on (a) maximum void ratio $e_{\max}$, (b) minimum void ratio $e_{\min}$, (c) intercept $\Gamma_{100}$, and (d) friction ratio $M$. Note: Symbols: refer to Table 3. A & SA: angular and subangular. R & SR: rounded and subrounded.
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Figure 10. Comparison between the normalized standard errors for the intercept $NSE_{i100}$ and the slope $NSE_j$. Note: PLA refers to the platinum tailings reported herein. All other symbols as indicated in Table 3.
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