Probabilistic assessment of urban runoff erosion potential

J.A. Harris and B.J. Adams

Abstract: At the planning or screening level of urban development, analytical modeling using derived probability distribution theory is a viable alternative to continuous simulation, offering considerably less computational effort. A new set of analytical probabilistic models is developed for predicting the erosion potential of urban stormwater runoff. The marginal probability distributions for the duration of a hydrograph in which the critical channel velocity is exceeded (termed exceedance duration) are computed using derived probability distribution theory. Exceedance duration and peak channel velocity are two random variables upon which erosion potential is functionally dependent. Reasonable agreement exists between the derived marginal probability distributions for exceedance duration and continuous EPA Stormwater Management Model (SWMM) simulations at more common return periods. It is these events of lower magnitude and higher frequency that are the most significant to erosion-potential prediction.

Key words: erosion, stormwater management, derived probability distribution, exceedance duration.

Résumé: Au niveau de la planification ou de la sélection en développement urbain, la modélisation analytique au moyen de la théorie de la distribution probabiliste dérivée est une alternative valable à la simulation continue car elle demande un effort computationnel beaucoup moindre. Dans le but de prédire le potentiel d'érosion par des eaux de ruissellement en milieu urbain, un nouvel ensemble de modèles probabilistes analytiques a été développé. Les distributions de probabilité marginales pour la durée d'un hydrogramme dans lequel la vitesse critique de courant dans le canal est dépassée (appelé durée de dépassement) sont calculées en utilisant la théorie de la distribution de probabilité dérivée. L'érosion potentielle est une fonction de la durée de dépassement et de la vitesse de pointe dans le canal. Une corrélation raisonnable entre les distributions de probabilité dérivée marginales quant à la durée de dépassement et les simulations continues SWMM à des fréquences de retour plus communes. Ce sont ces événements d'amplitude plus faible et à fréquence plus élevée qui ont le plus d'importance pour la prédiction du potentiel d'érosion.

Mots clés: érosion, gestion des eaux de ruissellement, distribution de probabilité dérivée, durée de l'excédent.

[Traduit par la Rédaction]

1. Introduction

The rapid expansion of urban areas into previously undeveloped land demands extensive drainage infrastructure. With changing land use, the impervious portion of the surface area of a catchment increases while infiltration and evapotranspiration, interflow, and baseflow decrease. To mitigate the effects of urbanization, one must first understand the behaviour of increased stormwater runoff (peak flow rate and duration) and how this relates to erosion potential. The goal of stormwater modeling is to predict catchment loads, such as runoff rates and volumes and pollutant concentrations and masses, from an existing or planned urban area. An analytical stormwater model for predicting stormwater erosion potential is developed in this paper.

1.1. Erosion potential and stormwater management

Increased stormwater discharge into a receiving channel often tends to proliferate the occurrence of downstream flooding and the erosion of the beds and banks of channels that result from higher discharge rates over longer durations and many other significant water quality problems. Erosion control emerges as one of the key issues of stormwater management.

Although accelerated soil erosion due to agricultural mismanagement has long been an issue in rural lands (Cooke and Doornkamp 1990), erosion is now a serious form of soil degradation in urban areas as well. Municipalities face unending development pressures and need a comprehensive long-term management strategy to deal with the problems of future development (Dillon Consulting Engineers and Planners 1982). An important objective of stormwater management is to sus-

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tain a fluvial system with its aquatic and aesthetic value and recreation potential while accommodating development needs within a watershed (MOE 1999).

Dry-weather flow and runoff from frequent rainfall events of moderate magnitude are conveyed in an active channel, and larger storm flows spill onto the floodplain. Increased peak flow rates and runoff volumes disrupt this dynamic balance. Urbanization substantially increases the occurrence of mid-bankfull flows (those which only partially fill the active channel), and it is thought that these frequent flow events perform the most work in shaping (i.e., eroding) the channel (MOE 1999). Leopold et al. (1964) generate an erosive work curve which indicates that the greatest quantity of material is transported by events with relatively high frequencies rather than by extreme events. MacRae (1997) further stresses that an erosion-control philosophy that does not address these high-frequency events may not adequately meet the intended purpose. A large contribution to the erosion of channels is from relatively frequent events of moderate magnitude (Leopold et al. 1964) as opposed to infrequent events of high magnitude.

1.2. Evolution of stormwater management modeling approaches

The practice of urban drainage modeling has seen a shift from the design-storm approach, whereby a modeled catchment is subjected to a hypothetical storm of a certain return period and the hydrologic response analyzed, to continuous simulation modeling, as it is generally accepted that, in the design of storage facilities and water quality control structures, long-term performance is more critical than single, design-event performance (Adams and Papa 2000). The computational burden of continuous simulation, however, particularly with short time steps, may be prohibitive to the practicing engineer in planning-level analysis where numerous alternatives require evaluation. Requiring considerably less computational effort is analytical probabilistic modeling using derived probability distribution theory. This alternative modeling methodology, developed for planning or screening level, returns simple mathematical relationships for system performance statistics such as runoff rates and volumes. Analytical models developed herein are planning-level models intended to predict the erosion potential associated with existing and proposed urban developments.

The fundamental principle behind this research is the derivation of the probability distribution of a random variable based on that of another variable on which the former is functionally dependent. The derivation is possible because the inherent randomness of the independent variable is imparted to the dependent variable (Benjamin and Cornell 1970). The probability density function (PDF) of the duration of a runoff event in which a critical discharge rate is exceeded, d , is derived from the PDFs of rainfall event volume, v , and total duration, t , the two random variables upon which d is functionally dependent. To illustrate the derivation of the PDF of a dependent variable, one-variable transformations are first described. The dependent variable, Y , and independent variable, X , are functionally related by $Y = g(X)$. The inverse of this function solves for the independent variable, $X = g^{-1}(Y)$. The cumulative density function (CDF) of the dependent variable Y is simply the probability that Y is less than some value y that is equal to the probability that X is less than the corresponding value $x = g^{-1}(y)$. The

Fig. 1. Derived probability distribution theory.

known CDF of X is used to obtain this probability, as illustrated in Fig. 1.

A random variable, Z , may be functionally dependent on two independent variables, X and Y , such that $Z = g(X, Y)$. Similar to the case with one independent variable, the CDF of Z , i.e., the probability that $Z \leq z$, describes the probability that X and Y lie in a region where $g(x, y) \leq z$. Mathematically, this is described by

$$\begin{aligned}
 [1] \quad F_Z(z) &= P[Z \leq z] \\
 &= P[X \text{ and } Y \text{ take on values } x, y \\
 &\quad \text{such that } g(x, y) \leq z] \\
 &= \int_{R_z} f_{X,Y}(x, y) dx dy
 \end{aligned}$$

where R_z is the region in the x - y plane where $g(x, y) \leq z$. Assuming independence, the joint probability density function of x and y is simply the product of their marginal distributions, that is,

$$[2] \quad f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

The probability of the occurrence of x and y falling in the region R_z is the volume beneath their joint PDF, $f_{X,Y}(x, y)$.

2. Hydrological applications of derived probability theory

Derived probability distribution theory has played an important role in water-resources and hydrological applications. A pioneer in this area of research, Eagleson (1972), used the functional relationship of the kinematic wave theory of hydrograph generation to derive the peak streamflow probability distribution, offering a method to calculate flood frequency in the absence of streamflow records. More recently, using the joint PDFs of meteorological inputs (rainfall event volume, duration, intensity, and inter-event time), probability distributions of event runoff volume and peak discharge rate were developed by Guo and Adams (1998, 1998a) using EPA stormwater management model (SWMM) type hydrology (Huber and Dickinson 1988) and triangular runoff hydrographs as illustrated in Fig. 2, where Q_p is the peak discharge rate (mm/h), Q_c is the critical flow rate (mm/h), t is the storm duration (h), t_c is the catchment

Fig. 2. Triangular hydrograph.

time of concentration (h), and d is the duration over which the critical discharge rate is exceeded.

Chan and Bras (1979) derived the frequency distribution of runoff volume above a critical discharge rate from a storm event (i.e., overflow volume) using the joint PDF of rainfall intensity and duration. This work assumed that rainfall intensity and duration may be represented by exponential PDFs; these exponential distributions for meteorological characteristics have been used extensively by hydrological researchers (e.g., Adams and Papa 2000) and are also adopted in this paper. The rainfall volume (v in mm) and duration (t in h) PDFs used to derive the exceedance probability are represented by exponential PDFs as follows:

$$[3] \quad f_V(v) = \zeta \exp(-\zeta v) \quad \zeta = 1/\bar{v}$$

$$[4] \quad f_T(t) = \lambda \exp(-\lambda t) \quad \lambda = 1/\bar{t}$$

where ζ is the parameter of the exponential probability density function for V , $\bar{v}$ is the mean rainfall event volume, λ is the parameter of the exponential probability density function for T , and $\bar{t}$ is the mean rainfall event duration. Because the random variables are considered statistically independent, the joint PDF of v and t is simply the product of their marginal PDFs as follows:

$$[5] \quad f_{V,T}(v, t) = \zeta \lambda \exp(-\zeta v - \lambda t)$$

Further developing analytical models for flood forecasting, Mukherjee and Mansour (1991) comment that not enough attention has been paid to using the joint probability of runoff flow and duration to forecast alarm-level floods. Their later work shows that a univariate analysis of flow alone underestimates the exceedance probability of flood flow and, by using the joint probability of flow and duration, more effective management of flood risk areas can be practiced (Mukherjee and Mansour 1996). Adams and Papa (2000) provide a more detailed account of the evolution of derived probability distribution and its role in water resources and hydrology.

2.1. Current practices in predicting erosion potential

A methodology for determining the erosion potential of a development plan was outlined by the Ontario Ministry of the Environment (MOE) based on the change in the hydrologic regime that results from the development of a subwatershed—watershed and changes in bank stratigraphy throughout various

reaches of a watercourse (MOE 1999). Collected streambank stratigraphy information is used to determine the threshold for erosion in each reach of a watercourse prior to development. The stormwater management plan can then be prepared. The MOE recommends that this analysis be conducted at the watershed or subwatershed level, rather than at incremental stages of development (MOE 1999).

Researchers also recommend using tractive force – permissible velocity analysis at the subwatershed plan level (Marshall Macklin Monaghan Limited 1994). This multicriterion concept considers discharge, duration that a critical discharge rate is exceeded, and boundary material characteristics (MacRae 1997). Assuming uniform material, the erodibility of a downstream channel can be expressed by the mean particle size that is subjected to a tractive force whereby the mean particle size corresponds to a mean velocity, above which erosion will occur, i.e., the critical velocity (Cooke and Doornkamp 1990). The erosion-potential impulse can then be calculated as the product of the difference between the channel velocity and the critical velocity and the exceedance duration. The summation is performed for the entire duration of the design storm event and can be extended to a continuous simulation application, since the active channel is not formed by a single event, but rather its form is a result of the sum of forces exerted on a channel by a continuum of flow events (MOE 1999).

In the absence of a subwatershed plan, a typical criterion for erosion control for a single development in the Province of Ontario is to design a storage facility that will detain runoff from a 25 mm storm event over 24 h. It is thought that this convenient approach may actually underestimate the required erosion control, since receiving stream conditions (such as stratigraphy, channel slope, and cross section) are not taken into consideration, and the use of a single event to represent erosion potential may be misleading, since erosion is a highly discontinuous but ongoing process resulting from a continuum of flow events (MOE 1999).

Alongside the 25 mm storm approach, another commonly applied method to determine erosion potential is the 2 year peak flow rate attenuation approach, whereby post-development flows are reduced to those of pre-development levels, resulting in a flat hydrograph with a long recession limb. The 2 year storm approach has met with some success, however, many researchers support the idea that it may actually aggravate erosion (MacRae 1997). The approach does not address the increased frequency of mid-bankfull to bankfull flows, the increased high discharge frequency caused by the peak flow rate attenuation due to storage, or the sensitivity of channel materials to erosion (MacRae 1997).

A univariate approach (i.e., considering peaks of flow rates only) to establish the erosion potential of an existing or proposed development does not consider low-magnitude, high-frequency events that many researchers believe to be key channel-forming events (Leopold et al. 1964; MacRae 1997; MOE 1999). A second variable, the duration that a critical tractive force is exceeded, must be considered when estimating erosion potential. As discussed earlier, this critical tractive force can be equated to a critical average channel velocity above which erosion occurs (Cooke and Doornkamp 1990). This paper advances the development of the joint probability distribution of peak channel velocity, u_p , and the duration of a streamflow event over which

the critical velocity is exceeded, d . Given that one method for predicting the erosion potential, E_p , is by taking the product of u_p and d , their joint PDF would be a valuable tool at the planning–screening level to predict the effect of land develop-

ment on the erosion potential of a downstream receiving water body. The marginal probability distribution of d is derived; however, further research is required to derive the joint PDF of these random variables, i.e., $f_{U_p, D}(u_p, d)$.

3. Analytical model development

Using SWMM-type hydrology, the triangular runoff hydrograph of Fig. 2 is generated from a rainfall input, with the runoff duration equalling the sum of the rainfall event duration and the catchment time of concentration. The exceedance duration, d , the length of time that the runoff discharge is greater than a critical discharge, Q_c , is determined by the similar triangles of Fig. 2. That is,

$$[6] \quad \frac{d}{t + t_c} = \frac{Q_p - Q_c}{Q_p} \quad \text{or} \quad d = (t + t_c) \left(1 - \frac{Q_c}{Q_p} \right)$$

The peak discharge rate is estimated from the runoff event volume, the duration of the rainfall event, and several catchment parameters. Following the work of Guo and Adams (1998, 1998a), the event runoff volume, v_r , is determined to be

$$[7] \quad v_r = \begin{cases} 0 & v \leq S_{di} & \text{in region a} \\ h(v - S_{di}) & S_{di} < v \leq S_{il} + f_c t & \text{in region b} \\ v - S_d - f_c(1 - h)t & v > S_{il} + f_c t & \text{in region c} \end{cases}$$

where h is the fraction of the impervious area of the catchment; S_{di} is the impervious area depression storage (mm); S_{dp} is the pervious area depression storage (mm); S_d is the area-weighted average depression storage of the pervious and impervious areas of the catchment (mm); f_c is the ultimate infiltration capacity of a soil (mm/h); and S_{il} is the initial loss in the pervious portion of the catchment (mm), where $S_{il} = S_{iw} + S_{dp}$ and S_{iw} is the initial soil wetting infiltration volume.

It is evident that the event runoff volume varies within the three mutually exclusive regions (a, b, and c) in the v – t plane. Based on the geometry of the hydrograph in Fig. 2, where the event runoff volume is the area under the triangular hydrograph, the peak discharge rate is determined to be

$$[8] \quad Q_p = \frac{2v_r}{(t + t_c)} \quad \text{or} \quad Q_p = \begin{cases} 0 & \text{in region a} \\ \frac{2h(v - S_{di})}{t + t_c} & \text{in region b} \\ \frac{2[v - S_d - f_c(1 - h)t]}{t + t_c} & \text{in region c} \end{cases}$$

Figure 2 shows that the exceedance duration is zero when the critical discharge is *not* exceeded. For the three ranges of Q_p , this relationship is described by

$$[9] \quad Q_p \leq Q_c \quad \text{or} \quad \begin{cases} v \leq S_{di} & \text{in region a} \\ v \leq \frac{Q_c}{2h}(t + t_c) + S_{di} & \text{in region b} \\ v \leq \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t & \text{in region c} \end{cases}$$

The relationship between d and v and t is then described by combining eqs. [7]–[9] as follows:

$$[10] \quad d = \begin{cases} 0 & \left\{ \begin{array}{l} \text{(region a) } v \leq S_{di} \quad \text{or} \\ \text{(region b) } S_{di} < v < S_{il} + f_c t \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} v \leq \frac{Q_c}{2h}(t + t_c) + S_{di} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \text{(region c) } v > S_{il} + f_c t \\ \text{and} \\ v \leq \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t \end{array} \right. \\ (t + t_c) \left(1 - \frac{Q_c(t + t_c)}{2h(v - S_{di})} \right) & \left\{ \begin{array}{l} S_{di} < v \leq S_{il} + f_c t \\ \text{and} \\ v > \frac{Q_c}{2h}(t + t_c) + S_{di} \end{array} \right. \\ (t + t_c) \left(1 - \frac{Q_c(t + t_c)}{2[v - S_d - f_c(1 - h)t]} \right) & \left\{ \begin{array}{l} v > S_{il} + f_c t \\ \text{and} \\ v > \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t \end{array} \right. \end{cases}$$

The exceedance duration transformation function of eq. [10] is plotted in Fig. 3.

3.1. Exceedance duration impulse probability distribution

The impulse probability of zero exceedance duration results from those storm events where the peak discharge rate is less than the critical rate, or $Q_p \leq Q_c$ (see eq. [10]). The impulse probability of the critical discharge exceedance duration, comprised of three parts, is described by the following relationship:

$$[11] \quad P[d = 0] = P[v \leq S_{di}] + P[S_{di} < v \leq S_{il} + f_c t] \\ \cap P[v \leq \frac{Q_c}{2h}(t + t_c) + S_{di}] + P[v > S_{il} + f_c t] \\ \cap P[v \leq \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t]$$

The probability distributions of the meteorological input to the catchment (i.e., rainfall event volume and duration) are satisfactorily represented by exponential PDFs for many climates (Adams and Papa 2000). The single-parameter exponential distributions, in which the mean and standard deviation are equal, best represent the distribution of meteorological parameters of lower magnitude and higher frequency (i.e., the left side of the PDF) and are not recommended to describe very low frequency, extreme events. Because it is evident that low-magnitude, high-frequency events have the most erosion potential (MacRae 1997), the rainfall volume and duration PDFs used to derive the exceedance probability are assumed to be represented by the exponential PDFs as in eqs. [3] and [4].

The impulse probability, $P[d = 0]$, can be found by integrating the joint PDF of v and t (eq. [5]) over the regions of integration described by eq. [10]. The mutually exclusive impulse integration regions are governed by the following inequalities in the v – t plane.

The impulse probability, $P[d = 0]$, can be found by integrating the joint PDF of v and t over the regions of integration described by eq. [11]. The mutually exclusive impulse integration regions are governed by these inequalities in the v – t plane:

Impulse integration region 1 (IP₁)

$$[12] \quad \{v \leq S_{di}\}$$

Impulse integration region 2 (IP₂)

$$[13] \quad \left\{ \begin{array}{l} S_{di} < v \leq S_{il} + f_c t \\ v \leq \frac{Q_c}{2h}(t + t_c) + S_{di} \end{array} \right.$$

Impulse integration region 3 (IP₃)

$$[14] \quad \left\{ \begin{array}{l} v > S_{il} + f_c t \\ v \leq \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t \end{array} \right.$$

Four straight lines are formed by the inequalities of eqs. [12]–[14] and are represented by the following equations:

$$[15] \quad \text{Line 0: } v = S_{di}$$

$$[16] \quad \text{Line 1: } v = \frac{Q_c}{2h}(t + t_c) + S_{di}$$

$$[17] \quad \text{Line 2: } v = S_{il} + f_c t$$

$$[18] \quad \text{Line 3: } v = \frac{Q_c}{2}(t + t_c) + S_d + f_c(1 - h)t$$

The definitions of integration regions IP₁, IP₂, and IP₃ indicate that IP₁ is the area in the v – t plane below line 0, IP₂ is the area in the v – t plane above line 0 and below lines 1 and 2, and IP₃ is the area in the v – t plane above line 2 and below line 3. By these definitions, the three impulse integration regions are mutually exclusive (Fig. 4).

Fig. 3. Exceedance duration transformation function. **Fig. 4.** Impulse integration regions IP_1 , IP_2 , and IP_3 for $P[d = 0]$ of type IV catchments.

It is a reasonable assumption that the impervious depression storage, S_{di} , will always be less than the initial hydrologic losses from the pervious area of a catchment, S_{il} , and therefore line 0 will always be below lines 1, 2, and 3. For this reason, IP_1 will remain unchanged for all scenarios. The algebraic nature of the lines is such that the slope and intercept of line 3 will always fall

between those of lines 1 and 2. The relative magnitudes of the slopes and intercepts of lines 1 and 2 will establish the impulse integration regions, IP_2 and IP_3 .

The difference between the slopes of lines 1 and 2 is

$$[19] \quad \frac{Q_c}{2h} - f_c$$

The difference between the intercepts of lines 1 and 2 is

$$[20] \quad S_{di} + \frac{Q_c t_c}{2h} - S_{il} = \frac{Q_c t_c}{2h} - S_{dd}$$

where $S_{dd} = S_{il} - S_{di}$. The impulse integration regions are determined from two sets of conditions: $Q_c < 2hf_c$ or $Q_c \geq 2hf_c$ (slope condition) and $Q_c < 2hS_{dd}/t_c$ or $Q_c \geq 2hS_{dd}/t_c$ (intercept condition 1). The conditions that hold true for a particular catchment are based on the characteristics of that catchment. Four possible combinations of these conditions result in the following four catchment types:

$$[21] \quad \text{Type I: } Q_c < 2hf_c \quad \text{and} \quad Q_c < \frac{2hS_{dd}}{t_c}$$

$$[22] \quad \text{Type II: } \frac{2hS_{dd}}{t_c} \leq Q_c < 2hf_c$$

$$[23] \quad \text{Type III: } 2hf_c \leq Q_c < \frac{2hS_{dd}}{t_c}$$

$$[24] \quad \text{Type IV: } Q_c \geq 2hf_c \quad \text{and} \quad Q_c \geq \frac{2hS_{dd}}{t_c}$$

The impulse exceedance probability is only developed for type IV catchments in this paper; however, fully developed analytical models for all catchments types are developed in Harris (2002) and summarized in Table 1. The joint probability density function, $f_{V,T}(v, t)$ of eq. [5] is integrated over the shaded area in Fig. 4.

The exceedance duration impulse probability for type IV catchments is the sum of the integration regions IP_1 , IP_2 , and IP_3 as follows:

$$[25] \quad P[d = 0] = \int_0^\infty \int_0^{Q_c t_c / 2h + S_d + [f_c(1-h) + Q_c/2]t} \zeta \lambda \exp(-\zeta v - \lambda t) dv dt$$

$$= 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp\left[-\zeta \left(\frac{Q_c t_c}{2} + S_d\right)\right]$$

3.2. Exceedance duration cumulative density function

Thus far, the impulse probability portion of the cumulative density function (CDF) for d , such that the critical discharge rate is *not* exceeded (i.e., $P[d = 0]$), is developed. The remaining portion of the CDF for d must incorporate those areas in the v - t plane where $Q_p > Q_c$ (i.e., the critical discharge is exceeded)

and $d \leq d_o$, where d_o is an arbitrary exceedance duration value that must be greater than zero.

The probability of $d \leq d_o$ (for any $d_o > 0$) is derived in the following two mutually exclusive parts:

$$P[d \leq d_o] = P_1[d \leq d_o] + P_2[d \leq d_o]$$

where v and t are such that $Q_p > Q_c$. P_1 and P_2 , mutually exclusive areas in the v - t plane, are the regions over which the probability density function, $f_{V,T}(v, t)$, is integrated. These regions are described by the following equations:

Region of integration, P_1

$$\begin{cases} (t + t_c) \left(1 - \frac{Q_c(t+t_c)}{2h(v-S_{di})} \right) \leq d_o \\ v \leq S_{il} + f_c t \quad \text{and} \\ v > \frac{Q_c}{2h}(t + t_c) + S_{di} \end{cases} \quad \text{or equivalently} \quad \begin{cases} v \leq \frac{Q_c(t+t_c)^2}{2h(t+t_c-d_o)} + S_{di} \\ v \leq S_{il} + f_c t \quad \text{and} \\ v > \frac{Q_c}{2h}(t + t_c) + S_{di} \end{cases}$$

Region of integration, P_2

$$\begin{cases} (t + t_c) \left(1 - \frac{Q_c(t+t_c)}{2[v-S_d-f_c(1-h)t]} \right) \leq d_o \\ v > S_{il} + f_c t \quad \text{and} \\ v > \frac{Q_c t_c}{2} + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t \end{cases} \quad \text{or equivalently} \quad \begin{cases} v \leq \frac{Q_c(t+t_c)^2}{2(t+t_c-d_o)} + S_d + f_c(1-h)t \\ v > S_{il} + f_c t \quad \text{and} \\ v > \frac{Q_c t_c}{2} + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t \end{cases}$$

Analogous to the impulse probability derivation, five lines are formed by these two inequalities, represented by the following equations:

$$\text{line 1} = v = \frac{Q_c t_c}{2h} + S_{di} + \frac{Q_c}{2h} t \quad (\text{equal to eq. [16]})$$

$$\text{line 2} = v = S_{il} + f_c t \quad (\text{equal to eq. [17]})$$

$$\begin{aligned} \text{line 3} &= v \\ &= \frac{Q_c t_c}{2} + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t \end{aligned} \quad (\text{equal to eq. [18]})$$

$$[26] \quad \text{line 4} = v = \frac{Q_c(t+t_c)^2}{2h(t+t_c-d_o)} + S_{di}$$

$$[27] \quad \text{line 5} = v = \frac{Q_c(t+t_c)^2}{2(t+t_c-d_o)} + S_d + f_c(1-h)t$$

According to the definitions of P_1 and P_2 , P_1 is the area in the v - t plane above line 1 and below both line 2 and line 4, and P_2 is the area in the v - t plane above lines 2 and 3 and below line 5. Line 0 of the impulse probability section will always be below lines 1 and 2 and therefore does not influence integration regions P_1 and P_2 .

3.3. Linearization

Due to the nonlinear nature of lines 4 and 5, closed-form expressions of exceedance duration probability distributions cannot be obtained. Although an approximation, closed-form expressions are more readily usable in planning-screening-level analyses to predict erosion potential. The nonlinear portion of lines 4 and 5 dominates when t is small; as t increases, the linear portion of the lines dominates. The linear approximation of line 4, termed line 4', is found by taking the difference between

lines 1 and 4 in the linear portion and then adding this value to line 1 as follows:

$$\begin{aligned} \text{line 4} - \text{line 1} &= \left(\frac{Q_c(t+t_c)^2}{2h(t+t_c-d_o)} + S_{di} \right) \\ &\quad - \left(\frac{Q_c}{2h}(t+t_c) + S_{di} \right) \\ &= \frac{Q_c(t+t_c)d_o}{2h(t+t_c-d_o)} \\ &\cong \frac{Q_c d_o}{2h} \quad \text{for } t \gg d_o \end{aligned}$$

$$\begin{aligned} [28] \quad \text{line 4'} &= \text{line 1} + \frac{Q_c d_o}{2h} \\ &= \frac{Q_c}{2h}(t_c + d_o) + S_{di} + \frac{Q_c}{2h} t \end{aligned}$$

Similarly, the linearized line 5, termed line 5', is found by taking the difference between lines 3 and 5 in the linear portion and then adding this value to line 3 as follows:

$$\begin{aligned} [29] \quad \text{line 5'} &= \text{line 3} + \frac{Q_c d_o}{2} \\ &= \frac{Q_c}{2}(t_c + d_o) + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t \end{aligned}$$

Analogous to the preceding impulse probability development for d , various combinations of the slopes and intercepts of lines 1-3, 4', and 5' determine the integration regions over which the joint probability density function $f_{V,T}(v, t)$ is integrated to determine the approximate CDF of d . The probability of $d \leq d_o$ (for any $d_o > 0$) is again in two mutually exclusive parts:

$$P[d \leq d_o] = P_1[d \leq d_o] + P_2[d \leq d_o]$$

where v and t are such that $Q_p > Q_c$. P_1 is the area in the v - t plane above line 1 and below lines 2 and 4'. P_2 is the area in the v - t plane above lines 2 and 3 and below line 5'. Linearization

Table 1. Summary of exceedance duration analytical probabilistic models for each of the four catchment types.

Type I catchments ($Q_c < 2hf_c$ and $Q_c < 2hS_{dd}/t_c$)	
For $0 < d \leq \frac{2hS_{dd}}{Q_c} - t_c$	$F_D(d) = 1 - \frac{2h\lambda}{\zeta Q_c + 2h\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2h} (t_c + d) + S_{di} \right) \right]$
For $d_o > \frac{2hS_{dd}}{Q_c} - t_c$	$F_D(d) = 1 + \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) - \left(\zeta f_c(1-h) + \zeta \frac{Q_c}{2} + \lambda \right) t^{**'} \right]$ $- \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) \right]$ $- \frac{2h\lambda}{\zeta Q_c + 2h\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2h} (t_c + d) + S_{di} \right) - \left(\zeta \frac{Q_c}{2h} + \lambda \right) t^{***'} \right]$
Type II catchments ($2hS_{dd}/t_c \leq Q_c < 2hf_c$)	
For $d > 0$	$F_D(d) = 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) \right]$ $+ \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) - \left(\zeta f_c(1-h) + \zeta \frac{Q_c}{2} + \lambda \right) t^{***'} \right]$ $- \frac{2h\lambda}{\zeta Q_c + 2h\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2h} (t_c + d) + S_{di} \right) - \left(\zeta \frac{Q_c}{2h} + \lambda \right) t^{***'} \right]$
Type III catchments ($2hf_c \leq Q_c < 2hS_{dd}/t_c$)	
For $0 < d \leq \frac{2hS_{dd}}{Q_c} - t_c$	$F_D(d) = 1 + \frac{2h\lambda}{\zeta Q_c + 2h\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2h} (t_c + d) + S_{di} \right) - \left(\zeta \frac{Q_c}{2h} + \lambda \right) t^{***'} \right]$ $- \frac{2h\lambda}{\zeta Q_c + 2h\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2h} (t_c + d) + S_{di} \right) \right]$ $- \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) - \left(\zeta f_c(1-h) + \zeta \frac{Q_c}{2} + \lambda \right) t^{***'} \right]$
For $d_o > \frac{2hS_{dd}}{Q_c} - t_c$	$F_D(d) = 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) \right]$
Type IV catchments ($Q_c \geq 2hf_c$ and $Q_c \geq 2hS_{dd}/t_c$)	
For $d > 0$	$F_D(d) = 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp \left[-\zeta \left(\frac{Q_c}{2} (t_c + d) + S_d \right) \right]$

leaves the slope condition and intercept condition 1 unchanged, and a new condition, intercept condition 2, emerges. The difference between the intercepts of lines 2 and 4' and the difference between the intercepts of lines 2 and 5' are equal and determine the following condition:

$$[30] \quad d_o \leq \frac{2hS_{dd}}{Q_c} - t_c \quad \text{or} \quad d_o > \frac{2hS_{dd}}{Q_c} - t_c$$

The intercept of line 4' is always greater than that of line 1, and in the same way the intercept of line 5' is always greater than that of line 3. Therefore, the ordering of the relative magnitudes of intercepts of importance includes lines 1 and 2, 2 and 4', and 2 and 5'. Again, the integration regions are defined by various combinations of the slope condition and intercept conditions 1 and 2. Not all combinations of these conditions exist, however. When $Q_c \geq 2hS_{dd}/t_c$, the term $2hS_{dd}/Q_c - t_c$ becomes negative, and d cannot assume a negative value. The same combinations of slope and intercept conditions used in the catchment types described earlier are used to determine the integration regions P_1 and P_2 .

Lines 1–5' are plotted in the v – t plane for the four catchment types, and the corresponding regions of integration are determined. Again, the exceedance probability distributions are only developed for type IV catchments in this paper; however, fully developed analytical models for all catchments types are summarized in Table 1. Referring to Fig. 5, the sum of integration

Fig. 5. Linearized integration regions P_1 and P_2 for $P[0 < d \leq d_o]$ of type IV catchments.

regions P_1 and P_2 is the sum of the following areas:

$$\begin{cases} 0 < t < \infty \\ \frac{Q_c t_c}{2} + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t < v \\ < \frac{Q_c}{2} (t + d_o) + S_d + \left[f_c(1-h) + \frac{Q_c}{2} \right] t \end{cases}$$

The probability that the exceedance duration is greater than an arbitrary amount, d_o , is found in only one part for $d_o > 0$ (because the term $2hS_{dd}/Q_c - t_c$ of intercept condition 2 is less

than zero and d cannot be less than zero). The joint probability density function, $f_{V,T}(v, t)$, is integrated over the shaded area in Fig. 5.

For any $d_o > 0$, the probability that d is between zero and d_o is

$$\begin{aligned}
 [31] \quad P[0 < d \leq d_o] &= \int_0^\infty \int_{Q_c t_c/2 + S_{dd} + [f_c(1-h) + Q_c/2]t}^{Q_c(t_c + d_o)/2 + S_{dd} + [f_c(1-h) + Q_c/2]t} \zeta \lambda \exp(-\zeta v - \lambda t) dv dt \\
 &= \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp\left[-\zeta \left(\frac{Q_c t_c}{2} + S_{dd}\right)\right] - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \\
 &\quad \times \exp\left[-\zeta \left(\frac{Q_c}{2}(t_c + d_o) + S_{dd}\right)\right]
 \end{aligned}$$

When the impulse probability for type IV catchments (eq. [25]) is added to eq. [31], the probability that $d \leq d_o$ is given by

$$\begin{aligned}
 [32] \quad F_D(d) &= P[d = 0] + P[0 < d \leq d_o] \\
 &= 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp\left[-\zeta \left(\frac{Q_c}{2}(t_c + d_o) + S_{dd}\right)\right]
 \end{aligned}$$

When $d_o = 0$, eq. [32] yields

$$\begin{aligned}
 F_D(0) &= P[d = 0] \\
 &= 1 - \frac{2\lambda}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp\left[-\zeta \left(\frac{Q_c}{2}t_c + S_{dd}\right)\right]
 \end{aligned}$$

which is in agreement with the impulse probability of eq. [25].

The PDF of d for a catchment with type IV characteristics is determined by taking the derivative of its CDF as follows:

$$\begin{aligned}
 [33] \quad f_D(d) &= \frac{dF_D(d)}{dd} \\
 &= \frac{\zeta \lambda Q_c}{2\zeta f_c(1-h) + \zeta Q_c + 2\lambda} \exp\left[-\zeta \left(\frac{Q_c}{2}(t_c + d_o) + S_{dd}\right)\right]
 \end{aligned}$$

The expected value of d per rainfall event for type IV catchments is found as follows:

$$\begin{aligned}
 [34] \quad E(d) &= 0 \cdot P[d = 0] + \int_0^\infty d \cdot f_D(d) dd \\
 &= \frac{4\lambda}{\zeta Q_c (2\zeta f_c(1-h) + \zeta Q_c + 2\lambda)} \exp\left[-\zeta \left(\frac{Q_c t_c}{2} + S_{dd}\right)\right]
 \end{aligned}$$

Without significant loss of accuracy, by linearizing the integration regions that determine the probability distribution of exceedance duration, an analytical model that is used with relative ease arises for planning–screening-level analyses. A summary of the analytical probabilistic exceedance duration models is presented in Table 1 for each of the four catchment types.

4. Comparison of results with continuous SWMM simulation results

The validity of the derived analytical model for predicting the probability distribution of exceedance duration is verified by comparing the model results with those from SWMM simulations. A continuous simulation model was chosen to compare results with those from the analytical model as opposed to using field data. Continuous gauging of runoff from urban areas is rare and often only provides short records. Guo (1998b) notes

that continuous simulation is the most reliable approach for estimating runoff quantities from small urban catchments in the absence of historical discharge data and is therefore used for comparison with the analytical models developed in this research. The 33 year historical rainfall record at the Pearson International Airport (years 1960–1992), Atmospheric Environment Service (AES) station 6158655 (Toronto, Ontario), was used as input to the SWMM RAIN module. Runoff quantities were then simulated using the RUNOFF module, and the simulated flow series were subjected to frequency analysis. The

Fig. 6. Analytical and SWMM simulation model comparison.

same rainfall record was used to extract parameter values for the PDFs of rainfall characteristics used in the analytical model.

Comparative SWMM simulations were performed with the PCSWMM 2002 interface. This version of PCSWMM allows the user to plot the historical runoff time series and define an “exceedance” or critical discharge, Q_c . The output summary includes the number of exceedances for the entire runoff time series. Because of the difficulties in correlating each runoff event from the SWMM simulation output with a specific input rainfall event (PCSWMM does not allow the user to define an inter-event time), a comparison of the annual duration that a critical discharge rate is exceeded was performed. Using the PCSWMM interface, the total critical discharge exceedance duration is returned for each year of simulation. The yearly total exceedance duration was recorded and ranked, and its corresponding Weibull plotting position determined.

The analytical model returns the exceedance duration probability *per rainfall event*. To convert the probability per rainfall event to a return period, the following expression is used:

$$[35] \quad T_R = \frac{1}{\theta \cdot P[d > d_o]} = \frac{1}{\theta \cdot (1 - P[d \leq d_o])}$$

where T_R is the return period of exceedance duration, d_o , in years; and θ is the average number of rainfall events per year.

Figure 6 shows a comparison between the analytical and SWMM model results for a type IV catchment. There was relatively close agreement between the analytical and SWMM results for the lower return periods. As the return period increases, large differences between the model results exist; however, the events of higher frequency (thus lower return period) are of the most interest in this research because it is thought that they are more significant from an erosion standpoint (Leopold et al. 1964; MacRae 1997). In addition, the reliability of the plotting position at high frequencies is reduced. It was noted earlier that the frequency distributions for meteorological parameters, rainfall volume and storm duration, were assumed to be exponential.

This one-parameter distribution is particularly useful to analyze those events of higher frequency, and an extreme-value distribution analysis is necessary for those more infrequent events. The same may be said for the analytical model developed here to predict the frequency distribution of erosion-causing runoff events.

The analytical model return period is consistently lower than that of the SWMM simulations for return periods of 30 years or less. The linearized integration regions in determining the cumulative probability density function may account for part of the discrepancies between the models. To some extent, linearization decreases the area of the integration regions, as the nonlinear portion is shaved off and the smaller integration regions underestimate the exceedance duration CDF. Given that return period is directly proportional to the CDF, the return period is also underestimated.

With a rather short rainfall record of 33 years, it would not be prudent to make hydrological decisions based on return periods of 10 years or greater. The highest exceedance duration that results from the SWMM simulation is given a return period of 34 years (slightly greater than the length of the rainfall record); however, this duration may actually be associated with a return period of much greater than 33 years. Outliers in the record may explain why the exceedance duration increases exponentially with increasing return period in the tail of Fig. 6.

5. Conclusions

Urbanization replaces previously pervious areas with impervious ones via pavements and rooftops. The resultant increased stormwater runoff brings about downstream flooding and the erosion of the beds and banks of drainage channels and water courses. With water as the eroding agent, soil particles are removed either uniformly by the impact of raindrops or irregularly by distributed flow in rills or gullies, i.e., runoff. Erosion control must be a priority to planners because of the economic impacts

of disappearing lands abutting waterways and the impairment of biological activity in channels as a result of transported and deposited sediment.

At the planning- or screening-level analysis of urban development, analytical modeling using derived probability distribution theory is a viable alternative to complement continuous simulation, offering considerably less computational effort. This paper provides an illustration of the methodology of this approach. Using the probability density functions (PDFs) of meteorological inputs (rainfall volume, duration, intensity, and inter-event time), screening-level analytical models for the evaluation of urban drainage system quantity and quality control have been developed for a range of control alternatives and system scenarios (Adams and Papa 2000).

Contributing to the analytical modeling research area, new probabilistic models are developed herein for predicting the erosion potential of stormwater runoff. Erosion potential is the product of peak channel velocity and the duration that a critical velocity is exceeded (termed exceedance duration). The joint PDF of these two random variables indicates with relative ease the erosion potential from existing or proposed developments, and this is not offered by continuous simulation. The marginal distribution for exceedance duration, d , is developed, and future research is required for the derivation of their joint probability density function with which the erosion potential of a rainfall event may be readily computed.

Reasonable agreement between the results of the analytical model and an equivalent continuous simulation model, SWMM, exists in the range of lower return period or higher frequency events. This is consistent with the concept behind the analytical probabilistic models developed with single-parameter exponential marginal PDFs for meteorological statistics which are largely influenced by the large number of small- and medium-magnitude rainfall events. It is thought that these events of smaller magnitude and higher frequency do most of the work in forming a water channel and are therefore most significant from an erosion standpoint.

The developed method for predicting these higher frequency events can be very useful when developing a watershed or sub-watershed plan at the screening or planning level. The closed-form mathematical solutions are easy to use, requiring few input parameters, making them attractive to the practicing engineer when compared to the computational burden often presented by the more conventional approach, continuous simulation.

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List of symbols

- d portion of a storm event when critical discharge rate is exceeded (h)
- d_o arbitrary value of d (h)
- $E(d)$ expected value of d
- E_p erosion potential
- f_c ultimate infiltration capacity of a soil (mm/h)
- $f_V(v), f_T(t)$ probability distributions of rainfall volume and intensity
- $f_{V,T}(v, t)$ joint probability density function of rainfall volume and intensity
- $f_X(x), f_Y(y)$ probability density functions of random variables X and Y
- $f_{X,Y}(x, y)$ joint probability density function of random variables X and Y
- $F_D(d)$ cumulative distribution function of D
- $F_X(x), F_Y(y)$ cumulative distribution functions of X and Y

$g(x)$	functional relationship	S_{iw}	initial soil wetting infiltration volume (mm)
h	fraction of impervious area of an urban catchment	t	rainfall event duration (h)
IP_1, IP_2, IP_3	impulse integration regions 1, 2, and 3	$t * t'$	intersection of lines 2, 4, and 5
P_1, P_2	integration regions 1 and 2	$\bar{t}$	mean rainfall event duration from an historical rainfall record (h)
Q_c	critical or threshold discharge rate (mm/h)	t_c	catchment time of concentration (h)
Q_p	peak discharge rate of a runoff event (mm/h)	T_R	return period (years)
R_z	region in the x - y plane where $g(x, y) \leq z$	u_p	a given peak channel velocity (m/s)
S_d	area-weighted average depression storage of a catchment = $hS_{dt} + (1 - h)S_{il}$ (mm)	v	rainfall event volume (mm)
S_{di}	depression storage of impervious portion of a catchment (mm)	$\bar{v}$	mean rainfall event volume from an historical rainfall record (mm)
S_{dd}	difference between the pervious portion initial losses and the impervious portion depression storage = $S_{il} - S_{di}$ (mm)	v_r	runoff event volume (mm)
S_{dp}	depression storage of the pervious portion of a catchment (mm)	X, Y, Z	random variables
S_{il}	initial losses of the pervious portion of a catchment = $S_{iw} + S_{dp}$ (mm)	λ	parameter of the exponential probability density function for $T = 1/\bar{t}$ (h^{-1})
		θ	average annual number of rainfall events
		ζ	parameter of the exponential probability density function for $V = 1/\bar{v}$ (mm^{-1})