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Radiation Damping of a Yang-Mills Particle Revisited

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Abstract

The problem of a color-charged point particle interacting with a four dimensional Yang-Mills gauge theory is revisited. The radiation damping is obtained inspired in Dirac's computation. The difficulties in the non-abelian case were solved by using an ansatz for the Liénard-Wiechert potentials, already used in the literature [40] for finding solutions to the Yang-Mills equations. Three non-trivial examples of radiation damping for the non-abelian particle are discussed in detail.

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1 Introduction

The problem of motion for a classical radiating particle has been studied for a long time in the non-relativistic and relativistic contexts [1, 2]. It is well known that this particle suffers a radiation damping due the self-force. In Dirac's article [2], based in a point-like model for the charged particle, a self-consistent relativistic equation of motion was obtained. This equation is the usual relativistic Lorentz force in addition of a term involving a third-order time derivative of the position of the particle. Solutions to this third-order equation have shown to posses unphysical behavior as runaway and preacceleration [3, 4, 5]. It is now known that this problematic behavior is a consequence of the point-like nature of the particle. An exhaustive study of the mentioned phenomena was performed by many authors and a systematic description of some divergences require to renormalize the electron charge and mass. For some recent overviews, see [6, 7, 8, 9]. The solution of the mentioned problem has been of great importance to understand the relativistic dynamics of the particles interacting with radiation. Particularly in the Ref. [2], Dirac calculated the self-force of the emitted electromagnetic radiation (or radiation damping) from an electron. He considered expressions for retarded and advanced potentials, in terms of a parameter σ that upon regularization is a function of the electron size, which obviously must be taken to zero at the end of the computation.

The equations of motion for a particle carrying isotopic-spin in a non-abelian Yang-Mills field has been originally worked out in the 70s by S.K. Wong [10]. Moreover the study of non-abelian particles and their equations of motion have been of great importance along the years. Later important developments were given in Refs. [11, 12]. In these papers Lagrangian and Hamiltonian for the charged particle in a non-abelian gauge field are proposed such that Wong's equations are obtained from them. The canonical quantization and some detailed cases were worked out in the second paper [12]. Further developments using Wong's equations and incorporating spin were described in [13]. The non-abelian particle is also described by using the world-line formalism in Ref. [14]. The application to transport properties of non-abelian fluids or plasmas and, in general, to non-abelian hydrodynamics was discussed in Refs. [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In particular the importance of Wong's equations in the problem of the quark-gluon plasma has been considered in Refs. [26, 27, 28, 29, 30]. More recently, in the context of the double copy approach [31], the consideration of Wong's equations of motion has been discussed in [32]. Moreover it also has been studied as a possible candidate of dark matter by the authors of Ref. [33].

It is natural to try to extend the results [2] to the case of non-abelian theories. However this simple idea has strong complications. The problem of motion for particles in a Yang-Mills fields have been discussed in a subsequent series of papers [34, 35, 36, 37]. In [34] the equations of motion were derived from the energy and momentum conservation. Moreover in this same reference some solutions were obtained in terms of the first order well known abelian Liénard-Wiechert potentials [38, 39]. In [35] it was studied the radiation damping of color in Yang-Mills theory. In Ref. [36] it was argued that the color charge can be changed for a suitable gauge group. If the gauge group is compact and semisimple then the color charge remains constant, however there is

a transfer of energy described by the Liénard-Wiechert (LW) potentials. Moreover in Ref. [37] the variation of the color charges with respect to the choice of the non-abelian waves, was studied.

One of the most important problems in this context is to find suitable Liénard-Wiechert's potentials as solutions to the Yang-Mills equations. However, it is possible to construct these potentials satisfying physical conditions and being also compatible with the Yang-Mills equations. The simplest case is to consider an expansion in the form $\frac{1}{R}$, where R is the retarded distance, for a retarded potential such that it vanishes at infinity. However it is possible to make an ansatz and write the LW potentials as a superposition of terms with different negative powers of R . The non-abelian dependence lies on certain functions α_i that take values in the Lie algebra of the Lie gauge group and also depend on the retarded proper-time [40]. In the present paper we use this ansatz and Dirac's procedure [2] in order to obtain the self-force and consequently the radiation damping of a Yang-Mills particle following Wong's equations.

This paper is organized as follows. In Section 2 we introduce the notation and conventions we will follow along the article. In particular we give a brief review of the Liénard-Wiechert potentials and the ansatz [40] that we will use to compute the radiation damping in subsequent sections. In this Section 2 we also compute the retarded Yang-Mills field strength and the associated radiated field strength. Section 3 is devoted to obtain the self-force correction to Wong's equations. We follow Dirac's method of computing the flux of momentum and energy, in the volume, of the energy-momentum tensor. In this Section 4 we find the equations of motion for different cases. In the first case we recover the abelian Dirac's modified equations of motion. Later we present three non-trivial cases of the radiation damping of the non-abelian particle. Finally in Section 5 we present our final comments. Three appendices were included in order to write down some relevant details of the computations and important results which are needed in Sections 2, 3 and 4.

2 Radiation Fields of a Non-abelian Yang-Mills Charge

In the present section we will give some preliminaries and the set up of the next sections. We also introduce the notation and conventions we will follow along the article.

Consider the Yang-Mills potential $A_\mu(x) = A_\mu^a(x)T_a$, where T_a $a = 1, \dots, \dim(G)$ are the generators of the compact and simple gauge group G , and x labels the points in the Minkowski space-time with signature $(+, -, -, -)$. These fields have a field strength given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu], \quad (1)$$

where g is the coupling constant. The action of the gauge field A_μ in the presence of an external source J_μ reads

$$S = -\frac{1}{4} \int d^4x \text{Tr} [F_{\mu\nu}(x)F^{\mu\nu}(x) - J_\mu(x)A^\mu(x)]. \quad (2)$$

The corresponding Euler-Lagrange equations for A_μ are the Yang-Mills equations

$$D_\mu F^{\mu\nu} = -J^\nu. \quad (3)$$

Here J_ν is known as the isotopic spin current [10], and it is expressed in the simple form

$$J_\nu = \int g I(s) \dot{z}_\mu(s) \delta^{(4)}(x_\mu - z_\mu(s)) ds, \quad (4)$$

which is the current associated to particles moving along the worldline with parameter s (proper time). Here $I(s)$ is a generic element of the Lie algebra of G and $z_\mu(s)$ stands for the position of the particle describing a smooth curve Γ in spacetime and x_μ is any point in Minkowski spacetime.

For a smooth curve Γ in space-time one has that the velocity and its respective derivatives in the proper time satisfy the following relations

$$\begin{aligned} \mathbf{v}^2 &= 1, \quad \mathbf{v}\dot{\mathbf{v}} = 0, \quad \mathbf{v}\ddot{\mathbf{v}} = -\dot{\mathbf{v}}^2, \quad \mathbf{v}\ddot{\mathbf{v}} = -3\dot{\mathbf{v}}\ddot{\mathbf{v}}, \\ \mathbf{v}\mathbf{v}^{(4)} &= -3\dot{\mathbf{v}}^2 - 4\dot{\mathbf{v}}\ddot{\mathbf{v}}, \quad \mathbf{v}\mathbf{v}^{(5)} = -5\dot{\mathbf{v}}\mathbf{v}^{(4)} - 10\ddot{\mathbf{v}}\ddot{\mathbf{v}}, \\ \mathbf{v}\mathbf{v}^{(6)} &= -10\ddot{\mathbf{v}}^2 - 6\dot{\mathbf{v}}\mathbf{v}^{(5)} - 15\ddot{\mathbf{v}}\mathbf{v}^{(4)}, \quad \mathbf{v}\mathbf{v}^{(7)} = -7\dot{\mathbf{v}}\mathbf{v}^{(6)} - 21\ddot{\mathbf{v}}\mathbf{v}^{(5)} - 35\ddot{\mathbf{v}}\mathbf{v}^{(4)}, \\ \mathbf{v}\mathbf{v}^{(8)} &= -8\dot{\mathbf{v}}\mathbf{v}^{(7)} - 28\ddot{\mathbf{v}}\mathbf{v}^{(6)} - 56\ddot{\mathbf{v}}\mathbf{v}^{(5)} - 35(\mathbf{v}^{(4)})^2, \end{aligned} \quad (5)$$

where we adopted Dirac's notation [2] for the scalar product: $\mathbf{v}\mathbf{v} \equiv \mathbf{v}^2 \equiv v^\mu v_\mu$, and similarly for the other products, for instance $\mathbf{v}\dot{\mathbf{v}} \equiv v^\mu \dot{v}_\mu$ with $v_\mu = \dot{z}_\mu(s)$. Moreover the velocity of the particle $\mathbf{v}^{(n)}$ in components reads as $v_\mu^{(n)}$ for $n \geq 4$ and they stand for the n -th derivative of v_μ with respect to parameter s . Notice that in general $\mathbf{v}^2 = \varepsilon$ with $\varepsilon = 0, \pm 1$ for null, time-like or space-like curves, respectively.

2.1 Liénard-Wiechert potential of a non-abelian particle

In this subsection we briefly review the Ref. [40] in order to introduce some procedure we will follow in future sections.

Let $(x_\mu - z_\mu(s))(x^\mu - z^\mu(s)) = 0$ be the light-cone condition, where $R = \dot{z}^\mu(x_\mu - z_\mu(s))$ is the retarded distance. The solution of this condition determines the advanced and the retarded proper time s , which we will use from now on.

In Ref. [40], the author proposed an ansatz in order to obtain some interesting classical solutions to the Yang-Mills equations in terms of the LW potentials. In the present article we will use this ansatz to study the radiation damping of an accelerated Yang-Mills point-particle following a singular trajectory $z^\mu(s)$ in the Minkowski four dimensional spacetime with metric $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. Consider the ansatz for the LW potential worked out in [40]

$$A_\mu^{(\text{ret})}(x) = H(R, c_i) \dot{z}_\mu(s) + G(R, c_i) \lambda_\mu(s), \quad (6)$$

where s is the *retarded* proper time and H and G are smooth functions of R and $c_i(s, a, a_k)$. Here c_i are functions of the acceleration of the particle $a \equiv \frac{1}{R} \ddot{z}^\mu(x_\mu - z_\mu(s))$ and additional scalars a_k defined by $a_k \equiv \lambda_\mu \frac{d^k \ddot{z}^\mu}{ds^k}$, $k = 0, 1, 2, 3$ with $a_0 = a$. Moreover

λ_μ is given by $\lambda^\mu = \frac{(x^\mu - z^\mu(s))}{R}$. R and λ_μ satisfy the conditions: $\lambda^\mu \dot{z}_\mu = 1$, $\lambda^\mu R_{,\mu} = 1$ and $\lambda^\mu a_{,\mu} = 0$ ($a_{,\mu} \equiv \frac{\partial a}{\partial x^\mu}$). This ansatz (6) was originally proposed in Ref. [41] as a proposal to find solutions to Einstein-Maxwell equations in D spacetime dimensions with a null fluid in the Kerr-Schild geometry. In this context the additional scalars a_k enter in the solution and generalizes the metric of Bonnor-Vaidya. Moreover, this ansatz was used to obtain some solutions to the source-free Yang-Mills equations in $D = 4$ with the requirement of finite total energy flux and total color charge at large distances ($R \rightarrow \infty$). Under these conditions non-trivial solutions require at least an expansion in $\frac{1}{R}$ of the form: $H = \frac{\beta'}{R} + \frac{\gamma'}{R^2} + O(R^{-3})$ and $G = \alpha + \frac{\beta}{R} + \frac{\gamma}{R^2} + O(R^{-3})$, where $\alpha, \beta, \gamma, \beta'$ and γ' are suitable functions. Taking these considerations into account the ansatz (6) is given by

$$A_\mu^{(\text{ret})}(x) = \left(\frac{\alpha_1}{R} + \frac{\alpha_2}{R^2}\right) \dot{z}_\mu + \left(\frac{\alpha_3}{R} + \frac{\alpha_4}{R^2} + \frac{\alpha_5}{R^3}\right) (x_\mu - z_\mu(s)) + O(R^{-4}), \quad (7)$$

where α_j with $j = 1, \dots, 5$ are Lie algebra-valued functions of the proper time and of the mentioned functions c_i , and they are R -independent functions. This expansion as a power series of $O(\frac{1}{R})$ generating non-trivial solutions to Yang-Mills equations has new implications on the physics of radiation by non-abelian particles as we will see in the further subsections. For large distances $R \rightarrow \infty$ in expansion (7), the term that survives is the usual $\frac{1}{R}$ term. Thus it is expected that the expansion will have implications for short distances from the source. It is well known that Yang-Mills theory describing the weak and strong interactions is valid at shorter distances than the electromagnetic interactions. Even though our treatment is done at the classical level it is important to study the radiation phenomena considering corrections to very near distances from the singular trajectory. By considering higher order terms in the expansion it will imply a better approximation, however it would make more complicated to study its consequences. In addition, for the Yang-Mills field we expect another type of corrections due to the non-linear terms of the field. It is clear that as soon as we have higher $1/R$ -corrections we will have more terms contributing due to the non-linearity.

As discussed in [2] for the abelian case, one can use the properties of Dirac's delta functions to reexpress the gauge potential and localize the LW potentials along the path of the particle. This procedure allows us to establish a relationship between the trajectory of the particle and the process of emission and absorbtion of radiation. Moreover it is easier to work in this representation given the possible divergences of the potential at $R = 0$. This physical situation also applies for non-abelian gauge theories, in which the potentials and the interaction is finite at long distances, at least within the classical description. The mentioned physical procedure of rewriting (7) in terms of Dirac's delta functions will be implemented in detail in the next subsection for the non-abelian theory.

2.2 Computation of the radiation fields

We follow the procedure introduced by Dirac in [2]. Thus we can express A_μ in the following form according to the ansatz (6) adapted for the retarded gauge field

$$A_\mu^{(\text{ret})} = 2 \int \left[\alpha_1 \dot{z}_\mu + \alpha_3 (x_\mu - z_\mu(s)) \right] \delta(\Omega) ds$$

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$$\begin{aligned}
& +4 \int \left[\alpha_2 \dot{z}_\mu(\bar{s}) + \alpha_4 (x_\mu - \bar{z}_\mu(\bar{s})) \right] \delta(\Omega) \delta(\bar{\Omega}) ds d\bar{s} \\
& +8 \int \alpha_5 (x_\mu - \tilde{z}_\mu(\tilde{s})) \delta(\Omega) \delta(\bar{\Omega}) \delta(\tilde{\Omega}) ds d\bar{s} d\tilde{s} + O(R^{-4}), \tag{8}
\end{aligned}$$

where $\Omega := (x_\mu - z_\mu(s))(x^\mu - z^\mu(s))$ vanishes as soon as x^μ coincides with the positions of the worldline of the particle $z^\mu(s)$. The variables s , $\bar{s}$ and $\tilde{s}$ stand for the integration variables in order to extract (after the integration) a factor $\frac{1}{R}$ in the vector potential for each δ -factor. Similarly for the factors $\bar{\Omega} := (x_\mu - \bar{z}_\mu(\bar{s}))(x^\mu - \bar{z}^\mu(\bar{s}))$ and $\tilde{\Omega} := (x_\mu - \tilde{z}_\mu(\tilde{s}))(x^\mu - \tilde{z}^\mu(\tilde{s}))$.

In order to calculate the corresponding retarded field strength we compute (8) and plugging it into Eq. (1), we obtain

$$\begin{aligned}
F_{\mu\nu}^{(\text{ret})} = & -\frac{\alpha_1}{R} \frac{d}{ds} \left[\frac{1}{R} \dot{z}_\mu (x_\nu - z_\nu) - \frac{1}{R} \dot{z}_\nu (x_\mu - z_\mu) \right] \\
& -\frac{2\alpha_2}{R^2} \frac{d}{ds} \left[\frac{1}{R} \dot{z}_\mu (x_\nu - z_\nu) - \frac{1}{R} \dot{z}_\nu (x_\mu - z_\mu) \right] \\
& + \left\{ \frac{-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3]}{R^3} \right\} \left[\dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \right] \\
& + \left\{ \frac{[\alpha_1, \alpha_5] + [\alpha_2, \alpha_4]}{R^4} + \frac{[\alpha_2, \alpha_5]}{R^5} \right\} \left[\dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \right] + O(R^{-6}). \tag{9}
\end{aligned}$$

In the process of getting Eq. (9) we used some relations between the α_i 's found in [40].

To find the Yang-Mills field strength in the neighborhood of the particle along its trajectory, we need to perform a Taylor expansion for $x_\mu = z_\mu(s_0) + \gamma_\mu$, such that γ_μ is a small function of the parameter σ . Here $s_0 - \sigma$ is the retarded proper-time. The Taylor expansion of the difference between the closed points near the trajectory and the trajectory itself can be written as

$$x_\mu - z_\mu(s_0 - \sigma) = \gamma_\mu + \sum_k \frac{(-1)^{k-1}}{k!} \sigma^k v_\mu^{(k-1)} \tag{10}$$

and

$$\dot{z}_\mu(s_0 - \sigma) = - \sum_k \frac{(-1)^k}{(k-1)!} \sigma^{k-1} v_\mu^{(k-1)}, \tag{11}$$

where $k = 1, 2, \dots, 9$. The possible values of k is due the last term in $F_{\mu\nu}$ is of order σ^{-8} , thus it is reasonable to take expansions up to order σ^9 to have all terms independent of σ , in analogy to the process carried out by Dirac in Ref. [2]. Using equations (10) and (11) we can calculate the expansions for R and the products $\dot{z}^\mu(x_\nu - z_\nu(s))$. This is done in appendix A.

The Yang-Mills field is singular at $R = 0$, thus in order to avoid these divergences it is convenient to introduce a regularization parameter ε which can be interpreted as

the size of the particle. Equations of motion cannot depend on ε , thus at the end of the computation we will have the finite and the divergent part. By taking the limit ε tends to zero we will take zero all the positive powers of ε and the remaining few negative powers of ε will be renormalized.

The field strength $F_{\mu\nu}^{(\text{rad})}$ of the radiating particle is related to the retarded and advanced field strength and to the *out* and *in* fields as follows

$$F_{\mu\nu}^{(\text{rad})} = F_{\mu\nu}^{(\text{ret})} - F_{\mu\nu}^{(\text{adv})} = F_{\mu\nu}^{(\text{out})} - F_{\mu\nu}^{(\text{in})}. \quad (12)$$

In analogy to the procedure performed by Dirac in Ref. [2] we can define the following relations

$$f_{\mu\nu} = F_{\mu\nu}^{(\text{act})} - \frac{1}{2} (F_{\mu\nu}^{(\text{ret})} + F_{\mu\nu}^{(\text{adv})}), \quad f_{\mu\nu} = F_{\mu\nu}^{(\text{in})} + \frac{1}{2} F_{\mu\nu}^{(\text{rad})}. \quad (13)$$

The possibility to define the field in this way will allow us to present the equations of motion in a compact and familiar form.

At this point it is worth highlighting the fact that in general these definitions were set out from first principles under the conditions of conservation of momentum and energy and in addition to the relativistic invariance of the Lagrangian. In our case all these conditions are satisfied classically, for this reason we have used these same tools. From Eq. (12) we can calculate $F_{\mu\nu}^{(\text{rad})}$ (for the details of the computation, see appendix B) which are expressed in the form

$$\begin{aligned} F_{\mu\nu}^{(\text{rad})} = & \left(-\frac{3}{4} \dot{\mathbf{v}}^2 A_2 - \frac{7}{12} \dot{\mathbf{v}}^2 A_4 - \frac{11}{12} \dot{\mathbf{v}} \ddot{\mathbf{v}} A_5 \right) (v_\mu \dot{v}_\nu - v_\nu \dot{v}_\mu) \\ & + \frac{1}{3} \left(\frac{3}{2} \dot{\mathbf{v}}^2 A_5 + 2A_3 + 4A_1 \right) (\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu) \\ & + \frac{1}{4} (A_4 + 3A_2) \left(v_\mu \ddot{v}_\nu + \frac{2}{3} \dot{v}_\mu \ddot{v}_\nu - \frac{2}{3} \ddot{v}_\mu \dot{v}_\nu - \ddot{v}_\mu v_\nu \right) \\ & + 2A_5 \left(-\frac{v_\mu v_\nu^{(4)}}{30} - \frac{\dot{v}_\mu \ddot{v}_\nu}{24} + \frac{\ddot{v}_\mu \dot{v}_\nu}{24} + \frac{v_\mu^{(4)} v_\nu}{30} \right), \end{aligned} \quad (14)$$

where $A_1 = -\alpha_1$, $A_2 = -2\alpha_2$, $A_3 = -2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3]$, $A_4 = [\alpha_1, \alpha_5] + [\alpha_2, \alpha_4]$, $A_5 = [\alpha_2, \alpha_5]$.

To calculate the advanced field strength $F_{\mu\nu}^{(\text{adv})}$ from $F_{\mu\nu}^{(\text{ret})}$ it is only required to change ε to $-\varepsilon$, however unlike Dirac's case [2] in the computation of $F_{\mu\nu}^{(\text{rad})}$ there will survive the even power terms in $F_{\mu\nu}^{(\text{rad})}$. This result is interesting because physically the radiated field has a subtle dependence on these terms which would involve divergences of the radiated field. However without these divergences the equations of motion would not be obtained correctly.

An important point is that in the second term of (14) there is a term that corresponds exactly to Dirac's radiative field

$$F_{\mu\nu D}^{(\text{rad})} = \frac{4}{3} A_1 (\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu), \quad (15)$$

which is the usual one for the radiation produced by an accelerated abelian charge.

3 The Equations of Motion

If a charged classical particle moves along its singular worldline it radiates and generates a flow of momentum and energy which travels through a tube surrounding the trajectory of the particle. This tube can be represented by a three-dimensional surface. In Dirac's analysis [2], it was shown that the three-dimensional surface split into the product of a two-dimensional surface S times its transverse part T . The mentioned flow can be calculated using the energy-momentum tensor of the field in the neighborhood of a point along the trajectory of the particle. The energy-momentum tensor is given by

$$T_{\mu\nu}^F = \text{Tr} \left(F_{\mu}^{\rho} F_{\rho\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right). \quad (16)$$

We can consider a spherical hyper-surface of radius ε covering the path. The integral over this surface is $4\pi\varepsilon^2$, and the expression between the braces is precisely the equation of motion outlined in appendix C

$$\begin{aligned} \int \varepsilon^{-1} T_{\mu\rho} \gamma^{\rho} dS |dx_{\parallel}| &= \int \text{Tr} \left\{ \frac{A_5 f_{\mu\nu} v^{\nu} + Q_{14} v_{\mu} + Q_{11} \dot{v}_{\mu} + Q_{17} \ddot{v}_{\mu} + Q_{16} \ddot{\ddot{v}}_{\mu}}{\varepsilon^2} \right. \\ &+ \frac{A_5 f_{\mu\nu} \dot{v}^{\nu} + Q_{36} f_{\mu\nu} v^{\nu} + Q_9 v_{\mu} + Q_3 \dot{v}_{\mu} + Q_{10} \ddot{v}_{\mu} + Q_{15} \ddot{\ddot{v}}_{\mu}}{\varepsilon} \\ &+ Q_{33} f_{\mu\nu} \ddot{v}^{\nu} + Q_8 v_{\mu} + Q_6 \dot{v}_{\mu} + Q_2 \ddot{v}_{\mu} + Q_{12} \ddot{\ddot{v}}_{\mu} - A_4 f_{\mu\nu} \dot{v}^{\nu} + Q_{35} f_{\mu\nu} v^{\nu} \\ &\left. + \frac{Q_{24} \dot{v}_{\mu}}{\varepsilon^5} + \frac{Q_{23} v_{\mu} + Q_{22} \dot{v}_{\mu} + Q_{20} \ddot{v}_{\mu}}{\varepsilon^4} + \frac{Q_{13} v_{\mu} + Q_{21} \dot{v}_{\mu} + Q_{19} \ddot{v}_{\mu} + Q_{18} \ddot{\ddot{v}}_{\mu}}{\varepsilon^3} \right\} ds, \quad (17) \end{aligned}$$

where the measure of the integral is the product of the measure dS on the 2-surface S and the measure on T which is called $|dx_{\parallel}| = (1 - \gamma \dot{\mathbf{v}}^2) ds$. Moreover the values of the functions Q_i 's are given in appendix C.

All terms depending on ε must disappear when we take the limit $\varepsilon \rightarrow 0$. This is an impediment to obtain the equations of motion directly from the calculation, due to the terms that arise as negative powers of ε . These terms induce divergences in the equation of motion that should be removed. One strategy is to use the fact that the braces is an exact differential and it can be written as the vector $\dot{B}_{\mu}$. However there is an obstruction to find Dirac's constraint in which $\mathbf{v}\dot{\mathbf{B}} \equiv v_{\mu} \dot{B}^{\mu} = 0$. A useful procedure is to consider

$$\begin{aligned} \dot{B}_{\mu} &= \left(\frac{Q_{23}}{\varepsilon^4} + \frac{Q_{13}}{\varepsilon^3} + \frac{Q_{14}}{\varepsilon^2} + \frac{Q_9}{\varepsilon} + Q_8 \right) v_{\mu} + \left(\frac{Q_{24}}{\varepsilon^5} + \frac{Q_{22}}{\varepsilon^4} + \frac{Q_{21}}{\varepsilon^3} + \frac{Q_{11}}{\varepsilon^2} + \frac{Q_3}{\varepsilon} + Q_6 \right) \dot{v}_{\mu} \\ &+ \left(\frac{Q_{20}}{\varepsilon^4} + \frac{Q_{19}}{\varepsilon^3} + \frac{Q_{17}}{\varepsilon^2} + \frac{Q_{10}}{\varepsilon} + Q_2 \right) \ddot{v}_{\mu} + \left(\frac{Q_{18}}{\varepsilon^3} + \frac{Q_{16}}{\varepsilon^2} + \frac{Q_{15}}{\varepsilon} + Q_{12} \right) \ddot{\ddot{v}}_{\mu} \\ &+ \left(\frac{A_5}{\varepsilon^2} + \frac{Q_{36}}{\varepsilon} + Q_{35} \right) f_{\mu\nu} v_{\nu} + \left(\frac{A_5}{\varepsilon} - A_4 \right) f_{\mu\nu} \dot{v}_{\nu} + Q_{33} f_{\mu\nu} \ddot{v}_{\nu}. \end{aligned} \quad (18)$$

We will determine a solution to the above expression in such a way that we can obtain the value of B_{μ} . In the present case the product $\mathbf{v}\dot{\mathbf{B}} \neq 0$ which complicates the

calculation of equations of motion. However a good strategy is to find, from our original set of vectors $\dot{B}_\mu, v_\mu, \dot{v}_\mu, \ddot{v}_\mu, \ddot{\ddot{v}}_\mu$, a new set of normalized vectors $\dot{B}_\mu, \hat{u}_{1\mu}, \hat{u}_{2\mu}, \hat{u}_{3\mu}, \hat{u}_{4\mu}$ (see appendix C). It is now easy to prove that $\hat{\mathbf{u}}_i \dot{\mathbf{B}} = 0, i = 1, \dots, 4$. Here $\hat{u}_{i\mu} = u_{i\mu}/|u_i|$ and we have $\hat{\mathbf{u}}_i \hat{\mathbf{u}}_i = 0$, due $\hat{\mathbf{u}}_i \hat{\mathbf{u}}_i = 1$. Thus a simple solution to the equation $\hat{\mathbf{u}}_i \dot{\mathbf{B}} = 0$ is given by

$$B_\mu = k\hat{u}_{4\mu}. \quad (19)$$

Now by substituting the value of $\hat{u}_{4\mu}$ in Eq. (19), we have a non-homogeneous first-order differential equation, whose coefficients are functions of the norm $|\dot{\mathbf{B}}| = \dot{B}$, that involves the limit $\varepsilon \rightarrow 0$ and which does not exactly cancel the divergences. A solution to this problem is to choose $\dot{B}_\mu$ so that it respects the general form of $\dot{B}_\mu$, i.e. $\hat{\mathbf{u}}_i \dot{\mathbf{B}} = 0$ and also eliminates the divergences. Therefore the best way to do this is to take the linear combination

$$\dot{B}_\mu = f_0(\varepsilon)v_\mu + f_1(\varepsilon)\dot{v}_\mu + f_2(\varepsilon)\ddot{v}_\mu, \quad (20)$$

for some suitable functions f_i . Moreover the choice of (20) is not arbitrary because $\hat{\mathbf{u}}_i \dot{\mathbf{B}} = 0$ requires the additional condition: $Q_{18} = 0$ (since $\hat{\mathbf{u}}_i \mathbf{v}^{(j)} = 0$ except for $j = 3$). This implies that there is an abelian subgroup of the Lie's algebra satisfying $[\alpha_2, \alpha_5]^2 = 0$. Consequently it reduces the equations to

$$\begin{aligned} \dot{B}_\mu = & \left[\frac{-\frac{2}{3}(A_4^2 + A_2^2 + A_2A_4)\dot{\mathbf{v}}^2}{\varepsilon^2} + \frac{(\frac{2}{3}A_1A_2 + \frac{1}{3}A_2A_3 + \frac{5}{6}A_1A_4 + \frac{7}{6}A_3A_4)\dot{\mathbf{v}}^2}{\varepsilon} \right. \\ & + \frac{(-\frac{1}{2}A_2^2 - \frac{5}{6}A_4^2 - \frac{5}{3}A_2A_4)\dot{\mathbf{v}}\ddot{\mathbf{v}}}{\varepsilon} - \frac{1}{24}(8A_2^2 + 9A_4^2 + 13A_2A_4)\dot{\mathbf{v}}^4 \\ & - \frac{1}{6}(2A_2^2 + A_4^2 + 3A_2A_4)\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{6}(2A_2^2 + A_4^2 + 3A_2A_4)\dot{\mathbf{v}}^2 \\ & - \frac{1}{2}(A_3^2 + A_1A_3)\dot{\mathbf{v}}^2 + \frac{1}{12}(9A_1A_2 + 17A_2A_3 + 9A_1A_4 + 13A_3A_4)\dot{\mathbf{v}}\ddot{\mathbf{v}} \left. \right] v_\mu \\ & + \left[\frac{-A_2^2 + A_4^2}{\varepsilon^3} + \frac{3A_1A_2 + A_1A_4 - A_2A_3 - 3A_3A_4}{2\varepsilon^2} + \frac{-A_2^2\dot{\mathbf{v}}^2 - A_2A_4\dot{\mathbf{v}}^2 + A_3^2 - A_1^2}{2\varepsilon} \right. \\ & + \frac{1}{8}(5A_1A_2 + 3A_2A_3 + 3A_1A_4 + A_3A_4)\dot{\mathbf{v}}^2 + \frac{1}{6}(A_4^2 - 5A_2^2 - 4A_2A_4)\dot{\mathbf{v}}\ddot{\mathbf{v}} \left. \right] \dot{v}_\mu \\ & + \left[\frac{-2A_2^2 + A_4A_2 + A_4^2}{3\varepsilon^2} + \frac{2A_1A_2 + A_1A_4 - 2A_2A_3 - A_3A_4}{3\varepsilon} - \frac{1}{6}(2A_2^2 + A_2A_4)\dot{\mathbf{v}}^2 \right] \ddot{v}_\mu \\ & + \left(\frac{A_2 - A_4}{\varepsilon} - A_1 + A_3 \right) f_{\mu\nu}v^\nu - A_4f_{\mu\nu}\dot{v}^\nu. \end{aligned} \quad (21)$$

We now can take $\dot{B}_\mu$ of the general form:

$$\dot{B}_\mu = f_0(\varepsilon)v_\mu + f_1(\varepsilon)\dot{v}_\mu + f_2(\varepsilon)\ddot{v}_\mu + k_3f_{\mu\nu}v^\nu, \quad (22)$$

where the last term in (22) is not consistent with (20). However, it can be proved that this term satisfies the condition $\dot{\mathbf{B}}\hat{\mathbf{u}}_i = 0$ but this is possible only if $L_\nu\dot{v}^\nu := 0$ where $L_\nu := f_{\mu\nu}\hat{u}_4^\mu$ is a null vector. The physical meaning of this is to have a privileged

direction of the Yang-Mills field, see Eq. (112) in appendix C, for which it is allowed to add it to Eq. (22). Then the equations of motion can be expressed in the form

$$\begin{aligned}
& \left[\rho_1 \dot{\mathbf{v}}^2 + \rho_2 \dot{\mathbf{v}} \ddot{\mathbf{v}} - \frac{1}{24} (8A_2^2 + 9A_4^2 + 13A_2A_4) \dot{\mathbf{v}}^4 \right. \\
& - \frac{1}{6} (2A_2^2 + A_4^2 + 3A_2A_4) \dot{\mathbf{v}} \ddot{\mathbf{v}} - \frac{1}{6} (2A_2^2 + A_4^2 + 3A_2A_4) \ddot{\mathbf{v}}^2 \\
& - \frac{1}{2} (A_3^2 + A_1A_3) \dot{\mathbf{v}}^2 + \frac{1}{12} (9A_1A_2 + 17A_2A_3 + 9A_1A_4 + 13A_3A_4) \dot{\mathbf{v}} \ddot{\mathbf{v}} \left. \right] v_\mu \\
& + \left[m + \mu_1 \dot{\mathbf{v}}^2 + \frac{1}{8} (5A_1A_2 + 3A_2A_3 + 3A_1A_4 + A_3A_4) \dot{\mathbf{v}}^2 + \frac{1}{6} (A_4^2 - 5A_2^2 - 4A_2A_4) \dot{\mathbf{v}} \ddot{\mathbf{v}} \right] \dot{v}_\mu \\
& + \left[\mu_3 - \frac{1}{6} (2A_2^2 + A_2A_4) \dot{\mathbf{v}}^2 \right] \ddot{v}^\mu + (q - A_1 + A_3) f_{\mu\nu} v_\nu - A_4 f_{\mu\nu} \dot{v}^\nu = 0, \tag{23}
\end{aligned}$$

where ρ_1 , ρ_2 , μ_1 , μ_3 , m and q are suitable renormalization constants. Furthermore the form of the functions $f_i(\varepsilon)$ that cancels the divergences in Eq. (21) is of the form

$$\begin{aligned}
f_0 = \rho_1 \dot{\mathbf{v}}^2 + \frac{-\frac{2}{3} (A_4^2 + A_2^2 + A_2A_4) \dot{\mathbf{v}}^2}{\varepsilon^2} + \frac{(\frac{2}{3} A_1A_2 + \frac{1}{3} A_2A_3 + \frac{5}{6} A_1A_4 + \frac{7}{6} A_3A_4) \dot{\mathbf{v}}^2}{\varepsilon} \\
+ \rho_2 \dot{\mathbf{v}} \ddot{\mathbf{v}} + \frac{(-\frac{1}{2} A_2^2 - \frac{5}{6} A_4^2 - \frac{5}{3} A_2A_4) \dot{\mathbf{v}} \ddot{\mathbf{v}}}{\varepsilon}, \tag{24}
\end{aligned}$$

$$\begin{aligned}
f_1 = m + \mu_1 \dot{\mathbf{v}}^2 + \frac{-A_2^2 + A_4^2}{\varepsilon^3} + \frac{3A_1A_2 + A_1A_4 - A_2A_3 - 3A_3A_4}{2\varepsilon^2} \\
+ \frac{-A_2^2 \dot{\mathbf{v}}^2 - A_2A_4 \dot{\mathbf{v}}^2 + A_3^2 - A_1^2}{2\varepsilon}, \tag{25}
\end{aligned}$$

$$f_2 = \mu_3 + \frac{-2A_2^2 + A_4A_2 + A_4^2}{3\varepsilon^2} + \frac{2A_1A_2 + A_1A_4 - 2A_2A_3 - A_3A_4}{3\varepsilon}. \tag{26}$$

Here k_3 is a correction to the charge of the non-abelian particle [4, 5]

$$k_3 = -q + \frac{A_2 - A_4}{\varepsilon}. \tag{27}$$

Now we give some remarks about the physical interpretation of the general equations of motion (23). It involves higher-order derivatives of the velocity with respect to the proper time s . All these terms are useful to study the classical electrodynamics of a non-abelian particle radiating color field near the source. One of the terms is, of course, the usual covariant Lorentz force. The radiation damping term also is present here. This term can be found from the term in the fourth line taking only the term inside the square bracket the one independent on the velocity, i.e. m . This will be worked out in Section 4.1. There are other higher-order terms in the velocity in (23) which correspond to Laue's terms that arise in the relativistic version [42]. In addition there are two types of contributions to the non-abelian case presented here. The first one comes from the obvious non-abelian generalization. That means that there are terms coming from the non-linear part of the field strength. The second kind of terms come from the expansion of the vector potential (7) in $\frac{1}{R}$. This expansion reminds a

truncated multipolar expansion (without the corresponding multipole sources) when the system has a not-point-like structure. In the present situation, this is not the case, since we are considering point-like particles. However this expansion produces additional higher-derivative terms in the general equation of motion (23). It looks like the equation of a charged sphere given in [43]. Thus we interpret the other terms in the equation of motion, besides the usual radiation damping, as terms that are related to the additional terms to the term $\frac{1}{R}$ in the ansatz (7).

Finally it is well known that non-linear higher-order derivative differential equations in time have a chaotic behavior [44, 45]. This is precisely the structure of the equation of motion (23). Thus in some sense it is expected a chaotic behavior in the higher-order terms of this equation. We will see that in Section 4.3 that the resulting equation has a fourth-order time derivative of the position of particle. This behavior is known as “snap” in the non-linear systems literature.

4 Some Examples

In this section we give some examples that correspond to some particular situations of the general equation of motion (23).

4.1 Dirac’s radiation damping effect

A particular case of Eq. (21) is when one considers $\alpha_2, \alpha_3, \alpha_4, \alpha_5 = 0$, which is justly the abelian case according to Eq. (9). For this case we have that Eq. (21) is of the form

$$\dot{B}_\mu = \frac{1}{2} \alpha_1^2 \varepsilon^{-1} \dot{v}_\mu - \alpha_1 f_{\mu\nu} v^\nu, \quad (28)$$

where $f_{\mu\nu}$ is determined by Eq. (13).

In this case $B_\mu = k v_\mu$ is a solution to Eq. (28), where $k = 1/2 \alpha_1^2 \varepsilon^{-1} + \beta$. Thus we have the traditional equation of a charged particle in an electromagnetic field with corrections given the emission/absorption of radiation or the Abraham-Lorentz-Dirac equation [1, 2]

$$m \dot{v}_\mu = -\text{Tr}(\alpha_1 f_{\mu\nu} v^\nu), \quad (29)$$

where $m = \text{Tr}(\beta)$.

4.2 The first non-abelian radiation damping effect

Now we consider the case in which $\alpha_2 = \alpha_5 = 0$. For this situation we find that $A_1 = -\alpha_1$, and $A_3 = [\alpha_1, \alpha_4]$. Plugging these expressions into Eq. (21) we have that the equations of motion take the simple form

$$\dot{B}_\mu = \frac{1}{2} (A_3^2 - A_1^2) \varepsilon^{-1} \dot{v}_\mu - \frac{1}{2} (A_3^2 + A_1 A_3) \dot{v}^2 v_\mu + (A_3 - A_1) f_{\mu\nu} v^\nu. \quad (30)$$

In order to obtain the equations of motion we have to take the trace in both sides of the previous equation

$$\text{Tr}(\dot{B}_\mu) = \text{Tr}\left(\frac{1}{2}(A_3^2 - A_1^2)\varepsilon^{-1}\dot{v}_\mu - \frac{1}{2}(A_3^2 + A_1A_3)\dot{\mathbf{v}}^2v_\mu + (A_3 - A_1)f_{\mu\nu}v^\nu\right). \quad (31)$$

The first term on the right-hand side can be renormalized by taking $B_\mu = kv_\mu$ and $k = \frac{1}{2}([\alpha_1, \alpha_4]^2 - \alpha_1^2)\varepsilon^{-1} + \beta$. Then we find the equation of motion

$$m\dot{v}_\mu + \mu_1\dot{\mathbf{v}}^2v_\mu = \text{Tr}(\tilde{\mu}f_{\mu\nu}v^\nu). \quad (32)$$

Thus we have the simplest non-abelian equation with $m = \text{Tr}(\beta)$ and $\mu_1 = \text{Tr}(\tilde{\mu}_1)$, where $\tilde{\mu}_1 = \frac{1}{2}([\alpha_1, \alpha_4]^2 - \alpha_1[\alpha_1, \alpha_4])$ and $\tilde{\mu} = [\alpha_1, \alpha_4] + \alpha_1$, respectively. The field $f_{\mu\nu}$ in (13) is modified only in the non-abelian charges of the radiation term which has the form

$$F_{\mu\nu}^{(\text{rad})} = \frac{2}{3}([\alpha_1, \alpha_4] - 2\alpha_1)(\ddot{v}_\mu v_\nu - v_\mu \ddot{v}_\nu). \quad (33)$$

We now study what conditions must be satisfied by the matrices α_i such that Eqs. (32) and (33) are fulfilled. From these equations, with $F_{\mu\nu}^{(\text{in})} = 0$, we have that Eq. (32) transforms into

$$\begin{aligned} m\dot{v}_\mu + \mu_1\dot{\mathbf{v}}^2v_\mu &= \frac{2}{3}\text{Tr}\left([\alpha_1, \alpha_4]^2 - \alpha_1[\alpha_1, \alpha_4] - 2\alpha_1^2\right)(\ddot{v}_\mu v_\nu - v_\mu \ddot{v}_\nu)v^\nu \\ &= \mu_2(\ddot{v}_\mu v_\nu - v_\mu \ddot{v}_\nu)v^\nu \end{aligned} \quad (34)$$

where

$$\begin{aligned} \mu_2 &= \frac{2}{3}\text{Tr}\left([\alpha_1, \alpha_4]^2 - \alpha_1[\alpha_1, \alpha_4] - 2\alpha_1^2\right) \\ &= \frac{4}{3}\tilde{\mu}_1 - \frac{4}{3}\text{Tr}\alpha_1^2. \end{aligned} \quad (35)$$

Moreover we have the relations

$$\mathbf{v}^2 = 1, \quad \mathbf{v}\dot{\mathbf{v}} = 0, \quad \mathbf{v}\ddot{\mathbf{v}} = -\dot{\mathbf{v}}^2. \quad (36)$$

Then we have the equation of motion reduces to the following two equations

$$m\dot{v}_\mu + \mu_1\dot{\mathbf{v}}^2v_\mu = \mu_2\ddot{v}_\mu + \mu_2\dot{\mathbf{v}}^2v_\mu, \quad (37)$$

$$m\ddot{v}_\mu - \mu_2\ddot{v}_\mu + (\mu_1 - \mu_2)\mathbf{v}^2v_\mu = 0. \quad (38)$$

Now from the above equations and using conditions (36) we have as solution $\mu_1 = 0$ or

$$\text{Tr}\left([\alpha_1, \alpha_4]^2 - \alpha_1[\alpha_1, \alpha_4]\right) = 0. \quad (39)$$

From this last equation we observe that if α_1 commutes with α_4 , then this case reduces to the previous abelian case in Section 4.1.

Let us now analyze the second non-abelian case of interest, such that k in B_μ are both functions of velocity and acceleration.

4.3 The second non-abelian radiation damping effect

This case is more general than the previous one since it includes divergences of order ε^{-1} , ε^{-2} and ε^{-3} in the mass and of order ε^{-1} in the color's charge. If we take $\alpha_2 = 0$ and $\alpha_4 = 0$, then equations of motion read

$$\begin{aligned} \dot{B}_\mu = & \left(-\frac{2A_4^2\dot{\mathbf{v}}^2}{3\varepsilon^2} - \frac{3}{8}A_4^2\dot{\mathbf{v}}^4 - \frac{1}{6}A_4^2\ddot{\mathbf{v}}^2 + \frac{3}{4}A_1A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{6}A_4^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{5A_1A_4\dot{\mathbf{v}}^2}{6\varepsilon} - \frac{5A_4^2\dot{\mathbf{v}}\ddot{\mathbf{v}}}{6\varepsilon} \right) v_\mu \\ & + \left(\frac{3}{8}A_1A_4\dot{\mathbf{v}}^2 + \frac{2}{3}A_4^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{6}A_4^2\mathbf{v}\ddot{\mathbf{v}} + \frac{A_4^2}{\varepsilon^3} + \frac{A_1A_4}{2\varepsilon^2} - \frac{A_1^2}{2\varepsilon} \right) \dot{v}_\mu + \left(\frac{A_4^2}{3\varepsilon^2} + \frac{A_1A_4}{3\varepsilon} \right) \ddot{v}_\mu \\ & - A_1f_{\mu\nu}v^\nu - \frac{A_4f_{\mu\nu}v^\nu}{\varepsilon} - A_4f_{\mu\nu}\dot{v}^\nu. \end{aligned} \quad (40)$$

The simplest expression for $\dot{B}_\mu$ is given by

$$\dot{B}_\mu = g_0v_\mu + k_1\dot{v}_\mu + k_2\ddot{v}_\mu + k_3f_{\mu\nu}v^\nu, \quad (41)$$

for suitable coefficients g_0 , k_1 , k_2 and k_3 . In this case we can simplify the previous equation by defining the non-abelian relations

$$k_1 := \varepsilon^{-3}A_4^2 + \frac{\varepsilon^{-2}}{2}A_1A_4 - \frac{\varepsilon^{-1}}{2}A_1^2 - m, \quad k_2 := \frac{\varepsilon^{-2}}{3}A_4^2 + \frac{\varepsilon^{-1}}{3}A_1A_4 - \mu_3, \quad k_3 := -\varepsilon^{-1}A_4 + q, \quad (42)$$

where m , μ_3 and q are some suitable renormalization parameters.

The first term in Eq. (41) depends on the function g_0 which is chosen as:

$$g_0 = - \left(\frac{2}{3}\varepsilon^{-2}A_4^2 - \frac{5}{6}\varepsilon^{-1}A_1A_4 + \rho_1 \right) \dot{\mathbf{v}}^2 - \left(\frac{5}{6}\varepsilon^{-1}A_4^2 + \rho_2 \right) \dot{\mathbf{v}}\ddot{\mathbf{v}}, \quad (43)$$

where now ρ_1 and ρ_2 are the renormalization parameters.

Once again the last term in Eq. (41) is not consistent with Eq. (20) but it satisfies the condition $\mathbf{\hat{B}}\hat{\mathbf{u}}_i = 0$. Thus similar conditions are also fulfilled for this term. Consequently the equations of motion can be expressed in the form

$$\begin{aligned} & \left[\rho_1\dot{\mathbf{v}}^2 + \left(\rho_2 - \frac{3}{4}\alpha_1[\alpha_1, \alpha_5] \right) \dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{3}{8}[\alpha_1, \alpha_5]^2\dot{\mathbf{v}}^4 - \frac{1}{6}[\alpha_1, \alpha_5]^2\ddot{\mathbf{v}}^2 - \frac{1}{6}[\alpha_1, \alpha_5]^2\dot{\mathbf{v}}\ddot{\mathbf{v}} \right] v_\mu \\ & + m\dot{v}_\mu + (\mu_1\dot{\mathbf{v}}^2 + \mu_2\dot{\mathbf{v}}\ddot{\mathbf{v}})\dot{v}_\mu + \mu_3\ddot{v}_\mu = (-\alpha_1 + q)f_{\mu\nu}v^\nu - [\alpha_1, \alpha_5]f_{\mu\nu}\dot{v}^\nu, \end{aligned} \quad (44)$$

where we have defined the quantities $\mu_1 := -\frac{3}{8}\alpha_1[\alpha_1, \alpha_5]$, $\mu_2 := \frac{2}{3}[\alpha_1, \alpha_5]^2$ in order to simplify the expression.

In order to find the equations of motion it is necessary to take the traces in both sides of the previous equation. This equation includes non-abelian corrections to the charge and a new coupling of the field strength with the velocity of the particle, where the tensor $f_{\mu\nu}$ is expressed through the radiated field Eq. (13) that now has the form

$$\begin{aligned} F_{\mu\nu}^{(\text{rad})} = & \frac{7}{12}[\alpha_1, \alpha_5]\dot{v}^2(v_\mu\dot{v}_\nu - v_\nu\dot{v}_\mu) - \frac{4}{3}\alpha_1(\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu) \\ & + \frac{1}{4}[\alpha_1, \alpha_5] \left(v_\mu\ddot{v}_\nu + \frac{2}{3}\dot{v}_\mu\ddot{v}_\nu - \frac{2}{3}\ddot{v}_\mu\dot{v}_\nu - \ddot{v}_\mu v_\nu \right). \end{aligned} \quad (45)$$

Given the equations of motion (44) and the radiation field strength (45), we proceed to obtain some consistency conditions. With $F_{\mu\nu}^{(\text{in})} = 0$ multiplying (44) by v^μ it is possible to show that the resulting equation is given by

$$(\rho_1 - \mu_3)\dot{\mathbf{v}}^2 + (\rho_2 - \frac{25}{12}\alpha_1[\alpha_1, \alpha_5])\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{9}{8}[\alpha_1, \alpha_5]^2\dot{\mathbf{v}}^4 - \frac{1}{6}[\alpha_1, \alpha_5]^2\ddot{\mathbf{v}}^2 - \frac{5}{12}[\alpha_1, \alpha_5]^2\dot{\mathbf{v}}\ddot{\mathbf{v}} = 0. \quad (46)$$

Now multiplying Eq. (44) by $\dot{v}^\mu$ and following a similar procedure as the one to obtain the above equation, we get

$$m\dot{\mathbf{v}}^2 + \left(\mu_1 + \frac{3}{4}(-\alpha_1 + q)[\alpha_1, \alpha_5]\right)\dot{\mathbf{v}}^4 + \left[\mu_3 + \frac{4}{3}(-\alpha_1 + q)\right]\dot{\mathbf{v}}\ddot{\mathbf{v}} + \mu_2(\dot{\mathbf{v}}\ddot{\mathbf{v}})\dot{\mathbf{v}}^2 + \frac{1}{4}(q - \alpha_1)[\alpha_1, \alpha_5]\ddot{\mathbf{v}}\dot{\mathbf{v}} = 0. \quad (47)$$

4.4 The third non-abelian radiation damping effect

In this case we choose the commutation relations to be $[\alpha_1, \alpha_5] = [\alpha_2, \alpha_4] = [\alpha_2, \alpha_5] = 0$. Then the equations of motion are

$$\begin{aligned} \dot{B}_\mu = & \left(-\frac{1}{3}A_2^2\dot{\mathbf{v}}^4 - \frac{1}{2}A_3^2\dot{\mathbf{v}}^2 + \frac{2A_1A_2\dot{\mathbf{v}}^2}{3\varepsilon} - \frac{1}{2}A_1A_3\dot{\mathbf{v}}^2 + \frac{A_2A_3\dot{\mathbf{v}}^2}{3\varepsilon} - \frac{1}{3}A_2^2\ddot{\mathbf{v}}^2 + \frac{3}{4}A_1A_2\dot{\mathbf{v}}\ddot{\mathbf{v}} \right. \\ & + \frac{17}{12}A_2A_3\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{3}A_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{2A_2^2\dot{\mathbf{v}}^2}{3\varepsilon^2} - \frac{A_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}}}{2\varepsilon} \Big) v_\mu \\ & + \left(\frac{A_3^2}{2\varepsilon} + \frac{5}{8}A_1A_2\dot{\mathbf{v}}^2 + \frac{3}{8}A_2A_3\dot{\mathbf{v}}^2 + \frac{3A_1A_2}{2\varepsilon^2} + \frac{1}{6}A_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{3}A_2^2\mathbf{v}\ddot{\mathbf{v}} - \frac{A_1^2}{2\varepsilon} - \frac{A_2^2\dot{\mathbf{v}}^2}{2\varepsilon} \right. \\ & - \frac{A_2A_3}{2\varepsilon^2} - \frac{A_2^2}{\varepsilon^3} \Big) \dot{v}_\mu \\ & + \left(-\frac{2A_2A_3}{3\varepsilon} - \frac{1}{3}A_2^2\dot{\mathbf{v}}^2 + \frac{2A_1A_2}{3\varepsilon} - \frac{2A_2^2}{3\varepsilon^2} \right) \ddot{v}_\mu + \left(\frac{A_2}{\varepsilon} + A_3 - A_1 \right) f_{\mu\nu}v^\nu. \end{aligned} \quad (48)$$

The vector $\dot{B}_\mu$ can be expressed in the form:

$$\dot{B}_\mu = g_0v_\mu + g_1\dot{v}_\mu + k_2\ddot{v}_\mu + k_3f_{\mu\nu}v^\nu, \quad (49)$$

for suitable parameters coefficients g_0 , g_1 , k_2 and k_3 that eliminate the divergences as ε goes to zero.

The renormalization of the equations of motion is carried out through the functions g_0 and g_1 depending on the velocities and accelerations, and the constants k_2 and k_3 that appear in the solution (49) are given by

$$g_0 := \frac{2A_1A_2\dot{\mathbf{v}}^2}{3\varepsilon} + \frac{A_2A_3\dot{\mathbf{v}}^2}{3\varepsilon} - \frac{2A_2^2\dot{\mathbf{v}}^2}{3\varepsilon^2} - \frac{A_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}}}{2\varepsilon} - \rho_1\dot{\mathbf{v}}^2 - \rho_2\dot{\mathbf{v}}\ddot{\mathbf{v}}, \quad (50)$$

$$g_1 := \frac{A_3^2}{2\varepsilon} + \frac{3A_1A_2}{2\varepsilon^2} - \frac{A_1^2}{2\varepsilon} - \frac{A_2^2}{2\varepsilon}\dot{\mathbf{v}}^2 - \frac{A_2A_3}{2\varepsilon^2} - \frac{A_2^2}{\varepsilon^3} - \mu_1 - \mu_2\dot{\mathbf{v}}^2, \quad (51)$$

$$k_2 := -\frac{2A_2A_3}{3\varepsilon} + \frac{2A_1A_2}{3\varepsilon} - \frac{2A_2^2}{3\varepsilon^2} - \mu_3, \quad k_3 := \left(\frac{A_2}{\varepsilon} + q\right), \quad (52)$$

where $\rho_1, \rho_2, \mu_1, \mu_2, \mu_3$ and q are the renormalization parameters.

In order to fulfill condition (112) we should have $A_4, A_5 = 0$. If we use the Eqs. (49)-(52), then the non-abelian equations of motion are

$$\begin{aligned} & \left(-\frac{4}{3}\alpha_2^2\dot{\mathbf{v}}^4 - \frac{1}{2}(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])^2\dot{\mathbf{v}}^2 + \frac{1}{2}\alpha_1(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])\dot{\mathbf{v}}^2 \right. \\ & + \rho_1\dot{\mathbf{v}}^2 - \frac{4}{3}\alpha_2^2\ddot{\mathbf{v}}^2 + \frac{3}{2}\alpha_1\alpha_2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{17}{6}\alpha_2(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{4}{3}\alpha_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \rho_2\dot{\mathbf{v}}\ddot{\mathbf{v}} \Big) v_\mu \\ & + \left(\frac{5}{4}\alpha_1\alpha_2\dot{\mathbf{v}}^2 - \frac{3}{4}\alpha_2(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])\dot{\mathbf{v}}^2 - \frac{10}{3}\alpha_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \mu_1 + \mu_2\dot{\mathbf{v}}^2 \right) \dot{v}_\mu \\ & + \left(-\frac{4}{3}\alpha_2^2\dot{\mathbf{v}}^2 + \mu_3 \right) \ddot{v}_\mu = (q - \alpha_1 + 2\dot{\alpha}_2 - [\alpha_1, \alpha_4] - [\alpha_2, \alpha_3]) f_{\mu\nu}v^\nu. \end{aligned} \quad (53)$$

In this expression the interaction term $f_{\mu\nu}v^\nu$ contains non-abelian corrections to the charge of the Yang-Mills particle. Analogously to the previous cases we can determine the radiated field $F_{\mu\nu}^{(\text{rad})}$ through Eq. (13). In this case it becomes

$$\begin{aligned} F_{\mu\nu}^{(\text{rad})} &= \frac{11}{6}\alpha_2\dot{\mathbf{v}}^2(v_\mu\dot{v}_\nu - \dot{v}_\mu v_\nu) + \frac{2}{3}(-2\dot{\alpha}_1 - 2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])(\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu) \\ & - \frac{3}{2}\alpha_2\left(v_\mu\ddot{v}_\nu + \frac{2}{3}\dot{v}_\mu\ddot{v}_\nu - \frac{2}{3}\ddot{v}_\mu\dot{v}_\nu - \ddot{v}_\mu v_\nu\right). \end{aligned} \quad (54)$$

If we take $F_{\mu\nu}^{(\text{in})} = 0$ and multiplying (53) by v^μ and consider (54) we find a consistency condition for the matrices α 's in this non-abelian case. Following a similar procedure as in the two previous cases we finally find

$$\begin{aligned} & \left[\rho_1 - \frac{1}{2}(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])^2 + [\alpha_2, \alpha_3] - \mu_3 \right. \\ & \quad \left. + \frac{1}{2}(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3])\dot{\mathbf{v}}^2 - \frac{4}{3}\alpha_1^2\ddot{\mathbf{v}}^2 \right. \\ & \quad \left. + \left[\frac{3}{2}\alpha_1\alpha_2 - \frac{17}{16}\alpha_2(-2\dot{\alpha}_2 + [\alpha_1, \alpha_4] + [\alpha_2, \alpha_3]) + \rho_2 \right] \dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{4}{3}\alpha_2\dot{\mathbf{v}}\ddot{\mathbf{v}} \right] = 0. \end{aligned} \quad (55)$$

5 Final Remarks

At the present time the classical problem of the dynamics of a radiating particle is still a subject of intense work. In the specific case of a color charged particle interacting with a Yang-Mills field there is a huge interest in the problem of non-abelian hydrodynamics and the study of the quark-gluon plasma.

In the present article we continue exploring the equations of motion of a non-abelian charge in an external Yang-Mills field, where the dynamics is described by

Wong's equations [10]. There has been much work in this subject as we mentioned in the introduction section. However a self-consistent analysis of the classical theory, as was given by Dirac [2] for the abelian case, has not been worked out in the literature for the Yang-Mills case. This analysis has been carried out in the present article.

The strategy we followed here was to adopt the ansatz (7) for the retarded Liénard-Wiechert potentials, given in Ref. [40]. This allowed us to compute the retarded, advanced and actual field strengths along the trajectory of the particle. A summary and the results of this derivation were collected in appendices A and B. Furthermore, using Dirac's method of integration over the energy-momentum tensor, in appendix C is outlined the computation of the equations of motion and its corrections due the radiation damping. These results were used to give precise formulas in Section 3 of the corrections due the self-force to Wong's equations. In Section 4 it is shown that under certain assumptions about the Lie algebra structure of the Dirac's equation of motion [2] can be recovered. We also were able to find three non-trivial examples for the non-abelian equations of motion with the properly cancelation of divergences.

As we mentioned at the end of Section 3, the general equation of motion (23) is a higher-derivative differential equation (with respect to time) what might have a chaotic behavior [44, 45]. It would be interesting to study some examples describing the phenomenon of 'snap' (in the context followed in this paper) and its possible generalizations.

Finally, as a possible perspective of this work, we point out that a generalized Lorentz-Dirac equation was obtained in the context of the AdS/CFT correspondence [46]. In the near future we would like to address the problem of finding a close correspondence between a gravity dual in the AdS space and some of the results we found in the present article for the Yang-Mills particle.

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A Series expansions of fields near the trajectory of the particle

In the following appendices we present some computations and results we used in the body of the article. In this appendix we will give some preliminary results that will be necessary in the subsequent appendices.

In order to perform the calculation of the field strength $F_{\mu\nu}$ in Eq. (9) we need to compute first the terms with different negative powers of R . R^{-1} is a function of x_μ and z_μ and therefore it can be expanded in powers of σ as is shown in Eqs. (10) and (11). Neglecting the higher positive powers in σ means that we are closer from the trajectory of the particle. Then the expansion reads

$$\begin{aligned} R^{-1} &= [\dot{z}^\mu (x_\mu - z_\mu)]^{-1} \\ &= \sigma^{-1} [1 + \gamma \mathbf{f}(\sigma)]^{-1} \\ &\times \left\{ 1 + \left[-\frac{1}{6} \dot{\mathbf{v}}^2 \sigma^2 + \sum_{p=3}^{10} (-1)^p \sum_{n=0}^p \frac{\mathbf{v}_n}{n!} \frac{\mathbf{v}_{p-n}}{(p+1-n)!} \sigma^p \right] [1 + \gamma \mathbf{f}(\sigma)]^{-1} \right\}^{-1}, \end{aligned} \quad (56)$$

where $\gamma \mathbf{f}(\sigma) = \sum_{k=1}^{10} (-1)^k \gamma \mathbf{v}_k \sigma^{k-1}$ and $\gamma \mathbf{v}_k = \gamma_\mu v_k^\mu$. The first factor can in turn be expanded as a power series of σ and it can be expressed as

$$\begin{aligned} [1 + \gamma \mathbf{f}(\sigma)]^{-1} &= 1 - \gamma \mathbf{f}(\sigma) + [\gamma \mathbf{f}(\sigma)]^2 - \dots - [\gamma \mathbf{f}(\sigma)]^{11} \\ &= h_{10} \sigma^{10} + h_9 \sigma^9 + h_8 \sigma^8 + h_7 \sigma^7 + h_6 \sigma^6 \\ &\quad + h_5 \sigma^5 + h_4 \sigma^4 + h_3 \sigma^3 + h_2 \sigma^2 + h_1 \sigma + h_0. \end{aligned} \quad (57)$$

In the process of expansion we have considered that the highest possible value of the powers of σ relevant in our computation is up to 11. Moreover we have assumed that γ is of the same order as σ , thus for instance we have $[\gamma \mathbf{f}(\sigma)]^{10} = (\gamma \dot{\mathbf{v}})^{10} - 5\sigma(\gamma \dot{\mathbf{v}})^9 \gamma \ddot{\mathbf{v}}$ and $[\gamma \mathbf{f}(\sigma)]^{11} = -(\gamma \dot{\mathbf{v}})^{11}$. The coefficients h_i of expansion in Eq. (57) are given by

$$h_0 = \gamma \dot{\mathbf{v}}^{11} + \gamma \dot{\mathbf{v}}^{10} + \gamma \dot{\mathbf{v}}^9 + \gamma \dot{\mathbf{v}}^8 + \gamma \dot{\mathbf{v}}^7 + \gamma \dot{\mathbf{v}}^6 + \gamma \dot{\mathbf{v}}^5 + \gamma \dot{\mathbf{v}}^4 + \gamma \dot{\mathbf{v}}^3 + \gamma \dot{\mathbf{v}}^2 + \gamma \dot{\mathbf{v}} + 1, \quad (58)$$

$$\begin{aligned} h_1 &= -5\gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^9 - \frac{9}{2} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^8 - 4\gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^7 - \frac{7}{2} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^6 - 3\gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^5 - \frac{5}{2} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^4 \\ &\quad - 2\gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^3 - \frac{3}{2} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^2 - \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}} - \frac{\gamma \ddot{\mathbf{v}}}{2}, \end{aligned} \quad (59)$$

$$\begin{aligned} h_2 &= \frac{9}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^8 + 9\gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^7 + \frac{8}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^7 + 7\gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^6 + \frac{7}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^6 + \frac{21}{4} \gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^5 \\ &\quad + \frac{6}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^5 + \frac{15}{4} \gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^4 + \frac{5}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^4 + \frac{5}{2} \gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^3 + \frac{4}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^3 + \frac{3}{2} \gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}}^2 \\ &\quad + \frac{3}{6} \gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}^2 + \frac{3}{4} \gamma \ddot{\mathbf{v}}^2 \gamma \dot{\mathbf{v}} + \frac{2\gamma \ddot{\mathbf{v}} \gamma \dot{\mathbf{v}}}{6} + \frac{\gamma \ddot{\mathbf{v}}^2}{4} + \frac{\gamma \ddot{\mathbf{v}}}{6}, \end{aligned} \quad (60)$$

$$\begin{aligned}
h_3 = & -\frac{1}{24}8\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^7 - \frac{28}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^6 - \frac{7}{24}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^6 - 7\gamma\ddot{\mathbf{v}}^3\gamma\dot{\mathbf{v}}^5 - \frac{21}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^5 - \frac{6}{24}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^5 \\
& - \frac{35}{8}\gamma\ddot{\mathbf{v}}^3\gamma\dot{\mathbf{v}}^4 - \frac{15}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^4 - \frac{5}{24}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^4 - \frac{5}{2}\gamma\ddot{\mathbf{v}}^3\gamma\dot{\mathbf{v}}^3 - \frac{10}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^3 - \frac{4}{24}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^3 \\
& - \frac{5}{4}\gamma\ddot{\mathbf{v}}^3\gamma\dot{\mathbf{v}}^2 - \frac{6}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^2 - \frac{3}{24}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^2 - \frac{1}{2}\gamma\ddot{\mathbf{v}}^3\gamma\dot{\mathbf{v}} - \frac{3}{6}\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}} - \frac{2\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}}{24} \\
& - \frac{\gamma\ddot{\mathbf{v}}\gamma\ddot{\mathbf{v}}}{6} - \frac{\gamma\ddot{\mathbf{v}}^3}{8} - \frac{\gamma\mathbf{v}^{(4)}}{24}, \tag{61}
\end{aligned}$$

$$\begin{aligned}
h_4 = & \frac{7}{120}\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}^6 + \frac{21}{36}\gamma\ddot{\mathbf{v}}^2\gamma\dot{\mathbf{v}}^5 + \frac{21}{24}\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^5 + \frac{6}{120}\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}^5 + \frac{15}{36}\gamma\ddot{\mathbf{v}}^2\gamma\dot{\mathbf{v}}^4 \\
& + \frac{105}{24}\gamma\ddot{\mathbf{v}}^2\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^4 + \frac{15}{24}\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^4 + \frac{5}{120}\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}^4 + \frac{35}{16}\gamma\ddot{\mathbf{v}}^4\gamma\dot{\mathbf{v}}^3 + \frac{10}{36}\gamma\ddot{\mathbf{v}}^2\gamma\dot{\mathbf{v}}^3 \\
& + \frac{15}{6}\gamma\ddot{\mathbf{v}}^2\gamma\mathbf{v}_3\gamma\dot{\mathbf{v}}^3 + \frac{10}{24}\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^3 + \frac{4}{120}\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}^3 + \frac{15}{16}\gamma\ddot{\mathbf{v}}^4\gamma\dot{\mathbf{v}}^2 + \frac{6}{36}\gamma\ddot{\mathbf{v}}^2\gamma\dot{\mathbf{v}}^2 \\
& + \frac{15}{12}\gamma\ddot{\mathbf{v}}^2\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}}^2 + \frac{6}{24}\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}}^2 + \frac{3}{120}\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}^2 + \frac{5}{16}\gamma\ddot{\mathbf{v}}^4\gamma\dot{\mathbf{v}} + \frac{3}{36}\gamma\ddot{\mathbf{v}}^2\gamma\dot{\mathbf{v}} \\
& + \frac{3}{6}\gamma\ddot{\mathbf{v}}^2\gamma\ddot{\mathbf{v}}\gamma\dot{\mathbf{v}} + \frac{3}{24}\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}\gamma\dot{\mathbf{v}} + \frac{2\gamma\mathbf{v}^{(5)}\gamma\dot{\mathbf{v}}}{120} + \frac{\gamma\ddot{\mathbf{v}}^4}{16} \\
& + \frac{\gamma\ddot{\mathbf{v}}^2}{36} + \frac{3}{24}\gamma\ddot{\mathbf{v}}^2\gamma\ddot{\mathbf{v}} + \frac{\gamma\ddot{\mathbf{v}}\gamma\mathbf{v}^{(4)}}{24} + \frac{\gamma\mathbf{v}^{(5)}}{120}, \tag{62}
\end{aligned}$$

where we adopted the notation $(\gamma\mathbf{v})^\alpha \equiv \gamma\mathbf{v}^\alpha = (\gamma_\mu\mathbf{v}^\mu)^\alpha$. Furthermore we have written only the first coefficients of the expansion since that up on computation one can see that the higher order terms will not contribute to the equations of motion.

The first term in the square brackets from the second factor of the right-hand side of Eq. (56) also can be expanded as powers of σ with coefficients given by certain functions p_i

$$\begin{aligned}
& \left[-\frac{1}{6}\dot{\mathbf{v}}^2\sigma^3 + \sum_{p=3}^{10} (-1)^p \sum_{n=0}^p \frac{\mathbf{v}_n}{n!} \frac{\mathbf{v}_{p-n}}{(p+1-n)!} \sigma^{p+1} \right] \\
& = p_3\sigma^3 + p_4\sigma^4 + p_5\sigma^5 + p_6\sigma^6 + p_7\sigma^7 + p_8\sigma^8 + p_9\sigma^9 + p_{10}\sigma^{10} + p_{11}\sigma^{11}, \tag{63}
\end{aligned}$$

where the important coefficients for our further computations are

$$p_3 = -\frac{1}{6}\mathbf{v}^2, \quad p_4 = \frac{5}{24}\dot{\mathbf{v}}\ddot{\mathbf{v}}, \quad p_5 = -\frac{1}{5!}(8\ddot{\mathbf{v}} + 9\dot{\mathbf{v}}\ddot{\mathbf{v}}), \quad p_6 = -\frac{1}{6!}(35\ddot{\mathbf{v}}\ddot{\mathbf{v}} + \dot{\mathbf{v}}\mathbf{v}^{(4)}). \tag{64}$$

$g_1 = h_2 - h_0^2 p_3$
$g_2 = -3h_0^4 p_3 + 3h_2 h_0^2 + 3h_1^2 h_0$
$g_3 = 3g_1 h_0^2 - 2h_0^4 p_3 + 2h_2 h_0^2 + 5h_1^2 h_0$
$g_4 = g_1 h_0^2 + 2h_0^4 p_3 - 2h_2 h_0^2 - h_1^2 h_0$
$g_5 = 2g_1 h_1 h_0 + 2h_1 h_0^3 p_3 - 2h_1 h_2 h_0$
$g_6 = -4h_0^5 p_3 + 4h_2 h_0^3 + 6h_1^2 h_0^2$
$g_7 = -4h_0^5 p_3 + 4h_2 h_0^3 + 6h_1^2 h_0^2$
$g_8 = -4h_0^5 p_4 - 20h_1 h_0^4 p_3 + 4h_3 h_0^3 + 12h_1 h_2 h_0^2 + 4h_1^3 h_0$
$g_9 = -5h_0^6 p_4 - 30h_1 h_0^5 p_3 + 5h_3 h_0^4 + 20h_1 h_2 h_0^3 + 10h_1^3 h_0^2$
$g_{10} = 4h_0^5 p_3 - 4h_2 h_0^3 - 6h_1^2 h_0^2$
$g_{11} = -5h_0^6 p_3 + 5h_2 h_0^4 + 10h_1^2 h_0^3$
$g_{12} = -5h_0^6 p_3 + 5h_2 h_0^4 + 10h_1^2 h_0^3$
$g_{13} = -5h_0^6 p_4 - 30h_1 h_0^5 p_3 + 5h_3 h_0^4 + 20h_1 h_2 h_0^3 + 10h_1^3 h_0^2$
$g_{14} = 5h_0^6 p_3 - 5h_2 h_0^4 - 10h_1^2 h_0^3$
$g_{15} = 15h_0^7 p_3^2 - 5h_0^6 p_5 - 30h_2 h_0^5 p_3 - 30h_1 h_0^5 p_4 - 75h_1^2 h_0^4 p_3 + 5h_4 h_0^4$
$g_{16} = 5h_0^6 p_4 + 30h_1 h_0^5 p_3 - 5h_3 h_0^4 - 20h_1 h_2 h_0^3 - 10h_1^3 h_0^2$
$g_{17} = -5h_0^6 p_3 + 5h_2 h_0^4 + 10h_1^2 h_0^3$
$g_{18} = -\frac{1}{2}5h_0^6 p_3 + \frac{5}{2}h_2 h_0^4 + 5h_1^2 h_0^3$

Table 1: Useful definitions to express the relevant observables such as $F_{\mu\nu}^{\text{ret}}$.

Moreover the expression $\dot{z}_\mu(x_\nu - z_\nu(s))$ is also expanded in powers of σ as follows

$$\begin{aligned}
\dot{z}_\mu(x_\nu - z_\nu(s)) = & \gamma_\nu v_\mu + \sigma (v_\mu v_\nu - \gamma_\nu \dot{v}_\mu) + \sigma^2 \left(k_{\mu\nu 2} + \frac{\gamma_\nu \ddot{v}_\mu}{2} \right) + \sigma^3 \left(k_{\mu\nu 3} - \frac{\gamma_\nu \ddot{v}_\mu}{3!} \right) \\
& + \sigma^4 \left(\frac{\gamma_\nu v_\mu^{(4)}}{4!} + k_{\mu\nu 4} \right) + \sigma^5 \left(k_{\mu\nu 5} - \frac{\gamma_\nu v_\mu^{(5)}}{5!} \right) + \sigma^6 \left(k_{\mu\nu 6} + \frac{\gamma_\nu v_\mu^{(6)}}{6!} \right) \\
& + \sigma^7 \left(k_{\mu\nu 7} - \frac{\gamma_\nu v_\mu^{(7)}}{7!} \right) + \sigma^8 \left(k_{\mu\nu 8} + \frac{\gamma_\nu v_\mu^{(8)}}{8!} \right) + \sigma^9 \left(k_{\mu\nu 9} - \frac{\gamma_\nu v_\mu^{(9)}}{9!} \right) \\
& + \sigma^{10} \left(\frac{\gamma_\nu v_\mu^{(10)}}{10!} + k_{\mu\nu 10} \right) + \sigma^{11} \left(k_{\mu\nu 11} - \frac{\gamma_\nu v_\mu^{(11)}}{11!} \right), \quad (65)
\end{aligned}$$

where $k_{\mu\nu i}$ are some functions depending on the velocities and their derivatives. Actually we need to compute the anti-symmetric version of the previous equation because this is the case that arises in the computation of $F_{\mu\nu}^{(\text{rad})}$, that is: $\dot{z}_\mu(x_\nu - z_\nu(s)) - \dot{z}_\nu(x_\mu - z_\mu(s))$. Thus, the relevant functions $k_{\mu\nu i}$ are antisymmetric tensors and they are given by

$$k_{\mu\nu 2} = \frac{v_\mu \dot{v}_\nu}{2} - \frac{\dot{v}_\mu v_\nu}{2}, \quad k_{\mu\nu 3} = \frac{\ddot{v}_\mu v_\nu}{3} - \frac{v_\mu \ddot{v}_\nu}{3}, \quad (66)$$

$$k_{\mu\nu 4} = \frac{v_\mu \ddot{v}_\nu}{8} + \frac{\dot{v}_\mu \ddot{v}_\nu}{12} - \frac{\ddot{v}_\mu v_\nu}{12} - \frac{1}{8} \ddot{v}_\mu v_\nu, \quad (67)$$

$$k_{\mu\nu 5} = -\frac{v_\mu v_\nu^{(4)}}{30} - \frac{\dot{v}_\mu \ddot{v}_\nu}{24} + \frac{\ddot{v}_\mu \dot{v}_\nu}{24} + \frac{v_\mu^{(4)} v_\nu}{30}. \quad (68)$$

Finally we give some definitions that will be useful later in order to simplify the relevant equations as the $F_{\mu\nu}^{\text{ret}}$ in the following appendices. These functions are organized in table 1.

B Computation of $F_{\mu\nu}^{(\text{rad})}$, $F_{\mu\nu}^{(\text{act})}$ and $f_{\mu\nu}$

In this appendix we will give some details of the computation of $F_{\mu\nu}^{(\text{ret})}$ and $F_{\mu\nu}^{(\text{adv})}$, through which the radiated field $F_{\mu\nu}^{(\text{rad})}$ is also obtained. To carry out the Taylor expansion we use Eqs. (10) and (11) and the first term in the $F_{\mu\nu}^{(\text{ret})}$ from Eq. (9), then it is written as

$$\begin{aligned}
& -R^{-1} \frac{d}{ds} \left[R^{-1} \{ \dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \} \right] \\
& = g_1 h_1 \gamma_\nu v_\mu - 2g_1 h_0 \gamma_\nu \dot{v}_\mu + 2g_1 h_0 v_\mu v_\nu + 3h_1 h_0 k_{\mu\nu 2} \\
& + 2h_0^2 k_{\mu\nu 3} + \frac{h_0^2 k_{\mu\nu 2} - h_1 h_0 \gamma_\nu \dot{v}_\mu + \frac{1}{2} h_0^2 \gamma_\nu \ddot{v}_\mu + h_1 h_0 v_\mu v_\nu}{\sigma} \\
& + h_0^3 (-p_4) \gamma_\nu v_\mu - 2h_1 h_0^2 p_3 \gamma_\nu v_\mu - \frac{2h_0^2 \gamma_\nu \ddot{v}_\mu}{6} - \frac{h_0^2 \gamma_\nu v_\mu}{\sigma^3} \\
& - \frac{h_1 h_0 \gamma_\nu v_\mu}{\sigma^2} + h_3 h_0 \gamma_\nu v_\mu - h_1^2 \gamma_\nu \dot{v}_\mu + \frac{3}{2} h_1 h_0 \gamma_\nu \ddot{v}_\mu + h_1^2 v_\mu v_\nu, \tag{69}
\end{aligned}$$

where the functions $k_{\mu\nu i}$ and h_i were given explicitly in appendix A. Similarly we can calculate the Taylor expansion for the following terms in the retarded field $F_{\mu\nu}^{(\text{ret})}$. It is given by

$$\begin{aligned}
& -R^{-2} \frac{d}{ds} \left[R^{-1} \{ \dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \} \right] \\
& = 3g_1 h_0^2 k_{\mu\nu 2} + 5h_0 h_1^2 k_{\mu\nu 2} + 2h_0^2 h_2 k_{\mu\nu 2} + 7h_0^2 h_1 k_{\mu\nu 3} + 3h_0^3 k_{\mu\nu 4} - 2h_0^4 p_3 k_{\mu\nu 2} + 4g_1 h_0 h_1 v_\mu v_\nu \\
& - 3h_0^4 p_4 v_\mu v_\nu - 8h_1 h_0^3 p_3 v_\mu v_\nu - \frac{h_0^3 \gamma_\nu v_\mu}{\sigma^4} + 3h_3 h_0^2 v_\mu v_\nu + 2h_1 h_2 h_0 v_\mu v_\nu + h_1^3 v_\mu v_\nu \\
& - \frac{2h_0^2 h_1 \gamma_\nu v_\mu}{\sigma^3} + g_1 h_1^2 \gamma_\nu v_\mu + 2g_1 h_0 h_2 \gamma_\nu v_\mu - h_0 h_2^2 \gamma_\nu v_\mu + 2h_0 h_1 h_3 \gamma_\nu v_\mu + h_0^2 h_4 \gamma_\nu v_\mu \\
& - 2g_1 h_0^3 p_3 \gamma_\nu v_\mu - 5h_0^2 h_1^2 p_3 \gamma_\nu v_\mu - 4h_0^3 h_1 p_4 \gamma_\nu v_\mu - h_0^4 p_5 \gamma_\nu v_\mu - 4g_1 h_0 h_1 \gamma_\nu \dot{v}_\mu \\
& - 3h_0^4 p_4 \gamma_\nu \dot{v}_\mu + 8h_1 h_0^3 p_3 \gamma_\nu \dot{v}_\mu - 3h_3 h_0^2 \gamma_\nu \dot{v}_\mu - 2h_1 h_2 h_0 \gamma_\nu \dot{v}_\mu + h_1^3 \gamma_\nu \dot{v}_\mu \\
& + \frac{3}{2} g_1 h_0^2 \gamma_\nu \ddot{v}_\mu + \frac{5}{2} h_0 h_1^2 \gamma_\nu \ddot{v}_\mu + h_0^2 h_2 \gamma_\nu \ddot{v}_\mu - h_0^4 p_3 \gamma_\nu \ddot{v}_\mu - \frac{7h_0^2 h_1 \gamma_\nu \ddot{v}_\mu}{6} + \frac{3h_0^3 \gamma_\nu v_\mu^{(4)}}{24} + \sigma^{-2} \left(g_1 h_0^2 \gamma_\nu v_\mu \right. \\
& + h_0^3 k_{\mu\nu 2} + 2h_0^4 p_3 \gamma_\nu v_\mu - 2h_2 h_0^2 \gamma_\nu v_\mu - h_1^2 h_0 \gamma_\nu v_\mu - h_1 h_0^2 \gamma_\nu \dot{v}_\mu + \frac{1}{2} h_0^3 \gamma_\nu \ddot{v}_\mu + h_1 h_0^2 v_\mu v_\nu \Big) \\
& + \sigma^{-1} \left(2g_1 h_1 h_0 \gamma_\nu v_\mu + 2g_1 h_0^2 v_\mu v_\nu + 4h_1 h_0^2 k_{\mu\nu 2} + 2h_0^3 k_{\mu\nu 3} - 2h_1 h_2 h_0 \gamma_\nu v_\mu + 2h_1^2 h_0 v_\mu v_\nu \right. \\
& \left. - 2g_1 h_0^2 \gamma_\nu \dot{v}_\mu + 2h_1 h_0^3 p_3 \gamma_\nu v_\mu - \frac{2h_0^3 \gamma_\nu \ddot{v}_\mu}{6} - 2h_1^2 h_0 \gamma_\nu \dot{v}_\mu + 2h_1 h_0^2 \gamma_\nu \ddot{v}_\mu \right), \tag{70}
\end{aligned}$$

where h_i , p_i , $k_{\mu\nu i}$ and g_i are given in appendix A.

Further terms of higher negative powers of R are given by

$$\begin{aligned}
 & R^{-3} \left[\dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \right] \\
 &= 3h_1 h_0^2 k_{\mu\nu 2} + h_0^3 k_{\mu\nu 3} \\
 &+ \frac{1}{\sigma} \left[h_0^3 k_{\mu\nu 2} - 3h_0^4 p_3 \gamma_\nu v_\mu + 3h_2 h_0^2 \gamma_\nu v_\mu + 3h_1^2 h_0 \gamma_\nu v_\mu - 3h_1 h_0^2 \gamma_\nu \dot{v}_\mu \right. \\
 &+ \left. \frac{1}{2} h_0^3 \gamma_\nu \ddot{v}_\mu + 3h_1 h_0^2 v_\mu v_\nu \right] - 3h_0^4 p_4 \gamma_\nu v_\mu - 12h_1 h_0^3 p_3 \gamma_\nu v_\mu \\
 &+ 3h_0^4 p_3 \gamma_\nu \dot{v}_\mu - 3h_0^4 p_3 v_\mu v_\nu - \frac{h_0^3 \gamma_\nu \ddot{v}_\mu}{6} \\
 &+ \frac{3h_1 h_0^2 \gamma_\nu v_\mu - h_0^3 \gamma_\nu \dot{v}_\mu + h_0^3 v_\mu v_\nu}{\sigma^2} + \frac{h_0^3 \gamma_\nu v_\mu}{\sigma^3} \\
 &+ 3h_3 h_0^2 \gamma_\nu v_\mu + 6h_1 h_2 h_0 \gamma_\nu v_\mu + h_1^3 \gamma_\nu v_\mu - 3h_2 h_0^2 \gamma_\nu \dot{v}_\mu \\
 &- 3h_1^2 h_0 \gamma_\nu \dot{v}_\mu + \frac{3}{2} h_1 h_0^2 \gamma_\nu \ddot{v}_\mu + 3h_2 h_0^2 v_\mu v_\nu + 3h_1^2 h_0 v_\mu v_\nu.
 \end{aligned} \tag{71}$$

The following term is written as

$$\begin{aligned}
 & R^{-4} \left[\dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \right] \\
 &= 10p_3^2 v_\mu \gamma_\nu h_0^6 - 4k_{\mu\nu 2} p_3 h_0^5 - 4p_4 v_\mu v_\nu h_0^5 - 4p_5 v_\mu \gamma_\nu h_0^5 \\
 &+ 4p_4 \dot{v}_\mu \gamma_\nu h_0^5 - 2p_3 \ddot{v}_\mu \gamma_\nu h_0^5 + k_{\mu\nu 4} h_0^4 - 20h_1 p_3 v_\mu v_\nu h_0^4 \\
 &- 20h_2 p_3 v_\mu \gamma_\nu h_0^4 - 20h_1 p_4 v_\mu \gamma_\nu h_0^4 + \frac{v_\mu \gamma_\nu h_0^4}{\sigma^4} + 20h_1 p_3 \dot{v}_\mu \gamma_\nu h_0^4 \\
 &+ \frac{v_\mu^{(4)} \gamma_\nu h_0^4}{24} + 4h_2 k_{\mu\nu 2} h_0^3 + 4h_1 k_{\mu\nu 3} h_0^3 + 4h_3 v_\mu v_\nu h_0^3 + 4h_4 v_\mu \gamma_\nu h_0^3 \\
 &- 40h_1^2 p_3 v_\mu \gamma_\nu h_0^3 - 4h_3 \dot{v}_\mu \gamma_\nu h_0^3 + 2h_2 \ddot{v}_\mu \gamma_\nu h_0^3 - \frac{4h_1 \ddot{v}_\mu \gamma_\nu h_0^3}{6} \\
 &+ 6h_1^2 k_{\mu\nu 2} h_0^2 + 12h_1 h_2 v_\mu v_\nu h_0^2 + 6h_2^2 v_\mu \gamma_\nu h_0^2 + 12h_1 h_3 v_\mu \gamma_\nu h_0^2 \\
 &- 12h_1 h_2 \dot{v}_\mu \gamma_\nu h_0^2 + 3h_1^2 \ddot{v}_\mu \gamma_\nu h_0^2 + 4h_1^3 v_\mu v_\nu h_0 + 12h_1^2 h_2 v_\mu \gamma_\nu h_0 \\
 &- 4h_1^3 \dot{v}_\mu \gamma_\nu h_0 \\
 &+ \frac{1}{\sigma} \left[-4p_3 v_\mu v_\nu h_0^5 - 4p_4 v_\mu \gamma_\nu h_0^5 + 4p_3 \dot{v}_\mu \gamma_\nu h_0^5 + k_{\mu\nu 3} h_0^4 \right. \\
 &- 20h_1 p_3 v_\mu \gamma_\nu h_0^4 - \frac{\ddot{v}_\mu \gamma_\nu h_0^4}{6} + 4h_1 k_{\mu\nu 2} h_0^3 + 4h_2 v_\mu v_\nu h_0^3 \\
 &+ 4h_3 v_\mu \gamma_\nu h_0^3 - 4h_2 \dot{v}_\mu \gamma_\nu h_0^3 + 2h_1 \ddot{v}_\mu \gamma_\nu h_0^3 \\
 &+ 6h_1^2 v_\mu v_\nu h_0^2 + 12h_1 h_2 v_\mu \gamma_\nu h_0^2 - 6h_1^2 \dot{v}_\mu \gamma_\nu h_0^2 + 4h_1^3 v_\mu \gamma_\nu h_0 \left. \right] \\
 &+ \frac{1}{\sigma^2} \left[-4p_3 v_\mu \gamma_\nu h_0^5 + k_{\mu\nu 2} h_0^4 + \frac{1}{2} \ddot{v}_\mu \gamma_\nu h_0^4 + 4h_1 v_\mu v_\nu h_0^3 \right] \\
 &+ 4h_2 v_\mu \gamma_\nu h_0^3 - 4h_1 \dot{v}_\mu \gamma_\nu h_0^3 + 6h_1^2 v_\mu \gamma_\nu h_0^2 \\
 &+ \frac{v_\mu v_\nu h_0^4 - \dot{v}_\mu \gamma_\nu h_0^4 + 4h_1 v_\mu \gamma_\nu h_0^3}{\sigma^3} + h_1^4 v_\mu \gamma_\nu.
 \end{aligned} \tag{72}$$

Finally the last expansion reads

$$\begin{aligned}
& R^{-5} \left[\dot{z}_\mu (x_\nu - z_\nu) - \dot{z}_\nu (x_\mu - z_\mu) \right] \\
& = 15p_3^2 v_\mu v_\nu h_0^7 + 30p_3 p_4 v_\mu \gamma_\nu h_0^7 - 15p_3^2 \dot{v}_\mu \gamma_\nu h_0^7 - 5k_{\mu\nu 3} p_3 h_0^6 - 5k_{\mu\nu 2} p_4 h_0^6 - 5p_5 v_\mu v_\nu h_0^6 \\
& + 105h_1 p_3^2 v_\mu \gamma_\nu h_0^6 - 5p_6 v_\mu \gamma_\nu h_0^6 + 5p_5 \dot{v}_\mu \gamma_\nu h_0^6 - \frac{5}{2} p_4 \ddot{v}_\mu \gamma_\nu h_0^6 + \frac{5p_3 \ddot{v}_\mu \gamma_\nu h_0^6}{6} + k_{\mu\nu 5} h_0^5 \\
& - 30h_1 k_{\mu\nu 2} p_3 h_0^5 - 30h_2 p_3 v_\mu v_\nu h_0^5 - 30h_1 p_4 v_\mu v_\nu h_0^5 - 30h_3 p_3 v_\mu \gamma_\nu h_0^5 - 30h_2 p_4 v_\mu \gamma_\nu h_0^5 \\
& - 30h_1 p_5 v_\mu \gamma_\nu h_0^5 + \frac{v_\mu \gamma_\nu h_0^5}{\sigma^5} + 30h_2 p_3 \dot{v}_\mu \gamma_\nu h_0^5 + 30h_1 p_4 \dot{v}_\mu \gamma_\nu h_0^5 - 15h_1 p_3 \ddot{v}_\mu \gamma_\nu h_0^5 \\
& - \frac{v_\mu^{(5)} \gamma_\nu h_0^5}{120} + 5h_3 k_{\mu\nu 2} h_0^4 + 5h_2 k_{\mu\nu 3} h_0^4 + 5h_1 k_{\mu\nu 4} h_0^4 + 5h_4 v_\mu v_\nu h_0^4 - 75h_1^2 p_3 v_\mu v_\nu h_0^4 \\
& + 5h_5 v_\mu \gamma_\nu h_0^4 - 150h_1 h_2 p_3 v_\mu \gamma_\nu h_0^4 - 75h_1^2 p_4 v_\mu \gamma_\nu h_0^4 - 5h_4 \dot{v}_\mu \gamma_\nu h_0^4 + 75h_1^2 p_3 \dot{v}_\mu \gamma_\nu h_0^4 + \frac{5}{2} h_3 \ddot{v}_\mu \gamma_\nu h_0^4 \\
& + \frac{5h_1 v_\mu^{(4)} \gamma_\nu h_0^4}{24} - \frac{5h_2 \ddot{v}_\mu \gamma_\nu h_0^4}{6} + 20h_1 h_2 k_{\mu\nu 2} h_0^3 + 10h_1^2 k_{\mu\nu 3} h_0^3 + 10h_2^2 v_\mu v_\nu h_0^3 + 20h_1 h_3 v_\mu v_\nu h_0^3 \\
& + 20h_2 h_3 v_\mu \gamma_\nu h_0^3 + 20h_1 h_4 v_\mu \gamma_\nu h_0^3 - 100h_1^3 p_3 v_\mu \gamma_\nu h_0^3 - 10h_2^2 \dot{v}_\mu \gamma_\nu h_0^3 - 20h_1 h_3 \dot{v}_\mu \gamma_\nu h_0^3 \\
& + 10h_1 h_2 \ddot{v}_\mu \gamma_\nu h_0^3 - \frac{10h_1^2 \ddot{v}_\mu \gamma_\nu h_0^3}{6} + 10h_1^3 k_{\mu\nu 2} h_0^2 + 30h_1^2 h_2 v_\mu v_\nu h_0^2 + 30h_1 h_2^2 v_\mu \gamma_\nu h_0^2 + 30h_1^2 h_3 v_\mu \gamma_\nu h_0^2 \\
& - 30h_1^2 h_2 \dot{v}_\mu \gamma_\nu h_0^2 + 5h_1^3 \dot{v}_\mu \gamma_\nu h_0^2 + 5h_1^4 v_\mu v_\nu h_0 + 20h_1^3 h_2 v_\mu \gamma_\nu h_0 - 5h_1^4 \dot{v}_\mu \gamma_\nu h_0 \\
& + \frac{1}{\sigma} \left[15p_3^2 v_\mu \gamma_\nu h_0^7 - 5k_{\mu\nu 2} p_3 h_0^6 - 5p_4 v_\mu v_\nu h_0^6 - 5p_5 v_\mu \gamma_\nu h_0^6 + 5p_4 \dot{v}_\mu \gamma_\nu h_0^6 \right. \\
& - \frac{5}{2} p_3 \ddot{v}_\mu \gamma_\nu h_0^6 + k_{\mu\nu 4} h_0^5 - 30h_1 p_3 v_\mu v_\nu h_0^5 - 30h_2 p_3 v_\mu \gamma_\nu h_0^5 - 30h_1 p_4 v_\mu \gamma_\nu h_0^5 \\
& + 30h_1 p_3 \dot{v}_\mu \gamma_\nu h_0^5 + \frac{v_\mu^{(4)} \gamma_\nu h_0^5}{24} + 5h_2 k_{\mu\nu 2} h_0^4 + 5h_1 k_{\mu\nu 3} h_0^4 + 5h_3 v_\mu v_\nu h_0^4 \\
& + 5h_4 v_\mu \gamma_\nu h_0^4 - 75h_1^2 p_3 v_\mu \gamma_\nu h_0^4 - 5h_3 \dot{v}_\mu \gamma_\nu h_0^4 + \frac{5}{2} h_2 \ddot{v}_\mu \gamma_\nu h_0^4 - \frac{5h_1 \ddot{v}_\mu \gamma_\nu h_0^4}{6} + 10h_1^2 k_{\mu\nu 2} h_0^3 \\
& + 20h_1 h_2 v_\mu v_\nu h_0^3 + 10h_2^2 v_\mu \gamma_\nu h_0^3 + 20h_1 h_3 v_\mu \gamma_\nu h_0^3 - 20h_1 h_2 \dot{v}_\mu \gamma_\nu h_0^3 \\
& + 5h_1^2 \ddot{v}_\mu \gamma_\nu h_0^3 + 10h_1^3 v_\mu v_\nu h_0^2 + 30h_1^2 h_2 v_\mu \gamma_\nu h_0^2 - 10h_1^3 \dot{v}_\mu \gamma_\nu h_0^2 + 5h_1^4 v_\mu \gamma_\nu h_0 \left. \right] \\
& + \frac{1}{\sigma^2} \left[-5p_3 v_\mu v_\nu h_0^6 - 5p_4 v_\mu \gamma_\nu h_0^6 + 5p_3 \dot{v}_\mu \gamma_\nu h_0^6 + k_{\mu\nu 3} h_0^5 - 30h_1 p_3 v_\mu \gamma_\nu h_0^5 - \frac{\ddot{v}_\mu \gamma_\nu h_0^5}{6} \right. \\
& + 5h_1 k_{\mu\nu 2} h_0^4 + 5h_2 v_\mu v_\nu h_0^4 + 5h_3 v_\mu \gamma_\nu h_0^4 - 5h_2 \dot{v}_\mu \gamma_\nu h_0^4 + \frac{5}{2} h_1 \ddot{v}_\mu \gamma_\nu h_0^4 \\
& + 10h_1^2 v_\mu v_\nu h_0^3 + 20h_1 h_2 v_\mu \gamma_\nu h_0^3 - 10h_1^2 \dot{v}_\mu \gamma_\nu h_0^3 + 10h_1^3 v_\mu \gamma_\nu h_0^2 \left. \right] + h_1^5 v_\mu \gamma_\nu \\
& + \frac{1}{\sigma^3} \left[-5p_3 v_\mu \gamma_\nu h_0^6 + k_{\mu\nu 2} h_0^5 + \frac{1}{2} \ddot{v}_\mu \gamma_\nu h_0^5 + 5h_1 v_\mu v_\nu h_0^4 + 5h_2 v_\mu \gamma_\nu h_0^4 \right. \\
& - 5h_1 \dot{v}_\mu \gamma_\nu h_0^4 + 10h_1^2 v_\mu \gamma_\nu h_0^3 \left. \right] \\
& + \frac{v_\mu v_\nu h_0^5 - \dot{v}_\mu \gamma_\nu h_0^5 + 5h_1 v_\mu \gamma_\nu h_0^4}{\sigma^4}.
\end{aligned} \tag{73}$$

Using these expansions obtained in this appendix we can calculate the field $F_{\mu\nu}^{(\text{ret})}$ in a

neighborhood near the trajectory of the non-abelian particle. It is given by

$$\begin{aligned}
F_{\mu\nu}^{(\text{ret})} = & \left(2k_{\mu\nu 3}h_0^2 - \frac{v_\mu\gamma_\nu h_0^2}{\sigma^3} + 3h_1k_{\mu\nu 2}h_0 - \frac{h_1v_\mu\gamma_\nu h_0}{\sigma^2} + \frac{k_{\mu\nu 2}h_0^2 + \frac{1}{2}\ddot{v}_\mu\gamma_\nu h_0^2 - h_1\dot{v}_\mu\gamma_\nu h_0}{\sigma} \right) A_1 \\
& + \left(k_{\mu\nu 3}h_0^3 + \frac{v_\mu\gamma_\nu h_0^3}{\sigma^3} + 3h_1k_{\mu\nu 2}h_0^2 + \frac{k_{\mu\nu 2}h_0^3 + \frac{1}{2}\ddot{v}_\mu\gamma_\nu h_0^3 - 3h_1\dot{v}_\mu\gamma_\nu h_0^2 + g_2v_\mu\gamma_\nu}{\sigma} \right. \\
& + \left. \frac{3h_0^2h_1v_\mu\gamma_\nu - h_0^3\dot{v}_\mu\gamma_\nu}{\sigma^2} \right) A_3 \\
& + A_5 \left(k_{\mu\nu 5}h_0^5 + \frac{v_\mu\gamma_\nu h_0^5}{\sigma^5} + 5h_1k_{\mu\nu 4}h_0^4 + g_9k_{\mu\nu 2} + k_{\mu\nu 3}g_{11} \right. \\
& + \frac{k_{\mu\nu 2}h_0^5 + \frac{1}{2}\ddot{v}_\mu\gamma_\nu h_0^5 - 5h_1\dot{v}_\mu\gamma_\nu h_0^4 + g_{12}v_\mu\gamma_\nu}{\sigma^3} + \frac{5h_0^4h_1v_\mu\gamma_\nu - h_0^5\dot{v}_\mu\gamma_\nu}{\sigma^4} \\
& + \frac{1}{\sigma} \left(k_{\mu\nu 4}h_0^5 + \frac{1}{24}v_\mu^{(4)}\gamma_\nu h_0^5 + 5h_1k_{\mu\nu 3}h_0^4 - \frac{5}{6}h_1\ddot{v}_\mu\gamma_\nu h_0^4 + k_{\mu\nu 2}g_{17} + g_{15}v_\mu\gamma_\nu \right. \\
& + \left. \dot{v}_\mu g_{16}\gamma_\nu + \ddot{v}_\mu\gamma_\nu g_{18} \right) \\
& + \left. \frac{k_{\mu\nu 3}h_0^5 - \frac{1}{6}\ddot{v}_\mu\gamma_\nu h_0^5 + 5h_1k_{\mu\nu 2}h_0^4 + \frac{5}{2}h_1\ddot{v}_\mu\gamma_\nu h_0^4 + g_{13}v_\mu\gamma_\nu - g_{14}\dot{v}_\mu\gamma_\nu}{\sigma^2} \right) \\
& + A_2 \left(3k_{\mu\nu 4}h_0^3 - \frac{v_\mu\gamma_\nu h_0^3}{\sigma^4} + 7h_1k_{\mu\nu 3}h_0^2 - \frac{2h_1v_\mu\gamma_\nu h_0^2}{\sigma^3} \right. \\
& + \left. g_3k_{\mu\nu 2} + \frac{k_{\mu\nu 2}h_0^3 + \frac{1}{2}\ddot{v}_\mu\gamma_\nu h_0^3 - h_1\dot{v}_\mu\gamma_\nu h_0^2 + g_4v_\mu\gamma_\nu}{\sigma^2} \right. \\
& + \left. \frac{2k_{\mu\nu 3}h_0^3 - \frac{1}{3}\ddot{v}_\mu\gamma_\nu h_0^3 + 4h_1k_{\mu\nu 2}h_0^2 - 2g_1\dot{v}_\mu\gamma_\nu h_0^2 + 2h_1\ddot{v}_\mu\gamma_\nu h_0^2 - 2h_1^2\dot{v}_\mu\gamma_\nu h_0 + g_5v_\mu\gamma_\nu}{\sigma} \right) \\
& + A_4 \left(k_{\mu\nu 4}h_0^4 + \frac{v_\mu\gamma_\nu h_0^4}{\sigma^4} + 4h_1k_{\mu\nu 3}h_0^3 + g_6k_{\mu\nu 2} \right. \\
& + \frac{k_{\mu\nu 2}h_0^4 + \frac{1}{2}\ddot{v}_\mu\gamma_\nu h_0^4 - 4h_1\dot{v}_\mu\gamma_\nu h_0^3 + g_7v_\mu\gamma_\nu}{\sigma^2} + \frac{4h_0^3h_1v_\mu\gamma_\nu - h_0^4\dot{v}_\mu\gamma_\nu}{\sigma^3} \\
& + \left. \frac{k_{\mu\nu 3}h_0^4 - \frac{1}{6}\ddot{v}_\mu\gamma_\nu h_0^4 + 4h_1k_{\mu\nu 2}h_0^3 + 2h_1\ddot{v}_\mu\gamma_\nu h_0^3 + g_8v_\mu\gamma_\nu + \dot{v}_\mu g_{10}\gamma_\nu}{\sigma} \right), \tag{74}
\end{aligned}$$

where A_1 , A_2 , A_3 , A_4 and A_5 are those given after Eq. (14).

Now to calculate the actual field we use the Eq. (13) and solve the equation for σ in terms of the parameter ε . These arguments were already presented in [2], to calculate $F_{\mu\nu}^{(\text{act})}$,

it is necessary to determine first σ as a function of ε for which we have

$$\begin{aligned} \sigma^2 = & \frac{\varepsilon^2}{1 - \gamma \dot{\mathbf{v}}} \left[1 - \frac{\gamma \ddot{\mathbf{v}} \varepsilon}{3} - \varepsilon^9 \left(\frac{\gamma \mathbf{v}^{(10)}}{19958400} + \frac{\mathbf{v}^{(4)} \mathbf{v}^{(5)}}{21600} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(6)}}{33600} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(7)}}{86400} + \frac{\dot{\mathbf{v}} \mathbf{v}^{(8)}}{453600} \right) \right. \\ & + \varepsilon^8 \left(\frac{\gamma \mathbf{v}^{(9)}}{1814400} + \frac{\mathbf{v}^{(4)2}}{8100} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(5)}}{5184} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(6)}}{11340} + \frac{\dot{\mathbf{v}} \mathbf{v}^{(7)}}{51840} \right) \\ & - \varepsilon^7 \left(\frac{\gamma \mathbf{v}^{(8)}}{181440} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(4)}}{960} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(5)}}{1728} + \frac{\dot{\mathbf{v}} \mathbf{v}^{(6)}}{6720} \right) + \varepsilon^6 \left(\frac{\gamma \mathbf{v}^{(7)}}{20160} + \frac{\ddot{\mathbf{v}}^2}{448} + \frac{\ddot{\mathbf{v}} \mathbf{v}^{(4)}}{315} + \frac{\dot{\mathbf{v}} \mathbf{v}^{(5)}}{1008} \right) \\ & - \varepsilon^5 \left(\frac{\gamma \mathbf{v}^{(6)}}{2520} + \frac{\ddot{\mathbf{v}} \ddot{\mathbf{v}}}{72} + \frac{\dot{\mathbf{v}} \mathbf{v}^{(4)}}{180} \right) + \varepsilon^4 \left(\frac{\gamma \mathbf{v}^{(5)}}{360} + \frac{\ddot{\mathbf{v}}^2}{45} + \frac{\dot{\mathbf{v}} \ddot{\mathbf{v}}}{40} \right) - \varepsilon^3 \left(\frac{\gamma \mathbf{v}^{(4)}}{60} + \frac{\dot{\mathbf{v}} \ddot{\mathbf{v}}}{12} \right) \\ & \left. + \varepsilon^2 \left(\frac{\gamma \ddot{\mathbf{v}}}{12} + \frac{\dot{\mathbf{v}}^2}{12} \right) \right]. \end{aligned} \quad (75)$$

This expression is obtained from expansion (10) and the condition $(x_\mu - z_\mu(s))(x^\mu - z^\mu(s)) = 0$. The substitution of these equations, in $F_{\mu\nu}^{(\text{act})}$ gives the field strength very close to the trajectory of the Yang-Mills particle

$$\begin{aligned} F_{\mu\nu}^{(\text{act})} = & f_{\mu\nu} - \frac{A_5 k_{\mu\nu} h_0^5}{\varepsilon U_1} - \frac{A_5 h_0^5}{24 \varepsilon U_1} [v_\mu^{(4)} \gamma_\nu - v_\nu^{(4)} \gamma_\mu] + \frac{A_4 h_0^4}{6 \varepsilon U_1} [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] \\ & - \frac{A_4 k_{\mu\nu} h_0^4}{\varepsilon U_1} + \frac{A_2 h_0^3}{3 \varepsilon U_1} [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] \\ & - \frac{A_3 k_{\mu\nu} h_0^3}{\varepsilon U_1} - \frac{2 A_2 k_{\mu\nu} h_0^3}{\varepsilon U_1} - \frac{A_3 h_0^3}{2 \varepsilon U_1} [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] - \frac{A_1 k_{\mu\nu} h_0^2}{\varepsilon U_1} \\ & - \frac{A_1 h_0^2}{2 \varepsilon U_1} [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] + \frac{A_1 h_1 h_0}{\varepsilon U_1} [\dot{v}_\mu \gamma_\nu - \dot{v}_\nu \gamma_\mu] \\ & + A_2 k_{\mu\nu} \left(\frac{2 a_1 h_0^3}{\varepsilon U_1^2} - \frac{4 h_0^2 h_1}{\varepsilon U_1} \right) + A_4 k_{\mu\nu} \left(\frac{2 a_1 h_0^4}{\varepsilon U_1^2} - \frac{4 h_0^3 h_1}{\varepsilon U_1} \right) + \frac{2 a_1 h_0^5}{\varepsilon U_1^2} A_5 k_{\mu\nu} \\ & - \frac{5 h_0^4 h_1}{\varepsilon U_1} A_5 k_{\mu\nu} + A_5 k_{\mu\nu} \left(-\frac{6 a_1^2 h_0^5}{\varepsilon U_1^3} + \frac{3 a_2 h_0^5}{\varepsilon U_1^3} - \frac{h_0^5}{\varepsilon^3 U_1^3} + \frac{10 a_1 h_1 h_0^4}{\varepsilon U_1^2} - \frac{g_{17}}{\varepsilon U_1} \right) \\ & + A_2 \left(-\frac{20 a_1^3 h_0^3}{\varepsilon U_1^4} + \frac{20 a_1 a_2 h_0^3}{\varepsilon U_1^4} - \frac{4 a_3 h_0^3}{\varepsilon U_1^4} - \frac{4 a_1 h_0^3}{\varepsilon^3 U_1^4} + \frac{12 a_1^2 h_1 h_0^2}{\varepsilon U_1^3} \right. \\ & \left. - \frac{6 a_2 h_1 h_0^2}{\varepsilon U_1^3} + \frac{2 h_1 h_0^2}{\varepsilon^3 U_1^3} - \frac{g_5}{\varepsilon U_1} + \frac{2 a_1 g_4}{\varepsilon U_1^2} \right) [v_\mu \gamma_\nu - v_\nu \gamma_\mu] \\ & + A_3 \left(-\frac{6 a_1^2 h_0^3}{\varepsilon U_1^3} + \frac{3 a_2 h_0^3}{\varepsilon U_1^3} - \frac{h_0^3}{\varepsilon^3 U_1^3} + \frac{6 a_1 h_1 h_0^2}{\varepsilon U_1^2} - \frac{g_2}{\varepsilon U_1} \right) [v_\mu \gamma_\nu - v_\nu \gamma_\mu] \\ & + A_4 \left(\frac{20 a_1^3 h_0^4}{\varepsilon U_1^4} - \frac{20 a_1 a_2 h_0^4}{\varepsilon U_1^4} + \frac{4 a_3 h_0^4}{\varepsilon U_1^4} + \frac{4 a_1 h_0^4}{\varepsilon^3 U_1^4} - \frac{24 a_1^2 h_1 h_0^3}{\varepsilon U_1^3} \right. \\ & \left. + \frac{12 a_2 h_1 h_0^3}{\varepsilon U_1^3} - \frac{4 h_1 h_0^3}{\varepsilon^3 U_1^3} - \frac{g_8}{\varepsilon U_1} + \frac{2 a_1 g_7}{\varepsilon U_1^2} \right) [v_\mu \gamma_\nu - v_\nu \gamma_\mu] \end{aligned}$$

$$\begin{aligned}
& +A_5 \left(-\frac{70a_1^4h_0^5}{\varepsilon U_1^5} - \frac{15a_2^2h_0^5}{\varepsilon U_1^5} + \frac{105a_1^2a_2h_0^5}{\varepsilon U_1^5} - \frac{30a_1a_3h_0^5}{\varepsilon U_1^5} + \frac{5a_4h_0^5}{\varepsilon U_1^5} - \frac{15a_1^2h_0^5}{\varepsilon^3 U_1^5} + \frac{5a_2h_0^5}{\varepsilon^3 U_1^5} \right. \\
& - \frac{h_0^5}{\varepsilon^5 U_1^5} + \frac{100a_1^3h_1h_0^4}{\varepsilon U_1^4} - \frac{100a_1a_2h_1h_0^4}{\varepsilon U_1^4} + \frac{20a_3h_1h_0^4}{\varepsilon U_1^4} + \frac{20a_1h_1h_0^4}{\varepsilon^3 U_1^4} - \frac{g_{15}}{\varepsilon U_1} + \frac{2a_1g_{12}}{\varepsilon U_1^2} \\
& \left. - \frac{6a_1^2g_{12}}{\varepsilon U_1^3} + \frac{3a_2g_{12}}{\varepsilon U_1^3} - \frac{g_{12}}{\varepsilon^3 U_1^3} \right) [v_\mu \gamma_\nu - v_\nu \gamma_\mu] \\
& +A_1 \left(\frac{6a_1^2h_0^2}{\varepsilon U_1^3} - \frac{3a_2h_0^2}{\varepsilon U_1^3} + \frac{h_0^2}{\varepsilon^3 U_1^3} - \frac{2a_1h_1h_0}{\varepsilon U_1^2} \right) [v_\mu \gamma_\nu - v_\nu \gamma_\mu] \\
& +A_3 \left(\frac{3h_0^2h_1}{\varepsilon U_1} - \frac{2a_1h_0^3}{\varepsilon U_1^2} \right) [\dot{v}_\mu \gamma_\nu - \dot{v}_\nu \gamma_\mu] + A_2 \left(\frac{2g_1h_0^2}{\varepsilon U_1} - \frac{2a_1h_1h_0^2}{\varepsilon U_1^2} + \frac{2h_1^2h_0}{\varepsilon U_1} \right) [\dot{v}_\mu \gamma_\nu - \dot{v}_\nu \gamma_\mu] \\
& +A_4 \left(\frac{6a_1^2h_0^4}{\varepsilon U_1^3} - \frac{3a_2h_0^4}{\varepsilon U_1^3} + \frac{h_0^4}{\varepsilon^3 U_1^3} - \frac{8a_1h_1h_0^3}{\varepsilon U_1^2} - \frac{g_{10}}{\varepsilon U_1} \right) [\dot{v}_\mu \gamma_\nu - \dot{v}_\nu \gamma_\mu] \\
& +A_5 \left(-\frac{20a_1^3h_0^5}{\varepsilon U_1^4} + \frac{20a_1a_2h_0^5}{\varepsilon U_1^4} - \frac{4a_3h_0^5}{\varepsilon U_1^4} - \frac{4a_1h_0^5}{\varepsilon^3 U_1^4} + \frac{30a_1^2h_1h_0^4}{\varepsilon U_1^3} - \frac{15a_2h_1h_0^4}{\varepsilon U_1^3} + \frac{5h_1h_0^4}{\varepsilon^3 U_1^3} \right. \\
& \left. - \frac{g_{16}}{\varepsilon U_1} - \frac{2a_1g_{14}}{\varepsilon U_1^2} \right) [\dot{v}_\mu \gamma_\nu - \dot{v}_\nu \gamma_\mu] \\
& +A_2 \left(\frac{a_1h_0^3}{\varepsilon U_1^2} - \frac{2h_0^3h_1}{\varepsilon U_1} \right) [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] + A_4 \left(\frac{a_1h_0^4}{\varepsilon U_1^2} - \frac{2h_0^3h_1}{\varepsilon U_1} \right) [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] \\
& +A_5 \left(-\frac{3a_1^2h_0^5}{\varepsilon U_1^3} + \frac{3a_2h_0^5}{2\varepsilon U_1^3} - \frac{h_0^5}{2\varepsilon^3 U_1^3} + \frac{5a_1h_1h_0^4}{\varepsilon U_1^2} - \frac{g_{18}}{\varepsilon U_1} \right) [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu] \\
& +A_5 \left(\frac{5h_0^4h_1}{6\varepsilon U_1} - \frac{a_1h_0^5}{3\varepsilon U_1^2} \right) [\ddot{v}_\mu \gamma_\nu - \ddot{v}_\nu \gamma_\mu], \tag{76}
\end{aligned}$$

where $U_1 \equiv (1 - \gamma \dot{\mathbf{v}})^{-1/2}$, and a_1, a_2, a_3 and a_4 are given by

$$\begin{aligned}
a_1 &= -\frac{\gamma \ddot{\mathbf{v}}}{6}; \quad a_2 = \frac{1}{72} (-\gamma \ddot{\mathbf{v}}^2 + 3\gamma \ddot{\mathbf{v}} + 3\dot{\mathbf{v}}^2) \\
a_3 &= \frac{1}{432} \left(-\gamma \ddot{\mathbf{v}}^3 - \frac{18\gamma \mathbf{v}^{(4)}}{5} + 3\gamma \ddot{\mathbf{v}} (\gamma \ddot{\mathbf{v}} + \dot{\mathbf{v}}^2) - 18\dot{\mathbf{v}}\ddot{\mathbf{v}} \right) \\
a_4 &= -\frac{1}{128} 5 \left(\frac{\gamma \ddot{\mathbf{v}}}{3} \right)^4 + \frac{3}{16} \left(\frac{\gamma \ddot{\mathbf{v}}}{12} + \frac{\dot{\mathbf{v}}^2}{12} \right) \left(\frac{\gamma \ddot{\mathbf{v}}}{3} \right)^2 - \frac{1}{4} \left(\frac{\gamma \mathbf{v}^{(4)}}{60} + \frac{\dot{\mathbf{v}}\ddot{\mathbf{v}}}{12} \right) \left(\frac{\gamma \ddot{\mathbf{v}}}{3} \right) \\
& - \frac{1}{2} \left(\frac{\gamma \mathbf{v}^{(5)}}{360} + \frac{\ddot{\mathbf{v}}^2}{45} + \frac{\dot{\mathbf{v}}\ddot{\mathbf{v}}}{40} \right) + \frac{1}{8} \left(\frac{\gamma \ddot{\mathbf{v}}}{12} + \frac{\dot{\mathbf{v}}^2}{12} \right)^2. \tag{77}
\end{aligned}$$

Now we can calculate the radiating field. In order to do that we calculate first the advanced field strength $F_{\mu\nu}^{(\text{adv})}$. Similarly as in the abelian case, it can be done by changing

ε to $-\varepsilon$. Thus the field strength $F_{\mu\nu}^{(\text{rad})} = F_{\mu\nu}^{(\text{ret})} - F_{\mu\nu}^{(\text{adv})}$ is written as

$$\begin{aligned}
 F_{\mu\nu}^{(\text{rad})} = & A_5 k_{\mu\nu 2} \left(-\frac{20a_1^3 h_0^5}{U_1^3} + \frac{24a_1 a_2 h_0^5}{U_1^3} - \frac{6a_3 h_0^5}{U_1^3} + \frac{30a_1^2 h_1 h_0^4}{U_1^2} - \frac{20a_2 h_1 h_0^4}{U_1^2} - \frac{2a_1 g_{17}}{U_1} + 2g_9 \right) \\
 & + A_2 k_{\mu\nu 2} \left(\frac{6a_1^2 h_0^3}{U_1^2} - \frac{4a_2 h_0^3}{U_1^2} - \frac{8a_1 h_1 h_0^2}{U_1} + 2g_3 \right) \\
 & + A_4 k_{\mu\nu 2} \left(\frac{6a_1^2 h_0^4}{U_1^2} - \frac{4a_2 h_0^4}{U_1^2} - \frac{8a_1 h_1 h_0^3}{U_1} + 2g_6 \right) \\
 & + A_5 k_{\mu\nu 3} \left(\frac{6a_1^2 h_0^5}{U_1^2} - \frac{4a_2 h_0^5}{U_1^2} - \frac{10a_1 h_1 h_0^4}{U_1} + 2g_{11} \right) \\
 & + A_1 k_{\mu\nu 2} \left(6h_0 h_1 - \frac{2a_1 h_0^2}{U_1} \right) + A_3 k_{\mu\nu 2} \left(6h_0^2 h_1 - \frac{2a_1 h_0^3}{U_1} \right) + A_2 k_{\mu\nu 3} \left(14h_0^2 h_1 - \frac{4a_1 h_0^3}{U_1} \right) \\
 & + A_4 k_{\mu\nu 3} \left(8h_0^3 h_1 - \frac{2a_1 h_0^4}{U_1} \right) + A_5 k_{\mu\nu 4} \left(10h_0^4 h_1 - \frac{2a_1 h_0^5}{U_1} \right) \\
 & + 2A_3 h_0^3 k_{\mu\nu 3} + 4A_1 h_0^2 k_{\mu\nu 3} + 2A_4 h_0^4 k_{\mu\nu 4} + 6A_2 h_0^3 k_{\mu\nu 4} + 2A_5 h_0^5 k_{\mu\nu 5}.
 \end{aligned} \tag{78}$$

If we take all the $A_i = 0$, $i = 2, 3, 4, 5$, except $A_1 \neq 0$, then we can calculate the radiated field by an abelian particle. Therefore the Yang-Mills field is reduced to the one of Dirac's radiation damping [2]

$$F_{\mu\nu}^{(\text{rad})D} = 4A_1 h_0^2 k_{\mu\nu 3} = 4A_1 h_0^2 \left(\frac{\ddot{v}_\mu v_\nu}{3} - \frac{v_\mu \ddot{v}_\nu}{3} \right). \tag{79}$$

Notice that in $F_{\mu\nu}^{(\text{rad})}$ there are two terms depending on A_1 . Moreover the terms depending on h_1 and a_1 will not contribute to the equations of motion, since they are linear in γ and they will be eliminated in the integral when the flow of radiation due to the particle is computed.

C Computation of equations of motion

In order to find the equations of motion we first should calculate the energy-momentum tensor (16) in terms of the radiation field at points near the trajectory of the particle. For this we make use of $F_{\mu\nu}^{(\text{act})}$ on the worldline. When we use these relations in (16), we can find the divergent terms in $T_{\mu\nu}$. In the process of computing and simplifying the energy-momentum tensor (16) it is useful to consider the following relations

$$\begin{aligned}
 [\gamma_\mu v_\nu - \gamma_\nu v_\mu] [\gamma^\nu v_\rho - \gamma_\rho v^\nu] &= -\gamma_\mu \gamma_\rho + \varepsilon^2 v_\mu v_\rho \\
 [\gamma_\mu v_\nu - \gamma_\nu v_\mu] [\gamma^\nu v_\rho^{(i)} - \gamma_\rho v^{(i)\nu}] &= \gamma \mathbf{v}^{(i)} \gamma_\rho v_\mu + \varepsilon^2 v_\mu v_\rho^{(i)} - \mathbf{v} \mathbf{v}^{(i)} \gamma_\mu \gamma_\rho, \\
 [\gamma_\mu v_\nu - \gamma_\nu v_\mu] [v^\nu \dot{v}_\rho - v_\rho \dot{v}^\nu] &= \gamma_\mu \dot{v}_\rho + (\gamma \dot{\mathbf{v}}) v_\mu v_\rho \\
 [\gamma_\mu v_\nu - \gamma_\nu v_\mu] [\ddot{v}^\nu \gamma_\rho - \ddot{v}_\rho \gamma^\nu] &= -\dot{\mathbf{v}}^2 \gamma_\mu \gamma_\rho - (\gamma \dot{\mathbf{v}}) v_\mu \gamma_\rho - \varepsilon^2 v_\mu \ddot{v}_\rho \\
 [v_\mu \dot{v}_\nu - v_\nu \dot{v}_\mu] [v^\nu \dot{v}_\rho - v_\rho \dot{v}^\nu] &= -\dot{\mathbf{v}}^2 v_\mu v_\rho - \dot{v}_\mu \dot{v}_\rho \\
 [\gamma_\mu v_\nu^{(i)} - \gamma_\nu v_\mu^{(i)}] [\gamma^\nu v_\rho - \gamma_\rho v^\nu] &= \gamma \mathbf{v}^{(i)} \gamma_\mu v_\rho + \varepsilon^2 v_\mu^{(i)} v_\rho - \mathbf{v} \mathbf{v}^{(i)} \gamma_\mu \gamma_\rho \\
 [\gamma_\mu v_\nu^{(i)} - \gamma_\nu v_\mu^{(i)}] [\gamma^\nu v_\rho^{(j)} - \gamma_\rho v^{(j)\nu}] &= \gamma \mathbf{v}^{(i)} \gamma_\mu v_\rho^{(j)} + \gamma \mathbf{v}^{(j)} \gamma_\rho v_\mu^{(i)} - \mathbf{v}^{(i)} \mathbf{v}^{(j)} \gamma_\mu \gamma_\rho + \varepsilon^2 v_\mu^{(i)} v_\rho^{(j)}.
 \end{aligned} \tag{80}$$

Now following [2] we are ready to compute the energy and momentum flow of the radiation through a 3-surface $\Sigma = S \times T$, where S is a 2-surface with differential element of area given by dS which is perpendicular to S . This surface is chosen a compact surface of area $4\pi\varepsilon^2$. Moreover T is its spatial transverse direction with differential element of length given by $|dx_{\parallel}| = (1 - \gamma\dot{\mathbf{v}}^2)ds$, where s is the worldline parameter. Thus the integration yields

$$\begin{aligned}
 & - \int \varepsilon^{-1} T_{\mu\rho} \gamma^{\rho} ds |dx_{\parallel}| = - \int 4\pi\varepsilon T_{\mu\rho} \gamma^{\rho} U_1^{-2} dS \\
 & = \int \text{Tr} \left\{ \frac{A_5 f_{\mu\nu} v^{\nu} + Q_{14} v_{\mu} + Q_{11} \dot{v}_{\mu} + Q_{17} \ddot{v}_{\mu} + Q_{16} \ddot{\ddot{v}}_{\mu}}{\varepsilon^2} \right. \\
 & + \varepsilon (A_3 f_{\mu\nu} \dot{v}^{\nu} + Q_{27} f_{\mu\nu} v^{\nu} + Q_{31} f_{\mu\nu} \dot{v}^{\nu} + Q_{34} f_{\mu\nu} \ddot{v}^{\nu} + Q_{29} f_{\mu\nu} \ddot{\ddot{v}}^{\nu} + Q_7 v_{\mu} + Q_1 \dot{v}_{\mu} \\
 & + Q_5 \ddot{v}_{\mu} + Q_4 \ddot{\ddot{v}}_{\mu}) + \frac{A_5 f_{\mu\nu} \dot{v}^{\nu} + Q_{36} f_{\mu\nu} v^{\nu} + Q_9 v_{\mu} + Q_3 \dot{v}_{\mu} + Q_{10} \ddot{v}_{\mu} + Q_{15} \ddot{\ddot{v}}_{\mu}}{\varepsilon} \\
 & - A_4 f_{\mu\nu} \dot{v}^{\nu} + Q_{35} f_{\mu\nu} v^{\nu} \\
 & + \varepsilon^2 (Q_{25} f_{\mu\nu} v^{\nu} + Q_{26} f_{\mu\nu} \dot{v}^{\nu} + Q_{28} f_{\mu\nu} \ddot{v}^{\nu} + Q_{32} f_{\mu\nu} \ddot{\ddot{v}}^{\nu} + Q_{30} f_{\mu\nu} v^{\nu(4)}) \\
 & + Q_{33} f_{\mu\nu} \ddot{v}^{\nu} + Q_8 v_{\mu} + Q_6 \dot{v}_{\mu} + Q_2 \ddot{v}_{\mu} + Q_{12} \ddot{\ddot{v}}_{\mu} + \frac{Q_{24} \dot{v}_{\mu}}{\varepsilon^5} \\
 & \left. + \frac{Q_{23} v_{\mu} + Q_{22} \dot{v}_{\mu} + Q_{20} \ddot{v}_{\mu}}{\varepsilon^4} + \frac{Q_{13} v_{\mu} + Q_{21} \dot{v}_{\mu} + Q_{19} \ddot{v}_{\mu} + Q_{18} \ddot{\ddot{v}}_{\mu}}{\varepsilon^3} \right\} ds, \quad (81)
 \end{aligned}$$

where we have integrated on a hyper-sphere of area $4\pi\varepsilon^2$ and used the line element $|dx_{\parallel}| = (1 - \gamma\dot{\mathbf{v}}^2)ds = U_1^{-2}ds$. If we remember that the linear terms in γ do not contribute to the equations of motion and we neglect all terms that depend on positive powers of ε (in the limit $\varepsilon \rightarrow 0$) we get Eq. (17). The functions Q_i are given by

$$\begin{aligned}
 Q_1 = & -\frac{35}{16} a_4 A_5^2 \dot{\mathbf{v}}^2 - \frac{5}{2} a_4 A_1 A_5 - \frac{5}{2} a_4 A_3 A_5 - \frac{17}{144} A_2 A_5 \dot{\mathbf{v}}^2 \mathbf{v} \ddot{\mathbf{v}} - \frac{17}{288} A_4 A_5 \dot{\mathbf{v}}^2 \mathbf{v} \ddot{\mathbf{v}} \\
 & + \frac{17 A_5^2 \dot{\mathbf{v}}^2 \mathbf{v} \mathbf{v}^{(4)}}{1152} + \frac{209 A_5^2 \dot{\mathbf{v}}^6}{9216} - \frac{25}{192} A_1 A_5 \dot{\mathbf{v}}^4 - \frac{1}{48} A_3 A_5 \dot{\mathbf{v}}^4 + \frac{21}{128} A_5^2 \dot{\mathbf{v}} \mathbf{v} \ddot{\mathbf{v}}^2 \\
 & - \frac{13}{288} A_2 A_5 \dot{\mathbf{v}} \mathbf{v} \ddot{\mathbf{v}}^2 - \frac{17}{72} A_4 A_5 \dot{\mathbf{v}} \mathbf{v} \ddot{\mathbf{v}}^2 - \frac{3}{16} A_1^2 \dot{\mathbf{v}}^2 - \frac{1}{16} A_3^2 \dot{\mathbf{v}}^2 + \frac{17}{576} A_5^2 \ddot{\mathbf{v}}^2 \dot{\mathbf{v}}^2 \\
 & - \frac{1}{4} A_1 A_3 \dot{\mathbf{v}}^2 - \frac{1}{96} A_4 A_5 \dot{\mathbf{v}} \mathbf{v} \ddot{\mathbf{v}}^2 - \frac{1}{12} A_1 A_2 \ddot{\mathbf{v}} \mathbf{v} - \frac{1}{12} A_2 A_3 V_2 \dot{\mathbf{v}} - \frac{1}{3} A_1 A_4 \ddot{\mathbf{v}} \mathbf{v} \\
 & - \frac{1}{3} A_3 A_4 \ddot{\mathbf{v}} \mathbf{v} + \frac{3}{16} A_1 A_5 \ddot{\mathbf{v}} \mathbf{v} + \frac{3}{16} A_3 A_5 \ddot{\mathbf{v}} \mathbf{v} + \frac{1}{8} A_1 A_5 \dot{\mathbf{v}}^2 + \frac{1}{8} A_3 A_5 \dot{\mathbf{v}}^2 \\
 & + \frac{1}{72} A_5^2 (\dot{\mathbf{v}} \mathbf{v})^2 + \frac{1}{36} A_2 A_5 \ddot{\mathbf{v}} \mathbf{v} + \frac{1}{72} A_4 A_5 \ddot{\mathbf{v}} \mathbf{v} - \frac{1}{288} A_5^2 \ddot{\mathbf{v}} \mathbf{v}^{(4)} - \frac{1}{6} A_1 A_2 \mathbf{v} \ddot{\mathbf{v}} - \frac{1}{6} A_2 A_3 \mathbf{v} \ddot{\mathbf{v}} \\
 & - \frac{1}{12} A_1 A_4 \mathbf{v} \ddot{\mathbf{v}} - \frac{1}{12} A_3 A_4 \mathbf{v} \ddot{\mathbf{v}} + \frac{1}{48} A_1 A_5 \mathbf{v} \mathbf{v}^{(4)} + \frac{1}{48} A_3 A_5 \mathbf{v} \mathbf{v}^{(4)}, \quad (82)
 \end{aligned}$$

$$\begin{aligned}
 Q_2 = & -\frac{10}{3} a_4 A_5^2 - \frac{1}{32} 3 A_5^2 \dot{\mathbf{v}}^4 - \frac{1}{3} A_2^2 \dot{\mathbf{v}}^2 - \frac{1}{6} A_2 A_4 \dot{\mathbf{v}}^2 - \frac{1}{4} A_1 A_5 \dot{\mathbf{v}}^2 + \frac{1}{4} A_5^2 \ddot{\mathbf{v}} \mathbf{v} - \frac{17}{24} A_2 A_5 \ddot{\mathbf{v}} \mathbf{v} \\
 & - \frac{55}{72} A_4 A_5 \ddot{\mathbf{v}} \mathbf{v} + \frac{1}{72} A_5^2 \ddot{\mathbf{v}} \mathbf{v} + \frac{2}{9} A_5^2 \ddot{\mathbf{v}}^2 - \frac{1}{3} A_2 A_5 \mathbf{v} \ddot{\mathbf{v}} - \frac{1}{6} A_4 A_5 \mathbf{v} \ddot{\mathbf{v}} + \frac{1}{36} A_5^2 \mathbf{v} \mathbf{v}^{(4)}, \quad (83)
 \end{aligned}$$

$$\begin{aligned}
Q_3 = & -\frac{5}{2}a_4A_5^2 + \frac{17}{192}A_5^2\dot{\mathbf{v}}^4 - \frac{1}{2}A_2^2\dot{\mathbf{v}}^2 - \frac{1}{2}A_2A_4\dot{\mathbf{v}}^2 - \frac{9}{16}A_1A_5\dot{\mathbf{v}}^2 \\
& + \frac{7}{16}A_3A_5\dot{\mathbf{v}}^2 + \frac{3}{16}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{11}{12}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{13}{12}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{8}A_5^2\ddot{\mathbf{v}}^2 \\
& - \frac{1}{3}A_2A_5\mathbf{v}\ddot{\mathbf{v}} - \frac{1}{4}A_4A_5\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{48}A_5^2\mathbf{v}\mathbf{v}^{(4)} + \frac{A_3^2}{2} - \frac{A_1^2}{2}, \quad (84)
\end{aligned}$$

$$\begin{aligned}
Q_4 = & -\frac{5}{8}a_4A_5^2 - \frac{1}{512}A_5^2\dot{\mathbf{v}}^4 - \frac{3}{64}A_1A_5\dot{\mathbf{v}}^2 - \frac{1}{64}A_3A_5\dot{\mathbf{v}}^2 + \frac{3}{64}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{48}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& - \frac{1}{12}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{24}A_5^2\ddot{\mathbf{v}}^2 - \frac{1}{24}A_2A_5\mathbf{v}\ddot{\mathbf{v}} - \frac{1}{48}A_4A_5\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{192}A_5^2\mathbf{v}\mathbf{v}^{(4)}, \quad (85)
\end{aligned}$$

$$\begin{aligned}
Q_5 = & \frac{10}{3}a_4A_2A_5 + \frac{5}{3}a_4A_4A_5 - \frac{1}{288}5A_2A_5\dot{\mathbf{v}}^4 - \frac{23}{576}A_4A_5\dot{\mathbf{v}}^4 + \frac{59}{576}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 \\
& + \frac{1}{4}A_1A_2\dot{\mathbf{v}}^2 + \frac{1}{12}A_2A_3\dot{\mathbf{v}}^2 + \frac{1}{8}A_1A_4\dot{\mathbf{v}}^2 + \frac{1}{24}A_3A_4\dot{\mathbf{v}}^2 + \frac{1}{9}A_2^2\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& + \frac{2}{9}A_4^2\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{2}A_2A_4\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{24}A_1A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{4}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{24}A_3A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& - \frac{1}{8}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{36}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{72}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{288}A_5^2\mathbf{v}^{(4)}\dot{\mathbf{v}} - \frac{2}{9}A_2A_5\dot{\mathbf{v}}^2 \\
& - \frac{1}{9}A_4A_5\dot{\mathbf{v}}^2 + \frac{2}{9}A_2^2\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{18}A_4^2\mathbf{v}\ddot{\mathbf{v}} + \frac{2}{9}A_2A_4\mathbf{v}\ddot{\mathbf{v}} - \frac{1}{36}A_2A_5\mathbf{v}\mathbf{v}^{(4)} - \frac{1}{72}A_4A_5\mathbf{v}\mathbf{v}^{(4)}, \quad (86)
\end{aligned}$$

$$\begin{aligned}
Q_6 = & 5a_4A_2A_5 + 5a_4A_4A_5 + \frac{17}{288}A_5^2\dot{\mathbf{v}}^2\mathbf{v}\ddot{\mathbf{v}} + \frac{37}{192}A_2A_5\dot{\mathbf{v}}^4 - \frac{5}{192}A_4A_5\dot{\mathbf{v}}^4 \\
& + \frac{125}{288}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 + \frac{5}{8}A_1A_2\dot{\mathbf{v}}^2 + \frac{3}{8}A_2A_3\dot{\mathbf{v}}^2 + \frac{3}{8}A_1A_4\dot{\mathbf{v}}^2 + \frac{1}{8}A_3A_4\dot{\mathbf{v}}^2 \\
& + \frac{1}{72}A_5^2\ddot{\mathbf{v}}\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{1}{6}A_2^2\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{2}{3}A_4^2\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{5}{6}A_2A_4\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{5}{12}A_1A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& - \frac{3}{8}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{3}A_3A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{3}{8}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{7}{24}A_2A_5\ddot{\mathbf{v}}^2 - \frac{7}{24}A_4A_5\ddot{\mathbf{v}}^2 \\
& - \frac{1}{72}A_5^2\ddot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{3}A_2^2\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{6}A_4^2\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{2}A_2A_4\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{12}A_1A_5\mathbf{v}\ddot{\mathbf{v}} \\
& + \frac{1}{12}A_3A_5\mathbf{v}\ddot{\mathbf{v}} - \frac{1}{24}A_2A_5\mathbf{v}\mathbf{v}^{(4)} - \frac{1}{24}A_4A_5\mathbf{v}\mathbf{v}^{(4)}, \quad (87)
\end{aligned}$$

$$\begin{aligned}
Q_7 = & -\frac{1}{288}31A_2A_5\dot{\mathbf{v}}^6 + \frac{41}{576}A_4A_5\dot{\mathbf{v}}^6 + \frac{1}{12}A_1A_2\dot{\mathbf{v}}^4 - \frac{1}{12}A_2A_3\dot{\mathbf{v}}^4 + \frac{11}{48}A_1A_4\dot{\mathbf{v}}^4 \\
& + \frac{7}{48}A_3A_4\dot{\mathbf{v}}^4 + \frac{1}{128}A_4A_5\dot{\mathbf{v}}^2\dot{\mathbf{v}}^4 - \frac{65A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^4}{1536} - \frac{979A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^4}{2304} - \frac{1}{9}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 \\
& - \frac{1}{18}A_2A_4\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{19}{24}A_1A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{1}{4}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{19}{24}A_3A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{1}{8}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 \\
& + \frac{23}{144}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{41}{288}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{17A_5^2\mathbf{v}^{(4)}\dot{\mathbf{v}}^2}{1152} + \frac{17}{24}A_2A_5\ddot{\mathbf{v}}^2\dot{\mathbf{v}}^2 + \frac{17}{48}A_4A_5\ddot{\mathbf{v}}^2\dot{\mathbf{v}}^2 \\
& - \frac{67}{384}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{1}{24}A_4^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{1}{12}A_2A_4\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{1}{64}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{1}{64}A_1A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 \\
& - \frac{3}{64}A_3A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{1}{6}A_1A_2\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{6}A_2A_3\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{12}A_1A_4\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{12}A_3A_4\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& - \frac{1}{48}A_1A_5\mathbf{v}^{(4)}\dot{\mathbf{v}} - \frac{1}{48}A_3A_5\mathbf{v}^{(4)}\dot{\mathbf{v}} - \frac{3}{64}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{48}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{12}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}\ddot{\mathbf{v}} \\
& + \frac{1}{3}A_1A_2\ddot{\mathbf{v}}^2 + \frac{1}{3}A_2A_3\ddot{\mathbf{v}}^2 + \frac{1}{6}A_1A_4\ddot{\mathbf{v}}^2 + \frac{1}{6}A_3A_4\ddot{\mathbf{v}}^2 + \frac{1}{24}A_2A_5\ddot{\mathbf{v}}^2 \\
& + \frac{1}{48}A_4A_5\ddot{\mathbf{v}}^2 - \frac{1}{9}A_2A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{18}A_4A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{16}A_1A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{16}A_3A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} \\
& + \frac{10}{3}a_4A_2A_5\dot{\mathbf{v}}^2 + \frac{5}{3}a_4A_4A_5\dot{\mathbf{v}}^2 - \frac{1}{4}A_1^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{4}A_3^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{96}A_5^2\dot{\mathbf{v}}^4\ddot{\mathbf{v}} - \frac{1}{2}A_1A_3\dot{\mathbf{v}}\ddot{\mathbf{v}} \\
& + \frac{1}{48}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{2}{9}A_2^2\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{18}A_4^2\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{2}{9}A_2A_4V_{23} + \frac{1}{36}A_2A_5\ddot{\mathbf{v}}\mathbf{v}^{(4)} + \frac{1}{72}A_4A_5\ddot{\mathbf{v}}\mathbf{v}^{(4)} \\
& - \frac{1}{192}A_5^2\ddot{\mathbf{v}}\mathbf{v}^{(4)} + \frac{5}{8}a_4A_5^2\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{24}A_5^2V_2^2\ddot{\mathbf{v}}\ddot{\mathbf{v}}, \tag{88}
\end{aligned}$$

$$\begin{aligned}
Q_8 = & -\frac{10}{3}a_4A_5^2\dot{\mathbf{v}}^2 + \frac{1}{16}A_4A_5\dot{\mathbf{v}}^2\mathbf{v}\ddot{\mathbf{v}} - \frac{5}{48}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{85}{144}A_5^2\dot{\mathbf{v}}^6 - \frac{1}{3}A_2^2\dot{\mathbf{v}}^4 \\
& - \frac{3}{8}A_4^2\dot{\mathbf{v}}^4 - \frac{13}{24}A_2A_4\dot{\mathbf{v}}^4 + \frac{7}{12}A_1A_5\dot{\mathbf{v}}^4 + \frac{19}{48}A_3A_5\dot{\mathbf{v}}^4 - \frac{1}{96}A_5^2\dot{\mathbf{v}}^2\dot{\mathbf{v}}^4 \\
& + \frac{1}{4}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 + \frac{13}{32}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 + \frac{245}{288}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{13}{288}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 - \frac{17}{24}A_5^2V_2^2\dot{\mathbf{v}}^2 \\
& - \frac{1}{9}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{5}{36}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 + \frac{1}{48}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 - \frac{1}{3}A_2^2\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{6}A_4^2\ddot{\mathbf{v}}\dot{\mathbf{v}} \\
& - \frac{1}{2}A_2A_4\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{12}A_1A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{5}{24}A_3A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{1}{24}A_2A_5\mathbf{v}^{(4)}\dot{\mathbf{v}} + \frac{1}{24}A_4A_5\mathbf{v}^{(4)}\dot{\mathbf{v}} - \frac{1}{3}A_2^2\ddot{\mathbf{v}}^2 \\
& - \frac{1}{6}A_4^2\ddot{\mathbf{v}}^2 - \frac{1}{2}A_2A_4\ddot{\mathbf{v}}^2 - \frac{1}{3}A_1A_5\ddot{\mathbf{v}}^2 - \frac{1}{3}A_3A_5\ddot{\mathbf{v}}^2 - \frac{1}{48}A_5^2\ddot{\mathbf{v}}^2 + \frac{1}{9}A_5^2(\dot{\mathbf{v}}\ddot{\mathbf{v}})^2 \\
& + \frac{1}{16}A_2A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{16}A_4A_5\ddot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{2}A_3^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{2}A_1A_3\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{3}{4}A_1A_2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{17}{12}A_2A_3\dot{\mathbf{v}}\ddot{\mathbf{v}} \\
& + \frac{3}{4}A_1A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{13}{12}A_3A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{3}A_2A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{6}A_4A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{36}A_5^2\ddot{\mathbf{v}}\mathbf{v}^{(4)}, \tag{89}
\end{aligned}$$

$$\begin{aligned}
Q_9 = & -\frac{5}{64}A_5^2\dot{\mathbf{v}}^2\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{4}(-5)A_2A_5\dot{\mathbf{v}}^4 - \frac{11}{12}A_4A_5\dot{\mathbf{v}}^4 + \frac{23}{72}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^3 + \frac{1}{9}A_5^2\ddot{\mathbf{v}}\dot{\mathbf{v}}^2 \\
& + \frac{1}{3}A_2A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} + \frac{3}{8}A_4A_5\ddot{\mathbf{v}}\dot{\mathbf{v}} - \frac{1}{48}A_5^2\mathbf{v}^{(4)}\dot{\mathbf{v}} + \frac{2}{3}A_2A_5\ddot{\mathbf{v}}^2 + \frac{1}{2}A_4A_5\ddot{\mathbf{v}}^2 \\
& - \frac{1}{16}A_5^2\ddot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{2}{3}A_1A_2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{3}A_2A_3\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{5}{6}A_1A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{7}{6}A_3A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{2}A_2^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{5}{6}A_4^2\dot{\mathbf{v}}\ddot{\mathbf{v}} \\
& - \frac{5}{3}A_2A_4\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{2}A_1A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{7}{6}A_3A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{9}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{8}A_1A_5\mathbf{v}\ddot{\mathbf{v}} - \frac{1}{8}A_3A_5\mathbf{v}\ddot{\mathbf{v}}, \tag{90}
\end{aligned}$$

$$Q_{10} = \frac{43}{72}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{4}A_2A_5\dot{\mathbf{v}}^2 - \frac{1}{8}A_4A_5\dot{\mathbf{v}}^2 + \frac{1}{9}A_5^2\mathbf{v}\ddot{\mathbf{v}} + \frac{2A_1A_2}{3} + \frac{A_1A_4}{3} - \frac{2A_2A_3}{3} - \frac{A_3A_4}{3}, \quad (91)$$

$$Q_{11} = \frac{1}{3}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{9}{16}A_2A_5\dot{\mathbf{v}}^2 - \frac{9}{16}A_4A_5\dot{\mathbf{v}}^2 + \frac{1}{12}A_5^2\mathbf{v}\ddot{\mathbf{v}} + \frac{3A_1A_2}{2} + \frac{A_1A_4}{2} - \frac{A_2A_3}{2} - \frac{3A_3A_4}{2}, \quad (92)$$

$$Q_{12} = \frac{5}{48}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{16}A_2A_5\dot{\mathbf{v}}^2 + \frac{1}{48}A_5^2\mathbf{v}\ddot{\mathbf{v}}, \quad (93)$$

$$Q_{13} = -\frac{1}{12}11A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{1}{2}A_4A_5\dot{\mathbf{v}}^2 + A_2A_5\dot{\mathbf{v}}^2 - \frac{1}{8}A_5^2\mathbf{v}\ddot{\mathbf{v}}, \quad (94)$$

$$Q_{14} = \frac{9}{16}A_5^2\dot{\mathbf{v}}^4 - \frac{2}{3}A_4^2\dot{\mathbf{v}}^2 - \frac{5}{24}A_5^2\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{3}A_5^2\dot{\mathbf{v}}^2 - \frac{2}{3}A_2^2\dot{\mathbf{v}}^2 - \frac{2}{3}A_2A_4\dot{\mathbf{v}}^2 - \frac{7}{6}A_1A_5\dot{\mathbf{v}}^2 - \frac{1}{3}A_3A_5\dot{\mathbf{v}}^2 + \frac{17}{12}A_2A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} + \frac{7}{4}A_4A_5\dot{\mathbf{v}}\ddot{\mathbf{v}} - \frac{1}{8}A_2A_5\mathbf{v}\ddot{\mathbf{v}} + \frac{1}{8}A_4A_5\mathbf{v}\ddot{\mathbf{v}}, \quad (95)$$

$$Q_{15} = \frac{1}{64}A_5^2\dot{\mathbf{v}}^2 - \frac{A_1A_5}{8} + \frac{A_3A_5}{8}, \quad Q_{16} = \frac{A_2A_5}{8} - \frac{A_4A_5}{8}, \quad (96)$$

$$Q_{17} = \frac{1}{6}A_5^2\dot{\mathbf{v}}^2 - \frac{1}{3}2A_2^2 + \frac{A_4A_2}{3} + \frac{A_4^2}{3} + \frac{2A_3A_5}{3} - \frac{2A_1A_5}{3}, \quad (97)$$

$$Q_{18} = \frac{A_5^2}{8}, \quad Q_{19} = -A_4A_5, \quad Q_{20} = \frac{2A_5^2}{3}, \quad (98)$$

$$Q_{21} = \frac{1}{2}A_5^2\dot{\mathbf{v}}^2 - A_2^2 + A_4^2 + A_3A_5, \quad Q_{22} = -\frac{1}{2}A_2A_5 - \frac{3A_4A_5}{2}, \quad (99)$$

$$Q_{23} = \frac{1}{6}A_5^2\dot{\mathbf{v}}^2, \quad Q_{24} = \frac{A_5^2}{2}, \quad (100)$$

$$Q_{25} = -5a_4A_5 + \frac{65}{192}A_5\dot{\mathbf{v}}^4 + \frac{1}{8}A_1\dot{\mathbf{v}}^2 + \frac{3}{8}A_3\dot{\mathbf{v}}^2 + \frac{3A_5\dot{\mathbf{v}}\ddot{\mathbf{v}}}{8} + \frac{A_5\dot{\mathbf{v}}^2}{3} - \frac{2A_4\dot{\mathbf{v}}\ddot{\mathbf{v}}}{3} - \frac{A_2\dot{\mathbf{v}}\ddot{\mathbf{v}}}{6}, \quad (101)$$

$$Q_{26} = -\frac{1}{3}A_2\dot{\mathbf{v}}^2 - \frac{13A_4\dot{\mathbf{v}}^2}{24} + \frac{7A_5\dot{\mathbf{v}}\ddot{\mathbf{v}}}{8}, \quad Q_{27} = \frac{5A_5\dot{\mathbf{v}}\ddot{\mathbf{v}}}{6} - \frac{1}{2}A_4\dot{\mathbf{v}}^2, \quad Q_{28} = \frac{17A_5\dot{\mathbf{v}}^2}{48} + \frac{A_1}{2} + \frac{A_3}{2}, \quad (102)$$

$$Q_{29} = \frac{A_5}{6}, \quad Q_{30} = \frac{A_5}{24}, \quad Q_{31} = -A_5\dot{\mathbf{v}}^2, \quad Q_{32} = -\frac{A_2}{3} - \frac{A_4}{6}, \quad Q_{33} = \frac{A_5}{2}, \quad (103)$$

$$Q_{34} = -\frac{A_2}{2} - \frac{A_4}{2}, \quad Q_{35} = \frac{5}{8}A_5\dot{\mathbf{v}}^2 - A_1 + A_3, \quad Q_{36} = A_2 - A_4. \quad (104)$$

Now the equations of motion (21) need to be renormalized, thus as we will need in Section 4, a set of four-vectors can be built in an orthonormal basis from the set $\{\dot{B}_\mu, v_\mu, \dot{v}_\mu, \ddot{v}_\mu, \ddot{v}_\mu\}$, where these vectors have the following form

$$\hat{\dot{B}}_\mu = \frac{\dot{B}_\mu}{\dot{B}}. \quad (105)$$

Here $\dot{B}$ is the norm of $\dot{B}_\mu$. The first orthonormal vector is given by

$$\hat{u}_{1\mu} = \frac{v_\mu - (\mathbf{v}\hat{\dot{B}})\hat{\dot{B}}_\mu}{u_1}, \quad (106)$$

with contraction of the form $|u_1|^2 = u_1^2 = 1 - \frac{\mathbf{v}\hat{\dot{B}}}{\dot{B}^2}$. For the second normalized vector in terms of $\dot{v}_\mu$ we find

$$\hat{u}_{2\mu} = \frac{1}{u_2} \left[\dot{v}_\mu - (\dot{\mathbf{v}}\hat{\dot{B}})\hat{\dot{B}}_\mu + \frac{(\mathbf{v}\hat{\dot{B}})(\dot{\mathbf{v}}\hat{\dot{B}})}{u_1^2} (v_\mu - (\mathbf{v}\hat{\dot{B}})\hat{\dot{B}}_\mu) \right], \quad (107)$$

whose magnitude can be expressed in the simple form

$$u_2^2 = \dot{\mathbf{v}}^2 - (\dot{\mathbf{v}}\hat{\dot{B}})^2 - \frac{(\mathbf{v}\hat{\dot{B}})^2(\dot{\mathbf{v}}\hat{\dot{B}})^2}{u_1^2}. \quad (108)$$

The vector $u_{3\mu}$ is expressed in terms of $\ddot{v}_\mu$ and it is given as follows

$$\begin{aligned} u_{3\mu} = & \ddot{v}_\mu - \frac{\mathbf{B}\ddot{v}_\mu}{B^2} - \frac{\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\hat{\dot{B}})(\ddot{\mathbf{v}}\hat{\dot{B}})}{u_1^2} (v_\mu - (\mathbf{v}\hat{\dot{B}})\hat{\dot{B}}_\mu) \\ & - \frac{1}{u_2^2} \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\hat{\dot{B}})(\ddot{\mathbf{v}}\hat{\dot{B}}) + \frac{(\mathbf{v}\hat{\dot{B}})(\dot{\mathbf{v}}\hat{\dot{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\hat{\dot{B}})\ddot{\mathbf{v}}\hat{\dot{B}}) \right] \\ & \times \left[\dot{v}_\mu - (\dot{\mathbf{v}}\hat{\dot{B}})\hat{\dot{B}}_\mu + \frac{(\mathbf{v}\hat{\dot{B}})(\dot{\mathbf{v}}\hat{\dot{B}})}{u_1^2} (v_\mu - (\mathbf{v}\hat{\dot{B}})\hat{\dot{B}}_\mu) \right]. \end{aligned} \quad (109)$$

Consequently we have

$$\begin{aligned} u_3^2 = & \ddot{\mathbf{v}}^2 - (\ddot{\mathbf{v}}\hat{\dot{B}})^2 - \frac{1}{u_1^2} [\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\hat{\dot{B}})(\ddot{\mathbf{v}}\hat{\dot{B}})]^2 \\ & - \frac{1}{u_2^2} \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\hat{\dot{B}})(\ddot{\mathbf{v}}\hat{\dot{B}}) + \frac{(\mathbf{v}\hat{\dot{B}})(\dot{\mathbf{v}}\hat{\dot{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\hat{\dot{B}})(\ddot{\mathbf{v}}\hat{\dot{B}})) \right]^2. \end{aligned} \quad (110)$$

All these vectors are constructed in such a way that the conditions of orthonormalization are fulfilled in all the expressions of vectors u_i . Here the index $i = 1, 2, 3, 4$ stands for a label of the basis. The index $j = 1, 2, 3, 4$ in $v^{(j)}$ stands to the derivative with respect to the proper-time of the particle. Analogously the last more general orthonormal vector that we

build in this basis is

$$\begin{aligned}
u_{4\mu} = & \ddot{v}_\mu - (\ddot{\mathbf{v}}\dot{\mathbf{B}})\dot{B}_\mu - \frac{[\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})]}{u_1^2} [v_\mu - (\mathbf{v}\dot{\mathbf{B}})\dot{B}_\mu] \\
& - \frac{1}{u_2^2} \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}}) + \frac{(\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})) \right] \\
& \times \left[\dot{v}_\mu - (\dot{\mathbf{v}}\dot{\mathbf{B}})\dot{B}_\mu + \frac{(\mathbf{v}\dot{\mathbf{B}})(\dot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (v_\mu - (\mathbf{v}\dot{\mathbf{B}})\dot{B}_\mu) \right] \\
& - \frac{1}{u_3^2} \left\{ \ddot{\mathbf{v}}\ddot{\mathbf{v}} - (\ddot{\mathbf{v}}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}}) - \frac{\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})) \right. \\
& - \frac{1}{u_2^2} \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}}) + \frac{(\mathbf{v}\dot{\mathbf{B}})(\dot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})\dot{\mathbf{v}}\dot{\mathbf{B}}) \right] \\
& \times \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}}) + \frac{(\mathbf{v}\dot{\mathbf{B}})(\dot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})) \right] \Big\} \\
& \times \left\{ \ddot{v}_\mu - (\ddot{\mathbf{v}}\dot{\mathbf{B}})\dot{B}_\mu - \frac{\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (v_\mu - (\mathbf{v}\dot{\mathbf{B}})\dot{B}_\mu) \right. \\
& - \frac{1}{u_2^2} \left[\dot{\mathbf{v}}\ddot{\mathbf{v}} - (\dot{\mathbf{v}}\dot{\mathbf{B}})(\ddot{\mathbf{v}}\dot{\mathbf{B}}) + \frac{(\mathbf{v}\dot{\mathbf{B}})(\dot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (\mathbf{v}\ddot{\mathbf{v}} - (\mathbf{v}\dot{\mathbf{B}})\dot{\mathbf{v}}\dot{\mathbf{B}}) \right] \\
& \times \left. \left[\dot{v}_\mu - (\dot{\mathbf{v}}\dot{\mathbf{B}})\dot{B}_\mu + \frac{(\mathbf{v}\dot{\mathbf{B}})(\dot{\mathbf{v}}\dot{\mathbf{B}})}{u_1^2} (v_\mu - (\mathbf{v}\dot{\mathbf{B}})\dot{B}_\mu) \right] \right\}. \tag{111}
\end{aligned}$$

Now with Eqs. (106), (107), (109), (111) and the conditions of orthonormalization, it can be proved that $v_\mu^{(i)}\hat{u}_4^\mu = v^{(i)}\hat{u}_4 = 0$ with $i = 0, 1, 2, 3$ and $v_\mu^{(0)} = v_\mu$. With Eqs. (111) it is easy to show that the product $\dot{\mathbf{B}}\hat{u}_4 = 0$, where $\dot{B}_\mu$ is taken from Eq. (21) and this lead us to

$$(\varepsilon^{-1}Q_{36} + Q_{35})L_\nu v^\nu - A_4 L_\nu \dot{v}^\nu = 0, \tag{112}$$

where $L_\nu := f_{\mu\nu}\hat{u}_4^\mu$. This expression is used in the subsections (4.3) and (4.4) to obtain the equations of motion.

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