Non-Vanishing of Modular $L$-Functions

with Large Level

by

Amir Akbary-Majdabadno

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

©Copyright by Amir Akbary-Majdabadno 1997
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author’s permission.

L’auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L’auteur conserve la propriété du droit d’auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-27593-0
ABSTRACT

Non-Vanishing of Modular L-Functions with Large Level
Amir Akbary-Majdabadno
Department of Mathematics, University of Toronto

This thesis studies the non-vanishing of the twisted modular L-function $L_f(s, \chi)$ for a fixed weight $k$, varying level $\mathcal{N}$ and a fixed Dirichlet character $\chi \pmod{q}$ where $(q, \mathcal{N}) = 1$. Here $f$ is a newform of level $\mathcal{N}$. Let $\mathcal{F}_\mathcal{N}$ be the set of newforms of weight $k$ and level $\mathcal{N}$.

(1) It is proved that

$$C_k \frac{\mathcal{N}}{(\log \mathcal{N})^2} \leq \sharp \{f \in \mathcal{F}_\mathcal{N} : L_f\left(\frac{k}{2}, \chi\right) \neq 0\}$$

for prime $\mathcal{N}$ large enough. Here, $C_k$ is a constant depending only on $k$.

(2) It is proved that for real Dirichlet characters $\chi_1$ and $\chi_2$ with $\chi_1 \chi_2(-\mathcal{N}) = 1$ and $k > 2$,

$$C'_k \frac{\mathcal{N}}{(\log \mathcal{N})^6} \leq \sharp \{f \in \mathcal{F}_\mathcal{N} : L_f\left(\frac{k}{2}, \chi_1\right)L_f\left(\frac{k}{2}, \chi_2\right) \neq 0\}$$

for prime $\mathcal{N}$ large enough. Here, $C'_k$ is a constant depending only on $k$.

(3) In the case $k = 2$, it is proved that under the assumption of the Generalized Riemann Hypothesis for $L_f(s)$ and the assumption of $L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll \mathcal{N}^{\frac{1}{2}-\eta}$ for some $\eta > 0$

$$c\mathcal{N} \leq \sharp \{f \in \mathcal{F}_\mathcal{N} : L_f(1) = 0 \text{ and } L'_f(1) \neq 0\}$$

for prime $\mathcal{N}$ large enough. Here $L'_f(s)$ is the derivative of $L_f(s)$ and $c$ ($0 < c < 1$) is an absolute constant.

During the course of the proof of (3), a “semi-orthogonality” relation between the Fourier coefficients of $\mathcal{F}_\mathcal{N}$ (newforms with root number $-1$) is given. Using this
relation and the symmetric square $L$-function properties, upper bounds for

\[ \sum_{f \in \mathcal{F}_N} \frac{r_f}{4\pi} < f, f >, \text{ and } \sum_{f \in \mathcal{F}_N} r_f^{\frac{1}{2}} \]

and asymptotic formula for

\[ \sum_{f \in \mathcal{F}_N} < f, f > \]

are obtained, where $r_f$ is the vanishing order of $L_f(s)$ at $s = 1$ and $< \ldots >$ denotes the Petersson inner product.
Acknowledgments

I would like to thank my supervisor, Professor Kumar Murty, for his support, guidance, and patience in the preparation of this thesis. It has been a privilege and a pleasure working with him and learning from him.

I would like to thank Kambiz Mahmoudian for his careful proof reading of the first manuscript of this thesis, and for his many helpful suggestions. I would also like to thank Vijay Patankar for many helpful conversations during the course of the preparation of this thesis. Ying Zheng’s help in reading this thesis and giving some helpful suggestions is also appreciated.

I would like to thank the Ministry of Culture and Higher Education of Iran and the Department of Mathematics of the University of Toronto for financial support during my graduate work. My thanks also go to the staff of the Department of Mathematics, particularly Ida, for their assistance in the past few years.

Many thanks go to all of my friends, especially Fariborz, Felipe, Heather, Heng, Kamran, Karl, Lucia, Madjid, Peter, Sasan and Stephen for their support and friendship and for many good times we shared together during the course of my study.

Finally, I am grateful to my parents and my sisters for their continuous support.
NOTATIONS

\( f(x) = o(g(x)) \) if \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \)
\( f(x) = O(g(x)) \) or \( f(x) \ll g(x) \) if there exists a constant \( C \) such that \( |f(x)| \leq Cg(x) \)
\( \mathcal{H} \): the upper half-plane
\( \mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\} \)
\( GL_2^+(\mathbb{R}) \): the group of \( 2 \times 2 \) matrices with real entries and positive determinant
\( SL_2(\mathbb{Z}) \): the group of \( 2 \times 2 \) matrices with integer entries and determinant equal to 1
\( \Gamma_0(N) \): the subgroup of \( SL_2(\mathbb{Z}) \) consist of matrices \((a_{ij})_{2 \times 2}\) which \( a_{21} \) is divisible by \( N \)
\( \Gamma_0(N) : \overline{\Gamma_0(N)} \mod \text{its center} \)
\( S_k(N) \): the space of cusp forms of weight \( k \) and level \( N \)
\( \langle f, g \rangle_N \): the Petersson inner product of \( f \) and \( g \) in \( S_k(N) \)
\( a_f(n) \): the \( n \)-th Fourier coefficient of the cusp form \( f \)
\( L_f(N) \): the \( L \)-function associated to the cusp form \( f \)
\( W_N \): the Atkin-Lehner involution
\( S_k^+(N) \): the \((-1)^{\frac{k}{2}}\)-eigenspace of \( W_N \) in \( S_k(N) \)
\( S_k^{-}(N) \): the \((-1)^{\frac{k}{2}+1}\)-eigenspace of \( W_N \) in \( S_k(N) \)
\( T_p (p \nmid N), U_q (q \mid N) \): the Hecke operators
\( \mathcal{F}_N \): the set of normalized newforms of weight \( k \) and level \( N \)
\( \chi \): a Dirichlet character
\( L_f(s, \chi) \): the twisted \( L \)-function associated to \( f \) and \( \chi \)
\( \varepsilon_f \): the root number of \( L_f(s) \)
\( \tau(x) \): the Gauss sum
\( \varepsilon_{\chi} \): the root number of \( L_f(s, \chi) \)
\( L_{sym}^2(f)(s) \): the symmetric square \( L \)-function associated to \( f \)
\( r_f \): the vanishing order of \( L_f(s) \) at \( s = \frac{k}{2} \)
\( P_n(z, k, N) \): the Poincaré series of weight \( k \) and level \( N \) for \( S_k(N) \)
\( P_n(m, k, N) \): the \( m \)-th coefficient of the Fourier expansion of \( P_n(z, k, N) \)
\( P_n(z, k, N) \): the Poincaré series of weight \( k \) and level \( N \) for \( S_k^-(N) \)

\( \Gamma_\infty \): the stabilizer of \( \infty \) in \( \Gamma_0(N) \)

\( \Gamma(s) \): the Gamma function

\( S(m, n; c) \): the Kloosterman sum

\( J_{k-1}(t) \): the Bessel function of order \( k - 1 \)

\( \delta_{mn} \): the Kronecker delta

\( T_p \ (p \nmid N), \ C_q \ (q \mid N) \): the Pizer operators

\( \mathcal{P}_N \): the Pizer basis

\( Tr(f) \): the trace function

\( tr(T_{e^2}) \): the trace of the \( e^2 \)-th Hecke operator

\( \zeta(s) \): the Riemann zeta function

\( \zeta_N(s) \): the Riemann zeta function with the Euler factors corresponding to \( p \mid N \) removed

\( d(n) \): the number of positive divisors of \( n \)

\( \phi(n) \): the Euler phi function

\( \Lambda(n) \): the Von Mangoldt function

\( \mu(n) \): the Möbius function

\( q^r \mid \mid N \): \( q^r \mid \mid N \) but \( q^{r+1} \nmid N \)

\( (m, n) \): the greatest common divisor of \( m \) and \( n \)
# Contents

1 **Introduction and Statement of Results** 1
   1.1 Modular forms .................................................. 1
   1.2 \(L\)-function of a cusp form .................................. 3
   1.3 Hecke operators .................................................. 5
   1.4 Oldforms and newforms .......................................... 6
   1.5 Problems .......................................................... 8
   1.6 Statement of results ............................................ 9

2 **Non-Vanishing of Weight \(k\) Modular \(L\)-functions** 12
   2.1 A semi-orthogonality relation ................................. 12
   2.2 A basis for \(S_k(V)\) ............................................ 15
   2.3 Critical values on average .................................... 18
   2.4 A lower bound for the Petersson inner product ............. 23
   2.5 Non-vanishing of product of twisted modular \(L\)-functions 28

3 **A "Semi-Orthogonality" Relation for \(S_k^-(V)\) and Its Applications** 34
   3.1 Poincaré series for \(S_k^-(N)\) .................................. 34
   3.2 A "semi-orthogonality" relation for \(S_k^-(V)\) ................ 36
   3.3 Non-vanishing of the derivative of modular \(L\)-functions .... 46
   3.4 An approximate trace formula for \(S_2^-(N)\) ................... 58
Chapter 1

Introduction and Statement of Results

We recall some basic facts about modular forms (see [12] and [15] for details).

1.1 Modular forms

Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{z = x + iy : x \in \mathbb{R}, y > 0\}.$$ 

Let $GL_2^+(\mathbb{R})$ be the group of $2 \times 2$ matrices with real entries and positive determinant. Then $GL_2^+(\mathbb{R})$ acts on $\mathcal{H}$ as a group of holomorphic automorphisms

$$\gamma : z \mapsto \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

Let $\mathcal{H}^*$ denote the union of $\mathcal{H}$ and the rational numbers $\mathbb{Q}$ together with a symbol $\infty$ (or $i\infty$). The rational numbers together with $\infty$ are called cusps.
Let $f$ be a holomorphic function on $\mathcal{H}$ and $k$ a positive integer. For 
\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}). \]
define the stroke operator "$|_k$" as 
\[ (f|_k \gamma)(z) = (\text{det}\gamma)^{\frac{k}{2}} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right). \]
Sometimes, we simply write $f|_{\gamma}$ for $f|_k \gamma$. Note that $(f|_{\gamma})|_{\sigma} = f|_{\gamma|\sigma}$.

Let $SL_2(\mathbb{Z})$ be the group of $2 \times 2$ matrices with integer entries and determinant 1 and let $\Gamma$ be a subgroup of finite index of it. Suppose $f$ is a holomorphic function on $\mathcal{H}$ such that $f|_{\gamma} = f$ for all $\gamma \in \Gamma$. Since $\Gamma$ has finite index, 
\[ \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)^M = \left( \begin{array}{cc} 1 & M \\ 0 & 1 \end{array} \right) \in \Gamma \]
for some positive integer $M$. Hence $f(z + M) = f(z)$ for all $z \in \mathcal{H}$. So $f$ has a "Fourier expansion at infinity" in the form of 
\[ f(z) = \sum_{n=-\infty}^{\infty} a_f(n) q_M^n, \quad q_M = e^{2\pi iz/M}. \]
We say that $f$ is holomorphic at infinity if $a_n = 0$ for all $n < 0$. We say it vanishes at infinity if $a_n = 0$ for all $n \leq 0$.

Let $\sigma \in SL_2(\mathbb{Z})$. Then $\sigma^{-1} \Gamma \sigma$ also has finite index and $(f|_{\sigma}|_{\gamma} = f|_{\sigma}$ for all $\gamma \in \sigma^{-1} \Gamma \sigma$. So $f|_{\sigma}$ also has a Fourier expansion at infinity. We say that $f$ is holomorphic at the cusps if $f|_{\sigma}$ is holomorphic at infinity for all $\sigma \in SL_2(\mathbb{Z})$. We say that $f$ vanishes at the cusps if $f|_{\sigma}$ vanishes at infinity for all $\sigma \in SL_2(\mathbb{Z})$.

Now for $N \geq 1$ let 
\[ \overline{\Gamma_0(N)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}): \ c \equiv 0 \text{ (mod } N) \right\} \]
and \( \Gamma_0(N) = \overline{\Gamma_0(N)}/\{\pm 1\} \). Note that \( \overline{\Gamma_0(N)} \) is of finite index in \( SL_2(\mathbb{Z}) \) (Here, we follow the unconventional notation of \([4]\) to be consistent with the results of \([5]\)).

A modular form of weight \( k \) and level \( N \) is a holomorphic function \( f \) on \( \mathcal{H} \) such that

(i) \( f|_\gamma = f \) for all \( \gamma \in \Gamma_0(N) \).

(ii) \( f \) is holomorphic at the cusps.

Such a modular form is called a cusp form if it vanishes at the cusps.

The modular forms of weight \( k \) and level \( N \) form a finite dimensional vector space \( M_k(N) \) and this has a subspace \( S_k(N) \) consisting of cusp forms. Note that since \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) is the same as \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in \( \Gamma_0(N) \), (i) shows that \( S_k(N) = \{0\} \) if \( k \) is odd. From now on we assume that \( k \) is even.

Moreover, one can define an inner product called Petersson inner product on \( S_k(N) \) by

\[
<f, g> = \int_{\Gamma_0(N) \backslash \mathcal{H}} f(z)\overline{g(z)}|y|^k \frac{dxdy}{y^2}.
\]

Note that if \( N_1 | N_2 \) then \( S_k(N_1) \subset S_k(N_2) \). However, the value of the Petersson inner product depends on \( N \). To emphasize this dependency sometimes we write \( <f, g>_N \).

### 1.2 \( L \)-function of a cusp form

Now if \( f \in S_k(N) \) since \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) \) its Fourier expansion at \( i\infty \) is of the form

\[
f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz), \quad e(z) = e^{i\pi z}.
\]

Attached to \( f \), we define the \textit{the \( L \)-function associated to} \( f \) by the Dirichlet series

\[
L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}.
\]
We can show that $L_f(s)$ represents an analytic function for $Re(s) > \frac{k+1}{2}$. This is a consequence of the fact that $a_f(n) = O(n^{\frac{k+1}{2}})$ (see formula (1.1)).

Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. It is not an element of $SL_2(\mathbb{Z})$ unless $N = 1$. However, $W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N)$.

Moreover, $f|W_N^2 = f$. $W_N$ is called the *Atkin-Lehner involution.*

More generally for any prime $q$ dividing $N$ with $q^r || N$ (i.e. $q^r | N$ but $q^{r+1} \nmid N$). let

$$W_q = \begin{pmatrix} q^r x & y \\ N z & q^r w \end{pmatrix}$$

where $x, y, z$ and $w$ are any integers satisfying $\det(W_q) = q^r$. $W_q$ is called the "$W_q$ operator" of Atkin and Lehner.

Since $W_N$ is a linear transformation of the vector space $S_k(N)$ and $W_N^2 = 1$, it decomposes the space of cusp forms (modular forms) to complementary subspaces corresponding to the eigenvalues $\pm 1$. Set

$$S_k^+(N) = \left\{ f \in S_k(N): f|W_N = (-1)^{\frac{s}{2}} f \right\}.$$  

$$S_k^-(N) = \left\{ f \in S_k(N): f|W_N = (-1)^{\frac{s}{2}+1} f \right\}.$$  

and so $S_k(N) = S_k^+(N) \oplus S_k^-(N)$.

Then the following Theorem of Hecke guarantees the analytic continuation of $f \in S_k(N)$.

**Theorem (Hecke)** Let $f \in S_k^\pm(N)$. Then $L_f(s)$ extends to an entire function and $\Lambda_f(s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L_f(s)$ satisfies the functional equation $\Lambda_f(s) = \pm \Lambda_f(k-s)$.

**Corollary** Let $f \in S_k(N)$. Then $L_f(s)$ extends to an entire function.

**Note** Our definition of $S_k^+(N)$ and $S_k^-(N)$ is slightly different from the conventional ones which denote them as subspaces corresponding to the eigenvalues $+1$ and $-1$ for operator $W_N$, so for $\frac{s}{2}$ odd our $S_k^\pm(N)$ is their $S_k^{\mp}(N)$. The root number of $L_f(s)$
is the sign appearing in the functional equation of \( L_f(s) \). In our notation \( S_k(\mathcal{N}) \) is the set of cusp forms whose \( L \)-functions have root number \( \pm 1 \), respectively.

### 1.3 Hecke operators

Let \( f \in M_k(\mathcal{N}) \). Let \( p \) and \( q \) be primes such that \( p \nmid \mathcal{N} \) and \( q \mid \mathcal{N} \). The Hecke operators \( T_p \) and \( U_q \) are defined by

\[
f \mid T_p = p^{\frac{k}{2} - 1} \left[ f \mid \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{e=0}^{p-1} f \mid \begin{pmatrix} 1 & e \\ 0 & p \end{pmatrix} \right].
\]

\[
f \mid U_q = q^{\frac{k}{2} - 1} \left[ \sum_{e=0}^{q-1} f \mid \begin{pmatrix} 1 & e \\ 0 & q \end{pmatrix} \right].
\]

We can show that \( f \mid T_p \) and \( f \mid U_q \) are also modular forms of weight \( k \) and level \( \mathcal{N} \). Furthermore, they are cusp forms if \( f \) is a cusp form.

Let \( f \in S_k(\mathcal{N}) \). We will say that \( f \) is an eigenform if \( f \) is an eigenfunction for all the Hecke operators \( \{ T_p \ (p \nmid \mathcal{N}), \ U_q \ (q \mid \mathcal{N}) \} \). The following theorem gives the main property of eigenforms.

**Theorem** The following conditions are equivalent:

(i) \( f \) is an eigenform and \( a_f(1) = 1 \).

(ii) \( L_f(s) \) has a product of the form

\[
L_f(s) = \prod_{q \nmid \mathcal{N}} \left( 1 - \frac{a_f(q)}{q^s} \right)^{-1} \prod_{p \mid \mathcal{N}} \left( 1 - \frac{a_f(p)}{p^s} + \frac{1}{p^{2s+1-k}} \right)^{-1}
\]

which converges absolutely for \( \text{Re}(s) > \frac{k+1}{2} \).

We call the product given in part (ii) of the above theorem an Euler Product. Inspired by the above theorems we may think of finding a basis for \( S_k(\mathcal{N}) \) consisting of eigenforms for all the operators \( \{ W_N, \ T_p \ (p \nmid \mathcal{N}), \ U_q \ (q \mid \mathcal{N}) \} \). We can show that there exists a basis for \( S_k(\mathcal{N}) \) consisting of eigenforms for all the operators \( \{ T_p \ (p \nmid \mathcal{N}) \} \) and the operator \( W_N \) (see [1] Lemma 27). The existence of such a basis
is the consequence of the fact that \( \{ T_p(p \nmid N), W_N \} \) form a commuting family of Hermitian linear operators (with respect to the Petersson inner product) and therefore from a theorem of linear algebra (see [10] p. 207, Theorem 8) the space of cusp forms is diagonalizable under these operators. Unfortunately the operators \( \{ U_q(q\mid N) \} \) are not Hermitian for \( S_k(N) \) and we can not diagonalize \( S_k(N) \) with respect to the operators \( \{ T_p(p \nmid N), U_q(q\mid N), W_N \} \). However, we may find such a basis for a certain subspace of \( S_k(N) \) (For a proof of the fact that the operator \( W_N \) is Hermitian, see Lemma 14. Chapter 3).

### 1.4 Oldforms and newforms

In [1] Atkin and Lehner construct a subspace of \( S_k(N) \) which is diagonalizable under the operators \( \{ T_p(p \nmid N), U_q(q\mid N), W_N \} \). More precisely they showed the existence of a subspace of \( S_k(N) \) whose \( \{ T_p(p \nmid N) \} \) eigenspaces are one dimensional. We call such a property "multiplicity one". Now since the \( \{ U_q(q\mid N), W_N \} \) commute with the \( \{ T_p(p \nmid N) \} \), the eigenform for the \( \{ T_p(p \nmid N) \} \) are eigenform for the \( \{ U_q(q\mid N), W_N \} \) too.

Let \( N' \mid N \quad (N' \neq N) \) and suppose that the \( \{ g_i \} \) is a basis of eigenforms for the \( \{ T_p(p \nmid N') \} \). Now if \( d \) is any divisor of \( \frac{N}{N'} \), then \( g_i(dz) \in S_k(N) \). Set

\[
S_k^{old}(N) = \text{span}\{g_i(dz); \text{ for any } N' \mid N \quad (N' \neq N), \quad d \mid \frac{N}{N'}\}
\]

We call \( S_k^{old}(N) \) the space of oldforms. Its orthogonal complement under the Petersson inner product is denoted \( S_k^{new}(N) \) and the eigenforms in this space are called newforms. So we have

\[
S_k(N) = S_k^{old}(N) \bigoplus S_k^{new}(N).
\]

If \( f \) is a newform then we can prove that \( a_f(1) \neq 0 \) and therefore we can normalize a newform such that \( a_f(1) = 1 \). Since the space of newforms has multiplicity one the set of normalized newforms of weight \( k \) and level \( N \) is uniquely determined. We
denote it by $\mathcal{F}_N$. From the above discussion it is clear that if $f \in \mathcal{F}_N$, $L_f(s)$ is given by an absolutely convergent series on the half plane $\text{Re}(s) > \frac{k+1}{2}$, it has an analytic continuation to the whole plane. Moreover, it satisfies a functional equation and has an Euler product on the half-plane $\text{Re}(s) > \frac{k+1}{2}$. For the Fourier coefficients of a newform $f$ we have the Deligne bound

$$|a_f(n)| \leq d(n)n^{\frac{k-1}{2}}$$  \hspace{1cm} (1.1)$$

where $d(n)$ is the divisor function.

Now let $f \in \mathcal{F}_N$ and $\chi$ be a primitive Dirichlet character mod $q$ with $(q, N) = 1$. The twisted $L$-function associated to $f$ and $\chi$ is defined by

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_f(n)}{n^s}.$$ 

The twisted $L$-function is given by an absolutely convergent series on the half-plane $\text{Re}(s) > \frac{k+1}{2}$ and has an Euler product valid there. Also it has an analytic continuation which satisfies the following functional equation

$$\left( \frac{q\sqrt{N}}{2\pi} \right)^s \Gamma(s)L_f(s, \chi) = \varepsilon_\chi(q\sqrt{N})^{-k-s} \Gamma(k-s)L_f(k-s, \chi)$$  \hspace{1cm} (1.2)$$

where $\varepsilon_\chi = \varepsilon_f\chi(N)\tau(\chi)^2q^{-1}$ with $\varepsilon_f = \pm 1$ (the root number of $f$) which depends only on $f$ and $\tau(\chi)$ is the Gauss sum (see [20] p. 93).

Now let $f \in \mathcal{F}_N$; then the symmetric square $L$-function associated to $f$ is defined by

$$L_{\text{sym}^2(f)}(s) = \frac{\zeta_N(2s+1)}{\zeta_N(s+1)} \sum_{n=1}^{\infty} \frac{(a_f(n))^2}{n^s}$$  \hspace{1cm} (1.3)$$

where $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p|N$ removed.

Since $f$ is a newform, we have

$$a_f(m)a_f(n) = \sum_{d|m,n} d^{k-1}a_f\left(\frac{mn}{d^2}\right)$$
Using this identity for \( m = n \) we have

\[
\frac{1}{\zeta_N(s + 1 - k)} \sum_{n=1}^{\infty} \frac{(a_f(n))^2}{n^s} = \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.
\]

Substituting this in (1.3) yields

\[
L_{\text{sym}^2(f)}(s) = \zeta_N(2s + 2 - 2k) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.
\]

Since \( \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s} \) is absolutely convergent for \( \text{Re}(s) > k \), \( L_{\text{sym}^2(f)}(s) \) is also absolutely convergent for \( \text{Re}(s) > k \). By [8] we know that \( L_{\text{sym}^2(f)}(s) \) extends to an entire function and satisfies a functional equation of the form

\[
R(s) = A^{s-1} \Gamma\left(\frac{s + k - 2}{2}\right) \Gamma\left(\frac{s + k - 1}{2}\right) \Gamma\left(\frac{s}{2}\right) L_{\text{sym}^2(f)}(s) = \omega R(3 - s)
\]

where \( |\omega| = 1 \), and \( A \) is a constant with \( \log A = O(\log N) \) (see [13] p. 337).

### 1.5 Problems

The \( L \)-function of a cusp form is one of the many \( L \)-functions which one studies in number theory. Specifically, the investigation of the non-vanishing of \( L \)-functions has been one of the main themes in modern number theory. For example, the distribution of prime numbers in arithmetic progressions is intimately connected with non-vanishing properties of various \( L \)-functions.

In this thesis we study the non-vanishing of the \( L \)-function associated to a cusp form of weight \( k \) and level \( N \). Specifically we consider the following problems.

**Problem 1** Find a lower bound for

\[
\{ f \in \mathcal{F}_N : L_f\left(\frac{k}{2}, \chi\right) \neq 0 \}
\]

*(here \( \chi \) is a primitive Dirichlet character)*.
Problem 2 Find a lower bound for
\[ \{ f \in \mathcal{F}_N; L_f \left( \frac{k}{2}, \chi_1 \right) L_f \left( \frac{k}{2}, \chi_2 \right) \neq 0 \} \]
(here \( \chi_1 \) and \( \chi_2 \) are distinct primitive Dirichlet characters).

Problem 3 Find a lower bound for
\[ \{ f \in \mathcal{F}_N; L'_f \left( \frac{k}{2}, \chi \right) \neq 0 \} \]
(here \( L'_f \) is the derivative of \( L_f \)).

1.6 Statement of results

In Problem 1, we expect that for a positive proportion of the newforms \( f \in \mathcal{F}_N \), \( L_f \left( \frac{k}{2}, \chi \right) \neq 0 \). However, it seems that we are still far from being able to prove this fact. The only known result concerning Problem 1 is one by W. Duke [5] for the case \( k = 2 \).

By comparing mean and mean square estimate for the twisted \( L \)-function \( L_f(s, \chi) \) attached to a newform \( f \) of weight 2, Duke proved that there is a positive absolute constant \( C \) and a constant \( C_q \) depending only on \( q \) such that for any prime \( N > C_q \) there are at least \( C N (\log N)^{-2} \) newforms \( f \in \mathcal{F}_N \) for which \( L_f(1, \chi) \neq 0 \).

Although this result does not give us a positive proportion of \( \mathcal{F}_N \) for which \( L_f(1, \chi) \neq 0 \), it is an important result and has certain applications. For example, if \( A \) is the factor of the Jacobian of the modular curve \( X_0(N) \) determined by \( f \in \mathcal{F}_N \), then \( L_f(1) \) is conjectured not to vanish if and only if the rank of the Mordell-Weil group of \( A \) over the set of rational numbers is zero. Thus, Duke’s result gives a lower bound for the frequency of this occurrence for a prime level \( N \).

The main difficulty in the generalization of the above result to the cusp forms of weight \( k \) is the contribution coming from oldforms of weight \( k \). In chapter 2, by using a special construction of a basis for the space of cusp forms of weight \( k \) and level \( N \), introduced by A. Pizer [17], we show that the contribution of oldforms is negligible.
and therefore we obtain a generalization of Duke's result to newforms of weight \( k \) and level \( \mathcal{N} \). More precisely, we prove the following result.

**Theorem** Suppose that \( \chi \) is a fixed primitive Dirichlet character mod \( q \) such that \((q, \mathcal{N}) = 1\). Then there are positive constants \( C_k \) (depending only on \( k \)) and constant \( C_{q,k} \) (depending only on \( q \) and \( k \)) such that for prime \( \mathcal{N} > C_{q,k} \) there exist at least \( C_k \mathcal{N}(\log \mathcal{N})^{-2} \) newforms \( f \) of weight \( k \) and level \( \mathcal{N} \) for which \( L_f(\frac{k}{2}, \chi) \neq 0 \).

By using similar techniques and an estimation of sums of Fourier coefficients due to W. Duke, J.B. Friedlander and H. Iwaniec [6], we have been able to prove the following theorem about the non-vanishing of the product of two distinct twist of a modular \( L \)-function.

**Theorem** Let \( k > 2 \) and \( \chi_1 \mod q_1 \) and \( \chi_2 \mod q_2 \) be fixed distinct real primitive Dirichlet characters such that \( \chi_1 \chi_2(-\mathcal{N}) = 1 \). Then there are positive constants \( C_k \) (depending only on \( k \)) and \( C_{q_1,q_2,k} \) (depending only on \( q_1, q_2 \) and \( k \)) such that for prime \( \mathcal{N} > C_{q_1,q_2,k} \) there exist at least \( C_k \mathcal{N}(\log \mathcal{N})^{-6} \) newforms \( f \) of weight \( k \) and level \( \mathcal{N} \) for which \( L_f(\frac{k}{2}, \chi_1)L_f(\frac{k}{2}, \chi_2) \neq 0 \).

If we set \( r_f = \text{ord}_{s=\frac{k}{2}} L_f(s) \) and consider

\[
\sum_{f \in \mathcal{F}_\mathcal{N}} r_f
\]  

we may find a solution for problem 1 (in the case that \( \chi \) is trivial) if we can find a good upper bound for (1.5). For example if we could prove that \( \sum_{f \in \mathcal{F}_\mathcal{N}} r_f \leq c \dim S^\text{new}_k(\mathcal{N}) + o(\mathcal{N}) \) for some \( c < 1 \).

In [16] R. Murty by applying the machinery of the Weil explicit formula to newforms of weight 2 and prime level \( \mathcal{N} \), showed that under the assumptions of the Generalized Riemann Hypothesis for the \( L \)-functions of the newforms \( f \) and the Lindelöf hypothesis for the symmetric square \( L \)-function of \( f \)

\[
\sum_{f \in \mathcal{F}_\mathcal{N}} r_f \leq \left( \frac{11}{6} + \epsilon \right) \dim S_2(\mathcal{N}) + o(\mathcal{N})
\]
as \( N \to \infty \) for any \( \epsilon > 0 \). Since \( \frac{11}{6} > 1 \) this result does not help us with Problem 1.

The main technical tool in the proof of Duke and R. Murty's results is the "semi-orthogonality" of the Fourier coefficient of an orthonormal basis of \( S_k(N) \) which is a consequence of the Petersson formulae about Poincaré series.

In chapter 3, we develop a new technical tool. We define a similar Poincaré series for \( S_k^-(N) \), the complex vector space of cusp forms with root number -1 as defined in section 1.2, and then by analogy with the classical case, we get a "semi-orthogonality" relation for \( S_k^-(N) \). As a consequence of this, by applying the methods developed in \([16]\), and under the assumption of the Riemann hypothesis, we obtain an upper bound for

\[
\sum_{f \in \mathcal{F}_N} \omega_f f
\]

where \( \omega_f = \frac{1}{4\pi \langle f, f \rangle} \). Also under certain assumption on \( L_{\text{sym}^2(f)}(s) \) on the line \( \frac{3}{2} + it \), we obtain an asymptotic formula for

\[
\sum_{f \in \mathcal{F}_N} \langle f, f \rangle.
\]

Finally, as a direct consequence of these two facts, we have

**Corollary:** Assume the Riemann hypothesis for \( L_f(s) \) and suppose that \( L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll N^{\frac{1}{2} - \eta} \), for some \( \eta > 0 \), then for prime \( N \) large enough, a positive proportion of elements of \( \mathcal{F}_N^- \) (and therefore \( \mathcal{F}_N \)) have order 1 at \( s = 1 \).
Chapter 2

Non-Vanishing of Weight $k$ Modular $L$-functions

2.1 A semi-orthogonality relation

We start by recalling some basic facts about Poincaré series (see [18] chapter 5 for more explanation).

We can show that $S_k(N)$ equipped with the Petersson inner product is a finite dimensional inner product space spanned by the Poincaré series

$$P_n(z, k, N) = \sum_{\gamma \in \Gamma_0(N)/\Gamma_0(N)} \frac{e(n \gamma z)}{(cz + d)^k}, \quad n \geq 1$$

where $e(z) = e^{2\pi iz}, \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, and $\Gamma_\infty$ is the stabilizer of $i\infty$ in $\Gamma_0(N)$. We know that for $k > 2$ the above series is absolutely convergent.

If $f \in S_k(N)$, we write the Fourier expansion of $f$ as

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

at $i\infty$. 
Petersson proved (see [11] p. 206)

\[ < P_n(\ldots, k, \mathcal{V}), f > = \frac{\Gamma(k - 1)}{(4\pi n)^{k-1}} a_f(n). \]  

(2.1)

Now if \( \{f_1, \ldots, f_r\} \) is an orthonormal basis for \( S_k(\mathcal{V}) \), and

\[ P_n(\ldots, k, \mathcal{V}) = \sum_i c_i f_i \]

we have

\[ c_i = < P_n(\ldots, k, \mathcal{V}), f_i >. \]

Therefore from (2.1)

\[ \frac{(4\pi n)^{k-1}}{\Gamma(k - 1)} P_n(\ldots, k, \mathcal{V}) = \sum_i a_{f_i}(n)f_i. \]

Now if \( \hat{P}_n(m, k, \mathcal{V}) \) is the \( m \)-th coefficient of the Fourier expansion of \( P_n(z, k, \mathcal{V}) \), by comparing the \( m \)-th coefficients on both sides we have

\[ \frac{(4\pi n)^{k-1}}{\Gamma(k - 1)} \hat{P}_n(m, k, \mathcal{V}) = \sum_i a_{f_i}(n)a_{f_i}(m). \]  

(2.2)

But by a formula of Petersson we have the following explicit representation (see [11] p. 206)

\[ \hat{P}_n(m, k, \mathcal{V}) = \left(\frac{m}{n}\right) \frac{J_{k-1}^{\frac{k-1}{2}}}{(4\pi \sqrt{mn})^k} \sum_{c \equiv 0 \pmod{\mathcal{V}}} c^{-1} \delta_{mn} + 2\pi i^{-k} S(m, n; c) \]

(2.3)

where \( \delta_{mn} \) is the Kronecker delta. \( J_{k-1}(x) \) is the Bessel function of order \( k - 1 \) which is defined by the following integral

\[ J_{k-1}(x) = \frac{1}{2\pi i} \int_{\gamma} \exp(i(z - z^{-1})) \frac{dz}{z^k} \]

13
and $S(m, n; c)$ is the Kloosterman sum

$$S(m, n; c) = \sum_{\substack{a \mod c \\ (a,c)=1}} e\left(\frac{ma + n\tilde{a}}{c}\right)$$

where $a\tilde{a} \equiv 1(\mod c)$.

From (2.2) and (2.3), one can get the following "semi-orthogonality" of the Fourier coefficients of an orthonormal basis of $S_k(N)$

$$\sum_i \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0(\mod N)} c^{-1} J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right) S(m, n; c) \right\}.$$  

(2.4)

Now for $0 \neq f \in S_k(N)$ set

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} < f, f >.$$

Then we have the following estimate.

**Proposition 1** If $\{f_1, \ldots, f_r\}$ is an orthogonal basis for $S_k(N)$, for $m$ and $n$ positive integers we have the inequality

$$|\sum_i \omega_{f_i} \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} - \delta_{mn}| \leq Md(N)N^{\frac{1}{2}-k}(m, n)^{\frac{1}{2}} \sqrt{(mn)^{k-1}}$$

where $M$ is a constant depending only on $k$ and $d(N)$ is the number of divisor of $N$.

**Proof:** The following expression for the Bessel function of order $k-1$ is known (see [23] p. 60)

$$J_{k-1}(z) = \frac{z^{k-1}}{2^{k-1} \Gamma(k - \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} \cos(z \cos \theta) \sin^{2k-2} \theta \, d\theta.$$

From this we get the following bound for $z \geq 0$;

$$|J_{k-1}(z)| \leq \frac{\sqrt{\pi} z^{k-1}}{2^{k-1} \Gamma(k - \frac{1}{2})}.$$  

(2.5)
Also we have Weil’s bound for the Kloosterman sum (see [7]).

\[ |S(m, n; c)| \leq (m \cdot n \cdot c)^{\frac{1}{2}}d(c)c^\frac{1}{2}. \tag{2.6} \]

Now the Proposition follows by applying (2.5) and (2.6) in (2.4). \(\square\)

Note Although in the case \(k = 2\), one does not have the absolute convergence of the Poincaré series, nevertheless, Proposition 1 is valid in this case as well.

### 2.2 A basis for \(S_k(N)\)

We are going to generalize Duke’s result to cusp forms of weight \(k\) and prime level \(N\) (see [5]. Theorem 1. and also 1.6 of the Introduction).

The first difficulty that we encounter is that \(\mathcal{F}_N\) is not a basis for \(S_k(N)\) when \(k\) is large (more precisely if \(k > 12\) and \(k \neq 14\)). So we must find a basis for \(S_k(N)\) with good analytic properties. A theorem of Pizer guarantees the existence of such basis for \(S_k(N)\).

In 1983 A. Pizer introduced the operators \(C_q\) on \(S_k(N)\) for \(q|N\), such that the action of \(C_q\) on the new part of \(S_k(N)\) is the same as the action of the classical \(U_q\) operators. More precisely he defined \(C_q\) as

\[
C_q = U_q + W_q U_q W_q + q^{\frac{k}{2} - 1} W_q \quad \text{if} \quad q \| N
\]

\[
C_q = U_q + W_q U_q W_q \quad \text{if} \quad q^2 \mid N.
\]

Then he showed that \(T_p (p \nmid N)\), \(C_q (q \mid N)\) form a commuting family of Hermitian operators. Using this, he proved ([17] Theorem 3.10) the following result.

**Theorem** There exists a basis \(f_i(z) (1 \leq i \leq \dim S_k(N))\) of \(S_k(N)\) such that each \(f_i(z)\) is an eigenform for all the \(T_p\) and \(C_q\) operators with \(p \nmid N\) and \(q|N\). Let \(f(z) = \sum_{n=1}^{\infty} a_f(n)e(z)\) be an element of this basis. Then \(a_f(1) \neq 0\) and assuming \(f(z)\) is normalized so that \(a_f(1) = 1\), we have \(f|T_p = a_f(p)f\) for all \(p \nmid N\), \(f|C_q = a_f(q)f\) for all \(q|N\), and \(a_f(nm) = a_f(n)a_f(m)\) whenever \((n, m) = 1\). Furthermore \(f(z)\) is an
eigenform for all $W_q$ operators, $q|N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the $T_p$ and $C_q$ operators with $p \nmid N$ and $q|N$, then $g(z) = cf_i(z)$ for some $c \in \mathbb{C}^*$ and some unique $i, 1 \leq i \leq \dim S_k(N)$.

Now let $P_N$ be the basis of $S_k(N)$ given by the above theorem. The elements of $P_N$ form an orthogonal basis for $S_k(N)$ and their $L$-functions have analytic continuation and satisfy certain functional equations. We can show that the action of $C_q$ on $S_k(N)^{\text{new}}$ is the same as the action of $U_q$ (see [17] Remark 2.9). This shows that $\mathcal{F}_N \subset P_N$.

In the sequel we need an estimation for the Fourier coefficient of an oldform in $P_N$. Suppose that $N$ is prime and $f \in P_N - \mathcal{F}_N$, then we can show the existence of $A \in \mathbb{C}$ such that

$$f(z) = h(z) + Ah(Nz)$$

where $h$ is the normalized newform of weight $k$ and level 1 associated to $f$ (see [17]. Proposition 3.6). From this we can get the following lemma.

**Lemma 1** With the above notations, $A = \pm N^{\frac{k}{2}}$.

**Proof:** Since $f \in P_N - \mathcal{F}_N$, we have

$$f(z) = h(z) + Ah(Nz).$$

Therefore, $f$ is in the space generated by $h$ and $h(Nz)$. From [17], Proposition 3.4, we know that this space is invariant under $C_N$. We can show that

$$h|C_N = c_h(N)h + N^kh(Nz)$$

$$h(Nz)|C_N = h + c_h(N)h(Nz)$$

where $c_h(N)$ is the $N$-th Fourier coefficient of $h$. We know that $a_f(N)$ is the eigenvalue of $C_N$ operator. The above identities show that the $C_N$ operator on the space generated by $h$ and $h(Nz)$ can be represented by the following matrix.
Therefore its characteristic polynomial is

\[
x^2 - 2c_h(N)x + (c_h(N)^2 - N^k) = 0.
\]

This shows that \( a_f(N) = c_h(N) \pm N^{\frac{k}{2}} \), and so \( A = a_f(N) - c_h(N) = \pm N^{\frac{k}{2}} \). □

Now by using Lemma 1, we give an estimation for the Fourier coefficient \( a_f(n) \).

**Lemma 2** Suppose \( N \) is a prime and \( f \in \mathcal{P}_N \). Then

\[
|a_f(n)| \leq c_0 n^{\frac{k}{2}}
\]

where \( c_0 \) is an absolute constant independent of \( f \).

**Proof:** If \( f \in \mathcal{F}_N \) we know that \( |a_f(n)| \leq d(n)n^{\frac{k+1}{2}} \) (Deligne’s bound) and therefore the result is clear.

If \( f \in \mathcal{P}_N - \mathcal{F}_N \) then from [17] Proposition 3.6 follows that there exists an \( A \in \mathbb{C} \) such that

\[
f(z) = h(z) + Ah(Nz)
\]

where \( h \) is the normalized newform of weight \( k \) and level \( 1 \) associated to \( f \). Since \( A = a_f(N) - c_h(N) \), where \( c_h(N) \) is the \( N \)-th Fourier coefficient of \( h \). Lemma 1 follows that \( |A| = N^{\frac{k}{2}} \).

Now if \( (n,N) = 1 \) then \( a_f(n) = c_h(n) \) and therefore the Deligne bound implies the result. and if \( (n,N) \neq 1 \) then \( n = mN \) and we can write

\[
a_f(Nm) = c_h(Nm) + Ah(m).
\]

By using the Deligne bound for the Fourier coefficients of \( h \) we get

\[
|a_f(Nm)| \leq d(Nm)(Nm)^{\frac{k-1}{2}} + N^k d(m)m^{\frac{k-1}{2}}
\]
\[ \frac{d(Nm)}{(Nm)^{1/2}} + \frac{d(m)}{m^{1/2}} \] 

The result follows from the fact that \( d(n) = O(n^{1/2}) \) with an absolute constant. \( \square \)

### 2.3 Critical values on average

Now we give a representation of \( L_f(k, \chi) \) as a sum of two convergent series for \( f \in \mathcal{P}_N \).

**Lemma 3** For any \( x > 0 \), let

\[ A(x) = \sum_{n \geq 1} \chi(n) a_f(n)n^{-\frac{k}{2}} \left\{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} \left( \frac{2\pi n}{x} \right)^j \right\} e^{-\frac{2\pi n}{x}}. \]

Where \( \chi \) is a fixed primitive Dirichlet character mod \( q \) with \( (q, N) = 1 \). Then we have

\[ L_f(k, \chi) = A(x) + \epsilon \bar{A}(Nq^2/x) \]

where \( \epsilon \chi \) is the root number of \( L_f(s, \chi) \) and \( \bar{A} \) is the conjugate of \( A \).

**Proof:** Define the function \( \mathcal{E}(x) \) by

\[ \mathcal{E}(x) = \frac{1}{2\pi i} \int_{(\frac{k}{2})} \frac{1}{x^s} \left( -\frac{1}{x} \right)^{\frac{k}{2}} \Gamma(s + \frac{k}{2}) \frac{ds}{s} \]

then

\[ \frac{1}{\Gamma(k/2)} \mathcal{E}(-\frac{1}{x}) = \left( \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} \left( -\frac{1}{x} \right)^j \right) e^{-\frac{1}{x}}. \quad (2.7) \]

This is true because

\[ \frac{1}{\Gamma(k/2)} \mathcal{E}(-\frac{1}{x}) = \frac{1}{\Gamma(k/2)} \int_0^\infty e^{-t/2} \left( \frac{1}{2\pi i} \int_{(\frac{k}{2})} (xt)^s \frac{ds}{s} \right) \frac{dt}{t}. \]

But we know that

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases} \]

18
Therefore
\[ \frac{1}{\Gamma\left(\frac{k}{2}\right)} E\left(-\frac{1}{x}\right) = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_{\frac{1}{x}}^{\infty} e^{-t} t^{\frac{k}{2}-1} dt = \left(\sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} \left(\frac{1}{x}\right)^j\right) e^{-\frac{1}{x}}. \]

Now by definition of $E(x)$, it is clear that
\[ A(x) = \frac{1}{2\pi i} \int_{\left(\frac{k}{2}\right)} L_f(s + \frac{k}{2}, \chi)(\frac{x}{2\pi}) s \Gamma(s + \frac{k}{2}) \Gamma\left(\frac{k}{2}\right)^{-1} ds \]

moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, and using the functional equation (1.2) for $L_f(s)$ yields
\[ A(x) = L_f\left(\frac{k}{2}, \chi\right) + e^{\chi} \int_{\left(-\frac{3}{4}\right)}^{\left(-\frac{1}{4}\right)} (2\pi x) s \Gamma(-s + \frac{k}{2}) \Gamma\left(\frac{k}{2}\right) L_f(-s + \frac{k}{2}, \chi) s^{-1} ds \]

Now changing variables $s \mapsto -s$ gives the result. \(\square\)

**Lemma 4** We have $\sum_{n=1}^{\infty} \frac{n^k}{(N^{2\pi})^n} = O(N^{-2\pi})$ and $\sum_{n=1}^{\infty} n^k e^{-\frac{N}{x}} = O(t^k)$. where $b > 1$.

**Proof:** We know that the geometric series $\sum_{n=1}^{\infty} x^n$ is uniformly convergent to $\frac{1}{1-x}$ on any closed sub-interval of $(-1,1)$. Now by using induction and term by term differentiation of the geometric series, we can show that
\[ \sum_{n=1}^{\infty} n^k x^n = \frac{(-1)^{k-1} x P(x)}{(x-1)^{k+1}} \]

where $P(x) = x^{k-1} + a_{k-2} x^{k-2} + \ldots + 1$ is a polynomial of degree $k - 1$.

Now the result easily follows by substituting $x = N^{-2\pi}$ and $x = e^{-\frac{1}{k}}$ in $(*)$. \(\square\)

From Proposition 1 and Lemma 3, we can get the following asymptotic formula.

**Proposition 2** Let $\chi$ be a fixed primitive character modulo $q$. Then we have
\[ \sum_{f \in \mathcal{P}_N} \omega_f L_f\left(\frac{k}{2}, \chi\right) = 1 + O\left(N^{-\frac{1}{2}}(\log N)^{k-1}\right) \]

for $N$ prime. The implied constant depends on $q$ and $k$. 

19
Proof: Choosing \( x = q^2 N \log N \) in Lemma 3 gives

\[
\mathcal{A}(\frac{Nq^2}{x}) = \sum_{n \geq 1} \frac{\chi(n) a_f(n)}{2^{2n \log N}} \left\{ \frac{1}{j! (2\pi n \log N)^j} \right\} (N - 2\pi)^n.
\]

Using Lemma 2 we get

\[
|\mathcal{A}(\frac{Nq^2}{x})| \leq \sum_{n \geq 1} |a_f(n)| n^{-\frac{3}{2}} \frac{2\pi n}{2} (2\pi n \log N)^{\frac{3}{2} - 1} (N - 2\pi)^n.
\]

\[
\leq c_0 \frac{k}{2} (2\pi)^{\frac{3}{2} - 1} (\log N)^{\frac{3}{2} - 1} \sum_{n \geq 1} \frac{n^{\frac{3}{2} - 1}}{(N^2 \pi)^n}.
\]

\[
\leq c_0 \frac{k}{2} (2\pi)^{\frac{3}{2} - 1} (\log N)^{\frac{3}{2} - 1} O(N^{-2\pi}).
\]

Therefore from Lemma 3 we get

\[
L_f(k, \chi) = \sum_{n \geq 1} \frac{\chi(n) a_f(n)}{2^{2n \log N}} \left\{ \frac{1}{j! (2\pi n \log N)^j} \right\} e^{-\frac{2\pi n}{N \log N} n^{-\frac{3}{2}}} + O(N^{-2\pi}).
\]

From this, we get

\[
\sum_{f \in \mathcal{P}_N} \omega_f L_f(k, \chi) = \sum_{n \geq 1} \chi(n) \left( \sum_{f \in \mathcal{P}_N} \omega_f \frac{a_f(n)}{\sqrt{n} 2^{2n \log N}} \delta_{1n} \left\{ \frac{1}{j! (2\pi n \log N)^j} \right\} \right) \frac{1}{\sqrt{n}} e^{-\frac{2\pi n}{N \log N} n^{-\frac{3}{2}}} + \sum_{f \in \mathcal{P}_N} \omega_f O(N^{-2\pi}).
\]

Note that

\[
\left( \sum_{j=0}^{\frac{3}{2} - 1} \frac{1}{j! (2\pi n \log N)^j} e^{-\frac{2\pi}{N \log N} n^{-\frac{3}{2}}} - 1 \right) = \left( \sum_{j=\frac{3}{2}}^{\infty} \frac{1}{j! (2\pi n \log N)^j} e^{-\frac{2\pi}{N \log N} n^{-\frac{3}{2}}} \right)
\]

From Proposition 1. with \( m = n = 1 \) we get

\[
\sum_{f \in \mathcal{P}_N} \omega_f = 1 + O(N^{1-k}).
\]
Also, by applying $m = 1$ in Proposition 1 and using the above identities and Lemma 4, we have

\[
| \sum_{f \in \mathcal{P}_N} \omega_f L_f \left( \frac{k}{2}, \chi \right) - 1 | \leq M_1 N^{\frac{k}{2} - \frac{1}{2}} \sum_{n \geq 1} n^{k-2} e^{-\frac{q}{2N \log N}} + \left( \sum_{j=\frac{1}{2}}^{e^{2N \log N}} \frac{1}{q^{j} \log N^{j}} \right) e^{-\frac{q}{2N \log N}}
\]

\[
+ M_2 N^{-\frac{6}{2} - \frac{1}{2}} \leq M_3 N^{-\frac{1}{2}} \log N^{k-1} + M_4 (N \log N)^{-\frac{1}{2}} + M_5 N^{-\frac{8}{2} - \frac{1}{2}} - 1
\]

where $M_1, M_2, M_3, M_4$ are constants. This completes the proof. \( \square \)

Now let $P_f(s) = L_f(s, \chi_1)L_f(s, \chi_2)$ where $\chi_1$ and $\chi_2$ are fixed primitive Dirichlet characters mod $q_1$ and $q_2$. Then we have $P_f(s) = \sum_{l \geq 1} b_f(l) l^{-s}$, where

\[
b_f(l) = \sum_{m=1}^{l} \chi_1(m) \chi_2(n) a_f(m) a_f(n).
\]

Define for $x > 0$

\[
g(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} (2\pi)^{-2s} \Gamma(s + \frac{k}{2})^2 \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma}\left( \frac{k}{2} \right) x^{-s} ds
\]

and set $B(x) = \sum_{l \geq 1} b_f(l) l^{-\frac{k}{2}} g(l^2)$. Then we have

**Lemma 5** Let $f \in \mathcal{P}_N$ for $N \geq 1$ and suppose that $\chi_1$ and $\chi_2$ are primitive Dirichlet characters mod $q_1, q_2$ with $(q_1 q_2, N) = 1$. For any $x > 0$, we have

\[
P_f \left( \frac{k}{2} \right) = B(x) + \delta_{\chi_1 \chi_2} B \left( \frac{(q_1 q_2)^2}{x} \right)
\]

where $\delta_{\chi_1 \chi_2} = \chi_1 \chi_2(N)(\tau(\chi_1) \tau(\chi_2))^2(q_1 q_2)^{-1}$ is the root number of $P_f(s)$ and $\bar{B}$ is the conjugate of $B$.

**Proof:** By the definition of $g(x)$, it is clear that

\[
B(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} x^s(2\pi)^{-2s} \Gamma(s + \frac{k}{2})^2 \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma\left( \frac{k}{2} \right)} P_f(s + \frac{k}{2}) s^{-1} ds.
\]

21
Now by moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, and using the functional equation for $P_f(s)$ which is a direct consequence of (1.2), we get

$$B(x) = P_f\left(\frac{k}{2}\right) + \frac{\xi_1 x_2}{2\pi i} \int_{\frac{3}{4}} \left(\frac{(Nq_1q_2)^2}{x}\right)^{-s} (2\pi)^{2s} \frac{\Gamma(-s + \frac{k}{2})}{\Gamma(\frac{k}{2})^2} P_f(-s + \frac{k}{2}) s^{-1} ds.$$ 

Now changing variables $s \mapsto -s$ yields the result. □

We come now to the following important proposition.

**Proposition 3** Let $\chi$ be a primitive Dirichlet character. Then

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f\left(\frac{k}{2}, \chi\right)| = \sum_{f \in \mathcal{P}_N} \omega_f P_f\left(\frac{k}{2}\right) = \prod_{p|q} (1 - p^{-1}) \log N + c + O(N^{-\frac{1}{2}} \log N)$$

for $N$ prime with $(q, N) = 1$, where $c$ and the implied constant depend on $q$ and $k$.

**Proof:** In Lemma 5, set $\chi_1 = \chi$, $\chi_2 = \chi$, we have $B = B$ and $\xi_1 = 1$. By Lemma 5 with $x = Nq^2$, we have

$$\sum_{f \in \mathcal{P}_N} \omega_f P_f\left(\frac{k}{2}\right) = 2 \sum_{f} \omega_f \sum_{l \geq 1} b_f(l) l^{-\frac{k}{2}} g\left(\frac{l}{Nq^2}\right)$$

$$= 2 \sum_{m,n \geq 1} \chi(m) \overline{\chi}(n) g\left(\frac{mn}{Nq^2}\right) \frac{1}{(mn)^{\frac{k}{2}}} \sum_{f} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}} \frac{a_f(n)}{\sqrt{n^{k-1}}}.$$  

By Proposition 1, it is clear that

$$\sum_{f \in \mathcal{P}_N} \omega_f P_f\left(\frac{k}{2}\right) = 2 \sum_{n \geq 1} |\chi(n)|^2 g\left(\frac{n}{Nq^2}\right)n^{-1} + R$$  

where

$$R \ll N^{\frac{1}{2} - k} \sum_{m,n \geq 1} g\left(\frac{mn}{Nq^2}\right)(m,n)^{\frac{k}{2}}(mn)^{\frac{k}{2} - 1}. $$

Now the first term on the right hand side of (2.10) is evaluated using the definition of $g$ as

$$\frac{1}{\pi i} \int_{\frac{3}{2}} L(2s + 1, \chi_0)(2\pi)^{-2s} \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} (Nq^2)^s ds$$

22
where \( \chi_0 \) is the principal character mod \( q \) and \( L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - \frac{1}{p^s}) \). Since the integrand has a double pole at \( s = 0 \), by moving the line of integration from \( \frac{3}{4} \) to \( -\frac{1}{2} \), we see that the above integral is equal to

\[
\prod_{p \mid q} (1 - p^{-1}) \log N + c + O(N^{-\frac{1}{2}}).
\]

(2.12)

Now in (2.11) we calculate \( \sum_{m,n \geq 1} g\left(\frac{mn}{Nq^2}\right)(m,n)^{\frac{1}{2}}(mn)^{\frac{s}{2}-1} \). It is

\[
\frac{1}{2\pi i} \int_{(\frac{3}{2}+\frac{1}{2})} (2\pi)^{-2s} \Gamma(s + \frac{k}{2})^2 \left( \sum_{m,n \geq 1} (m,n)^{\frac{1}{2}}(mn)^{-\frac{s}{2}} \right) \left( Nq^2 \right)^s \frac{ds}{s}
\]

because the integrand does not have any poles in the strip \( \frac{3}{4} < \text{Re}(s) < \frac{k+1}{2} \) and \( \sum_{m,n \geq 1} (m,n)^{\frac{1}{2}}(mn)^{-\left(s-\frac{s}{2}+1\right)} \) is absolutely convergent. Next we use the following identity

\[
\sum_{m,n \geq 1} (m,n)^{\frac{1}{2}}(mn)^{-\left(s-\frac{s}{2}+1\right)} = \frac{\zeta(2s - k + \frac{3}{2})\zeta(s - \frac{k}{2} + 1)^2}{\zeta(2s - k + 2)}
\]

(2.13)

(See [5] Lemma 4). By moving the line of integration from \( \frac{k+1}{2} \) to \( \frac{k}{2} - \epsilon (\epsilon > 0) \) we get

\[
\sum_{m,n \geq 1} g\left(\frac{mn}{Nq^2}\right)(m,n)^{\frac{1}{2}}(mn)^{\frac{s}{2}-1} \sim c_1 N^{\frac{s}{2}} \log N
\]

(2.14)

and by (2.11), \( R \ll N^{\frac{1}{4} - \frac{3}{4}} \log N \). This and (2.12) prove the Proposition. \( \square \)

### 2.4 A lower bound for the Petersson inner product

To obtain a lower bound for \( <f, f>_N \) when \( f \in \mathcal{P}_N \), we need to introduce the trace function which maps \( S_k(N) \) to \( S_k(1) \). More precisely suppose \( N \) is a prime. we can show that the elements

\[
\{ \gamma_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}, 0 \leq j < N \}
\]
are right coset representatives for $\Gamma = \Gamma_0(N) \backslash \Gamma_0(1)$. Then for $f \in S_k(N)$ we define

$$Tr(f) = \sum_{j=-1}^{N} f|_{\gamma_j}.$$ 

It is clear that $Tr(f) \in S_k(1)$. Let $W_N$ be the usual Atkin-Lehner involution. Since $W_N^{-1} \Gamma_0(N)W_N = \Gamma_0(N)$, it is clear that $f|_{W_N} \in S_k(N)$. We have the following lemma regarding the calculation of $Tr(f|_{W_N})$.

**Lemma 6** If $h$ is a normalized newform of weight $k$ and level 1. then

$$Tr(h|_{W_N}) = N^{1-\frac{k}{2}}c_h(N)h$$

where $c_h(N)$ is the $N$-th Fourier coefficient of $h$.

**Proof**: From the definition of trace we have

$$Tr(h|_{W_N}) = \sum_{j=-1}^{N} (h|_{W_N})|_{\gamma_j} = h|_{W_N} + \sum_{j=0}^{N-1} \left( h \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}$$

But $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 1 & 0 \end{pmatrix}$, since $h \in S_k(1)$

$$Tr(h|_{W_N}) = N^{\frac{k}{2}}h(Nz) + N^{\frac{k}{2}} \sum_{j=0}^{N-1} h\left(\frac{z+j}{N}\right) = N^{1-\frac{k}{2}}T_N(h)$$

where $T_N$ is the $N$-th Hecke operator, since $h$ is a normalized newform the result is clear. $\square$

Now we use the above lemma to evaluate $< h, h(Nz) >_N$.

**Lemma 7** If $h$ is a normalized newform of level 1. then

$$< h, h(Nz) >_N = N^{1-k}c_h(N)< h, h >_1.$$
Proof: Since \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \), and \( h \in S_k(1) \), and the operator \( W_N \) is Hermitian, we have

\[ < h, h(Nz) >_N = N^{-\frac{k}{2}} < h|W_N, h(z) >_N. \]

Now let \( F \) be a fundamental domain of \( \Gamma_0(1) \backslash \mathcal{H} \). Then since \( \Gamma_0(1) = \bigcup_{t=-1}^{N-1} \Gamma_0(N) \gamma_t \),

\[ F' = \bigcup_{t=-1}^{N-1} \gamma_t F \]

is a fundamental domain of \( \Gamma_0(N) \backslash \mathcal{H} \). This is because if \( z \in \mathcal{H} \) there exist \( z' \in F \) and \( \gamma \in \Gamma_0(1) \) such that \( z = \gamma z' \). We can write \( \gamma = \gamma' \gamma_t \) where \( \gamma' \in \Gamma_0(N) \), this shows that \( z = \gamma'(\gamma_t z') \). Therefore any \( z \in \mathcal{H} \) is equivalent to an element in \( \gamma_t F \) for some \( i \).

Now suppose that for \( z_1, z_2 \in \text{int} F' \) there exist \( \gamma' \in \Gamma_0(N) \) such that \( \gamma' z_1 = z_2 \), where \( z_1 \in \gamma_t F \) and \( z_2 \in \gamma_j F \). This shows that \( \gamma_j^{-1} \gamma'_t z_1 = z_2' \) where \( z_1', z_2' \in F \) and \( \gamma_j^{-1} \gamma'_t \in \Gamma_0(1) \) which is impossible. This shows that \( F' = \bigcup \gamma_t F \) is a fundamental domain for \( \Gamma_0(N) \backslash \mathcal{H} \).

So we have

\[ < h, h(Nz) >_N = N^{-\frac{k}{2}} \sum_{t=-1}^{N-1} \int_{\gamma_t F} (h|W_N)(z) \overline{h(z)}|y|^k \frac{dx dy}{y^2} \cdot \]

Using the change of variable \( z = \gamma_j w \), where \( w = u + iv \) we find that this is

\[ = N^{-\frac{k}{2}} \sum_{t=-1}^{N-1} \int_F ((h|W_N)|\gamma_j)(w) \overline{(h|\gamma_j)(w)}|y|^k \frac{du dv}{v^2} \]

But \( h \in S_k(1) \) and so \( h|\gamma_j = h \). Hence, by using Lemma 6 the above expression is

\[ N^{-\frac{k}{2}} \text{Tr}(h|W_N), h >_1 = N^{1-k} c_h(N) < h, h >_1. \]

Now we use the above Lemma to get a lower bound for \( < f, f >_N \).
Lemma 8 If \( f \in \mathcal{P}_N - \mathcal{F}_N \) and \( N \) is a prime then

\[
< f, f >_N \geq (N - 4N^{\frac{1}{2}} + 1) < h, h >_1.
\]

Proof: By applying Lemma 7 we have

\[
< f, f >_N = < h + Ah(Nz), h + Ah(Nz) >_N \geq \left( N + 1 + 2AN^{1-k}c_h(N) \right) < h, h >_1.
\]

By Lemma 1, \( |\mathcal{A}| = N^{\frac{k}{2}} \) and therefore

\[
|2AN^{1-k}c_h(N)| \leq 4N^{\frac{1}{2}} \tag{2.15}
\]

Applying (2.16) to (2.15) gives the desired result. \( \square \)

Now we are in a situation that we can establish an upper bound for

\[
\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} < f, f >_N.
\]

Proposition 4 If \( f \in \mathcal{P}_N - \mathcal{F}_N \) for \( N \) prime large enough

\[
\omega_f \ll_k \frac{1}{N}
\]

with implied constant depending on \( k \).

Proof: This is clear from Lemma 8. \( \square \)

Proposition 5 If \( f \in \mathcal{F}_N \) for \( N \) prime large enough

\[
\omega_f \ll_k \frac{\log N}{N}
\]

with implied constant depending on \( k \).

Proof: Set

\[
q(s) = \zeta_N(2s + 2 - 2k) \sum_{n=1}^{\infty} \frac{(af(n))^2}{n^s}.
\]
The Rankin-Selberg method shows that \( q(s) \) has a pole at \( s = k \) of residue

\[
\frac{\pi(4\pi)^k \phi(N)}{2\Gamma(k)N^2} < f, f >_N
\]

(see [21] p. 90). So from the definition of \( L_{\text{sym}^2(f)}(s) \) (see the Introduction) it is clear that for \( N \) prime

\[
L_{\text{sym}^2(f)}(k) = \frac{\pi(4\pi)^k}{2\Gamma(k)N^2} < f, f >_N.
\] (2.17)

But the extension of the Main theorem of [9] to holomorphic cusp forms, together with the fact that for prime \( N \) no \( f \in \mathcal{F}_N \) is a lift from \( GL(1) \), implies that

\[
L_{\text{sym}^2(f)}(k) \gg_k \frac{1}{\log N}
\] (2.18)

(see [9] p. 178, remark and paragraph following the Main Theorem). Now the result follows from (2.17) and (2.18). \( \Box \)

We are in the situation that we can prove the main theorem of this chapter.

**Theorem 1** Suppose that \( \chi \) is a fixed primitive Dirichlet character mod \( q \) such that \( (q, N) = 1 \). Then there are positive constants \( C_k \) (depending only on \( k \)) and \( C_{q,k} \) (depending only on \( q \) and \( k \)) such that for prime \( N > C_{q,k} \) there exist at least \( C_k N(\log N)^{-2} \) newforms \( f \) of weight \( k \) and level \( N \) for which \( L_f \left( \frac{k}{2}, \chi \right) \neq 0 \).

**Proof:** By the Cauchy-Schwarz inequality and Proposition 4, we have

\[
\left| \sum_{f \in \mathcal{P}_N} \omega_f L_f \left( \frac{k}{2}, \chi \right) \right|^2 \leq \left( \sum_{f \in \mathcal{F}_N : L_f \left( \frac{k}{2}, \chi \right) \neq 0} \omega_f \right)^2 \sum_{f \in \mathcal{P}_N} \omega_f \left| L_f \left( \frac{k}{2}, \chi \right) \right|^2
\]

\[
\ll \left( \# \{ f \in \mathcal{F}_N; L_f \left( \frac{k}{2}, \chi \right) \neq 0 \} \frac{\log N}{N} + 2 \dim \mathcal{S}_k \left( 1 \right) \frac{1}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f \left| L_f \left( \frac{k}{2}, \chi \right) \right|^2
\]

Now theorem follows from Propositions 2, 3 and 5. \( \Box \)
2.5 Non-vanishing of product of twisted modular $L$-functions

We may try to use the above trick to find a lower bound for the number of newforms $f$ for which $P_f(s) = L_f(s, \chi_1)L_f(s, \chi_2)$ is non-zero at the center of the critical strip. Here we assume that $\chi_1$ and $\chi_2$ are real and distinct such that $\chi_1 \chi_2(-N) = 1$. To do this we need to derive asymptotic formulae for $\sum_{f \in \mathcal{P}_N} \omega_f P_f(\frac{k}{2})$ and $\sum_{f \in \mathcal{P}_N} \omega_f |P_f(\frac{k}{2})|^2$.

**Proposition 6** Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ be distinct real primitive Dirichlet characters such that $\chi_1 \chi_2(-N) = 1$. Then for $N$ prime we have

$$\sum_{f \in \mathcal{P}_N} \omega_f P_f(\frac{k}{2}) = 2L(1, \chi_1 \chi_2) + O(N^{-\frac{1}{2}} \log N)$$

where the implied constant depend on $q_1$, $q_2$ and $k$.

**Proof:** In Lemma 5 we have $\hat{\chi_1} \chi_2$. This is because $(\tau(\chi_1))^2 = \chi_1(-1)q_1$ and $(\tau(\chi_2))^2 = \chi_2(-1)q_1$ (see [20] p.91). and therefore $\hat{\chi_1} \chi_2$ = $\chi_1 \chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2(q_1q_2)^{-1}$ = $\chi_1 \chi_2(-N) = 1$. So we may repeat the proof of Proposition 3 line by line. The result follows with the observation that

$$\frac{1}{\pi i} \int_{(\frac{1}{2})} L(2s + 1, \chi_1 \chi_2)(2\pi)^{-2s} \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} (Nq_1q_2)^s ds$$

is equal to

$$2L(1, \chi_1 \chi_2) + O(N^{-\frac{1}{2}}). \Box$$

We recall from (2.8) the definition of $g(x)$ as

$$g(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} (2\pi)^{-2s} \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} x^{-s} ds$$
Let for $x > 0$ and a non-negative integer $v$

$$K_v(x) = \frac{1}{2} \int_0^\infty e^{-\frac{t}{2}(u+\frac{1}{2})} u^{v+1} du.$$ 

be the $K_v$-Bessel function.

In the next lemma we give a representation of $g(x)$ as a sum of the $K$-Bessel functions.

**Lemma 9**

$$g(x) = \frac{2}{\Gamma(\frac{1}{2})} \sum_{j=0}^{\frac{1}{2}-1} \frac{1}{j!} (2\pi \sqrt{x})^{\frac{3}{2}+j} K_{\frac{3}{2}-j}(4\pi \sqrt{x})$$

**Proof:** From definition of $g(x)$ and $\Gamma$ function we have

$$I = \Gamma\left(\frac{k}{2}\right) g(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( \int_0^\infty \int_0^\infty t_1^{3/2-1} t_2^{3/2-v} e^{-(t_1+t_2)} dt_1 dt_2 \right) (4\pi^2 x)^{-s} ds.$$ 

By interchanging the order of integration we get

$$I = \int_0^\infty t_1^{3/2-1} e^{-t_1} \left( \int_0^\infty t_2^{3/2-v} e^{-t_2} dt_2 \right) dt_1.$$ 

Now by integration by parts we have

$$I = \Gamma\left(\frac{k}{2}\right) \sum_{j=0}^{\frac{1}{2}-1} \frac{1}{j!} (4\pi^2 x)^{j} \int_0^\infty t^{3/2-1-j} e^{-(t+4\pi^2 x)} dt.$$ 

But we know that

$$\int_0^\infty t^{3/2-1-j} e^{-(t+4\pi^2 x)} dt = 2(4\pi^2 x)^{j-\frac{1}{2}} K_{\frac{3}{2}-j}(4\pi \sqrt{x})$$


Substituting (2.20) in (2.19) yields the result. □

**Lemma 10**

$$g(x) \ll \begin{cases} 1 & \text{for } x \leq 1 \\ x^{\frac{3}{4}-\frac{3}{4}} e^{-4x^{\frac{1}{4}}} & \text{for } x > 1 \end{cases}.$$ 

**Proof:** By moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, we have

$$g(x) = 1 + O(x^{\frac{3}{4}})$$
which proves the Lemma if \( x \leq 1 \).

If \( x > 1 \), we know

\[
K_\nu(x) = (\frac{\pi}{2x})^{\frac{1}{2}} e^{-x} [1 + O(\frac{1}{x})]
\]

(see [24] p. 202). Now applying this identity to Lemma 9, yields the result. \( \square \)

**Lemma 11** Let \( f \in \mathcal{P}_N \) then

\[
a_f(m)a_f(n) = \sum_{d|\gcd(m,n)} d^{k-1}a_f(\frac{mn}{d^2})
\]

if \( (m, N) = 1 \).

**Proof:** We consider the collection of operators \( \{T_n, (n, N) = 1, n \in \mathbb{N}\} \) such that \( \{T_p, (p, N) = 1, p \text{ prime}\} \) is the collection of the classical Hecke operators as defined in section (1.3). Also we assume that \( T_n, (n, N) = 1 \) satisfies the following identities

(i) \( T_mT_n = T_{mn} \) if \( (m, n) = 1 \).

(ii) \( T_pT_p^n = T_{p^{n+1}} + p^{k-1}T_{p^{n-1}} \) if \( (p, N) = 1 \).

From here it is clear that if \( (m, N) = 1, T_m \) is the classical Hecke operator. Now if \( f \in \mathcal{P}_N, f \) is an eigenform for \( T_m \) and \( T_m(f) = a_f(m)f \). But we know that if 

\[
f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz) \text{ then } \\
a_f(m)f(z) = T_m(f)(z) = \sum_{n=1}^{\infty} \left( \sum_{d|\gcd(m,n), (d, N) = 1} d^{k-1}a_f\left(\frac{mn}{d^2}\right)\right)e(nz)
\]

(see [12] p. 163). Equating the \( n \)-th Fourier coefficient of the two sides of (2.21) and noting that \( (m, N) = 1 \) yields the result. \( \square \)

**Lemma 12** Under the assumption of Proposition 6, for \( f \in \mathcal{P}_N \) and \( X = Nq_1q_2(\log N)^2 \), we have

\[
P_f\left(\frac{k}{2}\right) = \sum_{l \leq X} c_la_f(l) + O(N^{-11})
\]

where \( c_l \ll \frac{d(j)}{l^2} \log N \) and the implied constants depends on \( q_1q_2 \) and \( k \).
Proof: In Lemma 5 set $x = Nq_1q_2$, then we have

$$P_f(\frac{k}{2}) = 2 \sum_{l=1}^{\infty} b_f(l) l^{-\frac{3}{2}} g(\frac{l}{Nq_1q_2}).$$

Now by using Lemma 10 and the fact that $b_f(l) \leq c_0^2 d(l) l^{\frac{3}{2}},$ we can estimate

$$2 \sum_{l > X} b_f(l) l^{-\frac{3}{2}} g(\frac{l}{Nq_1q_2})$$

using the integral

$$\int_{Nq_1q_2 \log N}^{\infty} \frac{1}{(Nq_1q_2)^{\frac{3}{2}}-\frac{3}{4}} t^{\frac{3}{2}}-\frac{1}{4} e^{-\frac{3\sqrt{t}}{2}} dt$$

which is $O(N^{-11}).$ Therefore

$$P_f(\frac{k}{2}) = 2 \sum_{l \leq X} b_f(l) l^{-\frac{3}{2}} g(\frac{l}{Nq_1q_2}) + O(N^{-11}). \quad (2.22)$$

In (2.22) the sum can be written as

$$\sum_{l \leq X} 2l^{-\frac{3}{2}} g(\frac{l}{Nq_1q_2}) \sum_{mn=l} \chi_1(m) \chi_2(n) a_f(m)a_f(n) = (*) + (\dagger) \quad (2.23)$$

where $(*)$ is the sum over the terms with $(m,N) = 1,$ and $(\dagger)$ is the sum over the terms with $N|m.$

Using Lemma 11 in (2.23) yields

$$(*) = \sum_{l \leq X} 2l^{-\frac{3}{2}} g(\frac{l}{Nq_1q_2}) \sum_{mn=l, (m,N) = 1} \chi_1(m) \chi_2(n) \sum_{d|m,n} d^{k-1} a_f(\frac{l}{d^2}).$$

Now by setting $j = \frac{t}{d^2}$ and rearranging the above sum, we have

$$(*) = \sum_{j \leq X} \left( \sum_{d \leq \sqrt{\frac{d}{j}}} \frac{2}{j^{\frac{3}{2}}} g(\frac{j d^2}{Nq_1q_2}) \sum_{mn=jd^2} \chi_1(m) \chi_2(n) \right) a_f(j) = \sum_{j \leq X} \alpha_j a_f(j) \quad (2.24)$$

where $\alpha_j \ll \frac{d(j)}{j^{\frac{3}{2}}} \log N$ by using Lemma 10.
Now suppose that \( N|m \). Since \( m \leq X = Nq_1q_2(\log N)^2 \), for \( N \) large enough we can assume that \( m = m_0N \) where \( (m_0, N) = 1 \). Using the multiplicative property of \( a_f(n) \)'s, we have

\[
(\dagger) = \sum_{l \leq X} 2l^{-\frac{\theta}{2}}g\left(\frac{l}{Nq_1q_2}\right) \left( \sum_{m = 1 \text{,m|m_0N}} \chi_1(m) \chi_2(n) a_f(N) \sum_{d \mid (m_0, n)} d^{k-1} a_f\left(\frac{l}{N^d} \right) \right).
\]

Now set \( \frac{l}{N^d} = j \). Rearranging (\dagger) yields

\[
(\dagger) = \sum_{j \leq \frac{X}{N}} \left( \sum_{d \leq \sqrt{\frac{X}{j}}} \frac{2N^{-\frac{\theta}{2}} a_f(N)}{j^2} g\left(\frac{jd^2}{q_1q_2}\right) \sum_{m = 1 \text{,m|m_0N}} \chi_1(m) \chi_2(n) \right) a_f(j) = \sum_{j \leq \frac{X}{N}} J_j a_f(j) \tag{2.25}
\]

where \( J_j \ll \frac{d(j)}{j^2} \log N \). Here again we are using Lemma 10 and the fact that \(|a_f(N)| \leq c_0N^{\frac{\theta}{2}}\).

Now the result follows from (2.22), (2.24) and (2.25). \( \square \)

We now employ the following mean value result.

**Lemma 13** For \( N \) prime and complex numbers \( c_n \) we have

\[
\sum_{f \in \mathbb{F}_N} \omega_f \left| \sum_{l \leq X} c_l a_f(l) \right|^2 = (1 + O(\sqrt{X \log X})) \sum_{l \leq X} l |c_l|^2
\]

with an absolute implied constant.

**Proof:** See [6] Theorem 1. \( \square \)

**Proposition 7** Under the assumption of Proposition 6 we have

\[
\sum_{f \in \mathbb{F}_N} \omega_f \left| P_f\left(\frac{k}{2}\right) \right|^2 \ll (\log N)^5
\]

for \( k > 2 \).

**Proof:** Apply Lemma 13 to Lemma 12. \( \square \)
Note: In the case $k = 2$, since $\sum_{l \leq N} d^2(l)l^{-1} \ll (\log N)^4$, we have

$$\sum_{f \in \mathcal{P}_N = \mathcal{F}_N} \omega_f |P_f(2)|^2 \ll (\log N)^9.$$ 

We can now state and prove the following theorem.

**Theorem 2** Let $k > 2$ and $\chi_1 \ (mod \ q_1)$ and $\chi_2 \ (mod \ q_2)$ be fixed real distinct primitive Dirichlet characters such that $\chi_1 \chi_2(-N) = 1$. Then there are positive constants $C_k$ (depending only on $k$) and $C_{q_1,q_2,k}$ (depending only on $q_1$, $q_2$ and $k$) such that for prime $N > C_{q_1,q_2,k}$ there exist at least $C_k N (\log N)^{-6}$ newforms $f$ of weight $k$ and level $N$ for which $P_f(2) = L_f(\frac{k}{2}, \chi_1) L_f(\frac{k}{2}, \chi_2) \neq 0$.

**Proof:** By the Cauchy-Schwarz inequality and Proposition 4, we have

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f P_f(2) \right|^2 \leq \left( \sum_{f \in \mathcal{P}_N} \sum_{P_f(\frac{k}{2}) \neq 0} \omega_f \right) \left( \sum_{f \in \mathcal{P}_N} \sum_{P_f(\frac{k}{2}) \neq 0} \omega_f |P_f(2)|^2 \right)$$

$$\ll \left( \{ f \in \mathcal{F}_N: P_f(\frac{k}{2}) \neq 0 \} \frac{\log N}{N} + 2 \text{dim} S_k(1) \frac{1}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f |P_f(2)|^2$$

Now theorem follows from Propositions 6, 7, 4 and 5. $\square$

**Note:** In the case $k = 2$ we get the lower bound $C_2 N (\log N)^{-10}$ for the number of non-vanishing $P_f(\frac{k}{2})$ (see [5] Theorem 2).
Chapter 3

A “Semi-Orthogonality” Relation for $S_k^-(N)$ and Its Applications

3.1 Poincaré series for $S_k^-(N)$

We know that if $k$ is odd, $S_k(N) = \{0\}$. So as we mentioned before we assume that $k$ is even and consider the following subspace of $S_k(N)$.

$$S_k^-(N) = \left\{ f \in S_k(N) : f|W_N = (-1)^{k+1} f \right\}$$

where $W_N$ is the Atkin-Lehner involution. We know that for every element $f$ of $S_k^-(N)$ if we set

$$L_f(s) = \sum_{n \geq 1} a_f(n)n^{-s}$$

then $L_f(s)$ has an analytic continuation to the whole plane and satisfies the functional equation $\Lambda(s) = -\Lambda(k-s)$ where

$$\Lambda(s) = (\frac{\sqrt{N}}{2\pi})^s \Gamma(s) L_f(s)$$

(see section 1.2 or [12] p. 140). In other words $S_k^-(N)$ is the subspace of cusp forms with root number $-1$.

In Chapter 2 we mentioned that $S_k(N)$ equipped with the Petersson inner product

34
is a finite dimensional inner product space spanned by the Poincaré series

\[ P_n(z, k, N) = \sum_{\gamma \in \Gamma = \Gamma_0(N)} \frac{e(n \gamma z)}{(cz + d)^k}, \quad n \geq 1 \]

where \( e(z) = e^{2\pi iz} \), \( \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \) and \( \Gamma_\infty \) is the stabilizer of \( i\infty \) in \( \Gamma_0(N) \). We know that for \( k > 2 \) the above series is absolutely convergent. From now on we assume that \( k \) is even and \( k > 2 \). We define

\[ P_n^-(z, k, N) = P_n(z, k, N) + (-1)^{\frac{k}{2} + 1} P_n(z, k, N)|W_N. \]

Note that since \( (W_N)^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \), then

\[ P_n^-(z, k, N)|W_N = P_n(z, k, N)|W_N + (-1)^{\frac{k}{2} + 1} P_n(z, k, N) \]

\[ = (-1)^{\frac{k}{2} + 1} \left( P_n(z, k, N) + (-1)^{\frac{k}{2} + 1} P_n(z, k, N)|W_N \right) = (-1)^{\frac{k}{2} + 1} P_n^-(z, k, N). \]

So, \( P_n^-(z, k, N) \in S_k^-(N) \).

As we mentioned before, the operator \( W_N \) is Hermitian with respect to the Petersson inner product. We continue with giving a proof of this fact.

**Lemma 14** For \( f, g \in S_k(N) \), \( < f|W_N|g > = < f, g|W_N > \).

**Proof:** Let \( F \) be a fundamental domain for \( \Gamma_0(N) \). Since \( W_N^{-1} \Gamma_0(N) W_N = \Gamma_0(N) \), we have

\[ < f, g|W_N > = \int_F f(z) \overline{g(W_N)(z)} y^k \frac{dxdy}{y^2} \]

\[ = \int_{W_N^{-1}F} (f|W_N)(z) \overline{(g|W_N)(z)} y^k \frac{dxdy}{y^2} \]

\[ = \int_{W_N^{-1}F} (f|W_N)(z) \overline{g(z)} y^k \frac{dxdy}{y^2} \]

\[ = \int_{W_N^{-1}\Gamma_0(N)\backslash W_N} (f|W_N)(z) \overline{g(z)} y^k \frac{dxdy}{y^2} = < f|W_N|g >. \]
Lemma 15 If \( \{f_i, \ldots, f_s\} \) is an orthonormal basis for \( S_k^-(N) \).

\[
< P_n^-(z, k, N), f_i > = \frac{2\Gamma(k-1)}{(4\pi n)^{k-1}} a_{f_i}(n).
\]

Proof: Let \( P_n^-(z, k, N) = \sum_i c_i f_i \). Then

\[
c_i = < P_n^-(z, k, N), f_i > = < P_n(z, k, N), f_i > + (-1)^{\frac{k}{2}+1} P_n(z, k, N)|W_N, f_i >.
\]

By Lemma 14.

\[
c_i = < P_n(z, k, N), f_i > + (-1)^{\frac{k}{2}+1} P_n(z, k, N), f_i|W_N > = 2 < P_n(z, k, N), f_i >.
\]

From (2.1) we know \( < P_n(z, k, N), f_i > = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_{f_i}(n) \). This completes the proof. \( \Box \)

From Lemma 15. we deduce that if \( \{f_1, \ldots, f_s\} \) is an orthonormal basis for \( S_k^-(N) \), then

\[
\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} P_n^-(z, k, N) = \sum_i a_{f_i}(n) f_i.
\]

Let \( \hat{P}_n^-(m, k, N) \) denote the \( m \)-th Fourier coefficient of \( P_n^-(..k..N) \). By comparing the \( m \)-th Fourier coefficients on both sides, we set

\[
\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} \hat{P}_n^-(m, k, N) = \sum_i a_{f_i}(n) a_{f_i}(m).
\]

This shows that to get a "semi-orthogonality" relation for \( S_k^-(N) \), we need to compute \( \hat{P}_n^-(m, k, N) \).

3.2 A "semi-orthogonality" relation for \( S_k^-(N) \)

Now we calculate \( \hat{P}_n(m, k, N)|W_N \). To do this, we set

\[
\Gamma_{\infty} = \left\{ U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N); \ l \in \mathbb{Z} \right\}
\]

36
and recall the definition of the Poincaré series

\[ P_n(z, k, N) = \sum_{\gamma' \in \Gamma_\infty \setminus \Gamma_0(N)} \frac{e(n \gamma' z)}{(c' z + d')^k} \]

where \( \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \).

If we set \( S = \Gamma_\infty \setminus \Gamma_0(N) \) then we can decompose \( S \) with respect to \( W_N \Gamma_\infty W_N^{-1} \) in the following way. Let \( R = \Gamma_\infty \setminus \Gamma_0(N)/W_N \Gamma_\infty W_N^{-1} \), and suppose \( (R) \) is a set of representatives for \( R \) in \( \Gamma_0(N) \). For any \( \gamma' \in \Gamma_\infty \setminus \Gamma_0(N) \) there exist \( \gamma \in (R) \) and \( l \in \mathbb{Z} \) such that

\[ \gamma' = \gamma W_N U^l W_N^{-1} = \begin{pmatrix} a - b N l & b \\ c - d N l & d \end{pmatrix}. \]

Here \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \). So we have

\[ P_n(z, k, N) = \sum_{\gamma \in (R)} \sum_{l=-\infty}^{+\infty} \frac{e(n \gamma W_N U^l W_N^{-1} z)}{(z + d')^k}. \]

Now we apply the \( W_N \) operator to \( P_n(z, k, N) \) to get

\[ P_n(z, k, N)|W_N = N^{\frac{k}{2}} (-N z)^{-k} P_n(W_N z, k, N) \]

\[ = N^{\frac{k}{2}} \sum_{\gamma \in (R)} \sum_{l=-\infty}^{+\infty} \frac{e(n \gamma W_N U^l W_N^{-1} z)}{(d N (z + l) - c)^k} = N^{\frac{k}{2}} \sum_{\gamma \in (R)} h_{n, \gamma}(z). \] (3.2)

Since the function \( h_{n, \gamma} \) is periodic with period 1, it has a Fourier expansion

\[ h_{n, \gamma}(z) = \sum_{m} b_{n, \gamma}(m) e(mz), \quad Imz \geq \alpha > 0. \]

Now we follow the method of [19] to calculate \( b_{n, \gamma}(m) \).
Considering the uniform convergence of the series defining $h_{n,\gamma}(z)$, we see that

$$b_{n,\gamma}(m) = \int_0^1 h_{n,\gamma}(z) e(-mz) \, dx = \int_{-\infty}^{+\infty} e\left(\frac{b_{Nz} - a}{d Nz - c}\right) e(-mz) \, dx.$$ 

If $w = dNz - c = u + iv$, then $v > 0$. This is because we can assume that $d > 0$. Note that $d \neq 0$ because $\gamma \in \Gamma_0(N)$. Therefore we have

$$b_{n,\gamma}(m) = \frac{1}{dN} e\left(\frac{nb}{d} + \frac{m\left(\frac{z}{N}\right)}{d}\right) \int_{-\infty}^{+\infty} e\left(-\frac{(nw - 1)}{d} + \frac{mw}{dN}\right) du.$$ 

Since $d > 0$ and $v = A$ (a fixed positive number), $b_{n,\gamma}(m)$ is defined by the above convergent integral, because

$$|b_{n,\gamma}(m)| \leq \frac{1}{dN} \int_{-\infty}^{+\infty} \exp\left(\frac{2\pi}{d} \frac{m\left(\frac{z}{N}\right)}{d} - \frac{u^2 + v^2}{2}\right) du < \infty.$$ 

The following lemma gives an exact expression for $b_{n,\gamma}(m)$.

**Lemma 16** We have $b_{n,\gamma}(m) = \begin{cases} M_{n,\gamma}(m) & m > 0 \\ 0 & m \leq 0 \end{cases}$, where

$$M_{n,\gamma}(m) = \frac{1}{dN} e\left(\frac{nb}{d} + \frac{m\left(\frac{z}{N}\right)}{d}\right) \int_{w = \frac{a}{m}} e\left(-\frac{(nw - 1)}{d} + \frac{mw}{dN}\right) dw.$$ 

**Proof:** First of all suppose $m > 0$ and consider the following diagram.

For $s = 1, 2, 3$, set

$$I_s(r) = \int_{C_s(r)} \frac{e(-\frac{m}{2}\frac{w + nw^{-1}}{d})}{w^k} dw.$$
We will show that for $s = 1, 2, 3$

$$\lim_{r \to \infty} |I_s(r)| = \lim_{r \to \infty} \left| \int_{C_s(r)} e^{(\frac{-\pi w + nw^{-1}}{d})} dw \right| = 0.$$

If $r > \max(1, \sqrt{\frac{n\gamma}{m}})$, then

$$\frac{m}{n} r - \frac{n}{r} > 0$$

and on the $C_1(r)$ and $C_2(r)$.

$$0 < \frac{n\gamma}{m} \frac{1}{r^2 + v^2} < 1.$$

So

$$|I_2(r)| \leq \frac{1}{r^{k-1}} \int_0^\pi \exp\left(-\frac{2\pi}{d} \left( \frac{m}{n} r - \frac{n}{r} \right) \sin \theta \right) d\theta \leq \frac{\pi}{r^{k-1}}$$

and for $s = 1, 3$

$$|I_s(r)| \leq \int_0^A \exp\left(\frac{2\pi m v}{d\gamma} \left(1 - \frac{n\gamma}{m} \frac{1}{r^2 + v^2} \right)^{\frac{3}{2}}\right) dv \leq \frac{1}{r^k} \int_0^A \exp\left(\frac{2\pi m v}{d\gamma} \right) dv.$$

Therefore, we can evaluate $b_n,\gamma(m)$ by integrating clockwise around a circle with center at origin, for example the circle $w\hat{w} = \frac{n\gamma}{m}$. This shows that $b_n,\gamma(m) = M_n,\gamma(m)$.

If $m = 0$, again we can show that $\lim_{r \to \infty} |I_s(r)| = 0$ for $s = 1, 2, 3$, and therefore

$$b_n,\gamma(0) = \frac{1}{d\gamma} \frac{e^{\frac{nb}{d}}}{d} \int_{C} \frac{e^{-\frac{n}{dw}}}{w^k} dw.$$

where $C$ is the unit circle. Calculation of the residue of $\frac{e^{-\frac{n}{dw}}}{w^k}$ at $w = 0$ shows that $b_n,\gamma(m) = 0$ if $m = 0$.

Now suppose $m < 0$ and consider the following diagram.
For $m = -\mu (\mu > 0)$, we have
\[
|I_4(\rho)| \leq \frac{1}{\rho^{k-1}} \int_{\beta}^{\pi - \beta} \exp\left( -\frac{2\pi}{d} \frac{\mu}{N} \rho + \frac{n}{\rho} \sin \theta \right) d\theta \leq \frac{\pi}{\rho^{k-1}}.
\]
Since the integrand is analytic in the region enclosed by $C_0(\rho)$ and $C_4(\rho)$, we deduce that $b_{n,\gamma}(m) = 0$ if $m < 0$. \(\square\)

From Lemma 16 and (3.2), we have
\[
P_n(z, k, N)|W_N = \mathcal{N}^{\frac{3}{2}} \sum_{\gamma \in (R)} \left\{ \sum_{m > 0} M_{n,\gamma}(m)e(mz) \right\}
\]
By using the integral representation
\[
J_{k-1}(t) = \frac{1}{2\pi i} \int_{\gamma}(z-z^{-1}) dz
\]
of the Bessel function of order $k - 1$ and substituting $t = \frac{4\pi\sqrt{mn}}{\sqrt{N}d}$ and $z = -i\sqrt{\frac{m}{nN}} w$, we get
\[
P_n(z, k, N)|W_N = \mathcal{N}^{\frac{3}{2}} \sum_{m > 0} \left\{ \sum_{\gamma \in (R)} \frac{1}{dN} e\left( \frac{nb}{d} + \frac{m(-\frac{c}{N})}{d} \right) \left( 2\pi i \right)^{1-k} \left( \frac{m}{n} \right)^{\frac{k-1}{2}} \left( \frac{4\pi\sqrt{mn}}{d\sqrt{N}} \right) J_{k-1}(\frac{4\pi\sqrt{mn}}{d\sqrt{N}}) \right\} e(mz).
\]
Now let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in (R)$. Since $(R)$ is a set of representatives for $\Gamma_\infty \Gamma_0(N)/W_N \Gamma_\infty W_N^{-1}$. we can assume that
\[
0 \leq b < d, \quad 0 \leq (-c) < dN, \quad (d, N) = 1, \quad N | (-c), \quad ad - bc = 1.
\]
So in (3.3) we can express the inner sum in terms of $d$. For $d \geq 1$ with $(d, N) = 1$, we are dealing with the following exponential sum
\[
\sum_{0 \leq b < d, (b, d) = 1, N | (-c)} e\left( \frac{nb + m(-\frac{c}{N})}{d} \right)
\]
Since \( ad - bc = 1 \) and the exponential sum (3.3) depends on \( b \) and \(-\frac{c}{N}\) only \( \bmod \) \( d \). we can substitute \( \mathcal{N}^{\phi(d)-1}b \) for \(-\frac{c}{N}\), where \( \phi \) is the Euler function and \( b \) is the inverse of \( b \) \( \bmod \) \( d \). Note that

\[
-\frac{c}{N} \equiv \mathcal{N}^{\phi(d)-1}b \quad (\bmod \ d). \quad (3.5)
\]

Since \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \), for given \( d \) and \( b \) \( (0 \leq b < d. (b, d) = 1) \) there exists \( c \) such that \( N|c \) and \( ad - bc = 1 \). If we assume that \( 0 \leq (-c) < dN \) then (3.5) shows that such \( c \) is unique.

Substituting (3.5) in (3.4) shows that (3.4) is actually a Kloosterman sum in the following way

\[
\sum_{0 \leq b < d, b \equiv 1 (\bmod \ d)} e\left( \frac{n\mathcal{N} + (m, \mathcal{N})^{-1}b}{d} \right) = S(n, m, \mathcal{N}^{\phi(d)-1}; d). \quad (3.6)
\]

Therefore (3.3) and (3.6) yields

\[
P_n(z, k; N)|W_N
= \sum_{m > 0} \left\{ N^{-\frac{1}{2}}(2\pi i)^{-k} \left( \frac{m}{n} \right)^{\frac{k-1}{2}} \sum_{d, (d, N) = 1} d^{-1} S(n, m, \mathcal{N}^{\phi(d)-1}; d) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{d\sqrt{N}} \right) \right\} e(mz). \quad (3.7)
\]

Now we are in the situation that we can derive a "semi-orthogonality" relation for the Fourier coefficients of an orthonormal basis of \( S_k^-(\mathcal{N}) \).

Let \( \{f_1, \ldots, f_i\} \) be an orthonormal basis for \( S_k^-(\mathcal{N}) \), and let \( P_n^- (z, k; \mathcal{N}) = \sum_c c_i f_i \).

From Lemma 15 we get

\[
\frac{(4\pi n)^{k-1}}{2\Gamma (k - 1)} P_n^- (z, k; \mathcal{N}) = \sum_i a_{f_i}(n) f_i.
\]

Now if \( \hat{P}_n^-(m, k; \mathcal{N}) \) is the \( m \)-th coefficient of the Fourier expansion of \( P_n^- (z, k; \mathcal{N}) \), we have

\[
\frac{(4\pi n)^{k-1}}{2\Gamma (k - 1)} \hat{P}_n^-(m, k; \mathcal{N}) = \sum_i a_{f_i}(n) a_{f_i}(m)
\]
and by definition of the Poincaré series for $S_k^-(N)$ we have

$$\hat{P}_n(m, k, N) = \hat{P}_n(m, k, N) + (-1)^{\frac{k}{2}+1}\hat{P}_n(m, k, N)|W_N$$  \hspace{1cm} (3.8)

so by applying (2.3) and (3.7) in (3.8) we get

**Theorem 3** Let $\{f_1, ..., f_s\}$ be an orthonormal basis for $S_k^-(N)$. Then

$$\sum_{i} \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} = \frac{(4\pi)^{k-1}}{2\Gamma(k-1)} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0 (mod N)} c^{-1} S(m, n; c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c}) \right\}$$

$$-2\pi N^{-\frac{1}{2}} \sum_{d, (d, N)=1} d^{-1} S(n, m; N^{\sigma(d)-1}; d) J_{k-1}(\frac{4\pi \sqrt{mn}}{d\sqrt{N}}) \right\}.$$

As a consequence of the above theorem we have

**Proposition 8** If $\{f_1, ..., f_s\}$ is an orthogonal basis for $S_k^-(N)$ and $m, n$ are positive integers, then we have the inequality

$$\left| \sum_{i} \omega_{f_i} \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} - \frac{1}{2}\delta_{mn} \right| \leq M d(N) N^{-\frac{5}{2}} (m, n)^{\frac{1}{2}} \sqrt{(mn)^{k-1}}$$

where $\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} <f, f>}$, $M$ is a constant depending only on $k$, and $d(N)$ is the number of divisors of $N$.

**Proof:** Similar to Proposition 1, the result follows easily from the following bound for the Bessel function $J_{k-1}(z)$ for $z \geq 0$

$$|J_{k-1}(z)| \leq \frac{\sqrt{\pi} z^{k-1}}{2^{k-1} \Gamma(k - \frac{1}{2})}$$

(see Proposition 1) and the Weil bound for the Kloosterman sum (see [7]), i.e.

$$|S(m, n; c)| \leq (m, n, c)^{\frac{1}{2}} d(c)c^{\frac{1}{2}}. \square$$

**Note** In the case of $k = 2$, one does not have absolute convergence of the Poincaré series. Nevertheless, Theorem 3 and Proposition 8 are valid in this case as well. To
see this, we use a method of Hecke and define

\[ P_n(z, 2 + 2s, N) = \sum_{\Gamma_0 \backslash \Gamma_0(N)} \frac{e(n \gamma z)}{(cz + d)^2 |cz + d|^{2s}} \]

for a positive real number \( s \).

We can show that as \( s \to 0^+ \) the above series tends to a cusp form of weight 2 (see [18] pp. 183-191 for details). So we define

\[ P_n(z, 2, N) = \lim_{s \to 0^+} P_n(z, 2 + 2s, N). \]

It can be shown that

\[ \hat{P}_n(m, 2, N) = (\frac{m}{n})^{\frac{1}{2}} \left\{ \delta_{mn} - 2\pi \sum_{c \equiv 0 \text{ (mod } N)} c^{-1} S(m, n; c) J_1(4\pi\sqrt{mn}/c) \right\} \]

(see [18] p. 188). To obtain a semi-orthogonality relation in the case \( k = 2 \), we need to calculate \( \hat{P}_n(z, 2, N)|W_N \), the \( m \)-th Fourier coefficient of \( P_n(m, 2, N)|W_N \). Since

\[ P_n(z, 2, N)|W_N = \lim_{s \to 0^+} (P_n(z, 2 + 2s, N)|W_N) \]

we start by finding the Fourier expansion of \( P_n(z, 2 + 2s, N)|W_N \).

Following the notation of the beginning of this section, let \((R)\) be a set of representative for \( R = \Gamma_\infty \backslash \Gamma_0(N)/W_N \Gamma_\infty W_N^{-1} \) in \( \Gamma_0(N) \). Then we have

\[ P_n(z, 2 + 2s, N) = \sum_{\gamma \in (R)} \sum_{l = -\infty}^{+\infty} \frac{e(n \gamma W_N U^l W_N^{-1} z)}{((c - dNl)z + d)^2 |(c - dNl)z + d|^{2s}} \]

where \( U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

Now we apply the \( W_N \) operator on \( P_n(z, 2 + 2s, N) \) to get

\[ P_n(z, 2 + 2s, N)|W_N = N^{1+2s} |z|^{2s} \sum_{\gamma \in (R)} \sum_{l = -\infty}^{+\infty} \frac{e(n \frac{bN(z+l)-a}{dN(z+l)-c})}{(dN(z+l) - c)^2 |dN(z+l) - c|^{2s}}. \]
Now in (*) set $z = \zeta + \frac{c}{d N}$, then we have

$$P_n(\zeta + \frac{c}{d N}, 2 + 2s, N) |W_N| = N^{1+2s}|\zeta + \frac{c}{d N}|^{2s} \sum_{x \in \mathbb{R}} \frac{e\left(\frac{nb}{d} \right)}{(d N)^{2+2s}} \sum_{l=-\infty}^{+\infty} \frac{e\left(-\frac{n}{d N} \zeta + l\right)}{(\zeta + l)^2|\zeta + l|^{2s}}. \quad (\dagger)$$

Let

$$F_s(\zeta) = \sum_{l=-\infty}^{+\infty} \frac{e\left(-\frac{n}{d N} \zeta + l\right)}{(\zeta + l)^2|\zeta + l|^{2s}}.$$

To sum this series we make use of the Poisson summation formula

$$\sum_{l=-\infty}^{+\infty} f(l) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(w) e(-mw) dw.$$

This is valid for any function $f$ defined on $\mathbb{R}$ for which $f''$ exists and is continuous, $f(x)$ and $f'(x)$ tend to zero as $|x| \to \infty$ and $|f|$, $|f'|$ and $|f''|$ are integrable over $\mathbb{R}$.

This is true for

$$f(w) = \frac{e\left(-\frac{n}{d N} (\zeta + w)\right)}{(\zeta + w)^2|\zeta + w|^{2s}}$$

for real $w$ and $s > -\frac{1}{2}$. Therefore by applying the Poisson summation formula to $F_s(\zeta)$ we get

$$F_s(\zeta) = \sum_{m=-\infty}^{+\infty} I_m(\zeta, s) e(m \zeta),$$

where

$$I_m(\zeta, s) = \int_{-\infty}^{+\infty} \frac{e\left(-\frac{n}{d N} (\zeta + w)\right) e(-mw)}{(\zeta + w)^2|\zeta + w|^{2s}} dw.$$

In this integral $s > -\frac{1}{2}$ and $\zeta$ is any point of $\mathcal{H}$.

Using $I_m(\zeta, s), (\dagger)$ can be written as

$$P_n(\zeta + \frac{c}{d N}, 2 + 2s, N) |W_N| = N^{1+2s}|\zeta + \frac{c}{d N}|^{2s} \sum_{x \in \mathbb{R}} \frac{e\left(\frac{nb}{d} \right)}{(d N)^{2+2s}} \sum_{m=-\infty}^{+\infty} I_m(\zeta, s) e(m \zeta).$$

Substituting $\zeta = z - \frac{c}{d N}$ in $I_m(\zeta, s)$ and noticing that $I_m(\zeta, s) = I_m(z + \frac{c}{d N}, s) = I_m(z, s)$ in the above identity yields to

$$P_n(z, 2 + 2s, N) |W_N| = N^{1+2s}|z|^{2s} \sum_{x \in \mathbb{R}} \frac{e\left(\frac{nb+mc}{d} \right)}{(d N)^{2+2s}} \sum_{m=-\infty}^{+\infty} I_m(z, s) e(m z).$$
Similar to the case $k > 2$, the above identity can be written as
\[ P_n(z, 2 + 2s, N)|W_N = N^{1+2s}|z|^{2s} \sum_{d, (d, N)=1}^{+\infty} \frac{S(n, m, \sigma_0(d)-1; d)}{(dN)^{2+2s}} I_m(z, s)e(mz). \hspace{1cm} (**) \]

Now suppose $c_0$ is any positive number and let $Im z \geq 2c_0$ then we can prove that
\[ |I_m(z, s)e(mz)| \leq \frac{2\Gamma(\frac{1}{2})\Gamma(s + \frac{1}{2})}{c_0^{2s+1}\Gamma(s + 1)} e^{-2\pi|m|^{-\frac{1}{2}} \frac{2c_0}{d^2N}} \]

(see [18] Theorem 5.7.1 for a proof).

By applying this upper bound and also the Weil bound for the Kloosterman sum, it is clear that the double series on the right of (**) is uniformly convergent for $s \geq 0$ and any $\eta = Im z \geq 2c_0 > 0$. Therefore
\[ P_n(z, 2, N)|W_N = \lim_{s \to 0^+} (P_n(z, 2 + 2s, N)|W_N) \]
\[ = \sum_{d, (d, N)=1}^{+\infty} \frac{S(n, m, \sigma_0(d)-1; d)}{Nd^{2s+1}} (\lim_{s \to 0^+} I_m(z, s))e(mz). \hspace{1cm} (***) \]

But it can be proved that
\[ \lim_{s \to 0^+} I_m(z, s) = I_m(z, 0) = \begin{cases} 0 & m \leq 0 \\ -2\pi dN^{\frac{1}{2}} (\frac{m}{n})^{\frac{1}{2}} J_1(\frac{4\pi \sqrt{nm}}{d\sqrt{N}}) & m > 0 \end{cases} \]

(see [18] Theorem 5.7.1 for a proof).

Now by substituting the above expression for $\lim_{s \to 0^+} I_m(z, s)$ in (***) we derive (3.7) in the case $k = 2$. Now it is clear that Theorem 3 and Proposition 3 are valid in the case $k = 2$ too.

**Note** From now on for simplicity we assume that $k = 2$ and $N$ is a prime.
3.3 Non-vanishing of the derivative of modular $L$-functions

Here we follow [16] and apply the explicit formula method to get an upper bound for

$$\sum_{f \in \mathcal{F}_N} r_f \frac{1}{2}$$

where $r_f = \text{ord}_s L_f(s)$. To start, let us write

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{q|N} \left(1 - \frac{a_f(q)}{q^s}\right)^{-1} \prod_{p|N} \left(1 - \frac{a_f(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

Now set

$$c_f(n) = \begin{cases} 
\alpha_n^a + \bar{\alpha}_n^a & \text{if } n = p^a \text{ and } p \nmid \mathcal{N} \\
(a_f(q))^a & \text{if } n = q^a \text{ and } q|\mathcal{N} \\
0 & \text{otherwise}
\end{cases}$$

This is true because $f$ is a newform and therefore it has an Euler product (see 1.3).

Here $\alpha_n + \bar{\alpha}_n = a_f(p)$, with $|\alpha_n| = \sqrt{p}$.

**Lemma 17 (Weil's explicit formula)** Let $F: \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(a) there is an $\varepsilon > 0$ such that $F(x) \exp((1 + \varepsilon)x)$ is integrable and of bounded variation,

(b) the function $\frac{F(x) - F(0)}{x}$ is of bounded variation.

Define

$$\Phi(\gamma) = \int_{-\infty}^{\infty} F(x) e^{i\gamma x} dx.$$ 

Then,

$$\sum_{L_f(1+it) = 0} \Phi(\gamma) = 2F(0) \log \frac{\sqrt{N}}{2\pi} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Gamma(1 + it) \Phi(t) dt}{(1 + it)\Pi(1 + it) \Phi(t)} - 2 \sum_{n \geq 1} \frac{c_f(n)}{n} \Lambda(n) F(\log n)$$
where \( \Lambda(n) \) is Von Mangoldt function and the sum on the left hand side is over \( \gamma \) such that \( L_f(1+i\gamma) = 0, 1 \leq \text{Re}(1+i\gamma) \leq \frac{3}{2} \).

**Proof :** See [14]. \( \square \)

We choose \( T > 0 \) and define

\[
F(x) = \begin{cases} 
2T - |x| & \text{if } |x| \leq 2T \\
0 & \text{otherwise}
\end{cases}
\]

Then \( F \) satisfies the conditions of Lemma 17 and

\[
\Phi(\gamma) = \left( \frac{2 \sin \gamma T}{\gamma} \right)^2.
\]

Moreover, the corresponding integral involving the logarithmic derivative of the gamma function is easily estimated to be \( O(T) \). To see this, let \( T > 1 \) and consider the following integral

\[
I = \int_0^\infty \frac{\Gamma'(1+it)(2 \sin tT)^2}{\Gamma(1+it)} dt = 4 \int_{\frac{1}{T}}^\infty +4 \int_{\frac{1}{T}}^\infty = I_1 + I_2.
\]

Since \( \sin x \leq x \) if \( x > 0 \), \( I_1 \) is \( O(T) \) as the gamma function is bounded in this range. Also we know that

\[
\frac{\Gamma'(1+it)}{\Gamma(1+it)} = O(\log(|t|+2))
\]

(see [3] p. 73). Therefore

\[
I_2 \ll 4 \int_{\frac{1}{T}}^\infty \frac{\log (t+2)}{t^2} dt = 4T \log \left( 2 + \frac{1}{T} \right) + 4 \int_{\frac{1}{T}}^\infty \frac{dt}{t(t+2)} \ll T.
\]

This shows that \( I = O(T) \). Also, we have

\[
\Phi(0) = T^2 \lim_{\gamma \to 0} \left( \frac{2 \sin \gamma T}{\gamma T} \right)^2 = 4T^2
\]

and therefore choosing \( T = \frac{(\log x)}{2} \), we will have \( \Phi(0) = (\log x)^2 \). On the assumption of the GRH (Generalized Riemann Hypothesis) \( \gamma \) is real and so \( \Phi(\gamma) \geq 0 \). Thus from
Before continuing we state some analytic estimations which will be used in future.

**Lemma 18**

1. \( \sum_{p \leq x} \log p \log \frac{x}{p} \ll x \).
2. \( \sum_{p^2 \leq x} \log p (\log \frac{x}{p^2}) \sim \frac{1}{2} (\log x)^2 \).

**Proof:** Set

\[
\begin{align*}
b(n) = \begin{cases} 
1 & \text{prime} \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Then by using partial summation we get

\[
\sum_{p \leq x} \log p = \sum_{n \leq x} b(n) \log n = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} \, dt \sim x - \int_2^x \frac{dt}{\log t} \ll x \tag{3.10}
\]

where \( \pi(x) \sim \frac{x}{\log x} \) is the number of prime less than \( x \). Now using (3.10) and partial summation yields

\[
\sum_{p \leq x} \log p \log \frac{x}{p} = \sum_{n \leq x} b(n) \log n \log \frac{x}{n} \ll x.
\]

The second formula is derived in a similar fashion by using the partial summation formula. \( \square \)

Now we have the following theorem

**Theorem 4** Let \( N \) be prime. Suppose that for each newform \( f \in \mathcal{F}_N^- \), \( L_f(s) \) satisfies the analogue of the Riemann hypothesis. Then

\[
\sum_{f \in \mathcal{F}_N^-} \omega_f r_f \leq \frac{3}{4} + O((\log N)^{-2}).
\]

**Proof:** From (3.9), we get

\[
(\log x)^2 \sum_{f \in \mathcal{F}_N^-} \omega_f r_f \leq 2(\log x)(\log \frac{\sqrt{N}}{2\pi}) \sum_{f \in \mathcal{F}_N^-} \omega_f
\]

48
\[-2 \sum_{n \leq x} \frac{\Lambda(n)}{n} \log \frac{x}{n} \left( \sum_{f \in \mathcal{F}_n} \omega_f c_f(n) \right) + O((\log x) \sum_{f \in \mathcal{F}_n} \omega_f).\]

Note that Proposition 8 with \( m = n = 1 \) gives

\[\sum_{f \in \mathcal{F}_n} \omega_f = \frac{1}{2} + O(N^{-1}).\]

Therefore,

\[(\log x)^2 \sum_{f \in \mathcal{F}_n} \omega_f r_f \leq (\log x) \log \frac{\sqrt{N}}{2\pi} - 2 \sum_{n \leq x} \frac{\Lambda(n)}{n} \log \frac{x}{n} \left( \sum_{f \in \mathcal{F}_n} \omega_f c_f(n) \right) + O(\log x).\]

We now study \( \sum_{f \in \mathcal{F}_n} \omega_f c_f(n) \).

In the case \( n = p \) prime, \( c_f(p) = a_f(p) \) and therefore by Proposition 8

\[\sum_{f \in \mathcal{F}_n} \omega_f a_f(p) = O(N^{-1} p),\]

which by Lemma 18 contributes

\[O(\sum_{p \leq x} \log p \log \frac{x}{p} N^{-1}) = O(N^{-1} x)\]

to the second sum in (3.11). The contribution from \( n = p^a \) with \( a \geq 3 \) is at most

\[\sum_{n=p^a \leq x, a \geq 3} \frac{\Lambda(n)}{\sqrt{n}} \log \frac{x}{n} \ll (\log x) \sum_{a \geq 3, p} \frac{\log p}{p^{2a/3}} \ll \log x.\]

We still have to deal with \( n = p^2 \). Note that

\[c_f(p^2) = \alpha_p^2 + \hat{\alpha}_p^2 = (\alpha_p + \hat{\alpha}_p)^2 - 2\alpha_p \hat{\alpha}_p = a_f(p)^2 - 2p.\]

But \( a_f(p)^2 = a_f(p^2) + p \). By the Rankin-Selberg method,

\[\sum_{p \leq x} \frac{\log p}{p} a_f(p)^2 \sim x.\]
Therefore, by partial summation, we deduce that

\[ \sum_{p \leq x} \frac{\log p}{p^2} a_f(p)^2 \sim \log x \]

and so,

\[ \sum_{p^2 \leq x} \frac{\log p}{p^2} a_f(p)^2 \sim \frac{1}{2} \log x. \]

Again, by partial summation,

\[ \sum_{p^2 \leq x} \frac{\log p}{p^2} (\log \frac{x}{p^2}) a_f(p)^2 \sim \frac{1}{4} (\log x)^2. \]

In addition by Lemma 18.

\[ -2 \sum_{p^2 \leq x} \frac{\log p}{p} (\log \frac{x}{p^2}) \sim -\frac{1}{2} (\log x)^2 \]

so that

\[ \sum_{p^2 \leq x} \frac{\log p}{p^2} (\log \frac{x}{p^2})(a_f(p)^2 - 2p) \sim -\frac{1}{4} (\log x)^2 \]

as \( x \to \infty \). Summing over \( f \) with weights \( \omega_f \), we obtain a contribution of

\[ \frac{1}{4} (\log x)^2 + O(N^{-1}(\log x)^2). \]

At last, we have

\[ \sum_{f \in F_N^-} \omega_f r_f \leq \frac{\sqrt{N}}{\log x} + \frac{1}{4} + O(N^{-1} + N^{-1} x (\log x)^{-2} + (\log x)^{-1}) \]

which simplifies to

\[ \frac{1}{4} + \frac{\log N}{2 \log x} + O(N^{-1} x (\log x)^{-2}). \]

Choosing \( x = N \), we get

\[ \sum_{f \in F_N^-} \omega_f r_f \leq \left( \frac{3}{4} + O((\log N)^{-2}) \right) \]
which completes the proof. □

**Note** In the Proof of Theorem 4 we estimated \( \sum_{f \in \mathcal{F}_N} \omega_f c_f(n) \) for \( n = p^a(p \nmid \mathcal{N}) \). We can show that the above estimations are valid in the case \( n = q^a(q \mid \mathcal{N}) \) too. Note that \( c_f(q) = \pm 1 \) (see [1] p. 147).

Now we are going to give an asymptotic formula for the Petersson inner product on average. To do this we start with reviewing some fact about the symmetric square \( L \)-function, which we denote by \( L_{\text{sym}^2(f)}(s) \). The value of \( L_{\text{sym}^2(f)}(s) \) at \( s = 2 \) and the Petersson inner product are related with each other as follows

\[
L_{\text{sym}^2(f)}(2) = \frac{8\pi^3}{\mathcal{N}} < f, f >
\]  

(see [21] p. 90). So to find an asymptotic formula for \( \sum_{f \in \mathcal{F}_N} < f, f > \) it is enough to find one for \( \sum_{f \in \mathcal{F}_N} L_{\text{sym}^2(f)}(2) \).

We start by recalling the following identity

\[
L_{\text{sym}^2(f)}(s) = \zeta_{\mathcal{N}}(2s - 2) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s} = \sum_{n=1}^{\infty} \frac{g_f(n)}{n^s}
\]  

(3.13)

where \( \zeta_{\mathcal{N}}(s) \) is the Riemann zeta function with the Euler factors corresponding to \( p \mid \mathcal{N} \) removed (see 1.4 for details).

Consider the integral

\[
\frac{1}{2\pi i} \int_{(2)} L_{\text{sym}^2(f)}(2 + s) T^s \Gamma(s) ds = \sum_{n=1}^{\infty} \frac{g_f(n)}{n^s} \exp\left(-\frac{n}{T}\right)
\]

and this is

\[
= L_{\text{sym}^2(f)}(2) + \frac{1}{2\pi i} \int_{(-\frac{1}{2})} L_{\text{sym}^2(f)}(2 + s) T^s \Gamma(s) ds.
\]  

(3.14)

In (3.14) the integral is easily estimated as

\[
O(\mathcal{N}^\theta T^{-\frac{1}{2}})
\]

on the assumption that \( L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll \mathcal{N}^\theta \).
From the Phragmén-Lindelöf theorem it follows that

$$L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll N^\frac{1}{2}(\log N)^3$$

(see [13] p. 336 for details). Also assuming the Lindelöf hypothesis (which is a consequence of the generalized Riemann hypothesis for $L_{\text{sym}^2(f)}(s)$), $\theta$ can be any positive number. Therefore from (3.14)

$$L_{\text{sym}^2(f)}(2) = \sum_{n=1}^{\infty} \frac{gf(n)}{n^2} \exp\left(-\frac{n}{T}\right) + O(N^\theta T^{-\frac{1}{2}}).$$

(3.15)

Now we derive an expression for $gf(n)$. From (3.13) we have

$$\sum_{n=1}^{\infty} \frac{gf(n)}{n^s} = \left( \sum_{n=1}^{\infty} \frac{n^2}{(n^2)^s} \right) \left( \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s} \right) = \left( \sum_{u=1}^{\infty} \frac{a_u}{u^s} \right) \left( \sum_{v=1}^{\infty} \frac{a_f(v^2)}{v^s} \right).$$

Here

$$a_u = \begin{cases} d^2 & \text{if } u = d^2, (d,N) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$gf(n) = \sum_{uv=n} a_u a_f(v^2) = \sum_{\frac{d^2 v^2}{n}} d^2 a_f(e^2).$$

Substituting the expression for $gf(n)$ in (3.15) gives

**Proposition 9** $L_{\text{sym}^2(f)}(2) = \sum_{d,e,(d,N)=1} \frac{a_f(e^2)}{d^2 e^2} \exp\left(-\frac{d^2 e^2}{T}\right) + O(N^\theta T^{-\frac{1}{2}})$ where $\theta$ is the positive number satisfying $L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll N^\theta$.

From Proposition 9, it is clear that to find an asymptotic formula for $\sum_{f \in \mathcal{F}N} L_{\text{sym}^2(f)}(2)$, we need an estimation for $\sum_{f \in \mathcal{F}N} a_f(e^2)$. By using the Selberg trace formula we have the following proposition.

**Proposition 10** For $N$ prime and $e$ prime to $N$

$$\sum_{f \in \mathcal{F}N} a_f(e^2) = \frac{N-1}{24} + O\left((e^2 d(e^2) + N^\frac{1}{2} e d(e^2)) \left(\log e.N\right)^2\right).$$
Proof: It is clear that
\[ \sum_{f \in \mathcal{F}_N} a_f(e^2) = \frac{tr(T_{e^2}) + tr(T_{-e^2}|W_N)}{2}. \]

Here \( tr(T_{e^2}) \) is the trace of the \( e^2 \)-th Hecke operator on \( S_2(N) \) and estimation of the trace of the Hecke operators as given in Proposition 2.8 of [2]. \( \square \)

We need the following two lemmas in the proof of our asymptotic formula for \( \sum_{f \in \mathcal{F}_N} L_{sym^2(f)}(2) \).

**Lemma 19** If \( d(n) \) is the number of divisors of \( n \), then

(i) \( \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta(s)\eta(s)}{\lambda(s)} \), where \( \zeta(s) \) is the Riemann zeta function and \( \eta(s) \) and \( \lambda(s) \) are Dirichlet series which are absolutely convergent for \( Re(s) > \frac{1}{2} \).

(ii) \( \sum_{n=1}^{\infty} d(n^2) \exp(-\frac{n}{T}) \ll T(\log T)^2 \).

(iii) \( \sum_{n=1}^{\infty} \frac{d(n^2)}{n} \exp(-\frac{n}{T}) \ll (\log T)^3 \).

**Proof:** Since \( d(n) \) is a multiplicative function, we have

\[ \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \prod_p (\sum_{j=0}^{\infty} \frac{2j + 1}{p^{js}}) = \prod_p (1 + \frac{3}{p^s} + \frac{5}{p^{2s}} + ...) \]

\[ = \prod_p (1 + \frac{3}{p^s})(1 + \frac{*}{p^{2s}} + \frac{**}{p^{3s}} + ... \prod_p (1 - \frac{3}{p^s})^{-1}(1 - \frac{9}{p^2})^{-1}(1 + \frac{*}{p^{2s}} + \frac{**}{p^{3s}} + ...) \]

\[ = \prod_p (1 - \frac{3}{p^s})^{-1} \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]

Let \( \eta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \), then \( a_n = 0 \) if \( n \) has a prime factor with multiplicity one. This shows that \( \eta(s) \) is absolutely convergent for \( Re(s) > \frac{1}{2} \). Now we have

\[ (\zeta(s))^3 = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-3} = \prod_{p} (1 - \frac{3}{p^s} + \frac{3}{p^{2s}} - \frac{1}{p^{3s}})^{-1} \]

\[ = \prod_p (1 - \frac{3}{p^s})^{-1}(1 + \frac{x}{p^{2s}} + \frac{x*x}{p^{3s}} + ...)^{-1} = \prod_p (1 - \frac{3}{p^s})^{-1} \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \]
Let \( \lambda(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \), again similar to \( \eta(s) \), \( \lambda(s) \) is absolutely convergent for \( Re(s) > \frac{1}{2} \).

Substituting

\[
\prod_p (1 - \frac{3}{p^s})^{-1} = \frac{(\zeta(s))^3}{\lambda(s)}
\]

in (†) yields (i).

(ii) Since

\[
e^{-\frac{s}{4}} = \frac{1}{2\pi i} \int_{(2)} T^* \Gamma(s) ds
\]

we have

\[
\sum_{n=1}^{\infty} d(n^2) e^{-\frac{n^2}{4}} = \frac{1}{2\pi i} \int_{(2)} \left( \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} \right) T^* \Gamma(s) ds.
\]

By part (i) this integral is

\[
= \frac{1}{2\pi i} \int_{(2)} \frac{(\zeta(s))^3 \eta(s)}{\lambda(s)} T^* \Gamma(s) ds.
\]

By moving the line of integration from 2 to \( \frac{3}{4} \) we get

\[
\sum_{n=1}^{\infty} d(n^2) e^{-\frac{n^2}{4}} = O(T(\log T)^2) + \frac{1}{2\pi i} \int_{(\frac{3}{4})} \frac{(\zeta(s))^3 \eta(s)}{\lambda(s)} T^* \Gamma(s) ds \ll T(\log T)^2.
\]

This completes the proof of (ii).

(iii) Similar to part (ii), we set

\[
\sum_{n=1}^{\infty} \frac{d(n^2)}{n} e^{-\frac{n^2}{4}} = \frac{1}{2\pi i} \int_{(2)} \left( \sum_{n=1}^{\infty} \frac{d(n^2)}{n^{s+1}} \right) T^* \Gamma(s) ds
\]

\[
= \frac{1}{2\pi i} \int_{(2)} \frac{(\zeta(s+1))^3 \eta(s+1)}{\lambda(s+1)} T^* \Gamma(s) ds.
\]

The result follows by moving the line of integration from 2 to \(-\frac{1}{4}\) and the calculation of the residue at \( s = 0 \). \( \Box \)

**Lemma 20** \( \dim S_{2}^{-}(N) = \frac{N}{24} + O(\sqrt{N}) \).

**Proof:** It is known that for \( N > 3 \) a prime the exact number of forms in \( \mathcal{F}_N \) is given by \( \# \mathcal{F}_N = \frac{1}{12} (N + \alpha(N)) \), where \( \alpha(N) = -13, -5, -7 \) or 1 according to whether
$N \equiv 1, 5, 7, \text{ or } 11 (\text{mod } 12)$. Now the result follows from the fact that

$$\dim S_2^-(N) = \frac{1}{2} \dim S_2(N) + O(\sqrt{N})$$

(see [16] p. 276). □

Now we can give an asymptotic formula for the Petersson inner product on average.

**Theorem 5** If we assume $L_{\text{sym}^2}((\frac{3}{2} + it) \ll N^{\frac{1}{2} - \eta}$ for some $\eta > 0$, then

$$\sum_{f \in \mathcal{N}} \langle f, f \rangle = \frac{\pi}{12} (\dim S_2^-(N))^2 + O(N^{2 - \frac{3}{2}}).$$

**Proof:** By Proposition 9 we have

$$\sum_{f \in \mathcal{N}} L_{\text{sym}^2}(f)(2) = \sum_{d,e,(d,N)=1} \frac{\exp \left(-\frac{d^2e}{T}\right)}{d^2e^2} \sum_{f \in \mathcal{N}} a_f(e^2) + O(N^{\frac{3}{2} + 1} T^{-\frac{1}{2}}).$$

Since $f$ is a newform with root number $-1$, we have $a_f(N) = -1$ and for $(e_0, N) = 1$, $a_f(N^2 e_0^2) = (a_f(N))^2 a_f(e_0^2) = a_f(e_0^2)$ (see [1] p. 147, Theorem 3). Therefore the above identity can be written as

$$\sum_{f \in \mathcal{N}} L_{\text{sym}^2}(f)(2) = \left(\sum_{d,e,(d,N)=1} \frac{1}{N^{2m}} \sum_{(d,e),(d,N)=1} \frac{\exp \left(-\frac{d^2e}{T}\right)}{d^2e^2}\right) \sum_{f \in \mathcal{N}} a_f(e^2) + O(N^{\frac{3}{2} + 1} T^{-\frac{1}{2}}).$$

By Proposition 10

$$\sum_{f \in \mathcal{N}} L_{\text{sym}^2}(f)(2) = \frac{N}{24} \left(\sum_{d,e,(d,N)=1,(e,N)=1} \frac{\exp \left(-\frac{d^2e}{T}\right)}{d^2e^2} + \sum_{m=1}^{\infty} \frac{1}{N^{2m}} \sum_{d,e,(d,N)=1,(e,N)=1} \frac{\exp \left(-\frac{d^2e}{T}\right)}{d^2e^2}\right) + \nu + O(N^{\frac{3}{2} + 1} T^{-\frac{1}{2}})$$

where

$$\nu \ll \left(\sum_{d,e} \frac{\exp \left(-\frac{d^2e}{T}\right)}{d^2e^2}\right)$$
\[
+ \sum_{m=1}^{\infty} \frac{1}{\sqrt{2m}} \sum_{d,e} \exp \left( -\frac{d^2 e N}{T} \right) \left( d(e^2) e^2 (\log e N)^2 + e d(e^2) N^{\frac{1}{2}} (\log e N)^2 \right).
\]

From Lemma 19 we know that

\[
\sum_{e} d(e^2) \exp \left( -\frac{d^2 e}{T} \right) \ll \frac{T}{d^2} (\log \frac{T}{d^2})^2
\]

and

\[
\sum_{e} d(e^2) \exp \left( -\frac{d^2 e}{T} \right) \ll (\log \frac{T}{d^2})^3
\]

Therefore by using the partial summation formula, we deduce that

\[
J \ll T(\log T)^4 + T(\log T)^2 (\log N)^2 + N^{\frac{1}{2}} (\log N)^5 + N^{\frac{1}{2}} (\log T)^3 (\log N)^2.
\]

The main term is deduced by an estimation of

\[
\sum_{(d,N) \neq 1} \frac{\exp \left( -\frac{d^2 e}{T} \right)}{d^2}.
\]

From the integral formula we have

\[
\sum_{(d,N) \neq 1} \frac{\exp \left( -\frac{d^2 e}{T} \right)}{d^2} = \frac{1}{2\pi i} \int_{(2)} \zeta_N(2 + 2s) T^s \Gamma(s) ds
\]

\[
= \zeta_N(2) + \frac{1}{2\pi i} \int_{(-\frac{1}{2} + \epsilon)} \zeta_N(2 + 2s) T^s \Gamma(s) ds
\]

which gives us

\[
\sum_{(d,N) \neq 1} \frac{\exp \left( -\frac{d^2 e}{T} \right)}{d^2} = \zeta_N(2) + O(T^{-\frac{1}{2} + \epsilon}). \tag{3.16}
\]

Therefore

\[
\frac{N}{24} \sum_{(d,N) \neq 1} \frac{\exp \left( -\frac{d^2 e}{2d^2 e^2} \right)}{d^2 e^2} = \frac{N}{24} \sum_{(e,N) \neq 1} \frac{1}{e^2} \left( \zeta_N(2) + O((\frac{T}{e})^{-\frac{1}{2} + \epsilon}) \right).
\]
This is easily seen to be equal to

\[ \frac{N}{24} \left( \frac{\pi^2}{6} \right)^2 + O(NT^{-\frac{1}{2}+\epsilon}). \]

Hence from (3.12), for any \( T > 0 \), we have

\[ \frac{8\pi^3}{N} \sum_{f \in \mathcal{F}_N^*} <f, f> = \frac{N}{24} \left( \frac{\pi^2}{6} \right)^2 + O(T(\log T)^4 + T(\log T)^2(\log N)^2) + N^{\frac{1}{4}}(\log T)^5 + N^{\frac{1}{2}}(\log T)^3(\log N)^2) + O(N^{\theta+1}T^{-\frac{1}{2}}). \]

Now by the assumption of theorem, \( \theta = \frac{1}{2} - \eta \). Therefore setting \( T = N^{1-\eta} \) we have

\[ \sum_{f \in \mathcal{F}_N^*} <f, f> = \frac{\pi}{12}(\text{dim} S_2^-(N))^2 + O(N^{2-\frac{\eta}{2}}). \]

Now from Theorems 4 and 5 we can deduce the following upper bound for the \( \sum_{f \in \mathcal{F}_N^-} r_f^{\frac{1}{2}} \).

**Corollary 1** Let \( N \) be prime. Assume the Riemann hypothesis for \( L_f(s) \) and suppose that \( L_{\text{sym}^2(f)}(\frac{3}{2} + it) \ll N^{\frac{1}{4} - \eta} \), for some \( \eta > 0 \), then

\[ \sum_{f \in \mathcal{F}_N^-} r_f^{\frac{1}{2}} \leq \frac{\pi}{2}\text{dim} S_2^-(N) + o(N) \]

as \( N \to \infty \).

**Proof**: By the Cauchy-Schwarz inequality we have

\[ \sum_{f \in \mathcal{F}_N^-} r_f^{\frac{1}{2}} \leq \left( \sum_{f \in \mathcal{F}_N^-} \omega_f r_f \right)^{\frac{1}{2}} \left( \sum_{f \in \mathcal{F}_N^-} \omega_f^{-\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ = \left( \sum_{f \in \mathcal{F}_N^-} \frac{r_f}{4\pi <f, f>} \right)^{\frac{1}{2}} \left( \sum_{f \in \mathcal{F}_N^-} 4\pi <f, f> \right)^{\frac{1}{2}}. \]
By Theorems 4 and 5, this is

\[ 2\sqrt{\pi} \left( \frac{3}{4} + o(1) \right)^{\frac{1}{4}} \left( \frac{\pi}{12} (\text{dim} S_2^{-}(N))^2 + O(N^{2-\frac{3}{2}}) \right)^{\frac{1}{2}} \]

which is

\[ \leq \frac{\pi}{2} \text{dim} S_2^{-}(N) + o(N) \]

as \( N \to \infty. \square \)

**Note** The upper bound given in Corollary 1 is actually weaker than what we can deduce from a result of Brumer. In [2] Brumer proved that under the assumption of the Riemann hypothesis for \( L_f(s) \),

\[ \sum_{f \in \mathcal{F}_N^{-}} r_f \leq \left( \frac{3}{2} + \epsilon \right) \text{dim} S_2^{-}(N) \]

for any \( \epsilon > 0 \) and \( N \) sufficiently large (see [2] Theorem 3.15). By using the Cauchy-Schwarz inequality, this yields

\[ \sum_{f \in \mathcal{F}_N^{-}} r_f^{\frac{1}{2}} \leq \left( \frac{3}{2} + \epsilon \right)^{\frac{1}{2}} \text{dim} S_2^{-}(N). \]

The following non-vanishing result is a direct consequence of Corollary 1.

**Corollary 2** Under the assumptions of Corollary 1 for \( L_{\text{sym}^2 f}(s) \) for any \( f \in \mathcal{F}_N^{-} \), and for prime \( N \) large enough a positive proportion of elements of \( \mathcal{F}_N^{-} \) (and therefore \( \mathcal{F}_N \)) have order 1 at \( s = 1 \).

### 3.4 An approximate trace formula for \( S_2^{-}(N) \)

In this section as another application of “semi-orthogonality” relation, we use Theorem 3 to derive a formula for

\[ \sum_{f \in \mathcal{F}_N^{-}} a_f(n) \]
where \( \mathcal{F}_N^- \) is the set of newforms in \( S_2^-(N) \). We follow \([16]\) closely.

**Proposition 11** Suppose that \( L_{\text{sym}^2(f)}(s) \ll N^\theta \), for some \( \theta > 0 \). If \( n \) is not a square, we have for any \( T > 0 \)

\[
\sum_{f \in \mathcal{F}_N^-} a_f(n) = O(n T d(n) + \sqrt{n} d(n) N^{1+\theta} T^{-\frac{1}{2}})
\]

where \( d(n) \) is the number of divisors of \( n \). If \( n \) is a square, we have

\[
\sum_{f \in \mathcal{F}_N^-} a_f(n) = \frac{N}{4\pi^2} \{ \zeta_N(2) + O(T^{-\frac{1}{2}} n^{\frac{1}{4}}) \} + O(n T d(n) + \sqrt{n} d(n) N^{1+\theta} T^{-\frac{1}{2}}).
\]

**Proof:** From (3.12) we have

\[
\sum_{f \in \mathcal{F}_N^-} a_f(n) = N \sum_{f \in \mathcal{F}_N^-} \frac{a_f(n)}{8\pi^2 < f, f >} L_{\text{sym}^2(f)}(2)
\]

Now from Proposition 9 and definition of \( \omega_f = \frac{1}{4\pi \langle f, f \rangle} \) we get

\[
\frac{1}{N}(2\pi^2) \sum_{f \in \mathcal{F}_N^-} a_f(n) = \sum_{d.e.(d,N)=1} \frac{\exp \left( -\frac{d^2 e}{d} \right)}{d^2 e^2} \left( \sum_{f \in \mathcal{F}_N^-} \omega_f a_f(n) a_f(e^2) \right) + \sum_{f \in \mathcal{F}_N^-} \omega_f a_f(n) O(N^\theta T^{-\frac{1}{2}}).
\]

Now by applying Proposition 8 and the Deligne bound for \( a_f(n) \), we get

\[
\frac{1}{N}(2\pi^2) \sum_{f \in \mathcal{F}_N^-} a_f(n) = \sum_{d.e.(d,N)=1} \frac{\exp \left( -\frac{d^2 e}{d} \right)}{d^2 e^2} (\delta_{d,e^2} \sqrt{n e} \frac{\sqrt{n e}}{2}) + O(N^{-1} n e^2(n, e^2)^{\frac{1}{2}}) + O(\sqrt{n d(n)} N^\theta T^{-\frac{1}{2}}))
\]
Here we are using the fact that $\sum_{f \in \mathcal{F}_N} \omega_f = \frac{1}{2} + O(N^{-1})$ and also $\mathcal{F}_N^{-1} = O(N)$ (see Lemma 20). The error term arising from the sum is

$$g(n) = \sum_{d|n} \mu(d)\sqrt{\frac{n}{d}}$$

where $\mu(d)$ is the Möbius function. From the Möbius inversion formula we know that

$$m^{\frac{1}{2}} = \sum_{d|m} g(d).$$

Now using the definition of $g(n)$, we can rewrite the above sum in the error as

$$N^{-1} n \sum_{\delta|n} g(\delta) \sum_{d, x, \delta|e^2} \exp\left(-\frac{d x}{T}\right) = N^{-1} n \sum_{\delta|n} g(\delta) \sum_{d} \frac{1}{d^2} \sum_{e, \delta|e^2} \exp\left(-\frac{d^2 x}{T}\right)$$

It is easily seen that the above expression

$$\ll N^{-1} n T \sum_{\delta|n} \frac{g(\delta)}{\sqrt{\delta}} = N^{-1} n T \sum_{\delta|n} \sum_{d|\delta} \frac{\mu(d)}{\sqrt{d}} = N^{-1} n T \sum_{d|n} \frac{\mu(d)}{\sqrt{d}}$$

which is

$$\ll N^{-1} n T d(n).$$

This proves the Proposition when $n$ is not a square.

When $n = e^2$, substituting (3.16) into the main term gives the desired result. $\Box$
Bibliography


