CLOSED MODEL CATEGORIES AND MONOIDAL CATEGORIES

by

Donald Stanley

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

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Abstract

In this thesis we explore some uncharted areas of the theory of closed model categories and monoidal categories. We give conditions under which a set of maps, in any category, determine a closed model structure. We then observe that if such a category is monoidal its monoidal objects also often can be given the structure of a closed model category. Next we look at how we can define chains on our model categories. In the last chapter the theory that has been built up is applied to stable homotopy theory. We demonstrate an equivalence of categories between differential graded algebras over a commutative ring $R$ and algebras over the Eilenberg-Mac Lane spectrum $KR$. In future work we will apply the theory to many other situations.
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Chapter 1

Introduction

1.1 Statement of Results and General Introduction

The concept of a model category goes back to Quillen [22]. They provide a nice framework in which to do homotopy theory. A forerunner of the model category is presented in Kan [17] and an alternative framework is presented in Baues [4]. Model categories have proven particularly useful in understanding equivalences of categories as in the classic work of Quillen on rational homotopy theory [23]. They have also proven useful in describing various localization functors [8]. Many papers include proofs that various categories have closed model category structures. See for example [11], [12], [22].

Another important concept in homotopy theory is that of a monoidal category (4.1.1). It turns out that many of the categories worked with by homotopy theorists are in fact monoidal categories. There even exist categories of spectra which can be given the structure of a monoidal category, [13] for example. Monoidal objects in these categories also play an important role. For example monoidal objects in topological spaces are topological monoids and group-like topological monoids are essentially the same as loop spaces.

It is to these two topics, closed model categories and monoidal categories, and to the interactions between them that this thesis has been devoted. We begin by introducing closed model categories and giving some of their basic properties. We want to see when a
set of maps in a category determine a closed model category. A cell category is a category together with a set of maps in the category. When this category satisfies certain hypotheses (3.2.14) we call it suitable. We prove the following theorem.

**Theorem 1.1.1** A suitable cell category can be given the structure of a closed model category.

Many examples of suitable cell categories are given. This is the first time in a general setting conditions have been given for a set of maps to determine a closed model category. All constructions of closed model categories to date are ad hoc or depend on having a closed model category already in some other category. To prove the theorem we need to construct two factorizations and show they have the desired lifting properties. The first, into a cofibration followed by an acyclic fibration, is similar to what has appeared in the literature ([7], [8], [6]). After the needed definitions and construction have been made the demonstration of its properties should be easy for the expert. The construction and demonstrations for the second factorization, however, are slightly more subtle.

We next turn to monoidal categories. We assume that the product is compatible with colimits (4.1.1) and describe colimits of monoidal objects by means of colimits in the underlying category. This results in the following theorem.

**Theorem 1.1.2** If $C$ is monoidally cocomplete (4.1.1) then the category of monoidal objects in $C$ is cocomplete.

This description also allows easier analysis of colimits in categories of monoidal objects. We are also able to construct a free functor $T : C \to C[Mon]$. This had previously been done in more restrictive circumstances in Mac Lane's book [19] and in Baues-Conduché [5]. We bring our two themes together in

**Theorem 1.1.3** If $C$ is a suitably monoidal cell category (4.4.4) then $C$ can be given a closed model category structure.
Our final direction takes us to look at under what condition we can define homology on our categories. We define a chain functor from cofibrant objects to differential graded modules which extends to a functor from cofibrant monoidal objects to differential graded algebras. For a ring $R$ let $K(R)$ denote the Eilenberg Mac Lane spectrum such that $\pi_0(K(R)) = R$. We construct spectra $KR$ for every $R$. This had been done previously by May ([20]). For commutative $R$ we get the following new result.

**Theorem 1.1.4 (6.7.7)** The homotopy category of differential graded algebras over $R$ is equivalent to the homotopy category of monoidal objects in the category of $K(R)$ modules.

As a special case we get

**Theorem 1.1.5 (6.7.8)** The homotopy category of differential graded algebras over $Q$ is equivalent to the homotopy category of rational $A^\infty$ ring spectra.

It should be noted that in this thesis an important technical theme is coherence. The theory would be somewhat simpler if such problems could be ignored.

Other results suggested by those here are in various states of completion. Three examples are:

1) non simply connected Adams-Hilton ([2]) and Husseini ([14],[15]) models in $Top_*$.

2) Algebras over the ring spectrum $E(1)$ equivalent to $DGA$'s over the graded ring $\mathbb{Z}_p[u, u^{-1}]$.

3) Making $C[Mon]$ into a monoidal category and thus getting models for $n$ fold loop spaces.

Many other directions have arisen begging to be explored. We give a few. We can also put homotopy diagonals on our monoidal objects and get Hopf objects up to homotopy, $Hoh$. In $DGA$ these are then Hopf algebras up to homotopy, $Hah([3])$. In $Top_*$ all monoidal objects are $Hoh$. We question whether

$$\Sigma^\infty : Hoh(Top_*) \to Hoh(\mathcal{M}_S)$$
is an equivalence for objects \( X \) with all primes less than the dimension of \( X \) divided by the connectivity of \( X \) inverted. This is an equivalence rationally by [3]. We also question the realizability of DGA's over \( \mathbb{Z} \) as \( C_* \) of \( S \) algebras and speculate that all are realizable.

1.2 Categories and conventions

We will use the following notations for categories. Let \( C \) be a category. We will denote morphisms in \( C \) in three different ways. The class of morphisms will be denoted by \( \text{Hom}_C \).

For \( X, Y \in C \) we will denote the morphisms from \( X \) to \( Y \) by \( \text{Hom}_C(X, Y) \) or if the category is clear from the context then we will allow ourselves to simply use the notation \( \text{Hom}(X, Y) \). A small category is one whose objects form a set. Let \( \text{Obj}(C) \) denote the objects of \( C \).

We will use the following notations for some commonly used categories. We use \(*\) to denote the category with one object and one morphism and for any object \( X \) in any category we also let \( X : * \to X \) denote the functor. \( \text{Top} \) will denote the category of topological spaces and continuous maps. \( \text{Top}_* \) will denote the category of pointed topological spaces. We will denote the category of differential graded modules over a commutative ring \( R \), by \( DGM(R) \).

In \( DGM(R) \) we will let \( R_n \) denote the differential graded module that is a free \( R \) module concentrated in degree \( n \). \( GM(R) \) will denote the category of graded modules over \( R \). For any category \( C \) and a small category \( D \) we will denote by \( C^D \) the category of functors from \( D \) to \( C \) with the morphisms being natural transformations.

\( \emptyset \) will denote the initial object in any category and \(*\) the terminal object. If the category has a zero object we will use \(*\) to denote it. \(*\) will also be used to denote any map that factors through the object \(*\). \( \emptyset \) will also be used to denote any map that factors through the object \( \emptyset \). We will use \( \vee \) to denote coproducts. \( X \vee Y \) \( Z \) will denote the push out of

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \\
Z
\end{array}
\]
when the maps are clear from the context. Also \( f + g \) will denote the map from the push out where \( f \) is a map from \( X \) and \( g \) is a map from \( Z \). We will also denote \( X \lor_Y Z \) by \( X \cup Z \) when \( Y \) is clear from the context.

### 1.3 Colimits

For a category \( C \) and a small category \( D \) a colimit is a functor, right adjoint to the constant functor, from \( C^D \) to \( C \). We state three lemmas about colimits that are easy to prove. They will be used repeatedly.

**Lemma 1.3.1** For any small categories \( A \) and \( B \) and any category \( C \) and functor \( F : A \times B \rightarrow C \)

\[
\text{colim}_{x \in A} \text{colim}_{y \in B} F(x, y) = \text{colim}_{(x, y) \in A \times B} F(x, y)
\]

whenever both sides are defined.

**Lemma 1.3.2** In the following diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \rightarrow C \\
\downarrow & & \downarrow \\
D & \rightarrow & F \rightarrow G \\
\end{array}
\]

if the left square and the outside square are pushouts then the right square is also.

**Lemma 1.3.3** In the following diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D \\
\downarrow & & \downarrow \\
E & \rightarrow & F \\
\downarrow & & \downarrow \\
G & \rightarrow & H \\
\end{array}
\]
if the back, left and right faces are pushouts then so is the front face.

**Definition 1.3.4** Let $D$ be a small category whose objects are a well ordered set and with one map from $a$ to $b$ if $a \leq b$. Let $G \to C$ be a functor. Then $\{G(x)\}_{x \in D}$ is referred to as a direct system and $\text{colim}_{x \in D}G(x)$ is referred to as a direct limit.

### 1.4 Orders

**Definition 1.4.1** Let $(S, \leq)$ be a partially ordered set. Assume that every nonempty $T \subseteq S$ has minimal elements. That is, there exists $x \in T$ such that there does not exist $y \in T$ with $y < x$. We call such an order a **well founded partial order**.

**Definition 1.4.2** Let $(S, \leq)$ be a well founded partial order. We define a function $\text{rank}$ that assigns to every $x \in S$ an ordinal. For $x \in S$ if $x$ is minimal define $\text{rank } x = 0$. Otherwise define $\text{rank } x = \sup_{y < x}(\text{rank } y + 1)$.

The next lemma says that every partial well ordering can be extended to a well ordering.

**Lemma 1.4.3** Let $(S, \leq)$ be a well founded partial order. Then there exists a well ordering $\leq'$ in $S$ such that $x \leq y$ implies that $x \leq' y$.

**Proof:** Easy. \(\square\)

**Definition 1.4.4** A **successor ordinal** is an ordinal of the form $\alpha + 1$. All other ordinals are called **limit ordinals**. We let $\omega$ denote the first successor ordinal. For a cardinal $\kappa$ we let $\kappa^+$ denote its successor cardinal. For any set $A$ we let $|A|$ denote the cardinality of $|A|$. We call a cardinal $\kappa$ **regular** if for subsets $\kappa(\alpha) \subseteq \kappa \cup_{\alpha \in A} \kappa$ implies that $|A| \geq \kappa$.

**Lemma 1.4.5** All successor cardinals are regular.

**Definition 1.4.6** Let $\alpha$ and $\beta$ be ordinals with orderings $\leq_\alpha$ and $\leq_\beta$. Then we define $\alpha + \beta$ to be the ordinal corresponding to the well ordered set $(\alpha \cup \beta, \leq)$ where if $a, b \in \alpha$ then $a \leq b$ if $a \leq_\alpha b$, if $a, b \in \beta$ then $a \leq b$ if $a \leq_\beta b$ and if $a \in \alpha$ and $b \in \beta$ then $a \leq b$.  


Note that + is not a symmetric operation. For example if $\beta$ is an infinite cardinal and $\beta > \alpha$ then $\alpha + \beta = \beta$. 
Chapter 2

Closed Model Categories

In this chapter we introduce closed model categories. Closed model categories were first introduced by Quillen [22]. A good exposition of the basics can be found in Dwyer-Sapinski [12] but the approach here is closer to that of Blanc [6]. We examine some of their basic properties that exist in the presence of a homotopy extension lifting property (HELP). To our knowledge this is the first time model categories with HELP have been studied. The first section gives us the definition of left model category, right model category and closed model category. Section 2 defines homotopies and HELP in a left model category. Composition is shown to preserve homotopies in this context. Section 3 introduces localizations and homotopy categories. It shows that a left model category with HELP has a homotopy category. The last section, section 4, defines the cylinder and suspension functors and introduces the coaction, giving its properties that will be needed later.
2.1 Closed Model Categories

Definition 2.1.1 Let the following be a diagram of maps in $C$

\[
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
| & & | \\
\downarrow f & \downarrow g & \downarrow f \\
K & \downarrow L & \downarrow K \\
| & & | \\
Y & \xrightarrow{id} & Y
\end{array}
\]

Then we say that $f$ is a retract of $g$.

Definition 2.1.2 Given the following commuting solid arrow diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i'} & C' \\
| & \searrow & | \\
h & & p' \\
\downarrow i & \downarrow & \downarrow p \\
C & \xrightarrow{p} & B
\end{array}
\]

if a map $h$ exists making both triangles commute then we say that the diagram has the extension lifting property and call $h$ an extension lift. Given $i$ and $p'$ if for any $i'$ and $p$ there exists such a $h$ then we say that $i$ has the extension property or left lifting property (LLP) with respect to $p'$ or equivalently that $p'$ has the lifting property or right lifting property (RLP) with respect to $i$.

Definition 2.1.3 Let $C$ be a category with three distinguished classes of morphisms: $C$, $F$ and $W$ all closed under composition. Assume that $W$ is closed under retracts, contains all isomorphisms and if for any two maps $f, g$ in $C$ such that $f \circ g$ is defined if two of $f, g$ and $f \circ g$ are in $W$ then the third is also. We call $W$ the weak equivalences.

Property 1:

Assume that for any $f : A \to B$ a morphism in $C$ there is a factorization $f = p \circ i$ where $p \in F \cap W$ and $i \in C$. Also if there exist another such factorization $f = p' \circ i'$ then the
The following diagram has the extension lifting property

\[
\begin{array}{ccc}
A & \xrightarrow{i'} & C' \\
| & | & | \\
\downarrow{i} & \uparrow{i} & \downarrow{p'} \\
C & \xrightarrow{p} & B
\end{array}
\]

Now we consider the property dual to property 1

**Property 2:**

Assume that for any \( f : A \to B \) a morphism in \( C \) there is a factorization \( f = p \circ i \) where \( p \in \mathcal{F} \) and \( i \in C \cap \mathcal{W} \). Also if there exists another such factorization \( f = p' \circ i' \) then the above diagram has the extension lifting property.

If \( C \) is closed under finite limits and satisfies property 2 it is called a right model category \((\text{RMC})\).

If \( C \) is closed under finite colimits and satisfies property 1 it is called a left model category \((\text{LMC})\).

If \( C \) is closed under both finite colimits and finite limits and both properties hold it is called a (closed) model category.

Note that an initial object is the empty colimit and a terminal object is the empty limit.

This definition is not the usual one. Presently we show it is equivalent to the usual one.

**Definition 2.1.4** Let the cofibrations be the closure of \( C \) under retracts and let the fibrations be the closure of \( \mathcal{F} \) under retracts. An acyclic cofibration is a cofibration and a weak equivalence and an acyclic fibration is a fibration and a weak equivalence.

**Definition 2.1.5** In an LMC an object \( X \) is called cofibrant if the map \( \emptyset \to X \) is a cofibration.

**Lemma 2.1.6** Let \( f \) and \( g \) be maps. If \( f \) has the LLP (resp. RLP) with respect to \( g \) then \( f \) has the LLP (resp. RLP) with respect to any retract of \( g \).

**Proof:** Easy. \( \square \)
Lemma 2.1.7 Let $C$ be a LMC. Then $i$ is a cofibration if and only if it has the LLP with respect to all $f \in \mathcal{F} \cap \mathcal{W}$.

Proof: We first prove the "only if" case. First assume that $i \in C$ and $p \in \mathcal{F} \cap \mathcal{W}$. Start with a diagram (*)

\[
\begin{array}{ccc}
A & \xrightarrow{g} & X \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{h} & Y
\end{array}
\]

now factor $g = p_1i_1$ and $h = p_2i_2$ with $i_j \in C$ and $p_j \in \mathcal{F} \cap \mathcal{W}$. Then in the following solid arrow diagram there exists a lift $f$.

\[
\begin{array}{ccc}
A & \xrightarrow{g} & X \quad \xrightarrow{i_1} & \xrightarrow{p_1} & Z \\
 & & \downarrow{p} & & \\
 & & W & & \\
 & & \xrightarrow{p_2} & & Y \\
B & \xrightarrow{h} & Y
\end{array}
\]

So we can take the required lifting to be $p_1 \circ h \circ i_2$.

Now let $i : A \rightarrow B$ be any cofibration. Then we have a retraction

\[
\begin{array}{ccc}
A & \xrightarrow{r} & A \\
\downarrow{i} & & \downarrow{i} \\
A' & \xrightarrow{i'} & B' \\
\downarrow{i} & & \downarrow{i} \\
B & \xrightarrow{r'} & B
\end{array}
\]
where \( i' \in C \). So if we again have the diagram at the beginning (*) , we can take the lifting to be \( f \circ j \) where \( f \) is a lifting in the following diagram

\[
\begin{array}{c}
A' \xrightarrow{g'} X \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B' \xrightarrow{r' h} Y.
\end{array}
\]

Now we prove the "if" part of the lemma. Let \( f : A \to B \) be a map that has the LLP with respect to every acyclic fibration. Factor \( f \) as \( i : A \to W \) composed with \( p : W \to B \) where \( i \in C \) and \( p \in \mathcal{F} \cap \mathcal{W} \). then by hypothesis there exists a \( g \) making the following diagram commute.

\[
\begin{array}{c}
A \xrightarrow{i} W \\
\downarrow \quad \downarrow g \quad \downarrow p \\
B \xrightarrow{id} B
\end{array}
\]

So we realize \( f \) as a retraction of \( i \in C \)

\[
\begin{array}{c}
A \xrightarrow{id} A \\
\downarrow id \quad \downarrow \quad \downarrow id \quad \downarrow \\
A \xrightarrow{i} W \\
\downarrow \quad \downarrow g \quad \downarrow p \\
B \xrightarrow{id} B
\end{array}
\]

\( \square \)

**Corollary 2.1.8** Let \( C \) be a LMC. Then a direct limit of cofibrations is a cofibration. coproducts of cofibrations are cofibrations and in the following pushout

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D
\end{array}
\]

if \( f \) is a cofibration then so is \( g \).
Proof: All three constructions preserve the LLP. □

Lemma 2.1.9 In a RMC any acyclic fibration is a retract of a map in $\mathcal{F} \cap \mathcal{W}$.

Proof: Let $p : X \to Y$ be an acyclic fibration. Factor $p$ as

$$
X \xrightarrow{i} X' \xrightarrow{p'} Y
$$

with $i \in \mathcal{C} \cap \mathcal{W}$ and $p' \in \mathcal{F}$. Then in the following solid arrow diagram

there exists an extension lift $h$ by the dual of Lemma 2.1.7. Also notice that $p' \in \mathcal{F} \cap \mathcal{W}$ by the two out of three condition. So we can exhibit $p$ as a retract of $p' \in \mathcal{F} \cap \mathcal{W}$.

□

Lemma 2.1.10 In a closed model category cofibrations are the maps that have the LLP with respect to all acyclic fibrations.

Proof: 2.1.7, 2.1.9 and 2.1.6. □

Lemma 2.1.11 In a LMC if $p$ has the RLP with respect to all maps in $\mathcal{C}$ then $p$ is an acyclic fibration.
**Proof:** Let $p : X \to Y$ have the RLP with respect to all maps in $C$. Factor $p$ as

$$X \overset{i}{\longrightarrow} X' \overset{p'}{\longrightarrow} Y$$

such that $i \in C$ and $p' \in \mathcal{F} \cap \mathcal{W}$. Then in the solid arrow diagram

$$\begin{array}{ccc}
X & \overset{=} \longrightarrow & Y \\
\downarrow^i & & \downarrow^p \\
X' & \overset{\sim} \longrightarrow & Y
\end{array}$$

there exists an extension lift $h$. So we can exhibit $p$ as a retract of $p' \in \mathcal{F} \cap \mathcal{W}$. So $p$ is an acyclic fibration. □

**Lemma 2.1.12** In a closed model category acyclic fibrations are the maps that have the RLP with respect to all cofibrations.

**Proof:** 2.1.7, 2.1.11, 2.1.9 and 2.1.6. □

We will refer to the following definition of closed model category as the usual one and the other one as the new one.

**Definition 2.1.13** Let $C$ be a finite complete, finite cocomplete category with three distinguished classes of morphisms: cofibrations (cof); fibrations (fib); weak equivalences (we). Assume that the following six conditions are satisfied:

1) For every map $f \in \text{Hom}_C$ there exists factorizations $f = ip'$ and $f = i'p$ such that $i$ and $i'$ are cofibrations, $p$ and $p'$ are fibrations and $i'$ and $p'$ are weak equivalences.

2) Cofibrations are the maps that have the LLP with respect to all acyclic fibrations.

3) Acyclic fibrations are the maps that have the RLP with respect to all cofibrations.

4) Fibrations are the maps that have the RLP with respect to all acyclic cofibrations.

5) Acyclic cofibrations are the maps that have the LLP with respect to all fibrations.

6) If $f = gh$ and any two of $f$, $g$ and $h$ are weak equivalences then all three are.

Then we say that $(C, \text{cof}, \text{fib}, \text{we})$ is a closed model category.
Lemma 2.1.14 The two definitions of closed model category are the same.

Proof: To see that the usual definition implies the new one let $C = \text{cof}$, $F = \text{fib}$ and $W = \text{we}$. Then the only nontrivial observation still to make is that the characterization by lifting properties implies that $C$ and $F$ are closed under composition.

So now assume we have a closed model category in the new sense. Define $\text{cof}$ and $\text{fib}$ to be the closure of $C$ and $F$ under retracts and $\text{we}$ to be $W$. With these definitions conditions 1 and 6 are clearly satisfied. Conditions 2 to 5 follows from Lemmas 2.1.10, 2.1.12 and their duals. □

For a definition of model category see [22] or [4]. The following corollary of the above lemma has never been made explicit in the literature.

Corollary 2.1.15 For every model category structure on $C$ there is a unique closed model category structure on $C$ whose cofibrations, fibrations and weak equivalences are all cofibrations, fibrations and weak equivalences in the model category structure.

The following lemma will be required later.

Lemma 2.1.16 In any model category let

\[
\begin{array}{ccc}
A(1,i) & \longrightarrow & A(2,i) \\
\downarrow & & \downarrow \\
A(3,i) & \longrightarrow & A(4,i)
\end{array}
\]

be pushouts and $f(j) : A(j,1) \rightarrow A(j,2)$ for $j \leq 3$ be acyclic cofibrations compatible with the maps in the above diagram. That is such that diagrams of the following form commute

\[
\begin{array}{ccc}
A(j,1) & \longrightarrow & A(j,2) \\
\downarrow & & \downarrow \\
A(j',1) & \longrightarrow & A(j',2).
\end{array}
\]
Let $Y$ denote the pushout 

\[
\begin{array}{ccc}
A(1,1) & \longrightarrow & A(2,1) \\
\downarrow & & \downarrow \\
A(1,2) & \longrightarrow & Y 
\end{array}
\]

and assume that the induced map $Y \rightarrow A(2,2)$ is a cofibration. Then the induced map $A(4,1) \rightarrow A(4,2)$ is an acyclic cofibration.

**Proof:** Use the characterization of acyclic cofibrations as having the LLP with respect to fibrations. □

### 2.2 Homotopies and HELP

In this section we let $C$ be a LMC.

**Definition 2.2.1** Let $f : A \rightarrow B$ be a map in $C$. Then given a factorization of $\nabla$ as in the following commutative diagram

\[
\begin{array}{ccc}
B \vee_A B & \xrightarrow{\partial_0 + \partial_1} & Cyl(B, A) \\
\downarrow & \sigma & \downarrow \\
B & \xrightarrow{\sigma} & B
\end{array}
\]

we call $Cyl(B, A)$ a cylinder object for the pair $(B, A)$ if $\partial_0 + \partial_1$ is a cofibration and $\sigma$ is a weak equivalence. If $A = \emptyset$ then we will also denote $Cyl(B, A)$ by $CylB$.

Note that $Cyl(B, A)$ depends on what map $f : A \rightarrow B$ we start with. Since it is usually clear which map we are using we have not included it in the notation.

**Definition 2.2.2** Let $f \in Hom(X, Y)$ and let $W \rightarrow Y$ be a cofibration. Then a cylinder of $f$ is any lift of the following diagram

\[
\begin{array}{ccc}
X \vee X & \xrightarrow{\partial_0 f + \partial_1 f} & Cyl(Y, W) \\
\downarrow & & \downarrow \\
CylX & \xrightarrow{f \sigma} & Y 
\end{array}
\]
We denote it by Cylf. If f is a cofibration and W = ∅ we will assume we have chosen CylY and Cylf so that Cylf is a cofibration.

Our definition of relative homotopy is the same as Baues Chapter I 1.6.

Definition 2.2.3 Let f, g : B → X be two maps such that for some cylinder object there is a commutative diagram.

\[
\begin{array}{ccc}
B \vee_{A} B & \xrightarrow{\partial_0 + \partial_1} & \text{Cyl}(B, A) \\
\downarrow{f+Ag} & & \downarrow{H} \\
X & & \\
\end{array}
\]

Then we say that H is a homotopy from f to g and that f is homotopic to g rel A, written \( f \simeq g \text{ rel } A \).

Definition 2.2.4 Let \( i : A \to B \) and \( j : Y \to X \). Denote the equivalence classes of maps from B to X under the relation generated by \( \simeq \text{ rel } A \) by \( \pi((B, A), X) \). If A = ∅ we will also use the notation \( \pi(B, X) \). It is clear that \( \pi((B, A), -) : C \to \text{ sets} \) is a functor. For a map \( f : X \to Y \) we will denote \( \pi((B, A), f) \) by \( \pi(f) \). We will denote equivalence classes of pairs of maps \( f : B \to X \) and \( g : A \to Y \) such that \( fi = jg \) under the equivalence relation generated by \( \simeq \text{ rel } A \) by \( \pi((B, A), (X, Y)) \). Note that \( \pi((B, A), (X, X)) = \pi((B, A), X) \).

Assume that we have maps \( j : A \to X \), \( j' : A \to Y \), \( f : X \to Y \) and \( g : Y \to X \) such that \( fj = j' \) and \( gj' = j \). Then if \( gf \simeq \text{id rel } A \) and \( fg \simeq \text{id rel } A \) we say X is homotopy equivalent to Y rel A, written \( X \simeq Y \text{ rel } A \). If \( A = ∅ \) then we also say that X is homotopy equivalent to Y, written \( X \simeq Y \).

Definition 2.2.5 If for every acyclic cofibration \( i : A \to B \) there exists an \( r : B \to A \) such that \( r \circ i = \text{id} \) and \( i \circ r \simeq \text{id rel } A \) then we call C good.

Good implies that all objects are fibrant. So we see immediately it cannot hold with the usual model category structure on simplicial sets. It then follows from Baues ([4]) Chapter II Corollary 1.12 that a proper model category is good if and only if all of its objects are fibrant.
We introduce a general definition of the homotopy extension lifting property (HELP).

**Definition 2.2.6** We say that $C$ has HELP if for all cofibrations $W \to A$ and $i : A \to B$ there exists choices of cylinders and a map $\text{Cyl}(i) : \text{Cyl}(A, W) \to \text{Cyl}(B, W)$ such that

$$\text{Cyl}(A, W) \cup_{AVwA} B \cup_W B \to \text{Cyl}(B, W)$$

is a cofibration and such that for every weak equivalence $f$ and any commuting solid arrow diagram of the following form there exist $g$ and $H$ making the diagram commute.

![Diagram](https://via.placeholder.com/150)

Our definition is the right one for a general model category. In a proper model category it is appropriate to let $W = \emptyset$ and not assume that $A$ is cofibrant. This is how HELP is usually defined.

For example HELP holds in $\text{Top}_*$ with the model category structure of chapter 3.

**Lemma 2.2.7** The definition of HELP is equivalent to the above definition but restricting the maps $A \to B$ to maps in $C$.

**Proof:** Let $f : A \to B$ be any cofibration. Let $r$ be any extension lift in the following diagram.

![Diagram](https://via.placeholder.com/150)

Let $\text{Cyl}(A, W), \text{Cyl}(\text{CW}(f), W)$ be cylinder objects and $\text{Cyl}(i(f))$ a map that makes HELP hold. Let $\text{Cyl}(B, W)$ be a cylinder object such that $\text{Cyl}(f) : \text{Cyl}(A, W) \to \text{Cyl}(B, W)$ induces a map $\text{Cyl}(A, W) \cup B \cup W B \to \text{Cyl}(B, W) \in C$ Let $\text{Cyl}r$ be any extension in the following
solid arrow diagram.

\[
\begin{array}{c}
Cyl(A, W) \cup B \vee W \xrightarrow{\text{Cyl}(i(f)) + \nabla \sigma + \nabla_1 r} Cyl(CW(f), W) \\
\downarrow \\
Cyl(B, W) \xrightarrow{\sigma} B
\end{array}
\]

one exists by lemma 2.2.8. Now assume that we are given a solid arrow diagram such that \( \phi \) is a weak equivalence.

We wish to have maps \( H \) and \( h \) that extend the diagram. If we replace \( f \) with \( i(f) \), \( B \) with \( CW(f) \), \( Cyl(f) \) with \( Cyl(i(f)) \) and \( Cyl(B, W) \) with \( Cyl(CW(f), W) \) we get a new solid arrow diagram and extensions \( H' \) and \( h' \). Defining \( H = H'Cylr \) and \( h = h'r \) it is easy to see the proof. \( \Box \)

**Lemma 2.2.8** HELP implies that given a solid arrow diagram

\[
\begin{array}{c}
A \longrightarrow B \\
\downarrow f' \downarrow g \\
C \longrightarrow D
\end{array}
\]

such that \( f \) is a cofibration and \( g \) is a weak equivalence there exists an \( r \) making the upper triangle commute.

**Proof:** Easy. \( \Box \)

**Lemma 2.2.9** HELP implies good.
Proof: Let \( i : A \to B \) be an acyclic cofibration. The map \( r \) and the homotopy, \( H \), are given by the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow i \\
B
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
Cyl(A, A) \\
\downarrow \partial_0 \\
\downarrow \partial_1 \\
A
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
Cyl(B, A) \\
\downarrow \partial_0 \\
\downarrow \partial_1 \\
B
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\partial_0 \\
\partial_1 \\
B
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\text{id} \\
\downarrow H \\
\text{id}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\text{id} \\
\downarrow r \\
\text{id}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\partial_0 \\
\partial_1 \\
A
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\partial_0 \\
\partial_1 \\
A
\end{array}
\end{array}
\end{array}
\]

\[\]

Lemma 2.2.10 Assume that HELP holds in \( C \). Then if \( X \) is cofibrant and \( f \) is a weak equivalence then \( \pi(X, f) \) is an isomorphism.

Proof: Easy consequence of HELP. \( \square \)

The next lemma says that in a good LMC all cylinder objects determine the same homotopies.

Lemma 2.2.11 Let \( C \) be good. Fix any \((B, A)\) cylinder \( Cyl(B, A)\). Let \( f, g : B \to X \) be maps such that \( f \simeq g \text{ rel } \ A \). Then there exists a homotopy from \( f \) to \( g \), \( H : Cyl(B, A) \to X \).

Proof: There exists a homotopy \( H' : Cyl'(B, A) \to X \). Factor \( \sigma' : Cyl'(B, A) \to B \) as a cofibration \( i : Cyl'(B, A) \to Cyl''(B, A) \) followed by an acyclic fibration \( \sigma'' : Cyl''(B, A) \to B \). Since \( \sigma' \) is a weak equivalence \( i \) must be one also and so by goodness \( i \) has a retract. Therefore there is also a homotopy \( H'' : Cyl''(B, A) \to X \) from \( f \) to \( g \). Finally since \( \sigma'' \) is an acyclic fibration we can get a commuting diagram.

\[
\begin{array}{c}
B \cup_A B \\
\downarrow \\
Cyl''(B, A) \\
\downarrow \\
Cyl(B, A)
\end{array}
\]

and therefore a homotopy \( H : Cyl(B, A) \to X \) between \( f \) and \( g \). \( \square \)

The following condition is also one required of a cofibration category of Baues ([4]).
**Definition 2.2.12** If for every diagram

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow i & & \downarrow j \\
B & \longrightarrow & D
\end{array}
\]

\(i\) being an acyclic cofibration implies that \(j\) is also then we say that \(C\) satisfies the **pushout** condition.

**Remark:** The pushout condition is always satisfied if \(C\) is a closed model category but in a good LMC it seems that only \(C\) being a retract of \(D\) is necessary.

**Lemma 2.2.13** If for every \(j \in C \cap \mathcal{W}\) in the following pushout

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow j & & \downarrow f \\
B & \longrightarrow & D
\end{array}
\]

\(f \in \mathcal{W}\) then the pushout condition holds.

**Proof:** Easy using the fact that weak equivalences are closed under retracts. \(\square\)

**Lemma 2.2.14** Assume that \(C\) satisfies the pushout condition. Then it is equivalent to define HELP by allowing the cofibration \(\text{Cyl}(i)\) to be any cofibration between any cylinder objects for \(A\) and \(B\).

**Proof:** Let \(i : \text{Cyl}(A, W) \rightarrow \text{Cyl}(B, W)\) be the cylinder objects for which HELP holds. Let \(i : \text{Cyl}'(A, W) \rightarrow \text{Cyl}'(B, W)\) be any map between cylinder objects for \(A\) and \(B\) compatible with the \(\partial_i\) such that

\[
\text{Cyl}'(A, W) \vee_{\mathcal{W}A} A \vee_{\mathcal{W}} B \rightarrow \text{Cyl}'(B, W)
\]

is a cofibration. Denote by \(\text{Cyl}''(A, W)\) the middle space in the factorization of \(\sigma + \sigma : \text{Cyl}(A, W) \vee_{\mathcal{W}A} \text{Cyl}(A, W) \rightarrow A\) into a cofibration followed by an acyclic fibration. Let \(Y\)
denote the pushout

\[
\begin{array}{ccc}
  Cyl(A, W) & \xrightarrow{(\sigma + \sigma)_0} & Cyl''(A, W) \\
  \downarrow & & \downarrow \\
  Cyl(B, W) & \xrightarrow{f} & Y
\end{array}
\]

Note that \( f \) is a weak equivalence by the pushout condition. Let \( Cyl''(B, W) \) denote the middle space in the factorization of the map \( Y \to B \) induced by the above diagram into a cofibration followed by an acyclic fibration. So \( i : Y \to Cyl''(B, W) \) is an acyclic cofibration and thus has a retract \( r : Cyl''(B, W) \to Y \).

Suppose we are given a solid arrow diagram with \( f \) a weak equivalence.

\[
\begin{array}{ccc}
  A & \xrightarrow{} & Cyl''(A, W) & \xleftarrow{} & A \\
  \downarrow & & \downarrow & & \downarrow \\
  Z & \xleftarrow{f} & X & \xleftarrow{} & \quad \\
  B & \xleftarrow{H''} & Cyl''(B, W) & \xrightarrow{h''} & B
\end{array}
\]

This gives us a solid arrow diagram

\[
\begin{array}{ccc}
  A & \xrightarrow{} & Cyl(A, W) & \xleftarrow{} & A \\
  \downarrow & & \downarrow & & \downarrow \\
  Z & \xleftarrow{H} & X & \xleftarrow{} & \quad \\
  B & \xleftarrow{h} & Cyl(B, W) & \xrightarrow{h} & B
\end{array}
\]

Since by assumption HELP holds for these cylinder objects, there exists \( h \) and \( H \). \( H \) together with maps already specified induces a map \( \tilde{H} : Y \to Z \). Let \( h'' = h \) and \( H'' = \tilde{H}r \). These extensions make the first diagram commute. Let \( j \) denote any lift in the solid arrow diagram.

\[
\begin{array}{ccc}
  B \cup_{\partial B} A & \xrightarrow{j} & Cyl''(B, W) \\
  \downarrow & & \downarrow \\
  Cyl'(B, W) & \xrightarrow{} & B
\end{array}
\]
and let $s$ denote a retract $Cyl''(A, W) \to Cyl''(A, W)$ compatible with the $\partial_i$. That is $s\partial_i = \partial_i$. Let a solid arrow diagram be given with $f$ a weak equivalence.

\[
\begin{array}{c}
A \quad Cyl'(A, W) \quad A \\
\downarrow \quad \downarrow \quad \downarrow \\
Z \quad f \quad X \\
\downarrow \quad \downarrow \quad \downarrow \\
B \quad Cyl'(B, W) \quad B
\end{array}
\]

Let the map $F'' : Cyl''(A, W) \to Z$ be given by $F'' = F's$ we get a corresponding solid arrow diagram

\[
\begin{array}{c}
A \quad Cyl''(A, W) \quad A \\
\downarrow \quad \downarrow \quad \downarrow \\
Z \quad f' \quad X \\
\downarrow \quad \downarrow \quad \downarrow \\
B \quad Cyl''(B, W) \quad B
\end{array}
\]

Then as was shown above the extensions $H''$ and $h''$ exist. Defining $h' = h''$ and $H' = H''j$ we can easily see they make the previous diagram commute. $\blacksquare$

**Lemma 2.2.15** Assume that $C$ satisfies the pushout condition. Let $A \to B$ and $h : B \to D$ be cofibrations. Let $f, g : D \to X$ be maps. Then $f \simeq g$ rel $B$ if and only if $f \simeq g$ rel $A$ by a homotopy constant on $B$. That is by a homotopy that restricted to $Cyl(B, A)$ factors through $\sigma : Cyl(B, A) \to B$.

**Proof:** Let $Y$ denote the pushout

\[
\begin{array}{c}
B \quad D \\
\downarrow \quad \downarrow \\
Cyl(B, A) \quad Y
\end{array}
\]

Let $Y'$ denote the pushout

\[
\begin{array}{c}
B \quad D \\
\downarrow \quad \downarrow \\
Y \quad Y'
\end{array}
\]
We have maps \( id : D \to D \) and \( h\sigma : Cyl(B, A) \to D \). So we get a map \( l : Y' \to D \). Let \( Y' \to Cyl(D, A) \to D \) be a factorization of \( l \) into a cofibration followed by an acyclic fibration. Then \( Cyl(D, A) \) is a cylinder object for \( D \) over \( A \) such that \( f : Y \to Cyl(D, A) \) is a cofibration compatible with the maps \( B \vee_A D \to Cyl(D, A) \). \( f \) is an acyclic cofibration so the pushout of

\[
\begin{array}{c}
Y \\
\downarrow \\
D
\end{array} \leftarrow \begin{array}{c}
f \\
\end{array} Cyl(D, A)
\]

is a cylinder object for \( D \) over \( B \). The statement of the Lemma follows easily. \( \square \)

**Lemma 2.2.16** Let \( C \) be good and \( i : A \to B, j : W \to X \) and \( k : Z \to Y \) any maps. Then we have a map induced by composition.

\[
\phi : \pi((B, A), (X, W)) \times \pi((X, W), (Y, Z)) \to \pi((B, A), (Y, Z))
\]

**Proof:** Let \( f, h : B \to X \) and \( g, l : X \to Y \) be maps such that \( f \simeq h \rel A \) and \( g \simeq l \rel W \). Then to prove the lemma it is sufficient to show that \( g \circ f \simeq g \circ h \rel A \) and \( l \circ f \simeq g \circ f \rel A \). The first statement is clear since if \( H \) is a homotopy from \( f \) to \( h \) then \( g \circ H \) is a homotopy from \( g \circ f \) to \( g \circ h \). To prove the second statement let \( H : Cyl(X, W) \to Y \) be a homotopy from \( g \) to \( l \). By Lemma 2.2.11 we can assume that \( \sigma : Cyl(X, W) \to X \) is an acyclic fibration. Then in the following diagram

\[
\begin{array}{c}
B \vee_A B \\
\downarrow \\
Cyl(B, A)
\end{array} \xrightarrow{f + \sigma f} \begin{array}{c}
X \vee_A X \\
\downarrow \\
Cyl(X, W)
\end{array} \xrightarrow{\epsilon} \begin{array}{c}
X \\
\downarrow \\
B
\end{array} \xrightarrow{f} X
\]

we have a lifting \( \phi : Cyl(B, A) \to Cyl(X, W) \) and so \( H \circ \phi \) is the desired homotopy from \( l \circ f \) to \( g \circ f \). \( \square \)

Since \( \phi \) is induced by composition it has nice properties that hold for composition such as associativity. That is \( \phi(\phi(f, g), h) = \phi(f, \phi(g, h)) \).
Definition 2.2.17 Let $C$ be good. Define $\pi(C)$ to be the following category: $\text{obj}(\pi(C)) = \text{obj}(C)$ and $\text{Hom}_{\pi(C)}(A,B) = \text{Hom}_C(A,B)/\simeq$. Note that composition is well defined by Lemma 2.2.16. Observe that $\text{Hom}_{\pi(C)}(A,B)$ is $\pi(A,B)$. We denote the functor $C \to \pi(C)$ by $\pi$.

2.3 Homotopy Categories

The proofs of the results in this section are easier with our set up than in a general closed model category since with goodness we can look at cofibrant objects instead of having to use cofibrant fibrant ones.

Definition 2.3.1 Let $C$ be a category and $S$ a class of maps in $C$. We call a category $S^{-1}C$ together with a functor $\Phi : C \to S^{-1}C$ the localization of $C$ with respect to $S$ if the following two conditions are satisfied.

1) If $f \in S$ then $\Phi(f)$ is an isomorphism.

2) If $F : C \to D$ is any functor such that for every $f \in S$ $F(f)$ is an isomorphism then there exists a unique extension $F' : S^{-1}C \to D$ such that the following diagram commutes.

\[
\begin{array}{ccc}
C & \xrightarrow{\Phi} & S^{-1}C \\
& \downarrow F^\prime & \\
& D & \\
& \downarrow F & \\
& B & \xrightarrow{g} Y
\end{array}
\]

Observe that the universal definition of $S^{-1}C$ makes it unique up to isomorphism.

Lemma 2.3.2 Let $C$ be a LMC. Let $h, h' \in \text{Hom}_C$ be extension liftings in the following diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
& \downarrow f & \\
B & \xrightarrow{g} & Y
\end{array}
\]

where $f$ is an acyclic fibration. Then $h \simeq h' \text{ rel} A.$
Proof: The homotopy can be taken to be any extension lift in the diagram

\[
\begin{array}{ccc}
B \vee_A B & \xrightarrow{h+Ah'} & X \\
\downarrow & & \downarrow f \\
Cyl(B, A) & \xrightarrow{go\sigma} & Y
\end{array}
\]

\[\square\]

**Lemma 2.3.3** In a good LMC let us be given a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow g & & \downarrow h \\
C & \xrightarrow{h} & B
\end{array}
\]

such that \(g\) and \(f\) are cofibrations and \(h\) is a weak equivalence. Then \(B \simeq C \text{ rel } A\).

Proof: Let \(B \vee_A C \rightarrow W \rightarrow B\) be a factorization of \(id + h\) into a cofibration followed by a weak equivalence. Then the maps \(B \rightarrow W\) and \(C \rightarrow W\) are both acyclic cofibrations. So by goodness and Lemma 2.2.15 we are done. \[\square\]

**Corollary 2.3.4** In a good LMC any weak equivalence between cofibrant objects is a homotopy equivalence.

**Definition 2.3.5** Let \(C\) be a good LMC. We define a functor \(Co : C \rightarrow \pi(C)\) and natural transformation \(\eta : Co \rightarrow \pi\) as follows. First if \(X\) is cofibrant let \(Co(X) = X\) and \(\eta_X = id\). Now for other \(X\) choose a factorization of \(\emptyset \rightarrow X\) as a cofibration followed by an acyclic fibration.

\[
\begin{array}{ccc}
\emptyset & \rightarrow & Co(X) \\
\eta_X & \downarrow & \gamma \\
& & X
\end{array}
\]
Let $f : X \rightarrow Y$ be a map then define $Co(f)$ to be the lift, unique in $\pi(C)$ by Lemma 2.3.2, in the following diagram.

\[
\begin{array}{c}
\emptyset \\
\downarrow \\
Co(X) \xrightarrow{fn_x} \rightarrow Y
\end{array}
\]

We call $Co(X)$ a cofibrant approximation for $X$.

We are now ready to define the homotopy category.

**Definition 2.3.6** Let $C$ be a good LMC. Define the homotopy category of $C$, $Ho(C)$, as follows.

\[
\text{obj}(Ho(C)) = \text{obj}(C)
\]

\[
\text{Hom}_{Ho(C)}(A, B) = \pi(Co(A), Co(B))
\]

$\text{Hom}_{Ho(C)}(A, B)$ is also denoted by $[A, B]$. Now define a functor $Ho : C \rightarrow Ho(C)$ by $Ho(X) = X$ and $Ho(f) = Co(f)$

**Remark:** To make the last two definitions we do not actually require goodness but we will see in Lemma 2.3.8 that it will imply that $Ho(C)$ is independent of the choices involved in defining the functor $Co$.

**Lemma 2.3.7** Let $C$ be a LMC and $F : C \rightarrow D$ be a functor such that if $f$ is a weak equivalence then $F(f)$ is an isomorphism. Then for every $f, g \in \text{Hom}_C(X, Y)$ such that $f \simeq g$, $F(f) = F(g)$.

**Proof:** Consider any cylinder object for $X$. We have a factorization of $\nabla$

\[
X \vee X \xrightarrow{\partial_0 + \partial_1} Cyl(X) \xrightarrow{\sigma} X
\]

We have that $\sigma \partial_0 = id = \sigma \partial_1$ but since $\sigma$ is a weak equivalence we get that $F(\partial_0) = (F(\sigma))^{-1} = F(\partial_1)$. For some cylinder object there is a map $H : Cyl(X) \rightarrow Y$ such that $\partial_0 H = f$ and $\partial_1 H = g$. This then implies that $F(f) = F(g)$. $\square$
Lemma 2.3.8 For $C$ a good LMC, $\text{Ho}(C)$ is the localization of $C$ with respect to the weak equivalences.

Proof We verify the two required properties.

1) Let $f \in C$ be a weak equivalence. We must show that $\text{Ho}(f)$ is an isomorphism. $\text{Co}(f)$ is a weak equivalence since $\eta_X$, $\eta_Y$ and $f$ are. Therefore by Lemma 2.3.4 $\text{Co}(f)$ is an isomorphism in $\text{Ho}(C)$.

2) Let $F : C \to D$ be a functor such that for every weak equivalence $f$ in $C$, $F(f)$ is an isomorphism. Then we wish to construct an extension

\[ C \xrightarrow{\text{Ho}} \text{Ho}(C) \xrightarrow{F} D. \]

To this end we define for $X \in \text{Ho}(C)$

\[ F'(X) = F(X) \]

and for $[f] \in [X, Y] = \pi(\text{Co}(X), \text{Co}(Y))$

\[ F'([f]) = F(\eta_Y) \circ F(f) \circ (F(\eta_X))^{-1} \]

This is indeed a definition since $f \simeq f'$ implies that $F(f) = F(f')$. Let $f : X \to Y$ be any map. All that remains to be shown is that $F(f) = F'(\text{Ho}(f))$.

\[ F'(\text{Ho}(f)) = F(\eta_X) \circ F(\text{Co}(f)) \circ (F(\eta_X)^{-1}) \]

and by the definition of $\text{Co}$

\[ f \circ \eta_X \simeq \eta_Y \circ \text{Co}(f) \]

so we are done. $\square$
2.4 Cones, Suspensions and Coactions

For this section assume $C$ is a good LMC. To do the contents of this section for a general model category Quillen ([22]) makes use of correspondences. The proofs in our setup are somewhat easier.

**Definition 2.4.1** A $X \in C$ together with a map $d : X \to \emptyset$ is called a **copointed space**. 

$d$ is called the **copointing**. A map between copointed spaces is any map compatible with the copointing. If $X$ is copointed then $CylX$ is naturally copointed as are the maps $\partial_i$ and $\sigma$.

We denote the map $X \xrightarrow{d} \emptyset \longrightarrow Y$ by $\emptyset$.

For a map $f : X \to Y$ of copointed spaces we let $Y/f(X)$ denote the pushout of

$$
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array}
\longrightarrow
\begin{array}{c}
\emptyset
\end{array}
\longrightarrow
Y
$$

with the natural copointing. If $f$ is clear from the context we will also denote it by $Y/X$.

For self maps $g : Y \to Y$ and $g' : X \to X$ such that $gf = fg'$ we denote the induced map $Y/X \to Y/X$ by $g/X$.

In a pointed category all spaces and maps are copointed.

**Definition 2.4.2** Let $X$ be a copointed space. We denote by $CX$ any copointed space such that $CX \to \emptyset$ is a weak equivalence and there exists a cofibration that is a map of copointed spaces $l_X : X \to CX$. We call $CX$ a **cone** on $X$. We call $l_X$ the inclusion of $X$ into its cone and will denote it as $l$ when it is clear which $X$ we are including.

Let $f : X \to Y$ be any map. We denote by $Cf$ any lift in the solid arrow diagram

$$
\begin{array}{c}
X \\
\downarrow l \\
CX
\end{array}
\longrightarrow
\begin{array}{c}
CY \\
\downarrow Cf \\
\emptyset
\end{array}
$$
We call \( C f \) a **cone** on \( f \). Denote \( CX/l(X) \) by \( \Sigma X \) and call it a **suspension** of \( X \). We denote the map induced by \( C f \) from \( \Sigma X \) to \( \Sigma Y \) by \( \Sigma f \) and call it the **suspension** of \( f \).

We will use \( Y \cup f CX \) to denote the push out of

\[
\begin{array}{c}
X \\
\downarrow \\
CX
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
Y
\end{array}
\]

If \( X \) is cofibrant then \( CylX/\partial_0(X) \) is an example of a cone on \( X \).

Next we show the uniqueness up to homotopy of cones and suspensions.

**Lemma 2.4.3** Let \( CX \) and \( C'X \) be two cones on \( X \). Then \( CX \simeq C'X \) rel \( X \). Let \( f : X \to Y \) be a map and let \( C f \) and \( C' f \) be two cones on \( f \). Then \( C f \simeq C' f \) rel \( X \).

The following proof of the lemma is quite pretty.

**Proof:** Let \( CX \) and \( C'X \) be two cones on \( X \). Let \( CX \cup_X C'X \to C"X \) be a cofibration such that \( C"X \to \emptyset \) is a weak equivalence. Then \( CX \to C"X \) is an acyclic cofibration so \( CX \simeq C"X \) rel \( CX \). Therefore \( CX \simeq C"X \) rel \( X \). Similarly \( C'X \simeq C"X \) rel \( X \). So \( C'X \simeq CX \) rel \( X \) and the first statement has been verified. For the second statement first assume that \( CY \to \emptyset \) is a fibration. Then the result follows from 3.2.7. Next let \( C'Y \) be any cone of \( Y \), \( f : X \to Y \) map and let \( C f \), \( C' f \) both denote two cones on \( f \) and also denote their classes in \( \pi((CX,X),(C'Y,Y)) \). Let \( \Theta \in \pi((C'Y,Y),(CY,Y)) \) be the homotopy class of the homotopy equivalence given by the first half of the lemma and let \( \Theta^{-1} \in \pi((CY,Y),(C'Y,Y)) \) be its inverse. We know that \( \Theta C f = \Theta C' f \) so we get \( C f = \Theta^{-1} \Theta C f = \Theta^{-1} \Theta C' f = C' f \). \( \square \)

**Corollary 2.4.4** Let \( \Sigma X \) and \( \Sigma'X \) be two suspensions of \( X \) then \( \Sigma X \simeq \Sigma'X \). Let \( f : X \to Y \) be a map and let \( \Sigma f \) and \( \Sigma' f \) be two suspensions of \( f \) then \( \Sigma f \simeq \Sigma f' \).
Definition 2.4.5 Let $f$ denote any extension lift in the following diagram.

\[
\begin{array}{ccc}
X \vee X & \rightarrow & CylX \\
\downarrow & & \downarrow \sigma \\
CylX \vee X & \xrightarrow{\sigma + \text{id}_{\sigma}} & X.
\end{array}
\]

Since $f$ is a weak equivalence by goodness we get a map $f^{-1} : CylX \rightarrow CylX \vee X$ by Lemma 2.3.3 such that

\[
\begin{array}{ccc}
X \vee X & \rightarrow & CylX \\
CylX \vee X & \xrightarrow{f^{-1}} & Cyl
\end{array}
\]

commutes. $f^{-1}$ induces a map

\[CylX/\partial_1(X) \rightarrow CylX \vee X CylX/\partial_1(X) \rightarrow CylX/\partial_1(X) \vee CylX/\partial_0(X) \vee \partial_1(X)\]

Identifying $CylX/\partial_1(X)$ as $CX$ and $CylX/\partial_0(X) \vee \partial_1(X)$. Let $\nabla : CX \rightarrow CX \vee \Sigma X$ denote this map.

Lemma 2.4.6 Let $p : CX \rightarrow \Sigma X$, $p_1 : CX \vee \Sigma X \rightarrow CX$ and $p_2 : CX \vee \Sigma X \rightarrow \Sigma X$ denote the projections. Then $\nabla$ has the following properties.

1) $p_1 \nabla \simeq \text{id rel } X$

2) $p_2 \nabla \simeq p \text{ rel } X$

Proof: Easy. □

Definition 2.4.7 Let $T : W \vee W \rightarrow W \vee W$ be the transpose map. Also let it denote any lift in the following diagram

\[
\begin{array}{ccc}
W \vee W & \xrightarrow{(\partial_0 + \partial_1)T} & CylW \\
\downarrow\sigma_0 + \partial_1 & & \downarrow \sigma \\
CylW & \xrightarrow{\sigma} & W
\end{array}
\]
Let $g : \Sigma W \to X$ be a map. Let $-g : \Sigma W \to X$ denote the map $g \circ (T/(W \lor W))$. Let $H : CylW \to X$ be a homotopy. Let $-H : CylW \to X$ denote the map $HT$. Also we denote the cylinder object

$$(\partial_0 + \partial_1)T : W \lor W \to CylW$$

by $-CylW$.

**Lemma 2.4.8** Assume the pushout condition holds in $C$. Let $X$ be cofibrant and $H : CylX \to Y$ any map. Then $H + -H \simeq id rel(X \lor X)$.

**Proof:** Let $Cyl'X$ denote $CylX \lor_X -CylX$. In other words the following diagram, which also defines the map $b$, is a pushout

$$
\begin{array}{ccc}
X \lor X & \xrightarrow{\partial_1 \lor \partial_1} & CylX \lor CylX \\
\downarrow & & \downarrow b \\
X & \longrightarrow & Cyl'X
\end{array}
$$

Let the following pushout define an object $Z$ and maps $s_0$ and $s_1$.

$$
\begin{array}{ccc}
X \lor X \lor X \lor X & \longrightarrow & Cyl'X \lor Cyl'X \\
\downarrow & & \downarrow s_0 + s_1 \\
CylX \lor CylX & \longrightarrow & Z
\end{array}
$$

There is a map $f : Z \to CylX$ induced by $(\partial_0 + \partial_1)(\sigma + \sigma)$ on $Cyl'X \lor Cyl'X$ and $id + id$ on $CylX \lor CylX$. Let $Cyl'(CylX)$ be the middle space in the factorization of $f$ into a cofibration followed by an acyclic fibration. Then $Cyl'(CylX)$ is a cylinder object for $CylX$.

The map

$$s_1 : Cyl'(X) \to Z$$
is a cofibration since $X$, and therefore $Cyl'X$, is cofibrant. Let $W$ be the pushout of

$$
\begin{array}{c}
Cyl'X \\
\downarrow^\sigma
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
Cyl'(CylX)
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
X.
\end{array}
$$

Then since $Cyl'(\partial_1(X))$ is an acyclic cofibration $X \to W$ is also by the pushout condition.

Let $Y = Cyl'X \vee_{X \vee X} Cyl'X$. Let $r_1 : Cyl'X \to Y$ be the inclusion into the two factors. A map $w : Z \to Y$ is determined by $r_0 + \partial_1 \sigma$ on $Cyl'X \vee Cyl'X$ and $r_1 b$ on $CylX \vee CylX$. Let $Q$ denote the pushout of

$$
\begin{array}{c}
Cyl'X \\
\downarrow
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
Z
\end{array}
$$

A map $Q \to Y$ is determined by $w$ on $Z$ and $\partial_1$ on $X$.

Next we construct a map $Y \to Q$. On the first $Cyl'X$ map using $s_0$ into $Z$ and then include into $Q$. On the second factor use the map induced by mapping $CylX \vee CylX$ to $Z$ by $\partial_0 + \partial_1$ and then including into $Q$. Easily we see $Q \cong Y$. So we get a pushout

$$
\begin{array}{c}
Z \\
\downarrow
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
Cyl'(CylX)
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
Y
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
W
\end{array}
$$

so $Y \to W$ is a cofibration and $W$ is a $Cyl(Cyl'X, X \vee X)$. Next let $H : CylX \to Y$ be any homotopy from $f$ to $g$. Note that $H \sigma : Cyl'CylX \to CylX \to Y$ is the identity homotopy. Then $H \sigma$ extends over $W$ and gives a homotopy $H + -H \simeq id \ rel(X \vee X)$. \qed

**Lemma 2.4.9** Assume that $C$ satisfies the pushout condition. Let $X$ be cofibrant. Let $f, g : X \to Y$ be maps such that $f \simeq g$. Then $Y \cup_f CX \simeq Y \cup_g CX \ rel Y$. 


**Proof:** Let $CylX \cup_X CX$ be the pushout of

\[
\begin{CD}
X @>>> CX \\
@VV \partial_1 V \\
CylX.
\end{CD}
\]

It is homotopy equivalent to $\emptyset$ and therefore a cone on $X$. So be Lemma 2.4.3 we have a map $\theta : CX \rightarrow Cyl X \cup_X CX$ such that the following diagram commutes.

\[
\begin{CD}
X @>>> CX \\
@VV \partial_0 V \downarrow \theta \\
CylX \cup_X CX
\end{CD}
\]

Let $H : f \simeq g$ be any homotopy then we have maps

\[
\phi : (id + H + id)(id + \theta) : Y \cup_f CX \rightarrow Y \cup_g CX
\]

and

\[
\phi^{-1} : (id + (-H) + id)(id + \theta) : Y \cup_f CX \rightarrow Y \cup_f CX
\]

Now

\[
\begin{CD}
CX @> (id + \theta) \phi >> CylX \cup CylX \cup CX \\
@VV id V \downarrow \sigma + \sigma + id \\
CX
\end{CD}
\]

commutes up to homotopy by Lemma 2.4.3. So we see using 2.4.8 and 2.2.15 that $\phi^{-1} \phi \simeq id$ and $\phi \phi^{-1} \simeq id$. $\square$

**Lemma 2.4.10** Assume that HELP and the pushout condition hold in $C$. Then if we have a pushout diagram

\[
\begin{CD}
A @>>> X @> f > Z @. \\
@VV V \downarrow \partial_1 \downarrow h \\
CA @>>> \partial_i Y
\end{CD}
\]
such that \(i\) has a retract and \(f\) is a weak equivalence, then \(Y \simeq X \lor \Sigma A \rel X\) and \(h\) can be extended to a weak equivalence \(h' : Y \lor \Sigma A \rightarrow Z\).

**Proof:** The first statement follows from 2.4.9. \(\pi(\Sigma A, f)\) is an isomorphism by Lemma 2.2.10. So the second statement follows by adjusting the homotopy equivalence \(Y \simeq \Sigma A\) so that \(h\) restricted to \(\Sigma A \lor X\) is homotopic to the trivial map. Then extension is then easy to construct and clearly a homotopy equivalence. \(\square\)

**Lemma 2.4.11** Assume that the pushout condition holds. Let \(j : W \rightarrow X\) be a cofibration, \(T\) cofibrant and let

\[
\begin{array}{ccc}
T & \xrightarrow{j} & X \\
\downarrow & & \downarrow \\
CT & \longrightarrow & Y
\end{array}
\]

be a pushout. Then the push out of

\[
\begin{array}{ccc}
CylT & \xrightarrow{Cylj} & Cyl(X, W) \\
\downarrow & & \downarrow^{Cylt} \\
CylCT & \longrightarrow & Cyl(Y, W)
\end{array}
\]

is a \(Cyl(Y, W)\)

**Proof:** Let the following diagram be a pushout

\[
\begin{array}{ccc}
CylT & \xrightarrow{Cylj} & Cyl(X, W) \\
\downarrow & & \downarrow \\
CylCT & \longrightarrow & Z
\end{array}
\]

In the push out

\[
\begin{array}{ccc}
X & \xrightarrow{\partial_0} & Cyl(X, W) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Cyl(X, W) \cup CT
\end{array}
\]
both horizontal maps are acyclic cofibrations. The same is true in the pushout.

\[
\begin{array}{c}
CylT \cup CT \xrightarrow{Cylt+\partial_0} CylCT \\
\downarrow \downarrow \\
Cyl(X,W) \cup CT \xrightarrow{} Z.
\end{array}
\]

Therefore \( Z \to Y \) is a weak equivalence. \( \square \)

**Lemma 2.4.12** Let \( A \) be a copointed space. Then in the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{H} \\
CA & & CW
\end{array}
\]

there exists a \( g \) if and only if \( f \simeq 0 \).

**Proof:** Easy. \( \square \)

**Lemma 2.4.13** Let \( W \) be cofibrant. Let \( f', g' : X \to Y \) be maps and \( f, g : X \cup_h CW \to Y \) be extensions of these maps. Let \( H' : f' \simeq g' \) be a homotopy. Let \( f - g \in [\Sigma W, Y] \) be the class of the map

\[
f|_{CW} + H' \circ Cylh + g|_{CW} : CW \vee W CylW \vee W CW \to Y
\]

Then \( f - g = * \) if and only if \( H' \) extends to a homotopy \( H : f \simeq g \)

**Proof:** To say \( H \) extends \( H' \) and that it's a homotopy from \( f \) to \( g \) is to say that \( H \) is an extension in the following diagram

\[
\begin{array}{c}
CW \vee W CylW \vee W CW \xrightarrow{f+H'Cylh+g} Y \\
\downarrow{H} \\
CylCW
\end{array}
\]
but since we have a pushout, the top map being a homotopy equivalence and not just a weak equivalence, since the right hand side is a suspension for $W$

\[
\begin{array}{ccc}
\Sigma W & \rightarrow & CW \vee_w CyW \vee_w CW \\
\downarrow & & \downarrow \\
C\Sigma W & \rightarrow & CyCW
\end{array}
\]

we are done. □
Chapter 3

Suitable Cell Categories

In this chapter we introduce some general conditions under which a set of maps determine a closed model category structure. We call such categories suitable cell categories. A cell category is a cocomplete category together with a set of maps. It is suitable if it is \( \kappa \) small (3.1.4) for some cardinal \( \kappa \), copointed (3.1.1), good (2.2.5) and satisfies the pushout condition (2.2.12). We then give examples of suitable cell categories. Section 1 defines cell categories and \( \kappa \) smallness. Section 2 defines homotopies in a \( \kappa \) small cell category and shows that such categories are LMC. Section 3 shows that suitable cell categories are naturally closed model categories. Section 4 shows that a number of standard model categories, and one unusual one, come from suitable cell categories. We then introduce the interval and show that categories with one satisfy HELP and the pushout condition. In section 5 we see that for any suitable cell category \( C \), \( C^D \), the category of diagrams from \( D \) to \( C \) is a suitable cell category.

3.1 Cell Categories

Definition 3.1.1 Let \( C \) be a cocomplete category and \( \{ \alpha : A(\alpha) \to B(\alpha') \}_{\alpha \in I} \) a set of maps in \( C \) such that for every \( \alpha \in I \) there exists \( \beta \in I \) such that \( A(\beta) = 0 \) and \( B(\beta) = A(\alpha) \). Then we call \( (C, I) \) a cell category and \( I \) the set of cells.
If for every $\alpha$, $B(\alpha)$ is copointed and $\alpha$ is a copointed map then we call $C$ copointed.

An example is $(\text{Top}, S)$ where $S = \{\emptyset \to \emptyset\} \cup \bigcup_{n \in \mathbb{Z}^+} \{S^n \to D^{n+1}\} \cup \{\emptyset \to S^n\}$

**Definition 3.1.2** Let $(C, I)$ be a cell category. We define a (relative) cell complex

$$(\kappa, \{X(i)\}_{i \leq \kappa}, \{K(i)\}_{i \leq \kappa}, \alpha, \{f(r)\}_{r \in \mathcal{K}(i)})$$

to be an ordinal $\kappa$, spaces $X(n)$, ordered sets $K(n)$, a map $\alpha : \bigcup K(n) \to I$ and maps for every $r \in K(n)$ $f(r) : A(\alpha(r)) \to X(n)$, such that the following diagram is a pushout

$$
\begin{array}{ccc}
V_{r \in K(n)} A(\alpha(r)) & \xrightarrow{V_{r \in K(n)} f(r)} & X(n) \\
\downarrow \alpha(r) & & \downarrow \\
V_{r \in K(n)} B(\alpha(r)) & \rightarrow & X(n + 1)
\end{array}
$$

and if $n \leq \kappa$ is a limit ordinal then

$$X(n) = \text{colim}_{m < n} X(m)$$

Let $X = X(0)$ and $Y = X(\kappa)$. We will also denote the (relative) cell complex by $j : X \to Y$ and leaving the other structure implicit. $\{X(i)\}_{i \leq \kappa}$ is called a decomposition of $i$ and $\kappa$ is called the length of the decomposition.

Let $(\kappa', \{X'(i)\}, \{K'(i)\}, \alpha', \{f'(r)\})$ be another cell complex such that $X(0) = X$. If for every $i < \kappa'$ we have $K'(i) \subset K(i)$, $\alpha' = \alpha|_{K'(i)}$ for every $i$ and for every $i$ a map $j(i) : X'(i) \to X(i)$ such that $j(i)f'(r) = f(r)$ for every $r \in K'(i)$ then $X \to X'(\kappa')$ is called a (relative) sub(cell) complex of $X \to Y$ and the canonical map $j : X'(\kappa') \to Y$ is called the inclusion of a (relative) sub(cell) complex.

Let $C(I)$ denote the class of maps that can be given the structure of a relative cell complexes. Let $K(Y)$ denote $\bigcup_{i \leq \kappa} K(i)$. $K(Y)$ is called the set of cells in $Y$ and $K(i)$ the set of cells of filtration $i$. 
A relative cell complex is just any object which has had "cells" attached to it in \( \kappa \) or fewer steps. For example in \((\text{Top}, S) \emptyset \to D^n\) is a cell complex and \( \emptyset \to S^{n-1}\) is a subcomplex of it. \( S^{n-1} \to D^n\) is a relative cell complex. In fact in any cell category if \( W \to X \) is a complex and \( W \to Y \) is a subcomplex then there exist a complex \( Y \to X \).

**Definition 3.1.3** Let \( i : X \to X(\kappa) \) be a cell complex. We define

\[
s(i) = |K(X(\kappa))|
\]

We call \( s(i) \) the size of \( i \).

The size of a cell complex is just the number of cells in it.

**Definition 3.1.4** Let \((C, I)\) be a cell category. We say that \( A \in C \) is \( \kappa \) small if for every \( i : X \to Y \in C(I) \) and every map \( f : A \to Y \) there exists a subcomplex of \( Y \), \( i' : X \to X' \) with \( s(i') < \kappa \) and a factorization \( A \to X' \to Y \). We say that \((C, I)\) is weakly \( \kappa \) small if for every \( \alpha \in I \) \( A(\alpha) \) is \( \kappa \) small. We say that \((C, I)\) is \( \kappa \) small for for every \( \alpha \in I \) \( B(\alpha) \) is \( \kappa \) small.

**Remark:** Clearly \( \kappa \) small implies weakly \( \kappa \) small.

In \( DGM(R) \) (3.4.1) the relative cell complexes are just the semi-free extensions. In commutative differential graded algebras over a (commutative) ring \( R \), \( CDGA(R) \), we can take

\[
I = \{ j_n : \Lambda(a_n) \to \Lambda(a_n, b_{n-1} : db = a) \} \cup \{ j'_n : 0 \to \lambda(a_n) \}
\]

where \( \Lambda(a(1), ..., a(n)) \) is the free graded commutative algebra on the \( a(i) \) and the subscript denotes the dimension. It is easy to see that both of these categories are \( \omega \) small.

The next definition is some variant of Quillen's small object argument. \( G(n) \) is an infinite gluing construction.
**Definition 3.1.5** Let $C$ be any category and $F$ a class of maps in $C$. Fix a map $f : X \to Y$ and define

$$S(f, F) = \{g, h : g \in F \text{ and } h \circ g = f\}$$

Let $X(g)$ denote the range of $g$ and if $S(f, F)$ is a set

$$G(f, F) = \text{colim}_{(g, h) \in S(f, F)} X(g)$$

where the only nontrivial maps in the system are $g : X \to X(g)$, one for every $(g, h) \in S(f, F)$. Also there are canonical maps

$$X \xrightarrow{i'(f)} G(f, F) \xrightarrow{p'(f)} Y$$

Define

$$i(1) = i'(f)$$

$$p(1) = p'(f)$$

$$G(1) = G(f, F)$$

$$G(n + 1) = G(p(n), F)$$

$$p(n + 1) = p'(p(n))$$

$$i(n + 1) = i'(p(n)) \circ i(n)$$

or if $n$ is a limit ordinal

$$G(n) = \text{colim}_{m < n} G(m)$$

$$p(n) = \text{colim}_{m < n} p(m)$$

and let $i(n) : X \to G(n)$ be the canonical map.
3.2 Weakly κ small cell categories

For this section assume that (C, I) is a weakly κ small cell category with κ a regular cardinal.

Definition 3.2.1 Let F be the class of all f ∈ C(I) such that s(f) = 1 and such that the length of f is 1. Let f : X → Y be any map. We then define

\[ CW(f) = G(\kappa)(f) \]
\[ CW(i)(f) = G(i)(f) \]
\[ p(f) = p(\kappa)(f) \]
\[ i(f) = i(\kappa)(f) \]

Lemma 3.2.2 For any f, \( i(f) \in C(I) \)

Proof: Clear. □

Lemma 3.2.3 Any solid arrow diagram of the following form has an extension lift

\[
\begin{array}{c}
A(\alpha) \xrightarrow{g} CW(f) \\
\downarrow^{a} & \downarrow^{p(f)} \\
B(\alpha) \xrightarrow{} Y
\end{array}
\]

Proof: \( i(f) \rightarrow CW(f) \in C(I) \) so there exists a factorization

\[
\begin{array}{c}
A(\alpha) \xrightarrow{g'} X' \\
\downarrow^{g} & \downarrow \\
CW(f) \xrightarrow{}
\end{array}
\]

such that \( X' \) is a subcomplex of \( CW(f) \) over \( X \) of size less than \( \kappa \). Each cell of \( X' \) must be in \( CW(\delta) \) for some \( \delta < \kappa \). Since \( \kappa \) is regular and \( X' \) is of size less than \( \kappa \) all of its cells must
be in $CW(\kappa')(f)$ for some $\kappa' < \kappa$. So we get a factorization

$$
A(\alpha) \xrightarrow{h} CW(\kappa') \xrightarrow{g} CW(f)
$$

Therefore it follows that we get a commutative diagram

$$
A(\alpha) \longrightarrow CW(\kappa')(f) \\
\downarrow \\
B(\alpha) \longrightarrow CW(\kappa' + 1)(f)
$$

and we are done. \(\Box\)

**Lemma 3.2.4** For every map $f$, every $i \in C(I)$ has the RLP with respect to $p(f)$.

**Proof:** First prove for $i$ such that $s(i) = 1$ using 3.2.3 and extend by taking colimits. \(\Box\)

**Definition 3.2.5** For every $X \in C$ we have a factorization of $id + id$

$$X \lor X \to Cyl(X) \to X$$

where we have denoted $CW(id + id)$ by $Cyl(X)$. Let $f, g : X \to Y$ be maps. If there exists a commutative diagram

$$
\begin{array}{ccc}
X \lor X & \xrightarrow{f+g} & Y \\
\downarrow & & \downarrow H \\
Cyl(X) & & 
\end{array}
$$

then we say that $f \sim g$ and that $H$ is a homotopy from $f$ to $g$. Let $\simeq$ be the equivalence relation generated by $\sim$. Denote $\text{Hom}(X, Y)/\simeq$ by $\pi(X, Y)$.

We will justify our notation by showing shortly that in a good category the above definition is compatible with the one in Chapter 2. In the present situation we are just picking a particular cylinder object for each $X$.
For a differential graded object $X$ let $Z_n(X)$ denote the cycles in degree $n$. Then in $DGM(R)$, $\text{Hom}(R(<a_n>, X) = Z_n(X)$ and $\pi_n = H_n$. In $CDGA(R)$ again $\text{Hom}(\Lambda(a_n), X) = Z_n(X)$. Let $\pi_n$ be $\pi_{j_n}$. Then the above equality extends to a Hurewitz homomorphism $h_n : \pi_n(X) \to H_n(X)$. $h_n$ is not necessarily an isomorphism.

For example let $R = \mathbb{Z}/p$. Then the two maps $\Lambda(a) \to \Lambda(a_0, a_1, b : db = a_0 - a_1)$ sending $a \mapsto a_i$ correspond to homologous cycles but are not homotopic maps if $|b|$ is even.

**Lemma 3.2.6** Fix $X \in C$ then $\pi(X, -) : C \to \text{Sets}$ is a functor.

**Proof:** Let $f, g : X \to Y$ and $h : Y \to Z$ be maps. If $H$ is a homotopy from $f$ to $g$ then $h \circ H$ is a homotopy from $h \circ f$ to $h \circ g$. □

**Lemma 3.2.7** Let $f : X \to Y$ be a map and $h, g : Z \to CW(f)$ be maps. Then $h \simeq g$ if and only if $p(f) \circ h \simeq p(f) \circ g$.

**Proof:** One direction is just the last lemma and the other follows from the fact that every map in $C(I)$, in particular $i(\nabla)$, has the RLP with respect to $p(f)$. □

**Definition 3.2.8** Let $\pi_\alpha(X) = \pi(\Lambda(\alpha), X)$ Let the weak equivalences, $W$, be the maps that induce a set isomorphism in $\pi_\alpha$ for every $\alpha$.

Our weak equivalences are a generalization of the usual weak equivalences in $\text{Top}_\alpha$ which are maps that induce isomorphisms on homotopy groups.

**Lemma 3.2.9** $W$ is closed under retracts and composition.

**Proof:** Follows from the functoriality of $\pi_\alpha$ and the corresponding fact for sets. □

**Lemma 3.2.10** For every $f$, $p(f)$ is a weak equivalence.

**Proof:** Let $f : X \to Y$ be a map. $p(1)$ being a $\pi_\alpha$ surjection implies that $p(f)$ is one. Let $g_i : A(\alpha) \to CW(f)$ be maps such that $p(f) \circ g_1 \simeq p(f) \circ g_2$. Then lemma 3.2.7 implies that $g_1 \simeq g_2$. So $p(f)$ is injective also. □
Lemma 3.2.11 Let $F$ be the class of all maps that are compositions of maps of the form $p(f)$ for some $f$. Then $(C, C(I), W, F)$ is a LMC.

Proof: $CW(f)$ gives the required factorization and lemma 3.2.4 above gives the required lifting. □

Lemma 3.2.12 Let $(C, I)$ be good. Then the two notions of homotopy 2.2.3 and 3.2.8 are the same.

Proof: $CW(id + id)$ is a cylinder object for $X$ so we just apply lemma 2.2.11 with $A = \emptyset$. □

Definition 3.2.13 Let $(C, I)$ be a cell category. We call it $\kappa$ acyclic small if for every map in $C(I) \cap W$, there exists a subcomplex $i \in C(I) \cap W$ such that $1 \leq s(i) < \kappa$.

We call $(C, I)$ strongly $\kappa$ acyclic small if for every map in $C(I) \cap W$ and every one of its subcomplexes, $i'$, of size less than $\kappa$, there exists a subcomplex $i \in C(I) \cap W$ containing $i'$ and such that $s(i) < \kappa$.

Recall that “good” is defined in 2.2.5

Definition 3.2.14 Let $(C, I)$ be a finite complete, copointed, for some $\kappa$, $\kappa$ small cell category that satisfies HELP 2.2.6 and the pushout condition 2.2.12. We call such a $(C, I)$ a suitable cell category.

The next theorem will be proved in the next section.

Theorem 3.2.15 A suitable cell category can be given the structure of a closed model category.

We actually prove the theorem for weakly suitable cell categories and observe that suitable cell categories are weakly suitable.
**Definition 3.2.16** We say that \( C \) satisfies the direct limit condition if for every direct system \( \{X(\alpha)\}_{\alpha \in \Gamma} \) such that for every \( \alpha \in \Gamma, X(\alpha) \to X(\alpha + 1) \in \mathcal{C}(I) \cap \mathcal{W} \) and such that if \( \alpha \) is a limit ordinal then \( \text{colim}_{\beta < \alpha} X(\beta) = X(\alpha) \). Then for every \( \alpha \in \Gamma, X(\alpha) \to \text{colim}_{\beta \in \Gamma} X(\beta) \in \mathcal{C}(I) \cap \mathcal{W} \)

Note that like the pushout condition the characterization of acyclic cofibration by lifting properties implies that the direct limit condition holds in every model category.

**Definition 3.2.17** Let \( \kappa \) be a regular cardinal. A weakly suitable cell category is a finite complete, \( \kappa \) small, \( \kappa \) acyclic small cell category such that the pushout and direct limit conditions hold.

**Lemma 3.2.18** If \( C \) is good then it satisfies the direct limit condition.

**Proof:** Let \( i_{\alpha\beta} : X(\alpha) \to X(\beta) \) for \( \alpha \leq \beta < \Gamma \) be given. Fix \( \gamma \) and assume that for every \( \delta \leq \alpha \leq \beta < \gamma \) there exists \( r_{\beta\alpha} : X(\beta) \to X(\alpha) \) and \( H_{\delta\alpha} : \text{Cyl}(X(\beta), X(\alpha)) \to X(\beta) \) satisfying the following four conditions.

1) \( r_{\beta\alpha} \circ i_{\alpha\beta} = id_{X(\alpha)} \)
2) \( H_{\beta\alpha} : id_{X(\beta)} \simeq i_{\alpha\beta} \circ r_{\beta\alpha} \) (rel \( X(\alpha) \))
3) \( r_{\delta\beta} = r_{\alpha\delta} \circ r_{\beta\alpha} \)
4) The following diagram commutes

\[
\begin{array}{ccc}
X(\alpha) \vee_{X(\delta)} X(\alpha) & \longrightarrow & X(\beta) \vee_{X(\delta)} X(\beta) \\
\downarrow & & \downarrow \\
\text{Cyl}(X(\alpha), X(\delta)) & \longrightarrow & \text{Cyl}(X(\beta), X(\delta)) \\
H_{\alpha\delta} \downarrow & & H_{\beta\delta} \downarrow \\
X(\alpha) & \longrightarrow & X(\beta)
\end{array}
\]

Condition 4) is just saying that \( H_{\beta\delta} \) extends \( H_{\alpha\delta} \).

Now we wish to construct \( r_{\gamma\alpha} \) and \( H_{\gamma\alpha} \) and demonstrate conditions 1)-4) for \( \beta = \gamma \). There are two cases.
Case 1) \( \gamma \) is a successor ordinal. Then since by hypothesis \( i_{\gamma-1} \), \( i_{\gamma-1} \) is an acyclic cofibration there is a retraction \( r_{\gamma-1} \). We define \( r_{\alpha} = r_{\gamma-1} \circ r_{\gamma-1} \). Let \( W \) denote the pushout in the following diagram.

\[
\begin{array}{ccc}
X(\alpha) \vee_{X(\delta)} X(\alpha) & \\ & \downarrow & \\
Cyl(X(\alpha), X(\delta)) & \longrightarrow & W.
\end{array}
\]

Notice that by the pushout condition of suitability all of the maps in the diagram are acyclic cofibrations. We also have an induced weak equivalence \( p : W \to X(\gamma) \). So \( CW(p) \) is a valid \( Cyl(X(\gamma), X(\delta)) \) and there also exists a retraction of \( i(p) \) which we will denote by \( r \). \( H_{\alpha\delta} \) together with \( id + i_{\alpha} \circ r_{\delta} : X(\gamma) \vee_{X(\delta)} X(\gamma) \to X(\gamma) \) determine a map \( H : W \to X(\gamma) \). Define \( H_{\alpha\delta} = H \circ r \). It is now easy to verify 1)-4).

Case 2) \( \gamma \) is a limit ordinal. Then define \( r_{\alpha} = \text{colim}_{\alpha<\delta<\gamma} r_{\alpha\delta} \) and \( H_{\alpha} = \text{colim}_{\alpha<\delta<\gamma} H_{\alpha\delta} \). These colimits are defined since conditions 3) and 4) imply compatibility. Again it is easy to verify conditions 1)-4).

Let \( X(\Gamma) \) denote \( \text{colim}_{\beta \in \Gamma} X(\beta) \). Then defining \( r_{\Gamma} \) and \( H_{\Gamma} \) as in case 2) we see that the conclusions of the lemma are verified. \( \square \)

The next lemma says that any \( \kappa \) small copointed cell category is \( \delta \) acyclic small for some \( \delta \).

**Lemma 3.2.19** Let \( (C, I) \) be \( \kappa \) small and copointed with HELP. For every \( \alpha \in I \) pick a cone \( l(\alpha) : A(\alpha) \to CA(\alpha) \). Then \( C \) is \( \max\{\sup_{\alpha}(s(l(\alpha))), \kappa \}^+ \) acyclic small.

**Proof:** All maps will be in the \( \kappa \) small category \( C \). Let \( \delta = \max\{\sup_{\alpha}(s(l(\alpha))), \kappa \}^+ \). Let \( j : X \to Y \) be a cell complex and \( j(1) : X \to Y(1) \) be a subcomplex with \( s(j(1)) \leq \delta \). Our plan is to add one cell at a time in the decomposition of \( Y(1) \) and then go to a \( Y(2) \) where all the new homotopy that has been added has died. This will lengthen the list of cells to be added. Since we have \( \kappa \) smallness this process will converge.

Assume that for all \( \beta < \gamma \) we have defined \( X(\beta), X'(\beta), Y(\beta), M(\beta) \), \( f(\beta) : X'(\beta) \to Y, \)
\( j(\beta) : X \to Y(\beta) \) and \( i(\beta) : X \to X(\beta) \). Such that \( f(\beta) \) is a weak equivalence, \( i(\beta) \)
is a subcomplex of \( j(\beta) \) and \( X(\beta) \) is a subcomplex of \( X'(\beta) \). Also \( Y(\beta) \) is the smallest subcomplex of \( Y \) such that there is a factorization.

\[
\begin{array}{c}
X(\beta) \longrightarrow Y(\beta) \\
\downarrow \quad \downarrow \quad \downarrow \\
X'(\beta) \longrightarrow Y
\end{array}
\]

Also there is a pushout

\[
\begin{array}{c}
A(\beta) \longrightarrow X(\beta) \\
\downarrow \quad \downarrow \quad \downarrow \\
CA(\beta) \longrightarrow X(\beta + 1)
\end{array}
\]

where \( A(\beta) \) is the minimal element of \( M(\beta) \) and \( i(\beta + 1) = i' \circ i(\beta) \). \( M(\beta) \) is the set of cells in \( X(\beta) \rightarrow Y(\beta) \). Well order \( M(\beta) \) so that if \( a, b \in M(\delta), \delta < \beta, a < b \) in \( M(\delta) \) and \( a, b \in M(\beta) \) then \( a < b \) in \( M(\beta) \), or if \( a \in M(\delta), b \notin M(\delta), \delta < \beta \) and \( a, b \in M(\beta) \) then \( a < b \) in \( M(\beta) \). This definition of the ordering is consistent since after being dropped off an element cannot be added again.

There are two cases:

Case 1) \( \gamma \) is a successor ordinal. Let \( W \) denote the pushout of

\[
\begin{array}{c}
X(\gamma - 1) \longrightarrow X'(\gamma - 1) \\
\downarrow \quad \downarrow \quad \downarrow \\
X(\gamma) \longrightarrow W
\end{array}
\]

and \( f : W \rightarrow Y \) the induced map. Then by Lemma 2.4.10 \( W \simeq X'(\gamma - 1) \vee \Sigma A(\gamma - 1) \).

Let \( X(\gamma) \) be \( W \cup C\Sigma A(\gamma - 1) \) where the attaching map is the identity on the wedged in suspension. Then, also be Lemma 2.4.10, there exists an extension of \( f \) to a weak equivalence \( X'(\gamma) \rightarrow Y \).

Case 2) \( \gamma \) is a limit ordinal. In this case define \( X'(\gamma), X(\gamma), Y(\gamma), i(\gamma), j(\gamma) \) and \( f(\gamma) \) as the usual colimits.
Clearly $s(i(\delta)) \leq \delta$. Let $f \in ker \pi_*(i(\delta))$. $f$ is represented by a map which factors through some $X(\beta)$. Since $|M(\beta)| < \delta$, $f = 0 \in \pi_*(X(\beta + |M(\beta)|))$. So $\pi_*(i(\delta))$ is injective. It is easily surjective. So it is a weak equivalence. 

**Lemma 3.2.20** Every suitable cell category is weakly suitable.

**Proof:** 3.2.18, 3.2.19 and 2.2.9. 

### 3.3 Suitable Cell Categories

For this section we fix a weakly suitable cell category $(C, I)$.

**Definition 3.3.1** Let $F$ be the class of all $f \in C(I) \cap W$ such that $s(f) < \kappa$ and such that the length of $f$ is $< \kappa$. Let $f : X \to Y$ be any map. Then we define

\[
\hat{X}(f) = G(\kappa)(f)
\]

\[
\hat{X}(i) = G(i)(f)
\]

\[
\hat{p}(f) = p(\kappa)(f)
\]

\[
\hat{i}(f) = i(\kappa)(f)
\]

**Lemma 3.3.2** Let $(C, I)$ be strongly $\kappa$ acyclic small. Then for every map in $C(I) \cap W$ every one of its subcomplexes, $i'$, of size less than $\gamma$, there exists a subcomplex $i \in C(I) \cap W$ containing $i'$ and such that $s(i) < \max\{\kappa, \gamma\}$.

**Proof:** Follows easily from the direct limit condition and that fact that for any regular cardinal $\kappa$ less than $\gamma$ things of size less than $\kappa$ are less than size $\max\{\kappa, \gamma\}$. 

**Lemma 3.3.3** For every $f : X \to Y$ and for every $\delta < \kappa$, $\hat{i}(\delta)(f)$ and $\hat{i}(f)$ are weak equivalences.
Proof: Follows from the direct limit condition. □

Lemma 3.3.4 For every \( f : X \to Y \), \( \tilde{p}(f) \) has the RLP with respect to any \( i \in C(I) \cap \mathcal{W} \)

Proof: Let \( j : E \to F \in C(I) \cap \mathcal{W} \). Let \( E(0) = E \) and assume that we have defined \( E(n) \) and a map

\[ j(n) : E(n) \to F \in C(I) \cap \mathcal{W} \]

If \( E(n) \neq F \) define \( E(n + 1) \) and

\[
\begin{align*}
E(n) &\xrightarrow{k} E(j + 1) \\
&\xrightarrow{j^{(n+1)}} F
\end{align*}
\]

in such a way that \( j(n) = j(n + 1) \circ k \), \( k \) is a weak equivalence and \( 1 \leq s(k) < \kappa \). We can do this since \((C, I)\) is acyclic \( \kappa \) small. If \( n \) is a limit ordinal let

\[ E(n) = \text{colim}_{m<n} E(n) \]

and

\[ j(n) = \text{colim}_{m<n} j(m). \]

Since \( s(j) \) is some cardinal there must exist an ordinal \( r \) such that \( E(r) = F \). So since we can lift a stage at a time and take colimits it is enough to consider the case \( s(j) < \kappa \).

Let us take a commuting square

\[
\begin{array}{ccc}
E & \longrightarrow & \hat{X} \\
\downarrow{i} & & \downarrow{\tilde{p}} \\
F & \longrightarrow & Y.
\end{array}
\]

Let the following pushout define \( W \) and \( j' \)

\[
\begin{array}{ccc}
E & \longrightarrow & \hat{X} \\
\downarrow{j} & & \downarrow{j'} \\
F & \longrightarrow & W.
\end{array}
\]
Then there exists a canonical map $p' : W \to Y$. We wish to define $r$ such that $r \circ j' = id$ and the following diagram commutes

\[
\begin{array}{c}
X' \\
\downarrow r \\
W \downarrow p' \\
\downarrow j' \\
Y
\end{array}
\]

$s(j') < \kappa$ so $j'$ can be decomposed in such a way that $W(0) = X$ and for every $n$

\[
\begin{array}{c}
\bigvee_{r \in K(n)} A(\alpha(r)) \\
\downarrow \\
\bigvee_{r \in K(n)} B(\alpha(r)) \\
\downarrow \\
W(n + 1)
\end{array}
\]

is a pushout. These being the cells used in constructing $B$ from $A$. $|K(n)| < \kappa$, and for $n$ a limit ordinal

\[W(n) = \text{colim}_{m < n} W(m).\]

and $W = W(\gamma)$ for some $\gamma < \kappa$.

Assume that for every $n < t$ there exists $k(n) < \kappa$. $W'(n)$ and a pushout

\[
\begin{array}{c}
\bar{X}(k(n)) \\
\downarrow \\
W'(n) \\
\downarrow q(n) \\
W(n)
\end{array}
\]

Case 1

$t$ is a successor ordinal. So then we have pushouts

\[
\begin{array}{c}
\bar{X}(k(t - 1)) \\
\downarrow \\
W'(t - 1) \\
\downarrow q(t - 1) \\
W(t - 1)
\end{array}
\]
and

\[ \forall_{r \in K(t-1)} A(\alpha(r)) \longrightarrow W(t-1) \]

\[ \forall_{r \in K(t-1)} B(\alpha(r)) \longrightarrow W(t) \]

define \( W'(t-1, l) \) by the pushout

\[ \tilde{X}(k(t-1)) \longrightarrow \tilde{X}(k(t-1) + l) \]

\[ W'(t-1) \longrightarrow W'(t-1, l) \]

Then \( W'(t-1, \kappa) = W(t-1) \) and because of \( \kappa \) smallness applied to the cell complex \( W'(i-1) \rightarrow W(i-1) \) for every \( r \in K(t-1) \) there exists \( h(r) < \kappa \) such that the map \( A(\alpha(r)) \rightarrow W(t-1) \) factors through \( W'(t-1, h(r)) \) as in the proof of 3.2.3. So

\[ \bigvee_{r \in K(t-1)} A(\alpha(r)) \rightarrow W(t-1) \]

factors through \( W'(t-1, h) \) for \( h = \sup_r(h(r)) \). But since \( \kappa \) is regular \( h < \kappa \). Let \( k(t) = k(t-1) + h \) and define \( W'(t) \) by the push out

\[ \forall_{r \in K(t-1)} A(\alpha(r)) \longrightarrow W'(t-1)(h) \]

\[ \forall_{r \in K(t-1)} B(\alpha(r)) \longrightarrow W'(t). \]

The map \( W'(t) \rightarrow W(t) \) is determined by the map \( W'(t-1)(h) \rightarrow W(t-1) \) and the inclusions of the cells \( B(\alpha(r)) \rightarrow W(t) \).

Case 2

Let \( t \) be a limit ordinal. Then

\[ k(t) = \sup_{m < t} k(m) < \kappa \]
since \( \kappa \) is regular. Let

\[
W'(t) = \bigcup_{m < t} W'(m)
\]

\( \bar{X}(k(t)) = \bigcup_{m < t} \bar{X}(k(m)) \) and it follows that

\[
\begin{array}{ccc}
\bar{X}(k(t)) & \longrightarrow & \bar{X} \\
\downarrow & & \downarrow \\
W'(t) & \longrightarrow & W(t)
\end{array}
\]

is a pushout and we are finished the induction.

So there exists a pushout

\[
\begin{array}{ccc}
\bar{X}(k(\gamma)) & \longrightarrow & \bar{X} \\
\downarrow j'' & & \downarrow j' \\
W'(\gamma) & \longrightarrow & W(\gamma)
\end{array}
\]

such that \( j'' \in \mathcal{C}(I) \) and \( s(j'') < \kappa \). The top map is an acyclic cofibration by Lemma 3.3.3 and so the bottom map is by the pushout condition. Also \( j'' \in \mathcal{W} \) since the other three maps in the square are so \( j'' \in \mathcal{F} \). Therefore there exists a map

\[
r' : W(\gamma) \to \bar{X}(k(\gamma) + 1)
\]

such that \( \bar{p}|_{\bar{X}(k(\gamma) + 1)} \circ r' = p'|_{W'} \). Recalling that \( W = W(\gamma) \) this determines a map \( r : W \to \bar{X} \) such that \( p' = \bar{p} \circ r \) and by composition a lifting \( h : F \to \bar{X} \).

**Theorem 3.3.5** Let \( \mathcal{F} \) be the class of all compositions of all maps of the form \( p(g) \) and \( \bar{p}(g) \). Then \((\mathcal{C}, \mathcal{C}(I), \mathcal{W}, \mathcal{F})\) is a closed model category.

**Proof:** It is enough to show that every \( i : A \to B \in \mathcal{C}(I) \) such that \( s(i) = 1 \) has the LLP with respect to every \( p \in \mathcal{F} \cap \mathcal{W} \). It suffices to prove this in the case \( i = i(\alpha) \) for \( \alpha \in I \). Let

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
\]
be a commutative diagram with \( i \) and \( p \) as above. Factor \( p \)

\[
\begin{array}{ccc}
X & \xrightarrow{=} & X \\
\downarrow^{i(p)} & & \downarrow^{p} \\
CW(p) & \longrightarrow & Y
\end{array}
\]

\( i(p) \in C(I) \cap \mathcal{W} \) so there is a lift \( h' \), gotten by lifting over each map \( p(g) \) or \( \tilde{p}(g) \) that were composed together to get \( p \). Also remembering the construction of \( CW(p) \), there exists a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X & \longrightarrow & X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & CW(1)(p) & \longrightarrow & CW(p) & \longrightarrow & Y
\end{array}
\]

and so by composition we have our required lifting \( h : B \to X \). □

An interpretation for the theorem is that giving a set of maps is often enough to give a category a closed model structure. The maps give us cells and the cell complexes are the cofibrations. Weak equivalences are maps that induce isomorphisms on homotopy classes of maps from the "boundaries" of cells. Certain smallness conditions are then enough to ensure that we can complete this setup to a closed model category. It is possible to refine the theorem to weaken the smallness and to be free to expand the class of weak equivalences. Notice that the theorem is also a characterization of \( \kappa \) small, \( \kappa \) acyclic small model categories whose weak equivalences are determined by maps that induce isomorphisms in homotopy sets.

### 3.4 Examples

In this section we give a few examples of suitable cell categories. Differential graded modules, topological spaces, and diagrams over a suitable cell category are all shown to be suitable cell categories. Remember that \( \text{Top} \) and \( \text{Top}_* \) are complete and cocomplete.
Definition 3.4.1 Fix a commutative ring $R$. We let $R < x_1, \ldots, x_n >$ denote the free $R$ module generated by the $x_i$. Let $DGM(R)$ be the category of differential graded modules over $R$. Let

$$I(R) = \{ j_n : R < a_n \mapsto (R < a_n, b_{n+1} >, db_{n+1} = a_n) \}_{n \in \mathbb{Z}} \cup \{ j'_n : 0 \mapsto R < a_n > \}_{n \in \mathbb{Z}}$$

where $|a_n| = |b_n| = n$.

We will denote by $I$ the $DGM(R) = (R < a_0, a_1, b >, db = a_0 - a_1)$ where $|b| = 1$

$DGM(R)$ is complete and cocomplete with the limits and colimits taken in each grading.

Definition 3.4.2 In $Top$ Let $M \in C(S)$ and pointed, that is a pointed cell complex. Define

$$I(M) = \{ j_n : \Sigma^n M \to C\Sigma^n M \}_{n \geq 0} \cup \{ j'_n : * \to \Sigma^n M \}_{n \geq 0}$$

also let

$$J(M) = \{ j : M \to CM \} \cup \{ j' : * \to M \}$$

and

$$K(M) = I(M) \cup \{ j : M \lor M \to I \times X \} \cup \{ j' : * \to M \lor M \}$$

where the map $M \lor M \to I \times M$ is the inclusion into the ends of the cylinder.

Remark: More generally $M$ could have been taken to be any set of cell complexes in $(Top, I(S^0))$.

Lemma 3.4.3 $DGM(R)$ and $Top$ are pointed.

Proof: $\square$

Definition 3.4.4 In $I(M)$ and $K(M)$ we will denote $\Sigma^n M$ by $S^n$ and $C\Sigma^n M$ by $D^{n+1}$ in $I(R)$ we will denote $R < a_n >$ by $S^n$ and $R < a_n, b_{n-1} >$ by $D^{n+1}$.
Lemma 3.4.5 \((DGM(R), I(R)), (Top_\ast, I(M)), (Top_\ast, J(M))\) and \((Top_\ast, K(M))\) are all \(\kappa\) small for some \(\kappa\)

Proof: Fix a set \(A\) and let \(V\) be the category of sets and \(I(V) = \{\emptyset \to \{\emptyset\}\}\). Not that \(C(I(V))\) is the inclusions. Then a point is \(\omega\) small in \((V, I(V))\) so \(A\) is \(|A|^+\) small in \((V, I(V))\). The result easily follows by letting \(A\) vary over all the cells. \(\Box\)

Lemma 3.4.6 Let \(M \in DGM(R)\) then the map

\[ p : M \otimes I \to M \]

has the RLP with respect to any cofibration.

Proof: Standard. \(\Box\)

Lemma 3.4.7 \(i_0 + i_1 : M \vee M \to M \otimes I\) is a cofibration in \((DGM(R), I(R))\)

Proof: Clear. \(\Box\)

Lemma 3.4.8 Let \(M \in C(Top_\ast, I(S^0))\) then the map

\[ p : I \times M \to M \]

has the RLP with respect to any cofibration.

Proof: Standard. \(\Box\)

Lemma 3.4.9 \(i_0 + i_1 : \Sigma^r M \vee \Sigma^r M \to I \times \Sigma^r M\) is a cofibration in \((Top, K(M))\) or if \(M = \Sigma N\) or \(r > 1\) in \((Top, I(M))\)

Proof: Clear. \(\Box\)

Remember in the next lemma that the notion of homotopy is that of 2.2.3 or 3.2.8. In effect then the lemma says that for a cofibrant domain the notion of homotopy is the same as the usual one.
Lemma 3.4.10 Let $M$ be cofibrant. Let $f, g : M \rightarrow K \in (DGM(R), I(R)), (Top_*, I(\Sigma X))$ or $(Top_*, K(X))$. Then $f \simeq g$ if and only if there exist a map $H$ making the following diagram commute

$$
\begin{array}{ccc}
M \vee M & \xrightarrow{f+g} & K \\
\downarrow_{i_0+i_1} & & \downarrow H \\
M \otimes I & \xrightarrow{H} & \\
\end{array}
$$

where in $Top_*$ $M \otimes I$ denotes $I \times M$.

Proof: In $DGM(R)$ for a cofibrant $M$ the maps $M \vee M \rightarrow Cyl(M)$ and $M \vee M \rightarrow M \otimes I$ are both cofibrations and the maps $Cyl(M) \rightarrow M$ and $M \otimes I \rightarrow M$ both have the LLP with respect to all cofibrations. The proof in the other categories is similar. □

Lemma 3.4.11 HELP holds in $(DGM(R), I(R)), (Top_*, I(\Sigma M))$ and $(Top_*, K(M))$.

Proof: We prove the Lemma in $DGM(R)$. First let us assume that we are given the solid arrows in a diagram of the following form

$$
\begin{array}{ccc}
S^n & \xrightarrow{f} & S^n \otimes I \\
\downarrow G & & \downarrow E \\
D^{n+1} & \xrightarrow{H} & D^{n+1} \otimes I \\
\downarrow h & & \downarrow h \\
S^n & \xrightarrow{g} & S^n \\
\downarrow F & & \downarrow F \\
D^{n+1} & \xrightarrow{H} & D^{n+1} \\
\end{array}
$$

where $\phi$ is a weak equivalence. Since $\phi$ is a weak equivalence there exists an extension $h'$ of $f$ over $D^{n+1}$. Denote the element induced by $\phi(h')(b) - g(b) + G(a \otimes b)$ in $\pi_{n+1}(E)$ by $c$. So letting $h = h' - \phi^{-1}(c)$ we can get an extension $H$ making the diagram above commute. The general case follows since if

$$
\begin{array}{ccc}
S^n & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
D^{n+1} & \xrightarrow{g} & Y \\
\end{array}
$$
is a pushout then

\[
\begin{array}{c}
S^n \otimes I \xrightarrow{f \otimes \text{id}} X \otimes I \\
\downarrow \\
D^{n+1} \otimes I \xrightarrow{g \otimes \text{id}} Y \otimes I
\end{array}
\]

is also a pushout.

The proofs for \((\text{Top}_*, I(\Sigma M))\) and \((\text{Top}_*, K(M))\) are similar. □

**Lemma 3.4.12** A weak equivalence in \((\text{Top}_*, J(S^n))\) is a map that induces a \(\pi_n\) equivalence.

**Proof:** Easy. □

**Lemma 3.4.13** In \(J(S^n)\) if

\[
\begin{array}{c}
S^n \rightarrow X \\
\downarrow \\
D^{n+1} \rightarrow Y
\end{array}
\]

is a pushout then

\[
\begin{array}{c}
\text{Cyl}S^n \rightarrow \text{Cyl}(X, W) \\
\downarrow \\
\text{Cyl}D^{n+1} \rightarrow \text{Cyl}(Y, W)
\end{array}
\]

is a pushout. If

\[
\begin{array}{c}
* \rightarrow X \\
\downarrow \\
S^n \rightarrow Y
\end{array}
\]

is a pushout then

\[
\begin{array}{c}
* \rightarrow \text{Cyl}(X, W) \\
\downarrow \\
\text{Cyl}S^n \rightarrow \text{Cyl}(Y, W)
\end{array}
\]

is a pushout.

**Proof:** Easy using 3.4.12. □

**Lemma 3.4.14** HELP holds in \((\text{Top}_*, J(S^n))\)
Proof: Lemma 3.4.13 means it is enough to show HELP in case the cofibration is $\ast \to S^n$ or $S^n \to D^{n-1}$. In the first case we are given a solid arrow diagram

\[
\begin{array}{c}
\ast \\
\downarrow \\
E \\
\downarrow \\
S^n & \xleftarrow{\phi} & CylS^n & \xrightarrow{h} & S^n
\end{array}
\]

with $\phi$ a weak equivalence. The maps $h$ and $H$ exist since $\phi$ is a $\pi_n$ equivalence.

The second case is easy to show once we have observed that $CylS^n \cup_{\partial_0+\partial_1} D^{n+1} \vee D^{n+1}$ is a $CylD^{n+1}$. \(\square\)

**Lemma 3.4.15** The pushout condition holds in $(DGM, I(R)), (Top_*, I(\Sigma M))$ and $(Top_*, K(M))$.

Proof: Let

\[
\begin{array}{c}
A \longrightarrow C \\
\downarrow i \\
B \longrightarrow D
\end{array}
\]

be a pushout with $j$ an acyclic cofibration. Let $r$ be the retraction of $j$ and $r'$ the induced retraction of $j'$. Since

\[
\begin{array}{c}
A \otimes I \longrightarrow C \otimes I \\
\downarrow \\
B \otimes I \longrightarrow D \otimes I
\end{array}
\]

is a pushout the homotopy $j \circ r \simeq id$ induces a homotopy $j' \circ r' \simeq id$ and so $j'$ must be a weak equivalence. \(\square\)

**Lemma 3.4.16** $(Top_*, J(S^n))$ satisfies the pushout condition

Proof: The pushout condition holds since 3.4.13 implies that if

\[
\begin{array}{c}
A \longrightarrow C \\
\downarrow j \\
B \longrightarrow D
\end{array}
\]
is a pushout and \( j \) a cofibration then

\[
\begin{array}{c}
A \longleftarrow C \\
\downarrow \quad \downarrow \\
Cyl(B, A) \longrightarrow Cyl(D, C)
\end{array}
\]

is a pushout. \( \square \)

**Theorem 3.4.17** \((DGM(R), I(R)), (Top_*, I(M)), (Top_*, K(M)) \) and \((Top_*, J(S^n))\) are all suitable cell categories.

**Proof:** Since \( DGM(R) \) and \( Top_* \) are complete and cocomplete we just need to use lemmas 3.4.5, 3.4.11, 3.4.14, 3.4.15 and 3.4.16. \( \square \)

**Corollary 3.4.18** \((DGM(R), I(R)), (Top_*, I(\Sigma M)), (Top_*, K(M)) \) and \((Top_*, J(S^n))\) are all model categories.

The following definition of an interval is similar to the one in the definition of I-category of Kan ([17]). The biggest difference is our assumption that HELP holds.

**Definition 3.4.19** Let \( C, I \) be a cell category. Let \( \_ \otimes I : C \to C \) be a functor that preserves colimits. Assume that for every \( \alpha \) we have a diagram

\[
\begin{array}{c}
A(\alpha) \otimes I \longrightarrow Cyl(A(\alpha)) \\
\downarrow \quad \downarrow \\
B(\alpha) \otimes I \longrightarrow Cyl(B(\alpha))
\end{array}
\]

with the horizontal maps being isomorphisms. Also assume that we have chosen functorial factorizations of \( id + id \).

\[
X \times X \xrightarrow{\partial_0 + \partial_1} X \otimes I \xrightarrow{\sigma} X
\]
We refer to homotopies that factor through the map $\sigma : X \otimes I \to X$ as constant homotopies.

Assume HELP holds for $- \otimes I$. That is given a solid arrow diagram

\[
\begin{array}{c}
A(\alpha) \xrightarrow{\partial_0} A(\alpha) \otimes I \xrightarrow{\partial_1} A(\alpha) \\
| & & | \\
| & E \phi & | \\
| & & | \\
| & & F \\
| & & | \\
| & B(\alpha) \xrightarrow{\partial_0} B(\alpha) \otimes I \xrightarrow{\partial_1} B(\alpha)
\end{array}
\]

such that $\phi$ is a weak equivalence there exist maps $H$ and $h$ extending the diagram.

We call such a functor an interval.

**Definition 3.4.20** Let $(C, I)$ be a $\kappa$ small cell category with an interval. Let $g, f : X \to Y$ and $h : A \to X$ be maps such that there exists a homotopy $F : X \otimes I \to Y$ between $f$ and $g$ that is constant on $A$. Then we write $f \simeq_A g$.

If now there exist maps $f : X \to Y$, $g : Y \to X$, $h : A \to X$ and $h' : A \to Y$ such that $fh = h'$ and $gh' = h$ and such that $fg \simeq_A id$ and $gf \simeq_A id$ then we write $X \simeq_A Y$.

**Lemma 3.4.21** Let $(C, I)$ be a $\kappa$ small cell category with an interval. Then if $X \simeq_A Y$ then $X$ is weakly equivalent to $Y$.

**Proof:** Easy since for every $\alpha \in I$, $A(\alpha) \otimes I$ is a $Cyl(A(\alpha))$. □

**Lemma 3.4.22** Let $(C, I)$ be a cell category with an interval. Let $D$ be a small category. Let $F, G \in C^D$, $f \in Hom_{C^D}(G, F)$ and $r \in Hom_{C^D}(F, G)$ such that $fr = id$ and there exists $H \in Hom_{C^D}(G \otimes I, G)$ such that for every $x \in D H(x) : r(x)f(x) \simeq_F (x)id$. Then $r' = colim r : colim F \to colim G$ and $f' : colim f : colim G \to colim F$ are such that $f'r' = id$ and $H' = colim H : colim G \otimes I \to colim G$ is a homotopy $r'f' \simeq_{colim F} id$.

**Proof:** Follows since $- \otimes I$ commutes with colimits. □
Lemma 3.4.23 Let \((C, I)\) be a cell category with an interval. Assume we are given a diagram in \(C\)

\[
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
\downarrow f & & \downarrow j' \\
X' & \xrightarrow{\cdot'} & Y' \\
\downarrow f' & & \downarrow j'' \\
X'' & & \\
\end{array}
\]

such that \(j\) is a weak equivalence and a cell complex with subcomplexes \(f\) and \(f'f\) and such that \(f'\) is a subcomplex of the complex \(j'\) with \(s(f') = 1\).

Assume that we have a map in \(C\), \(r : X' \to X\) such that \(rf = \text{id}\) and there exists a homotopy \(H\) between \(j'\) and \(jr\). Then there exists a map \(r' : X'' \to X\) extending \(r\) and a homotopy \(H'\) between \(j''\) and \(jr'\) extending \(H\).

Proof: Since \(s(f') = 1\) we have a pushout in \(C\).

\[
\begin{array}{ccc}
A(\alpha) & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \\
B(\alpha) & \longrightarrow & X'' \\
\end{array}
\]

By HELP for \(\_ \circlearrowright I\) we get extensions \(r'\) and \(H'\) of \(r\) and \(H\) over \(X''\).

Lemma 3.4.24 Let \((C, I)\) be a cell category with an interval. Let \(j : X \to Y\) be a cofibration in \(C\) and \(\mathcal{F}(j) \in \mathcal{W}\). Then there exists \(r : Y \to X\) such that \(rj = \text{id}\) and \(jr \simeq_A \text{id}\).

Proof: For \(j \in \mathcal{C}(I(C))\) the result follows from 3.4.23 by induction since \(\_ \circlearrowright I\) commutes with direct limits. If \(j\) is any cofibration the lemma follows from the fact that any cofibration
is a retract of a map $j'$ in $C$ as in the following diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \\
\end{array}
\]

$n$

Lemma 3.4.25 Let $(C, I)$ be a cell category with an interval. Assume we have $f : Y \to X$ and $r : X \to Y$ such that $fr = id$ and $rf \simeq_X id$. Then for every pushout in $C$

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
W & \rightarrow & Z \\
\end{array}
\]

$W \simeq_W Z$

Proof: Use the fact that $- \otimes I$ commutes with pushouts. $\Box$

Lemma 3.4.26 Let $(C, I)$ be a cell category with an interval. Let

\[
\begin{array}{ccc}
A(\alpha) & \longrightarrow & X \\
\downarrow & & \downarrow \\
B(\alpha) & \longrightarrow & Y \\
\end{array}
\]

be a pushout in $C$. Assume there exists $f : Cyl(X, W) \to X$ such that $f \partial_0(X) = id$ and assume there exists a homotopy $H : \partial_0(X)f \simeq_X id$. Then

\[
\begin{array}{ccc}
CylA(\alpha) & \longrightarrow & Cyl(X, W) \\
\downarrow & & \downarrow \\
CylB(\alpha) & \longrightarrow & Cyl(Y, W) \\
\end{array}
\]
is a pushout and there exists $f' : \text{Cyl}(Y, W) \to Y$ such that $f \partial_0(Y) = \text{id}$ and such that there
exists a homotopy $H' : \partial_0(Y)f' \simeq_Y \text{id}$ extending $H$.

**Proof:** The second statement implies the first so we proceed to prove the second.

Let $Z$ be the pushout of

$$
\begin{array}{c}
A(\alpha) \\
\downarrow \\
B(\alpha)
\end{array}
\quad \xrightarrow{g} \quad \begin{array}{c}
\mathcal{F}X \\
\end{array}
$$

and $U$ the pushout of

$$
\begin{array}{c}
\text{Cyl}(A(\alpha)) \\
\downarrow \\
\text{Cyl}(B(\alpha)).
\end{array}
\quad \xrightarrow{\text{Cyl} g} \quad \begin{array}{c}
\mathcal{F}\text{Cyl}(X, W)
\end{array}
$$

Let $\partial_0$ denote the map induced by $\partial_0$ on the corners of the pushout.

We observe that the map $\partial_0 : Z \to U$ is a weak equivalence. Let $Z'$ denote the pushout

$$
\begin{array}{c}
A(\alpha) \\
\downarrow \\
B(\alpha)
\end{array}
\quad \xrightarrow{\text{Cyl } \partial_0} \quad \begin{array}{c}
\mathcal{F}\text{Cyl}(X, W)
\end{array}
$$

Then we have a pushout

$$
\begin{array}{c}
\text{Cyl}A(\alpha) \vee_{A(\alpha)} B(\alpha) \\
\downarrow \\
\text{Cyl}B(\alpha)
\end{array}
\quad \xrightarrow{} \quad \begin{array}{c}
Z' \\
\downarrow \\
U.
\end{array}
$$

$X \to Z'$ is a weak equivalence by 3.4.25 and $Z' \to U$ is a weak equivalence by the pushout
condition for $C$ so $\partial_0$ is a weak equivalence.
Consider the solid arrow diagram

The dashed extensions $G$ and $g$ give extensions of $H$ and $f$. The extensions to $H'$ and $f'$ are so determined. □

Lemma 3.4.27 Let $C$ be a $\kappa$ small cell category with an interval. Then HELP and the pushout condition hold in $C$.

Proof: To see HELP use Lemmas 2.2.7, 3.4.24, 3.4.26 and 3.4.22. The pushout condition is easy from 3.4.24 and 3.4.22. □

3.5 Diagrams over a suitable cell category

For this section we fix a $\kappa$ small, strongly $\kappa$ acyclic small weakly suitable cell category, $(C, I)$ and a small category, $D$.

Definition 3.5.1 Define $C^D$ to be the category of diagrams from $D$ to $C$. So $F \in C^D$ is a functor $F : D \to C$ and $\phi \in \text{Hom}(F, G)$ is a natural transformation from $F$ to $G$.

Lemma 3.5.2 $C^D$ is cocomplete, finite complete and for $x \in D$

$$(\text{colim}F_i)(x) = \text{colim}(F_i(x))$$

$$(\text{lim}F_i)(x) = \text{lim}(F_i(x))$$

Proof: Easy. □

We now define cells in $C^D$. 
Definition 3.5.3 For \( \alpha \in I \) and \( x \in D \) define functors \( F(\alpha, x) \) and \( G(\alpha, x) \) and a natural transformation \( i(\alpha, x) : F(\alpha, x) \to G(\alpha, x) \) as follows.

\[
F(\alpha, x)(y) = \bigvee_{f \in \text{Hom}(x, y)} A(\alpha)_f
\]

where each \( A(\alpha)_f \) is a copy of \( A(\alpha) \) and with the empty wedge interpreted as the initial object.

For \( g \in \text{Hom}(y, z) \), \( F(\alpha, x) \) is determined by letting

\[
F(\alpha, x)(g)|_{A(\alpha)_f} = id : A(\alpha)_f \to A(\alpha)_{gof}
\]

\[
G(\alpha, x)(y) = \bigvee_{f \in \text{Hom}(x, y)} B(\alpha)_f
\]

where each \( B(\alpha)_f \) is a copy of \( B(\alpha) \). For \( g \in \text{Hom}(y, z) \), \( G(\alpha, x) \) is determined by letting

\[
G(\alpha, x)(g)|_{B(\alpha)_f} = id : B(\alpha)_f \to B(\alpha)_{gof}
\]

\( i(\alpha, x) \) is determined by the following equation

\[
i(\alpha, x)(y)|_{A(\alpha)_f} = i(\alpha) : A(\alpha)_f \to B(\alpha)_f
\]

When confusion is unlikely to arise we will write \( F \) for \( F(\alpha, x) \) and \( G \) for \( G(\alpha, x) \)

Denote \( \{F(\alpha, x) \to G(\alpha, x)\}_{(\alpha, x) \in I(C) \times D} \) by \( I(C^D) \)

Definition 3.5.4 Let \( i : X \to Y \in C(I(C^D)) \). We define the height if \( i \), \( ht(i) \), to be the minimum ordinal \( \alpha \) such that \( Y(0) = X, Y(\alpha) = Y \)

\[
\begin{array}{ccc}
\bigvee_{r \in K(i)} F(\alpha(r), x(r)) & \longrightarrow & Y(i) \\
\downarrow & & \downarrow \\
\bigvee_{r \in K(i)} G(\alpha(r), x(r)) & \longrightarrow & Y(i + 1)
\end{array}
\]

is a pushout and \( Y(i) = \text{colim}_{j<i} Y(j) \) if \( i \) is a limit ordinal.
ht(i) is the minimum number of steps required to get Y from X.

**Definition 3.5.5** Let $\check{\kappa}$ be the smallest regular cardinal $\geq \kappa + |\bigcup_{x,y \in D} \text{Hom}(x,y)|$

**Lemma 3.5.6** Let $i : X \rightarrow Y \in C(I(C^D))$. Then

1) $ht(i) \leq \check{\kappa}$

2) Let $A = F(\alpha, x)$ or $G(\alpha, x)$ for some fixed $\alpha \in I$ and $x \in D$. Then for every map

$$A \rightarrow Y$$

there exists a subcomplex $j : X \rightarrow Y'$ of Y such that $s(j) < \check{\kappa}$ and a factorization

$$
\begin{array}{c}
A \\
\downarrow \\
Y
\end{array} 
\xrightarrow{\exists} 
\begin{array}{c}
Y' \\
\downarrow \\
Y
\end{array}
$$

**Proof:** Let us assume that 1) and 2) are true for all $i$ such that $ht(i) < \gamma \leq \check{\kappa} + 1$. Assume that $i : X \rightarrow W \in C(I(C^D))$ such that $ht(i) = \gamma$ and prove that 1) holds. There are two cases to consider.

Case a) $\gamma$ is a successor ordinal. Let

$$
\begin{array}{c}
\forall \ F \\
\downarrow \\
\forall \ G
\end{array} 
\xrightarrow{\exists} 
\begin{array}{c}
\forall \ Y \\
\downarrow \\
\forall \ W
\end{array}
$$

be a push out such that $ht(Y) = \gamma - 1$. If $\gamma \leq \check{\kappa}$ then 1) is trivially satisfied. If $\gamma = \check{\kappa} + 1$ then 2) for $\check{\kappa}$ implies that all the new cells could have been added earlier and 1) is satisfied.

Case b) $\gamma$ is a limit ordinal. In this case 1) is trivially satisfied.

We next turn our attention to proving 2). Let $g : R \rightarrow Y$ be a map. We know that for every $y \in D$ and $f \in \text{Hom}(x, y)$

$$g(y)|_{A(\alpha)} : A(\alpha) \rightarrow Y(y)$$
factors through some subcomplex, \( j(Y'(f)) : X(y) \to Y'(f) \) of \( Y(y) \) such that \( s(j(Y'(f))) < \kappa \).

We will show that there exists a subcomplex \( \tilde{Y} \to Y \) such that for every \( y \) and \( f \), \( Y'(f) \to \tilde{Y}(y) \) is a subcomplex and \( s(\tilde{Y}) < \kappa \).

We have the usual decomposition of \( X \to Y \): \( Y(0) = X \), \( Y(\alpha) = Y \), \( Y(n) = \text{colim}_{\beta < n} Y(\beta) \) for \( n \) a limit ordinal and the following diagram a pushout

\[
\begin{array}{ccc}
\bigvee_{M(i)} F & \to & Y(n) \\
\downarrow & & \downarrow \\
\bigvee_{M(i)} G & \to & Y(n + 1).
\end{array}
\]

In the corresponding decomposition of \( X(y) \to Y'(f) \) denote the cells at the \( n \)th stage by \( M'(f)(n) \). Since \( Y'(f) \) is a subcomplex of \( Y(y) \) each cell in \( Y'(f)(n) \) is contained in some part of a cell of \( a(y) \) where \( a \in M(n) \). Let

\[
H(f)(n) = \{ a \in M(n) : a(y) \cap M'(f)(n) \neq \emptyset \}
\]

Let \( H = \bigcup_{n,f} H(f)(n) \). For any subset \( H \subset \bigcup_n M \) define

\[
ht(H) = \{ n : H \cap M(n) \neq \emptyset \}
\]

Since \( \kappa \) is regular and \( |ht(\bigcup_n M)| \leq \kappa \) and \( |H| < \kappa \), we get that \( |ht(H)| < \kappa \). Also \( ht(H) \) is well ordered since it is a subset of a well ordered set. Observe \( ht(H) \) is an invariant of \( g : F \to Y \) since we can take the possibility with the minimum \( ht \).

By the second induction hypothesis the boundary of every cell \( \alpha \in H \) factors through a subcomplex \( Z(\alpha) \) with \( s(Z(\alpha)) < \kappa \). Denote the complex with cells \( \bigcup_{\alpha \in H} Z(\alpha) \cup H \) by \( Z \). Then \( Z \) is a subcomplex of \( Y \) such that \( s(Z) < \kappa \) and we can factor the original map as \( F \to Z \to Y \).

\[\square\]

**Corollary 3.5.7** \( C^D \) is \( \kappa \) small.
Definition 3.5.8 Fix $x \in D$. For $X \in C^D$ and $Y \in C$ let $e : C^D \to C$ and $D : C \to C^D$ be defined by $e(X) = X(x)$ and $D(Y)(y) = \text{V}_{\text{Hom}(x,y)} Y$

Lemma 3.5.9 $e$ is right adjoint to $D$

Proof: Take the natural transformation

$$X \to X_{id} \to \bigvee_{\text{Hom}(x,x)} X$$

then by Chapter 6 Theorem 2 of [19] the result follows. □

Definition 3.5.10 Let $X \in C^D$ and let $j : X(x) \to Y \in C(I(C^D))$. $j$ can be decomposed as usual $X(x) = Y(0)$, $Y = Y(\gamma)$ and with pushouts

$$\bigvee_r A(\alpha(r)) \xrightarrow{V f(r)} Y(\beta)$$
$$\bigvee B(\alpha(r)) \longrightarrow Y(\beta + 1).$$

Define $D(j) : X \to Z \in C^D$ together with inclusions $j(\beta) : Y(\beta) \to Z(\beta)(x)$ by $X = Z(0)$, $Z = Z(\gamma)$, $Z(\beta) = \text{colim}_{\delta \leq \beta} Z(\delta)$ and $j(\beta) = \text{colim}_{\delta \leq \beta} j(\delta)$ for $\beta$ a limit ordinal and the following pushout

$$\bigvee_r F(x, \alpha(r)) \xrightarrow{(j(\beta) \sigma f(r))} Z(\beta)$$
$$\bigvee G(x, \alpha(r)) \longrightarrow Z(\beta + 1)$$

where for any $g \in \text{Hom}_C(A(\alpha), e(Z(\beta)))$ $g^a$ denotes the adjoint in $\text{Hom}_{C^D}(F, Z(\beta))$. Also the pushout determines $j(\beta + 1)$ and the map $D(j)$ is just the inclusion $Z(0) \to Z(\gamma)$.

Definition 3.5.11 Let $X \in C^D$ and $i : X(x) \to Y \in C$ be a cofibration. For $y \in D$ and $f \in \text{Hom}(x, y)$ let the following pushout define $Y_f$

$$X(x) \longrightarrow Y$$
$$f$$
$$X(y) \longrightarrow Y_f$$
Let \( Z'(y) = \text{colim}_{f \in \text{Hom}(x,y)} (X(y) \to Y_f) \). Let \( D'(i) : X \to Z'(i) \) be the canonical \( C^D \) map.

**Lemma 3.5.12** For \( j \in \mathcal{C}(I(C)) \) \( D(j) = D'(j) \)

**Proof:** The equality is clear if \( j \) is the attachment of a single cell. If we have cofibrations \( f : X(x) \to Y_1 \) and \( g : Y_1 \to Y_2 \), then define \( Y'_2 \) and \( g' \) by the push out

\[
\begin{array}{ccc}
X(x) & \xrightarrow{g \circ f} & Y_2 \\
\downarrow & & \downarrow \\
Z(f)(x) & \xrightarrow{g'} & Y'_2.
\end{array}
\]

It follows from the universal property of colimits that \( D'(g \circ f) = D'(g') \circ D'(f) \). A simple induction then completes the proof of the lemma. \( \square \)

**Lemma 3.5.13** Let \( i : X(x) \to Y \) be an acyclic cofibration, then for every \( y \in D \), \( D(i)(y) \) is an acyclic cofibration.

**Proof:** For every \( f \in \text{Hom}_{\mathcal{C}D}(X, Y) \) \( X(y) \to Y_f \) is an acyclic cofibration so, since acyclic cofibrations are determined by lifting properties, \( X(y) \to Z(y) \) is also. \( \square \)

**Lemma 3.5.14** Let \( C \) be good. Assume that in the following commutative diagram \( i : X \to Y \in \mathcal{C}^D \) is a cofibration and for every \( x \in D \) \( g(x) : X(x) \to C(x) \) is a weak equivalence.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & C \\
\downarrow i & & \downarrow \\
Y & \rightarrow & C.
\end{array}
\]

Then there exists a retraction \( r : Y \to X \) such that \( r \circ i = id \).

**Proof:** First assume \( i \in \mathcal{C}(I(C^D)) \). Decompose \( i : Y(0) = X, Y(\varepsilon) = Y \)

\[
\begin{array}{ccc}
F & \rightarrow & Y(\gamma) \\
\downarrow & & \downarrow \\
G & \rightarrow & Y(\gamma + 1)
\end{array}
\]
is a pushout. Notice that in this case we are only adding one cell at each stage.

Assume that for every \( \delta < \beta < \gamma \) we have defined \( X(\beta) \), compatible cofibrations \( i_{\delta \beta} : X(\delta) \to X(\beta) \) such that for every \( x \in D \) \( i_{\delta \beta}(x) \) is a weak equivalence. Assume we have defined maps \( p(\beta) : X(\beta) \to F \) such that for every \( x \in D \) \( p(\beta)(x) \) is a weak equivalence and compatible retractions \( r(\beta) : X(\beta) \to X(0) \) such that \( r(\beta) \circ i_{0 \beta} = id \). Also assume that we have commutative diagrams

\[
\begin{array}{ccc}
X(0) & \longrightarrow & X(\beta) \\
\downarrow & & \downarrow \\
Y(\beta) & \longrightarrow & F
\end{array}
\]

We wish to show the induction hypotheses for \( \gamma \). There are two cases.

Case 1) \( \gamma \) is a successor ordinal. Then we have a pushout

\[
\begin{array}{ccc}
F(\alpha, x) & \longrightarrow & Y(\gamma - 1) \\
\downarrow & & \downarrow \\
G(\alpha, x) & \longrightarrow & Y(\gamma).
\end{array}
\]

Decompose the map \( p(\gamma - 1)(x) : X(\gamma - 1)(x) \to F(x) \) into a cofibration, \( j \), followed by a fibration

\[
X(\gamma - 1)(x) \overset{j}{\longrightarrow} R \longrightarrow F(x)
\]

Let \( i_{\gamma - 1, \gamma} : X(\gamma - 1) \to X(\gamma) \) be \( D(i) \). \( i_{\gamma - 1, \gamma} \) is a weak equivalence by Lemma 3.5.13. Therefore we get a retraction \( X(\gamma) \to X(\gamma - 1) \) and therefore by composition \( r(\gamma) \). Also we have an extension

\[
\begin{array}{ccc}
A(\alpha) & \longrightarrow & R \\
\downarrow & & \downarrow \\
B(\alpha) & & 
\end{array}
\]

determined by the map \( B(\alpha) \to G(\alpha, x)(x)_{id} \to F(x) \). This extension determines an extension \( G(\alpha, x) \to X(\gamma) \) and so a map \( Y(\gamma) \to X(\gamma) \) extending the map \( Y(\gamma - 1) \to X(\gamma) \).

Case 2) \( \gamma \) is a limit ordinal.
Let \( X(\gamma) = \text{colim}_{\beta < \gamma} X(\beta) \) and clearly the induction hypotheses are satisfied.

\[
X \to Y \to X(\epsilon) \to X
\]

is the identity.

The case for general \( i \) follows easily from the definition of cofibrations as retractions of maps in \( C(I(C^D)) \). \( \square \)

**Lemma 3.5.15** Let \( C \) be good. Then for \( f, g : F(x, \alpha) \to X \in C^D \). \( f \simeq g \) if and only if \( f_{id} \simeq g_{id} \).

**Proof:** Assume \( f \simeq g \). Then we have a commuting diagram

\[
\begin{array}{ccc}
F(x) \lor F(x) & \xrightarrow{f+g} & X(x) \\
\downarrow & & \downarrow \\
(Cyl(F))(x) & & (Cyl(F))(x)
\end{array}
\]

but we also have a commuting diagram

\[
\begin{array}{ccc}
A(\alpha) \lor A(\alpha) & \xrightarrow{f+g} & F(x) \lor F(x) \\
\downarrow & & \downarrow \\
Cyl(A(\alpha)) & \xrightarrow{f_{id}+g_{id}} & (Cyl(F))(x)
\end{array}
\]

so \( f_{id} \simeq g_{id} \).

Assume that \( f_{id} \simeq g_{id} \). We have a commuting diagram

\[
\begin{array}{ccc}
D(A(\alpha)) \lor D(A(\alpha)) & \xrightarrow{f} & D(A(\alpha)) \\
\downarrow & & \downarrow \\
D(Cyl(A(\alpha))) & \xrightarrow{p} & D(A(\alpha))
\end{array}
\]
and know that $i$ is a $C^D$ cofibration and that $p(x)$ is a weak equivalence for every $x \in D$. We also have a cofibration $j : D(Cyl(A(\alpha))) \to Cyl(F)$ and a commuting diagram

\[
\begin{array}{ccc}
DCylA(\alpha) & \xrightarrow{} & CylF \\
\downarrow & & \downarrow \\
\to & & \to \\
F & \xrightarrow{} & F
\end{array}
\]

so we can use Lemma 3.5.14 to get a retraction $CylF \to DCylA(\alpha)$. Therefore $f \simeq g$. □

**Lemma 3.5.16** Let $C$ be good. Then $\phi \in C^D$ is a weak equivalence if and only if $\phi(x)$ is a weak equivalence for every $x \in D$.

**Proof:** Let $\phi : X \to Y \in C^D$. First assume that $\phi$ is a weak equivalence. Fix $x \in D$ and let $f \in \pi(A(\alpha), Y(x))$. We want to show there exists a unique $\phi^{-1}(f) \in \pi(A(\alpha), X(x))$ such that $\phi(\phi^{-1}(f)) = f$. By Lemmas 3.5.15 and 3.5.9 $f$ determines a unique element of $Hom(F(\alpha, x), Y)$ which by assumption determines a unique element of $\pi(F(\alpha, x), X)$ which, again by Lemmas 3.5.15 and 3.5.9, determines a unique element of $\pi(A(\alpha), X(x))$. The proof of the converse is almost identical. □

**Corollary 3.5.17** Let $C$ be good. Then the pushout and direct limit conditions hold in $C^D$.

**Lemma 3.5.18** Let HELP hold in $C$. Then HELP holds in $C^D$.

**Proof:** It is enough to show that given all the solid arrows in the following commutative diagram where the map $E \to E'$ is a weak equivalence, there exists the dashed maps making the diagram commute

\[
\begin{array}{ccc}
F & \xrightarrow{} & CylF & \xleftarrow{} & F \\
\downarrow & & \downarrow & & \downarrow \\
E' & \xleftarrow{} & E & \xrightarrow{} & E' \\
\downarrow & & \downarrow & & \downarrow \\
G & \xrightarrow{} & CylG & \xleftarrow{} & G.
\end{array}
\]
By HELP in $C$ and from the solid arrows in the diagram above we can get a commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{DCylA(\alpha)} & F \\
\downarrow & & \downarrow \\
E' & \xrightarrow{DCylB(\alpha)} & E \\
\downarrow & & \downarrow \\
G & \xrightarrow{DCylA(\alpha)} & CylF \\
\downarrow & & \downarrow \\
DCylB(\alpha) & \xrightarrow{W} & W
\end{array}
$$

This gives us the required map $G \to E$. Let $W$ be the pushout

$$
\begin{array}{ccc}
DCylA(\alpha) & \xrightarrow{CylF} & CylF \\
\downarrow & & \downarrow \\
DCylB(\alpha) & \xrightarrow{W} & W
\end{array}
$$

Then there is an acyclic cofibration $W \to CylG$ and a map $W \to E'$. Using lemmas 3.5.14 and 3.5.13 we get a retraction $CylG \to W$ and therefore our required map $CylG \to E'$. \(\Box\)

**Lemma 3.5.19** $C^D$ is strongly $\tilde{\kappa}^+$ acyclic small.

**Proof:** Let $i : A \to B \in C(I(C^D))$ and $i' : A \to B'$ a subcomplex such that $s(i') < \tilde{\kappa}^+$. Since $(C, I)$ is strongly $\kappa$ acyclic small and $\kappa \leq \tilde{\kappa}$ Lemma 3.2.19 gives us that for every $x \in D$ there exists $X_x \in C$ such that $j_x : A(x) \to X_x$ is a subcomplex of $i(x)$. $i'(x)$ is a subcomplex of $j_x$ and $s(j_x) \leq \tilde{\kappa}$.

Let $j : A(x) \to X(B')$ be the smallest subcomplex of $B$ containing $X_x$ for every $x \in D$. Let $j' : B' \to X(B')$ denote the inclusion. Then by Lemma 3.5.6 $s(j) \leq \tilde{\kappa}$ and by Lemma 3.5.15 if $z \not\in im\pi_\alpha(i')$ then there exists $z' \in Hom(A(\alpha), X(B'))$ such that $j'z \simeq z'$.

Denote $X(1) = X(B')$ and $X(i + 1) = X(X(i))$. For $i$ a limit ordinal let $X(i) = colim_{j<i}X(j)$. Since $C^D$ is $\tilde{\kappa}$ small, $A \to X(\tilde{\kappa})$ is a weak equivalence with size less than $\tilde{\kappa}^+$. \(\Box\)

**Theorem 3.5.20** Let $C$ be a (weakly) suitable cell category with HELP and $D$ a small category. Then $(C^D, I(C^D))$ is a (weakly) suitable cell category with HELP.
Proof: 3.5.7, 3.5.17, 3.5.18, 3.5.19. □
Chapter 4

Monoidal Categories

This chapter is concerned with monoidal categories. Our main goal is to show that under certain conditions the monoidal objects in a suitable cell category also form a suitable cell category. Along the way we also get a nice description of colimits of monoidal objects as colimits in the underlying category. Section 1 defines monoidal categories and monoidal objects. Section 2 introduces the monoid category on a small category and shows that the colimit of a diagram of monoidal objects is the same as a colimit of objects of its tensor diagram. This also allows us to give an adjoint to the forgetful functor from monoidal objects to objects. In section 3 we give another description of this colimit in a special case. In section 4 we show that with a few hypotheses the monoidal objects over a suitable cell category also give us a suitable cell category. In section 5 and 6 we give examples.

4.1 Monoidal Categories

Our definition of monoidal category has been taken from Mac Lane ([19]).

Definition 4.1.1 A monoidal category, $\mathcal{B} = (\mathcal{B}, \times, e, \alpha, \lambda, \rho)$ is a category $\mathcal{B}$, a bifunctor $\times : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ called the product, an object $e \in \mathcal{B}$ called the unit object and three natural isomorphisms $\alpha, \lambda, \rho$.

$$\alpha = \alpha_{abc} : a \times (b \times c) \to (a \times b) \times c$$
is natural for all \(a, b, c \in B\), and the following diagram commutes

\[
\begin{array}{c}
\begin{array}{ccc}
\alpha & : & a \times (b \times (c \times d)) \to (a \times b) \times (c \times d) \\
\downarrow \text{id} \times \alpha & & \downarrow \text{id} \times \alpha \\
\alpha & : & a \times ((b \times c) \times d) \to (a \times (b \times c)) \times d
\end{array}
\end{array}
\]

commutes for all \(a, b, c, d \in B\). Also \(\lambda\) and \(\rho\) are natural for all objects \(a \in B\).

\[
\lambda_a : e \times a \to a
\]

\[
\rho_a : a \times e \to a
\]

and the following diagram commutes for all \(a, b \in B\)

\[
\begin{array}{c}
\begin{array}{ccc}
\alpha & : & a \times (e \times b) \to (a \times e) \times b \\
\downarrow \text{id} \times \lambda & & \downarrow \rho \times \text{id} \\
\alpha & : & a \times b
\end{array}
\end{array}
\]

finally we require

\[
\lambda_e = \rho_e : e \times e \to e
\]

If \(B\) also comes with natural isomorphisms

\[
\gamma_{a,b} : a \times b \to b \times a
\]

such that \(\gamma_{a,b} \gamma_{b,a} = \text{id}\), \(\rho_b = \lambda_b \gamma_{b,e}\) and such that the following diagram commutes

\[
\begin{array}{c}
\begin{array}{ccc}
\alpha & : & a \times (b \times c) \to (a \times b) \times c \\
\downarrow \text{id} \times \gamma & & \downarrow \alpha \\
\gamma & : & a \times (c \times b) \to (c \times a) \times b
\end{array}
\end{array}
\]

then we say that \(B\) is symmetric.
We say that \((\mathcal{B}, \times)\) is strictly monoidal if \(e \times a = a = e \times a\) for every \(a \in \mathcal{B}\) and \((a \times b) \times c = a \times (b \times c)\) for every \(a, b, c \in \mathcal{B}\) and the maps \(\alpha, \lambda\) and \(\rho\) are the identity.

We call a monoidal category \((\mathcal{B}, \times, e, \alpha, \lambda, \rho)\) monoidally cocomplete if for all small categories \(\mathcal{D}_i\) and for every \(G_i \in \mathcal{B}^{\mathcal{D}_i}\) the natural map

\[
\theta : \text{colim}_{x \in \mathcal{D}_i} \text{colim}_{y \in \mathcal{D}_i} (G_1(x) \times G_2(y)) \to (\text{colim}_{x \in \mathcal{D}_i} G_1(x)) \times (\text{colim}_{y \in \mathcal{D}_i} G_2(y))
\]

is an isomorphism.

It has been shown by Isabel ([16]) that any monoidal category is equivalent to a strict monoidal category. Since we do not know if all our other constructions preserve this equivalence we choose to not assume that our monoidal categories are strict.

For example take the category of sets. Let the product be the Cartesian product, \(\times\), the unit object any one element set, \(e = \{\star\}\), \(\alpha\) the map which sends \((x, (y, z))\) to \(((x, y), z)\), \(\lambda\) the map which sends \((\star, x)\) to \(x\) and \(\rho\) the map which sends \((x, \star)\) to \(x\). Then \((\text{Set}, \times, e, \alpha, \lambda, \rho)\) is a monoidal category. It is easy to see it is monoidally cocomplete. Another simple example is that of \(R\) modules. Let the product be tensor product over \(R\), \(\otimes_R\), the unit object \(R\) itself and \(\alpha, \lambda\) and \(\rho\) the obvious maps. Then \((\text{DGM}(R), \otimes_R, R_0, \alpha, \lambda, \rho)\) is a monoidal category. This category is also monoidally cocomplete. The example of \(\text{Top}\) and \(\text{Top}_*\) will be looked at later in the chapter. We will see that the Cartesian product makes them monoidal but not monoidally cocomplete.

**Lemma 4.1.2** Let \(\mathcal{C}\) be monoidal and cocomplete. If for every \(a \in \mathcal{C}\) the functors \(a \times \_\) and \(\_ \times a\) have right adjoints. Then \(\mathcal{C}\) is monoidally cocomplete.

**Proof:** Easy. \(\square\)

**Definition 4.1.3** Let \((\mathcal{B}, \times, e, \alpha, \lambda, \rho)\) be a monoidal category. Then a monoidal object in \(\mathcal{B}\) is a \((C, \phi, e)\) where \(C \in \mathcal{B}\), \(\phi \in \text{Hom}(C \times C, C)\) is called the multiplication and
$e \in \text{Hom}(e,C)$ is called the unit, such that the following diagrams commute.

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) {$C \times (C \times C)$};
\node (b) at (0,1) {$(C \times C) \times C$};
\node (c) at (0,0) {$C \times C$};
\node (d) at (1,0) {$C$};
\node (e) at (1,-1) {$C$};
\draw[->] (a) to node {$\alpha$} (b);
\draw[->] (b) to node {$\phi$} (c);
\draw[->] (c) to node [swap] {$\phi$} (d);
\draw[->] (d) to node [swap] {$\phi$} (e);
\draw[->] (a) to node {$\text{id} \times \phi$} (c);
\end{tikzpicture}}
\end{array}
\]

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) {$e \times C$};
\node (b) at (0,1) {$C \times C$};
\node (c) at (0,0) {$C \times e$};
\node (d) at (1,0) {$C$};
\node (e) at (1,-1) {$C$};
\draw[->] (a) to node {$e \times \text{id}$} (b);
\draw[->] (b) to node {$\lambda$} (c);
\draw[->] (c) to node [swap] {$\rho$} (d);
\draw[->] (d) to node [swap] {$\rho$} (e);
\end{tikzpicture}}
\end{array}
\]

If $B$ is symmetric monoidal then we say $C$ is commutative if the following diagram commutes.

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) {$C \times C$};
\node (b) at (0,1) {$C \times C$};
\node (c) at (0,0) {$C$};
\node (d) at (1,0) {$C$};
\draw[->] (a) to node {$\tau_{C,C}$} (b);
\draw[->] (b) to node {$\phi$} (d);
\draw[->] (c) to node [swap] {$\phi$} (d);
\end{tikzpicture}}
\end{array}
\]

Monoidal maps are maps in $B$ that commute with the structures $\phi$ and $e$. We denote the category of monoidal objects and monoidal maps in $B$ by $B[\text{Mon}]$. In $B[\text{Mon}]$, $e$ is $\emptyset$ and if we have a terminal object in $B$ $*$ is $*$.

Let $F$ denote the forgetful functor $F : B[\text{Mon}] \to B$.

Let $X$ be a set. We let $M(X)$ denote the free (associative) monoid on $X$. Let $M^{\leq n}X$ denote the products of length $\leq n$ and $M^n$ denote the products of length exactly $n$.

In the category of sets the monoidal objects are monoids and in the category of $R$ modules the monoidal objects are $R$ algebras.

### 4.2 The Monoid Category on $D$

In this section for any small category we define its monoidal category and for any functor its monoidal functor. These are used in realizing colimits in $C[\text{Mon}]$ as colimits in $C$. 

Definition 4.2.1 Let $S$ be a set. We define sets $T^n(S)$. Define

\[ T^1(S) = S \cup \{e\} \]

\[ T^n(S) = \{(\alpha)(\beta) | \alpha \in T^i(S), \beta \in T^j(S), i + j = n\} \]

\[ TS = \cup_n T^n(S) \]

$(\alpha)(\beta)$ is called the product of $\alpha$ and $\beta$.

For a small category $D$, we define a category $TD$. First we let

\[ \text{obj}(TD) = T(\text{obj}(D)) \]

Next we construct the morphisms. Consider the set, $\hat{S}$, of symbols that is the closure under product and composition of the symbols $f, \alpha_{ABC}, \alpha^{-1}_{ABC}, \lambda_A, \lambda^{-1}_A, \rho_A$ and $\rho^{-1}_A$ such that $A, B, C \in TD$ and $f \in \text{Hom}_D$. For $f, g \in \hat{S}$ we denote the product by $(f)(g)$. We interpret the composition of zero maps as the identity. We define $\text{Hom}_{TD} = \hat{S} / \sim$ where $\sim$ is the equivalence relation generated by the relations that hold for all monoidal categories 4.1.1 together with the following six other relations which hold for all $g_i, h_i \in \text{Hom}_{TD}$ when both sides are defined.

\[ (g_1 \circ g_2)(h_1 \circ h_2) = (g_1)(h_1) \circ (g_2)(h_2) \]

\[ \alpha \circ (g_1)((g_2)(g_3)) = ((g_1)(g_2))(g_3) \circ \alpha \]

\[ \rho \circ (id)(g_1) = g_1 \circ \rho \]

\[ \lambda \circ (g_1)(id) = g_1 \circ \lambda \]

and for $g_1 \sim g_2$

\[ (g_1)(h_1) \sim (g_2)(h_1) \]
and

\[(h_1)(g_1) \sim (h_1)(g_2)\].

Let \(S\) be the closure under products and composition of the set \(\hat{S}\) together with the symbols \(e_x : e \to x\) and \(\phi_x : (x)(x) \to x\) where \(x \in D\). Then let \(\text{Hom}_{TD} = S/ \sim'\) where \(\sim'\) is the equivalence relation generated by the same relations that hold for \(\sim\) together with

\[\phi_y(f \times f) = f \phi_x\]

\[\phi_x \circ (id)(\phi_x) \circ \alpha_{xxx} = \phi_x \circ (\phi_x)(id)\]

\[\rho_x = \phi_x \circ (e_x)(id)\]

\[\lambda_x = \phi_x \circ (id)(e_x)\]

Let \(\hat{T}^n(D)\) denote the full subcategory of \(\hat{T}(D)\) whose objects are \(T^n(D)\). \(TD\) is called the monoid category on \(D\).

**Definition 4.2.2** Let \(R : \text{obj}(TD) \to M(\text{obj} D)\) be the associatization map. That is the one that sends \(x\) to \(x\) for \(x \in \text{obj} D \cup \{e\}\) and \((\alpha)(\beta)\) to \(R(\alpha)R(\beta)\).

**Definition 4.2.3** Let \(x \in T(M(\text{obj} D))\) then we define \(B(x) \in \text{obj} TD\) in the following way. If \(x \in MD\) and \(x = x_1 \ldots x_n\) then \(B(x) = (\ldots((x_1)(x_2))(x_3))\ldots(x_n))\). If we have already defined \(B(\alpha)\) and \(B(\beta)\) for \(\alpha, \beta \in T(MD)\) then we let \(B((\alpha)(\beta)) = (B\alpha)(B\beta)\).

Given a functor \(G : D \to C[\text{Mon}]\) or \(G : D \to C\) we wish to construct a functor \(TG : TD \to C\) or \(TG : \hat{TD} \to C\).

**Definition 4.2.4** Let \(C\) be a monoidal category. Let \(G : D \to C[\text{Mon}]\) or \(C\). We define a functor \(TG : TD \to C\) respectively \(\hat{T}G : \hat{TD} \to C\). On objects \(TG\) and \(\hat{T}G\) are determined by letting

\[\hat{T}G(x) = G(x)\]
if $x \in D$

$$ \tilde{T}G(x) = e $$

and

$$ \tilde{T}G((A)(B)) = \tilde{T}G(A) \times \tilde{T}G(B). $$

Let $T G(x) = \tilde{T}(F \circ G)(x)$. On maps $T G$ is determined by letting

$$ \tilde{T}G(f) = G(f) $$

if $f \in Hom_D$ and for $A, B, C \in TD$

$$ \tilde{T}G(\alpha_{ABC}) = \alpha_{\tilde{T}G(A)\tilde{T}G(B)\tilde{T}G(C)} $$

$$ \tilde{T}G(\rho_A) = \rho_{\tilde{T}G(A)} $$

$$ \tilde{T}G(\lambda_A) = \lambda_{\tilde{T}G(A)} $$

$$ \tilde{T}G(f \circ g) = \tilde{T}G(f) \circ \tilde{T}G(g) $$

$$ \tilde{T}G((f)(g)) = \tilde{T}G(f) \times \tilde{T}G(g). $$

These same rules together with the following two others determine $T G$

$$ TG(\phi_x) = \phi_{TG(x)} $$

$$ TG(\varepsilon_x) = e_{TG(x)}. $$

**Lemma 4.2.5** $T G : TD \to C$ and $\tilde{T}G : TD \to C$ are functors.

**Proof:** We have to show $T G$ and $\tilde{T}G$ are well defined. Since there are no relations on objects $\tilde{T}G = T G : objTD \to objC[Mon]$. On maps we still need to show that the relations in $TD$ and $\tilde{T}D$ are preserved under $T G$ and $\tilde{T}G$. The relations between the $\lambda, \rho$ and $\alpha$ are
preserved since \((C, \times, \lambda, \rho, \alpha)\) is a monoidal category. The first four relations of definition 4.2.1 follow from the naturality of \(\alpha, \lambda, \rho\) and \(\times\). The last two relations follow since \(\times\) is well defined in \(C\). This completes the proof for \(\bar{T}G\). The four extra relations that hold between maps in \(TD\) are preserved by \(TG\) since \(G(x)\) is a monoidal object for every \(x \in D\) and since \(G(f)\) is monoidal for every \(f \in Hom_D\). □

We now give \(colim_{TD} TG\) the structure of a monoidal object in \(C\).

**Definition 4.2.6** Let \(C\) be a monoidally cocomplete category. Let \(G\) be a functor from \(D\) to \(C\) or \(C[Mon]\). Let \(Z\) denote \(colim_{TD} TG\). A map

\[ \phi : colim_{TD} colim_{TD}(TG \times TG) \to Z \]

is determined by letting

\[ \phi(A, B) = id : TG(A) \times TG(B) \to TG((A)(B)). \]

\(\phi\) is then the composition

\[ \phi \circ \theta^{-1} : Z \times Z \to colim_{TD} colim_{TD}(TG \times TG) \to Z. \]

Let \(e_Z : e \to Z\) be the map

\[ e \to TG(e) \to Z \]

When \(Z\) is \(colim_{\bar{T}G} \bar{T}G\) the definitions are similar.

We will leave the subscripts off the \(e\) and \(\phi\) notation when the objects they refer to are obvious.

**Lemma 4.2.7** \((colim_{TD} TG, \phi, e), (colim_{\bar{T}G} \bar{T}G, \phi, e) \in C[Mon]\).
**Proof:** We need to show the identity and associativity. Let $Z$ denote $\text{colim}_{TD} TG$ and consider the following diagram.

\[
\begin{array}{c}
\text{colim}_{(x,y,z) \in TD^3} TG(x) \times (TG(y) \times TG(z)) \xrightarrow{\alpha} \text{colim}_{(x,y,z) \in TD^3} (TG(x) \times TG(y)) \times TG(z) \\
\xrightarrow{\phi(id \times \phi)} \downarrow \phi(\phi \times \text{id})
\end{array}
\]

The square commutes from the associativity of $\theta$ and the triangle commutes since its restriction to each $TG(x) \times (TG(y) \times TG(z))$ commutes. Therefore $\phi$ is associative.

For the left identity consider the following diagram.

\[
\begin{array}{c}
e \times Z \xrightarrow{e \times \text{id}} Z \times Z \xrightarrow{\text{id}} Z \times Z \xrightarrow{\text{id}} Z
\end{array}
\]

The square commutes since $\theta$ is natural and the triangle commutes since it commutes when restricted to each $e \times TG(x)$. Then the fact that $\lambda \theta = \lambda$ is enough to see that $e$ is a left identity. The proof that $e$ is a right identity is similar. So $Z$ is a monoidal object. The proof that $\text{colim}_{TD} TG$ is monoidal is similar. □

**Definition 4.2.8** Let $\eta \in \text{Hom}_{C[M]}(F,G)$. We define $T\eta \in \text{Hom}_{CTD}(TF, TG)$ in the obvious way. That is $T\eta(e) = id$, $(T\eta)(x) = \eta(x)$ for $x \in D$ and $T\eta((\alpha)(\beta)) = T\eta(\alpha) \times T\eta(\beta)$. For $\eta \in \text{Hom}_{CD}(F,G)$ we define $\tilde{T}\eta \in \text{Hom}_{CTD}$ in the same way.

Recall that for $X \in C$ we also denote $* \rightarrow X$ by $X$. Also recall 4.2.4.

**Definition 4.2.9** Let $X \in C$. Let $TX$ denote $(\text{colim}_{\tilde{T}} \tilde{T}X, \phi, e)$ and $T^n X$ denote $\text{colim}_{\tilde{T}^n} \tilde{T}X \in C$. 
In \( Top \), \( TX \) is just the James construction on \( X \). For a strictly monoidally cocomplete category in which \( e \) is the initial object Baues and Conduche ([5]) also constructed a free functor \( C \rightarrow C[Mon] \). This application of our more general construction allows one to drop the hypothesis on the unit.

The following four lemmas about \( T \) and \( \hat{T} \) are easy to check.

**Lemma 4.2.10** \( T : C[Mon]^D \rightarrow C^TD \) and \( \hat{T} : C^D \rightarrow C^TD \) are functors.

We let \( SC \) denote the category of small categories and functors between them.

**Lemma 4.2.11** \( T : SC \rightarrow SC \) and \( \hat{T} : SC \rightarrow SC \) are functors.

**Lemma 4.2.12** Let \( F \in \text{Hom}_{SC}(D, E) \). Then the following two diagrams commute

\[
\begin{array}{ccc}
C[Mon]^E & \xrightarrow{T} & C^TE \\
\downarrow{F^*} & \downarrow{T_{F^*}} \\
C[Mon]^D & \xrightarrow{T} & C^TD \\
\end{array}
\]

\[
\begin{array}{ccc}
C^E & \xrightarrow{\hat{T}} & C^{\hat{T}E} \\
\downarrow{F^*} & \downarrow{\hat{T}_{F^*}} \\
C^D & \xrightarrow{\hat{T}} & C^{\hat{T}D} \\
\end{array}
\]

The last lemma just says that \( T \) and \( \hat{T} \) are natural in the category variable.

**Lemma 4.2.13** \( \phi \) and \( e \) are natural in both the functor and category variables on the colimits.

Now follows a lemma supplied with a proof. It is similar to [19] chapter 7 theorem 2 and, unlike that theorem, holds for pointed topological spaces. This will be shown in a later section.

**Lemma 4.2.14** \( T : C \rightarrow C[Mon] \) is a functor left adjoint to \( \mathcal{F} \).
Proof: We have proved that $TX \in C[Mon]$. Functoriality follows from Lemmas 4.2.10 and 4.2.13 together with the functoriality of $\text{colim} : C^D \to C$.

Next we turn to adjointness. Let $X \in C$ and $(A, \phi_A, e_A) \in C[Mon]$. We let $f \in \text{Hom}(X, FA)$ and show that $f$ determines $f' \in \text{Hom}(TX, A)$. For every $\alpha \in T(*)$ let $\alpha$ denote the map $\tilde{T}X(\alpha) \to TX$. Then $f' e$ must be the map $e_A : e \to A$. $f'(*)$ is just $f$. All other maps are determined by taking products of these maps and composing with multiplications in $A$. This is both well defined and compatible with the relations in $\tilde{T}(*$) due to the fact that $A$ is a monoidal object. □

**Corollary 4.2.15** If $C$ is complete then $C[Mon]$ is complete.

**Proof:** This follows since $< FT, \eta, e >$ is a monad, $C[Mon]$ is the category of $FT$ algebras and $C$ is complete. A more direct proof follows easily also. □

**Theorem 4.2.16** Let $G : D \to C[Mon]$ be a functor then

$$\text{colim}_{D}^{C[Mon]} G = (\text{colim}_{D}^{C} TG, \phi, e).$$

**Proof:** First observe that again Lemmas 4.2.10 and 4.2.13 show that the association

$$G \to (\text{colim}_{D}^{C} TG, \phi, e)$$

is a functor, which we will denote by $F : C[Mon]^D \to C[Mon]$. Let $\{f(x) : G(x) \to X\}_{x \in D}$ be a set of compatible maps. Then the map on the $TG(x)$ for $x \in TD^n$ is determined recursively, thus determining a monoidal map $F f : FG \to X$. □

This theorem allows us to understand the properties of colimits in $C[Mon]$ through our understanding of colimits in $C$. This will be taken advantage of later to give a model category structure for some $C[Mon]$ and also used for calculating the homology of monoidal objects. Furthermore the theorem has the following useful corollary.

**Corollary 4.2.17** $C$ monoidally cocomplete implies that $C[Mon]$ is cocomplete.
4.3 Another Description of the Monoidal Category on D

This alternative description will be used later in our calculations of the chains on a monoidal object in Chapter 5.

**Definition 4.3.1** Let $A \in C$. In a product of two spaces we let $i_1$ denote the inclusion into the second factor. We make the following recursive definitions.

Let $U^0A = Y^0A = e$. Let the following pushouts define $Y^n(A)$, $U^n(A)$, $j_n : U^n(A) \to U^{n+1}(A)$ and $p_n : (A \vee e) \times U^n(A) \to U^{n+1}(A)$.

\[
\begin{array}{c}
U^{n-1}(A) \xrightarrow{i_1} (A \vee e) \times U^{n-1}(A) \\
\downarrow \quad j_{n-1} \\
U^n(A) \rightarrow Y^n(A)
\end{array}
\]

\[
\begin{array}{c}
Y^n(A) \xrightarrow{id+p_{n-1}} U^nA \\
\downarrow \quad i_1+(id \times j_{n-1}) \\
(A \vee e) \times U^nA \xrightarrow{p_n} U^{n+1}A
\end{array}
\]

where $j_0 : e \to A \vee e$ is the inclusion into the second factor.

**Lemma 4.3.2** There is a natural isomorphism $U^n(A) \cong T^n(A)$.

**Proof:** We exhibit maps in both directions. First observe that $U^1(A) = T^1(A) = A \vee e$.

Remembering that $\times$ commutes with colimits, let

\[p'_n : (A \vee e) \times T^nA \to T^{n+1}A\]
be the map that when restricted to \((A \lor e) \times TA(x)\), for any \((x, y) \in T^1(*) \times T^n(*)\), is just the map \(TA((x)(y)) \to T^{n+1}A\). \(p'_n\) makes the following diagram commute.

\[
\begin{array}{ccc}
(A \lor e) \times T^{n-1}A & \xrightarrow{p_{n-1}} & T^nA \\
\downarrow{id \times i} & & \downarrow{i} \\
(A \lor e) \times T^nA & \xrightarrow{p'_n} & T^{n+1}A \\
\end{array}
\]

First we construct the maps.

\[\phi_{n+1} : U^{n+1}A \to T^{n+1}A\]

Assume that we already have natural maps \(\phi_n : U^nA \to T^nA\) such that the following diagram \((*)_n\) commutes.

\[
\begin{array}{ccc}
(A \lor e) \times U^{n-1}A & \xrightarrow{p_{n-1}} & U^nA \\
\downarrow{id \times \phi_{n-1}} & & \downarrow{\phi_n} \\
(A \lor e) \times T^{n-1}A & \xrightarrow{p'_n} & T^nA \\
\end{array}
\]

The maps

\[p'_n \circ (id \times \phi_n) : (A \lor e) \times U^nA \to T^{n+1}\]

and

\[i \circ \phi_n : U^n \to T^{n+1}\]

would determine a natural \(\phi_{n+1}\) making \((*)_{n+1}\) commute if the two maps agree on \(Y^nA\). Since \(Y^nA\) is also described as a pushout it is sufficient to check agreement when restricting to the two spaces \(U^nA\) and \(A \times U^{n-1}A\). Agreement on these two spaces follows from the commutativity of the following two diagrams:

\[
\begin{array}{ccc}
U^nA & \xrightarrow{i_1} & T^nA \\
\downarrow{i_1} & & \downarrow{i_1} \\
(A \lor e) \times U^nA & \to & (A \lor e) \times T^nA
\end{array}
\]
and

\[
\begin{array}{c}
(A \vee e) \times U^{n-1} A^{p_{n-1}} \rightarrow U^n A \\
\downarrow \\
(A \vee e) \times T^{n-1} A^{p_{n-1}'} \rightarrow T^n A \\
\downarrow \text{id}_x \\
(A \vee e) \times T^n A \rightarrow T^{n+1} A.
\end{array}
\]

Soon we construct the maps

\[
\phi_{n+1}^{-1} : T^{n+1} \rightarrow U^{n+1}.
\]

Recall that for any \( A \in C \) we also use \( A \) to denote the functor \( * \rightarrow A \). The map is determined by the map restricted to \( (A \vee e) \times (\tilde{T}A)(x) \) where \( x \in T^n(*) \) is a product with \( n \) or fewer \(*'s\) in it. Assume that we have constructed all the \( \phi_j^{-1} \) for \( j \leq n \) naturally and compatibly with inclusion. Determine the map \( \phi_{n+1}^{-1} \) by the compositions

\[
(A \vee e) \times (\tilde{T}A)(x) \rightarrow (A \vee e) \times T^n A \xrightarrow{id \times \phi_n^{-1}} (A \vee e) \times U^n A \xrightarrow{p_n} U^{n+1} A.
\]

Clearly the definition is natural. It remains to be shown that the map is compatible with the \( \phi_j^{-1} \) previously defined. It is in fact enough to check compatibility with \( \phi_n^{-1} \).

Now let \( x \) have fewer than \( n \) \(*'s\) and therefore at least one \( e \). Let \( \tilde{x} \in T^{n-1}(*) \) denote \( x \) with some \( e \) removed. The following diagram then commutes

\[
\begin{array}{c}
(A \vee e) \times \tilde{T}A(x) \xrightarrow{\tilde{T}A(\tilde{x})} (A \vee e) \times \tilde{T}A(\tilde{x}) \rightarrow (A \vee e) \times T^{n-1} A \xrightarrow{id \times \phi_n^{-1}} (A \vee e) \times U^{n-1} A \\
\downarrow \\
(A \vee e) \times \tilde{T}A(x) \rightarrow (A \vee e) \times T^n A \xrightarrow{id \times \phi_n^{-1}} (A \vee e) \times U^n A.
\end{array}
\]

Finally we have to see that the two ways of going from \( e \times T A(x) \) agree. Let \( f \) denote the map

\[
e \times \tilde{T}A(x) \xrightarrow{\lambda} \tilde{T}A(x) \rightarrow (A \vee e) \times T^{n-1} A \rightarrow (A \vee e) \times U^{n-1} A \rightarrow Y^n A.
\]
Then we have

\[ \phi_{n+1}^{-1}|_{e \times TA(x)} = j_n(id + p_{n-1})f \]

and

\[ \phi_n^{-1} \lambda|_{e \times TA(x)} = p_n(i_1 + (id \times j_{n-1}))f \]

and so both are equal in the image, \( U^{n+1}.A \).

It remains only to show that \( \phi_n \phi_n^{-1} = id = \phi_n^{-1} \phi_n \). Assuming this equation for \( n \) it follows easily for \( n + 1 \) from the commutativity of the following three diagrams:

\[ \begin{array}{c}
\xymatrix{
U^n \ar[r]^-{\phi_n} \ar[d]_{\text{id}} & T^n \ar[r]^-{i} \ar[d]_{\phi_n^{-1}} & T^{n+1} \ar[d]_{\phi_n^{-1}} \\
U^n \ar[r]^-{\text{id}} & U^{n+1} & 
}
\end{array} \]

\[ \begin{array}{c}
\xymatrix{
(A \vee e) \times U^n \ar[r]^-{id \times \phi_n} \ar[d]_{id} & (A \vee e) \times T^n \ar[r]^-{p_n} \ar[d]_{id \times \phi_n^{-1}} & T^{n+1} \ar[d]_{\phi_n^{-1}} \\
(A \vee e) \times U^n \ar[r]^-{p_n} & U^{n+1} & 
}
\end{array} \]

\[ \begin{array}{c}
\xymatrix{
(A \vee e) \times T^n A \ar[r]^-{id \times \phi_n} \ar[d]_{id} & (A \vee e) \times U^n A \ar[r]^-{p_n} \ar[d]_{id \times \phi_n} & U^{n+1} \ar[d]_{\phi_{n+1}} \\
(A \vee e) \times T^n A \ar[r]^-{p_n} & T^{n+1} & 
}
\end{array} \]

\[ \square \]

4.4 \textbf{C[Mon], the suitable}

For this section let \((C, I)\) be a monoidally cocomplete suitable cell category. We wish to put a model category structure in \( \text{C}[\text{Mon}] \). Let

\[ I(\text{C}[\text{Mon}]) = \{ TA(\alpha) \to TB(\alpha) \}_{\alpha \in I} \]
and let $f$ be weak equivalence if and only if $\mathcal{F}f$ is. We will see that with appropriate hypotheses this determines a suitable cell category structure on $C[\text{Mon}]$.

**Definition 4.4.1** Let $f, g : TA(\alpha) \to X$ be maps in $C[\text{Mon}]$. Define $g \sim f$ if and only if there exists a commuting diagram in $C[\text{Mon}]$.

$$
\begin{array}{c}
TA(\alpha) \vee TA(\alpha)^{f_1 + f_2} \to X \\
TCylA(\alpha)
\end{array}
$$

**Lemma 4.4.2** $\pi(A(\alpha), FKN) = \text{Hom}(TA(\alpha), X)/ \sim$

**Proof:** Easy from Lemma 4.2.14. \qed

**Definition 4.4.3** Let $C$ have an interval. Assume there exist associative natural transformations

$$
\delta : (X \times Y) \otimes I \to (X \otimes I) \times (Y \otimes I)
$$

Then say that $C$ has a **multiplicative interval**. We will call $\delta$ the diagonal.

**Definition 4.4.4** Let $(C, I)$ be a monoidally cocomplete suitable cell category with a multiplicative interval. Also assume $(C[\text{Mon}], I(C[\text{Mon}]))$ is $\kappa$ small. Then we say that $C$ is suitably monoidal.

**Definition 4.4.5** Let $C$ have a multiplicative interval. Let $f, g \in \text{Hom}_{C[\text{Mon}]}(X, Y)$. Then a multiplicative homotopy from $f$ to $g$ is a map in $C$ $H : X \otimes I \to Y$ such that $H(\partial_0 + \partial_1) = \mathcal{F}f + \mathcal{F}g$ and

$$
\phi(H \times H)\delta = H(\phi \otimes I) : X \times X \otimes I \to Y.
$$

**Lemma 4.4.6** Let $C$ have a multiplicative interval. Let $D$ be a small category. Let $F, G \in C^D$, $f \in \text{Hom}_D(G, F)$ and $r \in \text{Hom}_D(F, G)$ such that $fr = id$ and there exists $H \in \text{Hom}_D(G \otimes I, G)$ such that for every $x \in D$ $H(x) : r(x)f(x) \simeq_F xid$. Then $r' = \text{colim} r$.
colimF \to \text{colim}G \text{ and } f' : \text{colim}f \colon \text{colim}G \to \text{colim}F \text{ are such that } f'r' = \text{id} \text{ and } H' = \text{colim}H : \text{colim}G \otimes I \to \text{colim}G \text{ is a homotopy } r'f' \simeq_{\text{colim}F} \text{id}. \text{ If } f, r \in \text{Hom}_{C[\text{Mon}]}\text{ and for every } x \in D \text{ } H(x) \text{ is multiplicative then } f', r' \in \text{Hom}_{C[\text{Mon}]} \text{ and } H' \text{ is multiplicative.}

\textbf{Proof:} \text{ Follows since } \_ \otimes I \text{ commutes with colimits. } \Box

\textbf{Lemma 4.4.7} \text{ Let } C \text{ have a multiplicative interval. Assume we are given a diagram in } C[\text{Mon}]

\begin{diagram}
  X & \rightarrow & Y \\
  \downarrow & & \downarrow \\
  X' & \rightarrow & Y
\end{diagram}

\text{ such that } j \text{ is a weak equivalence and a cell complex with subcomplexes } f \text{ and } f'f \text{ and such that } f' \text{ is a subcomplex of the complex } j' \text{ with } s(f') = 1.

\text{ Assume that we have a map in } C[\text{Mon}] \text{ } r : X' \rightarrow X \text{ such that } rf = \text{id} \text{ and there exists a multiplicative homotopy } H \text{ between } j' \text{ and } jr. \text{ Then there exists a map } r' : X'' \rightarrow X \text{ extending } r \text{ and a multiplicative homotopy } H' \text{ between } j'' \text{ and } jr' \text{ extending } H.

\textbf{Proof:} \text{ Since } s(f') = 1 \text{ we have a pushout in } C[\text{Mon}].

\begin{diagram}
  TA(\alpha) & \rightarrow & X' \\
  \downarrow & & \downarrow \\
  TB(\alpha) & \rightarrow & X''
\end{diagram}

Let \( g \) be the composition \( A(\alpha) \rightarrow TA(\alpha) \rightarrow X' \). \text{ Let } W \text{ be the pushout}

\begin{diagram}
  A(\alpha) \rightarrow & X' \\
  \downarrow & \downarrow \\
  B(\alpha) & \rightarrow W.
\end{diagram}

\text{ By HELP for } \_ \otimes I \text{ we get extensions } r(0) \text{ and } H(0) \text{ of } r \text{ and } H \text{ over } W. \text{ Let } V(0) = W(0) = W, U(0) = X', U(n + 1) = V(n) \times X' \times X', V(n + 1) = V(n) \times X' \times W \text{ and let } W(n + 1)
and \( h(n + 1) \) be defined by the pushout

\[
\begin{array}{ccc}
U(n + 1) & \xrightarrow{g(n)} & W(n) \\
\downarrow & & \downarrow \\
V(n + 1) & \xrightarrow{h(n + 1)} & W(n + 1)
\end{array}
\]

The map \( g(n + 1) \) is then determined by \( h(n + 1) \) and the multiplication in \( X' \). Observe that \( \text{colim} W(n) = X'' \) and that the extensions \( r' \) and \( H' \) are determined by \( r(0), H(0) \) and multiplicativity. □

**Lemma 4.4.8** Let \( C \) have a multiplicative interval. Let \( j : X \to Y \) be a cofibration in \( C[Mon] \) and \( F(j) \in W \). Then there exists monoidal \( r : B \to A \) such that \( rj = \text{id} \) and \( j r \simeq_A \text{id} \) by a multiplicative homotopy.

**Proof:** For \( j \in C(I(C[Mon])) \) the result follows from 4.4.7 by induction since \( - \otimes I \) commutes with direct limits. If \( j \) is any cofibration the lemma follows from the fact that any cofibration is a retract of a map in \( C \). □

**Lemma 4.4.9** Let \( C \) have a multiplicative interval. If \( X \) is cofibrant then \( TCX \simeq_e e \).

**Proof:** \( j : e \to CX \vee e \) is a weak equivalence by the pushout condition in \( C \). So using Lemma 4.4.6 and HELP for \( - \otimes I \) there exists \( r : CX \vee e \to e \) and \( H : (CX \vee e) \otimes I \to CX \vee e \) such that \( rj = \text{id} \) and \( H : j r \simeq_e \text{id} \). So we get unique multiplicative extensions \( r' : TCX \to e \) and \( H' : TCX \otimes I \to TCX \) such that \( r'e = \text{id} \) and \( H' : er' \simeq_e \text{id} \). □

**Lemma 4.4.10** Let \( C \) have a multiplicative interval. Then \( \times \) preserves weak equivalences between objects cofibrant over \( e \).

**Proof:** Let \( j : A \to A' \in C(I) \cap W \) then using HELP there exists \( r : A' \to A \) and \( H : A' \otimes I \to A' \) such that \( rj = \text{id} \) and \( H : jr \simeq_A \text{id} \). So for any \( B \) let \( G \) be the composition

\[
(A' \times B) \otimes I \xrightarrow{\delta} (A' \otimes I) \times (B \otimes I) \xrightarrow{H \times \text{id}} A' \times B
\]
where \(id\) denotes the identity homotopy. So we have \((r \times id)(j \times id) = id\) and \(G : (j \times id)(r \times id) \simeq_{A \times B} id\). So \(A \times B\) is weakly equivalent to \(A' \times B\). If we now let \(j\) be any acyclic cofibration then we get that \(A \times B\) is weakly equivalent to \(A' \times B\) since weak equivalences are closed under retracts. Finally if \(j\) is any weak equivalence between objects cofibrant over \(e\) then there exists \(A''\) such that we have acyclic cofibrations \(A \to A''\) and \(A' \to A''\). Therefore the result follows by composition. \(\square\)

**Lemma 4.4.11** Let \(C\) have a multiplicative interval. Let

\[
\begin{array}{ccc}
TA(\alpha) & \longrightarrow & X \\
\downarrow & & \downarrow \\
TB(\alpha) & \longrightarrow & Y
\end{array}
\]

be a pushout in \(C[Mon]\). Assume there exists \(f : Cyl(X, W) \to X\) such that \(f \partial_0(X) = id\) and assume there exists a multiplicative homotopy \(H : \partial_0(X)f \simeq X id\). Then

\[
\begin{array}{ccc}
CylTA(\alpha) & \longrightarrow & Cyl(X, W) \\
\downarrow & & \downarrow \\
CylTB(\alpha) & \longrightarrow & Cyl(Y, W)
\end{array}
\]

is a pushout and there exists \(f' : Cyl(Y, W) \to Y\) such that \(f \partial_0(Y) = id\) and such that there exists a multiplicative homotopy \(H' : \partial_0(Y)f' \simeq_Y id\) extending \(H\).

**Proof:** The second statement implies the first so we proceed to prove the second.

Let \(Z\) be the pushout of

\[
\begin{array}{ccc}
A(\alpha) & \xrightarrow{g} & FX \\
\downarrow & & \downarrow \\
B(\alpha) & \quad & 
\end{array}
\]

and \(U\) the pushout of

\[
\begin{array}{ccc}
Cyl(A(\alpha)) & \xrightarrow{Cyl \ g} & FCyl(X, W) \\
\downarrow & & \downarrow \\
Cyl(B(\alpha)). & \quad & 
\end{array}
\]
Let $\partial_0$ denote the map induced by $\partial_0$ on the corners of the pushout.

We observe that the map $\partial_0 : Z \rightarrow U$ is a weak equivalence. Let $Z'$ denote the pushout

$$A(\alpha) \xrightarrow{Cyl \ g \partial_0} Cyl(X,W) \xrightarrow{\partial_1} B(\alpha).$$

Then we have a pushout

$$Cyl A(\alpha) \vee_{A(\alpha)} B(\alpha) \longrightarrow Z'$$

$$Cyl B(\alpha) \longrightarrow U.$$ 

$X \rightarrow Z'$ is a weak equivalence by 3.4.25 and $Z' \rightarrow U$ is a weak equivalence by the pushout condition for $C$ so $\partial_0$ is a weak equivalence.

Consider the solid arrow diagram

The dashed extensions $G$ and $g$ give extensions of $H$ and $f$. The extensions to $H'$ and $f'$ are then determined by multiplicativity. □

**Lemma 4.4.12** Let $W \rightarrow X \in C(I(C[Mon]))$. Then there exists $f : Cyl(X,W) \rightarrow X$ and multiplicative $H : Cyl(X,W) \otimes I \rightarrow Cyl(X,W)$ such that $fr = id$ and $H : rf \cong_X id$.

**Proof:** Use 4.4.6 and 4.4.11. □

**Lemma 4.4.13** For every $X \in C[Mon]$ $\pi(TA(\alpha),X) = \pi(A(\alpha),\mathcal{F}X)$. 
Proof: Factor \( T Cyl(A(\alpha)) \to T(A(\alpha)) \) as in Chapter 3 by a cofibration followed by an acyclic fibration.

\[
\begin{array}{ccc}
TCyl(A(\alpha)) & \xrightarrow{\sigma} & TA(\alpha) \\
\downarrow \cong & & \downarrow \\
Cyl' TA(\alpha) & \xrightarrow{\sigma'} & \end{array}
\]

Then \( Cyl' TA(\alpha) \) is a cylinder object with the Chapter 3 structure. So \( \sigma' \) is a weak equivalence in that sense but that means that \( F\sigma' \) is a weak equivalence also. Therefore \( Fi \) is a weak equivalence and we can use 4.4.8 to get a monoidal retract \( r \) such that \( ri = id \). Therefore the two different notions of weak equivalence are the same. Therefore the two different notions of homotopy are the same.

Lemma 4.4.14 If \( C \) is suitably monoidal then the pushout condition holds in \( C[Mon] \).

Proof: Let the following diagram be a pushout in \( C[Mon] \)

\[
\begin{array}{ccc}
A & \to & C \\
\downarrow f & & \downarrow g \\
B & \to & D
\end{array}
\]

and \( f \) a weak equivalence.

Let \( (C,A,0) = C, (C,A,n) = ((C,A,n-1) \times A) \times C, (C,f,0) = f \) and \( (C,f,n) = ((C,f,n-1) \times f) \times id \). Let \( D^n \) denote the pushout of

\[
\begin{array}{ccc}
(C,A,n) & \to & C \\
\downarrow (C,f,n) & & \downarrow \\
(C,B,n)
\end{array}
\]

Then

\[
\begin{array}{ccc}
(C,B,n-1) \times A \times C & \to & D^{n-1} \\
\downarrow & & \downarrow \\
(C,B,n) & \to & D^n
\end{array}
\]

is a pushout. Since inverse isomorphisms are easy to construct, \( \text{colim}D^n = D \). The retract \( D \to C \) and multiplicative homotopy \( D \otimes I \to D \) are induced by the retract \( B \to A \) and
multiplicative homotopy $B \otimes I \to B$. Which are gotten from Lemma 4.4.8. □

**Lemma 4.4.15** For $C$ a suitably monoidal category, HELP holds in $(C[\text{Mon}], I(C[\text{Mon}]))$.

**Proof:** That HELP holds for maps in $I(C[\text{Mon}])$ follows from the following three facts:

1) $T$ and $\mathcal{F}$ are adjoint (Lemma 4.2.14)
2) $f$ is a weak equivalence if and only if $\mathcal{F}f$ is (Lemma 4.4.13)
3) HELP holds in $C$ ($C$ suitable).

Then we see that HELP holds in general by Lemmas 4.4.12, 4.4.11, 4.4.6 and 2.2.7. □

**Lemma 4.4.16** $T$ of any object is copointed and $T$ of any map is copointed.

**Proof:** Easy. □

**Theorem 4.4.17** Let $C$ be suitably monoidal. Then $(C[\text{Mon}], I(C[\text{Mon}]))$ is a suitable cell category.

**Proof:** 4.4.15, 4.4.14 and 4.4.16. □

### 4.5 An Example of a Monoidally Cocomplete Category

In this section we wish to work in both pointed and unpointed contexts. Therefore we will put $\text{resp.} \text{Top.}$ or $\text{resp.} \text{CG.}$ in brackets for things we wish to also state in those categories.

**Definition 4.5.1** $X \in \text{Top}(\text{resp.} \text{Top.})$ is compactly generated if $X$ is Hausdorff and if for every $A \subseteq X$, $A \cap K$ is closed for every compact $K \subseteq X$ implies $A$ is closed in $X$.

Let $\text{CG}$ denote the category of compactly generated spaces and $\text{CG.}$ denote the category of pointed compactly generated spaces.

**Definition 4.5.2** Let $X$ be a Hausdorff space. Define a compactly generated space $KX$ with the same underlying set as $X$ by requiring that $A \subseteq X$ be closed in $KX$ if and only if $A \cap K$ is closed in $X$ for all compact sets $K \subseteq Y$. 
The follow lemma shows that there are many examples of compactly generated spaces.

**Lemma 4.5.3** All locally compact Hausdorff spaces are compactly generated.

**Proof:** Easy. □

**Definition 4.5.4** Let $\times_C$ denote the Cartesian product in $\text{Top}(\text{resp.} \text{Top}_*)$ then in $\text{CG}(\text{resp.} \text{CG}_*)$ define $X \times Y$ to be $K(X \times_C Y)$.

**Lemma 4.5.5** With $\alpha$, $\lambda$ and $\rho$ being the same maps as in the Cartesian case, $(\text{CG}(\text{resp.} \text{CG}_*), \times, *, \alpha, \rho, \lambda)$ is a monoidal category.


**Lemma 4.5.6** Let $X$ be locally compact and $Y \in \text{CG}(\text{resp.} \text{CG}_*)$ then $X \times Y = X \times_C Y$

**Proof:** [24]. □

**Lemma 4.5.7** Let $X_i \in \text{CG}(\text{resp.} \text{CG}_*)$ and $\pi_i : X_i \to X_i/\simeq_i$ be projections. Then

1) $X_i/\pi_i \in \text{CG}(\text{resp.} \text{CG}_*)$

2) $X_1/\pi_1 \times X_2/\pi_2 = (X_1 \times X_2)/(\pi_1 \times \pi_2)$.

**Proof:** [24]. □

**Remark:** 2) does not hold for $\times_C$ ([9]).

**Lemma 4.5.8** Let $D$ be a small category, $X : D \to \text{CG}(\text{resp.} \text{CG}_*)$ a functor and $Y \in \text{CG}(\text{resp.} \text{CG}_*)$. Then whenever both sides exist the natural map

$$\text{colim}_{x \in D}(X(x) \times Y) \to \text{colim}_{x \in D}X(x) \times Y$$

is an isomorphism.

**Proof:** It is enough to prove the lemma for the two cases of a coequalizer and a coproduct.

Case 1) Coproducts.
First we will prove the following equality in $\text{Top}$.

$$\left( \prod_{x \in D} X(x) \right) \times Y = \prod_{x \in D} (X(x) \times Y)$$

As sets both sides are the same so we only need to show they have the same topology. We do that with a string of equivalences.

$$V \subseteq \left( \prod X_i \right) \times Y \text{ is closed}$$

if and only if

for every compact $K \subseteq \left( \prod X_i \right) \times Y$, $V \cap K$ is closed

if and only if for every

for every compact $K_1 \subseteq \prod X_i$ and $K_2 \subseteq Y$ $V \cap K_1 \times K_2$ is closed.

Since all compact $K_1 \subseteq \prod X_i$ are of the form

$$L_{i(1)} \cup \ldots \cup L_{i(n)}$$

where $L_{i(j)} \subseteq X_j$ is compact we have that the last condition is equivalent to

for every $i$ and every compact $K_1 \subseteq X_i$, $K_2 \subseteq Y$ $V \cap K_1 \times K_2$ is closed.

if and only if

for every $i$ and every compact $K \subseteq X_i \times Y$ $C \cap K$ is closed

if and only if

$V$ closed in $\left( \prod (X_i \times Y) \right)$
Case 2) Coequalizers. This case follows directly from 4.5.7 since in $\text{Top} \ coeq(f, g : A \to B)$ is a quotient of $B$.

To prove the lemma in $CG(\text{resp.} CG_\ast)$ note that colimits are the same in $\text{Top}(\text{Top}_\ast)$ and $CG(\text{resp.} CG_\ast)$. □

This lemma would be enough to make $CG$ and $CG_\ast$ monoidally cocomplete were it not for the fact that it is not complete.

### 4.6 Examples of Suitably Monoidal Categories

**Lemma 4.6.1** $(DGM(R), I(R)), (CG_\ast, I(\Sigma M))$ and $(CG_\ast, K(M))$ all have multiplicative intervals.

**Proof:** In $DGM(R)$ the functor is just $M \mapsto M \otimes I$ and $\delta : M \otimes N \otimes I \to M \otimes I \otimes N \otimes I$ can be taken to be the map such that

$$\delta(m \otimes n \otimes a_i) = m \otimes a_i \otimes n \otimes a_i$$

and

$$\delta(m \otimes n \otimes b) = m \otimes a_0 \otimes n \otimes b + m \otimes b \otimes n \otimes a_1.$$  

In the $CG_\ast$ categories the functor is just $X \mapsto I \ast X$ and $\delta : I \ast X \times Y \to I \ast X \times I \ast Y$ is the map such that

$$\delta(t, x, y) = ((t, x), (t, y)).$$

It is easy to verify that these definitions satisfy the required conditions. □

**Lemma 4.6.2** $(DGA(R), \{T\alpha\}_{\alpha \in I(R)})$, is suitably monoidal.

**Proof:** We already know it is a monoidally cocomplete suitable cell categories with multiplicative intervals. The $\kappa$ smallness condition follows from 3.4.5. □

**Corollary 4.6.3** $DGA(R)$ is a closed model categories.
Chapter 5

Spherically Generated Categories

In this chapter we look at what is required of a suitable cell category to define cellular homology. We define homology through chains. We then look at when we can extend our definition of chains to monoidal objects so that we get DGA's. We close by proving that if $H_* = \pi_*$ then we get certain equivalences of categories.

In Section 1 we define spherically generated categories and CW complexes. We also introduce the concept of irrelevant homotopies, so called because they do nothing to chains. That is irrelevantly homotopic maps induce the same map on chains. We prove necessary results for such homotopies. Section 2 gives a way in this setting of defining chains with reasonable properties. Section 3 examines how a cell structure on a product of $CW^e$ complexes is induced by a cell structure on a product of discs. We then examine chains on products of $CW^e$ complexes and maps between them. Section 4 shows how we can give monoidal objects in C cell structures and make the multiplication cellular in a nice way. We also introduce strongly irrelevant homotopies. They are irrelevant but are also homotopies in C[Mon] and not just C. In Section 5 we look at functors between such categories and what properties they need to satisfy to preserve the relevant structure. In Section 6 we show that chains are one such functor. Section 7 defines homology. In Section 8 we show that if the homology and homotopy functors are isomorphic then $C_*$ induces an equivalence of homotopy categories.
For the whole chapter we will assume that we will assume that \((C, I)\) is a spherically generated category, which is defined next, Definition 5.1.1.

5.1 Spherically Generated Categories

Definition 5.1.1 A spherically generated category \((C, I, \{\Theta_n\})\) is a pointed \(\omega\) small suitable cell category \((C, I)\) such that \(I\) is a set of the form

\[
I = \bigcup_{n \in R} \{i_n : S^n \to D^{n+1}\} \cup \{* \to S^n\}
\]

where \(R = \{j \in \mathbb{Z} | j > n\}\) for some fixed \(n \in \mathbb{Z} \cup \{-\infty\}\) and such that \(D^{n+1} \cong *\). together with a chosen isomorphism \(\Theta_{n+1} : D^{n+1}/S^n \to S^{n+1}\).

In this case using notation similar to 3.2.8 we denote \(\pi(S^n, X)\) by \(\pi_n(X)\).

Definition 5.1.2 In \(CG_*\) let \(R = \{j \in \mathbb{Z} | j > 0\}\), \(S^n = S^0 \wedge S^n\), \(D^{n+1} = I \wedge S^n\). Let \(f : S^0 \to I\) be the inclusion of the end points and \(i_n = f \wedge S^n\). Define \(\Theta_n\) by the composition

\[
I/S^0 \wedge S^n \xrightarrow{\cong} S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}
\]

For any localization we simply localize all the spaces and maps.

In \(DGM(R)\) we use Definition 3.4.1 and let \(\Theta_n\) be the map induced by \(b_n \mapsto a_n\).

Lemma 5.1.3 With the above definitions \(CG_*\) and \(DGM(R)\) are spherically generated.

Proof: Clear. □

Definition 5.1.4 A CW complex over \(W\), \((X, \{X_j\}_{j \in \mathbb{Z}}, \{f(r, j)\}_{r \in I(j, X), j \in \mathbb{Z}})\), is a cell complex, \(W \to X\) together with a decomposition into subcomplexes \(X_n\), sets of cells \(I(n, X)\) and attaching maps \(f(r, n + 1) : S^n \to X_n\) If \(x \in I(n, X)\) then we say \(x\) is of dimension \(n\)
and write $|x| = n$. We require that there be a pushout

$$
\begin{array}{c}
\bigvee_{r \in I(n+1,X)} D^{n+1} \\
\downarrow \\
\bigvee_{r \in I(n+1,X)} D^{n+1} \\
\end{array}
\xrightarrow{f_{r,n+1}}
\begin{array}{c}
\bigvee_{r \in I(n+1,X)} S^n \\
\downarrow \\
\bigvee_{r \in I(n+1,X)} S^n \\
\end{array}
\xrightarrow{f_{r,n+1}}
X_n
$$

that $X = \text{colim} X_n$ and that $W \to X_n$ have no cells of dimension greater than $n$. $X_n$ is called the n-skeleton of $X$.

A cellular map $f : X \to Y$ is a compatible set of maps $f_n : X_n \to Y_n$ such that $f = \text{colim} f_n$. For convenience we will denote $X_n/X_r$ by $X_{n/r}$ and $X_{n/r}$ by $f_{n/r}$.

We denote the category of cellular complexes over $\ast$ and cellular maps in $C$ by $\text{CW}(C)$.

In a monoidal category we define a CW* complex to be a cell complex over $e$, $(X, \{X_n\})$, such that $X_0 = Y \vee e$ where $Y$ is a CW complex. A CW* map $f : X \to Y$ between CW* complexes is a cellular map together with maps $f'_n : X'_n \to Y'_n$ for every $n \leq 0$ such that $(f'_n \vee \text{id}) \circ \delta_n(X) = \delta_n(Y) \circ f_n$. The category of CW* complexes and CW* maps in $C$ will be denoted $\text{CW}^*(C)$.

**Example:** Let $X \to Y$ be a cellular map of CW complexes over $W$ and $X \to Z$ a relative CW complex. Then the pushout of

$$
\begin{array}{c}
X \to Y \\
\downarrow \\
Z
\end{array}
$$

is canonically a CW complex over $W$.

**Definition 5.1.5** Let $X \in \text{CW}(C)$ (resp. $\text{CW}^*(C)$), and let $\leq$ be a partial well order on the set of cells, $K(X)$. We call $S \subset K(X)$ closed if for every $x \in S \{y | y < x \text{ and } |y| < |x|\} \subset S$. We say that $(X, \leq)$ is an ordered CW complex, $(X, \leq) \in \text{OrdCW}(C)$, if for every closed subset $S \subset K(X)$ we have chosen compatible subcomplexes of $X$ whose cells are $S$. We will denote this subcomplex by $X(S)$. We will often drop the ordering from the notation. Unless otherwise stated whenever we refer to a subset of $K(X)$ we will assume that it is closed.
We denote the category of ordered CW complexes and cellular maps by \(\text{Ord}(\text{CW}(C))\) (resp. \(\text{Ord}(\text{CW}^e(C))\)).

For \(x \in K(X)\) we let \(x^*\) denote \(\{y \mid y < x, |y| < |x|\}\). We will sometimes write \(x\) for \(x^* \cup \{x\}\). For closed \(T \subset K(X)\) we let \(T^+ = \{x \in K(X) \mid x^* \subset T\}\). We will refer to the decomposition of \(X\) into subcomplexes \(X(S)\) as the ordering filtration.

For example we let \(D^n \in \text{OrdCW}(C)\) in the following way. \(K(D^n) = \{a, b\}, |a| = n - 1, |b| = n, b > a\). Then \(D^n(\emptyset) = *, D^n(a) = S^{n-1}, D^n(b) = D^n, D^n(a^*) = *\) and \(D^n(b^*) = S^{n-1}\).

**Lemma 5.1.6** Let \(X \in \text{CW}(C)\) or \(\text{CW}^e(C)\) then there are canonical isomorphisms \(X_{n/n-1} \to \bigvee D^n/S^{n-1}\) and \(\phi_n : X_{n/n-1} \to S^n\). If \(X \in \text{CW}^e(C)\) there exist canonical isomorphisms \(X_0/X'_{-1} \to \bigvee D^0/S^{-1} \vee e\) and \(\phi_0^e : X_0/X'_{-1} \to S^0 \vee e\).

**Proof:** In the diagram

\[
\begin{array}{ccc}
\bigvee S^{n-1} & \rightarrow & \bigvee D^n \\
\downarrow & & \downarrow \rightarrow \\
* & \rightarrow & \bigvee D^n / \bigvee S^{n-1} \\
\downarrow & & \downarrow \\
X_{n-1} & \rightarrow & X_n \\
\downarrow & & \downarrow \\
* & \rightarrow & X_{n/n-1}
\end{array}
\]

the back, top and bottom faces are pushouts so the front is also. Observing that \(\bigvee D^n / \bigvee S^{n-1} = \bigvee (D^n/S^{n-1})\) and composing with \(\Theta_n\) completes the proof of the lemma in the \(\text{CW}\) case. The \(\text{CW}^e\) case is similar. □

**Lemma 5.1.7** Let \(X\) be an ordered CW complex. For a sequence of \(B(i) \subset K(X)\) such that \(B(i) \subset B(j)\) if \(i \leq j\).

\[
\text{colim}(\text{Cyl}(X(B(i)), W)) \cong \text{Cyl}(\text{colim}X(B(i)), W)
\]
Proof: Follows since $C$ is $\omega$ small. $\square$

We now define cellular and irrelevant homotopies. The concept of irrelevant homotopy is contained in the notion of 0-homotopy, $\simeq_0$, of Baues ([4]) Chapter III Section 3.

Definition 5.1.8 Let $X$ be an ordered CW complex over $W$. Let $Cyl(X, W)$ be an ordered CW complex over $W$ that is a cylinder object for $X$. Attach to it a set $\{Cyl(X(T), W)\}_{T \subseteq K(X)}$ of subcomplexes that are cylinder objects for the $X(T)$ together with for every $S \subseteq T$ a cofibration $i(S, T) : Cyl(X(S), W) \cup (T \setminus W) \to Cyl(X(T), W)$. Assume that $\text{colim} Cyl(X(T), W) = Cyl(X, W)$. Also assume that if if $T' = T \cup \{x\}$ then we have a pushout

$$
\begin{array}{ccc}
Cyl(X(x^+), W) & \longrightarrow & Cyl(X(T), W) \\
\downarrow & & \downarrow \\
Cyl(X(x), W) & \longrightarrow & Cyl(X(T'), W).
\end{array}
$$

Let $Cyl(X_n, W) = Cyl(X(T), W)$ where $T \subseteq K(X)$ is all cells of $X_n$. Then $Cyl(X, W)$ is called a filtered cylinder object and $\{Cyl(X(T), W)\}_{T \subseteq K(X)}$ is called its decomposition.

Let $f, g : X \to Y$ be cellular maps. Then a cellular homotopy, $H$, from $f$ to $g$ is a homotopy $H : Cyl(X, W) \to Y$ that is a cellular map. If we have $H_n : Cyl(X_n, W) \to Y$ from $i_n f_n$ to $i_n g_n$ such that $\text{colim} H_n = H$ and $H_n$ factors through $Y_n$ then we call $H$ an irrelevant homotopy.

Let $\simeq_i$ denote the equivalence relation generated by irrelevant homotopy.

For example in $\text{Top}_*$ the maps $id$ and $* : D^n \to D^n$ are homotopic but not irrelevantly homotopic. The idea of an irrelevant homotopy is that it has no effect on cellular chains. In other words irrelevance homotopic maps induce the same map on cellular chains. This will be proved later in 5.1.15

Lemma 5.1.9 Let $X$ be an ordered CW complex over $W$ and let $Cyl(X, W)$ and $Cyl'(X, W)$ be two filtered cylinder objects. Then there exists compatible homotopy equivalences

$$Cyl(X(T), W) \to Cyl'(X(T), W)$$
together with homotopy inverses.

**Proof:** Assume that for every subset $S$ of $T \subset K(X)$ the statement is true. Also assume that we have an irrelevant cylinder object for $X(T)$, $\text{Cyl}''(X(T), W)$ such that there are compatible weak equivalent cofibrations $i : \text{Cyl}(X(S), W) \to \text{Cyl}''(X(S), W)$ and $i' : \text{Cyl}'(X(S), W) \to \text{Cyl}''(X(S), W)$ with corresponding compatible retractions $r : \text{Cyl}''(X(S), W) \to \text{Cyl}(X(S), W)$ and $r' : \text{Cyl}''(X(S), W) \to \text{Cyl}'(X(S), W)$ and that the already defined homotopy equivalences are $r'i$ and $ri'$.

Now let $x \in K(X)$ be an element such that $x \notin T$ and $T \cup \{x\}$ is closed. Let $Z$ denote the pushout

$$
\begin{array}{ccc}
\text{Cyl}(X(x^*), W) & \longrightarrow & \text{Cyl}''(X(x^*), W) \\
\downarrow & & \downarrow \\
\text{Cyl}(X(x), W) & \longrightarrow & Z
\end{array}
$$

$Z'$ the pushout

$$
\begin{array}{ccc}
\text{Cyl}'(X(x^*), W) & \longrightarrow & \text{Cyl}''(X(x^*), W) \\
\downarrow & & \downarrow \\
\text{Cyl}'(X(x), W) & \longrightarrow & Z'
\end{array}
$$

and $Y$ the pushout

$$
\begin{array}{ccc}
\text{Cyl}''(X(x^*), W) \cup (X(x) \lor W X(x)) & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Z' & \longrightarrow & Y
\end{array}
$$

Then define $\text{Cyl}''(X(x), W)$ by the factorization of $Y \to X(x)$ by a cofibration followed by a weak equivalence.
The following example of HELP proves the induction hypotheses for $Cyl(X(x), W)$.

![Diagram]

where $g$ is determined by the pushout defining $Z$ and $f$ is determined by the previously defined homotopy on $Cyl''(X(x^*), W)$ and the identity homotopy on $Cyl(X(x), W)$.

For $S \subseteq T$ such that $S' = S \cup \{x\}$ is closed, $Cyl''(X(S'), W)$ is determined by the pushout

$$
\begin{align*}
Cyl''(X(x^*), W) &\to Cyl''(X(S), W) \\
\downarrow &\downarrow \\
Cyl''(X(x), W) &\to Cyl(X(S'), W)
\end{align*}
$$

The retractions are defined by the pushouts in the obvious way. The only remaining case of pushout is implied by Lemma 5.1.7. □

**Lemma 5.1.10** Let

$$
\begin{align*}
S^n &\to X \\
\downarrow &\downarrow \\
D^{n+1} &\to Y
\end{align*}
$$

be a pushout and $Cyl(X, W)$ an irrelevant cylinder object for $X$ over $W$. Then for a cellular $Cylf$ the pushout of

$$
\begin{align*}
CylS^n &\xrightarrow{Cydf} Cyl(X, W) \\
\downarrow &\downarrow \\
CylD^{n+1}
\end{align*}
$$

determines a filtered cylinder object for $Y$ over $W$.

**Proof:** Follows from Lemma 2.4.11. □

The following theorem says that irrelevant homotopies equivalences can be extended when cells are attached by irrelevantly homotopic maps.
Lemma 5.1.11 Let \( f : X \to Y \) be an irrelevant homotopy equivalence and \( g : S^n \to X \) and \( h : S^n \to Y \) be cellular maps such that \( f \circ g \simeq_{ir} h \) then any extension that is a weak homotopy equivalence \( f' : X_n \cup g D^{n+1} \to Y_n \cup h D^{n+1} \) is an irrelevant homotopy equivalence and there exist such extensions.

**Proof:** Let \( H \) be a homotopy \( H : f \circ g \simeq_{ir} h \). Also assume that we have chosen an extension \( f'(n) : X_n \cup g D^n \to Y_n \cup h D^n \) that is an irrelevant weak homotopy equivalence. Let \( Y_n \to X_n \) be the inverse and \( F : f f^{-1} \simeq_{ir} id \) and \( G : f^{-1} f \simeq_{ir} id \) be the homotopies. Consider the diagram

\[
\begin{array}{c}
\begin{array}{c}
S^n \\
\downarrow h \\
Y_n \cup \alpha \\
\downarrow i(\alpha) \\
D^{n+1}
\end{array} & \begin{array}{c}
CylS^n \\
\downarrow H \circ Cyl h \\
X_n \cup \alpha \\
\downarrow f'(n) \\
CylD^{n+1} \end{array} & \begin{array}{c}
S^n \\
\downarrow f^{-1} \circ h \\
Y_n \cup \alpha \\
\downarrow (f')^{-1} |_\alpha \\
D^{n+1}
\end{array}
\end{array}
\]

Since \( f'(n) \) is a weak equivalence by assumption, there exist dashed maps giving an extension \( f''^{-1} \) and using Lemma 5.1.10 a homotopy \( H' : f'(f')^{-1} \simeq_{ir} id \).

We now need to show that \( (f')^{-1} f' \simeq_{ir} id \). Let \( Z \) denote the pushout of the following diagram.

\[
\begin{array}{c}
X_n \cup X_n \\
\downarrow \\
X_n \cup D^n \cup X_n \cup D^n \\
\downarrow \\
Z
\end{array}
\]

and consider the following solid arrow diagram.

\[
\begin{array}{c}
Z \leftarrow CylZ & \rightarrow Z \\
\downarrow \\
Y_n \cup D^n & \leftarrow X_n \cup D^n \\
\downarrow \\
CylX \cup D^n & \rightarrow Cyl(CylX \cup D^n) & \leftarrow CylX \cup D^n \\
\downarrow \\
CylX \cup D^n & \rightarrow Z
\end{array}
\]

The map \( A \) is \((f'(n) \vee f'(n)) \circ \sigma\) on \( Cyl(CylX \cup D^n) \) and defined from the previous stage on \( CylCylX_n \). \( B \) is the map \( G_n +_{CylX_n} ((f'(n))^{-1} f'(n) + id) \). The dashed extensions
complete the proof of the first statement using lemmas 5.1.10 and 5.1.7.

Next we see that if $h = fg$ then the bottom row in the pushout

$$
\begin{array}{ccc}
X_n & \xrightarrow{f} & Y_n \\
\downarrow & & \downarrow \\
X_n \cup_g D^{n+1} & \rightarrow & Y_n \cup_h D^{n+1}
\end{array}
$$

is a weak equivalence. Let $X_n \vee_W Y_n \rightarrow Z \rightarrow Y_n$ be a factorization of $f +_W id$ into a cofibration followed be a weak equivalence. Then in the pushout

$$
\begin{array}{ccc}
X_n & \rightarrow & Z \\
\downarrow & & \downarrow \\
X_n \cup_g D^{n+1} & \rightarrow & Z'
\end{array}
$$

the top row and hence the bottom row is an acyclic cofibration. But so too it is in this pushout.

$$
\begin{array}{ccc}
Y_n & \rightarrow & Z \\
\downarrow & & \downarrow \\
Y_n \cup_h D^{n+1} & \rightarrow & Z'
\end{array}
$$

The fact that $f \simeq_r g$ implies that $Y_n \cup_f D^{n+1} \simeq Y_n \cup_g D^{n+1}$ rel $Y_n$ by 2.4.9 completes the proof of the second part of the lemma. □

**Lemma 5.1.12** Let $B \rightarrow C \rightarrow A$ be maps such that $A$ is an ordered CW complex over $B$ and $C$ is a closed subcomplex of $A$. Now let $Cyl(A, B)$ be a filtered cylinder object for $A$ such that for every $x \in K(A)$ we have a pushout

$$
\begin{array}{ccc}
CylS^{[x]} \rightarrow & \rightarrow & Cyl(A(x^*), B) \\
\downarrow & & \downarrow \\
CylD^{[x]} & \rightarrow & Cyl(A(x), B).
\end{array}
$$
If we let $\text{Cyl}(A(T), C)$ be defined by the pushout.

\[
\begin{array}{c}
\text{Cyl}(C, B) \\
\downarrow \\
C
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{Cyl}(A(T), B) \\
\downarrow \\
\text{Cyl}(A(T), C)
\end{array}
\]

Then we have determined the structure of an irrelevant cylinder object for $A$ on $\text{Cyl}(A, C)$.

Note that by Lemma 2.4.11 there is always a cylinder object for $A$ over $B$ satisfying the pushout condition of the lemma.

**Proof:** So as not to surprise anyone we use induction to prove the lemma. The lemma is clear if $A(T) = C$. So let $T$ be such that $K(C) \subset T \subset K(A)$ and assume the lemma is true for $A = A(T)$. Let $S \subset T$. In the cube

the front, back and left faces are pushouts so by Lemma 1.3.3 the right face is also. Therefore the outside of

\[
\begin{array}{c}
\text{Cyl}(S, B) \\
\downarrow \\
\text{Cyl}(T, B)
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{Cyl}(S, C) \\
\downarrow \\
\text{Cyl}(T, C)
\end{array}
\]

is a pushout. So by lemma 2.4.11 $\text{Cyl}(A(x), C)$ is a cylinder object for $A(x)$. For $S' = S \cup \{x\}$
closed another cube shows that

\[
\begin{align*}
\text{Cyl}(A(x^*), C) & \longrightarrow \text{Cyl}(A(S), C) \\
\downarrow & \downarrow \\
\text{Cyl}(A(x), C) & \longrightarrow \text{Cyl}(A(S'), C)
\end{align*}
\]

is a pushout. Lemma 5.1.7 is then enough to complete the proof that \( \text{Cyl}(A, C) \) is an irrelevant cylinder object for \( A \) over \( C \). \( \square \)

**Corollary 5.1.13** Let \( B \to C \to A \) be maps such that \( A \) is an ordered cell complex over \( B \) and \( C \) is a subcomplex. Let \( H : \text{Cyl}(A, C) \to X \) be an irrelevant homotopy then \( \text{Cyl}(A, B) \to \text{Cyl}(A, C) \to X \) is an irrelevant homotopy.

The following lemma has a nice proof.

**Lemma 5.1.14** \( \text{Cyl}(X_n, W)/\text{Cyl}(X_{n-1}, W) \) determines a \( \text{Cyl}X_n/X_{n-1} \).

**Proof:** Surprisingly we use an alternative to the tedious proof by induction. Let \( Z \) denote \( \text{Cyl}(X_n, W)/\text{Cyl}(X_{n-1}, W) \). First we show that \( \text{Cyl}(X_n/X_{n-1}) = Z \). To do this we show that the map

\[
(\partial_0)_{n/n-1} : X_{n/n-1} \to Z
\]

is an acyclic cofibration.

In the following diagram all the maps are cofibrations, the horizontal maps and the map \( Y \to \text{Cyl}(X_n, W) \) are equivalences and the square is a pushout.

\[
\begin{array}{ccc}
X_{n-1} & \overset{\partial_0}{\longrightarrow} & \text{Cyl}(X_{n-1}, W) \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & Y \\
\downarrow & & \downarrow \\
& & \text{Cyl}(X_n, W)
\end{array}
\]
Let $p : E \to B$ be a fibration and let a commutative diagram

\[
\begin{array}{ccc}
X_{n/n-1} & \xrightarrow{\pi} & E \\
\downarrow \text{(partial $n/n-1$)} & & \downarrow \\
Z & \xrightarrow{t} & B
\end{array}
\]

be given. Then we get a commutative solid arrow diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{\partial} & X_{n/n-1} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{h} & E \\
\downarrow & & \downarrow \\
Z & \xrightarrow{t} & B
\end{array}
\]

and so an extension lifting $h$ since we get an extension lifting in the following diagram and the map from $Y$ restricts trivially to $Cyl(X_{n-1}, W)$.

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & E \\
\downarrow & & \downarrow \\
Cyl(X_n, W) & \xrightarrow{t} & B.
\end{array}
\]

Thus $(\partial_n)_{n/n-1}$ is an acyclic cofibration and in particular a weak equivalence. So $Cyl(X_{n/n-1}) = Z$ and the lemma easily follows. □

**Corollary 5.1.15** $f \simeq g$ implies that $f_{n/n-1} \simeq g_{n/n-1}$.

**Definition 5.1.16** **Cellular approximation** holds in $C$ if every map in $C$ between CW complexes is homotopic to a cellular map and whenever $f$ and $g$ are cellular maps homotopic in $C$ then they are homotopic through a cellular homotopy.

**Lemma 5.1.17** **Cellular approximation** holds in $C$ implies that

\[
\pi_i(X_n \to X_{n+1})
\]

is an isomorphism if $i < n$ and a surjection if $i = n$. 
5.2 Chains

In this section we define cellular chains in a general setting. The reader is encouraged to look at Chapter III of Baues where he gives a very general definition of cohomology.

**Definition 5.2.1** Let $R$ denote a ring. Let $p_j : \bigvee_{j \in J} S^n \to S^n$ denote the $j$th projection and $i_j : S^n \to \bigvee_{j \in J} S^n$ denote the $j$th injection.

Assume that cellular approximation holds in $C$. We will say that $C$ has chains if for every $n$ we have chosen set maps $F_n : \pi(S^n, S^n) \to R$ such that $F_n(id) = 1$, $F_n(*) = 0$ and for any $f \in \pi(S^n, \bigvee_{j \in J} S^n)$, $g \in \pi(\bigvee_{j \in J} S^n, S^n)$, $h \in \pi(S^n, S^n)$, $F_{n+1}(\Sigma h) = F_n(h)$ and $F_n(g \circ f) = \Sigma_{j \in J} F_n(p_j \circ f)F_n(g \circ i_j)$. If $C$ is monoidal and $C$ has a $S^0$ we further assume that we have chosen a map $F_0^e : \pi(S^0, e) \to R$ such that $F_0^e(*) = 0$ and such that for every $g^e \in \pi(\bigvee_{j \in J} S^n, e)$ $F_0^e(g^e \circ f) = \Sigma_{j \in J} F_0(p_j \circ f)F_0^e(g^e \circ i_j)$.

Note that by smallness the sums are non zero for only finitely many $k$.

Remember we know that for $n \geq 1$, $\pi(S^n, S^n) \cong \mathbb{Z}$ as rings.

**Definition 5.2.2** In $(CG_*, I(S^1))$ choose $F_n$ to be the ring isomorphism $\pi(S^n, S^n) \to \mathbb{Z}$

**Lemma 5.2.3** With $F_n$ defined as above $(CG_*, I(S^1))$ has chains.

**Proof:** First notice that this category has no $S^0$ so we need not check any of the conditions on $F_0^e$. The other conditions follow from looking at (simplicial) homology for $n \geq 1$ $\pi_n(S^n) = H_n(S^n)$. □

**Definition 5.2.4** Let $C$ be a category with chains. Let $X$ and $Y$ be CW complexes or CW complexes and $f : X \to Y$ a cellular map. Remembering Lemma 5.1.6 we have an isomorphism

$$\phi_n(X) : X_{n/n-1} \to \bigvee_{j \in I(n, X)} S^n.$$
If \( X \in CW(C) \) then define \( C_n(X) = \bigoplus_{j \in l(n,X)} R \) and \( C_*(X) = \bigoplus_n C_n(X) \). If \( X \) is a \( CW^e \) complex then define \( C_0(X) = (\bigoplus_{j \in l(0,X)} R) \oplus R \), \( C_n(X) = \bigoplus_{j \in l(n,X)} R \) if \( n \neq 0 \) and \( C_*(X) = \bigoplus_n C_n(X) \). Let \( i_j \) and \( p_j \) denote the maps as in definition 5.2.1 and \( p : \forall S^n \forall e \rightarrow e \) be the projection. Define \( C_n(i_j) \) to be the \( j \) th inclusion \( R \rightarrow \bigoplus_{j \in l(n,X)} R \) and \( C_n(p_j) \) to be the \( j \) th projection \( \bigoplus_{j \in l(n,X)} R \rightarrow R \). If \( X \) is a \( CW^e \) complex let \( C_0(p) : C_0(X) \rightarrow R \) be the projection of the extra \( R \) and \( C_0(i) : R \rightarrow C_0(X) \) be the injection of the extra \( R \). Then define \( C_n(f) \) to be the unique linear map determined by the formula

\[
C_n(p_k)C_n(f)C_n(i_j)(1) = F_n(p_k\phi_n(Y)f_{n/n-1}(\phi_n(X))^{-1}i_j)
\]

If \( f \in Hom_{CW^e} \) we also require

\[
C_0(p)C_0(f)C_0(i)(1) = 1
\]

\[
C_0(p_k)C_0(f)C_0(i)(1) = 0
\]

\[
C_0(p)C_0(f)C_0(i_j)(1) = F_0(p\phi_0(Y)f_0/f^{-1}_0(\phi_0(X))^{-1}i_j)
\]

Define \( C_*(f) = \bigoplus_n C_n(f) \). We also denote \( C_*(f) \) by \( f_* \).

Lemma 5.2.5 \( C_* : CW(C) \rightarrow GM(R) \) and \( C_* : CW^e(C) \rightarrow GM(R) \) are both functors.

Proof: Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be cellular maps. We wish to show that \( C_*(g)C_*(f) = C_*(gf) \). This is demonstrated in the following string of equalities.

\[
C_n(p_k)C_n(g)C_n(f)C_n(i_j)(1)
\]

\[
= \sum_{l \in l(n,Y)} C_n(p_k)C_n(g)C_n(i_l)C_n(p_j)C_n(f)C_n(i_j)(1)
\]

\[
= \sum_{l \in l(n,Y)} (C_n(p_k)C_n(g)C_n(i_l)(1))(C_n(p_j)C_n(f)C_n(i_j)(1))
\]

\[
= \sum_{l \in l(n,Y)} F_n(p_k\phi_n(Z)g_{n/n-1}(\phi_n(Y))^{-1}i_l)F_n(p_l\phi_n(Y)f_{n/n-1}(\phi_n(X))^{-1}i_j)
\]
\[
= F_n(p_k \phi_n(Z) g_{n-1} f_{n-1}^{-1} \phi_n(X))^{-1} i_j
\]
\[
= C_n(p_k) C_n(gf) C_n(i_j)(1).
\]

That \( \Sigma_{j \in I(n, v)} C_n(i_j) C_n(p_j) = \text{id} \) implies the first equality. The second follows from the linearity of \( C_n \), the third and the fifth from the definition of \( C_* \) and the fourth from Definition 5.2.1. So the required formula has been proven. The proof for \( CW^*(C) \) is similar. \( \square \)

We give the suitable cell category \((DGM(R), I(R))\) chains as in Lemma 5.7.1.

**Lemma 5.2.6** \( f \simeq_i g \) implies that \( C_*(f) = C_*(g) \).

**Proof:** Follows from Lemma 5.1.15. \( \square \)

Recall the definition of \( \nabla \) and \( f - g \) from Section 2.4

**Lemma 5.2.7** For an irrelevant homotopy \( H \)

\[
(f - g)_{n-1} \simeq (f_{n-1} + (-g)_{n-1}) \nabla
\]

**Proof:** Easy. \( \square \)

**Lemma 5.2.8** Let \( g : X \cup D^n \to Y \) and \( f : S^n \to Y \) be any maps. Let \( \bar{f} : X \cup D^n \to S^n \to Y \) denote the composition of the pinch map with \( f \). Then for any \( \nabla C_*((g+f)\nabla) = C_*(\bar{f}) + C_*g \).

**Proof:** From the Lemma 2.4.6 and the linearity of \( C_* \). \( \square \)

**Lemma 5.2.9** \( C_*(-\text{id}) = -C_*(\text{id}) \)

**Proof:** Let \( H : Cyl \ S^{n-1} \to Cyl \ S^{n-1} \) be the identity, \( f : Cyl \ S^{n-1} / \partial_0 S^{n-1} \to D^n \) be a homotopy equivalence rel \( S^{n-1} \) (see 2.4.3) and \( p : D^n \to S^n \) the pinch map. Let \( g \) denote the composition

\[
S^n \xrightarrow{\simeq} Cyl \ S^{n-1} \sqcup S^{n-1} / (\partial_0 S^{n-1} + \partial_1 S^{n-1}) (f+f)(H+H) \xrightarrow{D^n \sqcup S^{n-1}} D^n \]
where \(-H\) is defined in 2.4.7. The following diagram commutes up to homotopy

\[
\begin{array}{ccc}
S^n & \xrightarrow{\cong} & D^n \vee S^{n-1} \\
\downarrow & & \downarrow \mathrm{id} + \mathrm{id} \\
S^n \vee S^n & \xrightarrow{\mathrm{id} + (-\mathrm{id})} & S^n
\end{array}
\]

and so \((\mathrm{id} + (-\mathrm{id}))\nabla \simeq \ast\) and so from Lemma 5.2.8 we get
\[0 = C_\ast((\mathrm{id} + -\mathrm{id})\nabla) = C_\ast(\mathrm{id}) + C_\ast(-\mathrm{id})\]. So \(C_\ast(-\mathrm{id}) = -C_\ast(\mathrm{id})\). \(\square\)

**Corollary 5.2.10** \(C_\ast(-g) = -C_\ast(g)\)

**Lemma 5.2.11** Assume that \(H\) is irrelevant. Then \(C_\ast(f - g) = C_\ast(f) - C_\ast(g)\)

**Proof:** Lemmas 5.2.7, 5.2.8 and 5.2.10. \(\square\)

**Definition 5.2.12** Let \(C\) have chains and \(X \in CW(C)\) or \(X \in CW^e(C)\). We define \(d_n : C_n \to C_{n-1}\) on module generators by \(d(i_j(1)) = f(j,n)_\ast(1)\). Where \(f(j,n)\) is the attaching map of the cell \(j\) as in Definition 5.1.4.

**Lemma 5.2.13** Let \(C\) have chains. Let \(f : X \to Y\) be a map of \(CW\) complexes or \(CW^e\) complexes. Then \(d \circ f_\ast = f_\ast \circ d\).

**Proof:** We will give a proof in the \(CW\) case; the \(CW^e\) case requires a small variation for the \(e\). It is sufficient to prove the equality for generators. So we can let \(X = D^n\) and assume \(Y_{n-1} = \vee_{i \in I(n-1)}(Y) S^{n-1}\). Let the following diagram be a pushout

\[
\begin{array}{ccc}
Vi \in I(n)(Y) S^{n-1} & \xrightarrow{g} & Y_{n-1} \\
\downarrow & & \downarrow \\
Vi \in I(n)(Y) D^n & \xrightarrow{} & Y_n
\end{array}
\]
Now consider the following commutative diagram

![Diagram](image_url)

The composition of the second row is a cone on $f_{n-1}$ (2.4.2) and so the composition of the bottom row is a suspension of $f_{n-1}$. But observing that

$$C_{*+1}(\Sigma(f_{n-1/n-2}))(1) = C_{*}(f_{n-1/n-2}))(1) = f_{*} \circ d(1)$$

and

$$C_{*+1}(\Sigma(g) \circ f_{n-1}))(1) = C_{*}(g) \circ C_{*+1}(f_{n-1}))(1) = d \circ f_{*}(1)$$

gives the desired result. □

**Lemma 5.2.14** $(C_*, d) : CW \to DGM(R)$ and $(C_*, d') : CW' \to DGM(R)$ are both functors.

**Proof:** Only that $d^2 = 0$ remains to be verified. Again it is enough to show this formula for generators. To this end assume that $X$ has a single $n$ cell. Then we have

$$X_{n/n-3} = \bigvee_{i \in I_{(n-2)}(X)} S^{n-2} \cup_{i \in I_{(n-1)}(X)} D^{n-1} \cup_f D^n.$$

Next we define a space $Y$ and a map $g : X_{n-3} \to Y$. Let $Y_{n-2} = X_{n-2/n-3}$ and $g_{n-2} : X_{n-2/n-3} \to Y_{n-2}$ be the identity. Let $Y_{n-1} = V_{i \in I_{(n-1)}(X)} D^{n-1}$ and $g_{n-1} : X_{n-1/n-3} \to Y_{n-1}$ be any map compatible with $g_{n-2}$. We know one must exist since $Y_{n-1} \simeq \ast$. Let $Y = V_{i \in I_{(n-2)}(X)} D^{n-1} \cup_{g_{n-1} \circ_f D^n}$. Then let $g$ be the extension induced on the pushouts by the identity map on $D^n$. We know that $d^2 = (g_{n-2})_* \circ d^2 = d^2 \circ (g_n)_*$ the first equality following
since $g_{n-2} = id$ and the second from the last lemma (5.2.13). But the map $(g \circ f)_{n-1/n-2}: S^{n-1} \to \vee S^{n-1}$ that defines $d$ in $Y$ factors as

\[
S^{n-1} \xrightarrow{f} \bigvee_{i \in \iota(n-2)(X)} D^{n-1} \xrightarrow{\bigvee_{i \in \iota(n-2)(X)}} S^{n-1}
\]

and so must be homotopic to 0 therefore completing the proof. □

**Lemma 5.2.15** Let $j: A \to B$ be a subcomplex and $f: A \to C$ a cellular map. Let

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & D
\end{array}
\]

be a pushout and give $D$ the natural cell structure. Then

\[
\begin{array}{ccc}
C_\ast(A) & \longrightarrow & C_\ast(C) \\
\downarrow & & \downarrow \\
C_\ast(B) & \longrightarrow & C_\ast(D)
\end{array}
\]

is a pushout in $DGM(R)$.

**Proof:** Easy. □

Recall Definition 5.1.4.

**Lemma 5.2.16** If $X$ is a CW or CW* complex then $I(C_\ast(X), n) = I(X, n)$.

**Proof:** Trivial. □

### 5.3 Products

For this section we also assume that $\mathcal{C} = (\mathcal{C}, \times, \alpha, \rho, \lambda)$ is a monoidally cocomplete category, see Section 4, with compatible monoidal and cell structures as follows.
Definition 5.3.1 Let $A, B \in \text{OrdCW}$ Let $(A(x) \times B(y))^*$ denote pushout of the following diagram.

$$\begin{array}{ccc}
A(x^*) \times B(y^*) & \longrightarrow & A(x^*) \times B(y) \\
\downarrow & & \downarrow \\
A(x) \times B(y^*) & & 
\end{array}$$

Definition 5.3.2 The monoidal and cell structures of $C$ are said to be weakly compatible if for every $r$ and $s$ we have chosen a map $h$ an isomorphism

$$\phi : (D^r \vee e) \times (D^s \vee e) \to ((D^r \vee e) \times (D^s \vee e))^* \cup_h D^{r+s}$$

where the right hand side is a CW complex with the cell structure of $((D^r \vee e) \times (D^s \vee e))^*$ compatible with other $\phi$. We give $(D^r \vee e) \times (D^s \vee e)$ the CW structure determined by $\phi$.

Let $id : \vee_{s \in S} S^r \to \vee_{s \in S} S^r$ be the identity. Assume that for every $f : \vee S^n \to \vee S^n$ $(f \times id)_{n+r/n+1} \simeq \vee_{s \in S} \Sigma^r f$ and $(id \times f)_{n+r/n+1} \simeq \vee_{s \in S} \Sigma^r f$ if $r \geq 0$ or $\Sigma^{-r}(id \times f)_{n+r/n+1} \simeq \vee_{s \in S} f$ and $\Sigma^{-r}(f \times id)_{n+r/n+1} \simeq \vee_{s \in S} f$ if $r < 0$. Also assume the map induced by

$$\alpha : S^n \times (S^r \times S^s) \to (S^n \times S^r) \times S^s$$

$S^{n+r+s} \to S^{n+r+s}$ is homotopic to the identity.

Definition 5.3.3 In $DGM(R)$ let $h$ be the map that sends $a_{r+s-1}$ to the image of $a_{r-1} \otimes b_s + (-1)^r b_r \otimes a_{s-1}$ under lower $\phi$.

Definition 5.3.4 $C$ is said to have compatible monoidal and cell structures if it has weakly compatible monoidal and cell structures and if

$$C_\ast((D^r \vee e) \times (D^s \vee e)) \cong (D^r \vee e) \times (D^s \vee e)$$

in $CW^\ast(DGM(R))$ with the isomorphism induced by a bijection of the canonical bases.
Lemma 5.3.5  With the above definitions $DGM(R)$ has compatible cell and monoidal structures.

Proof: Easy. □

Lemma 5.3.6  Let $C$ be a homology category with weakly compatible monoidal and cell structures. Then $C_*((D^s \vee e) \times (D^r \vee e)) \cong C_*(D^s \vee e) \otimes C_*(D^r \vee e)$.

Proof: Let $X = (S^s \vee e) \times (S^r \vee e)$. Then $X = (S^s \vee e \vee S^r) \cup D^{r+s}$. So

$$C_*(X) = (R_0 \oplus R_s \oplus R_r \oplus R_{r+s}, d).$$

Since $X$ has $S^s \vee e$ and $S^r \vee e$ as retracts $d = 0$. Now let $X = (D^s \vee e) \times (S^r \vee e)$. Since $C$ is suitably monoidal and using the fact that $D^s \vee e \simeq e$, we see that $X \simeq S^r \vee e$. Then it is easy to see that $C_*(X) = C_*(D^s \vee e) \otimes C_*(S^r \vee e)$. Finally let $X = (D^s \vee e) \times (D^r \vee e)$. Then $X \simeq e$. So again since $C$ is suitably monoidal we get that $C_*(X) = C_*(D^s \vee e) \otimes C_*(D^r \vee e)$ □

Definition 5.3.7  In $CG_* (D^n \times D^r)^* \cong S^{n+r-1}$. There are two non homotopy equivalent homeomorphisms. Choose $h$ to be the one so that the $C_*$ condition is satisfied. This can be done by Lemma 5.3.6. Choose $\phi^{-1}$ to be any homeomorphic extension for the inclusion $(D^r \times D^s)^* \to D^r \times D^s$. Note that we also have to make $(D^2 \times D^2)^*$ into a CW complex but that is easy.

Again for any localization we just localize the spaces and maps.

Lemma 5.3.8  With the above definitions $CG_*$ has compatible cell and monoidal structures.

Proof: Follows from looking at homology since we are only looking at maps between spheres of the same dimension. For localizations note that localization preserves isomorphisms and homotopies. □
Lemma 5.3.9 The map $S^{n+r+n} \to S^{r+n+n}$ induced by

$$\alpha : D^n \times (D^r \times D^s) \to (D^n \times D^r) \times D^s$$

is homotopic to the identity.

Proof: $\alpha$ is natural and homotopic to the identity on products of spheres by assumption. □

Lemma 5.3.10 Let $X_i, Y_i \in \text{OrdCW}^\ast(C)$ and $f_i : X_i \to Y_i$ maps such that

$\text{Diagram 1}$

is a pushout. Then

$\text{Diagram 2}$

is a pushout.

Proof: In the diagram

the left square is a pushout since in $C$ colimits commute with $\times$ and the right square is a pushout by definition. Therefore the outside square is a pushout. In the following two diagrams the left and outside squares are pushouts so the right one is also.
The outside square is a pushout since colimits commute with $\times$ in $C$. So it follows too in this diagram

$$
\begin{array}{cccc}
X_1(x_1^*) \times Y_2(f_2(x_2)) & (X_1(x_1) \times Y_2(f_2(x_2)))^* & \to & X_1(x_1) \times Y_2(f_2(x_2)) \\
\downarrow & \downarrow & & \\
Y_1(f_1(x_1))^* \times Y_2(f_2(x_2)) & (Y_1(f_1(x_1)) \times Y_2(f_2(x_2)))^* & \to & Y_1(f_1(x_1)) \times Y_2(f_2(x_2))
\end{array}
$$

The same argument in the other variable then shows that

$$
\begin{array}{cccc}
(X_1(x_1) \times X_2(x_2))^* & (Y_1(f_1(x_1)) \times Y_2(f_2(x_2)))^* & \to & Y_1(f_1(x_1)) \times Y_2(f_2(x_2)) \\
\downarrow & \downarrow & & \\
X_1(x_1) \times X_2(x_2) & Y_1(f_1(x_1)) \times Y_2(f_2(x_2))
\end{array}
$$

is a pushout which is the desired result.

**Lemma 5.3.11** Let $X_i \in \text{OrdCW}^e C$. Then $X_1 \times X_2 \in \text{OrdCW}^e C$ canonically with $K(X_1 \times X_2) = K(X_1) \times K(X_2)$ and $x \times y \leq x' \times y'$ if $x \leq x'$ and $y \leq y'$. For every pair $S_i \subset K(X_i)$ there are compatible isomorphisms

$$(X_1 \times X_2)(S_1 \times S_2) \cong X_1(S_1) \times X_2(S_2)$$

**Proof:** We prove the lemma by induction. We begin by doing the case of adding an element.

Fix $T \subset K(X_1) \times K(X_2)$. Assume that for every $S \subset T \subset K(X_1) \times K(X_2)$ we have defined $(X_1 \times X_2)(S)$ compatibly and for $S$ of the form $S_1 \times S_2 (X_1 \times X_2)(S) \cong X_1(S_1) \times X_2(S_2)$. Let $\beta \in K(X_1) \times K(X_2)$ such that $\beta \notin T$ and $T \cup \{\beta\}$ is closed. There are two cases.

Case 1) $\beta = a \times b$ where $a, b \neq e$.

$a^* \times b$ and $a \times b^*$ are both subsets of $T$ so there exist maps

$$f(a) \times b : S^{[a]-1} \times D^{|b|} \to X_1(a^*) \times X_2(b)$$

and

$$a \times f(b) : D^{|a|} \times S^{|b|-1} \to X_1(a) \times X_2(b^*)$$
where \( f(a) \) and \( f(b) \) are the attaching maps and \( a \) and \( b \) are the inclusions of cells. These two maps determine

\[
f : (D[a] \times D[b])^\bullet \to (X_1 \times X_2)(T).
\]

Let \( T' = T \cup \{ \beta \} \). Then let the following diagram be the push out defining \((X_1 \times X_2)(T')\)

\[
\begin{array}{ccc}
S^{[a]+[b]-1} & \longrightarrow & (D[a] \times D[b])^\bullet \\
\downarrow & & \downarrow f \\
D^{[a]+[b]} & \longrightarrow & D[a] \times D[b] \\
\downarrow & & \downarrow \\
& & (X_1 \times X_2)(T')
\end{array}
\]

We now need to see that if \( T' = T_1 \times T_2 \) then \((X_1 \times X_2)(T') \cong X_1(T_1) \times X_2(T_2)\). To see this first observe that

\[
(X_1 \times X_2)(x_1^* \times x_2^*) \longrightarrow (X_1 \times X_2)(x_1^* \times x_2)
\]

\[
(X_1 \times X_2)(x_1 \times x_2^*) \longrightarrow (X_1 \times X_2)((x_1 \times x_2)^*)
\]

is a pushout since in both vertical, or for that matter horizontal, cofibrations we are attaching the same sequence of cells. Then use lemma 5.3.10.

Case 2) \( \beta = a \times e \) or \( \beta = e \times a \).

We do \( \beta = a \times e \). \( \beta = e \times a \) being similar. In the following diagram the outside pushout defines \((X_1 \times X_2)(T')\).

\[
\begin{array}{ccc}
S & \xrightarrow{\rho^{-1}} & S \times e \\
\downarrow & & \downarrow \\
D & \xrightarrow{\rho^{-1}} & D \times e \\
\downarrow & & \downarrow \\
& & (X_1 \times X_2)(T')
\end{array}
\]

Clearly the induction hypothesis are satisfied. This finishes the proof of case 2).

It remains only to check what happens with direct limits. Let \( R \subset K(X_1 \times X_2) \) be an ordered subset. Let \( \Gamma \) be the supremum of the ranks (1.4.2) of the elements in \( R \). If \( R \) does not have an element of highest rank then define

\[
(X_1 \times X_2)(R) = \text{colim}_{\alpha \in \Gamma} (X_1 \times X_2)(R(\alpha))
\]
where \( R(\alpha) \) is all the elements of rank \( \alpha \). So suppose \( R \) has elements of rank \( \Gamma \). Let \( I \) be an ordinal and \( \{S(i)\}_{i \in I} \) be a sequence of proper subsets of \( R \) such that \( R(\alpha) \subseteq S(i) \) for every \( \alpha < \Gamma \) and \( i \in I \). Let \( S(i) \subseteq S(j) \) if \( i \leq j \) and \( \bigcup S(i) = R \). Then define \( (X_1 \times X_2)(R) = \text{colim}_i (X_1 \times X_2)(S(i)) \). It is easy to see that the definition is independent of the sequence of subsets chosen.

We next verify the condition on products in the direct limit case. Let \( \{R(\alpha) \subseteq K(X_1 \times X_2)\}_{\alpha \in \Delta} \) be such that if \( \beta > \alpha \) then \( R(\alpha) \subseteq R(\beta) \). Let \( R = \bigcup R(\alpha) \) and assume that the product condition of the lemma holds for all proper \( S \subseteq R \). Assume in addition that \( R = R_1 \times R_2 \). Let \( p_i : R_1 \times R_2 \to R_i \) denote the projection. Then

\[
\text{colim}_\alpha (X_1 \times X_2)(R(\alpha))
\]

\[
\cong \text{colim}_\alpha X_1(p_1(R(\alpha))) \times X_2(p_2(R(\alpha)))
\]

\[
\cong \text{colim}_\alpha \text{colim}_\beta X_1(p_1(R(\alpha))) \times X_2(p_2(R(\beta)))
\]

\[
\cong X_1(R_1) \times X_2(R_2).
\]

The first isomorphism is from the induction hypothesis, the second categorical and the last from the fact that the product commutes with colimits. We have thus demonstrated the induction hypothesis for \( R \). The proof is complete. \( \square \)

**Lemma 5.3.12** Assume that \( C \) is suitably monoidal. Let \( f_1, g_i : X_i \to Y_i \) be maps in \( \text{OrdCW}^*(C) \) such that \( f_i \simeq_{ir} g_i \) (rel \( e \)) then \( f_1 \times f_2 \simeq_{ir} g_1 \times g_2 \) (rel \( e \)).

**Proof:** Since \( \simeq_{ir} \) is preserved under composition it is enough to prove the lemma in the case \( f_2 = g_2 = id \). In this case we have an irrelevant homotopy \( H : Cyl(X_1, e) \to Y_1 \) from \( f_1 \) to \( g_1 \). Next we will observe that \( Cyl(X_1, e) \times X_2 \) is a \( Cyl(X_1 \times X_2, X_2 \vee e) \). \( C \) being monoidally cocomplete implies that

\[
(X_1 \vee e X_1) \times X_2 \cong (X_1 \times X_2) \vee_{X_2 \vee e} (X_1 \times X_2).
\]
Also using the fact that products of weak equivalences between cofibrant objects over \( e \) are weak equivalences (4.4.10) we see that \( Cyl(X_1, e) \times X_2 \to X_1 \times X_2 \) is a weak equivalence. Next we see that this cylinder object is irrelevant.

For \( T \subset K(X_1 \times X_2) \) let \( A(T) \) denote \( \text{colim}_{T_1, T_2 \subset T} Cyl(X_1(T_1), e) \times X_2(T_2) \). We wish to show that \( A(T) \simeq (X_1 \times X_2)(T) \cup (e \times X_2) \). Assume that this is true for every subset of \( T \).

Let \( x = x_1 \times x_2 \in K(X_1 \times X_2) \) be such that \( x \not\in T \) and \( T' = T \cup \{x\} \) is closed. Then we have pushouts

\[
\begin{array}{ccc}
(X_1 \times X_2)(x^*) & \longrightarrow & (X_1 \times X_2)(T) \\
\downarrow & & \downarrow \\
(X_1 \times X_2)(x) & \longrightarrow & (X_1 \times X_2)(T')
\end{array}
\]

and

\[
\begin{array}{ccc}
A(x^*) & \longrightarrow & A(T) \\
\downarrow & & \downarrow \\
A(x) & \longrightarrow & A(T').
\end{array}
\]

Let \( Y \) be the pushout of

\[
\begin{array}{ccc}
(X_1 \times X_2)(x^*) & \xrightarrow{\partial_b + id} & A(x^*) \\
\downarrow & & \downarrow \\
(X_1 \times X_2)(x).
\end{array}
\]

Since \( (X_1 \times X_2)(x_1^* \times x_2) \) and \( A(x_1^* \times x_2^* \) are subcomplexes of \( A(x_1^* \times x_2) \) whose intersection is \( X(x_1^* \times x_2^* \) we see that \( Y \to A(x) \) is a relative cell complex and therefore a cofibration. Therefore we see using Lemma 2.4.11 that \( A(T') \to X(T') \) is a weak equivalence. Direct limits are handled using Lemma 5.1.7.

The above means that by Lemma 5.1.12 the homotopy

\[ H \times id : Cyl(X_1 \times X_2, X_2 \vee e) \to Y_1 \times X_2 \]

gives us a homotopy

\[ H' : Cyl(X_1 \times X_2, e) \to Y_1 \times X_2 \]
between \( f_1 \times id \) and \( g_1 \times id \). Since by the definition above

\[
Cyl((X_1 \times X_2)_n, (X_2)_n \vee e) = \text{Colim}_{r \leq n}((Cyl((X_1)_r, e) \times (X_2)_s) \cup X_2)
\]

and the map

\[
Cyl((X_1 \times X_2)_n, e) \rightarrow Cyl((X_1 \times X_2)_n, X_2 \vee e)
\]
factors through \( Cyl((X_1 \times X_2)_n, (X_2)_n \vee e) \) it is seen that \( H' \) is irrelevant.

**Lemma 5.3.13** For \( S \subset K(X) \) there is a canonical isomorphism compatible with inclusion

\[
\left( C_\ast(X_1) \otimes C_\ast(X_2) \right)(S) \rightarrow C_\ast(X_1 \times X_2(S)).
\]

**Proof:** Using Lemma 5.3.6, Lemma 5.2.15 gives an isomorphism extending the already defined ones

\[
\left( C_\ast(X_1) \otimes C_\ast(X_2) \right)(T') \rightarrow C_\ast(X_1 \times X_2(T')).
\]

Clearly we then have that

\[
\left( C_\ast(X_1) \otimes C_\ast(X_2) \right)(T) \cong C_\ast(X_1 \times X_2(T)).
\]

\[\square\]

**Corollary 5.3.14** \( C_\ast(X \times Y) \cong C_\ast(X) \otimes C_\ast(Y) \) canonically.

**Proof:** \[\square\]

In the next lemma the isomorphism of the above corollary is considered implicit.

**Lemma 5.3.15** \( C_\ast(f) \otimes C_\ast(g) = C_\ast(f \times g) \)
Proof: First assume that \( f : S^s \to S^s \) and \( g = id \). The lemma is then demonstrated by the following string of diagrams each of which we know to commute.

\[
\begin{align*}
C_*(S^s) \otimes C_*(S^r) & \xrightarrow{C_*(f) \otimes C_*(id)} C_*(S^s) \otimes C_*(S^r) \\
C_*(S^{s+r}) & \xrightarrow{C_*(-r)(f)} C_*(S^{s+r}) \\
C_*(S^{s+r}) & \xrightarrow{C_*(\Sigma f)} C_*(S^{s+r}) \\
C_*(S^{s+r}) & \xrightarrow{C_*({\Theta}^{-1})} C_*(S^{s+r}) \\
C_*((S^s \times S^r)_{r+s/r+s-1}) & \xrightarrow{C_*({f} \times id)} C_*((S^s \times S^r)_{r+s/r+s-1})
\end{align*}
\]

The top square commutes always in DGM, the middle square from the definition of \( F \) and the bottom square since the monoidal and cell structures are compatible. The more general case of \( f : \bigvee S^s \to \bigvee S^s \) follows easily. If \( r < 0 \) then instead we use almost the same diagram to show \( \Sigma^{-r}(C_* f \otimes C_* id) = \Sigma^{-r} C_*(f \times id) \) which implies \( C_* f \otimes C_* id = C_*(f \times id) \).

The next case to deal with is \( f : X_1 \to Y_1 \) any map and \( g = id : X_2 \to X_2 \). First observe that the following diagram commutes from the previous case.

\[
\begin{align*}
C_*((X_1)_{s/s-1}) \otimes C_*((X_2)_{r/r-1}) & \xrightarrow{C_*({f}/s) \otimes C_* id_{r/r-1}} C_*((Y_1)_{s/s-1}) \otimes C_*((X_2)_{r/r-1}) \\
C_*((X_1)_{s/s-1} \times (X_2)_{r/r-1}) & \xrightarrow{C_*({f}/s \times id_{r/r-1})} C_*((Y_1)_{s/s-1} \times (X_2)_{r/r-1})
\end{align*}
\]

On any particular cell the pushout

\[
\begin{align*}
S^{r+s-1} & \longrightarrow (D^r \times D^s)^* \\
\downarrow & \\
D^{r+s} & \longrightarrow D^r \times D^s
\end{align*}
\]

defines the cell structures in \((X_1)_s \times (X_2)_r\) and \((X_1)_{s/s-1} \times (X_2)_{r/r-1}\) and so defines the maps

\[(X_1)_s \times (X_2)_r \to (X_1 \times X_2)_{r+s}\]
and

\[(X_1)_{r/r-1} \times (X_2)_{s/s-1} \to (X_1 \times X_2)_{r+s/r+s-1} \cdot \]

Therefore by Lemma 5.2.15 the following diagram commutes.

\[
\begin{array}{ccc}
(X_1)_{r} \times (X_2)_{s} & \longrightarrow & (X_1)_{r/r-1} \times (X_2)_{s/s-1} \\
\downarrow & & \downarrow \\
(X_1 \times X_2)_{r+s} & \longrightarrow & (X_1 \times X_2)_{r+s/r+s-1}
\end{array}
\]

Thus the diagram

\[
\begin{array}{ccc}
(X_1)_{r/r-1} \times (X_2)_{s/s-1} & \longrightarrow & (Y_1)_{r/r-1} \times (X_2)_{s/s-1} \\
\downarrow & & \downarrow \\
(X_1 \times X_2)_{r+s/r+s-1} & \longrightarrow & (Y_1 \times X_2)_{r+s/r+s-1}
\end{array}
\]

commutes since the horizontal maps are defined by what happens on individual cells and the vertical maps are induced by the identity on individual cells.

Together we get that

\[
\begin{array}{ccc}
C_\ast((X_1)_{s/s-1}) \otimes C_\ast((X_2)_{r/r-1}) & \xrightarrow{C_\ast((f_{s/s-1}) \otimes C_\ast(id_{r/r-1}))} & C_\ast((Y_1)_{s/s-1}) \otimes C_\ast((X_2)_{r/r-1}) \\
\downarrow & & \downarrow \\
C_\ast((X_1 \times X_2)_{r+s/r+s-1}) & \xrightarrow{C_\ast((f \times id)_{r+s/r+s-1})} & C_\ast((Y_1 \times X_2)_{r+s/r+s-1})
\end{array}
\]

commutes and the lemma has been demonstrated if one of the maps is the identity. The general case then follows easily by composition. □

5.4 Monoidal Objects

For this section assume that \( C \) is monoidally cocomplete with compatible monoidal and cell structures.

**Definition 5.4.1** A strong CW complex \((A, \phi)\) is a cell complex \( A \in C[Mon] \) together with a \( CW^\ast \) structure on \( F(A) \) such that \( F(\phi) \in Hom_{CW^\ast(C)} \) where \( \phi \) is the multiplication
on $A$ and $A \times A$ is given the canonical $CW^e$ structure. We denote the category of strong $CW$ complexes by $SCW(C[Mon])$. A map of strong $CW$ complexes is a monoidal map $f$ such that $F(f) \in \text{Hom}_{CW^e(C)}$.

This means a strong $CW$ complex is a cell complex in $C[Mon]$ together with a cell structure on $F(A)$ and a cellular multiplication. Let us look at an example in $DGM(R)$. $F(P(a) \otimes P(b)) \in CW^e(DGM(R))$ and $F(\phi) \in \text{Hom}_{CW^e(DGM(R))}$ but $P(a) \otimes P(b)$ cannot be given the structure of a cell complex in $DGA(R)$ since it is not a tensor algebra.

For example in the $C[Mon]$ pushout

$$
\begin{array}{ccc}
TS & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
TD & \rightarrow & Y
\end{array}
$$

$Y$ is SCW if $X$ is SCW and $Ff$ is cellular.

**Definition 5.4.2** A map $f : X \rightarrow Y \in \text{Hom}_{OrdCW^e(C)}$ together with an order preserving function $K(X) \rightarrow K(Y)$, also denoted $f$, and for every $S \subset K(X)$ and $T \subset K(Y)$ such that $f(S) \subset T$ compatible maps $f(S,T) : X(S) \rightarrow Y(T)$ is called a cellular isomorphic inclusion (cii) if for every $x \in K(X)$ the following square is a pushout

$$
\begin{array}{ccc}
X(x^*) & \rightarrow & Y(f(x)^*) \\
\downarrow & & \downarrow \\
X(x) & \rightarrow & Y(f(x))
\end{array}
$$

and if the induced map $S^{[x]} \rightarrow S^{[x]}$ is homotopic to the identity. The subcategories of $OrdCW^e(C)$ of cii will be denoted by $CII$.

**Lemma 5.4.3** 1) The composition of two cii is a cii.

2) The identity is a cii.

3) The product of two cii is a cii.
4) For $X_i \in OrdCW^*(C)$

$$\alpha : X_1 \times (X_2 \times X_3) \to (X_1 \times X_2) \times X_3$$

is cii.

**Proof:** We start by proving part three the first two being trivial. In that case the pushout condition follows from Lemma 5.3.10. Next we prove the other condition. The following diagram commutes

$$
\begin{array}{c}
X_1(x_1) \times X_2(x_2) \\
\downarrow \\
S|^{x_1} \times S|^{x_2} \\
\downarrow \\
S^{x_1+|x_2|} \\
\end{array}
\quad
\begin{array}{c}
\xrightarrow{f_1 \times f_2} \\
\xrightarrow{(f_1)|^{x_1}/|x_1|-1 \times (f_2)|^{x_2}/|x_2|-1} \\
\xrightarrow{(f_1 \times f_2)|^{x_1+|x_2|}/|x_1+|x_2|-1} \\
\end{array}
\begin{array}{c}
Y_1(f_1(x_1)) \times Y_2(f_2(x_2)) \\
\downarrow \\
S|^{x_1} \times S|^{x_2} \\
\downarrow \\
S^{x_1+|x_2|} \\
\end{array}
$$

and $(f_1)|^{x_1}/|x_1|-1 \times (f_2)|^{x_2}/|x_2|-1$ is the product of two maps irrelevantly homotopic to the identity. Therefore it is irrelevantly homotopic to the identity by Lemma 5.3.12. Therefore by Lemma 5.1.15 $(f_1 \times f_2)|^{x_1+|x_2|}/|x_1+|x_2|-1$ is homotopic to the identity.

Next we turn to part 4) which we show by induction.

Assume that for every $S'' \subset S' \subset S$ we have cii maps

$$TG(\alpha)(S') : TG(X_1 \times (X_2 \times X_3))(S') \to TG((X_1 \times X_2) \times X_3)(S')$$

such that the following two diagrams commutes

$$
\begin{array}{c}
TG(X_1 \times (X_2 \times X_3))(S'') \\
\downarrow \\
TG(X_1 \times (X_2 \times X_3))(S') \\
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\quad TG((X_1 \times X_2) \times X_3)(S'')} \\
\xrightarrow{\quad TG((X_1 \times X_2) \times X_3)(S')} \\
\end{array}
\begin{array}{c}
TG((X_1 \times X_2) \times X_3)(S'') \\
\downarrow \\
TG((X_1 \times X_2) \times X_3)(S') \\
\end{array}
$$
Remember that the isomorphisms $TG^+(a)$ is already determined. Let $\beta_i$ denote a cell in $X_i$ such that $\beta_1 \times \beta_2 \times \beta_3 \in S^+ - S$ and $S' \subset S$ such that $\beta_1 \times \beta_2 \times \beta_3 \in (S')^+$. Let $T = S' \cup \{\beta_1 \times \beta_2 \times \beta_3\}$ then we have a pushout

$$TG(X_1 \times (X_2 \times X_3))(S') \longrightarrow TG((X_1 \times X_2) \times X_3)(S')$$

$$TG(X_1 \times (X_2 \times X_3)) \longrightarrow TG((X_1 \times X_2) \times X_3)$$

The induction hypothesis and the naturality of $\alpha$ gives us two commuting diagrams

$$ TG(X_1 \times (X_2 \times X_3))(S') \longrightarrow TG((X_1 \times X_2) \times X_3)(S') $$

$$ (D|_{[\beta_1]} \times (D|_{[\beta_2]} \times D|_{[\beta_3]})) \longrightarrow TG(X_1 \times (X_2 \times X_3))(S') $$

$$ (D|_{[\beta_1]} \times (D|_{[\beta_2]} \times D|_{[\beta_3]})) \longrightarrow TG(X_1 \times (X_2 \times X_3))(T) $$

and

$$ (D|_{[\beta_1]} \times (D|_{[\beta_2]} \times D|_{[\beta_3]})) \longrightarrow (D|_{[\beta_1]} \times (D|_{[\beta_2]} \times D|_{[\beta_3]})) $$

$$ ((D|_{[\beta_1]} \times D|_{[\beta_2]} \times D|_{[\beta_3]})) \longrightarrow ((D|_{[\beta_1]} \times D|_{[\beta_2]} \times D|_{[\beta_3]})) $$

Therefore we get an isomorphism

$$ TG(\alpha)(T) : TG(X_1 \times (X_2 \times X_3))(T) \rightarrow TG((X_1 \times X_2) \times X_3)(T) $$

such that the following diagram commutes

$$ TG(X_1 \times (X_2 \times X_3))(S') \longrightarrow TG((X_1 \times X_2) \times X_3)(S') $$

$$ TG(X_1 \times (X_2 \times X_3)) \longrightarrow TG((X_1 \times X_2) \times X_3) $$

$$ TG(X_1 \times (X_2 \times X_3))(T) \longrightarrow TG((X_1 \times X_2) \times X_3)(T) $$
The induced map on spheres is homotopic to the identity since the cell and monoidal structures are compatible. This concludes the induction. □

**Definition 5.4.4** Let \( M \) be a monoid with a partial order \( \leq \). If the multiplication on \( M \) is order preserving then we say that \( \leq \) is **multiplicative**. This is equivalent to saying that whenever \( x_i \in M, x_1 \leq x_2 \) and \( x_3 \leq x_4 \) implies that \( x_1x_3 \leq x_2x_4 \).

Let \((A, \phi) \in SCW(C[Mon])\) and \((A, \leq) \in OrdCW^e(C)\) and

\[ \phi : K(A) \times K(A) = K(A \times A) \to K(A) \]

be a map that makes \( \phi \) cii and makes \((K(A) \cup \{e\}, \leq)\) into a free monoid with multiplicative partial order. We also require that the cell structure be given by the cell structure on generators. That is, if \( \beta \in K(A), \beta = x_1...x_r, \) with the \( x_i \) generators, then we have pushouts

\[
\begin{array}{c}
S^{n-1} \longrightarrow (A(x_1) \times (A(x_2)... \times A(x_r))...)^{\bullet} \longrightarrow A(\beta^*) \\
\downarrow \hspace{2cm} \downarrow \\
D^n \longrightarrow (A(x_1) \times (A(x_2)... \times A(x_r))...) \longrightarrow A(\beta)
\end{array}
\]

where the left square is the pushout that gives the cell structure of \( A \times (A \times (...A)...) \) and the right square commutes since \( \phi \) is cii. Our condition is to require that the cell structure of \( A(\beta) \) be given by the outside pushout. We then say that \( A \) together with the above structures is a **multiplicatively ordered CW complex**. We denote the category of such objects by \( MOrdCW^e(C) \) where the maps are \( SCW(C[Mon]) \) maps. The subcategory of \( MOrdCW^e(C) \) of cii maps that induce monoidal maps on the set of cells will be denoted \( MCII \).

**Lemma 5.4.5** \( X \in OrdCW(C) \) implies that \( TX \in MOrdCW^e(C) \) canonically such that \( K(TX) = MK(X) \) and with the minimal multiplicative order such that \( x > e \) for every \( x \in KT(X) \), and for \( x, y \in K(X) \), \( y > x \) in \( K(X) \) implies that \( y > z \) for every \( z \in M(K(X(x))) \).
Proof: In this proof we will use the notation of Definition 4.3.1. Let $X \in CW(C)$. Assume that for every $i \leq n$ that $U^i(X)$ is a CW complex over $e$ such that $K(U^i(X)) = M^{\leq i}(K(X))$. Then it is easy to see that $j : Y^n(X) \to U^1(X) \times U^n(X)$ is a relative CW complex such that $K(j) = M^{n+1}(K(X))$. So since $Y^n \to U^n$ is a cellular map $U^{n+1}(X)$ is a CW complex over $e$ such that $K(U^{n+1}) = M^{\leq n+1}(K(X))$ and so $K(T(X)) = M(K(X))$. In order to order $K(T(X))$ we let

1) $x, y \in K(X), y > x$ in $K(X)$ imply that $y > z$ for every $z \in M(K(X(x)))$.

2) $x > e$ for every $x \neq e$

and 3) $x_1 \ldots x_n \geq y_1 \ldots y_n$ if for every $i, x_i \geq y_i$.

All other pairs of elements are incomparable. To see that this order is well defined use induction on product length and observe that if we have $x_1, x_2, x_3, y_1, y_2, y_3$ such that $x_1 > y_1, x_2x_3 > y_2y_3, x_1x_2 < y_1y_2$ and $x_3 < y_3$ then $x_2 < y_2$. So $x_2x_3 < y_2y_3$ which gives a contradiction in lower product length.

Next we show by induction that closed subsets correspond to subcomplexes. Note first that $e$ is a subcomplex. Assume that the statement is true for all subsets of $T \subset K(T(X))$ and let $T' = T \cup \{x\}$ be a closed subset. There are two cases to examine.

Case 1) $x = ay$ where $a$ is a generator and $y \neq e$.

Then $TX(x)$ is the pushout

$$
\begin{array}{ccc}
TX((a \times y)^*) & \longrightarrow & TX(x^*) \\
\downarrow & & \downarrow \\
TX(a \times y) & \longrightarrow & TX(x)
\end{array}
$$

and so the induction step easily follows.

Case 2) $x$ is a generator.

Then we have a pushout

$$
\begin{array}{ccc}
X(x^*) & \longrightarrow & TX(x^*) \\
\downarrow & & \downarrow \\
X(x) & \longrightarrow & TX(x)
\end{array}
$$
and so again the induction step easily follows.

Next we want to see that φ is a cii map. Let α, β ∈ K(TX). α = x₁...xₙ, β = y₁...yᵣ. We need to see that

\[
\begin{array}{c}
TX((\alpha \times \beta)^*) \longrightarrow TX((\alpha \beta)^*) \\
\downarrow \\
TX(\alpha \times \beta) \longrightarrow TX(\alpha \beta)
\end{array}
\]

is a pushout that induces a map homotopic to the identity on S^{[αβ]}. To this end observe that both the left square and the outside square of

\[
\begin{array}{c}
TX((B(α) \times B(β))^*) \longrightarrow TX((\alpha \times \beta)^*) \longrightarrow TX((αβ)^*) \\
\downarrow \\
TX(B(α) \times B(β)) \longrightarrow TX(\alpha \times β) \longrightarrow TX(αβ)
\end{array}
\]

are pushouts so the right square is also. Next we show that φ induces maps homotopic to the identity on cells. The horizontal maps in the pushout diagram

\[
\begin{array}{c}
TX(B(αβ)^*) \longrightarrow TX(B(α)) \times TX(B(β))^* \\
\downarrow \\
TX(B(αβ)) \longrightarrow TX(B(α)) \times TX(B(β))
\end{array}
\]

are compositions of α and therefore cii. The pushout

\[
\begin{array}{c}
(TX(B(α)) \times TX(B(β))^*) \longrightarrow TX((α \times β)^*) \\
\downarrow \\
TX(B(α)) \times TX(B(β)) \longrightarrow TX(α \times β)
\end{array}
\]

is the way we define the attaching map of α × β in TX(α × β) and so induces the identity on S^{[αβ]}. Similarly it is in the pushout

\[
\begin{array}{c}
TX(B(αβ)^*) \longrightarrow TX((αβ)^*) \\
\downarrow \\
TX(B(αβ)) \longrightarrow TX(αβ).
\end{array}
\]
Putting these last three diagrams together we see that $\phi$ induces a map $S^{[\alpha\beta]} \to S^{[\alpha\beta]}$ homotopic to the identity on the cell $\alpha \times \beta$ mapping to $\alpha\beta$. Colimits are easy to handle. 

**Definition 5.4.6** Let $D$ be a small category and $G^+ : D \to \text{MCII}$ a functor. Let $G$ denote the underlying functor $G : D \to \text{C[Mon]}$. A cell structure for $T(G)$ is a functor $TG^+ : TD \to \text{CII}$ such that the following four conditions hold:

1) The following diagram commutes.

\[
\begin{array}{ccc}
D & \xrightarrow{G} & \text{MCII} \\
\downarrow & & \downarrow f \\
TD & \xrightarrow{TG} & \text{CII}
\end{array}
\]

2) $TG : TD \to \text{C}$ is the functor underlying $TG^+$.

3) For every $x, y \in TD$ $TG((x)(y)) = TG(x) \times TG(y)$ in the canonical way (Lemma 5.3.11).

4) For every $f, g \in \text{Hom}_{TD} (TG((f)(g)) = TG(f) \times TG(g)$ in the canonical way (Lemma 5.4.3).

Let $i : F^+ \to G^+$ be a natural transformation between functors $* \to \text{MCII}$. $Ti$ is called an inclusion of cell structures if $i(*) : F^+ \to G^+$ is a CII inclusion of cell complexes. $i$ induces an inclusion of monoids and as monoids:

$$K'\left(G^+(*\right)) = i(*) K F^+ (\ast) \prod B$$

where $B$ is a free monoid.

**Lemma 5.4.7** Let $D$ be a disjoint small category and $G^+ : D \to \text{MCII}$ a functor. Then there exists a canonical cell structure for $T(G)$.

**Proof:** We only need to verify that for every $f \in \text{Hom}_{TD} (TG(f)$ is a cii.

$\text{Hom}_{TD}$ is the closure under products and composition of $\alpha, \phi, \rho, \lambda$ and $id$. We know that every $TG(\phi)$ is a cii by hypothesis and it is obvious that $TG(\rho), TG(\lambda)$ and $TG(id)$ are
cii. Since by 5.4.3 cii are preserved by products and compositions, the only remaining case of $TG(\alpha)$ follows from 5.4.3.

Definition 5.4.8 Let $(A, \leq), (B, \leq)$ be monoids with multiplicative partial well orders. Then let $(A \sqcup B, \leq)$ be the order on the coproduct of $A$ and $B$ determined by $b > a$ for every $b \in B - \{e\}$ and $a \in A$ and $\alpha_i \leq \beta_i$ implies that $\alpha_1 \alpha_2 \leq \beta_1 \beta_2$ for every $\alpha_i, \beta_i \in A \sqcup B$.

Lemma 5.4.9 $(A \sqcup B, \leq)$ is a multiplicative well founded partial order.

Proof: Partial orders on $A$ and $B$ give a partial order on $M(A \cup B/(e_A = e_B))$ and the multiplicativity of the orders implies that it extends to its quotient $A \sqcup B$. It is easily seen to be multiplicative.

Now let $ab : A \sqcup B \to A \times B$ be the abelianization map and order $A \times B$ right lexicographically. Let $X \subset A \sqcup B$ and $x \in ab(X)$ be a minimal element. Then any of $(ab)^{-1}(x) \in X$ will be a minimal element of $X$.

Lemma 5.4.10 Let $D$ be the category

\[ a_1 \xrightarrow{f} a_2 \]

\[ \downarrow i \]

\[ a_3 \]

Let $F : D \to MOrdCW^+(\mathbb{C})$ be a functor such that $F(i)$ is an inclusion of cell structures and for some free monoid $B$, $K(F(a_3)) = B \sqcup K(F(a_1))$ ordered as in Definition 5.4.8. Then $G : * \to \text{colim}_{TD} TF$ has a canonical cell structure, $j : F(a_2) \to G$ is an inclusion of cell structures and with the ordering of 5.4.8

\[ K(G) = B \sqcup K(F(a_2)). \]

Proof: First we define a functor $G : * \to \mathbb{C}[\text{Mon}]$ and give a cell structure to $TG$. Let $K(G) = B \sqcup K(F(a_2))$ as required in the lemma. We make some recursive definitions.
Fix $\beta \in K(G)$ and assume that for every $x, y \in T(*)$, $f \in Hom(x, y)$, $S \subset K(TG(x))$, $T \subset K(TG(y))$. $S$ and $T$ closed.

$f(S) \subset T$ and that we have defined $TG(x)(S), TG(y)(T)$ and a cii

$$TG(f)(S,T): TG(x)(S) \to TG(y)(T)$$

such that $TG(g)(T, U) \circ TG(f)(S, T) = TG(g \circ f)(S, U)$.

Let $\beta' \notin T$ such that $T \cup \{\beta'\}$ is closed and $p(\beta') = \beta$. Let $T' = T \cup \{\beta'\}$ and $S' = S \cup A$ where $A \subset f^{-1}(\beta')$. If $y \neq *$ then $TG(y)(T')$ and $TG(x)(S')$ are determined by Definition 5.4.6 and Lemma 5.4.7 and $TG(g)(S', T')$ is a composition of products of cii maps and so is cii. So assume that $y = *$. Similarly we can also assume that $x = * \times *$. If $\beta$ is a generator then define $TG(*)(T')$ to be the pushout of

$$TG(*) (\beta^*) \to TG(*) (T)$$

$$\downarrow$$

$$TG(*) (\beta)$$

where the horizontal map has been defined by the induction hypothesis.

If $\beta$ is a product it is sufficient to look at the case $A = \{\beta_1 \times \beta_2\}$ with $\beta_1 \beta_2 = \beta$. the other cases following easily. Such are our assumptions.

Define $TG(*)(T')$ to be the pushout with induced cell structure

$$TG(B(\beta)*) \to TG(*) (T)$$

$$\downarrow$$

$$TG(B(\beta)) \to TG(*)(T').$$

We know $\beta_1 = a_1...a_r$ where $a_i$ are generators of $K(G)$ and that there exists a unique map $g \in Hom_{TD}(B(\beta),(* \times *) \times *)$ such that $TG(g)(B(\beta)) = (a_1 \times a_2...a_r) \times \beta_2$. $g$ is a composition of maps that are products of $\alpha$ and id and maps that are products of $\phi$ and id.

The maps $\alpha$ are cii by Lemma 5.4.7. Lemma 5.4.3 implies that if $TG(f_i)(S_i, T_i)$ are cii then
\(TG(f_1 \times f_2)(S_1 \times S_2, T_1 \times T_2)\) is also cii. The same lemma also shows that cii is preserved under products, so \(g\) is cii. Therefore the following diagram is a pushout

\[
\begin{array}{c}
TG(B(\beta)^*) \longrightarrow TG(((a_1 \times a_2...a_r) \times \beta_2)^*) \\
\downarrow \\
TG(B(\beta)) \longrightarrow TG((a_1 \times a_2...a_r) \times \beta_2).
\end{array}
\]

Also we know that

\[
\begin{array}{c}
TG((a_1 \times a_2...a_r) \times \beta_2)^* \longrightarrow TG(* \times *)(S) \\
\downarrow \\
TG((a_1 \times a_2...a_r) \times \beta_2) \longrightarrow TG(* \times *)(S')
\end{array}
\]

is a pushout. In the following diagram \(TG(\ast)(T')\) is determined by letting the outside square be a pushout.

\[
\begin{array}{c}
TG(B(\beta))^* \longrightarrow TG(* \times *)(S) \longrightarrow TG(\ast)(T) \\
\downarrow \\
TG(B(\beta)) \longrightarrow TG(* \times *)(S') \longrightarrow TG(\ast)(T')
\end{array}
\]

Therefore both the left and outside squares are pushouts so the right one is also. This then shows \(TG(\phi)(S', T')\) is cii thus completing the induction step. Inducting over direct limits is easy.

We now need to show that \(TG(\ast) = \text{colim}_{x \in TD} TF(x)\).

We first construct a compatible family of maps \(\{j(x) : TF(x) \to TG(\ast)\}_{x \in TD}\). To begin with we have compatible cii maps \(j(a_s)\). Now let \(x \in TD\) be any element. Let \(r(x) \in T(\ast)\) be the element gotten by replacing all the \(a_s\)'s by \(\ast\). Let \(f \in Hom_T\ast\) be the unique map that sends \(r(x)\) to \(\ast\) and \(g : TF(x) \to TG(r(x))\) be the map that is the product of \(j(a_s)'s\). Then let \(j(x) = TG(f) \circ g\). Clearly this is a compatible family. It only remains to show that given a compatible family of maps \(f(x) : TF(x) \to Y\) there exists a unique map \(f : TG(\ast) \to Y\) such that \(f(x) = f \circ j(x)\).
We define \( f \) on cells. Let \( \beta \in K(G) \) and let \( A = (B(\beta))^* \). Assume that we have defined

\[
f(\beta^*) : TG(\beta^*) \to Y
\]
such that \( f(x)|_{TF(x)(A)} = f \circ j(x)|_{TF(x)(A)} \) and that it is the unique map with this property.

Since, letting \( \beta \) determine \( x \), we have a pushout

\[
\begin{array}{ccc}
TF(x)(A) & \longrightarrow & TG(B(\beta)) & \longrightarrow & TG(\ast)((\beta)^*) \\
\downarrow & & \downarrow & & \downarrow \\
TF(x)(A \cup B(\beta)) & \longrightarrow & TG(B(\beta)) & \longrightarrow & TG(\ast)(\beta)
\end{array}
\]

we extend the map compatibly and uniquely over the cell \( \beta \).

\( KF(a_3) = KF(a_1) \amalg B \) and \( B \) has its own multiplicative partial well ordering. So we get a multiplicative partial well order on \( KF(a_3) \) from Definition 5.4.8 and Lemma 5.4.9. \( \square \)

We wish to define a special kind of cylinder object for strongly CW monoidal objects.

**Definition 5.4.11** Let \( A \in \text{MOrdCW}^*(C) \). Let \( Cyl(A, e) \) be a cylinder object for \( A \) in \( C[\text{Mon}] \) such that \( Cyl(A, e) \in \text{MOrdCW}^*(C) \). \( FCyl(A, e) \) is a filtered cylinder object for \( A \) over \( e \) and such that for every \( S, T \subset K(A) \) and \( U \subset K(A \times A) \) such that \( ST \subset U \) we have chosen \( CIL \) maps compatible with inclusion and compatible with the multiplication on \( Cyl(A, e) \)

\[
\phi : FCyl(A(S), e) \times FCyl(A(T), e) \to FCyl(A(U), e).
\]

We also require the following diagrams be pushouts

\[
(F(Cyl(A(\alpha), e) \times F(Cyl(A(\beta), e))^* \longrightarrow FCyl(A(\alpha\beta^*), e)
\]

\[
F(Cyl(A(\alpha), e) \times F(Cyl(A(\beta), e)) \longrightarrow FCyl(A(\alpha\beta), e)
\]

where \( F(Cyl(A(\beta), e))^* = FCyl(A(\beta)^*, e) \). Then we say that \( Cyl(A, e) \) is a strongly filtered cylinder object for \( A \).
Lemma 5.4.12 Let $Cyl(A, e)$ and $Cyl'(A, e)$ be strongly filtered cylinder objects for $A$. Then there exists compatible homotopy equivalences with compatible inverses for every $T \subset K(A)$

$$\Theta(T) : FCyl(A(T), e) \to FCyl'(A(T), e)$$

and on their colimits $\Theta : Cyl(A, e) \to Cyl'(A, e)$.

Proof: Let

$$\begin{array}{ccc}
TS & \longrightarrow & A \\
\downarrow & & \downarrow \\
TD & \longrightarrow & A'
\end{array}$$

be a pushout and assume that for a strongly filtered cylinder object $Cyl''(A, e)$, we have, for every $T \subset K(A)$, compatible cofibrations $i(T) : Cyl(A(T), e) \to Cyl''(A(T), e)$ and $i'(T) : Cyl'(A(T), e) \to Cyl''(A(T), e)$ with compatible retractions $r(T) : Cyl''(A(T), e) \to Cyl(A(T), e)$ and $r'(T) : Cyl''(A(T), e) \to Cyl'(A(T), e)$ such that $i = colim(i(T)), i' = colim(i'(T)), r = colim(r(T))$ and $r' = colim(r'(T))$ are all monoidal. Let $x$ denote the new generator in $A'$. Denote $Y, Y'$ and $W$ as the following pushouts.

Then let $W \to Cyl''(A'(x), e) \to A'(x)$ be a factorization into a cofibration followed by a weak equivalence. Let $r(x)$ and $r'(x)$ be any retraction extending the previously defined ones. Assume that for every $S \subset T \subset K(A')$ and $S_1S_2 \subset T$ we have defined compatible
Define maps

\[ \phi : Cyl''(A'(S_1), e) \times Cyl''(A'(S_2), e) \to Cyl''(A'(S_1S_2), e) \]

cofibrations \( i'(S) : Cyl'(A'(S), e) \to Cyl''(A'(S), e) \), \( i(S) : Cyl(A'(S), e) \to Cyl''(A'(S), e) \)
and retractions \( r(S) : Cyl''(A'(S), e) \to Cyl(A'(S), e) \), \( r'(S) : Cyl''(A'(S), e) \to Cyl'(A'(S), e) \)
all compatible with \( \phi \) and such that the required diagram is a pushout. Let \( \alpha \in K(A') \alpha \not\in T \). Then \( \alpha = y\beta \) where \( y \) is a generator and \( \beta \in K(A') \). For every closed \( S \cup \{\alpha\} \) we define \( Cyl''(A'(S), e) \) by the pushout

\[
\begin{array}{ccc}
(Cyl''(A'(y), e) \times Cyl''(A'(\beta), e)) & \to & Cyl''(A'(S), e) \\
\downarrow & & \downarrow \\
Cyl''(A'(y), e) \times Cyl''(A'(\beta), e) & \to & Cyl''(A'(S \cup \{\alpha\}), e).
\end{array}
\]

The extensions \( i'(S \cup \{\alpha\}), i(S \cup \{\alpha\}), r(S \cup \{\alpha\}), r(S \cup \{\alpha\}) \) and \( \phi \) are clear and it is easy to check that they are compatible.

We define \( FCyl''(A', e) \) to be \( colim_{T \in K(A')}Cyl''(A'(T), e) \) and \( \phi \) as the colimit of the restricted \( \phi \). Now we define the homotopy equivalences to be \( \theta(T) = r'(T)i(T) \) and \( \theta^{-1}(T) = r(T)i'(T) \) then \( \theta(T)\theta^{-1}(T) \simeq id \) and \( \theta^{-1}(T)\theta(T) \simeq id \) since \( r'(T)i'(T) = id \) and \( r(T)i(T) = id \) and \( i'(T)r'(T) \simeq id \simeq i(T)r(T) \). \( \square \)

For strongly \( CW \) monoidal objects we now construct a particular strongly filtered cylinder object.

**Definition 5.4.13** Define maps

\[ f(n + 1) : K(\text{TcylD}^{n+1}) \to M(K(D^{n+1})) \]

as follows. Let \( a \) denote the bottom cell and \( b \) denote the top cell of \( D^{n+1} \). For \( \alpha \in K(CylS^n) \)
\( f(n + 1)(\alpha) = a \), for \( \beta \in K(CylD^{n+1}) - K(CylS^n) \) \( f(n + 1)(\beta) = b \). Then extend such that \( f(n + 1) \) is multiplicative. That is, \( f(n + 1)(\alpha\beta) = (f(n + 1)(\alpha))(f(n + 1)(\beta)) \) for all other elements.
Lemma 5.4.14 Let $\text{Cyl}(A, e)$ be a strongly filtered cylinder object for $A \in M\text{OrdC}W^e(C)$ and let $S^n \to F A$ be a cellular map. Then for cellular $Cyl f$ the pushout of

$$
\begin{array}{c}
TCylS^n \twoheadrightarrow Cyl(A, e) \\
\downarrow \\
TCylD^{n+1}
\end{array}
$$

can canonically be given the structure of a strongly filtered cylinder object for $A \cup_{Tf} TD^{n+1}$.

Proof: Let $A'$ denote $A \cup_{Tf} D^{n+1}$. For $x = x_1 \ldots x_n$, $x_i$ generators and $T' = T \cup \{x\}$ closed let the following pushout define $Cyl(A'(T'), e)$.

$$(Cyl(A'(x_1), e) \times \ldots \times Cyl(A'(x_n), e))^* \longrightarrow Cyl(A'(T), e)$$

If $x$ is a generator corresponding to the new cell then we let this pushout define $Cyl(A'(T'), e)$.

$$
\begin{array}{c}
CylS^n \xrightarrow{Cyl f} Cyl(A(T), e) \\
\downarrow \\
CylD^{n+1} \longrightarrow Cyl(A'(T'), e).
\end{array}
$$

The fact that $Cyl(A'(T'), e) \to A'(T')$ is a weak equivalence follows by induction from Lemma 2.1.16 and the fact that in the diagram

$$(D^{n_1} \times \ldots \times D^{n_r})^* \longrightarrow (Cyl D^{n_1} \times \ldots \times Cyl D^{n_r})^*$$

both horizontal maps are weak equivalences and the map from the pushout of the first three spaces to the last is a cofibration. □
**Definition 5.4.15** Let $A, B \in MOrdCW^e(C)$ and $f, g \in Hom(A, B)$. Let $CylA$ be a strongly filtered cylinder object for $A$. Let

\[ \begin{array}{c}
A \oplus A \xrightarrow{f+g} B \\
\downarrow H \\
CylA
\end{array} \]

be a commuting diagram in $MOrdCW^e(C)$ such that we have compatible factorizations.

Then $H$ is a **strongly irrelevant homotopy** from $f$ to $g$. Let $\simeq_{sr}$ be the equivalence relation generated by strongly irrelevant homotopies.

A **strongly cellular homotopy** is a homotopy, $H : CylA \to B$, in $MOrdCW^e(C)$ out of a strongly irrelevant cylinder object such that $FH$ is cellular.

The following lemma says that an extension of a strongly irrelevant homotopy is determined by an irrelevant homotopy on the new generating cells.

**Lemma 5.4.16** Let $H : CylA \to B$ be a strongly irrelevant homotopy from $f$ to $g$. Then an extension to a strongly irrelevant homotopy $H' : Cyl(A \cup D^n) \to B$ is determined by an extension

\[ \begin{array}{c}
CylS^{r-1} \rightarrow Cyl(A_{r-1}) \rightarrow B_r \\
\downarrow \\
CylD^r
\end{array} \]

**Proof:** Easy. □

**Corollary 5.4.17** Let $f : X \to Y$ be a strongly irrelevant homotopy equivalence and $g : S^n \to X$ and $h : S^n \to Y$ be maps such that $fg \simeq_{ir} h$ then any extension $f' : X_n \cup g D^{n+1} \to Y_n \cup h D^{n+1}$ that is a weak homotopy equivalence induces a strongly irrelevant homotopy equivalence $F : X \cup_T D^{n+1} \to Y \cup_T D^{n+1}$. 
Lemma 5.4.18 Every map in $C[\text{Mon}]$ between objects in $MOrdCW^e(C)$ is homotopic to a $MOrdCW^e(C)$ map. If two $MOrdCW^e(C)$ maps are homotopic then they are homotopic by a strongly cellular homotopy.

**Proof:** Follows by using cellular approximation on generators since $C[\text{Mon}]$ maps and homotopies from objects in $MOrdCW^e(C)$ are determined by maps on generators. □

### 5.5 Functors

The categories $C$ and $C'$ in this section will be suitably monoidal categories with cellular approximation and with weakly compatible monoidal and cell structures.

For this definition we use the notation of 3.1.5.

**Definition 5.5.1** For the class $F$ in $OrdCW(C)$ of relative CW complexes $j$ with $s(j) = 1$ and length of $j = 1$ and $f$ the map $\emptyset \to X$ let $C_0'(X)$ denote $G(\omega)(f)$.

For the class $F$ in $MOrdCW^e(C)$ of inclusion of cell complexes $j$ with $s(j) = 1$ and length of $j = 1$ as a cell complex in $C[\text{Mon}]$ and $f$ the map $e \to X$ let $MCo(X) = G(\omega)(f)$.

In both cases they are given CW structures in the obvious way.

These are just taking CW approximation.

**Lemma 5.5.2** $C_0' : C \to OrdCW(C)$ and $MCo : C[\text{Mon}] \to MOrdCW^e(C)$ are functors.

**Proof:** Easy. □

Let $F$ denote the forgetful functor.

**Lemma 5.5.3** $F C_0' X \to X$ and $F MCo A \to A$ are weak equivalences

**Proof:** The lemma follows from cellular approximation, 5.1.17 and 5.4.18, and $\omega$ smallness. □

**Lemma 5.5.4** For $f \in Hom_{OrdCW(C)}$ or $Hom_{MOrdCW^e(C)}$ if $Ff$ is a weak equivalence then $f$ is a homotopy equivalence.
Proof: 2.3.3. □

Lemma 5.5.5 For \( f, g \in \text{Hom}_{\text{OrdCW}(C)} \) or \( \text{Hom}_{\text{MOordCW}^*(C)} \). If \( f \simeq g \) then \( \mathcal{F}f \simeq \mathcal{F}g \).

The following lemma depends on cellular approximation.

Lemma 5.5.6 \( \text{C}^0 \) and \( \text{MCo} \) induce equivalences of categories

\[ \text{Ho}(C) \to \text{OrdCW}(C)/ \simeq \]

and

\[ \text{Ho}(\text{C}[\text{Mon}]) \to \text{MordCW}^*(C)/ \simeq \]

where \( \simeq \) denotes cellular and strongly cellular homotopy.

Proof: Standard. □

Definition 5.5.7 Let \( G : \text{OrdCW}^*(C) \to \text{OrdCW}^*(C') \) be a functor. Let \( G \) preserve weak equivalences and assume the following identities: \( G(S^n \vee e) = S^n \vee e \), \( G(D^n \vee e) = D^n \vee e \) and \( G(i_n) = i_n \). Also assume that all the cell structure is given by the obvious pushouts. For example for the CW structure this means \( G(X, \{X_j\}, \{f(r, j)\}) = (G(X), \{G(X_j)\}, \{G(f(r, j))\}) \).

Assume that \( G(X) \times G(Y) \) is cti isomorphic to \( G(X \times Y) \) naturally as ordered CW\(^\ast\) complexes. We assume also \( G(\alpha) = \alpha \), \( G(\rho) = \rho \) and \( G(\lambda) = \lambda \). In other words the following three diagrams commute.

\[
\begin{align*}
G(A) \times (G(B) \times G(C)) & \xrightarrow{\alpha} (G(A) \times G(B)) \times G(C) \\
G(A) \times G(B \times C) & \xrightarrow{G(a)} G(A \times B) \times G(C) \\
G(A \times (B \times C)) & \xrightarrow{G(\alpha)} G((A \times B) \times C) \\
\end{align*}
\]

\[
\begin{align*}
e \times G(A) & \xrightarrow{\lambda} G(A) \\
g(\lambda) & \xrightarrow{G(\lambda)} G(e \times A)
\end{align*}
\]
Also assume that if we have a pushout in $\text{OrdCW}^e(C)$

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow^{i} & & \downarrow \\
Z & \to & W
\end{array}
\]

such that $i$ is an inclusion of ordered $CW^e$ complexes then $G(W)$ is the pushout of

\[
\begin{array}{ccc}
G(X) & \to & G(Y) \\
\downarrow^{G(i)} & & \downarrow \\
G(Z)
\end{array}
\]

and $G(i)$ is an inclusion of ordered $CW^e$ complexes. Finally assume that $G$ commutes with direct limits of $CW^e$ complexes. We call such a $G$ a **good functor**.

**Lemma 5.5.8** Let $G$ be a good functor. Let $Cyl(X, e)$ be an irrelevant cylinder object for $A$. Then $G Cyl(X, e)$ is an irrelevant cylinder object for $G A$. Therefore $G$ preserves homotopies.

**Proof:** Easy since $G$ preserves pushouts and weak equivalences. □

**Corollary 5.5.9** A good functor induces functors

\[
\text{OrdCW}^e(C)/ \simeq_{ir} \to \text{OrdCW}^e(C')/ \simeq_{ir}
\]

and

\[
\text{OrdCW}^e(C)/ \simeq \to \text{OrdCW}^e(C')/ \simeq .
\]

**Lemma 5.5.10** Let $G$ be a good functor and $f \in \text{Hom}_C$ a cii. Then $G(f)$ is cii.
Proof: Let \( f : X \to Y \) be a cii map. Then we have a pushout

\[
\begin{array}{c}
X(x^*) \longrightarrow Y(f(x)^*) \\
\downarrow \\
X(x) \longrightarrow Y(f(x))
\end{array}
\]

\( G \) of it will be a pushout also.

Next consider the pushouts

\[
\begin{array}{c}
S^{n-1} \vee e \longrightarrow X(x^*/(e \vee e) \longrightarrow e \\
\downarrow \\
D^n \vee e \longrightarrow X(x)/(e \vee e) \longrightarrow S^n \vee e
\end{array}
\]

and

\[
\begin{array}{c}
S^{n-1} \vee e \longrightarrow Y(f(x)^*/(e \vee e) \longrightarrow e \\
\downarrow \\
D^n \vee e \longrightarrow Y(f(x))/(e \vee e) \longrightarrow S^n \vee e
\end{array}
\]

Let \( f' \) denote the induced map \( f' : S^n \vee e \to S^n \vee e \). Then since \( G \) preserves pushouts of this kind we see that \( G(f') \) is the induced map on \( G \) of the two diagrams we are considering. So since \( f \) is cii \( f' \simeq id \) and since \( G \) preserves homotopies \( G(f') \simeq id \) and \( G(f) \) is cii. □

Lemma 5.5.11 Let \( G \) be a good functor then there is a unique monoidal extension of \( G \).

\( G' : M_{OrdCW^e}(C) \to M_{OrdCW^e}(C') \), such that \( FG' = G \) and \( G' \) induces the identity on \( K(A) \) as monoids. \( G' \) also has the following properties. \( G'(TS^n) = TS^n, G'(TD^n) = TD^n \) and \( G'(Ti_n) = Ti_n \). Also if we have a pushout in \( M_{OrdCW^e}(C) \)

\[
\begin{array}{c}
X \longrightarrow Y \\
\downarrow i \\
Z \longrightarrow W
\end{array}
\]
such that i is an inclusion of cell structures, then $G'(W)$ is the pushout of

$$
\begin{array}{c}
G'(X) \\
\downarrow^{G(i)} \\
G'(Z)
\end{array}
\rightarrow
\begin{array}{c}
G'(Y) \\
\downarrow \\
G'(Z)
\end{array}
$$

and $G'(i)$ is an inclusion of cell structures. Also $G'$ commutes with direct limits of inclusions of cell structures and preserves homotopies.

**Proof:** Let $A \in MOrdCW^*(C)$. We wish to define $G'(A)$. $FA \in OrdCW^*(C)$ so we define $FG' A = GF A$. We define $\phi : GA \times GA \rightarrow GA$ by

$$
GA \times GA \longrightarrow G(A \times A) \xrightarrow{G(\phi)} GA
$$

The condition on the cell structure of $GA$ is easy to verify. That $\phi$ is associative and has a right and left identity follows from the naturality of the isomorphism $G(X \times Y) \rightarrow G(X \times Y)$, the fact that $G(\alpha) = \alpha$, $G(\lambda) = \lambda$, $G(\rho) = \rho$ and $G(e) = e$ and since $A$ is a monoidal object.

For $f \in Hom_{MOrdCW^*(C)}$ we define $FG'(f) = G(f)$. $G(f)$ is monoidal since the isomorphism $G(A \times A) \rightarrow G(A) \times G(A)$ is natural and since $f$ is monoidal. The statement about direct limits follows since for monoidal objects $A_i$, $\mathcal{F} \text{dirlim} A_i = \text{dirlim} \mathcal{F} A_i$. That $G'$ preserves pushouts where one map is an inclusion of cell structures follows since $\mathcal{F}$ of an inclusion of cell structures is an inclusion of ordered cell complexes, the fact that $G$ preserves pushouts where one map is an inclusion of cell complexes, the fact that the product of two inclusions of cell complexes is an inclusion of cell complexes, the fact that $K(X) = K(G(X))$ and the description of a pushout in $C[Mon]$ by pushouts in $C$ (4.4.14). The statements about $S^n, D^n$ and $i_n$ follow from the construction of $TX$ by pushouts in $C(4.3.1)$.

**Lemma 5.5.12** Let $G$ be a good functor and $G'$ denote its monoidal extension. Let $Cyl(A, e)$ be a strongly irrelevant cylinder object. Then $G'Cyl(A, e)$ is a strongly irrelevant cylinder object for $GA$. Therefore $G'$ preserves homotopies.
Proof: From lemma 5.5.8 and since $\phi = G\phi$. □

**Corollary 5.5.13** For a good functor $G$, $G'$ induces functors

$$MOndCW^*(C)/ \simeq_{sr} \rightarrow MOndCW^*(C')/ \simeq_{sr}$$

and

$$MOndCW^*(C)/ \simeq \rightarrow MOndCW^*(C')/ \simeq .$$

### 5.6 Homology

For this section we assume that $C$ is a homology category as defined below. We wish to define a functor $H_*: C \rightarrow GM(R)$.

**Definition 5.6.1** If $C$ is a category with chains such that the canonical maps $C_*(CylS^n) \rightarrow C_*(S^n)$ and $C_*(CylD^n) \rightarrow C_*(D^n)$ are weak equivalences in $DGM(R)$ then we call $C$ a homology category.

**Definition 5.6.2** Let $H$ denote the homology functor and let $H_*$ denote the composition $HC_*.Co'$.

**Lemma 5.6.3** For $f, g \in Hom_{OndCW(C)} f \simeq g$ implies that $C_*(f) \simeq C_*(g)$.

**Proof:** Follows from 2.4.11 using the ordering filtration and since the hypothesis implies that $C_*.CylD^n$ is a $CylC_.D^n$ and that $C_*.CylS^n$ is a $CylC_.S^n$. □

**Corollary 5.6.4** $H_*$ is well defined.

**Corollary 5.6.5** For $f, g \in Hom_C f \simeq g$ implies that $H_*(f) = H_*(g)$

**Lemma 5.6.6** $H_*$ is a functor $C \rightarrow GM(R)$ that extends to a functor $Ho(C) \rightarrow GM(R)$. 
Proof: For the first part we need to show that $HC_*$, which we already know to be functorial on maps is functorial on homotopy classes of maps. This follows from Lemma 5.6.3.

For the second part we have to show that if $f$ is a weak equivalence then $H_*(f)$ is an isomorphism. This follows since if $f$ is a weak equivalence then $Cof$ is a homotopy equivalence and from Corollary 5.6.5. □

Definition 5.6.7 Define the Hurewitz map

$$h : \pi_*(X) \to H_*(X)$$

be letting $h(f) = H_*(f)(1)$ where $f \in \pi(S^n,X)$ and 1 is the identity in $H_n(S^n) = R$.

5.7 $C_*$ is Good

For this section let $C$ be a suitably monoidal homology category with compatible monoidal and cell structures.

Lemma 5.7.1 In $DGM(R)$ let $F : \pi(S^n,S^n) \to R$ be the map that sends the map $1 \to r$ to the element $r$. Then $DGM(R)$ is a homology category.

Proof: Trivial. □

Lemma 5.7.2 $C_* : C \to DGM(R)$ is a good functor.

Proof: $C_*$ preserves weak equivalences by 5.6.5. The facts that $C_*(S^n \vee e) = S^n \vee e$. $C_*(D^n \vee e) = D^n \vee e$ and $C_*(i_n) = i_n$ are clear. $C_*(\alpha) = \alpha$ follows since $\alpha : (S^{n_1} \times S^{n_2}) \times S^{n_3} \to S^{n_1} \times (S^{n_2} \times S^{n_3})$ induces a map homotopic to the identity on $S^{n_1+n_2+n_3}$. $C_*(\lambda) = \lambda$ and $C_*(\rho) = \rho$ follow from the way we give products cell structure. It is clear that $C_*$ commutes with direct limits. Lemmas 5.2.15, 5.3.14 and 5.3.15 are enough to complete the proof. □
Corollary 5.7.3 If

\[
\begin{array}{ccc}
F a_1 & \xrightarrow{f} & F a_3 \\
\downarrow{i} & & \downarrow{} \\
F a_2 & \xrightarrow{} & D
\end{array}
\]

is a pushout in $\text{MO}rd\text{CW}^*(C)$ such that $i$ is an inclusion of cell structures then

\[
\begin{array}{ccc}
C_*(F a_1) & \rightarrow & C_*(F a_3) \\
\downarrow & & \downarrow \\
C_*(F a_3) & \rightarrow & C_*(D)
\end{array}
\]

is a pushout in $DGA$

Proof: 5.5.11. □

Corollary 5.7.4 Let $X$ be $\text{MO}rd\text{CW}^*(C)$ and $f \in \text{Hom}_{\text{MO}rd\text{CW}^*(C)}$. Let $Y$ denote the pushout of the following diagram

\[
\begin{array}{ccc}
T S^n & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
T D^{n+1} & \rightarrow & Y
\end{array}
\]

then $C_*(Y) = (C_*(X) \amalg a^{n+1}da^{n+1} = C_*(f(i))).$

5.8 When Homotopy is Homology

For this section we assume that the homology functor and the homotopy functors are isomorphic. More precisely, for this section $C$ is a suitably monoidal homology category with compatible monoidal and cell structures such that the Hurewitz map $h : \pi_* X \to H_* X$ is an isomorphism for every $X \in C$. Recall the definition of $ht$ (3.5.4) and $X(r)$ be the subcomplex at the $r$th stage in the decomposition that realizes its height.

Lemma 5.8.1 Two cellular maps that induce the same map on $C_*$ are irrelevently homotopic.
Proof: Assume the statement is true for all \( X \) such that \( ht(X) < n \). Now let \( ht(X) = n \). Let \( a \) denote a cell in \( X(n) \). Let \( f, g : X(n - 1) \cup a \to Y \) be a cellular map such that \( f|_{X(n-1)} \simeq_{ir} g|_{X(n-1)} \). Denote the map \( Y_{|a|} \to Y_{|a|/|a|-1} \) by \( p \). Now \( p \circ (f - g) \in [S_{|a|}, Y_{|a|/|a|-1}] \) is independent of the irrelevant homotopy. Also \( C_\ast(f - g)(a) = 0 \) by assumption and lemma 5.2.11. Therefore \( p \circ (f - g) = * \). Since \( H_\ast(p) \) is injective, \( \pi_\ast(p) \) is injective and \( f - g = * \) and by Lemma 2.4.13 we are done. \( \square \)

**Lemma 5.8.2** \( f \) induces a \( C_\ast \) isomorphism implies that \( f \) is an irrelevant homotopy equivalence.

**Proof:** With the assumption that Hurewitz is an isomorphism the result follows easily from Lemma 5.1.11 using an induction argument. \( \square \)

Let \( \text{FreeDGM}(R) \) denote the full subcategory of \( \text{DGM}(R) \) whose objects are free \( R \) modules.

**Lemma 5.8.3** \( \text{OrdCW}(R) / \simeq_{ir} \) is equivalent to \( \text{FreeDGM}(R) \).

**Proof:** Easy. \( \square \)

Let \( \text{FreeDGA}(R) \) denote the full subcategory of \( \text{DGM}(R)[\text{Mon}] \) whose objects are free \( R \) algebras, in other words tensor algebras over \( R \).

**Lemma 5.8.4** \( \text{MOrdCW}(R) / \simeq_{ir} \) is equivalent to \( \text{FreeDGA}(R) \).

**Proof:** Easy. \( \square \)

**Lemma 5.8.5** There exists a functor \( X : \text{FreeDGM} \to \text{OrdCW} / \simeq_{ir} \) such that for every \( A \in \text{FreeDGM} \) and \( f \in \text{Hom}_{\text{FreeDGM}}, C_\ast(X(A)) = A \) and \( C_\ast(X(f)) = f \).

**Proof:** Assume that we have defined \( X \) on all \( A \in \text{FreeDGM} \) such that \( ht(A) \leq n \). Now let \( ht(A) = n + 1 \). We have a pushout

\[
\begin{array}{ccc}
\bigoplus_{a \in M(A)(n+1)} R & \xrightarrow{\Sigma d(a)} & A(n) \\
\downarrow & & \downarrow \\
\bigoplus_{a \in M(A)(n+1)} R & \xrightarrow{x_{|a|-1}} & A.
\end{array}
\]
For every a, d(a) ∈ H∗(A(n)|a|−1). Therefore d(a) determines f(d(a)) ∈ π(X(A(n))|a|−1 and therefore f(d(a)) ∈ π∗_\text{ir}(A(n)) so by Lemmas 5.1.11 and 5.8.1 we have uniquely determined X(A(n + 1)) up to irrelevant homotopy.

Now we deal with maps. Let f : A → B ∈ Free\text{DGM} be a map. Assume that if ht(A) ≤ n then there exists a cellular map unique up to irrelevant homotopy X(f)(n) : X(A) → X(B) such that C∗(X(f)) = f|A. Now assume that ht(A) = n + 1 and let a ∈ M(A)(n + 1) be any element. We know that the map

\[ X(f)(n) \circ X(da) : S^{|a|-1} \to X(B) \]

is irrelevantly inessential since

\[ C∗(X(f)(n) \circ X(da)) = f|_{A(n)} \circ da = 0. \]

Therefore by lemma 2.4.12 there exists an extension F : X(A)(n) ∪ a → X(B). There exists a cellular map α : S^n → X(B) unique up to irrelevant homotopy such that C∗(α) = f(a) − C∗(F)(a). Define X(f)(n + 1)|_{X(A)(n)∪a} = F + α by the coaction of Section 2.4. Then C∗(X(f)(n + 1)) = f and by Lemma 5.8.1 any other map with this property is irrelevantly homotopic to f. Therefore we have proved the induction step and the lemma. □

**Corollary 5.8.6** Ord\text{CW}(C) / ∼_ir is equivalent to Ord\text{CW}(\text{DGM}(R)) / ∼_ir.

**Corollary 5.8.7** Two strongly cellular maps that induce the same map on C∗ are strongly irrelevantly homotopic.

**Corollary 5.8.8** There exists a functor X : Free\text{DGA} → MOrd\text{CW}^e(C) / ∼_sir such that for every A ∈ Free\text{DGA} and f ∈ Hom_{Free\text{DGA}}, C∗(X(A)) = A and C∗(X(f)) = f.

**Corollary 5.8.9** Mord\text{CW}^e(C) / ∼_sir is equivalent to Free\text{DGA}(R).

**Theorem 5.8.10** Ho(\text{DGM}(R)) is equivalent to Ho(C) and Ho(\text{DGM}(R)[\text{Mon}]) is equivalent to Ho(C[\text{Mon}]).
Proof: We show equivalences between $OrdCW(DGM(R))/\simeq$ and $OrdCW(C)/\simeq$ and then use 5.5.6. To see this equivalence observe that the equivalence of Corollary 5.8.6 extends since both of the functors $X$ and $C_*$ take cylinder objects to cylinder objects. The proof in the monoidal case is the same. □

Observe that these equivalences do not come from adjoint Quillen functors. However the functors are nice enough that the extra structure of the model categories should be preserved and explicit enough that proofs should be possible.
Chapter 6

Stable Homotopy Theory

In this chapter we look at stable homotopy theory. We give the construction of the category of spectra of [13] and prove a few of its properties. We show it is suitably monoidal. We give a construction of Eilenberg-Mac Lane spectra $KR$ for any ring $R$. Finally we show that the category of algebras over $KR$ is the same as the category of differential graded algebras over $R$. In Section 1 we define the category of coordinate free spectra of [18] and give some of its basic properties. In Section 2 we introduce and give some facts about the half smash of a space with a spectrum, the category of linear isometries and the operad $L = \mathcal{L}(1)$. Section 3 deals with the properties of L-algebras and introduces a smash product on this category. It is observed that the homotopy category of L-algebras is equivalent to the stable category and that the equivalences take smash product to smash products. In section 4 we finally introduce S-modules. We show S-modules form a monoidal category under the smash product induced from L-algebras. It is also noted that the inclusion functor induces an equivalence of homotopy categories. Section 5 defines S-algebras and modules over S-algebras. $S = \Sigma^\infty S^0$ is an S-algebra and S-modules are just the modules over the S-algebra S. For any S-algebra R we introduce a smash product on the category of $(R,R)$ bimodules making the category of such into a monoidal category. In Section 6 we give the category of modules over any S-algebra the structure of a suitably monoidal suitable cell category that
also makes it spherically generated with compatible cell and monoidal structures. Section 7
deals with the Eilenberg-Mac Lane spectrum $KR$. In it we show that modules over them
form a homology category such that $\pi_* = H_*$. This allows us to use the machinery of Chapter
5 to get an equivalence of categories between algebras over $KR$ and $DGA(R)$.

6.1 Spectra

There are many categories of spectra whose homotopy category is equivalent to the stable
category. An early example is that Boardman, presented by Adams ([1]). Our starting
category of spectra will be the one of [18]. We will first define prespectra and then define
spectra as a full subcategory.

Definition 6.1.1 Let $R^\infty$ be given the structure of a real inner product space. It has count-
able dimension. A universe, $U$, is any isomorphic copy of $R^\infty$. Let $A(U)$ be the set of all
finite dimensional subspaces of $U$.

For $V, W \in A(U)$ with $V \subset W$ let $W - V$ denote the orthogonal compliment of $V$ in $W$
and $(W - V) \cup \{\infty\}$ its one point compactification with $\infty$ being the base point. Then for a
pointed space $X$ we let

$$\Omega^{W-V}(X) = \text{Map}_*(W - V \cup \{\infty\}, X)$$

and

$$\Sigma^{W-V}(X) = (W - V \cup \{\infty\}) \wedge X$$

Definition 6.1.2 Let $U$ be a universe. Give the set $A(U)$ the structure of a discrete category.
Then a prespectrum $(D, \sigma)$ is a functor $D : A(U) \to \text{Top}_*$ together with based maps for
every $V \subset W$

$$\sigma : \Sigma^{W-V}D(V) \to D(W)$$

such that the following two conditions hold:
1) \( \sigma : D(V) = \Sigma^0 D(V) \to D(V) \) is the identity map;

2) For \( V, W, X \in A(U) \) with \( V \subset W \subset X \), the following diagram commutes

\[
\begin{array}{ccc}
\Sigma^{X-W} \Sigma^{W-V} D(V) & \xrightarrow{\Sigma^{X-W} \sigma} & \Sigma^{X-W} D(W) \\
\downarrow & & \downarrow \\
\Sigma^{X-V} D(V) & \xrightarrow{\sigma} & D(X).
\end{array}
\]

Now let \((D, \sigma)\) and \((D', \sigma')\) be prespectra. Then a map \( f : D \to D' \) of prespectra is a natural transformation \( \{ f(V) : D(V) \to D'(V) \}_{V \in A(U)} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\Sigma^{W-V} D(V) & \xrightarrow{\Sigma^{W-V} f(V)} & \Sigma^{W-V} D'(V) \\
\downarrow & & \downarrow \\
D(W) & \xrightarrow{f(W)} & D'(W)
\end{array}
\]

We let \( PU \) denote the category of prespectra and maps of prespectra. A spectrum is a prespectrum such that the adjoints of the maps \( \sigma \) are homeomorphisms. The category of spectra is the full subcategory of \( PU \) and is denoted \( SU \).

We now give a few basic properties of spectra.

**Proposition 6.1.3 [18]** There exist functors

\[
L : PU \to SU
\]

and

\[
l : SU \to PU
\]

with the following two properties.

1) \( l \) is the forgetful functor and \( L \) is its right adjoint.

2) For every \( E \in SU \) the counit \( LlE \to E \) is an equivalence.

Many constructions made at the space level preserve prespectra but not spectra. To deal with this problem we forget we are working with spectra, do the construction, and then use
to get back to the category of spectra. The next definition is an example of this.

**Definition 6.1.4** Let \( D, D' \in PU, f : D \to D' \) a map and \( X \) a space. Then we define the cone on \( D \), \( CD \), the cofiber of \( f \), \( Cf \), and \( D \wedge X \) by the following formulae

\[
CD(V) = C(D(V))
\]

\[
Cf(V) = C(f(V))
\]

and

\[
(D \wedge X)(V) = D(V) \wedge X
\]

The connecting homomorphisms are the obvious maps. Now for \( E, E' \in SU, f : E \to E' \) a map, we define

\[
CE = L(C\{E\})
\]

\[
Cf = L(C\{f\})
\]

and

\[
E \wedge X = L(lE \wedge X)
\]

**Definition 6.1.5** Let \( E, E' \in SU \) and \( f, g : E \to E' \) be maps. Let \( i_0 : 0^+ \to I^+ \) and \( i_1 : 1^+ \to I^+ \) be the inclusions. Then we say that \( f \) is **homotopic** to \( g \), written \( f \simeq g \), if there exists a homotopy

\[
H : E \wedge I^+ \to E'
\]

such that \( H \circ (id \wedge i_0) = f \) and \( H \circ (id \wedge i_1) = g \)

**Definition 6.1.6** Let \( X \) be a space. Then we can define the prespectrum \( P\Sigma^\infty X \) by

\[
(P\Sigma^\infty(X))(V) = \Sigma^V X
\]
and $\Sigma^\infty X \in SU$ by

$$\Sigma^\infty X = L(P \Sigma^\infty X)$$

For convenience we will denote $\Sigma^\infty S^n$ by $S^n$ and $\Sigma^\infty S^0$ by $S$.

**Proposition 6.1.7 [18]** $SU$ is closed under all limits and colimits over any diagram.

**Definition 6.1.8** Let $U, U'$ be universes, $E \in SU, E' \in SU'$ and $f : U \to U'$ a linear isometry. Then we define $f^*(E')$ by

$$(f^*(E'))(V) = E'(f(V)), V \in A(U)$$

If $f$ is also an isomorphism we can define $f_*(E)$ by

$$(f_*(E))(V') = E(f^{-1}(V')), V' \in A(U').$$

**Definition 6.1.9** Again let $U$ and $U'$ be universes. Let $E \in PU, F \in PU'$. Then the external smash product $E \wedge F \in P(U \oplus U')$ is defined by, for $W \in A(U \oplus U')$

$$(E \wedge F)(W) = \Sigma^{W -(V_1 \vee V_2)} E(V_1) \wedge E(V_2)$$

for

$$W \in A(U \oplus U'),$$

$$V_1 = W \cap (U \oplus 0)$$

and

$$V_2 = W \cap (0 \oplus U).$$

Now for $E \in SU, F \in SU'$ we define $E \wedge F \in S(U \oplus U')$ by $E \wedge F = L(lE \wedge lF)$. Now let $f : U^2 \to U$ be an isomorphism. Then for $E, F \in SU$ the internal smash via $f$ is defined as $E \wedge_f F = f_*(E \wedge F) \in SU$
6.2 Half Smash, Linear Isometries, and L algebras

The smash product in the category of spectra defined above is neither associative nor commutative. It depends on some arbitrary map \( f : U^2 \to U \). We need some way to not give preference to any particular map. It turns out that there is a way to do this if we give our category of spectra extra structure. This allows us to take all the maps \( U^2 \to U \) and get a smash product that is both associative and commutative. As preliminaries we need to introduce the half smash product and linear isometries and state some of their properties.

**Definition 6.2.1** Let \( U, U' \) be universes. Then we define the space of linear isometries from \( U \) to \( U' \), denoted \( I(U, U') \), to be the subspace of \( \text{Map}(U, U') \) of maps that are linear isometries.

**Definition 6.2.2** For a fixed universe \( U \) we define \( L(n) = I(U^n, U') \).

**Lemma 6.2.3** [19] Given a diagram in any category

\[
\begin{array}{ccc}
A & \xrightarrow{d_0} & B \\
\downarrow{d_1} & & \downarrow{\epsilon} \\
C & &
\end{array}
\]

such that \( \epsilon d_0 = \epsilon d_1 \), assume there are maps \( h_\sim : C \to B \) and \( h_0 : B \to A \) such that \( \epsilon h_\sim = id_C, d_1 h_0 = id_B \) and \( d_0 h_0 = h_\sim \epsilon \). Then \( \epsilon \) is the coequalizer of \( d_0 \) and \( d_1 \).

**Definition 6.2.4** A coequalizer of the form in the above lemma is called a split coequalizer.

The next lemma is important for proving properties of the smash product and will be used later in the section.

**Lemma 6.2.5** Let \( n \geq 1, i_j \geq 1 \) be integers. Then the diagram

\[
L(n) \times (L(1))^n \times L(i_1) \times \ldots \times L(i_n) \longrightarrow L(n) \times L(i_1) \times \ldots \times L(i_n) \longrightarrow L(\Sigma i_j)
\]
is a split coequalizer with the map \( \gamma \) given by the formula

\[
\gamma(f, g_1, \ldots, g_n) = f \circ (g_1 \oplus \ldots \oplus g_n)
\]

for \( f \in \mathcal{L}(n) \) and \( g_j \in \mathcal{L}(i_j) \).

**Proof:** Select isomorphisms \( s_j : U_i^j \to U \). Then define

\[
h_{-1}(f) = (f \circ (s_1 \oplus \ldots \oplus s_n)^{-1}, s_1, \ldots, s_n)
\]

and

\[
h_0(f, g_1, \ldots, g_n) = (f, g_1 \circ s_1^{-1}, \ldots, g_n \circ s_n^{-1}, s_1, \ldots, s_n)
\]

It is then easy to verify the conditions of the lemma above.

Now in our slow plodding towards the half smash we need to introduce the following Thom complex construction.

**Definition 6.2.6** Let \( U, U' \) be universes, \( X \) a pointed space, \( \chi : X \to I(U, U') \) a map, and \( V \in A(U), V' \in A(U') \) be such that \( \chi(X)(V) \subset V' \). Then we get the following map of vector bundles over \( X \)

\[
\begin{array}{ccc}
X \times V & \longrightarrow & X \times V' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X
\end{array}
\]

Let \( \zeta(V, V') \) be the orthogonal complement of \( X \times V \) in \( X \times V' \). Now let \( T(X, V, V') \), or \( T(V, V') \) if \( X \) is clear, denote the Thom complex of \( \zeta(V, V') \).

The next concept is also used in the construction of the half smash.

**Definition 6.2.7** Let \( U, U' \) be universes. A **connection** \((\mu, \nu)\) is a pair of functions \( \mu : A(U) \to A(U') \) and \( \nu : A(U') \to A(U) \) such that for every \( V \in A(U), V' \in A(U'), \mu(V) \subset V' \) if and only if \( V \subset \nu(V') \).
Definition 6.2.8 Let $X$ be a compact space and $U, U'$ be universes. Let $\chi : X \to I(U, U')$ be a map. A connection $(\mu, \nu)$ is called a $\chi$ connection if for every $V \in A(U)$

$$\chi(x)(V) \subset \mu(V)$$

for every $x \in X$.

The following example shows that for every $\chi : X \to I(U, U')$ there exists a $\chi$ connection.

Example 6.2.9 Let $X$ be compact, $U, U'$ universes and $\chi : X \to I(U, U')$ a map. Then we can define for $V \in A(U), V' \in A(U')$

$$\mu(V) = \sum_{x \in X} \chi(x)(V)$$

and

$$\nu(V') = \bigcap_{x \in X} \chi(x)^{-1}(V')$$

.

Lemma 6.2.10 $(\mu, \nu)$ as defined above is a $\chi$ connection.

Proof: For $V \in A(U)$ we need to show that

$$\sum_{x \in X} \chi(x)(V)$$

is finite dimensional. So let $V(1)$ and $U'(1)$ denote the unit balls in $V$ and $U'$ respectively. So by restriction we get the map

$$X \times V(1) \to U'(1)$$

and the span of the image is finite dimensional if the image is compact. Therefore we get that $\mu(V)$ is finite dimensional. The rest is trivial. \(\square\)
We are now in a position to define the half smash with compact indexing space.

**Definition 6.2.11** Let \( U, U' \) be universes, \( X \) a compact space, \( \chi : X \to I(U, U') \) a map, \((\mu, \nu)\) any \( \chi \) connection and \((D, \sigma)\) \(\in PU\). Then we define \( X \ltimes D \in PU' \) by the following formula

\[
X \ltimes D(V') = T(\nu(V'), V') \land D(\nu(V')), V' \in A(U')
\]

and for \( V', W' \in A(U') \) with \( V' \subset W' \) we define the structure maps as the composition of the maps in the following diagram.

\[
\begin{align*}
\Sigma^{W' - V'} T(\nu(V'), V') \land D(\nu(V')) & \\
\cong & \\
\Sigma^{\nu(W') - \nu(V')} T(\nu(W'), W') \land D(\nu(V')) & \\
T(\nu(W'), W') \land D(\nu(W')) &
\end{align*}
\]

The next, compatibility lemma allows us the patch the half smashes with compact spaces together and lets us use any indexing space.

**Lemma 6.2.12** [18] Let \( U, U', X, \chi, D \) be as above then up to natural canonical isomorphism the spectrum \( X \ltimes D \) is independent of \( \chi \) connection.

**Definition 6.2.13** Now let \( U, U' \) be universes, \( X \) any space, \( \chi : X \to I(U, U') \) a map, and \( E \in SU \). Let \( K \) be the set of all compact subspaces of \( X \). Then define

\[
X \ltimes E = \text{colim}_{Y \in K} L(Y \ltimes l(E))
\]

A few relevant properties of \( \ltimes \) follow. Most follow easily from the definitions. For proofs of those stated and left unproved and for a more complete list of properties we refer the reader to [18].

**Proposition 6.2.14** [18] \( \ltimes \) is natural in both variables and commutes with colimits in both variables.
Lemma 6.2.15 [18] Let $A, B$ be spaces, $\chi_1 : A \to I(U_1, U'_1)$, $\chi_2 : B \to I(U_2, U'_2)$ be maps, $E \in SU_1$ and $F \in SU_2$. Then there exist natural canonical isomorphisms

1) $(A \times E) \wedge (B \times F) \to (A \times B) \times (E \wedge F)$

2) And if $U'_1 = U_2$ then

$$B \times (A \times E) \to (B \times A) \times E$$

We now go through the proof of 2) in detail.

Proof: Using the notation in the statement of the lemma and assuming that $A$ and $B$ are compact and for fixed $\chi_1, \chi_2$ connections $(\mu_1, \nu_1), (\mu_2, \nu_2)$. The map $\chi : B \times A \to I(U_1, U'_2)$ is given by $(b, a) \to \chi_2(b) \circ \chi_1(a)$ and we can take $(\mu_1 \circ \mu_2, \nu_2 \circ \nu_1)$ as our $\chi$ connection. Then we have

$$(B \times (A \times lE))(V') = T(\nu_1 \circ \nu_2(V'), \nu_1(V')) \wedge T(\nu_1(V'), V') \wedge lE(\nu_2 \circ \nu_1(V'))$$

$\to T(\nu_2 \circ \nu_1(V'), V') \wedge lE(\nu_2 \circ \nu_1(V'))$

$$= (B \times A) \times lE(V').$$

The equalities are just the definition and the map is an isomorphism because the two Thom spaces are naturally and canonically homeomorphic. Now let $A$ and $B$ be arbitrary spaces. Then since for every compact $K \subset A \times B$ there exists compact $K_1 \subset A$, $K_2 \subset B$ such that $K \subset K_1 \times K_2$ (and of course vice versa) the result follows. \(\Box\)

We will use the following notation for coequalizers.

Definition 6.2.16 Let $C$ be a category, $A, B, C \in C$, and $\phi_0 : B \times C \to C$ and $\phi_1 : A \times B \to A$ be maps. Then we let $A \times_B C$ denote the coequalizer of the following diagram.

$$A \times B \times C \xrightarrow{id \times \phi_0} A \times C.$$

We have now set up enough preliminaries to define the category of spectra of [13] and the smash product on it.
Definition 6.2.17 Let $L : SU \to SU$ be the functor defined by

$$E \mapsto \mathcal{L}(1) \times E.$$  

Define natural transformations 

$$\eta : I_{SU} \to L$$

by 

$$E \xrightarrow{\cong} \ast \times E \xrightarrow{f \circ \text{id}} \mathcal{L}(1) \times E$$

where 

$$f : \ast \mapsto (\text{id} : U \to U)$$

and 

$$\mu : L^2 \to L$$

by 

$$\mathcal{L}(\mathcal{L}(1) \times E) \xrightarrow{\cong} (\mathcal{L} \times \mathcal{L}) \times E \xrightarrow{\circ \circ \text{id}} \mathcal{L} \times E.$$  

Lemma 6.2.18 $L = \langle L, \eta, \mu \rangle$ is a monad. $\square$

Definition 6.2.19 Fix a universe $U$, let $X \in SU$ and let $\mathcal{L}(1) \times X \to X$ be a map such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{L}(1) \times (\mathcal{L}(1) \times X) & \longrightarrow & \mathcal{L}(1) \times X \\
\downarrow\cong & & \downarrow \\
(\mathcal{L}(1) \times \mathcal{L}(1)) \times X & \longrightarrow & \mathcal{L}(1) \times X \\
\downarrow & & \downarrow \\
\mathcal{L}(1) \times X & \longrightarrow & X
\end{array}
$$
Then $X$ is called a **$L$-module**. Now if $X$ and $Y$ are $L$-modules a map of $L$-modules is a map $f : X \to Y$ of spectra such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{L}(1) \times X & \overset{id \times f}{\longrightarrow} & \mathcal{L}(1) \times Y \\
\downarrow & & \downarrow \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
\]

Following monadic notation the category described above is referred to as $SU[L]$.

**Lemma 6.2.20** $SU[L]$ is cocomplete with colimits being created in $SU$.

**Proof:** Follows since $\text{colim}(\mathcal{L}(1) \times X_\alpha) = \mathcal{L}(1) \times \text{colim}(X_\alpha)$. □

### 6.3 Smash products

**Definition 6.3.1** Let $M_i \in SU[L]$ and let $\mathcal{L}(1) \times \mathcal{L}(1)$ act on the right on $M_1 \land M_2$ and on the left on $\mathcal{L}(2)$ in the obvious ways. Then we define $M_1 \land_L M_2$ to be the following coequalizer

\[
\mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \times M_1 \land M_2 \rightrightarrows \mathcal{L}(2) \times M_1 \land M_2 \longrightarrow M_1 \land_L M_2
\]

In general define

\[
M_1 \land_L \ldots \land_L M_n = \mathcal{L}(n) \times (\mathcal{L}(1))^n (M_1 \land \ldots \land M_n).
\]

The next lemma is used in the proof of the associativity of $\land_L$.

**Lemma 6.3.2** Let $M_i \in SU[L]$. Then there is a canonical natural isomorphism of spectra with $\mathcal{L}(1) \times \mathcal{L}(1)$ actions

\[
\mathcal{L}(i) \times (\mathcal{L}(1))^j M_1 \land \ldots \land M_i \land \mathcal{L}(j) \times (\mathcal{L}(1))^j M_{i+1} \land \ldots \land M_{i+j} \rightarrow \mathcal{L}(i) \times \mathcal{L}(j) \times (\mathcal{L}(1)^{i+j} M_1 \land \ldots \land M_{i+j}
\]

**Proof:** Let $A_i, B_i$ be spectra, $f_i, g_i : A_i \to B_i$ maps. Now let

\[
\begin{array}{ccc}
A_i & \overset{f_i}{\longrightarrow} & B_i \\
\downarrow{g_i} & & \downarrow{C_i}
\end{array}
\]
be coequalizers. It is a fact that
\[ \text{colim}(\alpha \wedge A) = \text{colim}(\alpha) \wedge A. \]

So from properties of colimits we get that
\[
\begin{array}{ccc}
A_1 \wedge A_2 & \xrightarrow{f_1 \wedge f_2} & B_1 \wedge B_2 \\
\downarrow{g_1 \wedge g_2} & & \downarrow \\
C_1 \wedge C_2
\end{array}
\]
is a coequalizer. The conclusions of the Lemma follow. \( \square \)

**Lemma 6.3.3** Let \( L, M, N, P \in SU[L] \). Then there exist natural isomorphisms in \( SU[L] \).

\[
\begin{align*}
\phi_1 : M \wedge_L (N \wedge_L P) & \to M \wedge_L N \wedge_L P \\
\phi_2 : (M \wedge_L N) \wedge_L P & \to M \wedge_L N \wedge_L P \\
\phi_3 : L \wedge_L (M \wedge_L N \wedge_L P) & \to L \wedge_L M \wedge_L N \wedge_L P \\
\phi_4 : (L \wedge_L M \wedge_L N) \wedge_L P & \to L \wedge_L M \wedge_L N \wedge_L P \\
\phi_5 : (L \wedge_L M) \wedge_L N \wedge_L P & \to L \wedge_L M \wedge_L N \wedge_L P \\
\phi_6 : L \wedge_L M \wedge_L (N \wedge_L P) & \to L \wedge_L M \wedge_L N \wedge_L P \\
\phi_7 : L \wedge_L (M \wedge_L N) \wedge_L P & \to L \wedge_L M \wedge_L N \wedge_L P
\end{align*}
\]
such that the following five diagrams commute.

\[
\begin{array}{ccc}
L \wedge_L (M \wedge_L (N \wedge_L P)) & \xrightarrow{\phi_1} & L \wedge_L M \wedge_L (N \wedge_L P) \\
\downarrow{\text{id} \wedge_L \phi_1} & & \downarrow{\phi_6} \\
L \wedge_L (M \wedge_L N \wedge_L P) & \xrightarrow{\phi_2} & L \wedge_L M \wedge_L N \wedge_L P
\end{array}
\]
Proof: As an example we construct $\phi_3$; the construction of the other six maps is similar.

\[
(L \wedge_L M) \wedge_L (N \wedge_L P) \xrightarrow{\phi_1} (L \wedge_L M) \wedge_L N \wedge_L P
\]
\[
\downarrow \phi_2 \quad \downarrow \phi_3
\]
\[
L \wedge_L M \wedge_L (N \wedge_L P) \xrightarrow{\phi_6} L \wedge_L M \wedge_L N \wedge_L P
\]
\[
\downarrow \phi_1 \quad \downarrow \phi_3
\]
\[
L \wedge_L ((M \wedge_L N) \wedge_L P)^{id \wedge_L \phi_2} \xrightarrow{\phi_7} L \wedge_L (M \wedge_L N) \wedge_L P
\]
\[
\downarrow \phi_7 \quad \downarrow \phi_3
\]
\[
L \wedge_L (M \wedge_L N) \wedge_L P \xrightarrow{\phi_4} L \wedge_L M \wedge_L N \wedge_L P
\]
\[
\downarrow \phi_4 \quad \downarrow \phi_3
\]
\[
((L \wedge_L M) \wedge_L N) \wedge_L P \xrightarrow{\phi_5} (L \wedge_L M) \wedge_L N \wedge_L P
\]
\[
\downarrow \phi_5 \quad \downarrow \phi_3
\]
\[
(L \wedge_L M \wedge_L N) \wedge_L P \xrightarrow{\phi_6} L \wedge_L M \wedge_L N \wedge_L P
\]

All the maps are natural canonical isomorphisms from the above lemmas.

Now we will prove that the first diagram in the lemma commutes. The proofs that the others commute are similar. We make use of the following temporary notation.

\[
A = \mathcal{L}(2) \times_{\mathcal{L}(1)^3} L \wedge \mathcal{L}(2) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(2) \times_{\mathcal{L}(1)^3} M \wedge N \wedge P
\]
Then the fact that the following four diagrams commute demonstrates the commutativity of the first diagram in the lemma.

\[
B = \mathcal{L}(2) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(2) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(2) \times_{\mathcal{L}(1)^3} L \land M \land N \land P
\]

\[
C = \mathcal{L}(2) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(2) \times_{\mathcal{L}(1)^3} L \land M \land \mathcal{L}(2) \times_{\mathcal{L}(1)^3} N \land P
\]

\[
D = \mathcal{L}(3) \times_{\mathcal{L}(1)^3} L \land M \land \mathcal{L}(2) \times_{\mathcal{L}(1)^3} N \land P
\]

\[
E = \mathcal{L}(2) \times_{\mathcal{L}(1)^3} L \land \mathcal{L}(3) \times_{\mathcal{L}(1)^3} M \land N \land P
\]

\[
F = \mathcal{L}(2) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(3) \times_{\mathcal{L}(1)^3} L \land M \land N \land P
\]

\[
G = \mathcal{L}(3) \times_{\mathcal{L}(1)^3} \mathcal{L}(1) \times \mathcal{L}(2) \times_{\mathcal{L}(1)^3} L \land M \land N \land P
\]

\[\square\]

**Theorem 6.3.4** [13] $\land_L$ is both associative and commutative.

We will only prove associativity.

**Proof of associativity:** The theorem is a corollary of the last lemma. $\square$
Lemma 6.3.5 Let $X$ be a space then

\[ \Sigma^\infty X = \mathcal{L}(0) \wedge X \]

Note that therefore $\Sigma^\infty X$ is an $L$-algebra by the obvious action of $\mathcal{L}(1)$ on $\mathcal{L}(0)$.

Proof: Trivial \( \square \)

Proposition 6.3.6 Let $E \in SU[L]$ then there exist natural maps

\[ r : S \wedge_L E \to E \]

and

\[ l : E \wedge_L S \to E \]

Proof:

\[ S \wedge_S M = C(2) \times_{C(1)} C(0) \times S^0 \wedge C(1) \times_{C(1)} M \]

\[ \cong C(2) \times_{C(1)} C(0) \times C(1) \times_{C(1)} S^0 \wedge M \]

\[ \to C(1) \times_{C(1)} M = M \]

The proof for $l$ is the same.

6.4 S-modules

We are now ready to introduce what we will use for our stable monoidal category.

Definition 6.4.1 An $L$ algebra is called an S-module if $\lambda$ is an isomorphism. $\mathcal{M}_S$ is the full subcategory of $T[L]$ consisting of $S$-modules.

Lemma 6.4.2 $C(2)/(C(1) \times C(1))$ is a point.
Proof: Let $f_1, f_2 \in \mathcal{L}(2)$

$$f_i : U_1 \oplus U_2 \to U$$

Observe that any isometric isometric isomorphism $W \to V$ with $V \subset W \subset U$ extends trivially to a linear isometry.

Also observe that the $C(1) \times C(1)$ equivalence class of the $f_i$ is determined by the two orthogonal subspaces $f_i(U_j) = W_{ij} \in U$ for $j = 1, 2$. Now there exist a set of four mutually orthogonal infinite dimensional inner product spaces

$$W'_{ij} \subset W_{ij}.$$ 

So it is easy to see that

$$(W_{i1}, W_{i2}) \sim (W'_{i1}, W'_{i2}) \sim (W'_{12} \oplus W'_{21}, W'_{12} \oplus W'_{22}).$$

□

Lemma 6.4.3 $r : S \wedge L S \to S$ is an isomorphism.

Proof:

$$\mathcal{L}(2) \cong \mathcal{L}(1) \times \mathcal{L}(1) \mathcal{L}(0) \times \mathcal{L}(0) \mathcal{S}^0$$

$$\cong \mathcal{L}(2) \times \mathcal{L}(0) \mathcal{S}^0$$

where the last isomorphism follows from the previous lemma. □

Corollary 6.4.4 $\lambda_r : \Sigma^\infty X \wedge_S \Sigma^\infty Y \to \Sigma^\infty(X \wedge Y)$ is a natural isomorphism

Lemma 6.4.5 Let $M, N \in \mathcal{M}_S$. Then

$$id \wedge_S l = r \wedge_S id : M \wedge_S S \wedge_S N \to M \wedge_S N$$
Proof:

\[ M \wedge_S S \wedge_S N \cong C(3) \times_{C(1) \times C(1) \times C(1)} C(1) \times C(0) \times C(1) \times_{C(1) \times C(1)} M \wedge N \]

\[ M \wedge_S N \cong C(2) \times_{C(1) \times C(1)} M \wedge N \]

It is sufficient to prove that \( r \wedge_S \text{id} \) is induced by the map

\[ C(3) \times_{C(1) \times C(1) \times C(1)} C(1) \times C(0) \times C(1) \rightarrow C(2) \]

defined by

\[ (f, g, h) \mapsto f \circ (g \oplus i \oplus h) \]

where \( i : \{0\} \rightarrow U \) is the inclusion, \( f \in C(3) \) and \( g, h \in C(1) \).

To prove this we note that

\[ (M \wedge_S S) \wedge_S N = C(2) \times_{C(1) \times C(1)} (C(2) \times_{C(1) \times C(1)} M \wedge C(0) \times S^0) \wedge N \]

\[ \cong C(2) \times_{C(1) \times C(1)} (C(1) \times_{C(1) \times C(1)} M \wedge C(0) \times S^0) \wedge C(1) \times_{C(1) \times C(1)} C(1) \times_{C(1) \times C(1)} N \]

\[ \cong C(2) \times_{C(1) \times C(1)} C(2) \times C(1) \times_{C(1) \times C(1) \times C(1)} C(1) \times C(0) \times C(1) \times_{C(1) \times C(1) \times C(1)} M \wedge S^0 \wedge N \]

and the map

\[ C(2) \times_{C(1) \times C(1)} C(2) \times C(1) \times_{C(1) \times C(1) \times C(1)} C(1) \times C(0) \times C(1) \times_{C(1) \times C(1) \times C(1)} M \wedge S^0 \wedge N \rightarrow C(2) \times_{C(1) \times C(1)} M \wedge N \]

is induced by the map

\[ (f_1, f_2, f_3, g, h) \mapsto f_1 \circ (f_2 \oplus f_3) \circ g \oplus i \oplus h \]
where \( f_1, f_2 \in C(2) \) and \( f_3, g, h \in C(1) \).

But now we remember that the equivalence

\[
C(3) \to C(2) \times C(1) \times C(1) \times C(2) \times C(1)
\]

is induced by the splitting

\[
C(3) \to C(2) \times C(2) \times C(1)
\]

given by

\[
f \mapsto (f \circ (s \oplus t)^{-1}, s, t)
\]

for some fixed isomorphisms \( s : U^2 \to U, t : U \to U \) and for any \( f \in C(3) \). So the proof of the lemma has been completed. \( \square \)

**Corollary 6.4.6**

\[
\lambda_r \circ (id \land \lambda_l) = \lambda_r \circ (\lambda_r \land id) : \Sigma^\infty M \land_S \Sigma^\infty L \land_S \Sigma^\infty N \to \Sigma^\infty (M \land L \land N)
\]

**Lemma 6.4.7** \( r = l : S \land_L S \to S \)

**Proof:** Again the method is the same as the last lemma so we will run through it more quickly.

\[
S^0 \land S^0 = C(2) \times_{C(1) \times C(1)} C(0) \times S^0 \land C(0) \times S^0
\]

\[
\cong C(2) \times_{C(1) \times C(1)} C(1) \times_{C(1) \times C(1)} C(0) \times S^0 \land C(0) \times S^0
\]

\[
\cong C(2) \times_{C(1) \times C(1)} C(1) \times C(0) \times_{C(1) \times C(1)} C(0) \times S(0) \land S(0)
\]

\[
\cong C(2) \times_{C(1) \times C(1)} C(0) \times C(0) \times S^0 \land S^0
\]

and the map

\[
C(2) \times_{C(1) \times C(1)} C(0) \times C(0) \times S^0 \land S^0 \to C(0) \times S^0
\]
is induced by the map
\[ C(2) \times_{C(1) \times C(1)} C(0) \rightarrow C(0) \]
so the lemma follows. \(\square\)

In view of the last two lemmas we will use the notation \(\lambda = \lambda_r = \lambda_l\). This makes sense since any time there is a choice between them the answer is the same.

**Lemma 6.4.8** [13] \(\mathcal{M}_S = (\mathcal{M}_S, \wedge_S, S, \phi_1, r, l)\) is a symmetric monoidal category.

**Proof:** [13]. \(\square\)

### 6.5 **R-Mod**

**Definition 6.5.1** An S-algebra. \(R = (R, \phi, e)\), is a monoid in S-mod. A left R module. \(M\), is an S-module together with a left R action \(\lambda_R : R \wedge_L M \rightarrow M\) such that the following diagram commutes.

\[
\begin{array}{ccc}
R \wedge_S R \wedge_S M & \xrightarrow{\phi \wedge_S \text{id}} & R \wedge_S M \\
\downarrow{\text{id} \wedge_S \lambda_R} & & \downarrow{\lambda_R} \\
R \wedge_S M & \xrightarrow{\lambda_R} & M
\end{array}
\]

Right R modules and \((R, R)\) bimodules are defined similarly. For a commutative R let \(\text{R-Mod}\) denote the category of left R modules.

**Remark:** S is an S-algebra.

**Lemma 6.5.2** [13] \(\text{R-Mod}\) is cocomplete with colimits being created in \(\text{SU}\).

**Proof:** Consider the composition

\[
R \wedge_S \text{colim}(M_\alpha) \cong \text{colim}(R \wedge_S M_\alpha) \rightarrow \text{colim}M_\alpha.
\]

\(\square\)

Elmendorf, Kriz, Mandell and May define the smash product of R modules similarly to the way that the tensor product over rings is defined.
Definition 6.5.3 [13] Let $N, M \in R - \text{Mod}$. Then define the smash of $N$ and $M$ over $R$, denoted $N \wedge_R M$ to be the coequalizer in the following diagram.

\[
\begin{array}{c}
N \wedge_S R \wedge_S M \xrightarrow{id \wedge_S \lambda_R} N \wedge_S M \xrightarrow{\lambda_R \wedge_S \text{id}} N \wedge_R M
\end{array}
\]

Lemma 6.5.4 [13] For $N, L, M \in R - \text{Mod}$ we have a canonical natural isomorphism.

\[
\alpha: (N \wedge_R L) \wedge_R M \to N \wedge_R (L \wedge_R M)
\]

Proof: In the following diagram if we first take colimits of the first two columns and then of the bottom row we get $X = (N \wedge_R L) \wedge_R M$. If we take colimits in the other order then we get $X = N \wedge_R (L \wedge_R M)$. So $\alpha$ is the map induced by taking the identity on $N \wedge_S L \wedge_S M$. \qed

Lemma 6.5.5 [13] Let $N, L, M, P \in R - \text{Mod}$. Then the following diagram commutes.

\[
\begin{array}{c}
N \wedge_R (L \wedge_R (P \wedge_R M)) \xrightarrow{\alpha} (N \wedge_R L) \wedge_R (P \wedge_R M) \xrightarrow{\alpha} ((N \wedge_R L) \wedge_R P) \wedge M \xrightarrow{\alpha^\wedge \text{id}} (N \wedge_R (L \wedge_R P)) \wedge_R M
\end{array}
\]

Proof: The conclusion follows from the fact that all the maps in the lemma are the identity when restricted to $N \wedge_S L \wedge_S P \wedge_S M$. \qed

Lemma 6.5.6 [13]

\[r \wedge id = (id \wedge l)\alpha: (M \wedge_R R) \wedge N \to M \wedge_R N\]
and
\[ r = l : R \wedge_R R \to R. \]

**Proof:** The first part follows since \( \alpha \) on \( \wedge_R \) is induced by \( \alpha \) on \( \wedge_S \) and since the following diagram commutes.

\[
\begin{array}{ccc}
(M \wedge_S R) \wedge_S N & \to & M \wedge_R N \\
\downarrow & & \downarrow \\
(M \wedge_R R) \wedge_R N
\end{array}
\]

The second statement is also easy to show. \( \square \)

**Corollary 6.5.7** ([13]) \((R - Mod, \wedge_R, R, \phi, r, l)\) is a monoidal category.

### 6.6 \( R - Mod \) the model category

**Lemma 6.6.1** Let \( N, M_\alpha \in M_S \). Then \( \text{colim}(M_\alpha \wedge_S N) = \text{colim}M_\alpha \wedge_S N \) and \( \text{colim}(N \wedge_L M_\alpha) = N \wedge_L \text{colim}M_\alpha \).

**Proof:**

\[
\text{colim}(M_\alpha \wedge_L N) = \text{colim}(\mathcal{L}(2) \otimes_{\mathcal{L}^1} M_\alpha \wedge N)
\]

\[
\cong \mathcal{L}(2) \otimes_{\mathcal{L}^1} \text{colim}(M_\alpha \wedge N)
\]

\[
\cong \mathcal{L}(2) \otimes_{\mathcal{L}^1} (\text{colim}M_\alpha) \wedge N
\]

\[
= (\text{colim}M_\alpha) \wedge_L N
\]

\( \square \)

**Lemma 6.6.2** Let \( N \) and \( M_\alpha \) be \( R \) modules. Then \( \text{colim}(M_\alpha \wedge N) = (\text{colim}M_\alpha) \wedge N \) and \( \text{colim}(N \wedge M_\alpha) = N \wedge (\text{colim}M_\alpha) \).

**Proof:** The lemma follows since taking colimits commutes with taking colimits and from the same lemma for \( L \) algebras. \( \square \)
Lemma 6.6.3 \( \mathcal{M}_S \) and \( R \)-Mod are monoidally cocomplete.

**Proof:** It follows from the last lemma. \( \square \)

Lemma 6.6.4 Let \( A, B \in SU[L] \) and \( X, Y \in \text{Top} \). then there is a natural associative isomorphism

\[
(A \wedge_L B) \wedge (X \wedge Y) \cong (A \wedge X) \wedge_L (B \wedge Y)
\]

**Proof:** Fix compact \( K \subset C(2), K_1, K_2 \subset C(1) \) with \( 0 \in K_1, K_2 \) and a \( K \times K_1 \times K_2 \) connection \( (\mu, \nu) \). Let \( K' \) denote the image of \( K \times K_1 \times K_2 \) in \( C(2) \). Then \( K' \) is compact and \( (\mu, \nu) \) is also a \( K' \) connection.

Now fix a finite dimensional subspace \( V \subset U \). Let

\[
D(K, K_1, K_2)(V) = \text{Colim}
\]

\[
T(K' \times K_1 \times K_2, \nu(V), V) \wedge (A \wedge B)(\nu(V)) \wedge (X \wedge Y)
\]

and

\[
D'(K, K_1, K_2)(V) = \text{Colim}
\]

\[
T(K' \times K_1 \times K_2, \nu(V), V) \wedge (A \wedge X) \wedge (B \wedge Y)(\nu(V))
\]

Then since

\[
(A \wedge B)(\nu(V)) \wedge (X \wedge Y) \cong (A \wedge X) \wedge (B \wedge Y)(\nu(V))
\]

we get that

\[
(A \wedge_L B) \wedge X \wedge Y = L(\text{colim}_{K,K_1,K_2} D(K, K_1, K_2))
\]

\[
\cong L(\text{colim}_{K,K_1,K_2} D'(K, K_1, K_2)) = (A \wedge X) \wedge_L (B \wedge Y)
\]

\( \square \)
Lemma 6.6.5 Let $A$ and $B$ be $R$ modules and $X$ and $Y$ spaces. Then there is a natural associative isomorphism

$$(A \wedge_R B) \wedge (X \wedge Y) \cong (A \wedge X) \wedge_R (B \wedge Y)$$

Proof: The lemma follows since for $R$ modules $M_\alpha$ and space $X$

$$\text{colim}(M_\alpha) \wedge X \cong \text{colim}(M_\alpha \wedge X)$$

and since the same result holds for $L$ algebras. \qed

Let $\Lambda^Z : SU \to SU$ be a desuspension functor and $\Lambda_Z : SU \to SU$ be a suspension functor. For precise definitions see [18] Chapter 1. We list two of their properties.

Lemma 6.6.6 [18]

$$\Lambda^Z \Lambda_Z = \Lambda_Z \Lambda^Z = id$$

Let $n$ be the dimension of $Z$. Then $\Lambda^Z(X) \wedge S^n$ is naturally isomorphic to $X$.

Definition 6.6.7 In $\mathcal{M}_S$ let

$$S^n = L\Sigma^\infty S^n, \quad D^{n+1} = L\Sigma^\infty D^{n+1} \quad \text{and} \quad i_n = L\Sigma^\infty i_n \quad \text{if} \quad n \geq 0$$

and

$$S^n = L\Lambda^Z(n)\Sigma^\infty S^0, \quad D^{n+1} = L\Lambda^Z(n)\Sigma^\infty D^1 \quad \text{and} \quad i_n = L\Lambda^Z \Sigma^\infty i_n \quad \text{if} \quad n < 0$$

where $Z(n) \subset U$ is a subspace of dimension $n$. Choose isomorphisms $\Theta_n$.

In $R - \text{Mod}$ let

$$S^n_R = R \wedge_S S^n, \quad D^n_R = R \wedge_S D^n \quad \text{and} \quad i_n = R \wedge_S i_n.$$ 

Lemma 6.6.8 With the above definitions $\mathcal{M}_S$ and $R - \text{Mod}$ are spherically generated.
**Proof:** Easy. □

The following lemma is used implicitly when taking $CW$ approximations of $R$ modules.

**Lemma 6.6.9** $R - mod$ is $\omega$ small.

**Proof:** Let $R = S$. Let $\{X_i \to X_{i+1}\}$ be a sequence of relative $CW$ complexes in $M_S$ and $Z \subset U$ a finite dimensional subspace. Then we get a sequence in $Top_*$,

$$\Omega^\infty \Lambda_Z X_i \to \Omega^\infty \Lambda_Z X_{i+1}$$

and $\text{colim} \Omega^\infty \Lambda_Z X_i \cong \Omega^\infty \Lambda_Z \text{colim} X_i$. Let $f : D^r \to \text{colim} \Omega^\infty \Lambda_Z X_i$ be any map. Then for some $s$ we get a factorization

$$D^r \xrightarrow{f'} \Omega^\infty \Lambda_Z X_s \xrightarrow{f} \text{colim} \Omega^\infty \Lambda_Z X_i.$$

Using the fact that $\Lambda^Z \Sigma^\infty$ and $\Omega^\infty \Lambda_Z$ are adjoint and then the fact that $\text{Hom}(D^k, FX) = \text{Hom}(F_R D^k, X)$ where $F$ is the forgetful functor we see that $M_S$ is $\omega$ small. The whole lemma is then easy. □

**Definition 6.6.10** Let $\_ \otimes I : R - Mod \to R - Mod$ be the functor such that $X \mapsto X \wedge I^+$ and

$$\delta : (X \wedge_R Y) \wedge I^+ \to (X \wedge_R Y) \wedge (I^+ \wedge I^+) \to (X \wedge I^+) \wedge_R (Y \wedge I^+)$$

the map induced by the diagonal map on $I$ composed with the isomorphism of Lemma 6.6.5.

**Lemma 6.6.11** $\_ \otimes I$ is a multiplicative interval in $R - Mod$.

**Proof:** The only two nontrivial things to check are HELP and the associativity of $\delta$. The first is proved in [13] and the second in 6.6.5. □

The following lemma is used implicitly in [13] in order to prove that cellular approximations are weakly equivalent to the things they are supposed to approximate.
Lemma 6.6.12 \((\mathcal{M}_S[\text{Mon}], \{T_i\})\) and \((R - \text{Mod}[\text{Mon}], \{T_i\})\) are \(\kappa\) small for some \(\kappa\).

Proof: Use adjointness to reduce the problem to the case of 3.4.5. \(\square\)

Lemma 6.6.13 \((\mathcal{M}, \{i_n\})\) and \((R - \text{Mod}, \{i_n\})\) are suitably monoidal suitable cell categories.

Proof: 6.6.9, 6.6.11, 6.6.3 and 6.6.12. \(\square\)

Again in \(\mathcal{M}_S\) we know that \(\pi(S^n, S^n) \cong \mathbb{Z}\) as rings and that \(\pi_0(e) \cong \mathbb{Z}\) as groups.

Definition 6.6.14 Choose \(F_n\) to be the ring isomorphism \(\pi(S^n, S^n) \to \mathbb{Z}\). Choose \(F_0^e\) to be (one of the two) group isomorphisms \(\pi_0(e) \to \mathbb{Z}\).

Lemma 6.6.15 With \(F_n\) and \(F_0^e\) defined as above \((\mathcal{M}_S, I)\) has chains.

Proof: Look at homology. \(\square\)

Definition 6.6.16 In \(CG\), make \(D^2 \wedge D^2\) into a CW complex satisfying the \(C_*\) condition of Definition 5.3.4. This makes \(LS^0 \wedge (D^2 \wedge D^2) = LD^2 \wedge_S LD^2\) into a CW complex satisfying the \(C_*\) condition. Then make \(D^n \wedge_S D^r\) into a CW complex in a consistent way and satisfying the \(C_*\) condition. The \(\phi\) and attaching maps \(h\) are then defined.

Lemma 6.6.17 With the above structure \(\mathcal{M}_S\) has compatible cell and monoidal structures.

Proof: Look at \(H_*\). \(\square\)

For the next construction we use the symmetry of \(\wedge_S\).

Definition 6.6.18 Let \(M \in R - \text{Mod}\) and \(X \in \mathcal{M}_S\). Then we define \(M \wedge_S X \in R - \text{Mod}\) to be \(M \wedge_S X \in \mathcal{M}_S\) with \(R\) actions

\[
R \wedge_S M \wedge_S X \to M \wedge_S X
\]

and

\[
M \wedge_S X \wedge_S R \to M \wedge_S R \wedge_S X \to M \wedge_S X.
\]
Lemma 6.6.19 For any $M, N \in R - \text{Mod}$ and $X \in \mathcal{M}_S$ there exists natural isomorphisms in $R - \text{Mod}$

$$(M \wedge_S X) \wedge_R N \to (M \wedge_R N) \wedge_S X,$$

$M \wedge_R (R \wedge_S X) \to M \wedge_S X$

and

$$(R \wedge_S X) \wedge_R M \to X \wedge_S M.$$

**Proof:** The first isomorphism is induced by the isomorphism in $\mathcal{M}_S M \wedge_S X \wedge_S N \to M \wedge_S N \wedge_S X$. The last two isomorphisms are easy since $M \wedge_R R \cong M \cong R \wedge_R M$. □

**Definition 6.6.20** In $R - \text{Mod}$ make $D^r_R \wedge_R D^S_R$ into the CW complex induced from the CW complex $D^r \wedge_S D^s$ in $\mathcal{M}_S$ and the isomorphism of the last lemma

$$(R \wedge_S D^r) \wedge_R (R \wedge_S D^s) \cong R \wedge_S (D^r \wedge_S D^s).$$

**Lemma 6.6.21** With the above structure $R - \text{Mod}$ has weakly compatible cell and monoidal structures.

**Proof:** We use Lemma 6.6.19. First we prove the condition on $\alpha$. Observe that the following diagram has commuting squares and that all the maps are isomorphisms.

Then use the fact that

$$S^r \wedge_S (S^n \wedge_S S^s) \xrightarrow{\alpha} (S^r \wedge_S S^n) \wedge_S S^s$$

$$S^{r+n+s} \to S^{n+n+s}$$
commutes up to homotopy and smashing with $R$ preserves homotopies.

The statement $(f \times id)_{n+r/n+r-1} \simeq \vee \Sigma^r f$ follows from the following commutative diagram with horizontal isomorphisms.

$$
\begin{array}{ccc}
\vee S^n_R \wedge R \vee S^n_R & \longrightarrow & \vee S^n_R \wedge S \vee S^r \\
\downarrow f \wedge \text{id} & & \downarrow f \wedge \text{id} \\
\vee S^n_R \wedge R \vee S^n_R & \longrightarrow & \vee S^n_R \wedge S \vee S^r \\
\downarrow \Sigma^r f
\end{array}
$$

The statement $(id \times f)_{n+r/n+r-1} \simeq \vee \Sigma^r f$ is similar. $\Box$

### 6.7 The $S$ algebra $KR$

For this section except for 6.7.1 and 6.7.2 we assume that $R$ is a commutative ring.

**Definition 6.7.1** Let $R$ be any ring. Assume $KR \in MOrdCW^e(M_S)$ is such that $\pi_0(KR) = R$ and $\pi_n(KR) = 0$ if $n \neq 0$. Also assume that the map $\pi_0(KR) \otimes \pi_0(KR) \to \pi_0(KR)$ is the multiplication in $R$. Then we call $KR$ an Eilenberg-Mac Lane spectrum for $R$.

The next lemma was proved by May ([20]) in the context of $A^\infty$ and $E^\infty$ ring spectra.

**Lemma 6.7.2** [20] Let $R$ be any ring. Then there exists an Eilenberg-Mac Lane spectrum for $R$.

**Proof:** Let $J$ be a generating set for $R$ and $K'$ be a relations set. Let $KR(1) = T(\vee_{j \in J} S^0)$ and $KR(2)$ be the pushout of

$$
\begin{array}{ccc}
T(\vee_{k \in K} S^0) & \longrightarrow & KR(1) \\
\downarrow & & \downarrow \\
T(\vee_{k \in K} D^1) & \longrightarrow & KR(2)
\end{array}
$$

where $k'$ represents $k \in \pi_0(KR(1))$. Then $\pi_0(KR(2)) = R$ and $\pi_n(KR(2)) = 0$ for $n < 0$. We kill the rest of the homotopy by adding cells. $\Box$

May shows that if $R$ is commutative then $KR$ can be constructed to be commutative.
Lemma 6.7.3  Cellular approximation holds in $(KR - Mod, I(KR - Mod))$.

Proof: Easy. □

We define chains in $KR - Mod$. We use the notation of Definition 5.2.1.

Definition 6.7.4  $\pi(KR \wedge_S S^n, KR \wedge_S S^n) \cong \pi_0(KR) \cong R$. Let $F_n$ be this composition of isomorphisms.

Lemma 6.7.5  With the above definition of the $F_n$, $(KR - Mod, I(KR - Mod))$ is a homology category.

Proof: Easy. □

Lemma 6.7.6  In $(KR - Mod, I(KR - Mod))$, $\pi_n = H_n$.

Proof: It is enough to check this for $X \in CW^c(KR - Mod)$. In this case it follows since both functors agree on spheres and commute with direct limits, cofibration sequences give long exact sequences on $\pi_*$ and $H_*$, since we have a Hurewitz homomorphism and from the five lemma. □

In the theorem we use the structure given by 6.6.13, 3.3.5 and 4.4.17.

Theorem 6.7.7  Let $R$ be any ring. Then $FreeDGA(R)$ is equivalent to $MOrdCW^c(KR - Mod[Mon])$ and $Ho(DGM(R)[Mon])$ is equivalent to $Ho(KR - Mod[Mon])$.

Note that the first equivalence is "finer" than the second.

Proof: 5.8.9, 6.7.6 and 5.8.10. □

The following corollary follows since $A^\infty$ algebras are equivalent to monoidal objects in $\mathcal{M}_S$ and since $KQ$ is the same as the rational localization of $S$.

Corollary 6.7.8  The homotopy category of differential graded algebras over Q is equivalent to the homotopy category of rational $A^\infty$ ring spectra.

The theorem should be able to be used to give extra structure to the Adams spectral sequence converging to $\pi_*(A)$ when $A \in \mathcal{M}_S$. 
Bibliography


