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Thermodynamic Quantities in the High Temperature Limit in the GUP Framework

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Abstract

We study the values of thermodynamic quantities of physical systems at high temperatures in the framework of the generalized uncertainty principle. First, we obtain the semi-classical partition function, internal energy and heat capacity for physical systems in a general case of deformed algebra. We show that at high temperatures limit the heat capacity vanishes and the internal energy is maximized. We calculate this temperature for the ultra-relativistic system in approximate form. Finally, the maximum temperature and maximum energy for the ultra-relativistic system are obtained in the framework of two different GUPs.

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I. INTRODUCTION

One of important prediction of various candidates of quantum gravity such as string theory, loop quantum gravity, doubly special relativity and black hole physics is the existence of a minimal observable length and/or a maximal observable momentum [1–9]. This fundamental minimal observable length is the order of Planck length $\ell_{\text{Pl}} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}\text{cm}$. The existence of such fundamental minimal observable length, leads to the Generalized Uncertainty Principle (GUP)[10, 11]. As a result of this modification, we expect that the Hamiltonian of physical systems will be modified, which results in the deformation of physical quantities in both classical and quantum levels.

In the GUP framework, various problems in classical and quantum mechanical levels such as one dimensional harmonic oscillator [12], gravitational quantum well, hydrogen atom, Lamb's shift, Landau levels, Casimir effect and particles scattering have been studied in Refs [13–21]. The composite N-particle system in GUP framework was studied in [22, 23]. In the field of statistical mechanics in the GUP framework some authors have been published several articles. In [24] Fityo modified a semiclassical method for the evolution of partition function based on the generalization of phase space elementary cells. The thermodynamical properties of some physical systems have been studied in the presence of general form of generalized commutation relations [25]. The effect of GUP on thermodynamical quantities in classical and quantum levels have been investigated in [17, 24–32]. Also, many research papers have been published on the thermodynamics of black holes and their relationship to GUP. In recent years, "GUP-blackhole thermodynamics" due to the recent observation of black hole, quantum gravity and thermodynamics, which are all in connection with GUP have gained much more importance [33–39].

In this paper, we investigate the limit values for thermodynamic quantities of physical systems at high temperatures within the GUP framework. We show that physical systems have a maximum temperature in the presence of generalized commutation relation. Using exact solutions [25], we will calculate the maximum temperature and internal energy of the physical system up to the first order approximation. Finally, we obtain the maximum temperature and internal energy of ultra-relativistic gas in two different GUP framework.

II. GUP AND THE MODIFIED THERMODYNAMICS OF PHYSICAL SYSTEMS

Consider the following one-dimensional GUP assumed by Kempf, Mangano and Mann in their seminal paper [12] in agreement with various theories of quantum gravity[4–9, 32, 40–44]

$$\Delta X \Delta P \geq \frac{\hbar}{2}(1 + \beta(\Delta P)^2 + \beta\langle P \rangle^2), \quad (1)$$

where X and P denote the modified position and momentum coordinate respectively and $\beta = \beta_0/(c^2 M_{\text{Pl}}^2)$ is the GUP parameter, β_0 is a dimensionless constant of order of unity M_{Pl} is Plank mass. It is easy to see that minimal observable length, which, as it is supposed, is of order of Plank's length $(\Delta X)_{\min} = \hbar\sqrt{\beta} = \sqrt{\beta_0}\ell_{\text{Pl}}$. The above GUP leads to generalized of the usual commutation relations. According to Kempf et al. [12, 32, 40] may read in three-dimensions

$$[X_i, P_j] = i\hbar((1 + \beta P^2)\delta_{ij} + \beta' P_i P_j) \quad , \quad (2)$$

$$[P_i, P_j] = 0 \quad , \quad (3)$$

$$[X_i, X_j] = i\hbar \frac{2\beta - \beta' + (2\beta + \beta')\beta P^2}{1 + \beta P^2} (P_i X_j - P_j X_i), \quad (4)$$

where $i, j = 1, 2, 3$, and β, β' are two deformation parameters. This algebra gives rise to minimal length $\Delta X_{\min} = \hbar\sqrt{\beta + \beta'}$. In connection with the idea of maximum momentum in deformed special relativity, Pedram presented a higher order GUP in Refs.[40, 41]

$$[X_i, X_j] = \frac{2i\hbar\beta}{(1 - \beta P^2)^2} (P_i X_j - P_j X_i), \quad (5)$$

$$[P_i, P_j] = 0 \quad , \quad (6)$$

$$[X_i, P_j] = \frac{i\hbar\delta_{ij}}{1 - \beta P^2} \quad , \quad (7)$$

this algebra implies a minimal length $\Delta X_{\min} = \frac{3\sqrt{3}}{4}\hbar\sqrt{\beta}$ and the maximum momentum of particle $P_{\max} = \frac{1}{\sqrt{\beta}}$. To incorporate this idea, Ali, and Vagenas proposed the deformed commutation relation in Ref. [13]. The effects of the GUP on physical systems have as well as thermodynamic quantities have been studied in several articles [3, 24, 45–52]. The partition function of a non-interacting N-particle system in GUP framework is [24]

$$Z_N = \frac{1}{h^{3N}} \int \exp\left(\frac{-H(X, P)}{k_B T}\right) \frac{d^{3N} X d^{3N} P}{J}, \quad (8)$$

where $H(X, P)$ is the Hamiltonian of the system, T and k_B are temperature and Boltzman's constant respectively. The Jacobian of transformation in D-dimension is J , which can be obtained from the modified Poisson brackets [24]. Now we consider the general forms of generalized commutation relations [46]

$$[X_i, P_j] = i\hbar f_{ij}(\beta, \beta', P), \quad (9)$$

$$[P_i, P_j] = i\hbar h_{ij}(\beta, \beta', P), \quad (10)$$

$$[X_i, X_j] = i\hbar g_{ij}(\beta, \beta', P), \quad (11)$$

where f_{ij}, g_{ij}, h_{ij} are deformation functions and β, β' are the deformation parameters. In the limit $\beta, \beta' \rightarrow 0$, this algebra tend to ordinary Haeisenberg algebra $[x_i, p_j] = i\hbar$, $[x_i, x_j] = [p_i, p_j] = 0$, where p_i and x_i are the ordinary momentum and position operators respectively. For the generalized commutation relation (5), the Hamiltonian of the physical systems can be written in the general following form

$$H = \alpha P^\varepsilon + U^{\varepsilon-1}(X), \quad (12)$$

here, $U(X)$ is external potential field, α is a constant and $\varepsilon = 1, 2$. For the system consisting of N identical non-interacting particles in the external potential field Hamiltonian is $H = \frac{P^2}{2m} + U(X)$, $\alpha = \frac{1}{2m}$ and $\varepsilon = 2$. For extreme relativistic particle, we have $H = Pc$, $\alpha = c$, $\varepsilon = 1$, and c is speed of light.

Using Hamiltonian (12) and relation (8) becomes

$$Z_N = \frac{1}{h^{3N}} \int d^{3N}X \exp \left[-\frac{U^{\varepsilon-1}(X)}{k_B T} \right] \int d^{3N}P \frac{\exp \left[-\frac{P^\varepsilon}{2mk_B T} \right]}{J(\beta, \beta', P)}, \quad (13)$$

for $\varepsilon = 2$ and $\alpha = \frac{1}{2m}$, we obtian [25]

$$Z_N = Z_N^0 \left(\frac{1}{2\pi mk_B T} \right)^{\frac{3N}{2}} \int d^{3N}P \frac{\exp \left[-\frac{P^2}{2mk_B T} \right]}{J(\beta, \beta', P)}, \quad (14)$$

and for $\varepsilon = 1$ and $\alpha = c$

$$Z_N = Z_N^0 \left[\pi \left(\frac{2k_B T}{c} \right)^3 \right]^{-N} \int d^{3N}P \frac{\exp \left[-\frac{cP}{k_B T} \right]}{J(\beta, \beta', P)}, \quad (15)$$

where Z_N^0 is the partition function for the non-deformed case. All modified thermodynamical quantities such as internal energy $E = -\frac{\partial}{\partial(k_B T)} \ln Z$, heat capacity $C = \frac{\partial E}{\partial T}$, Helmholtz free energy $A = -k_B T \ln Z$ and entropy $S = k_B \ln Z + \frac{E}{T}$ can be obtained in GUP framework.

The internal energy and heat capacity are given by [25]

$$E = E^0 - \frac{3}{2} N k_B T + \frac{N}{2m} \left(\frac{q_4}{q_2} \right), \quad (16)$$

$$C = C^0 - \frac{3}{2} N k_B + \frac{N}{4 k_B T^2 m^2} \left[\frac{q_6 q_2 - (q_4)^2}{(q_2)^2} \right], \quad (17)$$

and

$$E = E^0 - 3 N k_B T + c N \left(\frac{q_3}{q_2} \right), \quad (18)$$

$$C = C^0 - 3 N k_B + \frac{c^2 N}{k_B T^2} \left[\frac{q_4 q_2 - (q_3)^2}{(q_2)^2} \right], \quad (19)$$

where $\varepsilon = 2$, $\alpha = \frac{1}{2m}$ and $\epsilon = 1$, $\alpha = c$, respectively. Here,

$$q_n = \int dP P^n \frac{\exp \left[-\frac{\alpha P^\varepsilon}{k_B T} \right]}{J(\beta, \beta', P)} \quad (20)$$

and E^0 , C^0 are the internal energy and the heat capacity in the non-deformed thermodynamics, respectively.

III. GUP AND THE MAXIMUM TEMPERATURE AND ENERGY

At high temperature limit $T \gg \frac{\alpha}{k_B}$, when get $H \sim \alpha p^\varepsilon$, the internal energy in ordinary thermodynamic becomes

$$E^0 = \frac{3 N k_B T}{\varepsilon}, \quad (21)$$

thus, at this temperature, the modified energy (16) and (18) become

$$E = \frac{N}{2m} \left(\frac{q_4}{q_2} \right), \quad (22)$$

$$E = N c \left(\frac{q_3}{q_2} \right), \quad (23)$$

respectively. For $\varepsilon = 2$, the modified heat capacity becomes

$$C = \frac{N}{4 m k_B T^2} \left(\frac{q_6 q_2 - (q_4)^2}{(q_2)^2} \right), \quad (24)$$

and for $\varepsilon = 1$, is

$$C = \frac{Nc^2}{k_B T^2} \left(\frac{q_4 q_2 - (q_3)^2}{(q_2)^2} \right). \quad (25)$$

The modified heat capacity of system (24) and (25), goes to zero when $T \rightarrow \infty$ in agreement with Ref.[24]. Therefore, due to relationship between internal energy and heat capacity $C = \frac{\partial E}{\partial T}$, the internal energy of the system tends to a maximum value. This maximum value depends on the generalized algebra [25].

At high temperature limit, we consider extreme relativistic system of particles. The heat capacity of this system can be reduced to zero if we have in relation (25)

$$q_4 q_2 - (q_3)^2 = 0. \quad (26)$$

By solving relation (26), we can calculate numerically the value of maximum temperature and maximum energy for extreme relativistic system of particles, respectively.

For high temperature $T \gg \frac{c}{k_B}$, the relation (20) simplifies to

$$q_n \approx \left(1 - \frac{c}{k_B T} \frac{S_{n+1}}{S_n} \right) S_n, \quad (27)$$

where $S_n = \int dP \frac{P^n}{J}$. Using (26) and (27), we obtain the following relation to calculate the value of maximum temperature

$$T_{max} \approx \frac{c}{k_B} \left[\frac{S_5 S_2 - S_3 S_4}{S_4 S_2 - (S_3)^2} \right], \quad (28)$$

at this temperature, the maximum value of internal energy for the system becomes

$$E_{max} \approx cN \left(\frac{S_3}{S_2} \right) \left[1 - \left(\frac{(S_4 S_2 - (S_3)^2)^2}{(S_5 S_2 - S_3 S_4)(S_3 S_2)} \right) \right], \quad (29)$$

other modified thermodynamical quantities such as entropy $S = k_B \ln Z + \frac{E}{T}$ and Helmholtz free energy $A = -k_B T \ln Z$ can be obtained in the maximum temperature. Since the maximum temperature and maximum energy in relation (28) and (29) are function of S_n , it can be seen that their physical dimensions are proportional to $\frac{1}{\sqrt{\beta}}$. The Planck temperature $T_{Pl} = \frac{E_{Pl}}{k_B}$ and Planck length ℓ_{Pl} , have the following relation

$$T_{Pl} = \frac{c^4}{G k_B} \ell_{Pl}, \quad (30)$$

where c is speed of light and G is Newtonian gravitation constant. In one dimensional Kempf's GUP framework (1), the Planck temperature (30) can be written in term of GUP parameter as

$$T_{Pl} \approx \frac{c \sqrt{\beta_0}}{k_B \sqrt{\beta}}, \quad (31)$$

also, for three-dimensional Kempf's GUP (2 - 4) and (GUP^*) framework (5 - 7), the Planck temperature is $T_{Pl} \approx \frac{c}{k_B \sqrt{\beta + \beta_0}}$ and $T_{Pl} \approx \frac{4}{3\sqrt{3}} \left(\frac{c}{k_B \sqrt{\beta}} \right)$ respectively. So for the generalized commutation relation (9 - 11), the maximum temperature (28) and the maximum energy (29) are proportional to T_{Pl} and E_{Pl} respectively.

In the GUP framework, if the momentum has a maximum, we expect the temperature to show a maximum value.

IV. CALCULATION IN TERM OF PLANCK UNITS

In this section we calculate the maximum temperature and internal energy in two GUP's framework. The Kempf's GUP and the high order GUP framework (GUP^*).

The Jacobian of transformation for Kempf's GUP is given by [24]

$$J(\beta, \beta', P) = (1 + \beta P^2)^2 (1 + (\beta + \beta') P^2), \quad (32)$$

so, using (28) we obtain

$$T_{max}^{Kempf} = \frac{\pi(\beta'^2 \Omega_1 - \sqrt{\beta^3} \Omega_2)}{2\sqrt{\beta} (\Omega_3 - \Omega_4)}, \quad (33)$$

where

$$\begin{aligned} \Omega_1 &= \frac{(\beta' + \beta) \left(-\beta' + (\beta' - \beta) \ln \left(\frac{\beta}{\beta' + \beta} \right) \right) + \beta'^2 \ln(\beta' + \beta)}{(\beta' + \beta)(\beta' + 2(\beta + \sqrt{\beta} \sqrt{\beta' + \beta}))}, \\ \Omega_2 &= \left(\frac{\beta' - 2\beta}{\sqrt{\beta^3}} + \frac{2}{\sqrt{\beta' + \beta}} \right) \left(\beta' + \beta \ln \left(\frac{\beta}{\beta' + \beta} \right) \right), \\ \Omega_3 &= \frac{\pi^2 \beta'^2 \left(\beta' - 2\beta + 2 \frac{\sqrt{\beta^3}}{\sqrt{\beta' + \beta}} \right)}{4 \left(\beta' + 2(\beta + \sqrt{\beta} \sqrt{\beta' + \beta}) \right)}, \\ \Omega_4 &= \left(\beta' + \beta \ln \left(\frac{\beta}{\beta' + \beta} \right) \right)^2, \end{aligned}$$

in the limit $\beta', \beta \rightarrow 0$, the maximum temperature (32) tends to unacceptably infinitely negative. The Kempf's GUP framework don't implies a maximal momentum, so don't implies a finit maximum temperature. However, in high temperature limit $T \rightarrow \infty$, the exact maximum value of internal energy for extreme relativistic system (23) becomes

$$E_{max}^{Kempf} \approx \left(\frac{2Nc}{\pi\sqrt{\beta}} \right) \left[\frac{\beta' + \Gamma(\beta') + \beta' \ln(4) + \beta \ln \left(\frac{\beta}{\beta + \beta'} \right) + \beta' \Psi[0, \frac{1}{2}]}{\beta' + 2\beta - 2\sqrt{\beta(\beta' + \beta)}} \right], \quad (34)$$

where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ is the Euler Gamma function and $\Psi[n, z] = \frac{\partial^n}{\partial z^n} \Psi(z)$ is Polygamma function and $\Psi(z)$ is digamma function. The maximum energy (34) goes to infinity when $\beta, \beta' \rightarrow 0$, which is the maximum energy in ordinary thermodynamics.

Now, we consider the special case $\beta' = 2\beta$, wherein the position commutation relation in Kempf's algebra (2 - 4) to first order of GUP parameter becomes zero. In such as approximation the relation (34) becomes

$$E_{max}^{Kempf} \approx \left(\frac{\ln(3) - 2}{(\sqrt{3} - 2)\pi} \right) \left(\frac{Nc}{\sqrt{\beta}} \right), \quad (35)$$

by using $\frac{c}{\sqrt{\beta}} \approx \sqrt{3} E_{Pl}$ in (35) we obtain

$$E_{max}^{Kempf} \approx \left(\frac{(\ln(3) - 2)\sqrt{3}}{(\sqrt{3} - 2)\pi} \right) (NE_{PL}) \approx (1.854) (N E_{Pl}). \quad (36)$$

For GUP^* framework (5 - 7), the Jacobian of transformation becomes

$$J = \frac{1}{(1 - \beta P^2)^3}, \quad (37)$$

in this framework the momentum of particle cannot be more than $\frac{1}{\sqrt{\beta}}$. By using (28), we obtain the maximum temperature as

$$T_{max}^{GUP^*} \approx \left(\frac{1792}{1838} \right) \left(\frac{c}{k_B \sqrt{\beta}} \right), \quad (38)$$

the above relation can be obtained in term of Planck temperature

$$T_{max}^{GUP^*} \approx \left(\frac{1792}{1832} \right) \left(\frac{3\sqrt{3}}{4} T_{PL} \right) \approx (1.271) T_{Pl}, \quad (39)$$

at this temperature, E_{max} for the extreme relativistic system (29) becomes

$$E_{max}^{GUP^*} \approx \left(\frac{148899885}{322961408} \right) \left(\frac{cN}{\sqrt{\beta}} \right), \quad (40)$$

according to Planck energy, E_{max} can be written as

$$E_{max}^{GUP*} \approx \left(\frac{148899885}{322961408} \right) \left(\frac{3\sqrt{3}}{4} N E_{Pl} \right) \approx (0.5989) N E_{Pl}. \quad (41)$$

As a physical application for these corrections at high temperatures, we can refer to extra-Planck corrections universe [53, 54]. Also, for $\beta \rightarrow 0$, (35) and (40) become infinity. This is an expected result for temperature and energy in ordinary thermodynamics.

V. CONCLUSION

In this paper, we have investigated the effects of GUP on the thermodynamics of physical systems and obtained the semi-classical partition function for general Hamiltonian in a general case of deformed algebra. We have shown that the internal energy of physical systems reach to a maximum, whose value depends on chosen deformed algebra. We showed that the modified heat capacity in a generalized algebra, which implies the minimum length uncertainty and maximal observable momentum, can be vanished at a maximum temperature and the internal energy can be maximized. At very high temperature limit, we considered ultra-relativistic system, and in a general modified algebra framework, we have obtained the maximum temperature and energy for the system approximately. In particular we have calculated these quantities for GUP^* and Kempf's GUP. It is shown that the maximum temperature and energy are proportional to Planck temperature and energy, respectively. In the near future, we will investigate these corrections obtained in GUP framework and their possible relationship to the trans-Planckian corrections in inflationary cosmology.

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