SPIELRAUM:
HELMHOLTZ'S MANIFOLD THEORY OF PERCEPTION AND THE
LOGICAL SPACE OF WITTGENSTEIN'S TRACTATUS

by

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requirements for the degree of Doctor of Philosophy
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Abstract

The dissertation analyzes the theory of "logical space" developed by Ludwig Wittgenstein in his *Tractatus Logico-Philosophicus*. I show how this idea represents a development of arguments first put forward by Hermann von Helmholtz, the physicist and physiologist. Helmholtz--instead of honouring Kant's distinction between on the one hand time and space, and, on the other, empirical *qualia* (colour, taste, hardness, tone)--stretched the Kantian spatial manifold to cover the other *qualia* as well: the *qualia* are also organized in manifolds; and this new, extended manifold is the "space" of all possible human experience. He explained this *a priori* mesh as being a consequence of the physiological constitution of our bodies, and the physical constitution of the world in which they are situated. The subject is enclosed in an inner world whose structure is the network of possible sensation; outside of her is a world of unknown complexity; and separating the two is what Heinrich Hertz called the "no-man's-land" of sense-physiology, a border zone regulating all traffic between the two realms, common to both, yet proper to neither.

Wittgenstein, in his *Tractatus*, adapted the picture-theories of Helmholtz and Hertz to his analysis of logic. He too was confronted with the problem of defining a field of possible experience, of possible facts: his analysis of Russell's and Frege's logical theories had led him to the conclusion that the fundamental properties of logic could not be accounted for without assuming such an *a priori* space. Thus he assimilated Russell's types to the dimensions of a manifold, the elements of which were his *Sachverhalte*, or elementary facts. The truth-functions of our logic are defined on top of this space: it is the basis on which all symbolic systems, including those of the natural sciences, are erected.
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Chapter 1 - Introduction

In the following pages I develop a new reading of Wittgenstein's *Logisch-Philosophische Abhandlung*, or *Tractatus Logico-Philosophicus*, as it is now generally known. The reading is intended on the one hand to stand on its own: it clarifies terms and passages in that book whose central position has never been disputed, even if their exact significance has remained opaque. On the other hand my treatment is intended as a case study. I read the *Tractatus* as the culmination of a particular tradition within the philosophy of logic. I refer to this tradition as one of "naturalized" logic, because it treats logic as a kind of *physics* of thought. I do not discuss this tradition in the body of the thesis; however some knowledge of that background is helpful in understanding recurrent and characteristic features of the theories which I will elucidate. Of these, the most important is the following: to the question, What is logic?, the tradition mentioned above responds by saying that logic is a natural attribute of human thought. Logic is universally valid because it describes the necessary "behaviour" of our thoughts and representations, when, that is, we are employing them correctly, *i.e.* in accordance with their natures. And because our representations are copies of the objects they represent (though the interpretation given this notion differs from theoretician to theoretician), they will not lead us into error or permit invalid deduction, unless they have been in some sense denatured. A naturalized explanation of the ontological grounds of logic almost inevitably compels one to make statements such as the following: "We cannot think something illogical, for then we would have to think illogically." (*Tractatus* 3.03) This conclusion is inescapable, because logic is on the one hand identified with necessary and normal thinking (*i.e.* it is supposed to derive its authority from that fact), while it is on the other hand obvious that thinking is, in another sense, rarely logical.
This attitude is to be found in Frege's and Russell's works as well, for they were both inspired by Leibniz, the naturalizer *par excellence*. Both logicians thought that there was *one* logic, and that its structure could be determined once and for all. They differed primarily in Frege's reluctance to explain *why* the laws of thought are as they are. At the same time, however, Frege was far more willing than Russell to talk of his *Begriffsschrift* as if it described a perfect language which *already existed*. Frege, for instance, defines type-differences with recourse only to the syntactic properties of the propositions in which his proper-names occur. In doing so, he seemingly avoids ontological postulates; however, he is able to do so only because he assumes the existence of an interpreted language with a syntactic structure—and that is hardly an instance of ontological parsimony. It is indeed from Frege that Wittgenstein learned the habit (quite widespread among philosophers) of talking about logic on the one hand as if it were something we had not yet described fully, thus something we do not yet know, while at the same time talking about it as if it were something that already existed, something that is indeed the precondition for all our thinking. Wittgenstein took refuge from this contradiction in a position much the same as Leibniz's or Helmholtz's: he said that the ideal logic already existed *as the substrate* of our high-level thoughts and languages. The laws of everyday thought are merely a by-product of surface-grammar. But our thoughts and representation at the deepest level brook neither misuse nor error.

In this Introduction I sketch out the historical antecedents to this view briefly, which discussion is followed with an outline of the dissertation. Because the thesis depends on a large mass of technical material, it is hard to explain that development without trivializing the connections I draw. So for instance a large part of my treatment is devoted to giving the *Tractatus* term "logical space" a precise signification. However, what I have to say in the pages immediately following will not sound much more precise than what a seasoned reader of the *Tractatus* is already
accustomed to. For this reason Chapter 2, in which I give an extended account of Helmholtz’s manifold theory of perception, is particularly important: much of the technical vocabulary is introduced there in an exact context, and the constellation of ideas with which Helmholtz provides us play a controlling role in the development of the subsequent chapters.

The tradition in question regards logic as an attribute of the thinking human subject: it is a natural motion of the human organism. We can trace this idea at least as far back as Avicenna, although it might be argued to be, implicitly, Aristotle’s view as well. Thought is a motion or alteration of human thought, Avicenna argued, thus logic is the science of the laws governing thought when it moves validly, that is when it moves from true conceptions or assents concerning what is known (present to the mind) to true conceptions or assents concerning things not yet known. Avicenna’s view is naturalized in that it regards logical laws as flowing directly from the natures of the constituents of thought, which are themselves forms with cognitive, as opposed to real, being. The laws of combination and analysis of such intentions parallel those of their real objects because, quite simply, they are formally the same as their objects: the laws of cognitive being replicate those of real being.

Yet such a view inevitably also invites its own worst enemy: if logic is the science of how thoughts behave, then how is it to be distinguished from psychology, which has been, since Aristotle, the portion of physics concerned with thought? Not all thought is logical, after all. Avicenna sketched out the dilemma as follows in the section on logic in his Remarks and Admonitions:

CONCERNING THE PURPOSE OF LOGIC

Logic is intended to give the human being a canonical tool which, if attended to, preserves him from error in his thought.

I mean by “thought” here that which a human being has, at the point of resolving, to move from things present in his mind—conceptions or assents
(whether scientific, based on opinion, or postulated and already admitted)—to things not present in it.

This movement inevitably has order and form in the elements dealt with. Such order and form may occur in a valid or invalid manner. Often the invalid manner resembles the valid one, or gives the impression that it resembles it.

Thus logic is a science by means of which one learns the kinds of movements from elements realized in the human mind to those whose realization is sought, the states of these elements, the number of types and form in the movements of the mind which occur in a valid manner and the types which are invalid.

Logic is only logic in so far as it specifies how thought must move from the known to the unknown in a manner which is valid, i.e. truth-preserving. Since Avicenna’s intentions are cognitively existent forms, his logic is quite literally a description of the natural behaviour of forms within the mind. It specifies how we may manipulate representations in such a way (a valid way) that the consequences of these operations are also true conceptions or assents. The principal task of the logician is therefore to distinguish those movements of thought which constitute valid deduction from those which only appear to do so: an exhaustive description of those movements will be our “canonical tool.” When Ramon Lull assimilated Avicenna’s doctrine of intentions to the elements of his Ars Magna, which described the combinatorial possibilities of groups of primitive intentions by means of nested wheels, he gave Avicenna’s natural logic a mechanical figuration: the Ars Magna was the first “dynamical model” (the term is Hertz’s) of human thought.

To view logic as a physics of human thought is not yet to see it as governed by mechanical laws, if only for the simple reason that physics, for Lull or Avicenna, was neither mechanical, nor deterministic. Naturalized logic only becomes mechanical logic once physics becomes mechanical, one might say. The critical figure in this development is Leibniz: he read Lull, had learned Cartesian mathematics and physics,

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and he hit on the idea that logic could be seen as a mechanics of thought. The
Leibnizean monad aggregates pluralities into wholes by means of \textit{perception}: the
aggregation of a multiplicity in a unity. Both the identities which ground Leibniz’s
logical calculus, and those appearing in his integral calculus can be seen as
descriptions of this single perceptual act of the monad: both express containment
relations between concepts on the one hand, and their extensions on the other. The
fundamental difference between the propositions of physics and those of logic lies in
the \textit{kind} of containments expressed by our logical and physical propositions: the
containments expressed in physical propositions are infinite, thus not provably
necessary, whereas those occurring in logic are finite, and demonstrably necessary.
Perceptual acts of both sorts may be described by means of the calculi, which
describe perceptual motions internal to the monad as natural processes, in the sense I
outlined above. Leibniz’s calculi describe the fundamental laws of physics and logic
because the monad is a perfect copy of the external world: the calculi describe
simultaneously the \textit{internal} economy of the monad’s perceptual acts, and the external
economy of the systems giving rise to the “details” (the term is Leibniz’s, \textit{cf.}
\textit{Monadology}, §12) of those perceptions.

All the authors I discuss in this thesis--Frege, Russell, Wittgenstein, Helmholtz,
and Hertz--inherited this paradigm to some degree from Leibniz. Helmholtz, as I
discuss in Chapter 2, was from the beginning interested in analyzing the physiological
basis of our intuition of space. Frege and Russell developed logical calculi that were
intended to describe not \textit{a} logic, but \textit{the} logic of our thought, and Russell tried in his
\textit{Theory of Knowledge} to provide an epistemological account of thought from which
logic would arise as a feature of the world (of which thinking subjects are of course a
part). This is not to be confused with a shared interest in a simple correspondence
theory of truth: Wittgenstein and Russell had such a theory (a true proposition
asserted the existence of a complex), whereas Frege declined to say what made a
proposition true. But the very possibility of such a disagreement points to a deeper common ground among these thinkers: if you are interested in investigating the laws of thought, where this means, as for Avicenna above, the laws governing valid movement from assertion to assertion, then you are almost certainly attracted to the idea that these laws can be described, although not justified, without reference to the objects of thought. For that is the very essence of such an approach: our representations are presumed to be governed by laws proper to them—syntactic laws, we might say; and conversely their objects are governed by their laws (Hertz called the first Denknotwendigkeit and the second Naturnotwendigkeit). Valid inference schemata are those which bring the laws of the two domains in harmony, such that we can use the laws of the representations in order to make predictions about the objects. For Frege, logic was concerned only with the laws of the representations: the Begriffsschrift describes "a formal language of pure thought" without ever explaining what makes the thoughts in question true. Russell thought that this topic belonged to logic as well, but he was in full agreement as to the following: the laws of logic have to flow from the natures of the "logical objects," whatever these may turn out to be (thoughts, propositions, senses, functions, truth-functions, etc.). They were agreed, in other words, in their fundamental conviction that the laws of the one valid logic can be stated without saying what the fundamental constituents of reality are. They disagreed only in Russell's desire to give some sort of general semantics. Russell did not want to know exactly what the world was made of—it is irrelevant to his logic whether the objects are atoms or sensations—but he did think that one had an obligation to explain what makes a proposition true. And even here he still agreed with Frege that such questions were "philosophical" and not "logical," i.e. they are discussed in the Introduction to Principia Mathematica, and scarcely at all in the body of the text.
Wittgenstein was affected by both these concerns. He too was committed to the project of specifying the one natural logic, the logic which flows from the nature of thought. He believed, with his two mentors, that the mechanical laws of logic could be—indeed had to be—specified prior to any acquaintance with the fundamental constituents of either thought or reality. But he sided with Russell in wanting to give a semantic explanation of what truth is. This dual concern led him to the sort of account of cognition that was widely disseminated within the German scientific community: a picture-theory of the sort he knew from his reading of Hertz. The attraction of such a theory is in a word the following: it says that significant representations are always formulated against a background of intuitively imaginable (anschaulich vorstellbar) possibilities, against a space of possibilities. That space provides us with a stable semantics—the sensory manifold of each subject—which is, simultaneously, potentially related to an external world, an external physical manifold, of unknown complexity. It is a view which is simultaneously realistic, in that it gives a determinate (internal) semantics, and transcendently idealistic, in its allowing a naturalistic explanation of the connection between the internal manifold and the external one. I scarcely need add that it is a Leibnizean world-view, in the strictest sense.

This was the view to which Wittgenstein was driven, though the problems which led him to adopt the view expressed in the *Tractatus* derive almost exclusively from his reading of Russell and Frege. He did not set out to recast Frege’s and Russell’s work in a mechanical framework; indeed, as we shall see, the notion of logical space is a relatively late development. The theory of the *Tractatus*, although it claims to cover the whole space of significant discourse, is still primarily directed at the problems concerning types, quantification, judgment and inference that Wittgenstein inherited from *Principia Mathematica*, Russell’s *Theory of Knowledge*, the *Grundgesetze* and Frege’s related meta-theoretical essays. A reading of the
Tractatus still stands or falls, in my view, on its ability to relate the text to those topics. Therefore my discussion of the latter in the following chapters is confined to the work of Frege and Russell, for the most part. Once that problem-field has been defined, I relate it in the later chapters to the picture-theories of Hertz and Helmholtz.

The sequence of propositional theories leading up to the Tractatus began with Wittgenstein’s arrival in Cambridge in 1912. During that stay from 1912-14 Wittgenstein was enlisted by Russell in his project of developing a “theory of judgment.” The theory was to replace Frege’s metalogical theory of Sinn and Bedeutung with one relying on denotation only. Thus it was to eliminate Frege’s “third realm” in favour of an ontology in which there are neither senses nor mental objects, but only things. They were soon stymied by a series of difficulties, which centered on the following epistemological question: In the case that an elementary proposition (or judgment) is false, how are we to know what it means? How do we know the denotation of the judgment I-aRb in the case that the fact does not exist? The epistemological problem had a logical correlate, however: if the connection between judgment and (non-existent) fact was not secured, then the difference between “false” and “not-true” could not be explained.

Wittgenstein concluded that the symbol “aRb” must itself guarantee the possibility of the fact aRb. The symbol, he came to say, points to a “logical place,” which is analogous to the “spatial place,” in that “both are the possibility of an existence.” (Tractatus 3.411) The “location” aRb is not, however, an isolated logical place: it is one of a manifold (a Mannigfaltigkeit) of isomorphic “locations,” which consists of all possible facts of the form of the fact aRb (in this case an elementary relational fact). What corresponds to the dimensions of this manifold are the types of the objects that appear in the facts. The manifold itself is represented in the Tractatus by means of an Urbild. To every set of determinate values of the coordinates of a
manifold corresponds a location, and *vice versa*. Likewise to every combination of members of compatible types there corresponds an elementary fact. Quantified propositions select “slices” of manifolds by setting constants in elementary propositional expressions to variables: \( xRb \) selects all elementary facts in the “line” \( \{aRb, bRb, cRb, \ldots \} \). As I said, one can understand Wittgenstein’s reasons for adopting this theory without any reference to space and manifolds, for he developed it independently of any such considerations. However once he made the move to such a geometrical model in 1914, it quickly assumed a dominant role in his interpretation of his own results.

This fact is hardly surprising when one considers that the picture-theories of both Hertz and Helmholtz were geometric at heart: both of them tied the notion of a *Bild* to that of an internal manifold of experience. Helmholtz was a peculiar sort of Kantian: he believed on the one hand in a transcendental structure in which we order our experience; but he denied that this was the Euclidean manifold of spatial intuition. He was open to the possibility that there be an *a priori* component in the structure of our knowledge, but he maintained that it is our obligation, as natural scientists, to investigate and analyze all structures that appear to play such an *a priori* role. In Chapter 2, I show how Helmholtz applied this methodology to the manifold of spatial properties, concluding that while space (indeed Euclidean space) might appear to us as an *a priori* condition of experience, this is only because we ignore those regularities in our perceptions which make fundamental geometrical determinations possible in the first place: we fail to notice that even the concept “measuring-stick” presumes a spatio-temporal regularity in the internal field of perception. In order to formulate his argument precisely, Helmholtz connected his critique to the results of his research in sense-physiology, in particular to his project of analysing *Qualitätskreise*, or quality-fields, as *manifolds*. 
That work, Helmholtz argued, made it possible to regard all human experience as taking place within a “space” determined by the various Empfindungsweisen making up our primitive perceptions. Just as a spatial point can be determined by means of coordinate-values, so colours and aural tones can be determined as lying within spaces of possible values. Hence the perception of colours at locations in space can be given by means of an extended visual-space manifold, as is similarly the case for the other senses. As I just mentioned, Helmholtz did not say that he knew what the fundamental perceptual manifolds were—merely that 1) the sensibilia with which we are consciously acquainted could be organized in this manner, and 2) that we have every reason to believe that there are deeper manifolds, the regularities within which give rise to the manifolds of which we are consciously aware (thus our awareness of the complexity of the colour-manifold has evolved historically, and this has affected our conscious perception of colour today—we are now aware, as Helmholtz would argue, of a complexity which was always there, though not consciously perceived). The colour-manifold itself, Helmholtz argues, is almost certainly the product of regularities within a deeper, neuro-mechanical substrate. Helmholtz had this in mind when he said that our representations were “signs” for external objects: the geometric and mechanical properties of the representations in the internal manifolds were assumed to stand in a functionally symmetric relation to the systems in the external world; furthermore, those external systems cause changes in the internal ones—their functional symmetry can be given a realistic, natural explanation.

Hertz took over the fundamental role of the internal manifold from Helmholtz in writing his The Principles of Mechanics, the work which we know to have influenced Wittgenstein directly. Hertz’s book is divided into two parts: the first is concerned with describing the geometric properties of the internal space of our intuition; the second adds empirical postulates to the geometric definitions of the first book, thus giving it empirical content. The “pictures” of Hertz’s picture-theory are
thus also *pictures situated in a manifold*, in this case a classical Kantian one. The manifold of Hertz's mechanics can be seen, as I argue in Chapter 7, as resting on a Helmholtzian perceptual manifold. The indefinables of Hertz's *Mechanics*—even a concept as simple as "mass-point"—would qualify as defined regularities, or *Gesetzmäßigkeiten*, as Helmholtz called them. Indeed this relationship between the fundamental manifold of physical theory (that described in Hertz's *Mechanics*, in other words) and the fundamental manifolds of perception, is key to understanding the remarks on the relative positions of logic and natural science in the *Tractatus*. The aim of the *Tractatus* is to describe, one might say, the topological properties of *any* space on which the manifold of experience might be constructed. Such a description would define the space of all possible experience—thus a space of all possible significant language, if significant language is taken to be that which refers to what could possibly happen. The fundamental manifold of experience can support only one *elementary* language (one without truth-functions), but many equivalent logics. At the same time, it provides the means for defining other, higher-level manifolds of experience, made up of different types of complex objects. Some of these, those most rigorously defined, will serve as a basis on which we can define the various alternative systems of mechanics. Others will be constructed more loosely, by means of the natural processes of definition and aggregation that make up our everyday thoughts. "Colloquial language," says Wittgenstein *Tractatus* 4.002, "is a part of the human organism, and it is no less complicated than the latter." The surface structure of colloquial language results from a more haphazard compounding of the elements of the fundamental manifold. That compounding is determined by the nature of the organism, and it serves practical, as opposed to logical, purposes. But both natural language and physics rest upon the same possibility space—that is why physics holds in the world to which colloquial language refers.
In Chapters 3-5, I give a detailed analysis of Russell’s judgment-theory, the single most important predecessor to the language-theory of the *Tractatus*. Both Wittgenstein’s and Russell’s early theories represent an attempt to give what Wittgenstein described in 1913 as a “correct theory of the proposition.” Both try to define, in other words, what a proposition is, and why it is true or false. They try to explain, in a manner consonant with the naturalized view of logic I described above, how logic arises from the essential properties of things on the one hand, and judgments on the other. As such, the project undertaken by Russell and Wittgenstein was a direct reaction to Frege’s description of his *Begriffsschrift*. Russell wanted to avail himself of the division Frege had made between symbolism on the one hand, and semantics on the other. It was this conceptual distinction which allowed Frege to define his rules of inference and his truth-functions, as well as to provide a purely syntactic account of his “types.” It has become a commonplace that Frege was much more precise than Russell, that the latter failed to observe the distinction between object- and meta-language which is constitutive of modern logical theory. But Russell gave much time and thought to these issues in the period between 1903, when he wrote the appendix to the *Principles of Mathematics* in which he summarized Frege’s work, and the publication of *Principia Mathematica*. Russell understood and appreciated Frege’s achievement; however he thought that it all rested on assumptions which, although admissible on purely logical grounds, were philosophically unacceptable. Russell believed that Frege’s system was only workable if one was willing to admit mental entities (that was how he interpreted Frege’s third Reich) into one’s ontology, and to this he was adamantly opposed.

Russell’s judgment-theory soon ran into snags, however: it sought to remove entities corresponding to propositions by analyzing them as lists of components, or “classes of names,” as Wittgenstein later put it. Wittgenstein’s reaction to this and related problems resulted in his adopting a picture-theory of the sort he knew from
Hertz: Frege's senses cannot be dispensed with, however, we can consider the propositional symbols themselves to embody the information Frege ascribed to them. Wittgenstein thus restored the proposition (which Russell had sought to eliminate as an "incomplete symbol") as an independently existing structure, without, however, assuming a sense above and beyond that embodied in the propositional sign itself. The exact articulation of this theory is one of the two principle concerns of the *Tractatus*, the second being the related account of "molecular," *i.e.* truth-functional propositions, which I discuss in Chapter 6.

Chapters 2 and 6-7 concern themselves with the relation between the mature *Tractatus* theory and its antecedents in the work of Hertz and Helmholtz. I introduce these topics in Chapter 2 for purely expository reasons: the connections between Helmholtz and Wittgenstein are quite striking, and it is, I believe, helpful to have some idea of these before working through the often quite technical discussion of Russell and Frege. The exact significance of these connections can, however, only be seen once both accounts are in place--only then is the analogy between a type and a dimension, for example, fully comprehensible. The treatment of Helmholtz in Chapter 2 is fairly comprehensive, so I will not discuss him in greater detail here. This applies, unfortunately and of necessity, for a good portion of what follows: the account of Russell's theory of judgment, of Hertz's mechanics and of the Helmholtzian background are (regrettably) all new to this discussion, and they are thus in good measure exegetical. This may serve to obscure the larger point of the dissertation, which is to show how theories from two distinct domains--physics and logic--converged in a single theory of language. To make that discussion worthwhile I have to delve into detail, and the details in question have not--they being of a grey-rags-and-dust sort--occasioned much discussion in the *Tractatus* literature. Thus I ask for the reader's forbearance as I work through that material.
My reading is based on the new critical edition of the *Tractatus*.² All translations from this and other German texts are my own, except where otherwise noted. I generally place greater emphasis on terms relating to geometry and mechanics than do previous translators of Wittgenstein—the translations are glosses, and are not intended to be idiomatic. Translations from the *Tractatus* and the *Notebooks* are footnoted, owing to their number. Translations of other authors are included in the body text.

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Chapter 2 - Helmholtz's Perceptual Manifold

In this chapter I will describe Hermann von Helmholtz's manifold theory of perception, showing first its connections to the German psychophysical tradition, and, in conclusion, the influence it had on Heinrich Hertz's Principles of Mechanics, thus on the latter's "picture-theory" as well. My choice of Helmholtz as a central figure requires some justification: Wittgenstein never mentions Helmholtz in the Tractatus, even though he refers several times to Helmholtz's student Hertz; thus although the apparent connections between his work and Wittgenstein are striking, it may well remain impossible to prove a direct connection between the two writers. To that difficulty we can, however, oppose the following considerations: Helmholtz was from 1848 to 1871 Professor of Physiology in Berlin, Bonn and Heidelberg; from 1871 to his death in 1894, he was Professor of Physics in Berlin. During this period he published on topics ranging from frog-muscle physiology to mathematical physics: his research covered just about every area of contemporary natural science, indeed he is probably the last "renaissance" figure to have lived. His longevity and prolixity meant that he exerted tremendous influence within and without Germany, among his scientific colleagues as well as among the Bildungsbürger. Most important, however, his various investigations were unified around a single world-view--a species of naturalized Kantianism that he had learned in part from Müller and Herbart:

Meanwhile Herbart, who rejected post-Kantian philosophy, laid the foundations for a new beginning. Kant, he said, taught us that Space and Time are in us, but he did not say how they arise in us, he did not explain this fact. Out of his work arises a "Dualism" which ignores the psychological problem that lies at the root of Erkenntnistradition. But science may not have a dualist conception of nature. The modern talk of simple ideas stems from Kant. Uncountably many ontological, metaphysical consequences rest on an assumed simplicity of perceptual acts or intuitions, whereas on the other hand psychological science assumes that nothing is in consciousness that is not infinitely complex. Helmholtz regarded this analysis of the simplest perceptual acts into more elementary processes, which cannot be expressed with words, an analysis whose existence was unknown in Kant's time and of
which philosophy must remain ignorant so long as it occupies itself only with verbal or logical knowledge, as the greatest advance of modern times.¹

In his project of analyzing these "elementary processes" by means of physiological investigations, Helmholtz allied himself, at least in part, with the "psychophysical" work of Weber, Fechner, and their followers. But whereas they had concerned themselves with describing the mathematical relationships between the scalar values of stimulus and sensation (*Reiz und Empfindung*), Helmholtz was inspired in the 1860's and 70's to extend this project to *manifolds* of stimuli: Helmholtz is the first person to propose a *spatialization* of the entire field of primitive perceptions. As a result, much of his work seems to have been incorporated into the psychophysical programme, despite his disagreements with the psychophysicists on several points. Helmholtz did not pursue these physiological investigations merely for their own sake, however. He used the picture of the human subject that they implied as an epistemological basis on which to interpret the content of physical and mathematical theories in general. He investigated the physical properties of the world as a physicist, the physiological properties of the human body as physiologist, and the interface between the two—in Hertz’s terminology, the "no-man’s-land"—as an epistemologist.

Helmholtz’s influence was, in other words, universal. Although I cannot prove that Wittgenstein did in fact read his "The Facts in Perception" ("Die Tatsachen in der Wahrnehmung"), the essay I discuss at length in what follows, it is scarcely questionable that he grew up in an intellectual milieu suffused with Helmholtz’s views, and that these left an unmistakable mark on the *Tractatus*, even if only mediately. By analyzing Helmholtz’s formulation of these views, I seek on the one hand to make this context clear, and on the other hand to show how and why Helmholtz arrived at the epistemological position he did. If this analysis helps us to

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understand the background against which the *Tractatus* was written, and if it clarifies the doctrines of that text, then it will have served its purpose.

The essential characteristics of this view are the following:

1) It has a realist and an idealist interpretation, the first of which corresponds to the objective perspective of an observer, the second to the solipsistic perspective of a transcendental subject. Thus it is a monistic world-view with two interpretations, as I explain below.

2) It distinguishes between internal representations, which are “signs” (*Zeichen*) of external objects or systems, and the external entities themselves, whose actual nature is unknown.

3) It accords no particular significance to our being conscious of internal representations. Consequently it extends the notion of *thinking* to cover both conscious and unconscious, high- and low-level cognitive processes.

4) It uses an expanded notion of *Anschauung* to explain the field of “possible signs” that may make up our experience. This field is not necessarily given *a priori* (it may be learned), but it becomes fixed during our development in such a way that it later appears to us as being so (thus its structure is a function of our early environment). To conceive of something is to conceive of all the possible representations which that thing might bring about in us. Colours and sounds may be conceived as *manifolds* and can thus be spatialized: we can think of them as forming a space of possible primitive sense-data.

We can schematize this model as follows:
I said in point 1) that this picture allows of two interpretations, two kinds of monism. The first is the realist monism of the natural scientist (to which Gehlhaar alludes in the quotation above). It says that the mind in the above diagram is not distinct from the world, but part of it: it is one physical system embedded within another. From this realistic perspective, the system in question is a physical one, and the science which investigates it is that of sense-physiology or psychophysics: it investigates the mental life of the organism from the standpoint of physics.2 The connections between organism and environment are, however, structured: the perceptual organs and the nervous system of the organism determine how external causes map onto internal states; the boundary layer of nervous system and world (the classic instance is the retina, the "little net," which is an outgrowth of the optic nerve) determines how external states in the world get represented in terms of internal states. The external systems are always *transduced* by this boundary layer: on the one hand the internal signs capture real aspects of and relations among the external entities which are their causes, they translate the external causes into internal signs; on the other hand the sense organs, in transducing, say, air vibrations into motions of bones in the inner ear and, in turn, into nerve-impulses, not only alter the medium of representation, but also discard many aspects and properties of these external systems. The inner systems are only ever partial copies of the external ones, in that they faithfully replicate some, but by no means all aspects of the systems they represent.

But the realist monistic interpretation leaves room for a complementary, solipsistic one. Having given a realistic description of the perceiving and thinking organism, we can now ask how the world would appear from the point of view of such a subject. On the realistic picture, the threshold of the subject's consciousness—the lowest-level class of cognitive acts of which the subject is aware—is of no

2This is not a new idea: psychology was in the scholastic tradition a branch of physics.
particular ontological significance. From the point of view of the subject, however, this threshold will be the limit of experiential reality. We think of colours as primitive perceptual acts (or aspects of such) while at the same time acknowledging the existence of physiological processes of which we are unaware, which bring about the subjective experience of colour. But a given consciousness has no internal evidence for the existence and operation of these processes. It will experience colours as fundamental constituents of external reality; whereby "external" is no longer to be understood as "outside the internal world," but, instead, as "situated in space and independent of my will," in which definition we make no appeal to entities beyond the pale of experience. For this consciousness too there is only one world, although it is divided into portions called "external" and "internal." This consciousness can hypothesize an outside world whose laws are the causes of the regularities in the internal perceptual manifold. But all such hypotheses and theories will get their plausibility and validity only to the extent that they connect to the field of possible experience within which any conceivable entity must make itself known.

Many of Helmholtz's contemporaries adhered to some portion of the views discussed above. Specific to Helmholtz is the role of space and manifolds as the net or field within which experience plays out. Thus it may seem puzzling that Helmholtz cast them in that role not because he thought these manifolds were given a priori—not that is because he wanted to widen the field of a priori geometric truths to include, so to speak, chromatometric truth—but because he wanted to shrink this field as much as possible. He rejected the doctrine of psychophysical parallelism, which would account for the a prioricity of geometry by pointing to a pre-established harmony between mind and world, and insisted instead that the structure of the internal manifolds was a product of our early experiences. Thus the title to the second appendix to "The Facts in Perception" reads: "Space can be transcendental, without the axioms being so." Some aspects of the way we organize experience must be
given—that portion which is built into our physical natures—but we don’t know at what level these aspects are to be found. And space, colour and other primitive sensibilia are almost certainly not among those aspects: they are themselves merely ways of picturing or sensing (Empfindungsweisen, Helmholtz calls them), and high-level ones at that. Helmholtz postulates the internal manifold as a way of explaining just this contradiction: How can something (geometry) appear to us as an unshakable condition of all possible experience, and yet at the same time be contingent? If the doctrine of psychophysical parallelism were true, then geometry would appear transcendently true because it was doubly true: true of the external world, and true of the internal one as well. On Helmholtz’s picture it works the other way round: there are some unknown fundamental constraints on how we perceive, and these do yield us a primitive network of experience; we construct higher-level manifolds by observing regularities and relationships within the lowest-level sense-data, which manifolds we concretize at an early age. We then mistakenly take the properties of these manifolds to be necessary properties of both mind and world. The contingent properties of these manifolds go transcendental on us, without our being aware that this has happened. But these transcendental properties are by no means necessary. They are the deductively necessary consequence of premisses arrived at inductively.  

The notion of a manifold, which Helmholtz derived in part from Riemann, plays a critical role in this theory, because it suggests a way of assimilating spatial qualities to colours and other sensibilia, while at the same time spatializing those latter qualities. In his 1891 “Attempt at an Expanded Application of Fechner's Law in the Colour-system,” Helmholtz argues that the relation between stimuli and sensations known as Fechner’s Law can be made to apply in multi-dimensional sensation-...
manifolds such as colour. Fechner’s law states that the ratio of the intensities of two sensations equals the ratio of the logarithms of the two stimuli causing them:

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Die Gesamtheit der von unserem Auge empfundenen Farben ist nach Riemanns Ausdruck eine dreifache Mannigfaltigkeit, d. h. jede einzelne Farbe kann durch drei unabhängig veränderliche Größen bestimmt werden, und nicht durch weniger. Die Möglichkeit, das System der Farben durch räumliche oder flächenhafte Anordnungen übersehbar zu machen, beruht auf diesem Verhältnis, da auch ein Ort im Raume drei Variable zu seiner Festlegung verlangt. ... 


The totality of the colours perceived with the eye is, using Riemann’s expression, a three-fold manifold, i.e. every particular colour can be determined by means of three independently variable magnitudes, and not with fewer. The possibility of making the system of colour perspicuous by means of spatial or planar arrangements (Anordnungen) rests on this relationship, for a point in space demands three variables for its specification. ...

The measurements given up until now by E.H. Weber, Fechner and their followers relate, as far as I know, exclusively to changes that proceed in a single direction [i.e. to changes in a one-dimensional manifold]. The domain of colour perception offers us the opportunity to make such studies of a manifold extending in three dimensions.5

The notion under investigation is that of “smallest perceptible difference.” Helmholtz proposes to transfer this from a scalar definition to a spatial one: smallest difference in the colour field can be defined on analogy to smallest perceivable displacement; Helmholtz is looking for the minimum visible colour-differential. This passage is of interest for three reasons, the last two of which I will only touch on for the moment:

1) It makes clear what sort of truths would correspond to geometric axioms in the case of colours: relations of brightness, saturation and hue between the points of a colour-manifold appear as a priori truths about colours. When describing this paper in “Shortest Lines in the Colour-system,” Helmholtz remarks that the formula for shortest distance at which he arrives “would

4The Decibel is an example of a unit defined on the basis of Fechner’s Law. Interestingly enough, Fechner explicitly relates his law to the Law of Diminishing Return.

play the same role for the domain of colour-sensation as the formula for the length of a line element in geometry." And, as Helmholtz never tired of observing, it is the latter formula which determines the metric of space, hence the geometry of the physical (as opposed to the pure intuitive) space of our experience.

2) Helmholtz wrote the paper three years before his death in 1894, meaning that it was published sixteen years after "The Facts in Perception," at the very end of his career. Thus the view it describes is one we can assume him to have held throughout the second phase of his career, as Professor of Physics in Berlin.

3) Wittgenstein in 1929 is much vexed with the question of whether or not there can be a "geometry" of the visual field, whether there is such a thing as a "metric of colour." 7 I will not deal with the last point until the Conclusion of the thesis, where I argue that Wittgenstein's investigation of the geometry of the visual field--the topic with which he begins his 1929 notebooks on his return to Cambridge--is simply a regression to the origin of the view in the *Tractatus*: logical space did not get reinterpreted as a perceptual space, rather Helmholtz's doctrines, perhaps filtered through Hertz and later, psychophysical authors, are the source of the *Tractatus* 's logical space. The second point is of historical interest only, but it is important as regards much of my treatment: if the views I ascribe to Helmholtz were isolated oddities of his work in 1878, then it would be hard to see how they could have the importance I ascribe to them. If on the other hand they represent a position he held for the entire period in Berlin, a period in which his influence on the scientific community both at home and abroad was unsurpassed, then we would indeed expect these thoughts to surface in the work of many others (particularly in those of someone who, during his two years at the Technische Hochschule in Berlin, roomed with a professor of projective geometry). The first point is of importance in so far as it shows that the interpretation of

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"The Facts in Perception" and its appendices that I give here is correct: in those papers Helmholtz was not urging an analogy between space proper and quality spaces; he wanted to fuse them, quite literally. He thought, as Wittgenstein said in 1929, that "Space and colour saturate one another. And the way in which they penetrate one another makes the visual field."\(^8\) There are, for Helmholtz, no spatial intuitions disconnected from sensibilia, the colour-space is not an add-on, merely resembling the true Kantian spatial manifold: there is a single manifold of many dimensions (there are other ones for the different senses), and it is not possible to talk about any occurrence within this space without involving its entire Mannigfaltigkeit. This I will now explain.

a) "The Facts in Perception"

The two papers I discuss have an intricate publication history. The one is Helmholtz's lecture "Die Tatsachen in der Wahrnehmung," a Rektoratsrede held at the Friedrich-Wilhelms-Universität in Berlin on 3 August 1878. That paper was calculated, as he told his wife, to make even his colleagues "uncomfortable," for "it contained new thoughts ... and it is after all always better that they think me too educated than trivial."\(^9\) The second paper took its final form as the third technical appendix to the talk when it was published. The appendix was not, however, a simple addendum to the talk, for it had been published in a longer form and in English, in Mind in April 1878. In fact the appendix is the technical basis on which the talk rests: the talk presents an onto- and epistemological theory revolving around the manifold analysis of perception; the appendix sets this theory out in greater detail, shows how it may be realistically or idealistically interpreted, and, most important, how it overturns

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the thesis that the axioms of Euclidean geometry are given *a priori*. This last concern drove the whole project: the appendix originally appeared as a reply to one Professor Land of Leyden,¹⁰ who had objected to Helmholtz's contention in his “The Origin and Meaning of the Geometrical Axioms”¹¹ that the question of whether space was Euclidean or not was an empirical one. Land argued that the notion of “imaginability” used by Helmholtz in his “The Origin and Meaning” was “stretched far beyond what Kantians and others understand by the word.”¹² Helmholtz responded with a definition of “intuitively imaginable” (*anschaulich vorstellbar*), which not only leaves room for the possibility that our experience of space may be non-Euclidean, but also undermines the utility of any geometry that does not rest on empirical axioms--axioms derived inductively from measurements with rigid bodies. The theory of perceptual manifolds gives us a way of defining “rigid body” without making appeal to objects outside of our consciousness: a rigid body is an aggregate of primitive perceptions that we have--for whatever reason--designated as rigid, by means of which we have imposed a metric on the internal space.

The talk and appendix were influential: the concept of physical geometry is one essential part of Einstein's arguments in favour of relativity; the manifold theory of perception diffused into the work of later psychophysicists; the sign-theory and the connected doctrines of inner and outer worlds is the source of the fundamental doctrines of Hertz's *The Principles of Mechanics*; and, as I argue in this thesis, these ideas determined to a great extent, whether mediately or immediately, the ontology and epistemology of Wittgenstein's *Tractatus*. My discussion is naturally skewed


¹² Land, “Kant's Space and Modern Mathematics,” p. 41.
toward the latter connection, and so I begin by looking at Helmholtz's notion of *substance* in order to see how he distinguished between *objects*, of which we are consciously aware, and of which we make "finished pictures," and the elements of primitive sensations, which are not objects in any conventional sense, and which do not, as a consequence, get depicted (abgebildet). These elements are, in essence, coordinate values. As Wittgenstein later put it: "That a point in the plane can be represented by means of a pair of numbers ... shows that the represented object is not the point, but the point-fabric (Punktgewebe)."13 The components of our primitive sensations *make* our internal manifold, but they are not *objects* in the conventional sense of that word: they are like addresses in that space or point-fabric. What we experience as macroscopic objects are regularities, *Gesetzmäßigkeiten*, in that space:

> Weiβ wir aber unzweideutig und als Tatsache ohne hypothetische Unterschiebung finden können, ist das Gesetzliche in der Erscheinung. ... Alles, was in der Anschauung zu dem rohen Materiale der Empfindungen hinzukommt, kann in Denken aufgelöst werden, wenn wir den Begriff des Denkens so erweitert nehmen, wie es oben geschehen ist.

> Denn wenn "begreifen" heißt: *Begriffe* bilden, und wir im Begriff einer Klasse von Objekten zusammensuchen und zusammenfassen, was sie von gleichen Merkmalen an sich tragen: so ergibt sich ganz analog, daß der Begriff einer in der Zeit wechselnden Reihe von Erscheinungen das zusammenfassen suchen muß, was in allen ihren Stadien gleich bleibt. ... Wir nennen, was ohne Abhängigkeit von Anderem gleich bleibt in allem Wechsel der Zeit: die *Substanz*; wir nennen das gleichbleibende Verhältnis zwischen veränderlichen Größen: das sie verbindende *Gesetz*. Was wir direkt wahrnehmen, ist nur das letztere.

> But that which we may unambiguously, and may without any hypothetical supposition take to be a fact, is regularity in appearances ... Everything added in intuition to the raw materials of sensation can be resolved in thought, if we take the notion of thought to be so expanded as it was above.

> For if "conceptualising" means: forming *concepts*, and we in this conceptualising seek out and group together in a class of objects, those things which in themselves possess the same characteristics: so it follows analogously that the concept must seek to unify that which remains the same at all stages of a sequence of appearances changing in time .... We call that which remains the same in a change in time without any dependence on other things: *substance*; we call the constant relation between variable quantities:

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the law binding them together. That which we perceive directly is only the latter.\textsuperscript{14}

Helmholtz goes on to explain that entities such as light and heat were previously designated as substances in this sense, whereas we now know them to be species of motion—regularities in some more primitive substrate. Since his definition of thought includes sensory operations of the lowest order, there is no essential obstacle to our repeatedly applying our high-level scientific reasoning to the problem of decomposing the low-level cognitive operations which yield us the \textit{apparent} data of our perceptions: "Even the most elementary representations contain in themselves thinking, and proceed according to the laws of thought."\textsuperscript{15} Aristotle analyzed seeming substances like sugar cubes into unifications of qualities such as \textit{whiteness}, \textit{sweetness} and \textit{hardness}, and he gave these qualities a real correlate: the sensible form. From Helmholtz's point of view, he wrongly hypostatized the qualities in doing so: he confused potentialities of our sense organs with real entities. Helmholtz sees this progression, in which substance gets analyzed into temporary connections of, or regularities within, deeper substances, as ongoing: in analyzing colours and sounds into regularities of motion in the ether and in the air, he is simply carrying the project one step further. We cannot know in advance how deep this analysis may go, nor what the fundamental constituents, the fundamental manifolds, might look like. Nonetheless, at some point our nervous system makes contact with the world. It yields us signs (\textit{Zeichen}) of external objects. Helmholtz of course restricts his discussion to those signs of which he had a description: colours, tones and tactilia. These are differentiated from each other 1) as \textit{modalities}\textsuperscript{16} of perception, each


\textsuperscript{15}"Schon die ersten elementaren Vorstellungen enthalten also in sich ein Denken und gehen nach den Gesetzen des Denkens vor sich." Helmholtz, "Die Tatsachen in der Wahrnehmung," p. 240.

\textsuperscript{16}Helmholtz, "Die Tatsachen in der Wahrnehmung," p. 219.
modality corresponding to a species of sensation—hearing, sight, etc., and within each modality, 2) by having some value within a scale of possibilities. We are not usually conscious of these primitive perceptions, and certainly not of the conjunctions which make up even a simple perceptual judgment such as “that surface is hot.”

Helmholtz generally refers to sensory experience of the deepest sort as the Empfindung of Empfindungswesen. Among these he mentions colours, primitive tones, but also spatial locations—hence the quarrel with the Kantians; but he is clearly open to the possibility that these too may themselves be the objects of further analysis. Such analysis would, however, go beyond the domain of psychology, to the neuro-mechanical level, of which we are never conscious. Wahrnehmung, with its flavour of conscious perception, is of Tatsachen, and these are only ever regularities among Empfindungen. Even when he fails to observe this terminological distinction, Helmholtz is clear that our traditional conception of object is ill-suited to the elements of our fundamental perceptions:

Wir haben in unserer Sprache eine sehr glückliche Bezeichnung für dieses, was hinter dem Wechsel der Erscheinungen stehend auf uns einwirkt, nämlich: “das Wirkliche.” Hierin ist nur das Wirken ausgesagt; es fehlt die Nebenbeziehung auf das Bestehen als Substanz, welche der Begriff des Reellen, d.h. des Sachlichen, einschließt. In den Begriff des Objektiven andererseits schiebt sich meist der Begriff des fertigen Bildes eines Gegenstandes ein, welcher nicht auf die ursprünglichen Wahrnehmungen paßt.

We have in our language a fortunate expression for that which constantly affects us from behind the alternation of appearance, namely: “the actual.” This expresses only the action; missing is the concomitant relation to existing-as-substance, which the concept of the real, that is of the factual, includes. In the notion of the objective, on the other hand, there intrudes generally a notion of “the finished picture of an object,” which does not fit the most fundamental perceptions.  

The “objects” of our most fundamental sensations do not differ from the objects of our everyday language because of some absolute distinction between perception and cognition. Perceptions are symbols for Helmholtz, and thinking spans all

psychological activities. The concept “object” does not fit the elements of primitive perceptions because it invokes an entity, a real correlate of a representation, to which we ascribe attributes. The object unifies the attributes, and is thus conceived of as a thing existing independently of our perceptions, a thing in itself. We are not inclined to talk that way about colours, spatial coordinates or aural pitches.

b) “The Application of the Axioms to the Physical World”

So much for the “popular” part of Helmholtz’s talk. The paper in Mind, later the third appendix to the talk, takes this line of reasoning considerably further. In “The Facts of Perception,” the comparison of modalities of perceptions (colours, tones) to spaces has a familiar ring: we all know about musical scales and colour wheels. But Helmholtz runs his comparison both ways, and it is the second bit which is radical. Kantian spatial intuition, like the Aristotelian common sense it replaced, is a special mode of perception: in space objects are individuated and external; other external perceptible qualities attach to these things only as they are spatially situated. For Aristotle, the common sense unified the senses and was at the same time present in them all, and likewise for Kant, space and time are a priori conditions of all human perception (although other kinds of pure intuition are logically conceivable). It seems natural for us to think of space as a vessel existing prior to its contents. The contents must have a cause for being where they are; however, we do not conceive of the space itself, or consequently the geometric properties we may ascribe to it, as requiring a cause:

Eben deshalb sind Schopenhauer und viele Anhänger von Kant zu der unrichtigen Folgerung gekommen, daß in unseren Wahrnehmungen räumlicher Verhältnisse überhaupt kein realer Inhalt ist, daß der Raum und seine Verhältnisse nur transzendentaler Schein seien, ohne daß irgend etwas Wirkliches ihnen entspreche.

For just this reason Schopenhauer and many followers of Kant came to the incorrect conclusion that there is absolutely no real content in our
perceptions of spatial relations, that space and its relations are only a transcendental appearance, to which nothing actually corresponds\(^1\)

But colours are, Helmholtz has argued, arranged in a manifold of possibilities, and we do not have visual perceptions without their being coloured. Therefore it makes just as much sense to say that—at least within the visual field—colours are an \textit{a priori} condition for spatial perceptions as it does to say the converse. The spatial location of an object may be fixed by means of some arbitrary set of three independent coordinates, just as its colour may be fixed by means of its coordinates in a colour chart:


But we are in any case as justified in bringing such considerations to bear on our spatial perceptions as on other sensible symbols\(^2\) \textit{e.g.} colours. Blue is only a way of sensing (\textit{eine Empfindungsweise}); but that we see blue at a particular time and in a particular direction must have a real ground. If at another time we see there red, so must this real ground have changed.\(^3\)

As far as individual elementary perceptions are concerned, spatial coordinates are no more and no less properties than colour determinations; however, conceived of in this way, the \textit{thing} as bearer of properties becomes superfluous. If a fundamental visual perception is taken to be of the form “Blue at location \(x, y, z\)”, then we are just as entitled to regard \((x, y, z)\) as \textit{predicates of} Blue as we are to think of the point \((x, y, z)\) as \textit{having the predicate} Blue. And it is all the more natural to do so once we have admitted that Blue itself belongs to a \textit{Qualitätskreis}: a more neutral expression of the fact would read \((\text{Blue}, x, y, z)\), where the first slot of the expression contained a colour


\(^{19}\)\textit{cf.} \textit{Tractatus} 3.32, "Das Zeichen ist das sinnlich Wahrnehmbare am Symbol."

\(^{20}\)Helmholtz, "Die Anwendbarkeit der Axiomen auf die Physische Welt," p. 403. The last sentence has \textit{einer} instead of \textit{anderer}. I have emended it in conformity with the German text of the reply to Land cited in footnote 22 below.
coordinate, and the last three spatial coordinates. If anything corresponds to the
notion of object here, it is the determinate values of the coordinates; but they do not,
considered individually, look much like pictures of independent entities. That notion
forces itself on us only when we begin to assemble such sensations into complex
perceptions, by means of shared elements. I stress this point partly because of its
connection to the Tractatus terms Sachverhalt and Tatsache, the first being an
elementary fact, the second a group of the latter (remembering that an elementary fact
is always a "positive" fact), and partly to highlight Helmholtz's fusion of the notions
of space and quality. On the one hand, qualities are arranged in manifolds, they are
determinations of concepts, or modalities of perception; on the other hand, spatial
properties (locations) are treated as contingent predicates, or merely concomitants, of
these other modalities. There are no bearers of any of these properties--spatial,
chromatic, acoustic--beyond their elementary concatenations: [Blue, 2, 3] is perhaps
the bearer of Blue, 2, and 3; but none of the latter has an obvious claim to being
treated as a thing in itself.

It is worth mentioning that Helmholtz, like Kant and, indeed, most philo-
sophers, shows no interest in taste and smell, and, as far as I am aware, conducted no
physiological research on these sense faculties. These two sensibilia have always been
regarded as devoid of conceptual content: when Kant sought, in the third critique, to
devalue instrumental music, the worst he could think of was to compare it to perfume.
Helmholtz, living in the century of Schopenhauer, would never dream of such a thing;
however it is not a trivial objection to his undertaking that he takes no interest in the
other two faculties. Helmholtz's manifold theory makes little sense when applied
there, and this makes the general validity of the enterprise seem doubtful, and least in
the aggressively philosophical form it takes here. For one must remember that Helm-
holtz was above all a natural scientist. In his publications on optics and acoustics he
described the composition of colours and sounds as physical systems (waves and their
properties) and, simultaneously, he carried out physiological investigations of the structure of the retina and of the cochlea, as well as of the optical and mechanical properties of these structures. When he talks of internal signs and pictures, saying that they depict functionally systems in the external world, he is referring to the pathways described in this work. A vibrating string is a physical system with mechanical properties we can describe mathematically. The string sets air in motion, and the pressure waves in the air map some, although not all, properties of the motion of the string. The bones in the inner ear replicate these motions yet again, they set fluid in the cochlea in motion, and its spiral structure decomposes the waves in the liquid (essentially performing a Fourier transform) into their component frequencies. But Helmholtz and the psychophysicists had no equivalent theories for taste and smell; the suggestion that they too can be analyzed in terms of Qualitätskreise or perceptual manifolds is purely speculative. Since Helmholtz gave up physiological research after his move to Berlin, that is immediately after the publication of "The Facts in Perception," he never put this unified manifold theory to the test. Except for a sequence of papers on the colour metric in the early 1890’s (from which I quoted above), he did not develop it further as an empirical theory, using it instead as an erkenntnistheoretisches model. This suspicion of taste and smell is, I might add, as classic a philosophical prejudice as it is a false one, as Proust so well knew.

The assimilation of qualities and spatial locations is not intended to be metaphorical. The entire point of Helmholtz’s appendix is to refute the claim that the axioms of Euclidean geometry, being statements about the conditions of experience, are not empirical judgments, and thus cannot be subject to experimental verification. As such a claim would entail that non-Euclidean geometry be unimaginable (anschaulich unvorstellbar), Helmholtz argues first that he has no better criterion for the imaginability of an unknown object than that we be able to specify in advance all possible sense-impressions such an object would evoke under all possible conditions.
of observation; and, second, that Beltrami had already specified a way of mapping pseudo-spherical space onto portions of Euclidean space, thus that "this demand can be met for objects in spherical and pseudo-spherical spaces." The question as to which geometry is the geometry of our experience can be given empirical content, and it is therefore also subject to verification.

This criterion of conceivability is tightly coupled to the will of the Helmholtzian observer: we learn at an early age to distinguish sensations that result from our acts of will from those which simply obtrude on us, and this means that we come in short order to the realisation that we are able, by transporting our bodies and members to different locations, to bring about the presence of various "presentabilia." Even when we are not in their presence (or they not in ours) we can bring it about, through acts of will, that they become present to us again:

Nennen wir die ganze Gruppe von Empfindungsaggregaten, welche während der besprochenen Zeitperiode durch eine gewisse bestimmte und begrenzte Gruppe von Willensimpulsen herbeizuführen sind, die zeitweiligen Präsentabilien, dagegen präsent dasjenige Empfindungsaggregat aus dieser Gruppe, was gerade zur Perzeption kommt: so ist unser Beobachter zur Zeit an einen gewissen Kreis von Präsentabilien gebunden, aus dem er aber jedes Einzelne in jedem ihm beliebigen Augenblick durch Ausführung der betreffenden Bewegung präsent machen kann. Dadurch erscheint ihm jedes Einzelne aus dieser Gruppe der Präsentabilien als bestehend in jedem Augenblick dieser Zeitperiode.

Let us call the entire group of sensation-aggregates that may be brought about through a certain definite and finite group of will-impulses the temporary presentabilia, and consequently present, that sensation-aggregate from this group which is just coming to be perceived: so our observer is tied at the instant to a definite set of presentabilia, each of which he may make present by carrying out the appropriate motion in each and every moment he chooses. Thus each one of this group of presentabilia appears to him as obtaining in each moment of this period of time.

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The set of \textit{presentabilia} corresponds to a physical location, fixed by our awareness that we have not in the immediate past willed any motions, and the sensation-aggregate \textit{present} to us is the one on which we have just fixed our attention: the fact that we can choose at will which sensation-aggregate among the current presentabilia is present, means that we have the impression that \textit{all} are present during this period of time. So Helmholtz’s definition of \\textit{imaginability} appeals to just this capacity. We can imagine something if we can imagine all possible effects that it would have on our sense organs from all possible perspectives, \textit{i.e.} from all possible sets of presentabilia into whose immediate presence we can will ourselves.

Helmholtz’s argument against the transcendental nature of the axioms of Euclidean geometry depends on this ability of the observer to move about, and to be conscious of doing so. If neither I nor any objects in the room in which I find myself are in motion, then there is no way for me to establish \textit{physical} congruence between the objects in my visual or tactile field. If I am aware that I am moving, I will notice the curvature of space in the fact that objects which I believed to be at rest appear to expand and contract. And if I can manipulate the objects directly, I will note that \textit{perceived} congruence (between, say, the visual impressions of two rulers) need not agree with \textit{physical} congruence (the successful juxtaposition of the rulers). It would be possible to establish a deductive geometry based on generalisations of such actual measurements, a geometry whose statements always referred to the relationship between bodies being moved in space and in time. Such a geometry would be a \textit{physical geometry}, not a pure \textit{intuitive geometry}, whatever the latter may be. The only geometry which matters as far as our experience is concerned is the physical one. Obviously a geometry whose axioms are established empirically need not fear the question of why it applies to experience. It is instead intuitive geometry which needs to justify its existence.
Helmholtz had always taken the view that our intuitions of spatial relationships were learned and not native to human consciousness. While he admitted the latter as a possible hypothesis, he found it intuitively objectionable, and the unexpected developments in non-Euclidean geometry offered him a baculine argument of considerable weight. But it was not much more than that, above all because Darwinian theory had offered up a plausible explanation for the assumption of that “pre-established harmony” between mind and world which Helmholtz had so regularly lampooned. In these later works, we see Helmholtz adopting a far more conservative position: the axioms of Euclidean geometry would appear to be transcendental if our minds had a peculiar two-fold relationship to the objects of experience, but the question of whether or not there is such a relationship is still one which both can and must be tested. The relationship would be of the following sort: the causes responsible for the appearance of particular objects at particular locations would act 1) mediately on our consciousness, in other words temporally and in conjunction with sensible qualities, and 2) immediately as well, such that direct measurement (a process depending on both time and distinguishable qualities) was unnecessary. Helmholtz defines his terms in such a way that the argument need be interpreted neither realistically nor idealistically, although it is clear that he would take the former approach if forced to adopt a transcendental position. Either way, geometric propositions would have the peculiar characteristic of expressing time-independent regularities between all possible experiences. They would have that property because they described correlations between elements of our most fundamental perceptions, and they would thus appear to be conditions of these concrete and temporal experiences; but their necessity would not draw on a deeper

source than their independence of physical measurement, that is, from bodies moving in time and space—and those sorts of truths might well be found to hold for other Qualitätskreise of our most basic experiences, those of colours, pitches, etc.

Helmholtz makes one fundamental assumption in this argument: that our representations come to be according to some sort of fixed laws, and that from a diversity in these representations we may infer a diversity in the conditions giving rise to them. The causes of the perceptions in consciousness he terms moments, and he divides them along these lines:

Nun finden wir als Tatsache des Bewußtseins, daß wir Objekte wahrzunehmen glauben, die sich an bestimmten Orten im Raum befinden. Daß ein Objekt an einem bestimmten besonderen Orte erscheint und nicht an einem anderen, wird abhängen müssen von der Art der realen Bedingungen, welche die Vorstellung hervorrufen. Wir müssen schließen, daß andere reale Bedingungen vorhanden sein müssen, um zu bewirken, daß die Wahrnehmung eines anderen Ortes des gleichen Objekts eintrete. Es müssen also in dem Realen irgend welche Verhältnisse oder Komplexe von Verhältnissen bestehen, welche bestimmen, an welchem Ort im Raume uns ein Objekt erscheint. Ich will diese, um sie kurz zu bezeichnen, topogene Momente nennen. Von ihrer Natur wissen wir nichts, wir wissen nur, daß das Zustandekommen räumlich verschiedener Wahrnehmungen eine Verschiedenheit der topogenen Momente voraussetzt.


Now we find as a fact of consciousness that we believe ourselves to perceive objects which find themselves at particular locations in space. That an object appears at a particular definite location and not at another will have to depend on the kind of real conditions bringing about the representation. We must conclude that other real conditions would have to have been at hand in order to bring about the perception of the same object at a different location. There must therefore be in the Real some sort of relationship, or complex of relationships, which determine the location in space at which an object appears to us. I want to call these, in order to refer to them economically, topogeneous moments. We know nothing of their nature; we know only that the coming-to-be [das Zustandekommen] of spatially distinct perceptions requires a distinction of their topogeneous moments.

On the other hand there must be other causes in the realm of the real which bring about that we believe ourselves to perceive distinct material

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things at the same location at different times. I will allow myself to designate these with the name *hylogeneous moments*. Topogeneous moments are responsible for an object’s appearing at a given location, and not at another. The definition does not say how we identify an object, nor does it make any mention of time. In fact, it expresses a necessary connection between identity and location: if two things are identical with one another then they are, at any given time, at the same location. Hylogeneous moments account for the presence of qualitatively different things at the same place at different times. Here we assume that we do have such a criterion for identity—identical qualities, excluding spatial ones—and that two things not identical with each other must, if they are at the same place, be there at different times.

This all sounds more complicated than it is, however. From the example that Helmholtz gives us of “something that must have a ground”—Red at this location now—and the lengthier, less abstract account in “The Facts in Perception,” we know that he thinks of primitive perceptual judgments as points in a manifold of possible sensations. Space has a special role to play here, since spatial determinations show up in conjunction with all external *Qualitätskreise*: I can correlate a spatial location with tactile qualities as well as with visual ones—space is the common sense. Thus Kant, at the opening of the Transcendental Aesthetic, states that space “is not an empirical concept that can be derived from external experience,” because only by means of space can we represent objects to ourselves as being outside of us, individuated and alongside one another. Space grounds all of the last three aspects of experience for Helmholtz as well; however he denies the conclusion that spatial properties are consequently prior to experience. In essence he rejects the second part of Kant’s contention that, “one can never make oneself a representation that there is no space.

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while one can very well think to oneself that no objects are to be encountered in it.”

(Critique of Pure Reason, B38) To think of a space empty of *qualia* is, on Helmholtz’s account, possible *only if* we prescind from the qualitative part of the internal manifold: if we do so, we prescind also from the possibility of measurement. For the fact that every *Qualitätsskreis* is associated with the spatial manifold, but not conversely, it does not follow that the latter can be thought without reference to the former: I cannot imagine a meter without imagining a meter-stick. *Qualia* are as much an essential part of the perceptual manifold as spatial determinations are:

Nämlich die im Raume vorhandenen Objekte erscheinen uns mit den Qualitäten unserer Empfindungen bekleidet. Sie erscheinen uns rot oder grün, kalt oder warm, riechen oder schmecken u.s.w., während diese Empfindungsqualitäten doch nur unserem Nervensystem angehören und gar nicht in den äußeren Raum hinausreichen. Selbst, wenn wir dies wissen, hört der Schein nicht auf, weil dieser Schein in der Tat die ursprüngliche Wahrheit ist: es sind eben die Empfindungen, die sich zuerst in räumlicher Ordnung uns darbieten.

Namely the objects present in space appear to us vested with the qualities of our sensations. They appear to us red or green, cold or warm, they taste or smell etc., while in fact these sensations belong only to our nervous system and do not reach out into the external space. Even when we know this, the illusion does not cease, because this illusion is the original truth; it is indeed the *sensations* which present themselves from the beginning in a spatial order.27

The real correlates of such qualia, the hylogeneous moments, are unknown, although we have every expectation that these qualities will, at some level of analysis be

“resolved into mechanics,”28 *i.e.* into regular motions of particles in space. In actual perceptions, the effects of the hylogeneous moments are always conjoined with spatial determinations. So the perceptions “white at [3, 10],” “hard at [3, 10]” may combine to form the perception that something white and hard is situated at location [3, 10]. Helmholtz’s definitions serve to divide the attributes attached to a given complex of perceptions into two groups: the spatial data, on the one hand, and the

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other qualia on the other. Since the species of sensation are finite, we may assume, in accordance with Helmholtz’s definition of imaginability, that we are able to associate with each point in space a finite vector of qualities, say \([x, y][A, B, ..., Z]\), where the first two coordinates are spatial, the following ones indexes of various Qualitäts-kreise. The causes of the first two in a given judgment are its topogeneous moments, and those of the rest its hylogeneous moments. The properties he gives these moments follow directly from this characterisation, if we adopt the simple definition that two entities are identical if they consist of the same vector of qualia.

Consider first the hylogeneous moments. They account for the various material things with diverse qualities appearing at the same place at different times, that is, for the fact that we, when confining our attention to the location \([x, y]\) observe changes in some of the qualia \([A, ..., Z]\). Of course “to see another material thing” need mean nothing more than to see another set of qualities at that location: we never assumed a thing which was the bearer of these predicates except in so far as we considered the spatial location itself to be such a bearer. The matter is slightly more complicated when we come to the topogeneous moments. They account for the fact that a given object appears where it does: “other real conditions would have to have been at hand in order to bring about the perception of the same object at a different location.”\(^{29}\) In the first definition (of hylogeneous moments) Helmholtz does not speak of objects, but of material things, *stoffliche Dinge*, where the pairing of *hyle* and *Stoff* is deliberate. Aristotle’s matter is nothing other than the accidental properties, the qualities, of a thing, and it is their alteration which is motion, the essence of time (time is “the number of motion” for Aristotle). Changes in the hylogeneous moments associated with a topogeneous moment--changes in the qualities appearing at a given spatial location--are the changes that mark the passage of time. In this sense the

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\(^{29}\)Helmholtz, “Die Anwendbarkeit der Axiomen auf die Physische Welt,” p. 402.
hylogeneous moments correspond exactly to Aristotle’s *hyle*, with the important
difference that there are for Helmholtz no essences above and beyond the qualities. It
is the unifying and individualizing role of the topogeneous moments that makes the
hylogeneous moments appear to us as *things*. So if we associate with the concept
“object” the entity responsible for a *set* of elementary perceptions, thus treating it, as
in the quotation above, as the correlate of a *completed picture* (which does not fit the
elements of our simplest representations) then it is clear that “object” contains
“spatially situated” as part of its definition, that a determinate object is always
spatially situated. The object corresponding to \([x, y][A, B, ..., Z]\) is the entity respon-
sible for all the elementary perceptions making up this picture. However, that there is
one such entity is an unjustified inference, a consequence only of the perceptions
being unified by their simultaneous situation at a single location in space. The topo-
geneous moments are indeed responsible for our perception of *this* object’s being at
*this* location, for if it were somewhere else, it would not be the same object.

But if the unique spatial location is part of the definition of the object, then is
the definition not trivial? It says that the topogeneous moments are responsible for an
object’s appearing where it does, and I have just said that the concept “object”
includes spatial situation in its definition. And surely the fact that we see one and the
same object in motion overturns it on the first stroke? If an object’s definition
includes its location, then surely by changing the location we change the definition of
the thing. In answer to the first objection we should recall that the relation of the
spatial coordinates \([x, y]\) to the coordinates of the *Qualitätskreise* is asymmetric: the
other coordinates are unified in their all being associated with the single spatial
location--“white and hard” is a concept, not an elementary perception like “white at
[2, 3]” or “hard at [2, 3]”. So to say that the spatial location is an *essential* constituent
of the object, that real conditions would have to have been at hand for the same object
to appear at a different place is strained, but not trivial. Helmholtz’s definition refers
to a single moment in time—the Augenblick in which the observer directs her attention
to a group of presentabilia—and it expresses the following thought: entities which
exist simultaneously at different places are not, that is are not experienced, as being
identical. And to say that the same object could have appeared at a different place at
this moment means: an object qualitatively identical, indistinguishable with respect to
the consequences of its hylogeneous moments, but at another location. The value of
the definition becomes clear when we answer the second question, and turn our
attention to descriptions of moving objects, for the perception of a single object
moving through space can be arrived at only by means of a regular connection
between groups of the topogeneous and hylogeneous moments.

I will illustrate this connection by means of the following diagrams:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>B</td>
<td>R</td>
</tr>
<tr>
<td>$t_2$</td>
<td>G</td>
<td>B</td>
</tr>
<tr>
<td>$t_2'$</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

Each row represents two locations 1 and 2 in a finite one-dimensional space. B, R,
and G are three qualities from a single Qualitätskreis, which we will take, for
simplicity’s sake, to consist only of these three qualities. Each pair of cells in a row is
associated with a time $t_n$—I consider initially only the first two—$t_1$ and $t_2$. I refer to the
sum of these various elements as the perceptual manifold, the appropriateness of the
name being clearer if we represent these states as:
I will also assume in this example that there must be in each column exactly one filled square: each spatial point must have exactly one of R, B, or G. Helmholtz did not hold this position, for he believed that the manifold of colours as we experience them could be rendered as superimpositions of three primitive colours. However, this simplification does not affect the example materially, since indeed a "completed picture" of the points 1 and 2 would involve many more quality dimensions, each of which would require a separate coordinate system, whose values, furthermore, might well be continuous. Helmholtz himself does not provide a full account of his manifold theory, and I will not do so either--above all because there are excellent reasons for believing it will never fly. Helmholtz was in the grip of a space-mania quite as bad as Wittgenstein's, and he was willing, as my earlier remarks on smell and taste make clear, to ignore a lot of detail in setting it out. The theory as we have it makes excellent sense when applied to visual and tactile examples, less in the case of auditory ones (there we have a manifold of possible sensations, but the referral of these sensations to points in space is problematic), and none in the case of the last two senses.

We consider the diagrams for $t_i$ and $t_2$ to represent the movement of an object from location 1 to location 2. How are we to describe this in terms of topogeneous and hylogeneous moments? A physical process, according to Helmholtz, is the coming-to-be of certain hylogeneous moments and their consequences (that is, the perceptions they cause) and the procession (Ablauf) of these moments in conjunction...
with groups of distinct topogeneous moments. In my greatly simplified case, we will consider the one consequence of some hylogeneous moments (the sensation B) conjoined in succession to the group of two distinct topogeneous moments (the locations 1 and 2). At \( t_1 \) the topogeneous moments associated with 1 are conjoined to some hylogeneous moments giving rise to the sensation of B, and together they give us the perception "B at 1" or B(1), for short. At \( t_2 \) the hylogeneous moments associated with B are conjoined to the topogeneous moments of 2, and yield the perception B(2). But these conjunctions alone need not give rise to the perception that a single object has moved from 1 to 2. That perception will only result if the perceptions at 2 and 1, at times \( t_1 \) and \( t_2 \) respectively, are correlated with this process in such a way that a single quality appears to be displaced, i.e. that B was not perceptible at 2 at \( t_2 \), and is no longer perceptible at 1 at time \( t_2 \). If we had \( t'_2 \) instead of \( t_2 \), then there would be a perception of something stretching perhaps, but not of the displacement of a single thing. And if the perceptual manifold were to change randomly, like "snow" on a television screen, we would not be able to form coherent images of any stable objects, let alone see them move in a regular manner: "We could be living in a world in which every atom was different from every other, where there was nothing at rest. There would be not the slightest regularity to be found, and our thought-activity would be stilled."

The perception of a displacement of a B[ue] object from one location to another is the consequence of an intricate interaction of topogeneous and hylogeneous moments. "What we perceive directly" are only regularities in ever-changing associations of such. The movement of a B object appears as the sequence: \( t_1: B(1), R(2); \) and \( t_2: G(1), B(2); \) so the hylogeneous moments associated with the sensation B proceed from \( t_1 \) to \( t_2 \) in conjunction with the group of topogeneous moments (1, 2).

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and, simultaneously other hylogeneous moments come to be in conjunction, from \( t \), to \( t' \), with the group \((2, 1)\). Are the hylogeneous moments involved in \( B(1) \) the same as those involved in \( B(2) \)? Helmholtz's causal law says that we are entitled to infer a diversity in the causes of two perceptions if we experience a diversity in the perceptions, the signs. The diversity in the signs \( B(1) \) and \( B(2) \) has been ascribed here to a change in the association of some hylogeneous moment (\( B \)) with some topogeneous ones (\( 1 \) and \( 2 \)); and the transition at \( 1 \) from \( B \) to \( G \) has been ascribed to a change in the association of some topogeneous moments (those of \( 1 \)) with some hylogeneous ones (those of \( B \) and \( G \)). So the causes of \( B \) in \( B(1) \) and \( B \) in \( B(2) \) must in some sense be the same, but this does not entitle us to infer that they are identical in some deeper sense. It is safer to say that they are functionally identical, \( i.e. \) they might belong to a class of causes which produce similar, though never indistinguishable, results. To put it concretely: they might be the class of cells on the retina which all produce similar yet distinct (because always distinctly localised) sensations of blue. This is not a trivial problem, particularly not for Helmholtz. For he was, as I mentioned above, an opponent of nativism, which meant that he believed that all of our concepts were built up inductively, by means of judgments of similarity, as opposed to being given, if only in part, by a pre-established harmony between perceptual faculties and world. But if the causes of my sensations \( B(1) \) and \( B(2) \) are in fact identical--they are caused by the same external object--then their phenomenological similarity can only be explained by means of an appeal to such a capacity (a capacity to identify similarities), otherwise the process of association, thus of induction, could not begin at all.\(^{31}\)

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Now that we have an understanding of Helmholtz's definitions in a simple case, we can perhaps make better sense of his account of physically equivalent processes and their relation to on the one hand *physical*, and on the other hand *pure intuitive* geometry:

When we observe that divers physical processes can proceed in congruent spaces in equal periods of time, this means that, in the domain of the Real, equal aggregates and consequences of specific hylogeneous moments can come to be and unfold in conjunction with specific definite groups of different topogeneous moments—namely those which give us the perceptions of physically equivalent parts of space. And if experience then teaches us that *every* conjunction or *every* consequence of hylogeneous moments that can exist or unfold with that one group of topogeneous moments is also possible in conjunction with every physically equivalent group of *other* topogeneous moments—well this is in any case a proposition that has a real content, and the topogeneous moments thus doubtless influence the unfolding (*Ablauf*) of real processes.

“Physically equivalent” spatial magnitudes are ones in which the same processes may “exist and unfold” (*bestehen und ablaufen*) under the same conditions and in equal periods of time. The physical processes which Helmholtz has in mind are constructions and measurements. Now even something as simple as the movement of a ruler or the rotation of a compass depends on,

1) the ruler’s being assumed to be a rigid body—itself a postulate about the connection of some group of hylogeneous moments to certain definite groups of topogeneous moments (*i.e.* physically equivalent ones); and this requirement entails, as we have seen,

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2) the concomitant association of other groups of hyolgeneous and topogeneous moments—the ruler must move, and not stretch, as in our simple example above.

it follows that the concept “rigid body” already assumes a complicated association between hyolgeneous and topogeneous moments—between bodies and space. A statement about a rotation to the effect that, say,

```
\begin{center}
\begin{tikzpicture}
    \draw (0,0) node[anchor=north west] {A} -- (2,0) node[anchor=north west] {B} -- (1,1.73) node[anchor=north west] {C} -- cycle;
    \draw (0,0) -- (1,1.73) node[anchor=south west] {60°};
    \draw (1,1.73) -- (0,0) node[anchor=south west] {A'};
\end{tikzpicture}
\end{center}
```

“The rod A'B may be rotated about B in such a way that, without the rod AC moving, A' is made to coincide with C” will be a statement with real content, since, as we have seen, the very identity of the bodies AC and AB as well as, of course, their rigidity, depends on regularities in our perceptions—regularities in no way implied in the thin fabric of Helmholtz’s definitions. But this proposition would appear, from a pure intuitive perspective, as necessarily true, and, furthermore, as being quite independent of any facts concerning the actual physical behaviour of a rotating rod.

In fact the example Helmholtz gives involves a slight complication of that given above. He imagines the following construction:
Starting with an equilateral triangle ABC, we extend two sides to the points b and c, such that \( cA = bA \). The question is, is cb longer than, shorter than, or equal to \( cA \) and bA? If it is always equal, then the space is Euclidean. If, however, when \( bA < AB \), cb < bA, then the space is pseudospherical, and if cb > bA under the same condition, then it is spherical. This example involves no definitions beyond that of collinearity, which can itself be defined in terms of equal distance: c is collinear with A and C iff there is no point c' such that

\[
\begin{align*}
\text{c} & = \text{c'} \text{A and,} \\
\text{c} & = \text{c'} \text{C .}
\end{align*}
\]

Thus to state the problem, we need appeal only to:

1) ... Gleichheit oder Ungleichheit, d. h. physische Gleichwertigkeit oder Nicht-Gleichwertigkeit von Punktabständen; 2) ... Bestimmtheit oder Nicht-Bestimmtheit gewisser Punkte. 34

1) ... equality or inequality, that is physical equivalence or non-equivalence of point-distances; 2) ... determinateness or indeterminateness of specific points.

Euclidean geometry asserts that \( bc = cA = Ab \) independently of the relation between bA and AB. But to give this general assertion a concrete interpretation would be to carry out a sequence of measurements with physical objects in space and time. And

\[33\text{Helmholtz, "Die Anwendbarkeit der Axiomen auf die Physische Welt," p. 396.}\]

\[34\text{Helmholtz, "Die Anwendbarkeit der Axiomen auf die Physische Welt," p. 404.}\]
this, even on a purely idealistic interpretation, will involve making assertions about the law-like behaviour of points in our perceptual manifold, about law-like correlations among our elementary perceptions. These assertions will have imaginable content, because they will concern possible perceptions, in accordance with Helmholtz's definition. So, Helmholtz concludes, the axioms of physical geometry must have real content and are not \textit{a priori}.

c) Pre-established harmony

Helmholtz now proposes a shift in perspective that allows us to see how a particular kind of consciousness might come to experience Euclidean geometry as \textit{a priori}. He suggests, as I mentioned before, that our consciousness might be related to the topogeneous moments in a two-fold manner.

\begin{quote}
Man könnte z. B. annehmen, daß eine Anschauung von der Gleichheit zweier Raumgrößen ohne physische Messung unmittelbar durch die Einwirkung der topogenen Momente auf unser Bewußtsein hervorgebracht werde, daß also gewisse Aggregate topogener Momente auch in Bezug auf eine psychische, unmittelbar wahrnehmbare Wirkung äquivalent seien.
\end{quote}

One could for example assume that an intuition of the equality of two spatial magnitudes could be brought about directly by means of the action of the topogeneous moments on our consciousness, thus that determinate aggregates of topogeneous moments would be equivalent also with respect to a direct perceptual effect.\textsuperscript{35}

Physical measurement, as we have seen, always assumes regularities in our perceptions which manifest themselves as the appearance of stable, subsisting objects, which in turn may be moved, \emph{i.e.} put in association with groups of spatial points (the effects of the topogeneous moments). But the spatial points never appear divested of sensible qualities, just as the sensible qualities are always experienced as being located somewhere in space. If on the other hand the topogeneous moments could act on us directly, without our needing to measure their relationships, and, furthermore,

\textsuperscript{35}Helmholtz, "Die Anwendbarkeit der Axiomen auf die Physische Welt," p. 404.
they did so in such a way that their effects on our consciousness mirrored that of the
metric of Euclidean space, then we would intuit a pure geometry:

Nehmen wir an, daß die Intensität jener psychischen Wirkung, deren
Gleichheit als Gleichheit der Entfernung zweier Punkte im Vorstellen
erscheint, in derselben Weise von irgend welchen drei Funktionen der
topogenen Momente jedes Punktes abhängt, wie die Entfernung im
Euklidischen Raum von den drei Koordinaten eines jeden, so müßte das
System der reinen Geometrie eines solchen Bewusstseins die Axiome des
Euklid erfüllen, wie auch übrigens die topogenen Momente der realen Welt
und ihre physische Äquivalenz sich verhielten.

Let us assume that the intensity of the psychical effect whose equality
appears in Representation as the equality of the distance between two points
should depend in the same way on some three functions of the topogeneous
moments of every point, as the distance in Euclidean space depends on the
three coordinates of each point—then the system of pure geometry of such a
consciousness would have to fulfil Euclid’s axioms, no matter how the
topogeneous moments of the real world and their physical equivalence were
to behave.36

The physical equivalencies referred to in the last sentence are the metric of space as
determined by physical geometry, by measurement with rigid bodies in time. If both
that metric and the metric given by the direct effects of the topogeneous moments on
our consciousness (Helmholtz is most likely thinking of an energetic excitation in the
brain) were to agree with the Helmholtz/Riemann analytic description of the metric of
Euclidean space, then we would have a pre-established harmony between physical
and psychological space, a psycho-physical parallelism of the sort Helmholtz denied.

The kind of intuitive geometry which is here being described is a static,
atemporal one. Geometric statements are always statements about “regular (Gesetz-
mäßige) connections between topogeneous moments,”37 and in the scenario Helmholtz
envisages, we have direct access to such connections, although of course never
to the topogeneous moments themselves. It is as if I considered the space of my
perceptions at an instant, abstracting from the other properties distributed within it,


and thus also from the possibility of utilising them as a means of comparison. This form of imagination cannot peel this space off from that of my basic experiences entirely: I must be able to identify the direct and indirect effects of the same topo-
geneous moments with each other, otherwise the geometry based on pure intuitions of space would have no empirical referent, and I would literally be unable to apply it physically. There is here no question of the one or the other metric being “correct,” that is given by the actual relations of the topogeneous moments to one another outside of perception, for we have no direct knowledge of them. So the two-fold relation Helmholtz describes, though it lends itself to a straightforward realistic interpre-
tation, is, as he maintained at the outset, neutral on the question of realism or idealism. The agreement or disagreement between the two geometries could play out entirely within an idealistic and solipsistic consciousness, without any assumption of an independent reality. The one geometry would be an atemporal one, abstracted from the elementary perceptions that make up our experiences, the other a temporal one, derived from observed regularities among these elementary perceptions.

Helmholtz does not believe that this is the situation in which we find ourselves, but he also cannot dismiss it as a logical possibility. Nor does he adopt the position of later interpreters (e.g. Schlick or Reichenbach) who, partly under the influence of Wittgenstein, maintained that only the geometry of experience, of measurement, was meaningful. Indeed this latter interpretation results from the coincidence of two chains of influence originating in this article: the one remaining within the physical and mathematical problems discussed here, the other resulting from Wittgenstein’s application of these ideas in the Tractatus. For all the following concepts reappear in the Tractatus in a logical guise, supplemented there with the assumption that there is pre-established harmony between internal and external manifolds, between the space of experience, and our direct intuition of the latter:
1) The basic data of our experience consist of instantaneous judgments about the "bestehen"—the presence—of concatenations of qualities to consciousness.

2) The psychological operations which combine these primitive perceptions into higher-level ones are thought in exactly the same sense as our conscious thinking, meaning that we are also able to undo them by careful reflection and analysis—i.e. that conscious thinking is not the pre-eminent form of cognitive activity.

3) The elements of these basic experiences belong to manifolds of possible qualia. These elements are Zeichen, which mirror functionally important aspects of reality.

4) These elements are not objects in the colloquial sense, they require no real ground outside of such primitive judgments.

5) These primitive elements appear as substances which endure through time, and thus allow:

6) The definition of "conceivable" in terms of an anticipation of possible elementary perceptions.

7) The physical process by which these perceptions come about is beyond the boundary of what we experience.

8) Conscious perception of objects and motions of objects result from constellations of these elementary perceptions operating in regular connections in time.

9) The role of the Will of the observer: She can bring various presentabilia to consciousness by carrying out willed actions, where this process is likened to moving about a spatial manifold.

and, perhaps most important,

10) There is an intuitive a priori set of statements which describe fixed properties of the primitive manifold. These statements are such as define the most general topological properties of the space in question. It is to such propositions that Wittgenstein compares the fundamental propositions of logic.

In the following three chapters, I turn to a detailed account of the development of the Tractatus propositional theory. Once I have laid out the essential conditions that Wittgenstein set on such a theory, we will be able to see how Helmholtz's work provided him with a framework that met those conditions. Before turning to those topics, I should say a few words about Heinrich Hertz, the intermediate figure in this story.
It is not to Helmholtz but to Hertz that Wittgenstein directs the Tractatus's reader. We know he read Hertz, we can only suppose he read Helmholtz. As a student of Helmholtz's, Hertz was profoundly influenced by Helmholtz's work in sense-physiology and geometry; indeed, as I argue in Chapter 7, the fundamental division in Hertz's The Principles of Mechanics is only really understandable if one is familiar with that work. Hertz divides his mechanics into two books, the first one geometric, being a purely mathematical development of the properties of systems considered only as they appear in the Anschauung of the subject; the second an empirical kinematic portion, in which the range of "thinkable" systems described in the first book is reduced, by means of definitions and Hertz's "fundamental law," to those systems that are physically possible. An educated contemporary of Hertz's would have seen this connection immediately (Helmholtz survived Hertz, thus Hertz never escaped the shadow of his far more illustrious mentor, and Helmholtz of course wrote the introduction to the posthumous Principles of Mechanics), and one can safely say that Hertz's book belongs to a Helmholtzian tradition, the defining characteristics of which I have outlined in the preceding discussion.

The complex connection between these three thinkers will form the topic of the last two chapters of the thesis. For now it will suffice to summarize it: Helmholtz's perceptual manifold provides a basis (bearing in mind that we do not know the exact nature of the deepest manifold) on which the propositions of our language--logical, physical, everyday--can be defined. In this sense it provides, we might say, a stock of protocol sentences. The theory outlined in Hertz's Mechanics can be interpreted as resting on such a primitive data-space: he begins the first book by positing properties of mass-particles, each of which is "a characteristic by means of which we correlate uniquely a definite point in space at a given time with another point in space at every
other given time." And this is of course consistent with the doctrine that even the notion "movable object in space" is dependent on there being some kind of regularity within the space of our primitive perceptions: Hertz's "material particle" meets the defining criteria of a Helmholtzian hylogeneous moment. But the perceptual manifold could ground other languages and symbolic systems, given by means of other kinds of definitions: everyday languages, alternative formulations of mechanics, indeed musical languages. All of these semiotic systems will get their meaning with reference to the events and appearances in the fundamental manifold: its properties are the fundamental properties both of possible experience--thus of the world--and of course of the primitive representations that picture its points, thus of the various languages built on it. The Tractatus says: 1) that those fundamental properties are unsayable (Helmholtz would have said the same thing), but that 2) they are shown by the logical propositions. "Mechanics," says Wittgenstein in 6.343, "is an attempt to construct all true propositions that we need for the world-description according to one plan." Logic, he might just as well have added, is an attempt to construct all significant sentences according to one plan. That is the "relative position" of logic and mechanics alluded to in the Tractatus. Helmholtz provides us with the fixed point relative to which Wittgenstein's and Hertz's theories may be situated.

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Chapter 3 - Russell’s Judgment-theory I: The Contextual Elimination of the Proposition

In the next three chapters I define the group of problems which led Wittgenstein to the set of doctrines forming the basis of the propositional theory of the *Tractatus*. These doctrines are:

1) That an elementary proposition is a fact which, when it is true, stands in a depictive or mapping relation to a corresponding fact.

2) That groups of elementary propositions share a common form, as do the groups of corresponding facts.

3) That this form is determined by the objects making up the facts on the one hand, and the names making up the propositions on the other.

4) That the objects and names are consequently also organised in groups, such that all possible combinations of names on the one hand, and objects on the other, form isomorphic groups—the groups of propositions and facts of 2).

5) That the objects, names and their group-membership are invariant: if a name belongs to a given group and has a given denotation, then both denotation and group-membership are fixed.

The groups of which I speak may be equated with the types of *Principia Mathematica*: they are on the one hand significance ranges (the names), and, on the other, ranges of entities which may possibly combine with one another (the objects). Types of elementary facts and propositions are, *e.g.* predicative, two- and three-place relational, and so forth. Types of objects and names are individuals, predicates, relations, and so on. The claim that these “determine” the forms means that the objects on the one hand, and the names on the other, are so structured that they can only combine with each other in certain ways. The notion of structure is borrowed from Frege: the symbols for various objects either have slots (in the case of the predicates and relations), or they have none, and the slots allow or prevent their being filled with other names. What is important is that the forms of the propositions are a by-product of the forms of their constituents, and likewise for the facts. It is a major thesis of my interpretation that the senses of possibility and impossibility that are assumed in this account are, for want of a word, *ontological*: for Frege, Russell and
Wittgenstein, the notions of logical and physical impossibility coincide when we are talking about the properties of facts and propositions—how they can and cannot be structured. It will be my purpose in these chapters to make the preceding notions clearer by examining their application in the propositional theories of these three men.

These five planks of the *Tractatus* theory can each be seen as a reaction to specific difficulties in a "theory of judgment" that Russell developed between 1903 and 1913—initially on his own, and subsequently in collaboration first with Whitehead, then with Wittgenstein. The theory was prompted by Russell's study of Frege's propositional theory: it was to explain the notions of *true, false, proposition, assertion*, and *implication* in such a way as to dispose of Frege's *Sinn*, which Russell held to be dangerously idealist and psychologistic. Since, as I argue below, the judgment-theory was closely related to the theory of descriptions (it replaces the possibly non-denoting propositional sign with, in essence, a description of the fact which the sign supposedly denotes), it is difficult to assess Whitehead's contribution: he worked at length with Russell on the one theory (of definite descriptions), but apparently not on the other. The same goes for Wittgenstein: the paper trail leads almost exclusively through Russell's manuscripts, many of which have only been available in print in the last few years. Even Wittgenstein's proto-theory, which is the topic of Chapter 5, can be reconstructed only with the help of Russell's manuscripts, a few letters from Wittgenstein to Russell in 1912-1914, and a few entries in the *Notebooks* from 1914. Since Russell himself suppressed the one text, his 1913 *Theory of Knowledge*, in which he sought to give the theory a thorough exposition, the following discussion is at heart reconstructive: we have no final version of the theory which we can criticise; and the versions that we do have are either incomplete or indeed inconsistent. In order to draw the connections to the *Notebooks* and the *Tractatus* which are the aim of this investigation, I have engaged in as sympathetic as possible a reconstruction of Russell's views. I organise the discussion around a letter
from Wittgenstein to Russell in June of 1913, in which he states his "objection to your [Russell's] theory of judgment exactly."1 In unpacking the significance of Wittgenstein's criticism, I arrive at an inventory of the fundamental difficulties confronting the "theory of propositions." By combining that list with remarks from the early Notebooks, we can see how and why the notions of manifold and space, which were familiar to Wittgenstein from the German psychophysical tradition, came to play the role they did in the Tractatus.

Wittgenstein's shift in perspective hinged on a logical point: whatever the correct analysis of the elementary proposition, it must be such as to guarantee that such a proposition be exclusively either true or false. His conclusion was that the space of possible elementary facts, and thus its dimensions—the ranges of entities making these up—had to be given in advance. Acquaintance with the types of the components of an elementary proposition is in some sense prior to our understanding of that proposition. This slight logical turn entrains a revolution in the theory of knowledge, or Erkenntnistheorie, attached to the logic. It demands that the thinking agent have a priori knowledge of a network of possible facts or experiences as well as a capacity to group the elements of the network together (to generalise over groups of these possible experiences). The "objects" making up this world are not like Desdemona or this table, but are characteristics (Merkmale) or elements of these primitive experiences. There is no evidence to suggest that Wittgenstein made this move because he was already committed to such a transcendental outlook. Instead he rejected a theory of logic that failed, on his view, to distinguish between significant and non-significant language, and replaced it with one postulating a closed and

invariant possibility space, from which point on it was natural to think of it on the psychophysical model.

The letter to Russell has been the subject of sporadic commentary over the years; however only since the discovery of the Theory of Knowledge manuscript has its actual subject been clear. It reads:

... I can now express my objection to your theory of judgment exactly: I believe it is obvious that, from the proposition "A judges that (say) $a$ is in a relation $R$ to $b$", if correctly analyzed, the proposition "$aRb \lor \neg aRb" must follow directly without the use of any other premiss. This condition is not fulfilled by your theory.\(^2\)

We can and should give the letter as wide a reading as possible, beginning with the following remarks from the "Notes on Logic:"

When we say A judges that etc., then we have to mention a whole proposition which A judges. It will not do either to mention only its constituents, or its constituents and form, but not in the proper order. This shows that a proposition itself must occur in the statement that it is judged; however, for instance. "not-p" may be explained, the question of what is negated must have a meaning.\(^3\)

Only facts can express sense, a class of names cannot. This is easily shown.\(^4\)

Every right theory of judgment must make it impossible for me to judge that this table penholders the book. Russell's theory does not satisfy this requirement.\(^5\)

In the first two passages, Wittgenstein argues that a mere list of names or objects--say \{$a, R, b\}$--cannot distinguish among the various possible combinations of the latter; in the third, he emphasizes that not every group of such objects corresponds to a proposition--if, that is, there are types. Thus a theory that disassembles a propositional expression such as $aRb$ into a list of its components must somehow overcome these

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\(^2\)Notebooks, p. 122. The remainder of the letter concerns a luncheon-date.

\(^3\)Notebooks, p. 94.

\(^4\)Notebooks, p. 105.

\(^5\)Notebooks, p. 103.
difficulties. The letter does not say which of these concerns is foremost in Wittgenstein’s mind, and neither does the corresponding passage in the *Tractatus*:

5.5422 Die richtige Erklärung der Form des Satzes “A urteilt p” muß zeigen, daß es unmöglich ist, einen Unsinn zu urteilen (Russells Theorie genügt dieser Bedingung nicht). 5

Wittgenstein gave a lot of thought to the problem of type-matching (the requirement that the elements of an analysed proposition be such that they can combine to form a fact); however the *Tractatus* passage drops the specific reference to that problem, most likely because Wittgenstein, having adopted Frege’s solution of reflecting type differences in objects by means of different types of symbols, no longer felt this to be the critical difficulty. The other objection (about the ambiguity of propositions which are mere “classes of names”) may also stem from discussions with Frege, however I will defer that connection for the moment. As a preliminary I need only emphasise that both sorts of objection point at a single possible failing of a judgment-theory, as both the letter and the *Tractatus* make clear: the bivalence of the proposition must be implied in the analysis of judgment, and should this not be the case, then the theory is not a theory of judgment, whatever else it may be. That is, in the case that a) the state of affairs presented by a given proposition is not the case, it should follow that b) the proposition is a *false* proposition. It follows that,

1) a proposition represents a *possible* state of affairs, and,

2) a proposition represents a *unique* state of affairs.

If 1) does not hold, then we will not be able to distinguish between pseudo-propositions that assert combinations between mismatched elements, which could thus *never* combine, and actual propositions which assert combinations which, though possible, do not in fact obtain. If 2) does not hold, then we will not be able to say.

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65.5422 The correct explanation of the form of the proposition “A judges p” must show that it is impossible to judge a nonsense. (Russell’s theory does not fulfill this requirement).
when e.g. \( \{a, R, b\} \) are not combined, whether it follows that \( \sim aRb \) or that \( \sim bRa \). Thus "false" means: possibly true, but not true.

a) The Griffin/Somerville Reading

Nicholas Griffin\(^7\) and Stephen Somerville\(^8\) have published a series of papers in which they argue that the "other premiss[es]" to which Wittgenstein alludes in the letter are premisses stating that the objects of the judgment in question are appropriately type-matched to one another. This demand for supplementary premisses cannot be met, they argue, because these premisses would themselves involve further judgments. Since "it is the account of elementary judgment itself which is intended to provide independent support for classifying ranges of significance of functions into types,"\(^9\) it follows that, "There is nothing ... to preclude violation of type, unless type theory in invoked to confine judgments to significant propositions... ."\(^10\) Since the judgment-theory assumes type-theory, but, at the same time, the former is supposed to "support" the latter, Russell’s project is viciously circular. Griffin, in reworking Sommerville’s thesis, elides the last point, arguing that the circularity results from the order of the premisses in question:

Moreover, the further judgments required are of an extremely problematic character. For to judge that \( a \) and \( b \) are suitable arguments for a first-order relation is to make a judgment of higher than first-order. Yet, as


Russell makes quite clear in *Principia* (pp. 44-6), higher-order judgments are to be defined cumulatively on lower-order ones. Thus we cannot presuppose second-order judgments in order to analyze elementary judgments.\(^{11}\)

That difference aside, the two rebuttals take essentially the same form. The judgment-theory is to generate the significant elementary propositions, which in turn are to provide the base on which the theories of types and orders are to be built. Since, however, the significance of the elementary propositions can only be certain if type-constraints are put on the arguments of the judgment-relation, and these constraints must involve notions such as "type" or "all first-order functions," they assume the very concepts whose foundation they were to provide. Thus the judgment-theory is viciously circular.

Since I disagree with this reading on a number of points, I should begin by saying what large part of it I think right. Wittgenstein was much concerned with the possibility of type-mismatches in his own copula-theory; and, since that theory and Russell’s judgment-theory had a common origin, he certainly thought that the judgment-theory was vulnerable on this score as well. He informed Russell of this problem at the beginning of 1913, and, what is more, he told him how he had solved it: one has to assume that type-distinctions are reflected in our use of distinct types of signs for distinct types of things. So the claim that some form of the judgment-theory needed riders on the judgment-relation is unquestionably true. But there is no suggestion anywhere that such riders had to be eliminated because of the order of the propositions expressing them, let alone because the judgment-theory was conceived as the foundation of the type-theory. The problem is simpler: the notion of a function which defines types is, as I show in Chapter 5.a.i, incoherent, *whatever* the order of such a function, and whatever the purpose of the theory which invokes such

functions. Griffin and Sommerville have identified a major concern of Wittgenstein's during the period in question: the naive judgment-theory of Russell's "On the Nature of Truth and Falsehood" put all entities on the same level (it treats functions as if they were individuals), and thus it opens the door to expressions such as "The table penholders the book," in which an object has been illegitimately cast in the role of a relation, thus yielding nonsense. However they err both in explaining why such expressions cannot be eliminated (it is not that the premisses needed to exclude them are of the wrong order, but that they are simply impossible), and in arguing that Wittgenstein is simply reiterating cryptically in June a criticism he made clear in January. Lastly they do not appreciate "why Russell cannot permit belief (or even doubt) in nonsense." 12 As I will show in Chapter 4, the purpose of the judgment-theory was to provide a basis not for the theories of types and order, but for a much more pedestrian notion, that of implication. In the Principles of Mathematics, Russell defined proposition by means of the indefinable implication: a proposition is anything that implies itself. In Principia Mathematica, proposition is indefinable, and the primitive truth-functions are defined by means of the latter. 13 Since those definitions can only succeed if a proposition is of necessity something which is always either true or false, it follows that any definition of the notion proposition must show this to be an essential property of all propositions. The judgment-theory is just that: a definition of what a proposition is. Therefore from the analysis of an elementary judgment, the bivalence of the proposition corresponding to it must follow—and that is what Wittgenstein demands in his letter. This is of course not to deny that, if this


13 See Chapter 4.a for a discussion of these "definitions." They are not PM symbolic (syntactic) definitions, but semantic and metalogical ones.
requirement is not met, that the type-theory and the theory of orders might well collapse.\textsuperscript{14} However that would be a consequence of this more fundamental failing.

So the Griffin/Somerville interpretation puts great weight on the first of the two requirements I gave above: a proposition must represent a possible state of affairs, in that its components must be correctly type-matched. Since I think that 1) is only one of the concerns Wittgenstein had in mind, and since both he and Russell had working solutions to that problem in June 1913, I think it is not the main thrust of the letter, and I argue mostly against them in what follows. I do not, however, dispute that 1) and 2) together form the essential requirements that Wittgenstein placed on his own propositional theory, nor that Russell's theory, in its earlier versions, was vulnerable to both criticisms. Both Russell and Wittgenstein were trying to define the notion "elementary proposition" in such a way that the bivalence of the proposition followed essentially from their definition. In the course of developing his "copula-theory."

Wittgenstein hit on three separate postulates which would have to be assumed if a project like his and Russell's was to work. One of these was the demand that the components of the elementary propositions be type-matched; the other two were a) that the class of facts of the "form" in question had to be well-defined, and b) that the denotation of the names in the propositions (or judgments) had to be fixed.

Wittgenstein's letter to Russell in June of 1913 states only the most general requirement: the proposition must be bivalent, and this property cannot depend on other facts obtaining, on other propositions being true; it must be internal to the elementary proposition. Griffin and Sommerville correctly identify one of the conditions (the "other premisses") which Russell's and Wittgenstein's theories had failed to meet. However it was no longer the major concern of either of them in June 1913, I

\textsuperscript{14}I have not read Sommerville's unpublished thesis, "Types Categories and Significance," (McMaster University, 1979), so I cannot say whether this would follow on his reading or not.
contend, and thus I conclude that it is the requirement that the logical forms “exist” which proved critical. The requirement that some objects exist eternally, that there be atoms, was of course adopted by both men by 1919.

The very aim of Russell’s epistemology invited problems of just this sort. He wanted an account of knowledge which separated the subject from the objects of his knowledge as much as possible, and this meant freeing his theory of any appeals to the conditions under which a thing or fact might be known. He did not want to say, in other words, that judgments were relations between subjects and some set of objects specially suited to being elements of such judgments, such as mental objects. J.-P. Leyvraz has neatly summarized this early Russellian view as follows:

Ce qu’entend Russell par cette théorie de l’extériorité des relations, c’est donc que celui qui connaît peut d’emblée et doit pouvoir se distancer de ce qu’il connaît et donc se distancer d’une totalité imaginaire du monde ... qu’il connaîtrait déjà sans avoir jamais commencé à le connaître.15

Russell wanted his judgments to range across all the entities which do and could make up our world, so that my pen is just as much an element of the judgment “My pen is black” as the visual patch in the judgment, “That patch is red.” Neither pen nor patch is possessed of some property making it a possible subject of knowledge, beyond, that is, being an object. And the pen should be a possible constituent of a judgment even if it never has and never will have played such a role. This very liberality led Russell to a theory in which the proposition was reduced, in Wittgenstein’s language, to a “class of names” with “its constituents and form, but not in the proper order.” As a result, it became necessary to limit those classes of names, and their possible ordering, so as to exclude those such as \{table, penholder, book\} to which no fact could possibly correspond; however this had to be accomplished without any appeal to some things (or groups of things) being possible objects of

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knowledge, in contradistinction to others. This indifference to the possible facts of experience, and to the actual state of the world is essentially tied to Russell's realism: if my account of judgment has riders on it, then it is not judgment of the world-encompassing sort that Russell wanted.

Russell's anti-idealist stand, his commitment to a theory of external relations meant that his theory of judgment required that I be able to judge that, say, \( qa \), in complete ignorance of the status of the fact \( qa \). My connection to the objects of my world is always accidental: neither I nor they are essentially altered by my acquaintance with them. This fact leads Somerville and Griffin to argue that a theory of this sort must assume that the arguments to the judgment-function \( J(\hat{S}, \hat{\varphi}, \hat{x}) \)\(^{16}\) are correctly type-matched; thus that my independence from the objects of judgment introduces the possibility of type mistakes. But whereas type-matching is certainly a requirement of such a theory, and although Wittgenstein did voice such an objection to Russell's and his own proto-theory many times, it is a criticism that Russell dealt with successfully in later versions of the theory, in particular in that of Theory of Knowledge. My judgments will always involve some implicit knowledge about the nature of the objects of my judgment, and that knowledge will in turn be assumed implicitly in an ascription, to me, of that judgment: if someone else says of me that "He judges that 'the table penholders the book'" then she doesn't know what a judgment is. And if I say 'the table penholders the book' then I am not expressing a judgment. If Wittgenstein, in his letter of 1913, was demanding, as Griffin\(^{17}\) and Sommerville\(^{18}\) maintain, further judgments (either from me or from her) to the effect

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\(^{16}\) Here \( \hat{S} \) is the judging subject, and \( \varphi \) and \( x \) are the entities making up the content of the judgment that \( q \varphi x \).


that the elements of my judgment be correctly type-matched, then his demand is simply perverse: What theory could ever be safe from such an attack? Even if there were some way of judging that an object belonged to a particular type—and Wittgenstein, as we shall see in the chapter on his proto-theory, came to the correct conclusion that this was impossible—that judgment would always be open to the same criticism. Griffin’s and Sommerville’s error lies in identifying the “further premisses” that Wittgenstein demanded with judgments concerning types, instead of with premisses stating that entities such as Russell’s logical forms, or Wittgenstein’s $e$-copulae, exist.  

In order to give Wittgenstein’s criticism in the letter some plausibility, we should focus on the purpose of Russell’s meta-theory, that is on its role in explaining deduction, and connect that with Wittgenstein’s justifiable insistence that elementary propositions be bivalent. In order to do that, I shall consider the development of the judgment-theory from its first appearance in the early and unpublished work “On Fundamentals,” where Russell tried to reconcile Frege’s notions of Sinn and Bedeutung with his own denotative theory of meaning, through its intermediate stage in “On the Nature of Truth and Falsehood,” to its fullest, though abortive exposition in Theory of Knowledge. The first of these texts has been available in print only since 1994, the last since 1984, so it is no surprise that Russell’s judgment-theory has generally been seen as an isolated oddity, independent of his contemporaneous logical work. In fact it should be understood as a kind of meta-theory: it stands in the same

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relation to *The Principles of Mathematics* and *Principia Mathematica* as Frege's “Funktion und Begriff” or “Sinn und Bedeutung” do to the *Grundgesetze*. And it is above all as a development of these meta-theories that I believe Wittgenstein's *Tractatus* should be read.

b) The Development of the Judgment-theory

i) “On Fundamentals” - 1905

Russell started accommodating Frege's views on *Sinn, Bedeutung*, and judgment to his own before *The Principles of Mathematics* went to press. In the “Appendix A” of that book he assimilates Frege's *Bedeutung* (which he there translates as *indication*) to his *denotation*, and *Sinn* to his *meaning* (in the following discussion I shall use the terms *denotation* and *meaning* in Russell's senses). In that same appendix, he registers his disagreement with Frege's claims that proper names have a meaning (*a Sinn*), and that “assumptions are proper names for the true or the false.”

The second contention seems incomprehensible to him not because he objects to the idea that truth and falsity might be things, but because “It is almost impossible, at least for me, to divorce assertion from truth, as Frege does.” To separate the two notions would be to introduce the possibility of mistaken assertions, and Russell thinks that this leads unavoidably to a conception of assertion which is only psychological: “An asserted proposition, it would seem, must be the same as a true proposition.”

What worried Russell was that assertion divorced from truth would become merely Humean belief, a “lively idea” attached to our impressions. From the beginning he felt that Frege's notion of *Sinn* posed a threat to the realist

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philosophy he advocated. However this criticism of Frege's theory derives from a misunderstanding of its intent. Russell obviously read Frege's definition of a judgment as the "Anerkennung der Wahrheit eines Gedankens" as if the Anerkennung were a psychological event, as opposed to, as Frege clearly meant, an Erkenntnis of the truth of a thought. And Russell failed to see that his proposed union of assertion and truth—which makes it sound as if my assertions entailed their truth—was, in its incestuousness, at least as vicious as their divorce.

In the Principles of Mathematics, Russell talks of propositions as if they were entities whose defining characteristic is that they are true or false. Assertion is what we do when we ascribe truth to a proposition, thus it is indissolubly wedded to the concept of truth. For Frege, on the other hand, propositions are just a species of proper-name. A proposition presents us with a sense—it expresses a thought—and, when that proposition is asserted, we make a judgment that things are in fact as that thought represents them. For Frege, just as for Russell, an asserted proposition is a true one. So why does Russell claim this is not so? He thinks that Frege's Sinn is a mental object, corresponding to Russell's "propositional concepts," and so he reads Frege as saying: an assertion consists in predicating truth of a propositional concept, and if this is so, then the statement "The thought \( p \) is true" makes a statement not about the world, but about my thought. Such a conception is inevitably idealist, thus Frege's analysis must be wrong. But Frege would not have agreed that the assertion \( \neg \neg p \) means "the thought \( p \) is true." It means that "\( \neg p \)" denotes the True, and nothing more. Truth is not a predicate of a thought, and Frege avoids giving a semantic explanation of why a name denotes the True, because he feels that such an explanation is external to logic, to the laws of thought he wished to codify.

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In his 1905 "On Fundamentals" we see Russell dealing with both these disagreements with Frege—the denotation of names and propositions—more thoroughly. His basic insight in that paper was that the notion of denotation should be fundamental: a simple name only denotes, and, in contrast to Frege, the names of functions are to be understood as simple names as well. A complex name such as "F(a)" denotes as well, but its denotation is a complex. Complex names can appear in other complexes as "meaning" or as "denotation." He defines this distinction as follows:

3. When a complex is asserted, it occurs as meaning.

4. When a complex is said to be true, it occurs as being.

5. When a complex occurs as being, any other complex having the same denotation, or the denotation itself, may be substituted without altering the truth or non-truth of the complex in which the said complex occurs.

6. The way in which a complex occurs (i.e. whether as meaning or as being) depends on the nature of the complex in which it occurs.²⁶

In elaborating on the implications of these definitions, Russell enunciates for the first time his theory of descriptions, which is intended to determine the significance of a denotative phrase occurring in a functional complex. If "F(C)", where "C" is a complex name, is to be construed on the model of "F(a)", where "a" is simple, then what account are we to give of the case in which "C" fails to denote? Russell's well-known solution is to replace F(C) with a conjunction asserting, on the one hand, that there is a unique entity a corresponding to "C", and, on the other, that a has the property F. Two aspects of this paper are particularly striking: first, having hit upon this solution to the problem, Russell immediately turns to a parallel contextual elimination of classes; second, that he deals with the denotation of descriptions alongside that of propositions, indeed he sees the cases as being fundamentally the same.

"In every complex," says Russell, "at least one constituent occurs as meaning. It is the constituent occurring as meaning that gives form and unity to the complex; ... ."27 When I assert a proposition, I assert its meaning, and thus the correct interpretation of "l- F(a)" is, "Assert: 'the object denoted by a is F.'" That is, I assert a meaning, which is composed of a denotation (that of "a") and a meaning (that of "F"). Similarly, "In our symbolic system, '3' occurs as meaning."28 In "p ⊃ q", "p" and "q" must appear as entities (that is, as their denotations) because the latter proposition expresses "an indefinable relation of entities."29 It does not express a relation between expressions, but between facts:

...when a proposition is asserted, it is asserted quâ meaning, not quâ entity; for we cannot assert what is not a proposition... Now "p ⊃ q" has to be such that from l- p and l- p ⊃ q we can go to l- q. Thus if "l- q" simply asserts q, it will be necessary to hold that q occurs as meaning in "p ⊃ q" since it occurs as meaning when it is asserted. But if we regard "l- q" as asserting "q is true," then q may occur as entity in "p ⊃ q".

Russell analyses the derivation of l- q from l- p and l- p ⊃ q as follows:

\[
\begin{align*}
\text{l- p} & \quad \text{Assert: "p is true."} \\
\text{l- p ⊃ q} & \quad \text{Assert: "if p is true then q is true."} \\
\hline
\text{l- q} & \quad \text{Assert: "q is true."}
\end{align*}
\]

In this analysis, the expressions asserted are all meanings; p and q occur as entities; and "__ is true" and "if __ is true, then __ is true" are indefinable meanings giving unity to the asserted expressions. The following questions remain unanswered, however: What sort of entities are p and q? And in the case that they should not exist, what significance are we to attach to the expressions in which they occur? These questions are not analogous to those posed by our interpretation of descriptions—they are substantially the same, for "__ is true" and "if __ is true then __ is true" appear


here as predicate and relation, and the existence, or "being" of their arguments must be secured for exactly the same reasons.

That propositions are termed, along with descriptions and expressions for classes, "incomplete symbols" in the Introduction to *Principia Mathematica* is a natural consequence of this line of reasoning. Since the meaning of all expressions must ultimately reduce to combinations of denoting simple names, it follows that every symbol which occurs as an argument in an expression must denote *something*. The complex sign may itself fail to have a denotation, but, in stating that some particular entities have a particular predicate, or stand in some given relation to one another, we assume that these objects, predicates and relations exist. Frege said that, in those cases where this requirement is not met, these latter expressions are *bedeutungslos*, and, consequently, that the entire expression is *sinnlos*: every well-formed sign has to have a denotation. Russell was unwilling to assign symbols arbitrary denotations in deviant cases, e.g. to say that "my shirt = 2" is false. While accepting the utility of such notions as description, class, and proposition, he did not want to admit them to his ontology, and he therefore sought to eliminate them contextually. He did this most successfully in the first case, that of descriptions. To eliminate classes, he applied the same general approach, but required, for this purpose, his Axiom of Reducibility. 30 In the third, he sought to replace expressions of the form "p is true" by a relation between two complexes—the judgment-complex and the complex whose existence is asserted. The definition of this relation is the crux of what Wittgenstein called his "theory of propositions."

Russell’s theory of judgment was thus to give an account of the role of the symbol "p" in "\( \neg p \)." Since the latter is the symbol for an assertion, it would seem

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30 The Axiom of Reducibility is of course also part of the *Principia Mathematica* Theory of Descriptions, for identity is defined by means of the Axiom of Reducibility.
natural to say that "\( \neg \)" represents the act of asserting and "\( p \)" the thing whose existence is asserted. But "\( p \)" obviously cannot denote the fact \( p \). If it did, then the mere consideration of "\( p \)" would entail that it was true. Thus Russell starts "On Fundamentals" with the assumption that whatever is asserted must be a meaning. But this position is hardly tenable: 1) it tends away from realism, if not towards idealism, by putting in question the relation between the meaning and its denotation. Russell must separate what was supposed originally to be a single entity possessed of both meaning and denotation into two complexes, the meaning and the complex whose existence is meant. 2) It aggravates the difficulty of describing the process of deduction. For if we want to understand propositions involving truth-functions, such as "\( p \supset q \)" "as expressing an indefinable relation of entities,"\(^{31}\) then we must assume that "\( p \)" in this case denotes an entity or fact, from which it follows that "\( p \)" in "\( \neg p \)" would have to do so as well. The alternative would be to assume that all the components of "\( p \supset q \)" were meanings, with the consequence that the implication expressed a connection between thoughts, instead of things--an idealist conclusion of the worst sort. Russell's preliminary solution in "On Fundamentals" was to interpret all these assertions as predicative or relational statements between pseudo-entities, taking "\( \neg p \supset q \)" to signify "if \( p \) is true, then \( q \) is true."

This position seems scarcely an improvement. Faced with the problem of explaining the role of propositional symbols in our calculus of assertions (What is "\( p \)" in "\( \neg p \)?) we have replied that the actual structure of an assertion is that of "\( p \) is true," in which case the question poses itself again. But we now have new possibilities open to us, for in so rephrasing the problem, Russell has made the move to, if not a meta-language, at least a meta-explanation. Instead of having to find a denotation for a single symbol, we have now to analyse a proposition which, although in subject-

predicate form, is susceptible to other kinds of analysis. One approach—that of
Wittgenstein and Tarski—is to read statements such as “p is true” as expressing a
connection between a symbol and a state of affairs (cf. 5.542). But Russell opted for
another solution, still with a view to eliminating mental entities from his theory of the
propositional sign. He stipulated that “p is true” be understood as expressing a
conjunction of two facts: a judgment-complex and its corresponding fact, that is to
say, “someone’s believing that p” and p. This is the theory we find in the later
writings.

ii) Theory of Knowledge and the “logical forms”

In “On the Nature of Truth and Falsehood,” Russell presented the judgment-
theory as a desideratum in its own right, thus obscuring its role in the theory of
deduction, whereas Russell’s outlines for Theory of Knowledge restore this original
context, for there the theory of judgment is to feed into topics in the section on
inference. The earlier theory differs also in that its judgment-complex was to consist
only of the subject and the objects (in Russell’s wide sense of the term) of the fact to
which the belief was to refer, e.g. “aRb” would be rendered as J(S, xRy, a, b);
“\neg aRb” becomes “aRb is true”, which is interpreted as J(S, xRy, a, b).aRb. This
conception accords well with the account of “On Fundamentals,” in which one
component of a complex—the relation—was to give “form and unity” to it, in ap-
ppearing as meaning. However Russell abandoned this position while writing Theory
of Knowledge. There the judgment-complex includes, in addition to the objects,
predicates and relation, a “propositional form” among its elements. There are two
distinct reasons for making this move.

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32Russell, Bertrand. Theory of Knowledge. The 1913 Manuscript. The Collected Papers of
Appendices A.3, A.5, and A.6, pp. 184-85; 188-89; 190-91.
(1) Since the purpose of rendering "\(aRb \text{ is true}\)" as "\(J(S, xRy, a, b).aRb\)" was, above all, to restrict appearances of the terms of the propositions to denotative occurrences, Russell could hardly give the relation \(xRy\) a meaning-occurrence in the judgment complex. If the expression for a judgment-complex is supposed to represent a fact, then all its terms must appear as denotations. At the same time, there had to be some entity providing the necessary form and unity to the corresponding complex. To see why this is so, we need only recall the purpose and presuppositions of the theory; namely to explain the nature of propositions and deductions without making the assumption that there be a "third realm" of entities such as Frege's senses. The intermediate step was to regard "\(\neg p\)" as being of the form "\(p \text{ is true}\)," which gives us a way of describing assertion and implication in which it is not the meaning of the symbol "\(p\)" that gets asserted, but its denotation. The next step is to eliminate "\(p\)" and "\(\neg p \text{ is true}\)" as subject and predicate, and to replace them with two complexes. When both exist, the symbols "\(\neg p\)" and "\(p \text{ is true}\)" are appropriate, and, where the second (the fact) does not, then "\(p \text{ is false}\)" is appropriate, and "\(\neg p\)" is incorrect. This is contextual elimination in the best sense: we can make use of "\(\neg p\)" and "\(\neg p \supset q\)" as we please--which we wish to do, because they provide a convenient notation--and we can eliminate both of them, as well as the pseudo-entities their use appears to assume, at will. But in the case that the "corresponding complex" \(aRb\) does not obtain, it must be perfectly clear what would have to be the case for the judgment to be true. I must know what I am asserting, and a person observing me (henceforth, my observer) and asserting of me that I hold a particular belief, i.e. that I stand in a judgment-relation to some things, must know this as well. And this means that her expression of my judgment must make it unambiguously clear what the corresponding complex would be, what fact would obtain if my judgment were true. This lack of ambiguity must be reflected syntactically in her expression for my judgment: 1) if two facts \(aRb\) and \(bRa\) are distinct, then her statement that I judge the one or the other must make clear
which one is in question; and 2) it must be clear that neither "Rba" nor "baR" are possible candidates for the name of the corresponding complex, since these symbols are in fact ill-formed. If one thought that \( J(S, R, a, b) \) meant "\( S \) asserts that \( baR \)," then one would have to conclude that, since no fact corresponding to "\( baR \)" will ever be found, that the judgment was always, that is necessarily, false, whereas it is in fact senseless. I will use the terminology introduced by Griffin in referring to these two types of ambiguity as 1) the "narrow" and 2) the "wide direction problems."

By introducing a form to which the various components of the hypothetical fact might be related, Russell perhaps sought to palliate these difficulties somewhat: the form takes on the role of a "template into which various objects could be placed." But this only defers the problem at hand. If \( J(S, R, a, b) \) is ambiguous in both the senses given above, then \( J(S, R, a, b, x\chi y) \) is hardly any better. The mere appearance of the expression for the form of a relation \( (x\chi y) \) does not remove either problem, since I still have to have some way of mapping \( R, a, \) and \( b \) onto the appropriate slots in \( x\chi y \); and if I make appeal to ordering conventions in doing so, then I might just as well have availed myself of them in the first place.

(2) The second, and I think more important, reason for introducing the form into the judgment-complex is that without it, we would have no way of seeing two relational judgments (thus two relational propositions) as instances of the same type of judgments. Since molecular propositions (those involving truth-functions, as well as generalised propositions) were to be defined in terms of elementary ones, it is obviously necessary that expressions for the elementary propositions reflect relevant similarities and differences among the facts with which they are associated. "A given dual relation," says Russell, "is still one of a class of more or less similar entities.

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namely dual relations;" that is, we understand that \(aRb\) and \(cSd\) have something in common, that both are instantiations of \((\chi, x, y) x \chi y\), even though none of \(a, b, c, d, R,\) or \(S\) is common to them. But, he continues, ""dual relation' itself, although it might seem to be one of a class whose other members might be 'triple relation,’ etc., is really, in a very important sense, unique, and not a member of any class containing any terms other than itself.” The form is the hallmark of a class of complexes.

Without it, we would not be able to distinguish between complexes in which \(a, b,\) and \(R\) all appear as relata, and ones in which \(R\) is the relating relation.

We can summarize these various takes on the nature of the proposition, or propositional judgment, by means of the following table, which gives us the premisses for a single elementary inference:

\[
\begin{align*}
\text{1- qa} & \quad \text{"qa is true"} & J(S, q, a, q) & J(S, q, a, \chi \gamma). q a \\
\text{1- qa} \psi a & \quad \text{"if qa is true then qa is true"} & J(S, q, a, \psi a, \psi a) & J(S, q, a, \psi a, \psi a). q a \psi a
\end{align*}
\]

The left-most column gives us the premisses for a deduction in *Principia Mathematica*; the second one shows us the form these must take if the elements of the proposition are to be seen as denotation occurrences following the analysis of “On Fundamentals;” the third column gives us the contextual elimination of the first two in accordance with the theory of “On the Nature of Truth and Falsehood;” and the fourth does the same, adding the logical form as in the incomplete account of *Theory of Knowledge*. I do not give separate analyses of the implication in the last two cases, since Russell never got to this stage in either work. Following through with the deduction in the first column, we conclude that 1- \(\psi a\). Our (metalogical) justification

\[35\text{Theory of Knowledge, p. 97.}\]

\[36\text{Russell abandoned *Theory of Knowledge* before arriving at his analysis of molecular propositions, but it is clear that he hoped to analyse them along these lines--the logical connectives (which include, one should bear in mind, negation) are also objects with which we have acquaintance, and which connect complexes composed of elementary propositions. Thus the } \phi a \text{ and } \psi a \text{ in the full analysis of } \phi a \psi a \text{ must themselves be rendered into the appropriate judgment complexes. See the section following for a discussion of the molecular complexes.}\]
for drawing this conclusion is given by the expressions in the second column. And the analysis in the third and fourth columns gives us an interpretation of the second column which disposes of the pseudo-predicate "__ is true" and the non-denoting symbols from the first two columns. This analysis is given from an indirect or second-order perspective, but it is not given in a meta-language, for we never leave the one language whose usage is in question.

c) Objections to the Judgment-theory

There are two obvious difficulties with Russell's approach. The first is the seeming redundancy of the whole analysis. By what right do we make use in our contextual elimination of the very symbols we wanted to eliminate? Have we not simply returned to the dilemma with which we began, which was to remove the possibly non-denoting complex name? The second is that the theory does not explain the connection between judgment-complex and corresponding fact. Their correlation must be both unique and necessary if this analysis is to be plausible, let alone explain anything.

As far as the first point is concerned, we must be clear about the purpose of Russell's contextual elimination, and about its implicit assumptions. The two steps of the analysis were: 1) to get rid of the meaning involved in an assertion by replacing it with a denoting symbol and the pseudo-predicate "__ is true", and 2) to eliminate both of these with a statement of the form "The elements of a judgment-complex are in fact configured as they are judged to be configured," or more concisely, "there is a factual complex corresponding to a given judgment complex." Now it is clear that, in this latter analysis, the judgment-complex has taken on the role of a Sinn or meaning, and the correlated complex that of the Bedeutung or denotation. The replacement of our expression for an assertion with an expression predicating truth necessarily drags the Bedeutung along with it. What corresponds to the "p" in the assertion "I- p" is
exclusively the judgment complex, and the function \( \hat{p} \cdot aRb \) (where \( \hat{p} \) is to take the judgment-complex as argument) is what corresponds to the pseudo-function "\( \_ \) is true." So we have indeed eliminated the non-denoting propositional sign appearing in the first two columns. We must, on the other hand, make use of it again when we wish to state explicitly the conditions under which the judgment may be said to be true or false. But that does not vitiate the analysis. Indeed we could, if we wanted, eliminate this new use of "\( aRb \)" as well, were we to subject the entire conjunction to a further analysis, and this \( ad \ infinitum \). Far from being a vicious circle, however, this only shows that in stating the truth-conditions for a given expression, we will always have to make use of some expression which has itself the very characteristics we are analysing. In the case where the conditions were at hand, we could explain the meaning of "\( p \) is true" by pointing to the fact \( p \). The reappearance of the symbols \( aRb \) in Russell's analysis results only from our needing to say what the truth conditions of the original expression are; and in this sense, Russell's judgment-theory is in good company with Tarski's Convention T.

If we read statements about my judgments as statements on the part of an observer, in which I appear as an object ("He judges that \( p \)"), then the legitimacy of this approach becomes clearer. She, the observer, is trying to explain what my propositional acts consist in. Under what circumstances is it correct to say that I judge rightly? Under which is it correct to say that I judge falsely? Clearly hers would be a bad account of judgment if she said that judging was a relation between a fact and me. Instead she says that my judging is a relation between me, the objects of my judgment, and the form of the fact whose existence is being judged. At the same time, however, it is clearly a goal of this analysis to give the meanings of my higher-level, molecular judgments \( in \ terms \ of \) the elementary ones. When I judge that \( pvq \), I adopt a propositional attitude towards myself that mirrors that of an observer towards me. For if I am ignorant of the status of \( p \) and \( q \), that is, in Russell's terminology, I am not
acquainted with them, I would have to express my meaning, to myself or to others, as
"Either a judgment to the effect that p, or one to the effect that q, is true." If pressed
on the meanings of the disjuncts, I would reply that "p is true" means "the complex p
exists" and I would have to spell that out along the lines given above. In the following
discussion I will distinguish between "her" (the observer) and "me" (the judging
subject), in order to keep these two levels of Russell's analysis clear. The claim that a
particular true proposition (aRb v ~aRb) must follow from one ascribing a judgment to
someone ("A judges that aRb") should thus be analysed with reference to the person
making those judgments, that is, in the manner of speaking that I just proposed, with
reference to her statements about me. What is she in fact saying about me when she
ascribes to me relations such as "He judges that p", "He judges that not-p", and "He
judges that pvq"?

She regards my uttering "pvq" as the expression of a judgment, and she
expresses her judgment to that effect as: "David believes that either 'p' is true or 'q'
is true." Her statement about me is, as always, true or false independently of the truth
or falsity of "pvq." But in order for her to ascribe a propositional (in this case
disjunctive) attitude to me, certain conditions must be met. If what I am doing is
judging that "pvq", then certain facts about me must obtain, for instance that I do not
hold that belief or judgment simultaneously with those that both p and q are false.
Now assume that she is right in saying of me that I judge p, but I am wrong in having
this belief, that is, it is not the case that p, the complex p does not obtain. On Russell's
interpretation, where p is not part of my mind, but a fact, we would get an empty
denotation. Since the full expression for p is in every case complex, e.g. qa, Russell
suggests that J(D, p) should instead be interpreted as J(D, qa), such that she may be
right in saying that of me, even though I am wrong in asserting what I do, because qa
and a are not in fact so combined. If on the other hand they are combined, then
obviously the nature of my relation to qa and a remains same: my act of judging is
completely independent of, though of course not indifferent to, the status of the fact which I judge to be the case. Consequently when my observer says of me, "David believes that $p \land q$" she must use one and the same expression both in the case that $p \land q$ obtains and in the case that it does not, otherwise it would not meet the critical requirement of independence from the world that Russell wanted--my judgment must remain external to the facts with which it is concerned.

We don't know how Russell planned to analyse molecular judgments; but as his remarks in the outlines for Theory of Knowledge show, he expected to have rehabilitated the term "proposition" by this point in the book:

_Molecular Propositional Thought [Not "Inference"]_

The essence of this is that and, or, not, all, some, etc. come in. We are in the region of logic with these notions.

_Observable. A Judgment requires acquaintance with one form of atomic complex. An inferential consciousness requires acquaintance with such terms as or and not, i.e. with a form of complex in which propositions are constituents._

Similarly we see Wittgenstein, in the summer of 1912, grappling with the question of what "$p \land q$" means (denotes), and he makes use, in an otherwise incomprehensible letter, of an inference-copula, which "copulates complexes." The complexes in question are themselves ε-copulae, Wittgenstein's version of elementary judgment-complexes. Both Russell's and Wittgenstein's theories were intended first to eliminate elementary propositional names by means of descriptions--in Russell's case they were to be replaced by judgment complexes--and, second, to define the molecular complexes in terms of these descriptions. There is little point in guessing at how the two men thought they could work this out; however we can say with some certainty what they wanted the theory to do.

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37Russell, Theory of Knowledge. Appendix A.5, p. 189. The remark "[Not 'Inference']" is Russell's, and refers to his earlier working title for this section.

38Notebooks, p. 120. Italics in original.
If, as in "On Fundamentals" my judgment that "p or q" can be understood as "either p is true or q is true," thus for Russell as "either a judgment that p, or a judgment that q, is true," then I, in making such a judgment, adopt the same position with respect to the elementary judgments as the observer in the previous examples did to me. She didn’t need to know whether the fact p obtained when she asserted “He believes that ‘p’ is true”; although we must assume that both she and I were clear on when my judgment was, in fact, a true one. And if she says of me “He believes that at least one of either ‘p’ or ‘q’ is true”, then this can be read as a statement about a complex relation between me and all the various entities possibly composing the facts p and q. If someone attributes such a belief to me, then she expects that should I believe that “‘p’ is not true”, i.e. that “p is not the case”, I will of necessity agree that “‘q’ is true”, that “q is the case”. Whatever analysis we give of my disjunctive judgment-complex, it must be such that it is impossible for me to hold p \lor q, not-p and not-q simultaneously, and we will have to interpret this impossibility in the strongest sense. If the analysis of my judgment is such that it even seems to allow for the simultaneous judgment of all three, then we will have to conclude either that a) the analysis is only partially realised, or b) the complex being analysed--my relation(s) to the objects of my judgments--was something else, perhaps some sort of psychological state, but not in any case a propositional judgment.

Now one might ask how we are to understand this kind of necessity. Is it a logical necessity, that is, a necessity born of what she can conceive about me? Or is it perhaps a physical one, given by the combinatorial possibilities inherent in me and the objects of my beliefs? It lies at the heart of Russell’s project that these two types of impossibility are, in the analysis, to converge. For me, my inability to judge inconsistently is grounded in the nature of judgment, in my nature as a judging subject, and in the natures of the objects about which I judge. On the other hand my observer’s descriptions of my belief must reflect that necessity. If she cannot conceive how I, on
the basis of her analysis, could possibly hold those inconsistent beliefs, then she will have grounds for believing her analysis to be correct. There is, I think, no better criterion for the accuracy of her analysis to be had, at least not without making endless and regressive appeals to other observers observing her, *et cetera*. The objects related in her judgments about *my* judgments are the same objects related with me in mine--the same *a, b, and R*. My inability to hold inconsistent beliefs and her inability to judge that I do so will be grounded in the same properties of the world, namely in the properties of the objects themselves, including subjects, relations, terms, *et cetera*; and it therefore becomes a matter of taste whether we call these properties physical or logical.

As I said, we do not know what the analysis of \( J(D, pvq) \) would actually look like; however, we do know what it is supposed to predict about *me*. When confronted with evidence of the fact that *q* does not obtain, I judge that *p, J(D, p)*. In other words I have to understand that the absence of *q* means that ‘*q*’ is not true, where this understanding will manifest itself in my being able to conclude that *p*. Now if the full analysis of ‘*q*’ is ‘*Fa*’ (remembering that the whole point of the undertaking was to get rid of non-denoting propositional signs such as ‘*q*’) then according to the Russell/Wittgenstein complex-theory I am supposed to conclude that *p* when I see that not-*Fa*, that *F* and *a* are not combined in a complex. As Wittgenstein put it to Russell in the “Notes on Logic”:

> To understand a proposition *p* it is not enough to know that *p* implies “‘*p*’ is true”, but we must also know that ~*p* implies “*p* is false”. This shows the bi-polarity of the proposition.\(^39\)

In other words, from my awareness that *Fa* does not obtain, I should *judge* that ~*Fa*. The fact that *F* and *a* are not combined does seem to imply that ‘‘*Fa*’ is false”, but in fact this follows only if additional conditions, to which I turn in a moment, are met.

\(^{39}\text{Notebooks, p. 94. cf. my discussion of Frege’s “Die Verneinung” in Chapter 5.a.i.}\)
Before doing so, however, I should emphasise that this point gets its importance only from the connection of elementary judgments to molecular ones in the process of inference, that is in the context in which Russell first addressed it nine years before in "On Fundamentals." Russell wanted an explanation of what takes place when a thinking subject goes from fact to proposition, moves deductively from proposition to proposition, and lastly returns to the facts, as when, e.g. I see that not-$Fa$; I assert "$Fa$ is false"; I combine that with "Either $Fa$ or $p$ is true" to conclude that "$p$ is true"; and lastly turn with confidence to the fact $p$. Just as an observer can describe my expressions (the incomplete symbols of "On Fundamentals" and the "Introduction" to *Principia Mathematica*)—so can I regard them myself: "not-$Fa$" is a reflection of my judgment that not-$Fa$; "$Fa \lor p$" of my judgment that either $Fa$ or $p$ are true, that is, that one of the two complexes $Fa$ or $p$ obtains; and the formal laws that I give myself for manipulating those symbols are such as capture the relations between my judgment acts and the entities, and combinations thereof, which those judgments engage. In *The Principles of Mathematics* Russell was unable to conceive of such a process clearly, because he literally did not distinguish between assertion and truth, nor, as a consequence, between proposition and fact:

In the first place, it seems doubtful whether the introduction of truth-values marks any real analysis. If we consider, say, "Caesar died," it would seem that what is asserted is the propositional concept "the death of Caesar," not "the truth of the death of Caesar." The latter seems to be merely another propositional concept, asserted in "the death of Caesar is true," which is not, I think, the same proposition as "Caesar died." ...  

It is almost impossible, at least to me, to divorce assertion from truth, as Frege does. An asserted proposition, it would seem, must be the same as a true proposition.40

I quote this remarkable passage not to bury Russell, but to emphasise that he made no clear division between the world of facts and that of propositions at this time, and that it is Frege who put him on the right track. The judgment-theory is an attempt to make

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the notion of deduction within a system of thoughts or symbols—a system both
distinct from and yet connected to the world—a cogent one. But whereas Frege was
able to describe his system in a sort of meta-language (he had terms for “true” and
“false”, and he used them to define his truth-functions), Russell had deliberately
denied himself this resource. And Frege had also failed to explain the difference
between propositions and names in general, with consequences Russell (and
Wittgenstein) could not accept. For instance, Frege’s definition of implication did not
demand that either antecedent or consequent of a conditional be significant (i.e.
denote a truth-value), but only that they have denotations. This meant that when
applying, for instance, Modus Tollens to “p implies q” we would literally not know
whether the inference from not-q to not-p meant that the proper-name p (on its own,
without the content-stroke) denoted the False, or simply had a denotation other than
the True. Frege allowed this ambiguity because he wanted to let his quantifiers range
over values for which the functions in a conditional might not have truth-values, and
of course he had to allow for this possibility in the definition of the conditional itself.
He could afford to do so because the symbol for the antecedent or the consequent of
an implication remained significant for him, even when it was false. Russell and, in
consequence, Wittgenstein would not allow such non-denoting symbols. At the same
time they expected the theory of judgment to yield a definition of “proposition,” by
means of which they would then be able to define the logical properties. But the
theory with which they were working could not satisfy both demands simultaneously.
In order to work out this opposition fully, I turn in the next chapter to the relation
between the notions of proposition and implication in the Grundgesetze, Principles of
Mathematics, and Principia Mathematica.
Chapter 4 - Russell's Judgment-theory II - Judgment and Implication

The judgment-theory was intended to provide an account of the concepts proposition, true and false that would explain the following properties of propositions: i) they do not denote; ii) they are always either true or false. It explains i) with the relation between judgment-complex and corresponding complex. The judgment-complex does not name or denote the corresponding complex: it describes it by specifying its components and, somehow, the manner of their arrangement. The theory explains ii), the bivalence of the proposition, by means of the ontological bivalence of the corresponding complex: when it exists, the descriptive judgment-complex finds a referent, and the proposition associated with that complex (the content of that judgment) is said to be true; when it does not, the description fails, and the associated proposition is said to be false. Once the bivalence of the elementary propositions is secured, we can define truth-functions, indeed quantified propositions, as predicates and relations of and among the elementary propositions or judgments. Wittgenstein's critique of the theory lays great weight on the point with which I concluded Chapter 3: it must be the case that, from our knowledge that F and a are not combined, the proposition ""Fa' is false" follows. Why is this requirement essential to the theory, and why does the theory fail to meet it?

In answering the first question, we need simply point to the postulate of bivalence, and the theory's explanation of the latter. The proposition is bivalent because the fact is "bivalent"--either it exists or it does not. The theory associates one of two possible states of affairs--existence and non-existence--with two possible truth-states of the proposition--true and false. If the non-existence of the fact were not equivalent to the falsehood of the proposition describing it, then clearly the duality of the fact would not ground the bivalence of the proposition. Therefore the correlation of non-existence and falsehood must be definite.
In answering the second, we must consider how this connection between non-existence and falsehood might break down. That is, we must explain the doctrine I mentioned at the opening of Chapter 3: Wittgenstein's belief that "Fa' is false" means not simply that "F and a are not combined," but "F and a may possibly be combined, but are not in fact combined." This qualification is needed only on the assumption that there are kinds of things which cannot combine with each other—if there are, to use Wittgenstein's language, "different Types of things." Consider the statements:

   a) x and y are not combined,
   b) 'xy' is false
   c) |- ~xy

If c) is valid, it follows that x and y are significant arguments for one another (they saturate one another); however c) is on the judgment-theory merely a paraphrase of b). The question is: Does a) entail b) for all x and y? Since the variables in a) could refer to anything, we must reply as follows: if there is only one type, such that every object may combine with every other, then statement a) entails statement b) for all x and y; however, if there are classes of objects that may not combine, then the passage from a) to b) must be blocked in their case. Suppose this were not so: I assume two types of two members each—\{a, b\} and \{F, G\}—such that a and b are significant arguments for F and G, but neither F nor G, nor a nor b can be arguments for each other. We have,

   a) F and G are not combined,
   b) 'FG' is false
   c) |- ~FG

From c) I can infer the existence of a function,

   \[ p \alpha = \sim Fx \text{ Def.}. \]
for which \(a, b, G\) are all significant arguments (\(a\) and \(b\) are postulated as significant arguments for \(F\), thus also for \(\neg F\); \(G\) is significant for \(\varphi\) by c) and the definition). It follows that \(a, b, G\) are of the same type, contradicting our initial assumption. Since a) is true in this case however, it follows that the move from a) to b) was illegitimate; i.e. that

\[
\begin{align*}
a) & \quad x \text{ and } y \text{ are not combined } \quad \text{b) } \quad 'xy' \text{ is false}
\end{align*}
\]
does not hold of all \(x\) and \(y\) in a world with types, unless we assume additional constraints on the arguments in “\(x\) and \(y\) are not combined.” They will be such as to ensure that \(x\) and \(y\) can indeed possibly combine with one another.

But there is a further complication. Asymmetric relations generate distinct complexes out of the same set of components, but the components of the judgment-complex are given as a list--we cannot tell, without ordering conventions, whether \(\{a, R, b\}\) is a description of the fact \(aRb\) or \(bRa\). For that matter, how do we know that \(a, R,\) and \(b\) exist, that “\(a\)” has a denotation at all? Since we are concerned with a false proposition (the complex \(aRb\) does not exist), how can we know by inspecting the list \(\{a, R, b\}\)

1) That its elements may possibly combine?
2) Which of their combinations is intended?
3) That all its elements exist?

Without that information, we are not entitled to go from the fact of their not being combined to the assertion \(\vdash \neg aRb\). We have no Fregean senses which preserve, in the absence of the objects and complexes denoted by our language, the intension of our assertion. The conditions 1-3 are of great importance to the following discussion; indeed, I would argue, to any understanding of the *Tractatus*. For they are all instances of propositions on whose truth an elementary proposition would depend for its sense--if such a thing were possible, which Wittgenstein denies. Thus it is essential to the *Tractatus* that all elementary propositions satisfy 1-3 from the outset; and, as
we shall see in Chapter 5, a good part of the picture theory can be derived from them alone.

All three requirements have been discussed in the literature: 1) is the point of the Griffin/Somerville reading; 2) is connected to the "logical forms" discussed by Pears; and 3) leads to the requirement that there be simple objects—the one of these theses that is both well-known and well-understood. Before taking them up in detail, however, I want to look at the relation between, on the one hand, the judgment theory's definitions of proposition and truth, and, on the other, the truth-functions (above all negation and implication) that get defined in terms of these notions. For it is only because of that use of the theory—the connection between judgment and the theory of deduction—that the requirement of bivalence, thus requirements 1-3 and the related Tractatus doctrines, get their urgency.

a) The Complementarity of Proposition and Implication

In the system of Frege's Grundgesetze, True and False are indefinables, and propositions are simply a sub-class of proper-names: those which denote either the True or the False. "My shirt is red" denotes the True if and only if my shirt is red; however, Frege studiously avoids giving an explanation of the relation between the expression and its truth-value. That is, he does not define the circumstances which licence the connection between expression and truth-value, as does Russell when he says that an expression is true when its corresponding complex exists. Clearly there are conditions under which we rightly say "p is true" and "p is false"; however to specify these conditions generally (assuming such a thing is possible) is not to do logic. Logic describes the laws of thought according to which we move from true statements to other true statements. How we arrive at the truth of the premisses, and how we interpret the truth of the conclusion is irrelevant to those laws. Russell advanced a similar view in his Principles of Mathematics: there, implication is an
indefinable stating a logical relationship between entities of a specific sort, that is, propositions.

To make this point clear, we need to look at Frege's definitions of implication and negation. For Frege "\(\neg \varphi \alpha\)" (the "\(\neg\)" is the content-stroke) denotes the False when either 1) "\(\varphi \alpha\)" denotes the False, or 2) "\(\varphi \alpha\)" denotes something which is neither the True nor the False (call it "N"). Negation in the *Grundgesetze* is,

\[
\begin{array}{ccc}
\varphi \alpha \\
T & N & F \\
\end{array}
\]

\[
\begin{array}{ccc}
\neg \varphi \alpha \\
F & T & T \\
\end{array}
\]

where the lower row gives the denotations of "\(\neg \varphi \alpha\)" for the denotations of the proper-name "\(\varphi \alpha\)"—without the content-stroke. Implication is,

\[
\begin{array}{ccc}
\varphi \alpha \\
T & N & F \\
T & F & F \\
N & T & T \\
F & T & T \\
\end{array}
\]

for the implication "\(\psi \alpha \text{ implies } \varphi \alpha\)". That is, the implication denotes the False if the antecedent is the True and the consequent is *not the True*; otherwise it denotes the True. For Frege the judgment that the consequent is false, "\(\varphi \alpha\) is False", holds only when \(\varphi \alpha\) (on its own, without the content stroke) denotes the True. If it denotes something else, then not-\(\varphi \alpha\) is true, so Modus Tollens cannot be applied, and no inferences can be drawn. For Russell and Wittgenstein "\(\varphi \alpha\) is False" meant: \(J(S, \varphi, a)\) and \(\varphi \alpha\) does not obtain. So if all we know is that \(\varphi\) and \(a\) do not form a
complex, we will not know whether this means that "\(qa\) is false", or that, following
the above schema, "\(qa\) is N(ot significant)". Thus it is critical that we have a
definite way of getting from the lack of a connection between \(q\) and \(a\) to the judgment
that "\(qa\) is false". That was the point of Wittgenstein's criticisms of both his own
and Russell's theories.

Consider, for instance, the Fregean conditional:

\[
\frac{2=3}{\text{My shirt}}
\]

This conditional is true because "my shirt" does not denote the True. "2=3" on the
other hand denotes the False. So, applying Modus Tollens, we get the result:

\[
\frac{\text{My shirt}}{-\text{My shirt}}
\]

Meaning that

\[
\text{My shirt}
\]

denotes the True, and "--my shirt" denotes the False. From this it does not follow,
however, that "my shirt" (without the content-stroke) denotes the False---quite the
contrary, it refers to my shirt. Frege allowed this ambiguity because he wanted the
variable in an expression such as \((x). \psi x \supset \phi x\) to range not only over the type for
which \(\phi\) and \(\psi\) are significant predicates, but over all proper-names, many of which
will of course not belong to \(\phi\)'s and \(\psi\)'s significance range (note that if there is an
argument for which \(\psi\) is true and \(\phi\) is not significant, then \((x). \psi x \supset \phi x\) is false).
Clearly this can only work if implication has a truth-value under all possible sub-
stitutions: if we know that

\[
\text{\lnot qa}
\]

denotes the True, then we must be able to move directly to
from the general conditional; otherwise the inference would require that we know what "a" refers to. Modus Tollens would then not be a logical law of inference, but would depend instead in each case on our knowing something specific about the objects with which the premisses of the inference were concerned.

So Frege’s unconstrained quantification necessitates truth-functions taking arguments with denotations other than the True and the False. The inference:

\[ \neg \psi a \]

is a valid logical inference even in the case that "\(\neg \psi a\)" and "\(\psi a\)" do not denote truth-values, in other words even when they are not propositions at all. One virtue of this approach is that it frees us from any obligation to define what propositions are, in distinction to other sorts of proper-names. The only constraint on legitimate symbols in the Begriffsschrift of the Grundgesetze is that every proper-name denote. What they denote is not essential to the significant use of the calculus.

Wittgenstein and Russell agreed that all symbols had to denote; however their complex-theory of truth made “false” a property of (non-denoting) propositional symbols: “‘p’ is false” means “p” might denote, but in fact does not. The “might” of “might denote” has to cash out somehow, otherwise “p does not denote” and “p is false” will coincide. Now when I make the inference: \(\neg p \supset q, \neg q \) therefore \(\neg \neg p\), I have, on both Frege’s and Russell’s accounts, argued from the falsity of “\(\neg q\)” to the falsity of “\(\neg p\)”. For Frege, “‘\(\neg q\)’ denotes the False” or “‘\(\neg q\)’ denotes the true” are perfectly unexceptionable propositions. They say that the thought expressed by “\(\neg q\)”
denotes the false. For Russell, however, there are no thoughts or senses. So what does it mean to assert that "'p' is false" or "'~p' is true"?

We can see Wittgenstein's remark in the letter as a simple inversion of one of Russell's early objections to Frege's definition of implication. Russell maintained in *The Principles of Mathematics* that "a definition of implication is quite impossible," because any attempt to state its meaning would itself make use of the notion of implication. On the other hand,

It may be observed that, although implication is indefinable, *proposition* can be defined. Every proposition implies itself and whatever is not a proposition implies nothing. Hence to say "'p is a proposition" is equivalent to saying "'p implies p"; and this equivalence may be used to define propositions.¹

In *Principia Mathematica* this dependence is inverted:

The idea of implication, in the form in which we require it, can be defined. ... The essential property that we require of implication is this: "What is implied by a true proposition is true." It is in virtue of this property that implication yields proofs.²

This "essential property" is PM*1.1, the first proposition of the book. We "cannot express this principle symbolically"³ for the same reasons Russell gave in *The Principles of Mathematics*, to which passages Russell and Whitehead direct the reader. However *elementary proposition, assertion, disjunction and negation* are primitive ideas in *Principia Mathematica*, and they have symbolic expressions---expressions whose definitions are essentially those given in "On Fundamentals:" ¹⁻¹⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻مادة 90

¹ *The Principles of Mathematics*, p. 15.


³ *Principia Mathematica*, p. 94.
properties; but Russell cautions that the definition of \( \neg p \) as "it is true that \( p \)" should not be taken at face value, for "philosophically this is not exactly what it means." His reason for making this qualification should be clear enough from my discussion in the last chapter. \( p \) does not in fact denote.

I should emphasise that this sense of "define" is not that used within the body of *Principia Mathematica*. A proper PM definition is (supposed to be) purely syntactic: it gives a way of translating sentences using a defined symbol into sentences which do not make use of it. The truth-functions "defined" at the beginning of PM are of course indefinables in this sense. At the same time, however, Russell and Whitehead wish to give a philosophical explanation of what those terms mean, which desire is intimately connected with the problem of explaining what inference—as opposed to implication—is. Consider, for comparison's sake, Frege's statement of his rule of detachment in the *Grundgesetze*:

\[
\text{Aus den Sätzen } '\Gamma' \text{ und } '\Delta' \text{ kann geschlossen werden: '}\neg \Gamma' \text{; denn, wäre } \Gamma \text{ nicht das Wahre, so wäre, da } \Delta \text{ das Wahre ist, } \Gamma \text{ das Falsche.}
\]

\[
\text{From the propositions '}\Gamma' \text{ and '}\Delta' \text{ one can conclude '}\neg \Gamma' \text{; for, if } \Gamma \text{ were not the True, so would be '}\Gamma', since } \Delta \text{ is the True, the False.}
\]

The rule explains how we are to move from two symbolic expressions to a third. The expressions in question express judgments—for Frege as well as for Russell. It says, in

other words: from a true implication and a true proposition which is also the antecedent of the implication, we can conclude that the conclusion is true, which judgment we symbolize by means of an expression, \textit{i.e.} "$\rightarrow \Gamma"$. Frege explains why the rule holds in semantic terms, that is by means of the terms "True" and "False", and by means of the conditions under which an implicational symbol is to be called True or False. If \( A \) is true and \( \Gamma \) were false, then the implication would not denote the true (by the definition of implication) which is absurd; thus the consequent \( \Gamma \) must be true.

Now compare this to PM*1.1: it says that "Anything implied by a true elementary proposition is true."\textsuperscript{5} This is clearly not an expression that has a meaning within the system of \textit{Principia Mathematica}. But if we interpret it from the point of view of the judgment-theory, we can see that it is in fact the rule of detachment, defined metalogically on analogy with Frege. For if "follows from \( x \)" means "appears as the consequence of \( x \) in a conditional," then *1.1 says: if an implication is true and its antecedent is true, then the consequent is true as well. The relation cannot be \textit{defined} within \textit{Principia Mathematica}, since PM definitions are purely paraphrastic. They merely unpack propositions into synonymous ones consisting of more basic symbols, so that they never take us from one proposition to another, different one. The section on "Primitive Ideas," and Pp. *1.1 of \textit{Principia Mathematica} are supposed to fill the same gap in the system as are Frege's rule of detachment and his definition of implication. But Russell thought (and I take it that this was mainly Russell's preoccupation) that the metalogical terms that Frege deliberately left undefined---True, False, proposition, individual, function, \textit{etc.}---could themselves be given a philosophical reduction, by means of a theory of judgment. Jean van

\textsuperscript{5}\textit{Principia Mathematica}, p. 94.
Heijenoort\textsuperscript{6} argues that Russell is in some sense confused about the distinction between object- and meta-language, even at the time of writing \textit{Principia Mathematica}. But while that may be true of the discussion in \textit{The Principles of Mathematics}, in \textit{Principia Mathematica} Russell is clearly aware of the distinction: \*1.1 is singled out as a proposition of a completely different sort from the axioms, and the fact that it is flagged "Pp." (primitive proposition) is thus of subsidiary importance. Russell had understood Frege's rule (for how could he not have, with seven years to think about it?), and he thought he could give a more thorough analysis of Frege's semantic notions. Thus he \textit{declined} a metalogical explanation of inference, just as Frege declined a semantic explanation of truth and falsity.

Although the essential property of implication cannot be expressed symbolically,\textsuperscript{7} we can, Russell and Whitehead believe, express it in terms of the notions "either ___ is true, or ___ is true" and "not ___". Of course the connections between "not-\(p\)", "\(p\) is false" and "\(p\) is true" are not explained. One needs to add to this account that the last two are mutually exclusive (non-contradiction), and that the first two are the same (the excluded middle); for, as we have seen, the identification of false and not-true does not go without saying. The cumbersome exposition of \textit{Principia Mathematica} really only makes sense if we read it as an inversion of the definition of proposition by means of implication in \textit{The Principles of Mathematics}. There a proposition is whatever may truly be both the antecedent and consequent of a single implication, whereas in \textit{Principia Mathematica} implication is the relation that a proposition always has to itself, and may have to other propositions. In neither case does Russell raise the question of negation, or of the meaning of false propositions--


indeed the essential property of implication "by no means determines whether anything, and if so what, is implied by a false proposition." Why do Russell and Whitehead say this? Because Russell knew from his reading of Frege that the "essential property" of implication that he uses here—the relation between two entities permitting one to infer the truth of the second from the truth of the first—is far weaker than that which he in fact defines. Implication is defined in terms of both negation and disjunction as: when \( p \) and \( q \) are propositions, either \( p \) is false or \( q \) is true. From this it follows that implication as it is symbolized in *Principia Mathematica* holds exclusively between propositions, and also that, from the falsehood of the consequent of an implication, the falsehood of the antecedent follows as well. We see Russell make the move from the weaker to the stronger definition in the following sentences:

But this property by no means determines whether anything, and if so what, is implied by a false proposition. What it does determine is that, if \( p \) implies \( q \), then it cannot be the case that \( p \) is true and \( q \) is false. i.e. it must be the case that either \( p \) is false or \( q \) is true. [italics added]

The first clause in italics is Frege’s definition of implication: it leaves open the possibility that either \( p \) or \( q \) might be senseless, that is that they might be neither true nor false. The second clause is the *Principia Mathematica* definition: it secures a connection between the falsehood of \( q \) and the falsehood of \( p \). Since the connective "or" always takes *propositions* as arguments, the definition of "\( p \) implies \( q \)" as "\(~pq\)" assumes that \( p \) and \( q \) are always either true or false.

Russell knew that these two definitions of implication were not the same, and he had explained the difference himself seven years earlier:

The relation which Frege employs as fundamental in the logic of propositions is not exactly the same as what I have called implication: it is a relation which holds between \( p \) and \( q \) whenever \( q \) is true or \( p \) is not true, whereas the relation which I employ holds whenever \( p \) and \( q \) are propositions, and \( q \) is true or \( p \) is false. That is to say, Frege’s relation holds

---

8 *Principia Mathematica*, p. 94.

9 *Principia Mathematica*, p. 94.
when \( p \) is not a proposition at all, whatever \( q \) may be; mine does not hold unless \( p \) and \( q \) are propositions. His definition has the formal advantage that it avoids the necessity of hypotheses of the form "\( p \) and \( q \) are propositions"; but it has the disadvantage that it does not lead to a definition of proposition and of negation.\(^{10}\)

Since Russell takes "not-\( p \)" to be the same as "\( p \) is false", it follows for him that it can be defined as "\( p \) implies all propositions". This definition assumes that it makes sense to speak of the totality of propositions, which possibility Russell soon had to reject. But he stuck with the reciprocal relation between material implication and the notion of a proposition. If I have a definition of "proposition," then I can, as Russell and Whitehead do in *Principia Mathematica*, define my various primitive truth-functions in terms of it. The notion of a "significant proposition" must include that of bivalence, otherwise the definition of negation would not hold, that is "not-\( p \)" and "\( p \) is false" must mean the same:

\[ (5) \text{Negation. If } p \text{ is any proposition, the proposition "not-\( p \)" or "\( p \) is false", will be represented by "} \neg p \text{".} \]

This definition, coupled with that of disjunction (if \( p \) and \( q \) are propositions, either \( p \) is true or \( q \) is true) yields a definition of implication (in the metalogical sense discussed above). So a proposition is something which must be either true or false, otherwise the definition will not do the job. If on the other hand I understand a proposition as something which can be true and can be false, although not both simultaneously, then "not-\( p \)" need not be the same as "\( p \) is false", and it need not be that from, e.g. \( p \lor q \) and not-\( p \), \( q \) follows. Thus for Frege,

\[ \vdash p \lor q \text{ is} \]

\[ \begin{array}{c}
q \\
\hline
p \\
\end{array} \]

that is: \( \vdash \neg p \) implies \( q \); and from

\(^{10}\)The Principles of Mathematics, pp. 518-519.

\(^{11}\)Principia Mathematica, p. 93.
\[ \sim q \text{ we get} \]

\[ \vdash \sim p \]

that is: \[ \vdash \sim \sim p. \]

A judgment which may be true even when "p" does not denote a truth-value. Since the theory of judgment is supposed to explain what a proposition is, and under what circumstances we say that a proposition is true and under which that it is false, it is critical to its role in Russell's theory of inference that the proposition be bivalent.

Wittgenstein's demand that from the analysis of a judgment, the bivalence of the propositional content should follow, would have been perfectly understandable to Russell, since it was in essence Russell's own point of view to which Wittgenstein was alluding.

b) Wittgenstein's "Other Premisses"

As we have seen, however, the bivalence of elementary propositions would be an immediate consequence of the judgment-theory only if three requirements are met:

1) the objects related in my judgment must be appropriately matched,
2) it must be clear which combination of them I intend, and
3) the objects in question must exist.

Requirement 3) is not a problem for Russell's theory, because it assumes that I can only make judgments about objects with which I am acquainted. Indeed the objects of my judgment are actual constituents of the latter, so the very possibility of a judgment's occurring depends on their existence: judgment carries existential import, so long as what gets judged is not merely a thought. The first requirement is trickier. It says, in essence, that only those propositions whose elements can possibly combine are properly propositions. Since propositions are abstractions from judgments, it follows that only those sets of objects which can possibly combine can occur as legitimate constituents of a judgment: only judgments about such sets are properly
judgments. On the Griffin/Sommerville reading, this was the critical flaw in the judgment-theory: without type-restrictions, the range of the judgment-relation is too wide, meaning that there will be judgments, and consequently propositions, that are not bivalent. But this is only an apparent difficulty, as we shall see in a moment, for the judgment-relation is also a relation, and it takes arguments of various types (it is "heterogeneous"). Thus its type-restrictions carry over to the corresponding complex: the judgment-theory carries type-import, we might say. Russell tried to meet the second requirement by introducing a "form" into the judgment-complex. It was to reflect the structure of the (possibly non-existent) fact whose existence the judgment asserts, thereby preserving the information he lost by rendering, e.g. asymmetric complexes as an unordered list. The ontological status of such forms wobbles in Russell's manuscript; on the one hand they are like fully abstract facts, whose existence is asserted in propositions such as $(\exists \varphi, x) \varphi x$, on the other hand they are likened to (logical) "objects" with which we are acquainted. Since, however, they are necessary components of the judgment-complexes, they must exist if the elementary propositions are to have sense. It is this requirement that proves fatal to the theory.

c) The Type-constraints

Wittgenstein proposed to solve the problem of type-mismatches by postulating a symbolism with Fregean properties: illegitimate propositions would be avoided by means of a symbolism whose syntax prevents their formation in the first place. The combinatorial possibilities of the objects would be reflected in the structure of their names. He did this because he thought that type-constraints could not do what they were supposed to.\textsuperscript{12} In Russell's theory, there is no independent symbolism, and such a move is therefore not possible: what corresponds to a proposition is a judgment, and

\textsuperscript{12}In Chapter 5.a.i I discuss his reasons in detail.
judgments are made up of objects, not of names. If we are to eliminate nonsense propositions, we must do so by eliminating nonsense judgments: it must be impossible for me to judge a nonsense. Once again we must consider this impossibility from two points of view:

1) How do I, the judging subject, avoid making such judgments? What knowledge do I make use of in avoiding them?

2) How is this knowledge explicit in an observer's description of me, that is in her expression for my judgment complex?

One must keep these two aspects distinct in order to avoid the following mistake. In order to judge that $Fa$ without being acquainted with that fact, I must know something about $F$ and $a$, namely, I must know that they could form a complex, even if they do not. Obviously someone describing me and my judgments must be able to describe that knowledge, and she would be perfectly entitled to call that knowledge a condition of my making that judgment. She might then be tempted to ask, "How does he know that $F$ and $a$ could form a complex? Could he not be mistaken?" And a plausible conclusion would be that if I made a judgment in the absence of such knowledge, then I might be judging nonsense. That is, if we conceive of knowledge of the type-relationships as contingent knowledge, then it would seem that I would be capable, in its absence, of forming nonsense judgments. But my observer could just as well invert the relationship, concluding instead that, "If he didn't have that knowledge, then he couldn't make a judgment at all, that is, whatever he were doing, I wouldn't call it judgment." My implicit knowledge of the type-relationships may accordingly be assumed: it will be reflected in her description of my judgment by the correct type-matching of the arguments to the judgment relation. What I do psychologically when I judge is not important. I am said to be judging when I stand in a particular relation to certain kinds of things, and when I fail to stand in that relation to those kinds of things, then I am not said to be judging. For example, if $J(S, \varphi, x, \psi x)$ is the predicative judgment-relation (the last element being the form of predicative facts),
then we can demand that the arguments \( \varphi \) and \( x \) be filled by predicates and individuals only. If they are, then \( J(S, F, a, \psi x) \) is well-formed, and so is \( Fa \); if they are not, then the first expression is nonsense anyhow (it fails to describe a judgment) so the question of \( Fa \)'s significance is moot.

If we do not keep these two perspectives distinct, we fall into the trap of thinking that the conditions of my judgment are *antecedent* to my act of judgment, that I first determine that the objects *could* be related and subsequently judge that they are or are not. This danger is aggravated if we do not keep distinct the act of *judging*, a fact about me, which falls under that category only if significance conditions are met, from other relational complexes I might enter into (these might look like judgments, but would be *e.g.* mere psychological relations). Griffin, in his most recent and clearest statement of his reading of Wittgenstein's letter, sums up his position as follows:

If the theory of orders is to be supported by the theory of judgment it must be the case that there are no facts about the world of the form "A judges that \( aRb \)" where \( aRb \) is non-significant; similarly, if there are to be no higher-order judgments of the kind "\( aRb \) is true". In order to ensure that there are no such facts as these, it is necessary that certain propositions be true. In particular, it is necessary that the following all be true: (1) '\( a \) is an individual', (2) '\( b \) is an individual', (3) '\( R \) is a first-order dyadic relation'. Whether we say these propositions are logically required for a derivation of '\( aRb \lor \neg aRb \)' or are merely presupposed by it does not matter. For these propositions, like all others, are logical fictions and their truth must be derived from the truth of corresponding judgments. Thus, in order to ensure that any given judgment is possible, it must be possible to ensure that some other judgment is true. But if these judgments are to be true they must be possible, and thus will require that further judgments be true. ... Thus before we can make the elementary judgment that '\( aRb \)' is true we must be able to make the higher-order judgments that (1), (2) and (3) are true, the exact opposite of what Russell intended.\(^{13}\)

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\(^{13}\)Griffin, Nicholas. "Was Russell Shot or Did He Fall?" *Dialogue*, 30 (1991). pp. 551-552.
But as both Tully\textsuperscript{14} and Wrinch\textsuperscript{15} have observed, Russell is entitled to assume that a judgment involves implicit type restrictions—the very ones that her judgment that I make that judgment assumes. Griffin himself remarks in his earlier paper that, Wrinch is alone in noting that there are already type restrictions on the terms of the judging relation—in this it doesn’t differ from any other relation of Russell’s formal system. It is clear such restrictions could be used to ensure that nonsense could not be judged were it desirable to ensure this.\textsuperscript{16}

Having said this, he drops the matter, although it seems reasonable to suppose that Russell would indeed have found it desirable to ensure this. But Wrinch’s point is given good support from Wittgenstein himself. In the letter of June 1913, he says that, “... from the proposition ‘A judges that (say) $a$ is in a relation $R$ to $b$’ ... the proposition ‘$aRb \lor \neg aRb$’ must follow directly ...” (italics added). That is, he uses the weaker, Fregean notion of implication that I outlined above: on the assumption that the antecedent is a true, thus significant, expression, it must follow that the consequent is one as well. Clearly if the antecedent is not significant, then nothing follows, good or ill, from the implication. Since $J(S, a, R, b)$ is itself a proposition, in that it is a significant statement made by our observer, it follows of necessity that the various arguments are correctly type-matched: that is her job, and if she fails in it, then she has failed to make a significant statement about me, namely that I am making a judgment about $a, R$ and $b$, and their connection with one another. She has to know enough about the world, that is, about the objects in it, about other people, about judging, to ensure that she does not ascribe incoherent beliefs to other people. When she fails to do so, then she fails to make significant utterances, and in that case the antecedent condition of Wittgenstein’s implication is not met. His point was that


even in the case where it was met, the consequent would still not follow directly. The judging subject, on the other hand, in no sense judges that objects are of appropriate types: I either judge significantly or I fail to do so, and that is the end of the matter. The work of type-matching is, so to speak, done not by me, but by the judgment-relation itself.

d) The Logical Form

We know, from Wittgenstein’s criticisms in the *Notebooks* of his own proto-theory, which facts he thought were needed to secure the significance of a false judgment: there had to be some sort of referent for the “form” occurring in the expression for the judgment-complex. Logical forms—the different types of elementary propositions—were common both to Wittgenstein’s early theory and to *Theory of Knowledge*, indeed I would conjecture that Russell expected his own “epistemological” theory in this period to mesh with Wittgenstein’s “logical” one once both were completed. In essence, Russell would have assigned Wittgenstein the technical task of making his theory of judgment work as a meta-theory for *Principia Mathematica*, while he turned to greener fields of knowledge.  

17 *e.g.* “The nature of consciousness and the varieties of realism. Our knowledge of space, time and matter.”  

18 In a letter of December 12, 1912, Wittgenstein alludes to that collaboration when he tells Russell that he has talked with Frege about “our theory of symbolism,”  

19 and at the very beginning of his 1914 notebook he remarks:

Also können wir uns fragen: Gibt es die Subjekt-Prädikat Form? Gibt es die Relationsform? Gibt es überhaupt irgend eine der Formen, von denen

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19 *Notebooks*, p. 121.
Russell und ich immer gesprochen haben? (Russell würde sagen: "ja! denn das ist einleuchtend." Jaha!)  

These logical forms were missing from the earlier versions of Russell’s theory of judgment, because Russell thought that relational or predicative elements of the judgment-complex would “appear as meanings” (“On Fundamentals”) or as “universals” (“Knowledge by Acquaintance and by Description”). But in passing from Part I to Part II of Theory of Knowledge, Russell introduced the form of an elementary complex as an additional component of the judgment complex. It corresponds to the “copula” of Wittgenstein’s proto-theory, which I discuss in the next chapter. Russell refers to the form variously, but most tellingly as a *summum genus* resulting “from a process of generalisation which has been carried to its utmost limit”—virtually the same language used by Wittgenstein when talking about the tractarian *Urbild*. Its function in Russell’s theory is similarly multifarious, but its most obvious role is to explain how and why two relational propositions whose “relating relations” differ (for example, $aRb$ and $aSb$) are nonetheless both *relational* propositions. This motive is once again not epistemological in origin: if Russell could not explain why the two complexes corresponding to these propositions were of the same sort, then he would have to conclude that they belonged to distinct classes of facts, and this would have had serious consequences when he turned to the analysis of generalised judgments, that is of quantified propositions.

Griffin suggests that Russell was also pleased to use the form as a solution to the “wide direction problem” (the type-matching problem), since it would allow one

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20 *Notebooks*, pp. 2-3. So we can ask ourselves: Is there a subject-predicate form? Is there a relational form? Are there indeed any of the forms about which Russell and I always spoke? (Russell would say: “yes! for that is self-evident.” *Ha*)


22 *Theory of Knowledge*, p. 97.
to rule out incorrect substitutions by means of "different styles of variable;" however, I see no evidence for this claim in Russell's text; indeed once one makes use of such symbolic distinctions—a suggestion made to Russell by Wittgenstein already in January of that year—the type-matching problem is essentially resolved. Conversely, as I argued in the preceding chapter, the mere introduction of the form into the judgment-complex does not resolve any essential syntactic ambiguities: if the form $x\chi y$ is supposed to exclude illegitimate combinations of $a, R, \text{ and } b$, then it can only do so if the observer knows that $R$ cannot be put in the $x$-slot of $x\chi y$, i.e. that $R$ is of the wrong type for this slot. But if she already has that knowledge, then she could reflect it by symbolizing $R$ as "$xRy$", making the form superfluous. If she does not, then the form is of no help in any case.

It helps in understanding this peculiar doctrine if one remembers that the judgment-theory discards information contained in the propositional sign by replacing the sign (as it had to, to avoid empty denotations) with a mere list of its elements. If the list is untyped, ambiguities result; however, as we just saw, we may recover those relationships by assuming that the list is typed. If the list consists of elements that may combine in several ways, then the manner of their combination must be specified. Here as well the form can only help if ordering conventions are already given: if they are, then we can dispense with it once again, and if they are not, then it makes no difference. If the analysis of $\downarrow-aRb$ is to be $J(S, a, R, b, x\chi y)$, I need to know that $a$ correlates to $x$, and so forth. Since the role of $R$ is distinct (it is of a different type, which knowledge we now assume as given), the problem is only of ordering $a$ and $b$. But we cannot do this without either giving the variable bindings, by saying, in essence, that $x = a$ and $y = b$, or assuming an ordering convention. In either case, the form drops out as unnecessary.

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e) The Judgment-theory and Descriptions

The problem of analysing asymmetric relational propositions is the last one Russell dealt with before abandoning his book. It is followed in the manuscript by two chapters (on self-evidence and degrees of certainty), but neither of these contribute to the central theory of judgment. In the chapter entitled “Truth and Falsehood” Russell proposes replacing elementary asymmetric relational propositions with a peculiar sort of quantified proposition. His reasons for doing so are those we have just discussed: even assuming a form and built-in type-restrictions, we still have no way of knowing whether the class of names \( \{a, xRy, b\} \) appearing in a given judgment-complex corresponds to the complex \( aRb \), or the complex \( bRa \).

His solution was to replace “\( aRb \)” with “There is an \( R \)-relation complex in which \( a \) is the first component, and \( b \) the second.” If we call the complex \( \gamma \), and we symbolize the relations “being first in an \( R \)-relation complex” and “being second in an \( R \)-relation complex” as \( xC, \gamma \) and \( yC, \gamma \), then we can symbolize the statement as:

\[
(i\gamma): aC, \gamma . bC, \gamma \text{ 24}
\]

The relations \( C_n \) are uniquely associated with the relation \( R \): they are not positional relations for relational complexes in general. They are “heterogeneous” relations, meaning that their arguments are of differing types; therefore a list of their components does determine uniquely the complexes to which they refer. Furthermore, the form can now be dispensed with, for the relations \( C_n \) themselves determine that the complex in question is of the relational form, with relating relation \( R \). So construed, an elementary proposition is a description of a fact; for the above proposition says the same as,

\[
(\exists x, \chi, y): x\chi y. x = a. y = b. \chi = R .
\]

\[24\] I use “\( i \)” instead of Russell and Whitehead’s inverted iota to denote a description.
that is: there is a relational complex with the relating relation $R$, with $a$ in the first slot, and $b$ in the second. The positional relations of the variables in the matrix or $Urbild$ $xyr$ reflect the ordering relations in the fact being described, and the identities bind the variables to the individuals and relations in question.

This formulation of the elementary judgment as an existentially quantified proposition makes explicit the approach that Russell and Wittgenstein had been pursuing for some time. This was to analyse the elementary proposition $Fa$, say, as $(\exists x)Fx.x=a$, or, at the "utmost limit" of analysis as $(\exists \phi, x)\phi x.\phi = F.x = a$. The first of these crops up in Wittgenstein's early letters to Russell, as well as throughout the "Notes on Logic," the Notebooks, and the Tractatus itself. The point of the analysis is that it separates the discrete propositional sign into its components, and makes a statement about the connections among these. As in Russell's analysis above, it says, "there is a function $\phi x$ which holds for the arguments $F$ and $a$." In the complex theory this becomes "there is a complex of the form $\phi x$, and $F$ is in the $\phi$-slot and $a$ is in the $x$-slot." Of course neither of these formulations can be the terminus of the analysis. Certainly I can't mean something like this when I assert $Fa$. Griffin is perfectly right in his claim that the judging subject cannot make use of quantification in his primitive judgments, for that would indeed be circular. What makes it an attractive approach, however, is that it dissects the elementary proposition into members with stable denotations. My assertion that $Fa$ is the case is not a molecular proposition involving identity, but it may well reflect a relation holding between $F$, $a$ and me, as well as, unfortunately, a more troublesome relation to the form $\phi x$. In the observer's statement about me--$J(D, F, a, \phi x)$--there is no mention of quantification or identity; we have instead a simple list of the components, each matched to the appropriate argument-place in the expression for my judgment.
Thus the expression for a simple judgment-complex is, in essence, a quantified expression stripped of its quantifier and variable-bindings: \((\exists \varphi, x) \varphi x. \varphi = F. x = a\) becomes \(J(S, F, a, \varphi x)\). So long as the variable-bindings are unambiguous—as they are in any "heterogeneous" or any asymmetric proposition—no information is lost by doing this: we can reconstruct unambiguously the fact being described. But homogeneous asymmetric propositions cannot be analysed in this fashion. The description by means of positional relations and quantification is not, therefore, a further development of the theory, but a regression to its roots: the analysis of non-denoting symbols by means of descriptions, an approach going back to 1904. It makes explicit the thinking that underlies all the talk about forms, type-constraints, *et cetera*. And it brings the contradiction in the theory to a head: one can only quantify over things that exist, so the variable in Russell's description must have a range. Since, however, it ranges over complexes, it is by definition possible that this range be empty. What happens to the description then?

f) Asymmetry and the Logical Form

In the chapter on "Truth and Falsehood," Russell describes these complications as follows:

We may now generalise this solution, without any essential change. Let \(y\) be a complex whose constituents are \(x_1, x_2, \ldots, x_n\) and a relating relation \(R\). Then each of these constituents has a certain relation to the complex. We may omit the consideration of \(R\), which obviously has a peculiar position. The relations of \(x_1, x_2, \ldots, x_n\) to \(y\) are their "positions" in the complex; let us call them \(C_1, C_2, \ldots, C_n\). As soon as \(R\) is given, but not before, the \(n\) relations \(C_1, C_2, \ldots, C_n\) are determinate, though unless \(R\) is non-permutative, it will not be determinate which of the constituents has which of the relations. If \(R\) is symmetrical with respect to two constituents, two of the \(C\)'s will be identical. If \(R\) is heterogeneous with respect to two constituents, two of the \(C\)'s will be incompatible. But in any case there are these relations \(C_1, C_2, \ldots, C_n\), and each constituent has one of these relations to \(y\).

When \(C_1, C_2, \ldots, C_n\) are given, conversely, \(R\) is determinate. Thus our complex \(y\) can be described unambiguously without mentioning \(R\), as simply

\[25\text{Theory of Knowledge, p. 146}\]
"the complex \( \gamma \) in which \( x_1 C_1 \gamma, x_2 C_2 \gamma, \ldots, \) and \( x_n C_n \gamma. \)" If we have decided, once and for all, that when \( R \) is the relating relation, the term which has the \( C_i \)-position is to be mentioned first, then the one with the \( C_2 \)-position, and so on, we can denote the complex \( \gamma \) by the symbol

\[
R(x_1, x_2, \ldots, x_n).
\]

But this symbol, though it has a certain notational convenience, is not sufficiently explicit for philosophical purposes. For philosophical purposes, the symbol

\[
(i\gamma). x_1 C_1 \gamma, x_2 C_2 \gamma, \ldots, x_n C_n \gamma\]

is preferable because it does not make more or less concealed use of the spatial order of \( x_1, x_2, \ldots, x_n. \)

By means of the above account of "positions" in a complex we can give a non-permutative complex associated with the complex \( \gamma, \) namely:

"there is a complex \( \alpha \) in which \( x_1 C_1 \alpha, x_2 C_2 \alpha, \ldots, \) and \( x_n C_n \alpha. \)" Here "\( \alpha \)" is an apparent variable. Instead of the one relation \( R, \) we now have the \( n \) relations \( C_1, C_2, \ldots, C_n. \) The new complex is molecular, and is non-permutative as regards its atomic constituents \( x_1 C_1 \alpha, x_2 C_2 \alpha, \ldots, \) and \( x_n C_n \alpha; \) also each of these atomic constituents is non-permutative because it is heterogeneous.\(^{27}\)

In taking this step Russell returned, one might say, to the true source of his judgment theory: the theory of descriptions. In the analysis of symmetric relations, the form played the role of a definite description, whereas here Russell solves the problem of the non-denoting propositional sign by using a definite description in the strict sense of the term. This was an unfortunate step, as I shall now show; however it should be clear that it was not the "wide" but the "narrow direction problem," that is the difficulty of disambiguating asymmetric complexes, which pushed Russell into taking it.

The step is regrettable because it postulates a class of facts of a given form, which class can fail to exist, and this violates the fundamental requirement of the theory of descriptions. That theory was developed not to solve problems about

\footnote{26} substitute 'i' for Russell's and Whitehead's inverted \( i, \) the symbol for a description in Principia Mathematica.

\footnote{27} Theory of Knowledge, p. 147.
Scottish novelists and French kings, but to give some sort of meaning to expressions about classes, among them unit classes, even when those classes do not, in *Principia Mathematica* terms, *exist*, i.e. even when there is no entity at hand to which one may significantly attribute—truly or falsely—a given property. In tracing the judgment-theory back to "On Fundamentals" I have sought to emphasise its genetic connection to *Principia Mathematica*’s contextual elimination of classes—a connection which will be more obvious when we turn to Wittgenstein’s theory. It suffices for now to point out that the purpose of Russell’s “contextual eliminations” of troublesome entities such as classes, descriptions and propositions is to displace properties of pseudo-entities onto entities whose denotation is secure. Given a function of an incomplete symbol (or its pseudo-denotation), we substitute an unspecified entity meeting a defining condition, and state that the function applies to the latter. The stand-in must belong to a type for which the question of denotation is already settled, for the fully analysed proposition contains a quantifier, and a quantified expression assumes the existence of arguments for which it is at least significant, even if false.

So the use of a description in the above case is disastrous. If there are no complexes of the form in question, then the apparent variable has no range, and the entire expression is senseless. Even without knowing how Russell hoped to analyse quantified propositions, we can see that this general approach would not work. On it, \( \text{I} - \text{aRb} \) is equivalent to \( \text{I} - (\text{i} \gamma \text{;} \text{aC}_1 \text{;} \text{bC}_2 \text{;} \gamma) \).\(^{28}\) Thus \( \text{I} - \text{aRb} \) means "there is a \( \gamma \) standing in the \( C \)-relations to \( a \) and \( b \)." But if there is no fact of the form of \( \gamma \), then this statement is not false but senseless. From the fact that \( \text{aRb} \) does not obtain, in other words, it does not necessarily follow that \( (\text{i} \gamma \text{;} \text{aC}_1 \text{;} \text{bC}_2 \text{;} \gamma) \) is false: the description might simply be senseless, meaning that the proposition would be not-true without being false, the very conclusion to be avoided. However we analyse the description, it will not meet

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\(^{28}\)"\( \text{i} \)" is once again the inverted iota, the description operator, of *Principia Mathematica*. 
the requirement that I be able to run implications of the form \( p \supset aRb \) backwards by means of Modus Tollens when the fact \( aRb \) does not obtain: we will always require the supplementary assumption that some complex of the form \( x\forall y \) exists, otherwise the quantifier will have no range. This premise is, I take it, the one Wittgenstein required.

This is, in a nutshell, the flaw in Russell’s analysis of judgment: propositional entities can only be eliminated descriptively on the assumption that the class of all such propositions is well defined, and this meant, for Russell and Wittgenstein, that the class had to contain at least one member. As Wittgenstein said in criticism of his earlier view,

Ich dachte, die Möglichkeit der Wahrheit eines Satzes \( \varphi(a) \) ist an die Tatsache \( (\exists \varphi, x)\varphi x \) gebunden. Aber es ist nicht einzusehen, warum \( \varphi a \) nur dann möglich sein soll, wenn es einen anderen Satz derselben Form gibt.\(^{29}\)

Wittgenstein’s “theory of the proposition” postulated relational forms as primitive elements of facts, and it also assumed that, since the elements of the proposition must have fixed denotations, that these forms exist. David Pears comments on this passage that, “Here Wittgenstein makes it clear that, unlike Russell, he assumed that, if \( (\exists \varphi, x)\varphi x \) is true, it is only contingently true.”\(^{30}\) Russell hoped to make sense of the notion of a logical form by making it a “Platonic” one, such that we are always acquainted with something like \( (\exists \varphi, x)\varphi x \), whether we take this to be a logical form or a necessary truth. In the case of asymmetric homogeneous complexes he was forced to make this claim explicit, since the only way to ensure that the objects were correctly bound to the appropriate slots in the relating relation was to introduce new

\(^{29}\)Notebooks, p 17. I thought the possibility of the truth of a proposition \( \varphi(a) \) was tied to the fact \( (\exists \varphi, x)\varphi x \). But it is not evident why \( \varphi a \) should only be possible when there is another proposition of the same form.

relations for this purpose, and those relations had to have a range of arguments for which they were significant—there had to be at least one fact of the form in question.

So of the three requirements with which I opened section b) of this chapter, it is the second one that proves critical: I must have some acquaintance with the form of the judged fact in order to make a judgment, otherwise I will be unable to analyze asymmetric propositions in the proposed manner. Russell spends a lot of time in *Theory of Knowledge* wrestling with the nature of this sort of acquaintance, as well as with that of its object. Logically it was intended to replace the notion "class of propositions of the form $\gamma$", or, in class-free terms, of the ultimate relation holding between the terms of the elements of that class.\(^{31}\) We have seen how, in the case of permutative homogeneous complexes, the attempt to quantify over this class presupposes its having at least one member. Wittgenstein's example in the letter makes reference to just such a complex, and in this choice of example, he struck the Achilles' heel of Russell's entire analysis. Since we cannot know just which passages of his manuscript Russell showed Wittgenstein, we cannot know whether the dart was so deliberately aimed; however, it exposes an assumption with which Russell was already extremely uncomfortable: if it is the case that no facts of a given form exist, that is that $(\varphi, x) \sim qx$, then the class of such facts is empty, and, according to *Principia Mathematica*, it does not exist. So how are we to make sense of statements which appeal to its existence for their significance? The example of the non-permutative relational complex makes explicit an assumption implicit in the other kinds of judgment complexes as well: namely, that in order to analyse propositional names, we have to assume names of forms, and that there is no compelling reason to

\(^{31}\)Griffin makes this point in his "Russell's Multiple Relation Theory of Judgment," p. 230, but he does not develop it.
assume that their denotaiion is more stable than that of the names which were to be analysed in the first place.

Russell could have stuck to his approach in the case of non-permutative complexes, but he would have had to admit that propositions about asymmetric complexes could not be given a stable significance. For we can imagine a world in which, although no positive elementary propositions were true, nonetheless objects, functions and relations existed, and were appropriately grouped into types. According to Russell's analysis, we would, in such a world, be incapable of forming negative judgments such as $\neg aRb$, because we would have no way of forming, of understanding, positive ones; for Russell's approach assumes that negative judgments rest on the significance of positive ones. So the claim that the forms must exist seems arbitrary. That (1) and (3) must hold (that the components of an elementary propositions must exist and be properly type-matched to one another) is on the other hand a plausible requirement. They amount to a postulate which was familiar to both Wittgenstein and Russell from Frege's *Sinn und Bedeutung*:

Von einer logisch vollkommenen Sprache (Begriffsschrift) ist zu verlangen, daß jeder Ausdruck, der aus schon eingeführten Zeichen in grammatisch richtiger Weise als Eigenname gebildet ist, auch in der Tat einen Gegenstand bezeichne, und daß kein Zeichen als Eigenname neu eingeführt werde, ohne daß ihm eine Bedeutung gesichert sei.\(^3^2\)

One must demand of a logically perfect language (concept-script) that every expression for a proper-name which is grammatically correctly constructed from signs that have already been introduced does, in fact, denote an object, and that no sign be introduced as a proper name without its denotation being secured.

Frege explained in this article that the possibility that the determinate denotation of a proposition (that is its truth or falsity, thus in the context of this discussion, its significance) depend on the truth of another proposition (stating that its constituents existed) had to be removed. Wittgenstein, as is well-known, took up this doctrine and

all that it implied; and Russell’s theory of descriptions was intended to immunise Whitehead’s and his concept-script against this problem’s many strains. Indeed his theory of judgment is an attempt to meet just this requirement for the case of false propositions, and it is in turn designed to deal with this possibility as far as the *constituents* of the propositions are concerned. For whether I consider myself to be a solipsistic entity alone in my private world of sense-data (the schema of Part I of *Theory of Knowledge*), or I am considered by someone else as an objective entity related to other, publicly accessible, objects and universals (that of Part II), it is still the case that I will only be capable of making elementary judgments about things with which I am acquainted, and that it will only be possible to ascribe those judgments to me which are themselves concerned with things that actually exist. The capability and possibility in question will be both physical and logical: it will be whatever possibility is contained in the natures of the elements of the world, of which I and the judgment relation are merely some among many. If *my* judgment that *Fa* is nonsense, then so is my observer’s judgment that I judge that *Fa*. But the same doesn’t hold in the case of requirement (2): if I am not acquainted with some relational complex of a given form, but my observer is, then it would follow that *I could not make* the significant judgment that a given fact of that form obtained, although she could significantly ascribe such a belief to me (falsely), could understand such a proposition herself, and indeed see whether or not things were as she thought they were. These considerations lead to a conclusion that would have been devastating to Russell: his theory of judgment does not allow the subject to form all thoughts which may be objectively true for that same subject, for on it the stocks of facts and of expressible thoughts are not coextensive, *i.e.* the logic it yields is incomplete.
Chapter 5 - The Breakdown of Wittgenstein's Copula-theory

In the preceding chapters, I took issue with Griffin and Sommerville's claim that type ambiguities proved the downfall of Russell's Theory of Knowledge. But I also said that the problem in question was a real one, that their analysis is essentially correct, even if it is directed at the wrong target. In this chapter I will show why it is a problem, and how Wittgenstein sought to solve it. My main concern is to collect the various doctrines—the five "planks" I listed at the beginning of Chapter 3—that we find in the Tractatus, among the ruins of both Russell’s judgment theory and Wittgenstein’s related “copula-theory.” I start by sketching the historical connection between Wittgenstein’s and Russell’s work, with particular attention to Frege’s (presumed) criticisms of their theories. From there I turn to a discussion of a) type-restrictions, b) the role of the “copula” in Wittgenstein’s theory, and c) the need for giving unambiguous or stable denotations for names. These topics correspond to the “other premises” I listed in Chapter 4, section b): if these matters are not settled for the elementary propositions, Wittgenstein maintains, then they will not be bivalent. He solves these problems by arguing that the types of objects and the possible combinations of their members are given a priori. The form is replaced with the Urbild, which is the class of facts of a similar form. Corresponding to these elements is an isomorphic symbolic structure, a manifold of elementary propositions. This conception, which Wittgenstein concretised shortly after leaving Cambridge in 1914, is from the outset described in terms of coordinates, dimensions and spaces: it is the source of the spatialized view of language that we get in the Tractatus.

Wittgenstein’s proto-theory and Russell’s Theory of Knowledge both stem from some earlier, shared version of the judgment theory. Wittgenstein’s work was probably intended to mesh with the encyclopedic theory of knowledge that Russell was pursuing at the time. In December 1912, Wittgenstein writes Russell that,
... I had a long discussion with Frege about our theory of symbolism of which, I think, he understood the general outline. He said he would think the matter over. The complex-problem is now clearer to me and I hope very much that I may solve it.1

But in June, 1913 his criticisms are of “your theory of judgment:” between December and June, Wittgenstein abandoned their common ground. The impetus for this break was almost certainly his meeting with Frege, for in January of 1913 (shortly after the meeting), he proposes a typically Fregean solution to the problem of type-matching:

... I have changed my views on “atomic” complexes: I now think that qualities, relations (like love) etc. are all copulae! That means I for instance analyse a subject-predicate proposition, say, “Socrates is human” into “Socrates” and “something is human”, (which I think is not complex). The reason for this is a very fundamental one: I think that there cannot be different Types of things! In other words whatever can be symbolized by a proper name must belong to one type. And further: every theory of types must be rendered superfluous by a proper theory of symbolism: For instance if I analyse the proposition Socrates is mortal into Socrates, mortality and $(\exists x, y) R(x, y)$ I want a theory of types to tell me that “mortality is Socrates” is nonsensical, because if I treat “mortality” as a proper name (as I did) there is nothing to prevent me from making the substitution the wrong way round. But if I analyse (as I do now) into Socrates and $(\exists x) x$ is mortal or generally into $x$ and $(\exists x) \varphi x$ it becomes impossible to substitute the wrong way round because the two types of symbols are now of a different kind themselves. What I am most certain of is not however the correctness of my present way of analysis, but of the fact that all theory of types must be done away with by a theory of symbolism showing that what seem to be different kinds of things are symbolized by different kinds of symbols which cannot possibly be substituted in one another’s places. I hope I have made this fairly clear!

Propositions which I formerly wrote $e, (a, R, b)$ I now write $R(a, b)$ and analyse them into $a, b$ and $(\exists x, y) R(x, y).$2

$[(\exists x, y) R(x, y)$ is bracketed underneath, with the rubric “not complex”]

Shortly after this break Wittgenstein would have rejected the analysis in the last sentence of the letter as well: it still renders the analyzed proposition as a list (a class of names), and it is open to exactly those criticisms with which I concluded the last chapter--it depends for its sense on the truth of the generalised proposition $(\exists x, y) R(x, y)$. Since this position has also been abandoned at the latest by the Fall of

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1Notebooks, p.121.

2Notebooks, pp. 121-22.
1913, when the "Notes on Logic" were dictated, we can suppose the following developments: Wittgenstein talks to Frege at the end of 1912, while working on a collaborative "theory of symbolism" with Russell, whose main feature was the elimination of meaning in favour of denotation, Sinn in favour of Bedeutung. The theory requires the analysis of the elementary propositional sign, which they propose to do by means of "contextual elimination." Frege criticises the theory 1) because it puts all types of objects on a single level, with the consequence that nonsensical propositions are possible, and, 2) because it suggests that a proposition can be understood even when its elements are not assembled in a structured whole, as were Frege's thoughts and senses. Wittgenstein accepts the first criticism, informs Russell of his changed views, but does not yet see the point of 2). Realising some time in the Spring that 2) can only be addressed by assuming the existence of some sort of paradigm structure or class (the form), whose existence can be no logical requirement, he adopts the view that the proposition is an organized collection of typed symbols. He continues to make both criticisms of Russell's theory, not least because both point to similar flaws in the judgment- and copula-theories: certain antecedent conditions must be met if the stock of possible propositions is not going to exceed that of possible facts, that is, if there are to be no senseless propositions.

a) The copula-theory

We know the outline of the copula theory from Wittgenstein's letters of 1912. It analyzed propositions such as Fa into F, a and ε₁(x, y), where ε₁(x, y) is the membership relation between predicates and individuals.³ The choice of notation points to the source of the theory: in Principia Mathematica *62, Russell and Whitehead redefined x ε α as a relation between object and function, replacing their earlier

³cf. letters dated 1.7.12, August 1912, Jan. 1913 in Notebooks, pp. 120-122.
definition of it in *20 as a propositional function. The new definition "requires, strictly speaking, a change in meaning ... but it is a change that does not falsify any of the previous propositions." Instead of reading $x \in \alpha$ as a function of $x$ and $\alpha$ ("$x$ is a member of $\alpha$"), we take it to express a fact about the couple $(x, \alpha)$, namely that it belongs to the class $\hat{x} \hat{\alpha} \varepsilon(x, \alpha)$, which is of course the class of couples $(x, \alpha)$ such that $\alpha(x)$ is true.

Like its Russellian cousin, Wittgenstein's copula-theory decomposed facts into complexes of forms and objects, the indefinables of both theories falling into two groups:

1. A set of primitive membership relations (Russell's "logical forms") $\varepsilon_,(x, y)$, $\varepsilon_,(x, y, z)$, corresponding to the different kinds of possible elementary facts.

2. The objects--individuals, predicates, relations--whose permutations within the strictures of 1, would trace out the range of possible facts.

The fully articulated theory of elementary propositions, in specifying both the fundamental elements of facts (objects and forms), and the ways in which they may combine with each other (the types in which these are ordered), would in consequence specify the range of possible facts and propositions--for it would say for any group of names (or, correspondingly, objects) whether, and in what ways, they might be significantly combined. In doing so, it would allow us to dispense with non-denoting propositional signs by reducing these to lists of their elements: the type-theory, which would need be no subtler than a phone book, would then tell us how to construe the latter. Lastly, it would allow us to define truth-functions as relations holding between such complexes, with the consequence that the bivalence of the elementary

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4 *Principia Mathematica*, p. 395.

5 "The theory of types, in my [Russell's] view, is a theory of correct symbolism: a simple symbol must not be used to express anything complex: more generally, a symbol must have the same structure as its meaning." Russell quoted by Wittgenstein in Letter to Russell, 19.9.19, *Notebooks*, pp. 130-1.
proposition had to follow from the theory. If implication is a relation which may hold between complexes—judgment complexes for Russell, and copulative complexes for Wittgenstein—then those complexes must be such that: whenever their related elements are not related (once again, the $F$ and $a$ of $\varepsilon_{e}(a,F)$ are not combined) that the falsity of $\varepsilon_{e}(a,F)$ is implied, as opposed to its senselessness. The one difficulty—that $\varepsilon_{e}$ denote, that the form exist—I will return to presently, and I will focus, to begin with, on the types of the arguments.

i. Type Restrictions

The $\varepsilon$-relation does not, as Wittgenstein first conceived it, make any distinction among the types of its arguments: it allows one to “treat ‘mortality’ as a proper name.”6 This means that any two entities may appear as its arguments: there is no reason to rule out either of $\varepsilon_{e}(a,F)$ or $\varepsilon_{e}(F,a)$, each of which says that the couple $(a,F)$ or $(F,a)$ belongs to the class $\varepsilon_{e}$, respectively. However in the second case $(F,a)$ is not a possible member of $\varepsilon_{e}$, because $aF$ is not a possible fact. The question thus arises: How can we ensure that the $\varepsilon$-relation hold only among appropriate arguments, that is among those which might, on their own, combine to form a fact? From $\varepsilon_{e}(a,F)$ in other words, it should follow that $Fav~Fa$—a demand with which we are now well-familiar.

The importance of constraining the range of the variables in such contexts was noted by Frege in the *Begriffsschrift*. Since a statement concerning the natural numbers, say $x + y = y + x$, not only does not hold for other objects, but is in fact senseless for them, restrictions on the variability of $x$ and $y$ in e.g. a statement of the

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6The language is English, but the terminology is Frege's.
law of commutation will be "incorporated into the judgment." As Jean van
Heijenoort points out, Russell took the same position as Frege on this point in The
Principles of Mathematics, and that because both logicians were devoted to the task
of constructing a universal language:

Mais tous deux sont d'accord pour penser que la logique repose sur un
univers unique et ne devrait pas s'abaisser à considérer, successivement, de
soi-disant univers de discours, univers desséchés dont on peut changer à
volonté. ... Une première conséquence d'une telle conception, c'est que les
quantificateurs liant des variables individuelles vont s'étendre à tous les
objets, c'est-à-dire à tous les objets dans l'univers.

If an implication is to be a formal one, then any implicit restrictions on the variables
in a generalized judgment must be made explicit as an initial hypothesis. Thus the law
of commutation becomes:

\[ Nx \land Ny \supset x + y = y + x. \]

where \( N x \) restricts \( x \) and \( y \) to the range of numbers, outside of which addition is not
significant. So in the case of the \( \varepsilon \)-relation, the question becomes: Can we restrict the
variables in \( \varepsilon (a, F) \) in such a way that \( \varepsilon (a, F) \supset Fa \land \sim Fa \) becomes a formal
implication, with no implicit restrictions? What we need are functions, say \( T_1 \) and \( T_2 \),
which pick out all and only those entities which are appropriate arguments for \( \varepsilon \). The
full conditional would then read,

\[ C \colon T_1(a) \land T_2(F) \supset \varepsilon (a, F) \supset Fa \land \sim Fa. \]

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7 Frege, Gottlob. *Begriffsschrift, eine der arithmetischen nachgebildete Formalsprache des
§11, p. 19.

113. The ideas and references in this paragraph are borrowed in part from this short and excellent
collection.


i.e. "if \(a\) is an individual and \(F\) is a predicate, then if they stand in the \(\epsilon_i\)-relation, then \(\text{`Fa'}\) is a proposition." In adding these type restrictions we have not disallowed the possibility of complexes such as \(\epsilon_i(F,a)\). We have said only that they do not correspond to significant propositions.

Griffin and Somerville take this problem schema to be that to which Wittgenstein referred in the June letter. They argue that Russell could not possibly bring his theory into the above form because any judgment that "\(a\) is an individual" would involve quantification over higher-order functions. This must lead to a contradiction, since the proposition being analysed is an elementary proposition, and its analysis cannot make appeal to second-order judgments (this because the second-order judgments are to be defined on the basis of the first-order ones). I have already argued that the charge of circularity here is moot: when describing how I or anyone else goes about judging, I am entitled to make use of all the resources at my disposal, and this apparent circularity is no more vicious than that which explains the truth of '\(aRb\)' as consisting in the existence of the fact \(aRb\). In both cases I make use of an expression whose own analysis would depend on the success of the analysis at hand—if, that is, we were to carry out such subsequent analyses. That point aside, it is clear that this schema is certainly one which Wittgenstein used to critique his own theory, as well as that of Russell. But what Wittgenstein found troubling is that the very notion of a function, elementary or otherwise, which is on the one hand true of all the members of a type, and on the other false of things not in the type, is, whatever their order, absurd: if \(T_o\hat{x}\) is such a function, and, say, \(~T_oA\) is true—\(A\) does not belong to the type \(T_o\hat{x}\)—then \(A\) is a significant argument for \(T_o\hat{x}\); thus \(A\) does belong to the type \(T_o\hat{x}\). \(T_o\) and \(T_i\) must therefore be true of all arguments for which they are significant. For all others, they are not false, but senseless. The problem is not whether or not \(T_o\) and \(T_i\) are possible (they are), or that they must be of a higher order (they might be).
It is rather that even if we assume their existence, they will not in fact advance our analysis. Rewriting \( C \) as a single implication, we get,

\[
C': T_0(a) \quad T_1(F) \quad \varepsilon_1(a, F) \vdash F a \sim Fa
\]

At first glance, it would appear as if the conditions under which the antecedent is true are more restricted than those under which \( \varepsilon_1(x, y) \) alone is true. But this appearance is deceiving: if \( xT_o\hat{x} \) is a type, and the functions \( T_o\hat{x} \) and \( \varepsilon_1(\hat{x}, y) \) are sometimes both significant for some \( x \), then there cannot be an \( x \) for which \( \varepsilon_1(\hat{x}, y) \) is significant and not \( T_o\hat{x} \), and similarly for \( T_1\hat{y} \). If \( \varepsilon_1(\hat{x}, y) \) can take an argument that does not belong to \( xT_o\hat{x} \), then either:

1) \( xT_o\hat{x} \) is not a type, or

2) \( x\varepsilon_1(\hat{x}, y) \) and \( xT_o\hat{x} \) never have common members, i.e. \( xT_o\hat{x} \) is not the type of \( x \) in \( \varepsilon_1(x, y) \).

Thus the supposition that type restrictions could be added to \( \varepsilon_1(x, y) \) leads to a contradiction, regardless of how the types are defined. The contradiction can be avoided only if we assume that the range of significance of \( x\varepsilon_1(\hat{x}, y) \) and \( xT_o\hat{x} \) coincide. That would make the introduction of \( T_o \) and \( T_1 \) superfluous.

Wittgenstein's reaction to this difficulty is given in the long letter of January cited above: the symbols for the various elements in the proposition must, through their structure, guarantee the significance of the sign, since no supplementary function will be able to do the job. That was Frege's solution to the problem of types: we define the various kinds of things not by differentiating among a universe of objects, but by syntactic operations on sentences which we know already to have meaning, and we demand that all possible combinations of these have meanings as well.

Functions that straddle the types cannot arise, and functions coextensive with the types, while perfectly possible, are not needed as hypotheses for limiting the range of the former, polytypic (and thus impossible) functions. Thus a definition of a type is
impossible, if by definition we understand the setting of limits to a concept. The types are maximal and discrete classes, and they cannot be delimited. As Wittgenstein put it to Russell: “We can never distinguish one logical type from another by attributing a property to members of the one which we deny to members of the other.”\(^{11}\)

This approach presupposes that the significant (for Frege *bedeutungsvolle*) propositions, whose analysis yields the function names and proper names of our logical language, have a recognizable internal structure: if they did not, we could not very well parse them. Consequently the Fregean *thought* and its correlate, the saturated propositional sign, must be assumed as given. So a theory suggesting that the proposition, as it is *judged*, must dissociate its elements from one another is, on this view, absurd. That Frege may have criticized this fundamental move in Russell’s and Wittgenstein’s “theory of symbolism” is rendered plausible by those passages of “Die Verneinung” where he inveighs against a theory of negation in which “the negation of a thought is to be grasped as the dissolution of the thought into its components.”\(^{12}\) He rejects such a theory with the argument that the persons judging (in this case, a jury) must recognize what they are denying, when they judge that something is not the case:

> Ist nun das Verneinen eines Gedankens als ein Auflösen des Gedankens in seine Bestandteile aufzufassen? Die Geschworenen können durch ihr verneinendes Urteil an dem Bestande des in der ihnen Vorgelegten Frage ausgedrückten Gedankens nichts ändern. Der Gedanke ist wahr oder falsch ganz unabhängig davon, ob sie richtig oder unrichtig urteilen. Und wenn er falsch ist, ist er eben ein Gedanke. Wenn sich, nachdem die Geschworenen geurteilt, gar keine Gedanke [sic] vorfindet, sondern nur Gedankentrümmer, so ist derselbe Bestand schon vorher gewesen; ihnen ist in der scheinbaren Frage gar kein Gedanke, sondern ihnen sind nur Gedankentrümmer vorgelegt worden; sie haben gar nichts gehabt, was sie hätten beurteilen können.\(^{13}\)

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\(^{11}\)“Notes on Logic,” *Notebooks*, p. 98.


So is the negation of a thought to be understood as the dissolution of a thought into its components? The jurors cannot through their judging change anything in the constitution of the thought expressed in the question set before them. The thought is true or false quite independently of their judging correctly or incorrectly. And if it is false, it is indeed a thought. If, after the jurors have judged, there is no more thought to be found, but only thought-rubble, then the same situation must have obtained beforehand; they were not proffered a thought, but only thought rubble; they had indeed nothing on which they could have passed judgment.

This argument is very close to those made by Wittgenstein in the “Notes on Logic,”14 where he insists that “however, for instance, ‘not-p’ may be explained, the question of what is negated must have a meaning.” Although Wittgenstein’s and Russell’s theory was not only a theory of negation, it was, as we have seen, most troublesome on just this point: What is it that we understand to be true when we assert a proposition which is in fact false? Since the entity whose existence is asserted is not in fact in existence, we must have a clear notion of the conditions under which it might exist. For Russell and Wittgenstein, in January 1913 “¬p” means “‘p’ is false,” thus in turn, “the fact described by ‘p’ does not obtain.” Their theory faces precisely the dilemma sketched out above by Frege, and reiterated by Wittgenstein in the “Notes on Logic:” we have to know what would be the case if “p” were true, when we assert that “‘p’ is false.” But if “‘p’ is false” gets analysed into an expression in which the symbol ‘p’ is reduced to “thought rubble,” (one should think of the case of asymmetric relations), then this demand will be hard to meet.

We cannot possibly establish the exact content of Frege’s and Wittgenstein’s discussions at the end of 1913. That Wittgenstein’s break with the judgment theory was motivated by the above concerns seems to me to be beyond question, as does the conclusion that his views on types were taken over from Frege. The first connection (the point about the integrity of the proposition) is less certain than the second one (about types); however there is no question that Wittgenstein’s remarks in the “Notes

14cf. the second set of quotations in Chapter 3.
on Logic" are easier to understand when looked at from this point of view. Conversely the somewhat incongruous passage in "Die Verneinung"--an essay that Wittgenstein recommended to Peter Geach as particularly worthy of attention--makes good sense when read as a reaction to the Russell/Wittgenstein project.

In short, by construing all the components of his propositions as "simple names," Wittgenstein obliterated type distinctions among them, and with them the implicit structure of the proposition being analyzed. This led, as we have seen, to ambiguities. We must know what would be the case if a false proposition were true; and if \{F, a, (\exists x, y) \epsilon,(x, y)\} is supposed to render a proposition, we must know which of the various possible combinations of its elements is supposed to be its fact, or "corresponding complex," as Russell put it. Furthermore we should be able to exclude \textit{a priori} those combinations which are senseless, to which no fact could possibly correspond. This means that when we judge that a proposition \(Fa\) is true, we must be certain that \(F\) and \(a\) could in fact form a complex. Whence does this certainty arise? Since we have only the symbol and the denotations of its components before us (assuming that we do not know whether \(Fa\) obtains), we cannot know this by looking to the fact in question. If we needed supplementary knowledge to the effect that \(F\) and \(a\) belonged to the appropriate categories of things, then this knowledge would have to be expressible. We would have to say, in essence, that "although \(a\) might not be an appropriate argument for \(F\), in this case it is one." But no significant function can say this of any given pair \((a, F)\): this knowledge cannot be \textit{said}, as Wittgenstein later put it. Either we know already which objects can be possible argument for \(\epsilon,(x, y)\) or we don't, and, in the latter case, no further information will help us make the decision. The conclusion is that our symbolism must be such as to allow us to settle such questions merely by inspecting the signs; they must \textit{show} this possibility.
Wittgenstein sometimes made this point more obscurely, by insisting that an attempt to talk about types could only result in talk about symbols, thus that type theory was impossible:

That's exactly what one can't say. You cannot prescribe to a symbol what it may be used to express. All that a symbol can express, it may express.\(^{15}\)

If I can significantly ask of a thing whether it shares a property with all members of a given type, then it does, and if it does not, then I cannot even ask the question—thus any attempt to circumscribe the range of significance of a symbol by means of assertions about things will be either redundant or self-defeating. This rather dramatic formulation of the dilemma has occasioned more commentary than the one I give above, largely because it puts such weight on the saying/showing distinction. That serves, in my view, to obscure the cogency of Wittgenstein's objection: types are not classes like any other, if there are indefinable functions coextensive with them, then this is a contingent fact. And even if we had such functions, we could not use them as a basis on which to categorize the names of our language, for they assume for their own significance that such type distinctions be observed.

b) The \(\varepsilon\)-copula and the Propositional Form

The other, more tentative solution referred to in the January letter, is that the \(\varepsilon\)-copula itself must be done away with. What reasons did Wittgenstein have for suggesting this? The description of the fact as a relation—of the most abstract sort—holding between its components suggests an alternative way of analyzing the propositional sign: "\(Fa\)" may or may not have a denotation; the components of \(\{F, a, (\exists x, y)\varepsilon f(x, y)\}\) on the other hand do, if we read the last one as stating some fixed fact about the world—that some fact of this form obtains. This is equivalent to the

\(^{15}\)Letter to Russell, 19.9.19, Notebooks, pp. 130-1.
condition that the relation \( \varepsilon \), in the language of *Principia Mathematica*, exist.\(^{16}\) So Pears’s claim that, for Wittgenstein, “if \((\exists \varphi, x) q \varphi\) is true, it is only contingently true,”\(^{17}\) is undoubtedly correct: Wittgenstein was just as skeptical about the existence of such facts as he was about the fact that \( a = a \). One must bear in mind that Russell’s theory denied independent existence to propositions; their real correlates, the judgment-complexes, are composed not of names, but of the same objects as the facts to which they refer. Similarly in the copula-theory, the propositional sign does not denote its fact directly, but asserts instead that its components are related to one another by means of the \( \varepsilon \)-relation, and this relation must therefore exist, if the analysis is to achieve its goal. Russell, Pears argues, went the high road, and said that the \( \varepsilon \)-relation or logical form was a necessarily existing logical object—but that still didn’t solve the problem of asymmetric complexes. Wittgenstein eventually concluded that such an assumption was unacceptable, for it is perfectly conceivable that, at any given time, no complex of a given form obtains.\(^{18}\) Even if we assumed that each type of fact had a stable (permanent) member—a propositional-type paradigm—we would still run into the ordering ambiguities with which I closed the last chapter. I contend that it is these difficulties that pushed Wittgenstein into reestablishing the propositional sign as an independent entity corresponding to Frege’s *thought*. In short, even if we assume type-paradigms, we will still need to reflect ordering relations; these can only be captured by means of explicit variable bindings of the slots in the “form” to the names; however the form’s existence is a contingent fact, and thus, as in

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\(^{16}\)The relation \( \varepsilon \), *exists* whenever “there is at least one pair of terms between which it holds,” *Principia Mathematica* *25*, p. 228, thus whenever one fact of the form in question obtains.


\(^{18}\)cf. *Notebooks*, p. 17, cited in Chapter 4, footnote 29.
Russell's theory, the analysis will not hold for asymmetric relations when no relational complexes obtain.

For let us assume there were a fact \( Fa \) which always held, so that "\( Fa \)" was always true, and with it, "\( (\exists \varphi, x)q\varphi \)". Since \( Fa \) can now anchor the class of propositions of the form \( \varphi x \), we can analyze propositional signs with that form into their constituents, one of which will be \( (\exists \varphi, x)q\varphi \). Thus far the only difficulty is that the significance of all these others depends on the truth of "\( Fa \)". Now since "\( Fa \)" is a proposition of the same form as its doubles, it ought to be susceptible to analysis in the same way, leading to the conclusion that the truth of \textit{this} proposition is a condition for the sense of its own propositional sign: as long as the fact \( Fa \) obtains, then \( (\exists \varphi, x)q\varphi \) will have a referent, so the sign \{ \( F, a, (\exists x, y)\varepsilon, (x, y) \) \} will be significant, and indeed, true. If on the other hand \( Fa \) were \textit{not} to obtain, then the proposition "\( Fa \)" would be nonsensical. On this model, a proposition such as \( Gb \) would take the form: "\( G \) and \( b \) are combined in the same manner as \( F \) and \( a \) in \( Fa \)."

We must of course know that \( a \) and \( b \), and \( F \) and \( G \) correspond to one another (that they are of the same types); however we can solve that problem by assuming that the type-membership of the latter is given \textit{a priori}. This was, as we saw in Chapters 3-4, an option open to Russell as well: one can assume that type-mismatches are excluded by the fundamental ontological properties of the objects (which of course include the judgment-relations), and therefore the type-matching problem need not force us to adopt the position that there are independently existing propositional signs. But since Wittgenstein had already concluded, at Frege's prodding, that the type-matching problem was best met by assuming symbolic elements whose type-membership was reflected in their syntax, it was only a short step to the conclusion that objectively existing proto-types could be dispensed with as well. Whether we construe such a proto-type as a paradigm fact, \( Fa \), or a necessarily existing class, \( \varphi x(\hat{\varphi}x) \), we can dispose of it by displacing its role onto the propositional sign itself. For what does the
proto-type get us? The class \( \varphi(x(\bar{x})) \) is made up of all the possible combinations of the elements of the types of \( F \) and \( a \). As with Russell's logical forms, the purpose of the proto-type was to show us how they combine; and in the predicative case, that is uniquely given by the structure of the signs in question ("a" fits in "\( F\bar{x} \)"), but not vice versa). With asymmetric relations, as we saw, this is not the case. If \( cSd \) means "\( c, S \) and \( d \) are combined as \( aRb \)," we need to know not only that a) "\( a, b, c, \) and \( d \) are of the same type," but also that b) "\( c \) fits the a-slot and \( d \) the b-slot." That was the purpose of Russell's C-relations: they gave the positions of the individuals in a hypothetical (because descriptively specified) complex. Even if we assume that there are no type-ambiguities, we still need the ordering relations of b); however we do not need the existence either of the class of facts in question, nor of a paradigm for that class. Wittgenstein's conclusion was that the propositional sign already reflects these relations, and that, once again, to state them would only be possible by means of a) and b) above. Since however a) and b) make use of the same properties of \( aRb \) as we sought to analyse in \( cSd \), this procedure is either redundant, or it leads to an infinite regress, for the referent of \( aRb \) is unclear. If it denotes a complex, then the complex could fail to exist (that was Pears's point); and if it does not, then by what right do we use a complex symbol to denote it? That is surely to devolve on the symbol a role which the entity denoted by the symbol cannot play. We might just as well stick with the symbol "\( cSd \)". It is assumed to show the types of its components, as well as their arrangement.\(^{19}\)

There are no sources which allow us to say with certainty when Wittgenstein arrived at this conclusion. The analysis at the end of the January letter still does not disambiguate between asymmetrical relations, and it cannot have been seriously entertained for long; by the Fall, however, the propositional sign is an entity in its

\(^{19}\)This is plank 1) of Chapter 3.
own right: “the symbol for ‘a fact’ is a proposition, and this is no incomplete symbol.” This conclusion represents a more complete break with the earlier theory than does that concerning types and syntax. As we just saw, the requirement that the objects of our acquaintance be organized in types could be made to fit the earlier approach without assuming the independent existence of propositional signs, or Fregean thoughts. However this last chain of arguments, which corresponds to Russell’s attempts to analyse asymmetric relations, leads either, 1) to the result that there must be proto-types and structural specifications in every such judgment—which means that, in the eminently imaginable situation where no such proto-type exists, the propositions become senseless, or, 2) to the conclusion that the propositional sign itself plays the role of such a proto-type, by reflecting, in its internal structure, the structure of the fact whose existence is predicted by the judging subject. That is however to return to a position much like Frege’s, to the very position from which Russell departed, in other words:


The world of thoughts has its depiction in the world of propositions, expressions, words, symbols. The construction of the thought corresponds to the composition of the proposition out of words, whereby the order is in general not indifferent.

Since Frege’s thought corresponds to his sense, we can see that Wittgenstein’s terminal position--as expressed, for instance, in Tractatus 3: “Das logische Bild der Tatsachen ist der Gedanke.”--wholly rejects the analytic position from which he and Russell set out.

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20 Letter from Wittgenstein to Russell, Fall 1913, in which he answers questions about the typescript of the “Notes on Logic.” Notebooks, p. 124.

c) The Early Picture Theory

Thus the analysis at the end of the January letter—"Propositions which I formerly wrote \( \varepsilon_1(a, R, b) \) I now write \( R(a, b) \) and analyse them into \( a, b \) and (\( \exists x, y)R(x, y) \),"\(^{22}\) in which the last term is supposedly "not complex"—must be rejected as well. From 1914 onward, Wittgenstein's preferred analysis of the elementary proposition is: "'fa' says the same as '(\( \exists x \). fx. x = a \)'."\(^{23}\) That is, the elementary proposition 'fa' says that there is a function \( f \bar{x} \) which is true for (at least) one object, which object is \( a \). This analysis makes clear:

1) The type relationship of \( f \bar{x} \) and \( a \).

2) The form of the complex whose existence is being asserted: \( f \bar{x} \).

3) That \( a \) exists.

1) The type relationship is reflected in the relation of the symbols 'fx' and 'x = a'. The former makes clear that \( f \bar{x} \) is a function taking an individual as argument, and the latter gives the identity of that argument. In the "Notes Dictated to Moore," Wittgenstein puts this as:

\[ \text{The symbol for identity expresses the internal relation between a function and its argument: i.e. } \varphi a = (\exists x). \varphi x. x = a. \] \(^{24}\)

The internal relation is: "\( a \) is an appropriate argument for \( f \bar{x} \)." It is displayed typographically by the form of the symbols \( f \bar{x}, x \) and \( a \). In the relational case, the analysis would read \( a R b = (\exists x, y). x R y. x = a. y = b. \) Here the identities, which were missing from the account in the January letter, display the ordering of the individuals in the functional symbol, for they specify the variable bindings of the latter.

\(^{22}\)Notebooks, p. 122.

\(^{23}\)Tractatus 5.47.

\(^{24}\)Notebooks, p. 117.
2) The condition \( f^x \) specifies the class to which the fact whose existence is being asserted belongs: it makes clear, should no such fact obtain, what structure the fact in question must have. The symbol \( f^x \) thus plays the role of the proto-type or paradigm in the preceding discussion:

\[
\text{Damit also ein Satz einen Sachverhalt darstelle, ist nur nötig, daß seine Bestandteile die des Sachverhalts repräsentieren und daß jene in einer für diese möglichen Verbindung stehen.}^{25}
\]

The symbol \( x^R y \) in \( x^R y \cdot x = a \cdot y = b \) makes clear that \( abR \), and \( Rab \) are not among the "possible connections" of \( a \), \( b \), and \( R \); \( bRa \) is of course a possible connection, however it is excluded not by the functional sign alone, but by the order of \( a \) and \( b \) in \( aRb \), an order which is made explicit by the identities. The function symbol reverts to the copulative role which it had in Russell’s “On the Nature of Truth and Falsehood:” it provides form and unity to the proposition. But this is no longer an appeal to some special ontological category, for the function-name is an independent entity, unlike the “meaning” of the judgment complex, which was to be something like an aspect under which a subject could consider a thing (recall that in “On Fundamentals” things could appear in complexes as “meaning” or “being”).

3) That \( 'a' \) has a denotation is reflected in the expression \( 'x = a' \), which Wittgenstein would have understood, in part through his acquaintance with Moore’s work, as expressing the requirement that \( a \) exists. Moore, as I discuss below, analysed "\( a \) is necessarily \( F \)" as "necessarily: anything identical to \( a \) is \( F \)", that is as, \( L(x = a \supset Fx) \). He did so because he wished to allow for cases in which \( a \) did not in fact exist, and he read \( 'x = a' \) as embodying that requirement. Similarly in Wittgenstein’s analysis, the condition \( x = a \) gives a second way in which \( fa \) might fail to be true: there might be no entity answering to \( 'a' \).

\[^{25} \text{Notebooks, p} \ 27. \text{Thus in order for a proposition to represent a state of affairs, it is only necessary that its components represent those of the state of affairs, and that the former stand in a connection that is possible for the latter.} \]
Wittgenstein concluded that the first two of these conditions could not be explicit or *external* conditions on the elementary proposition, because any such conditions would be either insufficient or redundant. Type constraints or proto-typical facts can only do their job if the very conditions they are intended to secure—that the elements of the proposition are correctly type-matched, and that there is some prototype with reference to which we can conceive the fact in question—are already met. Thus these conditions are built into the elementary propositional sign: any possible (assertable) elementary proposition entails, by the fact of its being assertable, that they are met. Since,

\[ aRb \models (\exists x, y) xRy \cdot x = a \cdot y = b, \]

it follows, from the assertability of \( aRb \), that \( a \) and \( b \) are individuals, that \( R \) is a binary relation, and that they combine in the structure \( xRy \), such that \( a \) is in the \( x \)-slot, *et cetera*. Since it is the symbols themselves which display these properties, and since many symbols might have them, we end up with a curious possibility: Might there not be signs such as "\( Jc \)" which have the first two of the above characteristics (the structural ones), and yet do *not* depict possible facts? The answer must be yes: the class of symbols like "\( Jc \)" might exceed the class of possible facts of that form; however, that could occur only if the names in these symbols had no denotations:

5.4733 Frege sagt: Jeder rechtmäßig gebildete Satz muß einen Sinn haben; und ich sage: Jeder mögliche Satz ist rechtmäßig gebildet, und wenn er keinen Sinn hat, so kann das nur daran liegen, daß wir einigen seiner Bestandteile keine Bedeutung gegeben haben.\(^\text{26}\)

In Frege’s system, names have syntactic properties which determine their combinations, and they have denotations. A proposition is well-formed when it meets the various syntactic requirements outlined at the beginning of the *Grundgesetze*: the names must be appropriately matched to one another and must fill out each other’s

\(^{26}\text{5.4733 Frege says: every well-formed sentence must have a sense; and I say: every possible sentence is well-formed, and if it has no sense, then this can only stem from the fact that we have not given some of its components a meaning.}\)
slots, so that the whole symbol is saturated. If these requirements are met, and the names have denotations, then the propositional sign must denote a truth value, which Wittgenstein equated with being *significant*. If it does not have a denotation, then it must be given one, and this to ensure the significance of quantified sentences in the concept-script. From Wittgenstein's point of view, any symbol with a given syntactic structure whose elements have been given denotations simply does have a truth-value; for if it didn’t, then the whole story about the sign’s “guaranteeing” the possibility of the fact would collapse, and with that all the problems about type-matching and proto-types would reemerge. Thus there must be a pre-established harmony (no weaker word will do) between the possible combinations of objects on the one hand, and names on the other. Once the elements of the sign have been given denotations, and a mapping between the two domains is established, then any combination of the members of one group must find its correlate in one of the other.\textsuperscript{27}

The denotation we give to the name “\(a\)” in “\(aRb\)” is on the one hand arbitrary, and on the other hand constrained. It is constrained in so far as the denotation must be selected from among a given type of things; however, we may assign it any arbitrary denotation within that type. At the same time the denotation we give it must be *stable*. The cleavage between essential structural properties and arbitrary denotations entails that any indeterminacy in the sense of a well-formed sign must be traceable to a failure in denotation (that is the exact meaning of 5.4733). So if we demand that a well-formed elementary propositional sign be one whose significance is at all times determinate, then it follows that the denotations of its components must be secured under all circumstances and at all times. The type-membership of the elements of the

\textsuperscript{27}This is plank 4) of Chapter 3.
elementary facts (the denotations of the elements of the elementary propositions) must be permanent.\textsuperscript{28}

\textbf{i. Functions of Facts}

This worry about the stability of denotation is addressed at the outset of the \textit{Notebooks}:

\begin{quote}
Gibt es Funktionen von Tatsachen? Z.B. "Es ist besser, wenn dies der Fall ist, als wenn jenes der Fall ist."\textsuperscript{29}
\end{quote}

Since a symbol such as $fa$ could be seen as the name of an individual, just as for Frege it was, being saturated, a proper name, one must of necessity consider whether or not it can be the argument for a first-level function. Wittgenstein considers a function $\varphi \hat{x}$ taking as its argument the fact $\psi x$, and concludes that his proposed analysis--

$\varphi(\psi x) = \psi x$ Def.--fails to capture the sense of such propositions, if they have one at all. On the next day, he remarks:

\begin{quote}
Es scheint mir jetzt plötzlich in irgend einem Sinne klar, daß eine Eigenschaft eines Sachverhaltes immer intern sein muß.
\end{quote}

\begin{quote}
Es scheint mir jetzt klar, daß es keine Funktionen von Sachverhalten geben kann.\textsuperscript{30}
\end{quote}

Wittgenstein's thinking in these pages proceeds from his concern that we have no right, not being acquainted with any simple subject-predicate or relational propositions, to stipulate that any given propositional form is a valid one, to demand that our language, when fully analyzed, does make use of propositions of that form,\textsuperscript{31} and to expect that the facts of our most basic experiences are of a corresponding

\textsuperscript{28}This is plank 5) from Chapter 3.

\textsuperscript{29}\textit{Notebooks}, p. 5. Are there functions of facts? e.g. 'It is better for this to be the case, than for that to be the case.'

\textsuperscript{30}\textit{Notebooks}, p. 6. Now it suddenly seems to me in some sense clear that a property of a fact must always be internal. It now seems clear to me that there cannot be functions of facts. [Note that there is no need to read \textit{Sachverhalt} as a technical term in this context.]

\textsuperscript{31}\textit{Notebooks}, pp. 1-2.
structure. On the one hand it is the symbols alone that are supposed to guarantee the sense of propositions, on the other hand we have no instance of such a simple bivalent proposition available. Such questions cannot be resolved with an appeal to "self-evidence," but they seem to be beyond empirical confirmation, for that would require "some kind of experience, and that I regard as out of the question."\footnote{Notebooks, p. 3.} This leads to the awkward conclusion that our mere acquaintanceship with unanalyzed subject-predicate sentences must provide the grounds on which to "complete logic," and settle "the syntactical rules for functions"\footnote{Notebooks, p. 1.} conclusively. But is this in fact a hopeless result?

\begin{quote}
Kann man nicht sagen: Es kommt nicht darauf an, daß wir es mit nicht analysierbaren Subjekt-Prädikat Sätzen zu tun haben, sondern darauf, daß unsere Subjekt-Prädikat Sätze sich in jeder Beziehung so wie solche benehmen, d.h. also, daß die Logik unserer Subjekt-Prädikat Sätze dieselbe ist, wie die Logik jener anderen. Es kommt uns ja nur darauf an, die Logik abzuschließen, und unser Haupteinwand gegen die nicht-analysierten Subjekt-Prädikat Sätze war der, daß wir ihre Syntax nicht aufstellen können, solange wir ihre Analyse nicht kennen. Muß aber nicht die Logik eines scheinbaren Subjekt-Prädikat Satzes dieselbe sein wie die Logik eines wirklichen? Wenn eine Definition überhaupt möglich ist, die dem Satz die Subjekt-Prädikat Form gibt ... ?\footnote{Notebooks, p. 4. Can't we say: It all depends, not on our dealing with unanalysable subject-predicate propositions, but on the fact that our subject-predicate propositions behave in the same way as such propositions in every respect, \textit{i.e.} that the logic of our subject-predicate propositions is the same as the logic of those. The point for us is simply to complete logic, and our main objection against unanalyzed subject-predicate propositions was that we cannot specify their syntax so long as we don't know their analysis. But must not the logic of an apparent subject-predicate proposition be the same as the logic of an actual one? If a definition giving the proposition the subject-predicate form is possible at all ... ? [ellipsis in original]} [ellipsis in original]
\end{quote}

What counts is not whether we have acquaintanceship with simple objects, but that our unanalyzed subject-predicate propositions, at least within their defining assumptions, behave as though they were fully analyzed; that the sentence "The chair is red," as long as the complex denoted by "the chair" exists, exhibits the properties of a simple

\footnote{Notebooks, p. 3.}

\footnote{Notebooks, p. 1.}

\footnote{Notebooks, p. 4. Can't we say: It all depends, not on our dealing with unanalysable subject-predicate propositions, but on the fact that our subject-predicate propositions behave in the same way as such propositions in every respect, \textit{i.e.} that the logic of our subject-predicate propositions is the same as the logic of those. The point for us is simply to complete logic, and our main objection against unanalyzed subject-predicate propositions was that we cannot specify their syntax so long as we don't know their analysis. But must not the logic of an apparent subject-predicate proposition be the same as the logic of an actual one? If a definition giving the proposition the subject-predicate form is possible at all ... ? [ellipsis in original]}


subject-predicate sentence. To see what these properties might be, consider the following definitions:\(^{35}\)

\[ A = aRb \text{ Def.} \]

\[ \varphi[xRy] = \varphi x. \varphi y. xRy \text{ Def.} \]

The definitions permit us to put the proposition \( \varphi a. \varphi b. aRb \) in the form \( \varphi(A) \). And as long as the condition \( aRb \) holds, then the "syntax" of \( \varphi(A) \) will indeed be like that of an unanalyzable subject-predicate sentence: from \( \varphi(A) \) it will follow that \( (\exists x) \varphi x \), meaning that one member of the class of propositions \( \varphi(x) \) is true,\(^ {36}\) and from \( \neg \varphi(A) \) will follow \( (\exists x) \neg \varphi x \), that is, one member of that class is false. If the analysis of \( Fa \) is \( (\exists x). Fx. x = a \), then it is clear that both of these implications must hold for elementary propositions; otherwise we would never know, in the case that they were false, whether this was due to a fact’s failing to obtain, or to a symbol’s failure to denote. Should the condition \( aRb \) not hold, the complexity of \( \varphi(A) \) will reveal itself through a breakdown in the syntax of the latter.

The problem with propositions such as \( \varphi(A) \), where "A" may be complex, is that we are ignorant of their syntax—we cannot tell by inspection how they are to be understood. Such propositions do not have determinate sense. The difference between the syntax of simple predicative propositions and that of complex ones was already the subject of Russell’s theory of descriptions: he developed it in order to eliminate expressions that appear to be proper names, but whose referents may or may not exist. If \( A = " \text{the } x \text{ such that } \varphi x" \) then \( \neg F(A) = \neg (\exists x)[Fx.(y)(\varphi y \equiv x = y)] \), which is indeed significant even when no such \( A \) exists. But whereas from the simple proposition \( \neg Fa \),

\(^{35}\text{cf. Notebooks, entry of 5.9.14, p. 4., Appendix 4, Plate 7, "Notes Dictated to Moore," p. 112, for similar examples.}\)

\(^{36}\text{This is Principia Mathematica *9.1: the primitive proposition effecting "the passage from elementary to first order propositions." Principia Mathematica, p. 131.}\)
\((\exists x) ~ Fx\) ought to follow, it needn’t follow from \(\neg F(A)\).\(^{37}\) It ought to follow because \(\neg Fa\) is \((x) ~ Fx \lor x \cdot a\), and \((\exists x) ~ Fx\) must follow from the latter unless ‘a’ has no denotation. So long as it does, then the statement that "‘Fa’ is false" (that being the Principia Mathematica explanation of \(\neg Fa\)) entails the statement that some instance of \(Fx\) is not the case. But if ‘A’ has a complex denotation, then \(F(A)\) can be false in two ways: there may be no unique entity answering to ‘A’, or there is one, but it is not \(F\). Of necessity, a complex entity may fail to obtain, and whenever we predicate something of a complex, we postulate implicitly that it does obtain. Should our statement be false there will be two possible grounds—either the complex didn’t exist, or it didn’t have the predicate in question. This is the "main objection" to unanalyzed subject-predicate sentences to which Wittgenstein refers above: a complex subject-predicate proposition looks like a simple one, but its syntax is different, because this depends not only on its surface structure, but also on the complexity of the facts which it is supposed to describe.

In answering the question with which I began this section—Can there be functions of facts?—Wittgenstein follows a similar line:

"\(\varphi(\psi x)\)". Nehmen wir an, uns sei eine Funktion eines Subjekt-Prädikat Satzes gegeben, und wir wollen die Art der Beziehung der Funktion zum Satz dadurch erklären, daß wir sagen: Die Funktion bezieht sich unmittelbar nur auf das Subjekt des Subjekt-Prädikat Satzes, und was bezeichnet ist das logische Produkt aus dieser Beziehung und dem Subjekt-Prädikat Satzzeichen. Wenn wir das nun sagen, so könnte man fragen: wenn du den Satz so erklären kannst, warum erklärst du dann nicht auch seine Bedeutung auf die gleiche Art und Weise? Nimmt die es keine Funktion einer Subjekt-Prädikat Tatsache, sondern das logische Produkt einer solchen und einer Funktion ihres Subjektes? Muß nicht der Einwand, der gegen diese Erklärung gilt, auch gegen jene gelten?\(^{38}\)

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\(^{37}\)Consider for instance that \((x) Fx\), \(qa\), and \(qb\) all hold.

\(^{38}\)Notebooks, pp. 5-6. "\(\varphi(\psi x)\)". Suppose we are given a function of a subject-predicate proposition and we try to explain the way the function refers to the proposition by saying: The function only relates immediately to the subject of the subject-predicate proposition, and what signifies is the logical product of this relation and the subject-predicate propositional sign. Now if we say this, it can be asked: If you can explain the proposition like that, then why not given an analogous explanation of what it stands for? Namely: ‘It is not a function of a subject-predicate fact, but the logical product of
The case considered here differs slightly from the example given above \((F(A))\), where \(A = \text{"the } x \text{ such that } \varphi x\)" in that here we assume the individual symbols in the proposition \((\varphi, \psi, \text{ and } x)\) to be simple, and thus in one sense we do "know its analysis." We still do not know its syntax, however—what follows from it, how it may be combined with other propositions. In the case where "A" stood for a description, we saved the significance of our seeming subject-predicate sentence by replacing it with a quantified proposition—that is, with a proposition asserting the applicability of a given function. Here the case is somewhat different: should \(\psi x\) be false, then \(\varphi x\) lacks an argument, and the proposition \(\varphi(\psi x)\) is accordingly not significant; but we cannot very well replace \(\psi x\) with a description, and thus we must seek another alternative.\(^{39}\) Wittgenstein proposes distributing the function \(\varphi x\) over the subject of \(\psi x\), and replacing the original propositions with a conjunction of the two resulting propositions \(\varphi x\) and \(\psi x\). Since the elements of the latter are assumed to be simple, we know both of these propositions to be significant, and the conjunction thus makes clear what must be the case if it is false. Now, Wittgenstein objects, why can't we analyze the meaning (denotation) of \(\varphi(\psi x)\) in the same way? The answer is, briefly put, that Wittgenstein's analysis construes \(\varphi x\) and \(\psi x\) as independent facts; in particular it allows us to consider the state of affairs in which \(\varphi x\) held, but not \(\psi x\).

That possibility cannot be expressed by means of the original symbol "\(\varphi(\psi x)\)", for the latter was to represent three distinct possibilities \({\sim \psi x, \psi x, \sim \varphi(\psi x), \text{and } \psi x, \varphi(\psi x)}\), whereas our analysis gave us four (the four rows of the truth-table for conjunction). What we need is a restriction of the form "\(\varphi x\) entails \(\psi x\)" so that \(\varphi x, \sim \psi x\) would drop out as impossible; however such an option is not open to

\[^{39}\]Of course we can replace, as did Russell in Theory of Knowledge, \(\psi x\) with a proposition of the form "there exists a complex \(y\) such that \(y\) is of the form \(\varphi x\) and \(\psi\) and \(x\) make up \(y\)." This form of analysis leads, however, to the requirement that some proposition of the form \(\varphi x\) be true, as we saw in Chapter 4.e-f.
Wittgenstein, above all because he believes that necessary implications can only be explained by means of the containment of the consequent in the antecedent,\textsuperscript{40} which containment must show itself structurally once the implication is properly analyzed. That is to say that such necessary statements may follow from the syntactic analysis Wittgenstein attempts here, but they cannot be used to shore it up, and this is directly connected to the thesis that all aspects of the proposition's sense that do not depend on the denotations of the symbols must be reflected in its structure.

Having rejected his best candidate for the analysis at hand, Wittgenstein concludes that there cannot be one, and he expresses this by saying that if a fact can have a property, then it must be an internal property of such a fact. "Internal property" is a key concept in the \textit{Tractatus}: type-membership is an internal property of things and names, as well as of complexes and propositions, the logical propositions show internal properties of logical space, and so on. Here Wittgenstein asserts that complexes (facts) can have only internal properties, which properties he later calls "structural." Any thing that can have external properties must accordingly be not-complex, thus an indestructible simple. To see why this is so, we need a definition of "internal property," which I shall now give. It will serve as a basis for the discussion in the following chapters as well.

\section*{ii. Internal Properties}

Wittgenstein had the concept of an internal property from G.E. Moore. He never offers a systematic definition of the term himself, and although Moore's "External and Internal Relations"\textsuperscript{41} gives a detailed one, it was also published after the war, so

\textsuperscript{40}\textit{Notebooks}, p. 90. Entweder eine Tatsache ist in einer anderen enthalten, oder sie ist unabhängig von ihr.

that we have no contemporary account of Wittgenstein’s understanding of this term. The *Tractatus* definition, which I discuss below, equates internal properties with structural properties, although that refinement is a consequence of the work discussed here. However, since Moore’s essay counts among his most mature work, and since we see Wittgenstein using the term internal relation in discussion with Moore in April 1914 (the passage at hand being from September of that year), we can assume at the very least a shared understanding of the word, which ought not differ substantially from that given by Moore in 1919.

In that paper Moore gives an informal and a formal definition of the concept *internal property*:

\[ P \text{ is an internal property of } A \text{ implies that “anything which were identical with } A\text{ would, in any conceivable universe, necessarily have } P. \]

or, \[ Px \supset (x = y \text{ entails } Py) \]

This may seem a cumbersome way of expressing the proposition “Necessarily: \( Px. \)” But what is striking about Moore’s definition (and Wittgenstein’s appropriation of it) is the connection between the modality of logical entailment and that of containment. Moore clearly wants to leave open the possibility that \( A \) may never exist, as can be seen in the spatial example he uses: if \( a \) is a part of \( b \), such that \( C(b, a) \) (\( b \) Contains \( a \)), then \( \psi x = C(b, x) \text{ Def.} \) is clearly an external property of \( a \) (for the part \( a \) may exist without the whole \( b \)), and \( q x = C(x, a) \text{ Def.} \) is an internal property of \( b \) (for whenever \( b \) exists, \( a \) does too). However what of the cases when \( b \) does not exist? Moore would reply that his analysis remained valid: nothing is identical with \( b \), so nothing need have its internal properties. So it does not follow from Moore’s definition that, if

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42Moore, "External and Internal Relations," p. 293.

43Moore, "External and Internal Relations," p. 298.

$P$ is an internal property of $x$, then "$Px$" is true in all possible worlds—it need be true only in those in which $x$ in fact exists.

To apply this reasoning to the case at hand, we need only simplify Moore's definition somewhat. We say that $\varphi x$ is an internal property of $\psi x$ if:

$$\varphi(\psi x) \supset (\psi x \text{ entails } \varphi(\psi x))$$

This formulation is simpler because the antecedent of the entailment, which for Moore posited the existence of the subject by means of an identity, is here replaced with the condition that $\psi x$ be true (though this amounts for Russell and Wittgenstein to the claim that "the complex $\psi x$ exists"). Whenever $\psi x$ exists, then it has the property $\varphi$, and the statement that it does not have that property implies that it does not exist. By stipulating that the truth-grounds of $\varphi(\psi x)$ be identical with those of $\psi x$, Wittgenstein avoids both the problem of ambiguous sense and that of mismatched truth-possibilities of fact and proposition. The sense of $\varphi(\psi x)$ is determinate, for $\sim \varphi(\psi x)$ entails $\sim \psi x$, and thus there is no second way for $\varphi(\psi x)$ to be false, whereas the analysis initially proposed by Wittgenstein, in replacing $\varphi(\psi x)$ with $q \times \psi x$, allowed for possible situations ($q \times. \sim \psi x$) which had no correlate in the sense of the statement which he had hoped to analyse.

So we have two ways of considering the sign "$\varphi(\psi x)$": either it is necessarily equivalent to $\psi x$, and expresses an internal property of $\psi x$, or it is simply an alternative expression for $q \times. \psi x$. In the first case $\varphi \times$ is just what Wittgenstein calls a "property of structure" (4.122), thus not a real property at all. In the second case, $\varphi \times$ is a property like $\psi \times$, and $\varphi(\psi x)$ exhibits the syntax of a predicative proposition, under the constraint that $\psi x$ exist: from its truth or falsity it will follow that $(\exists x).q x$ and $(\exists x).\sim q x$, respectively. And that is the result that we both wanted and expected, given the doctrine with which we began (that the sign plus the specified denotation guaranteed the significance of the proposition): so long as ' $\psi x$' denotes, then $\varphi(\psi x)$ is
bivalent, as an elementary proposition must be; and should it not be bivalent, then this is to be explained by the fact that 'zar' fails to denote. If there are propositional signs whose bivalence does not depend on some further specification to the effect that their elements have a denotation, then this will be because their denotation is permanent. Lastly, since all the analyses of complex propositions put forward by Russell and Wittgenstein result in expressions compounded out of elements which are themselves assumed to be bivalent (other, simpler, propositions), it follows that some set of signs with stable denotations must exist. If bivalence is the same as determinateness of sense, then this is the meaning of 3.23:

3.23 Die Forderung der Möglichkeit der einfachen Zeichen ist die Forderung der Bestimmtheit des Sinnes.45

As this is one of the best understood Tractatus doctrines, I will not develop this point at greater length in what follows. However since this thesis has not yet been looked at in its connection to the arguments concerning types and forms, I should mention two connected points:

1) The requirement that objects exist has no compelling force if one does not assume that certain symbols are inherently possible propositional signs.

2) The entities to which this argument infers are not simple objects, but simple objects with given structural potentialities (the combinatory possibilities referred to in 2.0123).

With regard to the first point, we should recall that Wittgenstein’s and Russell’s travails during 1912-1913 were very much concerned with establishing the bivalence of elementary propositions, but paid little attention to the question of complex objects. Their concern was to eliminate that huge class of pseudo-propositions which results from unrestrictedly combining the different entities making up our world, and thus to ensure that a judgment or a proposition unambiguously correlate with that combination of objects which is intended to be its corresponding fact. Wittgenstein

45 3.23 The requirement of the possibility of the simple signs is the requirement of the determinateness of sense.
realised that this can only be achieved by assuming a parallelism between the facts and the signs for these facts; if we cannot assume this, then we cannot eliminate the pseudo-propositions from our logic. Once we assume it, the failure of a given propositional sign to be significant can only result from a failure in denotation. Without that assumption, however, the reductio alluded to in 3.23 will not go through.

The second point relates to a passage complementary to 3.23, which stipulates that there be simple objects (as opposed to the simple names of 3.23) in order to secure the sense of propositions:

2.021 Die Gegenstände bilden die Substanz der Welt. Darum können sie nicht zusammengesetzt sein.

2.0211 Hätte die Welt keine Substanz, so würde, ob ein Satz Sinn hat, davon abhängen, ob ein anderer Satz wahr ist.46

As we saw, \(q(A)\) is bivalent just in case \(A\) exists, so the truth of that last statement is the condition for \(q(A)\)'s having a sense. The passages following emphasize the form of the objects in question just as much as their simplicity:

2.022 Es ist offenbar, daß auch eine von der wirklichen noch so verschieden gedachte Welt Etwas—eine Form—mit der wirklichen gemein haben muß.

2.023 Diese feste Form besteht eben aus den Gegenständen.

2.024 Die Substanz ist das, was unabhängig von dem was der Fall ist, besteht.

2.025 Sie ist Form und Inhalt.47

When Wittgenstein rejected his copula-theory, he also rejected the notion of the propositional form as something above and beyond the forms of the objects—those properties which determine their combinatorial possibilities. The forms of the objects are,

462.021 Objects make up the substance of the world. Therefore they cannot be complex. I 2.0211 If the world had no substance, then whether a proposition had sense would depend on whether another were true.

472.022 It is obvious that even a conceived world quite different from the real one must have something—a form—in common with the real one. I 2.023 This stable form derives from the objects. I 2.024 Substance is that which subsists independently of what is the case. I 2.025 It is form and content.
from this new perspective, the correlates of the forms of the symbols denoting them: both determine the possible combinations of elements in each domain, and they must be such that the two domains are isomorphic.\textsuperscript{48} 3.23 and the 2.02x sequence do indeed give Wittgenstein's argument for the existence of logical atoms; however this argument is only complete if it is seen as an argument for the existence of groups of objects, which groups are furthermore connected to each other: elements of one group may combine with elements of a second, but not with those of some third.

d) Wittgenstein's "Correct Theory of Propositions"

Elementary propositions in the \textit{Tractatus} are no longer incomplete symbols whose relations to other propositions must be secured by means of a catalogue of propositional forms and a type-directory. They must instead ground these various logical properties \textit{themselves}: we cannot construct the types assertorically, for to do so would violate sense restrictions, and the assumption of proto-types for possible propositions results either in circularity or redundancy, for a complex proto-type might fail to exist, and a simple one will not be able to reflect the ordering of the corresponding fact's elements. The question of whether two names are of the same, of adjacent or of incompatible types must be settled, as with Frege, by attention to their structural properties alone. This gives us an indication as to why Wittgenstein began to describe his new theory from the point of view and in the language of mechanics. For consider the fully generalized propositional form $\varphi x$--an \textit{Urbild} in the language of the \textit{Tractatus}. "$\varphi x$" picks out a class of propositions, each of which is obtained by replacing the variables with names of the appropriate type. The result of the preceding discussion is that all elements of this system must in some sense be given together--the class $\hat{p}\langle(\exists \varphi, x)\varphi(x) = p\rangle$, the elements which may be put in place of $\varphi$, and $x$, as

\textsuperscript{48}These amount to planks 2) and 3) in Chapter 3.
well as the ensuing constraints on substitutions: this system is that defined by the five "planks" of the Tractatus theory that I listed at the beginning of Chapter 3. Having made such a move, we can clearly dispense with the notion of the relational form and its paradigmatic fact. We need not assume e.g. that there are facts of the form in question, that "the possibility of the truth of the proposition $\phi a$ [is] tied up with the fact $(\exists x, \varphi)\psi x$." The range of the Urbild is a manifold whose individual elements are determined by setting the variables to values among some pre-determined sets—the types.

The results of these early considerations can be read off the first pages of the Tractatus, where Wittgenstein demands that the class of all possible facts of a given form be given together with those of all objects which may occur in those facts (the types of their elements). He makes this demand plausible by means of a spatial analogy: the spatial object cannot be conceived without a space in which it lies; similarly the logical object is held to be inconceivable without a space of possible facts in which it may occur. What is the point of this comparison? Wittgenstein's dilemma was that you can't say what the types of things (or symbols) and the forms of complexes are. But in the absence of such specifications, you cannot decompose the complex propositional sign in such a way that it remains significant even when no fact of that form obtains. The conclusion is that acquaintance with an object must entail acquaintance with the type of the object, with the types of facts in which it may occur, and with the types of the objects which occur in these facts:

2.012 In der Logik ist nichts zufällig: Wenn das Ding im Sachverhalt vorkommen kann, so muß die Möglichkeit des Sachverhaltes im Ding bereits präjudiziert sein.

2.0123 Wenn ich den Gegenstand kenne, so kenne ich auch sämtliche Möglichkeiten seines Vorkommens in Sachverhalten. ...

49Notebooks, p. 17.
Wittgenstein's analogy in 2.0131 between "logical" and phenomenological spaces is potentially misleading, because it leads naturally to the thought that objects are grouped in different perceptual spaces, consisting of series of predicates denoting degrees of colour, hardness, etc. But that is not the point of the comparison. To be an object capable of being coloured means belonging to the group of potentially coloured things. Wittgenstein adds to this the requirement that, if an object can be coloured then it must have some colour—having a colour is, to use Moore's language, an "internal property" of that thing:


(Statt Eigenschaft der Struktur sage ich auch "interne Eigenschaft;" statt Relation der Strukturen "interne Relation." ... )

Das Bestehen solcher interner Eigenschaften und Relationen kann aber nicht durch Sätze behauptet werden, sondern es zeigt sich in den Sätzen, welche jene Sachverhalte darstellen und von jenen Gegenständen handeln.51

The coloured object is unthinkable without its having a colour: in this sense the thing is in a "space" of possible colours. What corresponds here to the class of potentially coloured things is, once again, a type, and what corresponds to the coloured thing is an object belonging to that type. As we have seen, if type membership is conceived as being a property that holds of the objects in a type, then it is nonsense to ask of something not in the type whether it has that property, and it is likewise

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502.012 In logic nothing is accidental: If a thing can occur in an elementary fact, then the possibility of the elementary fact must already be constitutive of the thing. | 2.0123 If I am acquainted with an object, then I am also acquainted with all possibilities of its occurrence in elementary facts. ... | 2.0124 If all objects are given, then with them are given all possible elementary facts as well.

514.122 We can in a certain sense speak of formal properties of objects and elementary facts, or of properties of the structure of facts, and in the same sense of formal relations and relations of structures. | (Instead of properties of structure I also say "internal property;" Instead of relation of structures "internal relation."... ) | The existence of such internal properties and relations cannot however be asserted by means of propositions, rather it shows itself in those propositions which represent those elementary facts, and deal with those objects.
inconceivable, or at least inexpressible, that one of the type-members not have that property. The distinctions among types must thus be captured in the symbols by means of "structural properties" in the sense of 4.122. To entertain the possibility that a thing might belong to some type other than its own would be like asking whether a musical tone might not be red: "... an illogical language would be one in which, e.g. you could put an event into a hole."52 A language not meeting the requirement that type-membership be internal would be illogical in this sense, for it would permit us to entertain such pseudo-possibilities.

The defining characteristic of Wittgenstein's "correct theory of propositions"53 lies in its rehabilitation of the propositional sign: the $Sar$54 is an independent entity that is structurally isomorphic to its corresponding fact. Through its structural properties (the number, type, and ordering of its elements), it picks out a class of possible facts with the same structure. At the same time, however, it is itself a member of a class of similar signs: on the one hand, the very possibility of distinguishing the elements of, say, "$aRb$" depends on our being acquainted with other signs of the form "$xRb$";55 on the other hand the rigid distinction between the arbitrary determinations of the proposition's sense (the denotations we have given the names) and its essential, internal determinants (the types and arrangement of its components) means that any member of a class of signs having those essential properties is potentially a propositional sign for every element of the corresponding class of complexes. To say that a complex has a particular structural characteristic is to assume that there are other complexes of that structure, and this goes for the

52 "Notes Dictated to Moore," in Notebooks, p. 108.


54 Der Satz was of course Wittgenstein's working title for the Tractatus.

55 I return to this point in Chapter 7.a.
sentences as well as for their corresponding facts. This is, as we shall see in Chapter 6, of particular importance for Wittgenstein’s account of quantification.

As we specify the denotations of the elements of the propositional sign, we constrain its potential denotation to subsets of the class of facts with an isomorphic structure. This process culminates in a single fact when all denotations have been given. Reversing this sequence, we go from the individual elementary proposition to the Urbild, the symbol for the class of all propositions of that type:

If we change a constituent \( a \) of a proposition \( \varphi(a) \) into a variable, then there is a class

\[
\hat{\varrho}(\exists x). \varphi(x) = p
\]

This class in general still depends on what, by an arbitrary convention, we mean by “\( \varphi(x) \)”. But if we change into variables all those symbols whose significance was arbitrarily determined, there is still such a class. But this is not dependent upon any convention, but only upon the nature of the symbol “\( \varphi(x) \)”. It corresponds to a logical type.\(^{56}\)

This remark appears in a somewhat different form as Tractatus 3.315, where it is used to explain how we go about generating sentential variables. That context may make it appear as if the possibility of there being an Urbild depended on an ancillary assumption: that sentences are organized in classes. In fact this connection between Urbilder and names, and logical forms and objects is essential to Wittgenstein’s conception of the individual elementary proposition: since everything “characteristic of the sense” of the latter—except the denotations—is contained in its structure, it follows that any symbol with that structure could refer to any fact sharing it:

\[3.31\] Jeden Teil des Satzes, der seinen Sinn charakterisiert, nenne ich einen Ausdruck (ein Symbol).

(Der Satz selbst ist ein Ausdruck.)

Ausdruck ist alles, für den Sinn des Satzes wesentliche, was Sätze miteinander gemein haben können.

Der Ausdruck kennzeichnet eine Form und einen Inhalt.

\(^{56}\)“Notes of Logic.” Notebooks, p. 101.
3.311 Der Ausdruck setzt die Formen aller Sätze voraus, in welchen er vorkommen kann. Er ist das gemeinsame charakteristische Merkmal einer Klasse von Sätzen.57

To be collected with other symbols (and other facts) of the same structure is not an accidental but an essential characteristic of both the elementary propositional sign, and its correlate the Sachverhalt. The elementary fact, like a spatial point, can be conceived as an element in a manifold of similar elements, each of which shares its essential constitution; and the elementary proposition is also situated in an isomorphic manifold of structurally identical propositions. The names in the symbol picking out a fact belong to types of names, and in determining these denotations of these names we uniquely identify the element in the manifold (defined by the types of things) to which the symbol refers: to set those determinations is to specify the mapping (Abbildung) between the two manifolds. As such the types serve as the coordinates of the manifold in question, and the Urbild corresponds to the totality of that manifold, the set of all such facts. In the next chapter, I will develop this correspondence between types and dimensions on the one hand, and facts and manifolds on the other. This correspondence is what moved Wittgenstein to coin the term “logical space.”

573.31 I call each part of a proposition that characterises its sense an expression (a symbol). 1 (The proposition is itself an expression.) 1 Everything which is essential to the sense of a proposition, and which propositions can have in common with one another, is an expression. 1 The expression characterises a form and a content. 1 3.311 The expression presumes the forms of all the sentences, in which it can appear. It is the common characteristic mark of a class of sentences.
Chapter 6 - Logical Space

In this chapter I give a partial account of Wittgenstein's "class-theory" of propositions, and show how it connects first to his account of simple truth-functions, and, in turn, to that of quantified propositions. In the process, I show that the complex propositions of the Tractatus get their meaning only through their connection to what I call the "core logical space." The latter is the space of elementary propositions determined by the types. I use the term in distinction to "logical space" in general, since Wittgenstein sometimes uses this term to refer to the whole set of propositions which can be formed in a given logical notation, whereas the core logical space is the space given by the elementary propositions alone. From this I turn to a discussion of the relation between Urbilder (proto-types) and their relation to the class of propositions of which they are the Urbild. The quantifier notation depends on there being an internal relation between Urbild and class, for otherwise it would be impossible to pick out all and only those elementary propositions having the form in question. In explicating sections 4.04-.0411, I argue that Wittgenstein conceives this connection as being like that between the model of a physical system, and the class of states which that model represents. Lastly, I connect this spatialized view of the generalized proposition with the view of objects that it entails. The objects of the Tractatus have little in common with the objects of Russell's original theory: they are indices recording salient aspects of a class of facts, and they are indeed only definable on the assumption that there be several such facts.

a) Truth-functions and the "Core Logical Space"

i. Logical constants and representation

Two remarks in the Tractatus lay particular weight on the connection between the "logical space" and truth-functions:
3.42 Obwohl der Satz nur einen Ort des logischen Raumes bestimmen darf, so muß doch durch ihn schon der ganze logische Raum gegeben sein.

(Sonst würden durch die Verneinung, die logische Summe, das logische Produkt, etc. immer neue Elemente—in Koordination—eingeführt.)

(Das logische Gerüst um das Bild herum bestimmt den logischen Raum. Der Satz durchgreift den ganzen logischen Raum.)

4.0641 Man könnte sagen: Die Verneinung bezieht sich schon auf den logischen Ort, den der verneinte Satz bestimmt.

Der verneinende Satz bestimmt einen anderen logischen Ort als der verneinte.

Der verneinende Satz bestimmt einen logischen Ort mit Hilfe des logischen Ortes des verneinten Satzes, indem er jenen als außerhalb diesem liegend beschreibt.

Daß man den verneinten Satz wieder verneinen kann, zeigt schon, daß das, was verneint wird, schon ein Satz und nicht erst die Vorbereitung zu einem Satze ist.

The remarks forming 4.0641 are from the beginning of November 1914, and those of 3.42 are from the end of that month. 3.42(1) has the first use of the term “logical space.” Together they give a good explanation of what Wittgenstein called his “fundamental thought:”


Mein Grundgedanke ist, daß die “logischen Konstanten” nicht vertreten. Daß sich die Logik der Tatsachen nicht vertreten läßt.

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13.42 Although the proposition may only determine one place in the logical space, still the whole of the logical space must be given with it. (Otherwise through the introduction of negation, the logical sum, the logical product, etc., new elements would be introduced—in coordination.) (The logical framework around the proposition determines logical space. The proposition reaches through the whole logical space.)

24.0641 One could say: Negation does indeed refer to the logical place determined by the negated proposition. (The negated proposition determines another logical place than that of the negated one. The negated proposition determines a logical place with the help of the logical place of the negated proposition, in that it describes the former as lying outside the latter. That one can once again negate the negated proposition shows that which is denied is already a proposition, and not merely the preparation for a proposition.

34.0312 The possibility of the proposition rests on the principle of the representation of objects through signs. My fundamental thought is that the “logical constants” do not represent. That the logic of facts cannot be represented.
Wittgenstein’s purpose in introducing the concept of a “logical place” was to explain how an elementary proposition might be significant. In comparing the names to coordinate values, and the possible existence of a fact to a location that might be filled, he sought to capture two essential results of his critique of his and Russell’s earlier theories: that names are organized into groups and have stable denotations; and that a proposition, when not true, is false. The first of these finds its analogue in the coordinates of a manifold; the second in the notion of a location in this manifold, which is either filled or empty. If we think back to the original, Russellian project, we see that this conception meets its primary aim: it provides an account of elementary judgments which will secure bivalence as an essential characteristic of the proposition; thus it provides a domain of propositional entities, the relations among which can be defined. The relations are the logical constants, which “do not represent.”

In the earlier theories, the correlate of the elementary proposition is the elementary judgment, and those of molecular propositions—negations, disjunctions, generalizations—are higher-order judgments, some of whose elements will be the elementary judgments, and some of which will be “logical objects.” Whether one interprets these, with Russell, as constituents of complexes, or, on the other hand, as classes,⁴ the result is the same: if we are to explain the nature of such logical objects, then we must assume that their arguments, the propositions, are bivalent. Just as, however, Wittgenstein sought to eliminate the form of the proposition as an entity whose existence must be separately supposed, he regarded the forms of molecular propositions with the same suspicion: their sense depends on nothing more than the truth-possibilities of the elementary propositions; thus to posit them as “elements”

⁴Recall Russell’s intensionalist and Wittgenstein’s extensionalist interpretations of the logical form: Russell saw it as an abstract object, Wittgenstein as a class which might fail to exist.
above and beyond those of the entities named by the elementary propositions is redundant.

Bivalence is essential for the following reasons. We want to define "¬" as follows: \( \{ ¬p \} \) is the class of propositions such that when \( p \) is true, then \( ¬p \) is false, and when \( p \) is false, then \( ¬p \) is true. Obviously \( ¬p \) can be bivalent only in the event that \( p \) is. In the elementary case, this property of \( p \) is explained by means of the notions of logical place and existing (bestehend) complex. They explain it by saying that \( p \) is true when a complex exists at the corresponding logical place, and that it is \textit{false} when there is no complex there. Thus if \( p \) is an elementary proposition, then:

1) \( ¬p \) is true means: there is no complex at location \( p \);
2) \( ¬p \) is false means: there is a complex at location \( p \);
and we have,

3) the class of propositions of the form \( ¬p \) is the class of propositions which are true when there is no complex at \( p \) (when "\( p \)" is false), and false when there is one at \( p \) (when "\( p \)" is true).

If the range of \( p \) in 3) exceeds that of bivalent sentences, then expressions such as "not the table penholders the book" will be judged true: we will end up with a Fregean account of logical constants and propositions. Thus if "¬" is to have the necessary logical properties, it must be defined only for \( p \)'s which are true/false propositions, and this is what the spatial theory purports to do.

Similarly disjunction can be defined as the shared characteristic of symbols \( p \lor q \) which have the following properties:

1) When either \( p \) or \( q \) is true, then \( p \lor q \) is true.
2) When \( p \lor q \) is false, then \( p \) is false and \( q \) is false.

In the definition of negation, I assumed that \( p \) was an elementary proposition, but clearly that can only be a partial definition. In fact both definitions depend only on \( p \), and \( q \), always having a truth-value—we need not suppose they are elementary. The
"core logical space"--the manifold of elementary propositions determined by the types--grounds truth and falsity for the elementary propositions, and the molecular propositions must be analyzed recursively until we reach the elementary sentences. Their bivalence is the ground for the bivalence of the molecular ones, so long as the truth-functions are introduced in such a way that they depend for their truth-values only on the truth-values of their arguments. The introduction of functions such as "v" leads in a sense to an expansion of the core space, if we take the view that the logical space is the space of all possible propositions. But if we are to take that view, we must still hold to the requirement that this addition be no more than a superstructure, and an arbitrary one at that: the introduction of $p \lor q$ into the elementary language does not add new entities to the ontology (the function "or") and it does not introduce new possible facts either. It introduces new classes of propositions, but their sense can be exhaustively described in terms only of the elementary ones.

ii. The class-theory of logical functions

So the "class-theory," as Wittgenstein called it, defines truth-functions in terms of primitive and essential characteristics of the elementary propositions. Unlike the proto-theory, which supposed that the elements of say, " $p \lor q$" would have denotations, thus that "v" must designate a form determining the class of disjunctive propositions (one need only think of Wittgenstein's "inference copula"), this approach makes no assumption that "v" denotes anything. For if we know what " $p$" and " $q$" in " $p \lor q$" refer to, and we know that "v" in such a propositional sign relates the components of the sign in a particular manner (disjunctively, in accordance with the definition given above), then we know exactly under which conditions " $p \lor q$" is to be judged either true or false: whenever there is a complex at least one of the

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5Letter to Russell of 1.7.12, Notebooks, p. 120.
locations $p$ and $q$, then it is true, otherwise it is false. In order to determine that, we need only know the denotations of the names in "$p$" and "$q$": they are the only symbols in "$p$" and "$q$" that represent (vertreten), even though they are not the only symbols in "$p \lor q$" which qualify as expressions, in being "essential to their sense," for "$\lor$" meets that criterion as well. Those elements of a propositional sign which "represent" are only those which have a correlate in the system of coordinates (the types) that define the spaces of elementary facts and propositions. That space constitutes the core of our (natural) logic, for the truth of every complex proposition depends only on the truth or falsity of these. The molecular propositions do of course have a sense, but it is nothing more than the connections postulated between locations in this core space:

4.2 Der Sinn des Satzes ist seine Übereinstimmung, und Nichtübereinstimmung mit den Möglichkeiten des Bestehens und Nichtbestehens der Sachverhalte.  

The sense of a proposition is thus whatever distinguishes those truth-value assignments on which the proposition is true from those on which it is false. But we can define a truth-function such as disjunction without knowing the sense of the components of the disjunctive proposition, indeed without knowing whether the components are themselves elementary or complex, because the definition depends only on the truth-values of the latter. The significance of "disjunction" can be reflected by a sign displaying the connections between the truth-values of the component propositions and that of the complex one: both the $ab$-notation and the truth-tables of the Tractatus are notations which display these connections. The fact that we can do this shows that our understanding of this notion does not depend on some feature of the core logical space (the elementary facts and propositions) beyond the bivalence of its elements. Conversely the number and kinds of logical functions

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64.2 The sense of a proposition is its agreement and disagreement with the possibilities of the existence and non-existence of elementary facts.
that can be defined in this way is unbounded: the core logical space will support any superstructure of complex sentences, providing that expressions appearing in these sentences—the logical operators—are defined in such a way that the truth-value of such sentences depends on those of the elementary propositions alone. Since there can be arbitrarily many such superstructures (as long as there are arbitrarily many elementary propositions), each determined by the kind and number of functions that have been defined, we must conclude that each has an equal claim to validity; that is, an expression such as ‘p * q’, which may be defined in a language L₁ but not in L₂ does not necessitate an entity corresponding to “*”, let alone a ‘logical fact’ corresponding to ‘p * q.’ When Wittgenstein says in 3.42 that, through the proposition, the whole of logical space must be given, for otherwise our introducing truth-functions would involve the introduction of “new elements—in coordination,” he is alluding to just this relation between the core language (the core space) and the logical superstructure, whose structure and definitions are parasitic on the former. Just as knowing an object means knowing all of its possible occurrences in possible elementary facts (2.0123), so knowing an elementary proposition means knowing all of the sentences in which it could occur as a constituent, i.e. if I know what “p” and “q” signify, and I know the role of “v” in disjunctive propositions, then I must know the significance of “p v q.” Otherwise I would have to suppose that “p v q” referred to something beyond the possible references of “p” and “q”; that is, beyond the facts p and q. If this were so, then “p v q” would point to a further logical place, of which “v” was a coordinate, just as F and a are coordinates of the logical place Fa. The connection between “p” and “p v q” would have to be defined, in order that the logical relationship between the two sentences be secured: each logical function would necessitate the introduction of new “elements” into the logical space, and the relationship between these elements and the others would have to be stated—they would have to be introduced “in coordination” with the elements of the old logical space.
A brief comparison of this molecular theory with Wittgenstein's analysis of the elementary propositions is in order here, for the class-theory of molecular propositions and the picture-theory of elementary ones are, as I have already suggested, in their motivation very close, and the points in which they differ mark fundamental divisions within the logic Wittgenstein envisages. The common origin is of course the judgment-theory, which associated a copulative complex with each possible true judgment: every proposition was to be analyzed into its immediate components (which might be complexes themselves) and some relating relation which specified how they were combined. The proposition $aRb$ becomes $e_2(a, R, b)$, a complex related by the relation copula $e_2$, and made up of $a$, $R$, and $b$. Similarly $p \lor q$ could be analysed first as $v(p, q)$, where the relating relation would be disjunction. The class of pairs of propositions $(p, q) \lor (\bar{p}, \bar{q})$ would be those pairs of which one or the other were true, and to say $p \lor q$ would be to say that $p$ and $q$ belonged to that class, that $(p, q) \in v$. Following the pattern of the elementary proposition, we could use the object $v$ to form the complete analysis $e_8(p, q, v)$, where $e_8$ is the sumnum genus, to use Russell's definition of the propositional form, of binary truth-functions. This latter expression says: "the pair of propositions $(p, q)$ belongs to $v$, the class of all pairs of propositions of which at least one is true."

But we know this story all too well. The analysis assumes that the class $v$ always exists, and it depends on our having a working theory of elementary propositions. Wittgenstein seems to have abandoned this approach even before his rejecting the copula-analysis of the elementary proposition:

I believe that our problems can be traced down to the atomic propositions. This you will see if you try to explain precisely in what way the Copula [sic] in such a proposition has meaning.7

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7Letter to Russell from the Fall of 1912. Notebooks, p. 121.
The decisive move in Wittgenstein's break with the elementary copula-theory was his demand that the class of elementary facts of a given form was determined by the classes of objects which complement (saturate) one another, such that they may form complexes. The class of possible predicative facts is all possible combinations of predicates and individuals. We demand a notation in which the symbols for the objects reflect these possible combinations. The signs must show us what facts there could be, even when those facts do not obtain. So in answer to the question, "How do we know what the proposition 'Fa' predicts about the world when we assert it?" we answer: "We know what $Fx$ and $a$ are, we know what classes they belong to, and we know how the elements of these classes can combine." The class-theory of molecular propositions follows essentially the same lines.

It is supposed to answer the question, What is the meaning of a proposition such as $p \lor q$? It rejects the idea that such an expression brings a logical relation into connection with two complexes, and says instead that it belongs to a class of propositions of a particular sort: those consisting of two propositional components, at least one of which is true when the complex proposition is true. This is not the class $\lor$, however. That was a class of pairs of propositions, each pair of which contained one member which was actually true. The class of propositions to which we are here referring is defined only by means of the interconnection between the possible truth-values of its component parts. The definition picks out a class of signs, in other words, and it is clear that the class of signs which could have this characteristic (which could be so construed) could be larger than the class of symbols which are in fact used in this way. Recall the equivocal nature of the symbols for predicative complexes: they had to have certain structural characteristics in order to serve in that capacity, indeed the structural characteristics had to be essential properties of the signs in question, for otherwise they would fail to embody the type restrictions; conversely in assigning denotations to the names appearing in them, we fixed
arbitrary connections between the elements of the sign and the elements of the possible fact it was to denote. Signs not given such coordinations would be of necessity possible propositional signs, but, lacking a coordination with the world, they would be facts and not pictures.

The situation in the molecular case is much the same. Assume that $p$ and $q$ have been coordinated with possible facts. Then there will be symbols in which $p$ and $q$ appear which have the requisite characteristics for depicting the disjunctive relation, just as there would be others which would lend themselves to being symbols of the conjunction of $p$ and $q$. Wittgenstein's $ab$-notation (which appears as the TF-notation in the Tractatus) is one way of displaying the essential characteristics of such a sign, as are the truth-tables. The logical relation of disjunction is thus the shared characteristic of those signs which we use in a disjunctive fashion. This conception in no way implies the existence of a class of disjunctive facts. What it says is rather the following: if "$p$" and "$q$" in "$p \lor q$" are both true/false propositions, and

$$\{p, q\} \vdash p \lor q$$

and

$$\neg (p \lor q) \vdash \{\neg p, \neg q\}$$

(this latter being a statement about how we use the sign "$p \lor q$" in conjunction with "$p$", "$q$", and "$\neg$"), then "$p \lor q$" is a sign for the disjunction of $p$ and $q$. Similarly our analysis of $Fa$ said: if "$a$" and "$Fx$" are names with fixed denotations, such that we say that "$Fa$" is true whenever "$a$" and "$Fx$" are combined, and false whenever they are not, then "$Fa$" is a propositional sign asserting the existence of $Fa$. The two analyses share the following features:

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8For an exact account of the properties a logic must have in order to conform to the demands of Wittgenstein's class-theory, see Hacking, Ian. "What Is Logic?" The Journal of Philosophy, vol. 86, no. 6 (1979). pp. 285-319.
1) the possibility of the propositional signs "Fa" and "p \lor q" is a by-product of the characteristics of their elements "a" and "Fx", and "p" and "q";

2) the possibility of the facts Fa and p \lor q is a possibility inherent in the constitutions of Fx and a, and p and q.

1) Says that the symbols for the components have to display the possibility of their being combined to form the signs "Fa" and "p \lor q"; and 2) says that the possibility of whatever fact or facts in the world correspond to the truths of "Fa" and "p \lor q" is already contained in the natures of their elements—in a's and Fx's belonging to the types they do, and in p's and q's being facts which either exist or do not. But there is a fundamental difference as well: the definition of each kind of truth-function adds new propositions to our language, but it does not add further elements to the logical space; instead it allows us to construct infinitely many complex sentences which nonetheless depend for their significance only on the elementary propositions. The possibility of a sign like "p \lor q" is given once "p" and "q" are given, just as that of "Fa" was given when "a" and "Fx" were given. But nothing corresponds to "p \lor q" aside from p and q, whereas to "Fa" corresponded not only Fx and a, but, when "Fa" is true, the complex Fa as well. The addition of a new name, or indeed a new type, would expand the manifold of possible facts; however the addition of a new truth-function leaves it as it is.

I will illustrate this point by means of a little-understood argument of Wittgenstein's in the "Notes Dictated to Moore." There he observes that:

What is unarbitrary about our symbols is not them, nor the rules we give; but the fact that, having given certain rules, others are fixed = follow logically.\(^9\)

As I have been arguing, the property of being a possible symbol of a predicative fact, or a possible symbol for a disjunction, is a structural property of some facts, and is in no way arbitrary, since other facts cannot serve this purpose. What is arbitrary.

however, is that we are using just these symbols among the possible ones, and that we have given their components the referents we have. The internal structural characteristics translate into logical properties once this reference has been fixed--up until that point they are merely \textit{Züge} which may form the basis for \textit{Zeichenregel}. Wittgenstein compares the definitions of $p \lor q$ and $p.q$ to make this clear:

Take $p.q$ and $q$. When you write $p.q$ in the $ab$-notation, it is impossible to see from the symbols alone that $q$ follows from it, for if you were to interpret the true-pole as the false, the same symbols would stand for $pq$, from which $q$ doesn't follow. But the moment you say \textit{which} symbols are tautologies, it at once becomes possible to see from the fact that they are and the original symbol that $q$ does follow.$^{10}$

The symbols for $p.q$ in the $ab$-notation is:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {a};
  \node (b) at (2,-2) {b};
  \draw (a) -- node[above] {a-p-b} (b);
  \draw (a) -- node[below] {a-q-b} (b);
\end{tikzpicture}
\end{center}

The diagram correlates the outer $a$-pole with both the inner $a$-poles, and the outer $b$-pole with all the other pairings of the inner poles. If we stipulate that the $a$-poles are the true-poles, and the $b$-poles the false-poles, then this is the symbol for conjunction. With the opposite correlation, it becomes the symbol for disjunction. If we choose the first course, then the symbol shows the fact that whenever it is true, then $q$ is true. If we choose the second, then it does not. So the sign in question has certain characteristics making it serviceable as the symbol for the truth-function of two elementary propositions; however \textit{how} it is to do this, indeed which function it is to represent, still depends on our making arbitrary decisions about the use we will make of the poles. Similarly the truth-tables for disjunction and conjunction have the same

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$^{10}$\textit{Notebooks}, p. 113-114.
structural characteristics: it is only once we have fixed the meanings of “T” and “F” that we know which functions they describe. On its own the sign is a symbol neither for conjunction nor disjunction. Depending on how we settle its interpretation, it may serve as a sign for either.

Disjunction then is a property of some signs being used in a certain way. The way in which such signs are used is this: they are said to be true when at least one of their two (disjoined) components is true, and when they are false, then both their components are false. Clearly the signs in question must make clear what their two components are, otherwise we would have no way of knowing how the truth-value of the disjunction was determined. But what weight should we attach to their “showing” the internal connection between component and whole? The \( ab \)-notation and truth-tables go a long way to displaying this relationship, but they are also ambiguous in their sense: we had to fix the interpretations of \( a \) and \( b \), \( T \) and \( F \), before we knew how they were to be read. Wittgenstein’s demand that these signs show their application is analogous to the claim that the elementary propositional signs show the type and arrangement of their elements. The basic thought is that the symbols of a perfect logic (that to which I alluded in the Introduction) are like models: they have certain physical or structural characteristics which determine the sorts of things we can do with them. The sign for disjunction in the \( ab \)-notation can be seen as a small mechanism: if we set the outside poles in one orientation (say \( b \)), then the inside poles are also set to \( b \); if either of the inside poles is set to \( a \), then so is the outside one; and if the outside pole is set to \( a \), then the inside ones can take several, but not all values. These mechanical characteristics are to be conceived of as inherent possibilities of this kind of sign; however it is clear that we can use many signs in such a way as to correspond exactly to this schema, while not every sign will lend itself to being used in such a way. These inherent possibilities—whether they arise from material constraints, or the artificial constraints of convention—do not yet amount to logical
properties, however, for, as we just saw, they exist and can be described without any mention being made of truth, falsity, or the world. Once we have, on the other hand, introduced the relation between the components and their senses—whether these be elementary facts, or other molecular propositions—these properties will reflect connections between the truth-values of the components, and that of the whole. We will then be able to use these mechanical properties to draw actual conclusions about the truth or falsity of other propositions. The signs and the rules—whose cooperation yields the purely structural or syntactic properties—are arbitrary. But what follows from them once they have been fixed is not. So “p v q” can also serve as the sign for disjunction, providing it is used in a manner consistent with the truth-table schema: the symbol for “disjunction” (e.g. “v”) in the disjunctive propositional sign is arbitrary, providing that the latter be used as a disjunction; however the signs “p” and “q” are essential, for they make clear how this sign is to connect to the world.

The virtue of Wittgenstein’s ab-notation is that it displays graphically the connection between what I have called the logical superstructure—the molecular sentences of the logic—and the core logical space—the elementary sentences. The ab-symbols for conjunction and disjunction are of course indifferent to the complexity of their components; however the full analysis of a complex proposition in this notation will connect the outer poles of the complex proposition to the innermost poles—those of the elementary components. In doing so it establishes a reciprocal relation between the components (the elements of the core logical space), and the (perhaps nested) group of complex symbols in which they are embedded: the truth-value assignments of the innermost components, which point directly to points in the core logical space, determine the truth-value of the outermost sign; and the truth-value of the latter determines the set of truth-values which the innermost signs may take. In setting the truth-value of the complex propositions, we limit the allowable truth-value
assignments of the innermost ones, and, as a result, we constrain the possibilities of
the existence and non-existence of the elementary facts:

4.463 Die Wahrheitsbedingungen bestimmen den Spielraum, der den
Tatsachen durch den Satz gelassen wird.

(Der Satz, das Bild, das Modell, sind im negativen Sinn wie ein fester
Körper, der die Bewegungsfreiheit der anderen beschränkt; im positiven
Sinne, wie der von fester Substanz begrenzte Raum, worin ein Körper Platz
hat.)...

The complex propositional sign brings about this constraint by means of two distinct
properties:

1) Its identification of the elementary propositions on whose truth-values its
   truth-value depends.

2) Its showing, explicitly or by convention, how these truth-values are
   connected to each other, i.e. what the logical connections among these
   elementary propositions must be.

To these two aspects of the complex propositional sign correspond two of the
elementary one, both of which we have encountered several times: the elementary
proposition must say which objects are related, and it must show how they are related.
In both cases, the first property is fixed by coordinating the elements of the sign with
elements in reality, and the second depends on our conventional interpretation of
various inherent structural properties of the sign itself, without regard to its ap-
application (i.e. the ordering of aRb and bRa, the arrangement of the T's and F's in a
truth-table).

b) Complex and Elementary Picturing

The elementary proposition is a picture in that each of its elements is correlated
with an element of reality, and in that it uses its own structural arrangement to
replicate the structure of the fact it depicts. The complex proposition pictures as well,
but it does so by means of the elementary propositions. They fix a region in logical space, and they do so in turn because their elements have been given denotations. If we restrict the picturing elements to the names and their combinations, then it is clear that what is actually depicted are only the logical places to which the elementary propositions refer. Only the properties of 1) involve depiction, in other words. Those of 2), the logical properties, determine the "free play left to the facts" depicted. The ab-functions, the "logical scaffolding around the picture" divide the possible truth-value assignments of the components: on the one hand are those possible when the complex proposition is true; on the other, those possible when it is false. They determine the logical space, but they do not depict a distinct location in it: they do not increase its dimensionality or Mannigfaltigkeit. That is the significance of the remark on negation in 4.0641. Both p and \( \neg p \) refer to the same location in logical space, for \( \neg p \) does not introduce an element beyond that involved in p. If we want to talk as though complex propositions actually did expand the logical space (and Wittgenstein often does talk this way in the Notebooks) such that p and \( \neg p \) had their own distinct referents, then we must still concede that the latter is parasitic on the former: "The negating sentence determines another logical place with the help of the negated sentence, in that it describes the former as lying outside of the latter." This passage from the Notebooks 3.11.14 appears verbatim in 4.0641, and suggests that complex propositions do, in some sense, increase the multiplicity of the space. However the remarks in 3.42 (from the middle and end of the same month) put matters straight: negation, disjunction and the other logical functions do not introduce new elements into the logical space; the whole of logical space is given with the elementary propositions, and any propositions defined on top of these refer only to possible situations in the core space.

These possible situations Wittgenstein refers to as Sachlagen, punning on the meaning of Lage not only as position, but also, as in Lagrangian mechanics, as the
state of a system in space. In such a representation of mechanics (say that of Heinrich Hertz, to which I turn in a moment), we represent the state of one point by means of three coordinates, that of two by six, *et cetera*. We can regard a system of two points in three dimensions as being equivalent to a system of one point in six dimensions. Each possible configuration of the two points in three-space corresponds to one possible position or Lage in six-space. Thus each state of the first system can be called a Lage of the system. The Sachlage depicted by an elementary proposition is simply the state of affairs that would make it true: the existence of a complex at the specified location. The Sachlage depicted in *p v q* is all the possible situations at positions *p* and *q* which are consistent with *p v q* being true. Left to themselves, *p* and *q* can have four possible truth values, but if *p v q* is true, then they can take only three. This limitation of their possibilities determines (*bestimmt*) the logical space, and it limits the free play (*den Spielraum*) of the facts in question. Consequently it sets a constraint on logical space: only certain states are possible when it is true, and these states form the Sachlage it depicts. Since the complex propositional sign constrains the truth-values of its components by means of its internal structural characteristics and their associated conventions, we can compare it to a mechanism coupled to the elements of the logical space (in the same sense in which one speaks of coupled mechanical systems): changes in the state of the mechanism map into predictions about the state of the world, and changes in the state of the world map onto changes in the state of the mechanism. The complex proposition is thus also an Abbildung, but of a different sort: it describes possible changes in the states of many points in the logical space. It does so by means of the logical scaffolding around the elementary pictures, in other words by means of the elements of 2).

As we have seen, a proposition must make two things clear:

1) The point or points in logical space with which it is concerned.
2) The connections between the states of these points, i.e. whether they must be filled or empty, whether if the one is filled another must be empty, and so on.

The first of these is indispensable if the sign in question is to qualify as a proposition at all; for if we don't know which aspects of reality are being connected with each other, then we cannot possibly know how the sign is to determine logical space. On the second score we have somewhat more latitude. On the one hand Wittgenstein talks about classes of symbols (the ab-symbols and truth-tables) which show through their structure the exact connections between the truth-values of their components and that of the whole: they are precise descriptions of the truth-functions in question. On the other hand he must allow that there are other classes of signs which may, although they do not spell out the truth-functions graphically, be used in conformity with the use of those symbols. What is the essential difference between the conventional signs and the ideal symbols? Wittgenstein distinguishes them as follow:

3.31 Jeden Teil eines Satzes, der seinen Sinn charakterisiert, nenne ich einen Ausdruck (ein Symbol). ...

3.326 Um das Symbol am Zeichen zu erkennen, muß man auf den sinnvollen Gebrauch achten.

3.327 Das Zeichen bestimmt erst mit seiner logisch-syntaktischen Verwendung zusammen eine logische Form.\footnote{3.31 I call each part of a proposition that characterizes its sense an expression (a symbol) ... \textellipsis} \footnote{3.326 In order to recognize the symbol in the sign, one must attend to the significant use. \textellipsis} \footnote{3.327 Only together with its logico-syntactic application does the sign determine a logical form.}

A symbol is, we might say, an interpreted sign: an elementary propositional sign whose names have been give denotations, and which as a consequence makes a statement about a location in the logical space; or an ab-sign whose poles have been given an interpretation ("the a-pole means that the elementary proposition \( p \) is true"), and which therefore asserts a connection between the states at the relevant locations. However both of these cases represent an idealized extreme: on the one hand the elementary propositions with their built-in syntactic properties (which defy
misinterpretation); and on the other the \(ab\)-symbols or truth tables, which are conceived of as quasi-mechanical devices. The \(ab\)-sign for disjunction is like a logic gate, which maps inputs into outputs in a determinate fashion. We decide how the inputs and outputs are to be interpreted, but the mapping relation is fixed.

Wittgenstein must, however, also allow for a larger class of signs that get \textit{used} in conformity with such symbols. The difference between the two kinds of signs can be defined as follows: both \(p.q\) and the \(ab\)-symbol for conjunction have the same consequences when the first is used in conformity with the second, however the \(ab\)-sign spells out how it is to be used, that is, what a usage conforming to it must be; and, as we have seen several times, predicative propositions whose names denote complexes can obey the logic of idealized predicative propositions, so long as the complexes in question exist. We can think of \(p.q\) as a shorthand for the \(ab\)-sign, in that it elides certain structural characteristics and displaces their functional role onto tacit conventions of use.

But even the \(ab\)-sign only approximates to being a symbol. No sign on its own may be one, for even the \(ab\)-notation must be interpreted in order to be correctly used—it is not so pellucid that one can understand it without an explanation, and the same goes for truth-tables. What is missing in the sign is in some sense present in the symbol:

3.262 Was in den Zeichen nicht zum Ausdruck kommt, das zeigt ihre Anwendung. Was die Zeichen verschlucken, das spricht ihre Anwendung aus.

3.32 Das Zeichen ist das sinnlich Wahrnehmbare am Symbol.\textsuperscript{13}

That missing component—the use or application of the sign—is the substructure of the latter. In the truth-functional case the sign is related to the symbol in the same way that the schematic for an and-gate is related to the actual gate: for we cannot interpret

\textsuperscript{13}3.262 That which is not expressed in the sign is shown by its application. That which the signs elide is declared by their application. 3.32 The sign is the sensibly perceptible in the symbol.
the schematic for an and-gate without making use of something that *actually* functions as an and-gate. The distinction is, once again, that between a sign-language that "obeys logical grammar" and one which excludes most, "but not all mistakes" (3.325). In the case of such a perfect logic, this substructure consists in nothing more that the correlations set up between facts and signs by means of names; however in the case of high-level languages, it will consist in a good deal more, for the tacit conventions governing the use of a symbol such as "p v q" must be accounted for somehow, just as the tacit assumptions involved in a pseudo-predication such as q(A) (where A is complex) must also be captured in some substructure to my thought.

Think of Helmholtz’s perceptual manifold, or Hertz’s dynamical models: both thinkers argued that the truth of our physical propositions rests on observed regularities in a manifold whose full resolution is not—consciously—known to us. The processes which secure the connection between high- and low-level representations are no less real for our being unconscious of them: indeed without assuming that such a translation takes place, the entire point of the *a priori* manifold would go out the window.\(^{14}\)

I will return to this important distinction in the next chapter. What matters for the moment is only the conclusion that there must be a language whose structure completely governs its use, although not its reference (for there are always arbitrary, denotative correlations to be made). The signs for complex propositions can display their interpretations to varying degrees: some are completely inappropriate to a given interpretation, others (like "p . q") are minimally suited, and still other maximally so (the *ab*-signs). All of these tend toward an ideal *symbol* for the truth-function, which

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\(^{14}\)To take this position is to give the notion of substructure (that which "die Zeichen verschlucken") a realistic interpretation; however there is also the idealist solipsistic one to consider: from that perspective there is no *explanation* of the logical properties of our fundamental language. We experience them as necessary properties of thought in general, in other words as logic. I discuss these points in Chapter 7.d-e.
can be used in at most two ways, these depending on which outputs get called the true and the false ones. The logic of our actual thinking—that which makes it impossible to think illogically (3.03)—makes use of such symbols. The logic of our artificial, written symbolisms is governed by that logic to the extent that we use those signs in conformity with the rules implicit in those symbols. Thus in investigating high-level or artificial symbolisms, the question will always be: Do the signs we use in this symbolism have the characteristics needed for their being used in conformity with the symbols of our thought?

This approach has its antecedent in the requirement that type-membership be mirrored symbolically: the names for individuals and relations are supposed to be formed in such a way that they "cannot possibly be substituted in one another's places." But the force of this cannot has haunted us from the beginning of this essay: it is the same cannot to which I appealed in saying that my observer cannot ascribe a nonsense judgment to me, because objects can only combine with each other in certain ways. Obviously when Wittgenstein claims that "Fx" and "a" can only be combined in a particular way, he cannot be talking about the ink-marks on his page—those ink-marks are at best members of classes of signs which share the characteristic of denoting Fx, and a, respectively; however some signs in this class must literally have the characteristics he demands, or the symbols could not ground the type distinctions. Similarly the appeal to truth-functional symbols with the mechanical properties described above cannot rest on a metaphorical interpretation of the necessity in question.

One must keep in mind the relation between the pen-and-paper sign-languages of Principia Mathematica, the Grundgesetze, and the Tractatus, and the logical

language of our thoughts clear when reading all these authors. Since they all understand their project as that of specifying Logic and not a logic, with the consequence that they have no use for a meta-language (the very existence of a meta-language would show that the logic in question was not Logic), the only methodology open to them is that of gradual approximation: one selects a notation, some inference rules, and sees whether or not the system in question agrees with one’s intuitions and yields the results it ought to. As van Heijenoort says of Frege’s and Russell’s attitude towards the completeness of their logics:

La notion même de complétude n’a pas de sens, et nous voyons que le système est adéquat en déduisant dans lui autant de théorèmes de logique et de mathématiques que nous pouvons. La seule complétude à laquelle nous puissions aspirer, c’est ... une “complétude expérimentale.”

What we are investigating are laws of thought, and the symbolic systems we contrive must be tested against our intuition of these actual laws. And so Wittgenstein, in his Notebooks and in the Tractatus, approximates to the properties of the symbols of Logic by successive examination of different kinds of signs. The aim is to arrive at a conception of the essential properties of such symbols by seeing what sorts of signs can and cannot serve in conformity with those ideal symbols. To understand sections 4.04-0411, in which Wittgenstein enquires into the multiplicity of the propositional sign and the connection of this multiplicity to that of various quantifier notations, one must keep this “experimental” approach in mind. In asking after the properties of such a written notation, he is looking for evidence of the essential properties of all such signs.

c) Manifolds and Quantification: 4.04-4.0411

The three sections in question read as follows:

4.04 Am Satz muß gerade soviel zu unterscheiden sein, als an der Sachlage, die er darstellt.

Die beiden müssen die gleiche logische (mathematische) Mannigfaltigkeit besitzen. (Vergleiche Hertz's Mechanik, über Dynamische Modelle.)


4.0411 Wollten wir z.B. das, was wir durch "(x)fx" ausdrücken, durch Vorsetzen eines Indexes vor "fx" ausdrücken—etwa so: "Alg.fx", es würde nicht genügen—wir würften nicht, was verallgemeinert wurde. Wollten wir es durch einen Index "a" anzeigen—etwa so: "fx_a"—es würde auch nicht genügen—wir würften nicht den Bereich der Allgemeinheitsbezeichnung.


Alle diese Bezeichnungsweisen genügen nicht weil sie nicht die notwendige mathematische Mannigfaltigkeit haben.

4.04 There must be as much to distinguish in the proposition as in the Sachlage that it represents.

Both must have the same logical (mathematical) multiplicity [Mannigfaltigkeit]. (Compare Hertz's Mechanics, on dynamical models.)

4.041 Of course one cannot depict this multiplicity itself. One can't get outside of it in depicting.

4.0411 If we wanted to express that which we express with "(x)fx" by prefixing an index before "fx"—roughly so: "Gen.fx", it would not suffice—we would not know what was being generalized. If we wanted to display it with an index "a"—roughly so "fx_a"—it would also not suffice—we wouldn't know the scope of the generality.

If we wanted to try it by introducing a mark in the argument-slot—roughly so: "(A, A).F(A, A)"—it would not suffice—we couldn't determine the identity of the variables, etc.

All of the methods of symbolizing do not suffice, because they do not have the necessary mathematical multiplicity.

Wittgenstein is appealing here not only to our intuition that a proposition concerning four things ought to contain four names, and thus also to our feeling that a generalisation of such a proposition ought to reflect this multiplicity as well. He is thinking of the mathematical description of a point in space: because the point \( P(x, y, z) \) is located in a three-dimensional space, we need three coordinates to situate it uniquely, that is to distinguish it from all other points in that space. The choice of
these coordinates, as well as some aspects of the sign displaying them, is arbitrary—but that there must be three is a necessary requirement for all such signs. 4.0411 emphasises this point by considering generalised propositions as analogous to descriptions of sub-domains of a given manifold: \( P(1,2,z) \) says, e.g. that \( P \) lies somewhere on the line \((1,2,z)\)—\( P \) has 1 degree of freedom;\(^{17}\) similarly the function name \( f_\alpha \) can be used to construct propositions asserting either that some logical location on the “line” \( f_\alpha \) is filled, or that all such locations are filled. If a function-name in a generalised proposition is to pick out all elementary propositions that belong within the range of the proposition’s sense, then it must faithfully reflect all essential characteristics of the elementary propositions, those common characteristics of the elementary symbols being what Wittgenstein calls their logical form (3.311-315). So 4.0411 is indeed a comment on 4.041 (contra Black): if function-names are considered to be generalised pictures of domains of facts, then 4.0411 highlights the impossibility of “getting outside” of the multiplicity inherent to both the manifold and its elements.

Before unpacking this passage further I want to emphasise that Wittgenstein’s fundamental claim—that an \( n \)-dimensional manifold can be mapped (abgebildet) determinately only onto a manifold of equal multiplicity—is false; and, moreover, that Wittgenstein had to be aware of this, since Cantor’s proof of the contrary in e.g. his “Beitrag zur Mannigfaltigkeitslehre”\(^{18}\) is explicitly directed against the work of Helmholtz and Riemann in this area. Furthermore this fact is acknowledged in the Tractatus itself, both in the admission that every variable, including the variable name, can be used as a sentential variable (i.e. “\( p \)”, 3.314), and in 5.501 where

\(^{17}\)cf. passage from the Notebooks 29.10.14 quoted in footnote 27, below.

Wittgenstein allows for the "direct enumeration" of the arguments for his $N$-operation as an alternative to specifying them by means of a function-name. It simply is not true that we can map only between equal-dimensional manifolds, and this fact throws light on the highly idiosyncratic conceptions of both name and object in the tractarian system.

The basic idea of 4.0411 can be found already in the "Notes on Logic:"

What is essential to a correct apparent-variable notation is this:-- (1) it must mention a type of propositions; (2) it must show which components of a proposition of this type are constants.\footnote{Notebooks, p. 96}

This remark appears in the \textit{Tractatus} as 5.522, where it serves to highlight the role of the function-symbol in the generality notation. The thought is worked out in detail in the \textit{Notebooks} 23.10.14, where Wittgenstein addresses the question of the scope of the variables in such a notation--a consideration he simply ignored in the above remark, presumably because it was concerned only with elementary open sentences. The wording there is essentially the same as in 4.0411, with one salient difference: the various ways of symbolising are rejected not by appeal to the notion of multiplicity, but because "they do not have the necessary logical properties:"

\begin{quote}
Alle diese Bezeichnungsweisen genügen nicht, weil sie nicht die notwendigen logischen Eigenschaften haben. Alle jene Zeichenverbindungen vermögen den gewünschten Sinn--auf die vorgeschlagene Weise--nicht abzubilden. [italics in original]\footnote{Notebooks, p. 18. All of these methods of designating do not suffice, because they do not have the necessary logical properties. All these sign-aggregates are not capable of representing the desired sense--in the proposed manner. [italics in original]}
\end{quote}

To have the necessary logical properties is to be able to express the desired \textit{sense}; however at this date Wittgenstein has not yet begun to use the notion of a "logical place" to capture that of \textit{Sinn}. He does that one month later in a sequence of remarks which later appear as \textit{Tractatus} 3.4-3.42, of which 3.411 is perhaps the simplest expression:
Having dropped both Frege's notion of Sinn as well as Russell's complex-correspondence theory, Wittgenstein needed a way of describing what we mean when we assert or consider a false proposition. His answer is that we consider a potential existence, a "location" where a complex, later a Sachverhalt, may obtain:

Wenn auch kein Komplex in dem logischen Ort ist, so ist doch Einer: nicht in dem logischen Ort.²¹

This location, later a Sachlage, must in some sense exist. We needn't associate anything more mysterious with these doctrines than the conclusions at which we arrived when considering Russell's judgment theory; namely that the elementary propositions cannot require ancillary propositions in order to give them determinate sense, e.g. that their primitive expressions, the names, have a determined reference, are appropriately type-matched to one another, and are ordered in the sign in such a way that they uniquely identify the possible fact to which the sign refers.

i. The internal relation of sign and fact

The problem of securing the potential meaning of the propositional sign was in Russell's theory the problem of securing a necessary or internal connection between judgment complex and the corresponding complex: there can be no ambiguity in the correlation of the two complexes, for any further specifications, in stating further conditions for the significance of the proposition, would undermine the bivalence of the proposition. The mere possibility that there be riders on the elementary proposition would indicate that the connection was not internal. This idea is so deeply rooted in Wittgenstein's thought that we find him voicing it fifteen years later, still in opposition to a (later) theory of Russell's:

²¹Notebooks, p. 24. Even when there is no complex at the logical place, still there is one—not in the logical place.
Das Wesentliche an der Intention, an der Absicht ist das Bild. Das Bild des Beabsichtigten.

... Der wesentliche Unterschied der Bild-Auffassung von der Auffassung Russells, Ogden und Richards ist aber, das jene das Wiedererkennen als das Erkennen einer internen Relation sieht, während diese das Wiedererkennen für eine externe Relation hält.22

Both in 1914 and 1930 Wittgenstein is adamant that if we are to understand assertion as a correlation of two complex entities—sentence or thought on the one hand, and fact or event on the other—then this correlation must be nailed down. A strict analogy to the later work is not necessary for the argument at hand, but it does help in understanding Wittgenstein’s motivation in 1914, when he attempts to define truth along these lines:


If Wittgenstein found the earlier remark bad, nonetheless the thought it tried to express—that Russell’s judgment theory was to be replaced with one defining truth as a correspondence between isomorphic complexes—wasn’t so bad that he left it out of the Tractatus:


22Wittgenstein, Ludwig. Philosophische Bemerkungen. Schriften 2. ed. R. Rhees. Frankfurt am Main: Suhrkamp Verlag. 1964. p. 63. The essential in an intention, in an expectation, is the picture. The picture of the expected. ... The essential difference between the picture-conception and the conception of Russell, Ogden and Richards is however that the former sees recognition as the cognition of an internal relation, while the latter takes recognition to be an external relation.

23Notebooks, p. 23. The proposition "'p' is true." is synonymous with the logical product of 'p' and a proposition "'p'" which describes the proposition 'p', and a coordination of the components of the two propositions—the internal relation of proposition and meaning are depicted by means of the internal relations between 'p' and "'p'". (Bad remark).

245.542 It is however clear that "A believes that p", "A thinks p","A says p" are of the form "'p' says p": And here it is not a matter of coordinating a fact with an object, but rather of coordinating facts by means of a coordination of their objects.
The remark in the Notebooks is bad because it seek to state the truth conditions of, say, ‘$aRb$’ by means of yet another symbol “‘$aRb$’” whose relation to ‘$aRb$’ is like that of ‘$aRb$’ to the fact $aRb$. It is not just that we get hopelessly wound up here in questions of multiple indirection—What are $aRb$, ‘$aRb$’, and “‘$aRb$’”, and who is asserting them?—more fundamentally, this approach makes it look as though I needed a language beyond the one which I am speaking in order to give meaning to sentences such as “‘$aRb$’ is true” or, on 10.11.14 “$aRb$ is possible”:

Wenn ich sage “‘p ist möglich”, heißt das “‘p” hat einen Sinn”? Redet jener Satz von der Sprache, so daß also für seinen Sinn die Existenz eines Satzzeichens (“‘p”) wesentlich ist? (Dann wäre er ganz unwichtig.) Aber will er nicht vielmehr sagen, was “pv~p” zeigt?25

“$aRb$ is possible” does not, on the face of it, make a statement about the propositional sign ‘$aRb$’. just as “‘$aRb$’ is true” does not involve the notion of some meta-symbol “‘$aRb$’” whose similarity to ‘$aRb$’ is like the latter’s similarity to the fact $aRb$. What Wittgenstein wants to say is:

1) “‘$aRb$’ is significant” and “the complex $aRb$ is possible” are equivalent; and

2) “‘$aRb$’ is true” and “the complex $aRb$ exists” are so as well.

2) is of course the same “definition” of truth that we have in Russell’s judgment-theory, as well as in the Introduction to Principia Mathematica: the only difference is that the judgment-complex has been replaced by the complete propositional symbol “‘$aRb$’”. Truth is defined by 2) alone, however falsity requires both, and it is here that Wittgenstein’s account differs from Russell’s: “the complex $aRb$ is possible” is not, for Wittgenstein, something that can be said. The propositions that would pretend to establishing that truth are either necessarily true, or senseless—they can only be stated, we might say, in a meta-language. But Wittgenstein, like Frege and Russell, could not accept such a conclusion, because the existence of such a meta-language would show

25Notebooks, p. 28. When I say “p is possible” does that mean “‘p” has a sense”? Does that proposition concern language, such that the existence of a propositional sign (“‘p”) is essential to its sense? (In that case it would be quite unimportant.) But doesn’t it in fact want to say that which “pv~p” shows?
the logic under consideration to be incomplete—it would govern only a portion of significant language, meaning that it was not the universal logic they sought. Commenting on Russell's belief that implication cannot be defined, Jean van Heijenoort's remarks: "Nous avons ici devant nos yeux un homme qui s'avance sur un plancher gluant, incapable de lever un pied sans s'y coller de nouveau." The remark applies just as well to Wittgenstein here, and for the same reasons: each time Wittgenstein attempts to state the conditions of a complex symbol's significance or truth, he needs a symbol of similar complexity, thus of insecure denotation. Instead of converging on ""aRb"' says that aRb" he gets bogged down in ever more indirect formulations. Repelled by the need for a meta-language, he concludes in the Tractatus that such definitions—of truth and significance—are impossible: "Diese mathematische Mannigfaltigkeit kann man natürlich nicht selbst wieder abbilden. Aus ihr kann man beim Abbilden nicht heraus." It is often and correctly said that the notion of a meta-language can be traced back at least to Russell's "Introduction" to the Tractatus, so it should come as no surprise that the latter's author had had it out with that idea along the way.

Not having Wittgenstein's scruples in these matters, I accept the equivalencies given above, and I will not worry about their denotations any more than Tarski did: when "p" is true, then the complex p exists; and the sense of "p" is the potential existence of that complex—a notion which, however, remains empty if that potential existence is not in some sense given, comprehensible, vorstellbar:27

Man könnte zwei Koordinaten a_p und b_p als einen Satz auffassen, der aussagt, der materielle Punkt P befinde sich im Ort (ab). Und damit diese Aussage möglich sei, müssen die Koordinaten wirklich einen Ort bestimmen.


27 cf. Helmholtz's definition of same, cited in the Conclusion, footnote 16.
The notions of location and space agree nicely with the postulates of this theory not the least because they allow for an intuitive generalisation of the notion of *Sinn*. The sense of an elementary proposition is the existence of a complex (a *Sachverhalt*) at a point in logical space, and that of the compound sentence expressing a *Tatsache* (a conjunction of *Sachverhalte*) is the region of such points such that when the sentence is true then they all contain complexes. If we expand this account to include the assertion of negated sentences, it still holds: the sense of a negated elementary proposition is the absence of a complex; that of “\( p \sim q \)” is the presence of a complex at the location associated with “\( p \)” and the absence of one at the location of “\( q \)”.

In our discussion of truth-functions, we saw that it was an essential characteristic of such signs that they make clear which *Sachlagen* they were to tie together: if the components of “\( p \lor q \)” are elementary, then “\( p \)” and “\( q \)” must refer unambiguously to some points in the core logical space; and if they are molecular, then it must be clear how they are compounded—what their truth-functions and elements are (these conditions may in practice be violated, but only at the price of indeterminate sense). The complex propositional sign must also say how the elements are connected to one another (what their truth-table looks like); however this connection will in most cases be given conventionally—we write “\( p \lor q \)” and not the \( ab \)-sign for the latter. 4.0411 examines quantifier notation from the same point of view, indeed Wittgenstein sees quantification as, in essence, a shorthand for conjunctions and disjunctions of possibly infinite length. A universal proposition \((x).Fx\) says the same as \(Fa.Fb.Fc.\ldots\) : both specify a range of locations in logical

\[28\textit{Notebooks,} \text{ pp. 20-21.} \text{ One could interpret two coordinates} \, a_p \, \text{and} \, b_p \, \text{as a proposition, which says that the material point} \, P \, \text{is located at position} \, (ab) \. \text{And in order for this statement to be possible, the coordinates must really determine a place. In order for a statement to be possible, the logical coordinates must really determine a logical place!} \]
space, the values of which locations are connected by disjunctions (the existential proposition) or conjunctions (the universal one). On the same day that he notes down 4.0411, Wittgenstein observes that "all things" "is so to speak a description, instead of 'a and b and c':" \(^{29}\) and on 9.7.16, he admonishes himself not to forget that \((\exists x).f_x\) does not mean "there is an \(x\) such that \(f_x\), but rather: there is a true proposition \(f_x\)." \(^{30}\)

The sentential variable of the \(N\)-operation is similarly determined (5.501) as a "description of the sentences which the variable represents." The description can take three forms: 1) direct listing, in which case the variable can be replaced with constant values; 2) the use of a function-name, whose values are the sentences to be described; 3) the use of a formal law, which stipulates how the sentences are to be formed. The molecular functions we have treated so far have finite numbers of arguments, and the sentences they conjoin are given by a list. Quantified sentences differ from these in only one respect: the sentences which they conjoin (or disjoin) are given not by means of a list, but by means of a function-name. So it will be just as critical to the sense of such propositions that we know unambiguously which sentences are picked out by that variable, as it was to know the disjoints involved in "\(p \lor q\)." If \((x).F_x\) means, all sentences \(F_x\) are true, then the function-name \(F_x\) must pick out all and only sentences of the list \(Fa.Fb.Fc.\ldots\).

The function-name in \((x).F_x\) is thus a description of a range of sentences. On this reading, the relation between 4.04 and 4.0411 becomes clearer: 4.04 says that a propositional sign must share certain structural characteristics with the fact it depicts; and 4.041-.0411 argue that a depiction of several such propositions must have the same characteristics—a depiction of a group of facts of a given multiplicity must have that same multiplicity, or it would fail to pick out the facts that we intended. As with

\(^{29}\)Notebooks, p. 18.

\(^{30}\)Notebooks, p. 75.
many of the passages discussed in this chapter, the original formulation of 4.04 in the *Notebooks* seems to have been conceived with elementary propositions in mind, and to have been altered for inclusion in the *Tractatus* in such a way as to apply to complex ones as well. In its original version, the first paragraph reads:

Am eigentlichen Satzzeichen muß geradesoviel zu unterscheiden sein, als am Sachverhalt zu unterscheiden ist. Darin besteht ihre Identität.\(^{31}\)

*Sachverhalt* is not at this point a technical term for Wittgenstein—the passage might refer to complex or elementary facts. However the identity to which Wittgenstein refers is a property he required of propositional signs on 22.10.14, when he is still focusing on the problem of elementary forms:

Im Satz muß etwas mit seiner Bedeutung identisch sein, der Satz darf aber nicht mit seiner Bedeutung identisch sein, also muß etwas in ihm mit seiner Bedeutung nicht identisch sein. (Der Satz ist ein Gebilde mit den logischen Zügen des Dargestellten und mit noch anderen Zügen, diese nun werden willkürlich sein und in verschiedenen Zeichensprachen verschieden.) Es muß also verschiedene Gebilde mit denselben logischen Zügen geben; das Dargestellte wird eines von diesen sein, und es wird sich bei der Darstellung darum handeln, dieses von anderen Gebilden mit denselben logischen Zügen zu unterscheiden (da ja sonst die Darstellung nicht eindeutig wäre). Dieser Teil der Darstellung (die Namengebung) muß nun durch willkürliche Bestimmungen geschehen. Es muß dann also jeder Satz Züge mit willkürlich bestimmten Bedeutungen enthalten.\(^{32}\)

The argument runs as follows: There must be an internal connection between the sign for a fact and the fact itself—they must be in some sense identifiable with one another, without, of course, their being numerically the same. Their identity consists in their isomorphism: they contain the same number of elements, they are of the appropriate categories (types), and they are arranged in the same way. The coordination of the

\(^{31}\) *Notebooks*, p. 37. There must be just as much to distinguish in the actual propositional sign as is to be distinguished in the situation. That is what their identity consists in.

\(^{32}\) *Notebooks*, p. 17. Something in the proposition must be identical with its reference, however the proposition may not be identical with its reference, thus something in it must be not identical with its reference. (The proposition is a structure with the logical features of the represented and with still other features, these will be arbitrary and they will differ in different sign-languages.) There must therefore be various structures with the same logical features; the represented will be one of these, and, when representing, it will be a matter of distinguishing that one from other structures with the same logical features (for otherwise the representation would not be unambiguous). This part of the representation (the assignment of names) must be effected by means of arbitrary determinations. Thus every proposition must contain features with arbitrarily determined references.
two groups of elements by means of naming makes the relationship determinate and unique.

ii. The shared multiplicity of manifold and fact

The original entry for 4.0411 is from the next day, and it is preceded by a series of criticisms of Wittgenstein’s current approach to generalized propositions, one of which we have already discussed:

Einerseits scheint meine Theorie der logischen Abbildung die einzige mögliche, andererseits scheint in ihr ein unlöslicher Widerspruch zu sein!

Wenn der ganz allgemeine Satz nicht ganz entmaterialisiert ist, so wird ein Satz durch die Verallgemeinerung wohl überhaupt nicht entmaterialisiert, wie ich glaubte.

Ob ich von einem bestimmten Ding oder von allen Dingen, die es gibt, etwas aussage, die Aussage ist gleich materiell.

"Alle Dinge," das ist sozusagen eine Beschreibung statt "a und b und c."

Wie, wenn unsere Zeichen ebenso unbestimmt wären, wie die Welt welche sie spiegeln?33

Having decided that the propositional sign itself embodies the form on whose model we conceive the fact it represents, Wittgenstein thought of treating general propositions as partially “dematerialized” elementary ones. On Frege’s and Wittgenstein’s approach, a function name is the saturated portion of a propositional sign: when we remove the proper name, we get an unsaturated function name—the function name has a “hole” in it, and that hole makes clear the type of arguments that may fill it. So we get function names by eliminating parts of significant signs. But if we carry out this process to the limit—following Wittgenstein’s recipe for making an Urbild,

33Notebooks, pp. 17-18. On the one hand my theory of logical depiction seems to be the only possible one, on the other there seems to be an insoluble contradiction in it. If the completely general proposition is not completely dematerialized, then a proposition does not get dematerialized through generalization at all, as I thought. Whether I say something about a particular thing or about all things that there are, the statement is equally material. "All things," that is so to speak a description, instead of "a and b and c." What if our signs were just as undetermined as the world which they mirror?
which is supposed to subsume all propositions of a given form—then we will be left with nothing but a hole: the propositional sign will be completely dematerialized, and we shall have no way of knowing the form of the proposition from which it was derived, thus the range of propositions of which it is supposedly the Urbild. The trick is to get the signs to be "just as indeterminate as the world they mirror;" but on this theory, the Urbild is more indeterminate than the world it mirrors. 4.0411 is a solution to this problem, as the Prototractatus version of 4.041 makes clear:

Prototractatus 4.0741 Diese mathematische Mannigfaltigkeit kann man natürlich nicht selbst wieder abbilden, da jedes Bild von ihr diese Mannigfaltigkeit selbst besitzen muß. Aus ihr kann man beim Abbilden nicht heraus.34

The matrix in a generalized proposition is supposed to serve as a description of the propositions that the generalized proposition conjoins or disjoins. The assumption that such functions are formed by dematerializing saturated propositional signs leads to the conclusion that these dematerialized signs cannot adequately circumscribe the range of sentences they are to depict: how do I know, when I eliminate $a$ and $R$ and $b$ from $aRb$ that what I end up with is a sign for the class of relational and not predicative propositions? Wittgenstein cannot appeal to a form above and beyond that given by the arrangement of the names; however the names are gone, and so must be the form as well.

In 4.0411 Wittgenstein argues to the following conclusion: any satisfactory generality notation must make use of a prefixed list of variable names, which, in stating the number, type and identity of the variables over which we are quantifying, restores to the function symbol within the quantifier the very information that would, in the earlier theory, have been destroyed when the function symbol was produced. He considers three possible notations which do not have this characteristic, and

34Prototractatus 4.0741 Of course one cannot depict this multiplicity itself, because every picture must have this multiplicity in itself. One can't get outside of it in depicting.
concludes that each is insufficient to the task of “depicting the desired sense in the proposed manner.”\textsuperscript{35} In the first case, we prefix the symbol “\( f^x \)” with a generality sign. This notation specifies the type of proposition in question (predicative) but it does not say which of \( f \) and \( x \) is constant, and which variable. Thus we introduce in the second case an index flagging the variable: \( f_{x^s} \). Here the type of proposition is given, as are the constant and the variable; however it is not clear whether the proposition \( f_{x^s} \supset g_{x^s} \) says that \( (x).f^x \supset g^x \) or that \( (x).f^x \supset (x)g^x \); we cannot determine the scope of the generalization. We can solve that problem by prefixing the function name with a list of generality-marks, in order that the scope be clear: \( (A, A).f^A \supset g^A \) cannot mean \( (x).f^x \supset (x)g^x \). But here we cannot distinguish between \( (x,y).\varphi(x,y) \lor \varphi(x,y) \) and \( (x,y).\varphi(x,y) \lor \varphi(y,x) \) (the example is from Anscombe\textsuperscript{36}). The prefix must reflect the identity and non-identity of the various variables.

To generate a generalized proposition, we do the following: we start with a significant proposition, and we successively turn names into variables. Each time we do so, we add a name to a prefixed list of variables. The matrix with which we are left, together with the prefix, constitutes a description of the class of propositions over which the matrix ranges. Lastly we introduce a symbol to show whether the propositions in question are to be conjoined or disjoined. This method does not dematerialize the sign with which we began, for it introduces a new element into the sign for each one it eliminates--indeed there is no requirement that the names be eliminated, for we could go from \( R(a,a) \) to \( (a).Raa \) instead of replacing \( a \) with \( x \). By reiterating \( a \) in the matrix, we show that \( R(a,a) \) ranges over all identical pairs of

\textsuperscript{35}Notebooks, p. 18.

arguments of the type of $a$. Since the sign $(a)$ $Raa$ captures all the relevant information—the type of proposition in question, the identity of the variables and constants, and the ordering of these in the sign—it has the same multiplicity or logical form as $R(a,a)$. We cannot depict all the facts of that form without making use of symbols whose multiplicity is shared by all those facts.

This is not to say that $R(a,a)$ and $(a)$ $Raa$ are the same type of symbol: the prefix "$(a)$" shows that $a$ is a variable, that the picture in question picks out a submanifold of the class of proposition of type $\varphi(x,y)$—those in which $\varphi = R$ and $x = y$.

If $Fa$ is to be conceived as one point in a manifold of related points $Fb$, $Ga$, ..., then $Fx$ is a slice through that manifold, e.g.

Wittgenstein's reference to Hertz emphasises the connection between his logical space and the mathematical manifolds of mechanics. In *The Principles of Mechanics*, Hertz introduces the notion of a "dynamical model," which is a physical system standing in a particular relation to another.\(^{37}\) The most important aspect of that relation for our purposes is given by Hertz's requirement that both systems have the same degree of freedom, *i.e.* that each have the same number of free coordinates, such that the values of the one system of coordinates may be mapped onto those of the

other. The class of systems standing in this transitive and symmetric relation will be vast, for there is no requirement that the actual physical components of the two systems be equal in number, nor indeed that the motions described by the one resemble those of the other: the one could be a system of strings and pulleys, the other a system of fluids and pipes. All that matters is that we be able to correlate each state \((Lage)\) of the one with a state of the other, and \textit{vice versa}. This account of Hertz's has little to do with the main mathematical development of his book. Its importance lies in its giving him a precise definition of the notion of a "picture," which he had from Helmholtz. Our thoughts are pictures of systems in the external world, argues Hertz in the Introduction to his book.\footnote{Hertz, Heinrich. \textit{Die Prinzipien der Mechanik}. pp. 1-5.} They must be so constituted that they allow us to predict events in the world by manipulating those dynamic pictures. In the section on dynamic models, he gives that earlier account substance: the pictures we form of the external world stand in the same relation to the systems they depict as do two systems that are dynamical models each other.

Obviously this conception agrees well with the Wittgensteinian picture theory as I have interpreted it: a dynamical model of a system stands in a necessary structural relation to a whole class of systems, each of which may serve as a model for the others, and it stands in this relation \textit{whether or not we use it as a model}. Once we specify a mapping between the two sets of coordinates (those of the model and the modeled system) it becomes a picture of the latter: each state that it takes corresponds to a state of the modeled system. To specify one of these states, we must give each of the free coordinates a determinate value from the range of values that coordinate may take (remembering that there are no essential restrictions on the kinds of coordinates—cylindrical, Cartesian, spherical—that we may use). In determining some of the coordinates while leaving others free, we constrain the degree of freedom: \textit{e.g.} \((x, y, z)\)
has a degree of freedom of 3 (the point may lie anywhere in 3-space). \((x, y, 3)\) has one of two (it lies somewhere on the plane \(z = 3\)). Furthermore by specifying connections between the coordinates, we have the same effect on the degree of freedom: \((x, y, x)\) says that the point must lie on the plane \(x = z\), giving the system a degree of freedom of two as well. Just as Wittgenstein's elementary and quantified propositions have the same multiplicity, so do the expressions for a single point in 3-space, and the expressions for sub-domains of 3-space. And just as a quantified expression picks out certain subsets of a group of (for argument's sake) elementary propositions by using on the one hand a picture of the facts in question (which specifies the space, \(e.g.\) predicative, relational, in which the fact lies) and on the other a list showing which variables are free; so the expression for points whose motions are constrained state on the one hand the dimensionality of the space within which they move, and on the other hand the constraints on these coordinates.

A fully generalized Urbild such as \(qa\) has a degree of freedom equal to the multiplicity of each point \(Fa\) in that manifold—in this case two; and \(R(x, x)\), although it has three separate components, also has a degree of freedom of two, for the identity of the variable names \(x\) and \(x\) reduces the number of independently variable co-ordinates by one. The structural relation between the signs \("(q, x).qax"\) and \("Fa\) mirrors that between the manifold \(qax\) and the element \(Fa\): we can see that the second is an instance of the first so long as we know how the generality notation works, and that \("F"\) and \("a"\) denote. If our symbolism is to capture these relations, then the equal multiplicities of generalized proposition and instantiation is an unavoidable postulate: the symbolism must group together the elementary propositions in a manner which reflects the internal relations, the essential connections, between the elementary facts.

I said earlier that the claim that a manifold of dimension \(n\) can be mapped only onto a second of the same dimensionality is false: we can map the plane of pairs of
real numbers onto the real numbers themselves, since both are of the same power. Similarly there can be no objection to mapping the facts $Fa, Fb, Ga, Gb, \ldots$, onto a list of sentential names $p, q, r, s, \ldots$. Since Wittgenstein is openly indifferent to the way in which we specify the arguments for his primitive truth-operations, nothing argues against this procedure beyond the fact that we cannot necessarily write down the series of sentential names corresponding to $(x).Fx$—it may be infinite. This is an important limitation, however there is one connected to it is that far more so: the symbol " $Fx$" makes use of the constant " $F$" to select a class of propositions, and it is presumably not an arbitrary class. The presence of the sign " $F$" in all those propositional signs reflects a shared characteristic of the facts for which those signs stand—they all have the feature (Zug) $F$. If our names for those facts were all sentential names, then they would have nothing in common beyond all being names of facts. We would have no means of selecting groups of them, thus either of quantifying over relevant subsets of them, or of constructing definitions such as $\varphi(\psi x) = \text{def } \exists x. \psi x$, which make appeal to multiple occurrences of the same feature in distinct facts. It is, as Ramsey observed, the quantified proposition $(\exists x).xRb$ which points to the importance of the fragment " $Rb$" in signs of that form: if we did not use propositions of that sort, then it wouldn't matter whether we represented the fact $aRb$ as ' $aRb$' or ' $p$'.

This is an unexpected conclusion: the names of the elementary propositions actually do work only in quantified propositions and definitions; and the work that they do is merely to pick out common features of elementary facts. But on reflection one can see that this position is already implicit in Frege's definitions of proper name and function name:

---

Ein Name einer Funktion erster Stufe mit einem Argumente hat dann eine Bedeutung (bedeutet etwas, ist bedeutungsvoll), wenn der Eigennamen, der aus diesem Funktionsnamen dadurch entsteht, daß die Argumentstellen mit einem Eigennamen ausgefüllt werden, immer dann eine Bedeutung hat, wenn dieser eingesetzte Name etwas bedeutet.

The name of a first-level function with one argument has a meaning [Bedeutung] (means something, is meaningful), if the proper-name that is formed out of this function-name when its argument-places are filled with a proper name has a meaning for all such inserted names which themselves have a meaning.40

Similarly a proper name has a reference if and only if all propositions formed by combining it with a first-level function name have references as well. Both object and function are defined with respect to the other, and, more importantly, with respect to a class of functions and objects, respectively: if there were only one elementary proposition, we would have no criterion for deciding which part of it was the proper name and which the function name--any such division would be possible, since nothing could speak against it. To be a proper name is always to be a significant argument for all first-level functions, and to be a function name is always to be significant in combination with all proper names: the concept “being an object” is thus for Frege essentially connected to the concept “being a possible argument to some functions;” and similarly for “being a function.” Generalized propositions highlight this essential feature of names precisely because they depend on it: a function name is necessarily connected to a range of arguments for which it is significant, since it could not be constructed without there being such a range; the quantifier notation makes use of this essential property to cull out facts from within such a range--those facts of which the function is a constituent. The objects that the names denote are consequently no more than characteristic features of classes of facts--if such classes did not exist, the names’ utility as handles for such classes would disappear, and there would be no point in our referring to them at all. Such

objects, as we shall see in the next chapter, have little in common with the self-
sufficient things of our experience—with Russell’s objects, in other words—and
everything in common with the Helmholtz/Riemann definition of an *element* in a
*manifold*: the objects are determinations of a concept, such that the concept is
inconceivable without the possibility of its determinations, and the determinations are
inconceivable without the concept—that is, without their being several such
determinations: “Each thing is, so to speak, in a space of possible elementary facts. I
can conceive this space being empty, but not the thing without the space.” (2.013)
Chapter 7 - The Picture-theories of Helmholtz, Hertz and Wittgenstein

Taken together, the class-theory of truth-functions and quantification, and the Abbildung-theory of the elementary proposition (Abbildung in the dual sense of mapping and depiction) respond to the question with which Wittgenstein began his 1914 notebook: How is it possible that logic should take care of itself, when we have no acquaintance with the primitives of that logic? In Chapters 3-6 I have shown what this “taking care of itself” consists in: we cannot make appeal to logical judgments about type-memberships, or the existence of “logical forms” or “logical objects” when laying down our theory of judgment (Russell) or theory of the proposition (Wittgenstein). If our theory appealed to such entities, then it would follow that the subjects whose judgments the theory described might literally fail to use their language in a meaningful sense. In failing to secure the grounds of its own significance, logic would lose its a priori character, it would fail to take care of itself. The picture theory explains the significance of elementary propositions by positing a primitive language with a self-determining syntax; and the class-theory then widens the scope of “significant proposition” to include all those symbols that are used to make definite statements about the state (die Lage) of the propositions of that primitive language.

Thus there are two distinct topics to consider here: the first is the relation between what I will call macroscopic languages (natural languages, the Begriffsschriften of Frege, Russell and Whitehead) and the ideal microscopic language whose symbols embody all the various syntactic characteristics required to guarantee significance; the second is the congruence between the outer manifold of the Sachverhalte and the inner manifold of the elementary propositions which mirror them. We have already discussed the first of these from the point of view of the symbol/sign distinction: signs that have certain essential structural properties get used
on the model of symbols whose structure completely determines their use. The former correspond (when used correctly) to the symbols of the macroscopic languages, the latter to those of the microscopic ones. Clearly the possibility of using the signs in accordance with the inherent functional properties of the symbols rests on the symbols’ existing, but that need not imply our having conscious acquaintance with the elements of the microscopic language. An interpreted (as opposed to compiled) language need share no symbolic elements with the machine-language; however without the latter, the logical properties of the former hover without support. Only on interpretation and execution will these properties be realised.

The second consideration—the relation between the inner and outer manifold—is best thematized by comparing Wittgenstein’s manifold theory with those of his most predecessors, Hertz and Helmholtz. The fundamental connection consists in a dichotomy between kinds of truth. Both Helmholtz and his student, Hertz, conceived of the connection between mind and world as a kind of mesh or grating. The outer world, whose complexity is unknown, is channeled into our experience only through the grate: the possible appearances in the gridwork (the points in the manifold) constitute the sum of our possible experiences. The inner world is that of appearances (Erscheinungen) and representations (Vorstellungen): to each possible primitive appearance corresponds a primitive representation; and we can think of states of affairs that are not present to us by means of these representations. Obviously this account depends on the coordination between representations and primitive experiences being complete and unambiguous. The grate belongs neither to mind nor world alone, for just as the world beyond our perceptual manifold is perhaps of an unknowable complexity, or perhaps simply non-existent, the inner world consists of aggregate upon aggregate of the primitive appearances (yielding high-level, perhaps conscious perceptions) and parallel combinations of the primitive representations (yielding high-level, perhaps conscious thoughts). The grate is the boundary between
inner and outer, a G~rzestreife or no-man’s-land, as Hertz called it in an essay on
Helmholtz’s life work:

In unserem Bewußtsein finden wir eine innere geistige Welt von
Anschauungen und Begriffen, außerhalb unseres Bewußtseins liegt fremd
und kalt die Welt der wirklichen Dinge. Zwischen den beiden zieht sich als
schmaler Grenzstreifen das Gebiet der sinnlichen Empfindung hin. Kein
Verkehr zwischen diesen beiden Welten ist möglich, als über diesen
Grenzstreifen hinüber; keine Änderung in der Außenwelt kann uns
bemerklich machen, als indem sie auf ein Sinnesorgan wirkt und Kleid und
Farbe dieses Sinnes erborgt, keine Ursachen unserer wechselnden Gefühle
cönnen wir uns in der äußeren Welt vorstellen, als nachdem wir denselben,
wenn auch noch so ungern, sinnliche Attribute beigelegt haben. Von höchster
Wichtigkeit für jede Erkenntnis der Welt und unser selbst ist es also, daß uns
jener Grenzstreifen gründlich bekannt sei, damit wir nicht das, was ihm
angehört, für das Eigentum der einen oder der anderen der durch ihn
geschiedenen Welten halten. ...

Welche Rolle spielt bei der Bildung der geistigen Vorstellungen das
Auge selbst, die Gestalt der Bilder, welche es entwirft, die Art seiner
Farbenempfindung, die Akkomodation, die Augenbewegung, der Umstand,
däß wir zwei Augen haben? Genugt die Mannigfaltigkeit dieser
Beziehungen, um alle denkmöglichen Mannigfaltigkeiten der inneren Welt
durchsichtig zu machen? ¹

We find in our consciousness an internal psychical world of intuitions
and concepts--outside our consciousness lies, cold and foreign, the world of
real things. The region of sensory perception runs like a narrow no-man’s-
land between the two. No traffic between the two worlds is possible except
through this no-man’s-land; no alteration in the external world can make
itself known to us, without acting on a sense-organ and taking on the colour
and clothing of that sense, no cause of our changing feelings can we
represent to ourselves in the external world, until we have attached--however
unwillingly--sensible properties to the latter. Thus it is of the greatest
importance for all knowledge of the world and of ourselves that we be
thoroughly acquainted with this no-man’s-land, in order that we do not
mistake that which belongs properly to it for a property of the one or the
other of the worlds that it divides. ...

What role is played in the formation of psychical representations by
the eye itself, by the form of the pictures it produces, by the nature of its
colour sensitivity, by accommodation [focusing], by eye-movement, by the
fact that we have two eyes? Does the multiplicity [Mannigfaltigkeit] of these
relations suffice to justify all thinkably possible manifolds [alle
denkmöglichen Mannigfaltigkeiten] of the inner world?

The multiplicity of this no-man’s-land is the multiplicity of the world as it enters
human consciousness, and this notion clearly cannot be separated from those of world
on the one hand, and consciousness on the other. As we saw in Chapter 2, Helmholtz

¹Hertz, Heinrich. “Zum 31. August 1891,” from the Münchener Allgemeine Zeitung, August 31,
first formulated his theory of perceptual manifolds in order to prove that geometrical propositions had empirical content: his spatial manifold is assumed to have no metrical properties, since these get defined only by observing how bodies behave when they are moved about in space and time. The physical geometry that we derive in doing so does not make statements about relations among the components of the properly spatial part of the perceptual manifold (the topogeneous moments), but about regularities (Gesetzmäßigkeiten) in the manifold as a whole (i.e. including the property-manifolds of colour, et cetera) over time. There could be, Helmholtz concedes, a geometry that was independent of matter and time, if we suppose that we could intuit the metric of the manifold directly. If that were the case, then we would have to admit two kinds of geometric truths: the “pure intuitive” truths of a geometry which prescinds from metrical properties and time; and those of a physical geometry, whose predictions always cash out in statements asserting certain regularities in the temporal and material process of measurement. The first geometry would consist of truths about the spatial manifold itself, about its inner structure. These truths would allow us to make predictions about the results of measurements not because they were defined in terms of measurements (as in the physical geometry), but because they describe the a priori structure of a portion (the properly spatial portion) of the perceptual manifold— they would be statements about necessary regularities in the underlying structure of all possible experience. The second geometry would state nothing more than the results of observed regularities in actual measurements. It would have no more than inductive certainty, and its prognostications would also always be open to revision, for they would not describe the framework, the manifold itself, but regularities of events within the manifold.

Helmholtz did not believe that there was a pure intuitive geometry, thus he did not believe in two kinds of geometric truths. But his framing of the problem leads unavoidably to the conclusion that the manifold must have some properties which
have the intuitive *a priori* character he wanted to deny: the dimensionality of the manifold, the number of its elements, and the ordering relations holding among them are internal aspects of any given manifold—they could be otherwise, but then we would not be dealing with the same manifold. And so statements such as, “Space has three dimensions,” “Crimson is darker than pink,” and “(1, 2, 3) and (3, 4, 5) are distinct points,” have precisely the force of the geometry whose existence he denies. This dichotomy between statements which assert regularities within the manifold, which are contingently true, and statements about inherent properties of the manifold, which are necessarily true, is taken up by Wittgenstein in his account of the “logical propositions.” Such propositions do not treat of the existence and non-existence of complexes at given logical places, but instead of the properties of the structure which supports and defines the logical places themselves. Since that structure is conceived as the boundary layer between mind and world, these pseudo-propositions “show” the topology of two congruent structures: that of possible experience and that of the possible representations of experience. As Wittgenstein put it, they show us the limit of our language and, simultaneously, of our world.

a) The No-man’s-land

I concluded the last chapter with the observation that Wittgenstein’s extensional definitions of truth-functions and quantification lead to the conclusion that every expression making use of the components of an elementary proposition (the names) could be replaced with one that did not make use of them, if we allow the possibility of infinite conjunctions and disjunctions. The names are handles which allow us to pick out relevant propositions, and indeed their identification as *parts* of the elementary propositional symbols depends always on the possibility of combining them with other such parts to form significant propositions: if there were only one possible fact, we would have no criteria according to which we could parse its expression, and
thus there would be no point in giving it a complex expression at all—we might just as well call it "p". In general, any finite truth-function of elementary propositions can be rewritten without making use of object-names: \( Fa.Fc.Rac \) can be rendered as \( p.q.r \), and, so long as we know which logical places "p", "q", and "r" point to, the second expression will say exactly the same thing as the first. Of course the first expression shows that the three elementary propositions share elements; but this only matters once we introduce quantified expressions, or definitions such as:

\[
\varphi(A) = aRb.p.q.qb \text{ Def.}, \text{ or }
\]

\[
(\exists x, y): Fx.Fy.xRy
\]

In making use of these features, such expressions point to a particular "topology" of the manifold of elementary facts: in this case that there are at least individuals, predicates and two-place relations in the world. The high-level propositions of our natural language are supposed to be resolved in truth functions of the elementary propositions, and this must be done by means of definitions and quantified expressions like those given above. If there were no discernible elements in our fundamental language—if the manifold of that language were one-dimensional—such definitions would not be possible. The high-level languages depend in other words on the primitive languages capturing a multiplicity of relations among the elementary facts: the Sachverhalte are not uniform, but are grouped together in classes. The existence and structure of the high-level language depends on the elementary propositions having handles, "characteristics of a class of propositions (3.311)."

Both Helmholtz and Riemann\(^2\) had defined an \( n \)-fold manifold as a collection of entities for each of whose determinations \( n \) variable values are required. When they

gave this definition, they assumed that the cardinality of the points in the plane, for instance, was greater than that of the real numbers; thus that the unique specification of all the points in the first would only be possible by means of two variable quantities. As I mentioned earlier, this assumption was proved false by Cantor, as Wittgenstein of course knew. But what he would have known equally well is that the topological relations between the points in an \( n \)-fold manifold remain well-defined only under those transformations that preserve at least the multiplicity (even if not the metric) of the manifold in question.\(^3\) Now in the case of the \textit{Tractatus} manifolds, we have the following situation: on the one hand Cantor's result, taken together with Wittgenstein's extensionalism, suggests that the elementary propositions can be adequately identified by means of only \textit{sentential} variables; on the other hand the fact that we can and do select sub-manifolds of the logical space means that we must make use of shared and recognizable characteristics of the elementary facts—characteristics which both distinguish the latter from one another, while at the same time permitting them to be grouped together. These characteristics permit us to identify shared features of distinct facts, as well as to differentiate formally identical facts by means of distinct determinations of their elements. If the primitive language is to preserve the "topological" relations between the elementary facts, then it must capture the relations of similarity and distinctness that hold between them: their similarity consisting in the facts sharing either forms (\( aRb \) and \( cSd \)) or elements (\( aRb \) and \( Fa \)); their distinctness in the non-identity of the determinations of a category or type (\( Fa \) and \( Fb \) are both determinations of \( F\check{x} \), and their relation to one another is reflected in our use of two \textit{distinct} symbols of the \textit{same} type). The class of facts \( xF\check{x} = \{Fa, Fb, Fc, \ldots\} \) is a cut through the predicate-manifold, and the essential

relations between these facts is expressed once again by their being distinct determinations of the same variable structure—the Urbild for Wittgenstein, a Begriff for Helmholtz and Riemann. Although we could represent the predicative manifold by means of a one-dimensional sentential manifold, we would obliterate this information by doing so, and with it the topological relations between these facts: they are all determinations of the concept \( F \), they form a one-dimensional sub-manifold of a two-dimensional manifold—a line.

If we had direct acquaintance with elementary propositions, then their structural characteristics would show something about the topologies of the world of experience and of the elementary language which mirrors it. Of course we are not acquainted with that language, in the sense that we are conscious of it, that we can give instances of an elementary proposition. That is once again the difficulty with which the Notebooks begin. In the remainder of this section, I will assume that we do have such acquaintance, and will return to this question in the next section on the macro- and microscopic languages. Making this assumption is equivalent to assuming that our language is composed of symbols and not merely signs. Such a language is one in which all sense is determinate: type mismatches, and names with ambiguous reference are a fortiori excluded, and the symbols for truth-functions and quantification have no ambiguity in their application. In this language, though not in the macroscopic one, it is impossible to think illogically. It sets the standard with which the macroscopic languages must agree—it is a natural logic of the sort I described in the Introduction.

The topology of this language will be identical to that of the space of elementary facts it represents: each will be determined by the types of elements of which it is composed, and the manner in which these may combine with each other. The only requirements are 1) that a possible combination of names correspond to
every possible combination of objects, and vice versa; and 2) that whenever two
elementary facts share a common element, this relationship be mirrored in the ap-
pearance of a common characteristic (Merkmal) in the two elementary propositions
corresponding to the facts. All of the following are possible core spaces, or portions
of the core space:

Fig. 1

<table>
<thead>
<tr>
<th>f e d c b a</th>
<th>f e d c b a</th>
</tr>
</thead>
<tbody>
<tr>
<td>* (ab)</td>
<td>* (ab)</td>
</tr>
</tbody>
</table>

Fig. 2

<table>
<thead>
<tr>
<th>Fξ Eξ Dξ Cξ Bξ Aξ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fξ Eξ Dξ Cξ Bξ Aξ</td>
</tr>
</tbody>
</table>

Fig. 3

<table>
<thead>
<tr>
<th>Fξu Eξu Dξu Cξu Bξu Aξu</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fξu Eξu Dξu Cξu Bξu Aξu</td>
</tr>
</tbody>
</table>

Figure 1 represents a combinatory logic like Leibniz's, with the difference that
Leibniz allowed arbitrarily long concatenations of his names, whereas in this
language they are restricted to concatenations of two elements. Figure 2 is that of
first-order predicate logic: there are no elementary relations, although we could of
course define relations in terms of the predicates. One should note that the symbols on
the vertical axis look like Frege's unsaturated function-names. They have a slot which
fits the symbols of the horizontal type, but not those of their own type. But this does
not make them the predicates and the other the individuals, for we could secure the
same combinatory possibilities if the horizontal axis symbols had the slot, and those
on the vertical axis had none. Figure 3 represents a world of only individuals and
relations. Here such a reversal of the type relationships is not possible: two axes
consist of members of one type (the individuals a, b, c, ...), the third of a second type
(the relations Aξu, Bξu, Cξu, ...), and this relationship must be reflected in the
syntactical characteristics of the elementary symbols. If what are here represented as
individuals had two argument-slots, and the predicates none, then each of (aξu, bξu,
would combine with two members of \((A, B, C, \ldots)\), and the range of the possible elementary facts would differ accordingly.

As we saw in the last chapter, a sentence which is significant is one which determines the core logical space. When an elementary proposition is true, the element of the world-manifold that it depicts exists; and when it is false, it does not. So a complete truth-value assignment to the elementary propositions will correspond to a distribution of points in the logical space: this is a possible *Sachlage*. The truth-conditions of a molecular sentence will be all those truth-value assignments under which it is true, whereby we will of course be able to ignore those points whose symbols are not among the expressions that are "essential to its sense:" in giving the truth-conditions of \(p \lor q\), we can confine ourselves to the locations \(p\) and \(q\). To say that a significant proposition determines the logical space is to say that it divides the class of possible truth-value assignments (first of the sub-manifold to which it directly refers, and, in consequence, of the entire manifold as well) into two sub-classes: those under which it is true, and those under which it is false. Propositions that do not do this have no empirical content, they are *sinnlos* (since we are dealing with the pure symbolic language, all *unsinnige* signs have been supposed impossible). There are two sets of such propositions:

4.463 Die Wahrheitsbedingungen bestimmen den Spielraum, der den Tatsachen durch den Satz gelassen wird.

(Der Satz, das Bild, das Modell, sind im negativen Sinne wie ein fester Körper, der die Bewegungsfreiheit der anderen beschränkt; im positiven Sinne, wie der von fester Substanz begrenzter Raum, worin ein Körper Platz hat.)

Die Tautologie läßt der Wirklichkeit den ganzen—unendlichen—logischen Raum; die Kontradiktion erfüllt den ganzen logischen Raum und läßt der Wirklichkeit keinen Punkt. Keine von den beiden kann daher die Wirklichkeit irgendwie bestimmen.  

4.463 The truth-conditions determine the free play left to the facts by the proposition. (The proposition, the picture, the model, are in the negative sense like a rigid body, which constrains the freedom of motion of the other; in the positive sense, like a space bounded by solid substance in which
The "free play" (Spielraum) in question is the play of the truth-value assignments. The proposition, conceived of negatively, excludes the possible Sachlagen that would make it false; conceived of positively, it defines the group which make it true. Both these conceptions apply equally well to elementary and to complex propositions.

Logical propositions allow every Sachlage, and contradictions exclude them all. Quantified propositions allow us to select "slices" of the manifold, and to posit connections between the elements of such slices. A fully generalized proposition such as \((\exists \varphi, x) \varphi x\) selects the entire manifold of predicative elementary propositions and says that one of them is true. Thus there is a complementary relation between each quantified proposition and the elementary propositions it subsumes:

\[5.5262\] Es verändert ja die Wahr- oder Falschheit jedes Satzes etwas
am allgemeinen Bau der Welt. Und der Spielraum, welcher ihrem Bau durch
die Gesamtheit der Elementarsätze gelassen wird, ist eben derjenige, welchen
die ganz allgemeinen Sätze begrenzen. ...\(^5\)

A truth-value assignment to all elements of the core manifold will of course determine the truth-values of all generalized propositions, hence it will determine the truth-values of the fully generalized ones as well; they, conversely, will determine a Spielraum for the elementary propositions in exactly the manner described in 4.463.

Propositions such as \((\exists \varphi, x) \varphi x\) are thus also contingent, unlike in the proto-theory, in which they had to be true. But they differ from other contingent propositions in one critical respect. At the beginning of the Notebooks, Wittgenstein still conceived of the fully generalized propositions as logical forms. They were to describe the most general constituents of reality:

\[28.10.14\] Das, was die ganz allgemeinen Sätze beschreiben, sind
allerdings in gewissem Sinne strukturelle Eigenschaften der Welt. Dennoch

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\(^5\)5.5262 The truth or falsity of every proposition does indeed alter something in the general construction of the world. And the free play that the totality of elementary propositions leaves to that construction is in fact the one which the entirely general propositions delimit. ...
können diese Sätze noch immer wahr oder falsch sein. Auch nachdem sie *Sinn haben*, bleibt der Welt noch immer jener Spierraum.

Schließlich verändert ja die Wahr- oder Falschheit *jedes* Satzes etwas an der allgemeinen Struktur der Welt. Und der Spierraum, der ihrer Struktur durch die Gesamtheit aller Elementarsätze gelassen wird, ist eben derjenige, welchen die ganz allgemeinen Sätze begrenzen.6

At this transitional stage, Wittgenstein is still searching for a way of interpreting 

\((\exists \phi, x). \varphi x\) which will on the one hand make it contingent, and, on the other, still do justice to his older Russellian view of this proposition as saying, “There is a predicative complex,” or, “Predicative complexes are possible.” He admits that the proposition is not a logical truth, but he still wants to read it as saying something about the fundamental structure of things, i.e. “There are predicates and individuals.”

When the remarks are rewritten as 5.5262, *Struktur* is replaced by *Bau*, because the structure of logical space is invariant and independent of the truth or falsity of the elementary propositions. The structure of the world is the structure of *possible* truth, that is of significance, and the confusion in the passage from the *Notebooks* stems from Wittgenstein’s failure to distinguish this “structure of the world” from that determined by the totality of existing complexes, the structure of *actual* truth. This confusion is resolved three days later (on 3.11.14) when Wittgenstein hits on the notions of logical place and picture in a long sequence of remarks, several of which I discussed in Chapter 5. It is that shift, however, together with the thesis that quantified propositions are not dematerialized, which leads to the *Tractatus* view of the fully generalized proposition:

5.5261 Ein vollkommen verallgemeinerter Satz ist, wie jeder andere Satz zusammengesetzt. (Dies zeigt sich darin, daß wir in “(∃ϕ, x). ϕx”

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6*Notebooks*, p. 20. That which the entirely general propositions describe are indeed in a certain sense structural qualities of the world. Nonetheless these propositions can still be true or false. Even after they *have sense*, the world retains this free play. 1 For the truth or falsity of each sentence does in the end change something in the general structure of the world. And the free play that the *totality* of elementary propositions leaves to its structure is in fact the one which the entirely general propositions delimit.
In the Tractatus, such propositions differ from others only in their making use of an Urbild: "Fa" says something about the point Fa; (∃x).Fx about the line Fx, and (∃ϕ, x).ϕx about the surface ϕx. Although the last proposition states no necessary fact about the manifold ϕx, it does make appeal to the whole of the manifold, and so, by making use of an Urbild, it reflects something about that structure in its expression. But in the Tractatus there are still propositions that have the characteristics which Russell and Wittgenstein attributed to (∃ϕ, x).ϕx, (∃χ, x, y).χxy, et cetera, namely the logical propositions: (∃ϕ, x).ϕx v ~ ϕx, (ϕ, x).ϕx ⊃ ϕx, et cetera. These propositions try to express "structural properties" of the world:

5.5351 Es gibt gewisse Fälle, wo man in Versuchung gerät, Ausdrücke von der Form "a = a" oder "p ⊃ p" u. dgl. zu benützen. Und zwar geschieht dies, wenn man von dem Urbild: Satz, Ding, etc. reden möchte. So hat Russell in den "Principles of Mathematics" den Unsinn "p ist ein Satz" in Symbolen durch "p ⊃ p" wiedergegeben und als Hypothese vor gewisse Sätze gestellt, damit deren Argumentstellen nur von Sätzen besetzt werden könnten.8

Wittgenstein goes on to explain that such a condition is Unsinn because "the hypothesis will not be false, but Unsinn, for an argument which is not a proposition," a point which we have seen before in connection with types. Similarly propositions such as (ϕ, x).ϕx ⊃ ϕx, although they do not determine logical space, do describe internal properties of that space:

6.121 Die Sätze der Logik demonstrieren die logischen Eigenschaften der Sätze, indem sie sie zu nichtssagenden Sätzen verbinden.

75.5261 A perfectly generalized proposition is, like every other proposition, composite. (This shows itself in the fact that we have to mention "ϕ" and "x" separately in (∃ϕ, x).ϕx. Both stand independently in a denoting relation to the world, as in the ungeneralized proposition.) ....

86.5351 There are certain cases in which one tries to use expressions of the form "a = a" or "p ⊃ p" etc. And this happens when one wants to talk about the proto-type: proposition, thing, etc. Thus Russell rendered the nonsense "p is a proposition" in symbols as "p ⊃ p" in the "Principles of Mathematics," and set it as a hypothesis before certain propositions, in order that their argument-places might be occupied only by propositions.
In showing how propositions may be connected to each other in such a way as to create a proposition which says nothing (which is either necessarily true, or necessarily false), we highlight the properties of those propositions: in the case of \((\varphi, x).\varphi x \supset \varphi x\) we highlight the bivalence of propositions; in \((\varphi, \psi, x).\varphi x. \psi x \supset \psi x\) we bring out the containment relation between a region of the space and its parts. The propositions that try to talk about Urbilder of elementary propositions are attempts to talk about the coordinates which define the logical space: individuals, predicates, relations. Such propositions will have to make use of identities, and they will also always depend for their significance on their being true; they are not even well-formed propositions according to the Tractatus conception of identity (identity is merely a way of expressing connections between variables in quantified propositions). Logical propositions involving truth-functions, on the other hand, are as well-formed as any other proposition; however they make no statement about the existence of complexes in the logical space. The first sort of propositions concern the number and kinds of things there are, and the second sort show the binary nature of the Sachverhalte, of the facts, and the connections between the part of the space. A sentence such as

\((\exists x, y, z).\varphi x v \sim \varphi x. \varphi y v \sim \varphi y. \varphi z v \sim \varphi z\)

is a mixture of the two, showing that there are at least three individuals that are significant arguments to \(\varphi x\). If there are three such objects, then it is necessarily true, and if there are not, then it is senseless. Thus it states an internal property of the space.

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9.121 The propositions of logic demonstrate the logical properties of propositions by connecting them together into propositions saying nothing. One might call this a zero method. In the logical proposition, propositions are brought into equilibrium, and the state of equilibrium indicates how these propositions must be constituted logically.
it describes, in just that sense of internal property that we developed in the last chapter.

The fully generalized propositions such as $(\exists \varphi, x). \varphi x$, which are still contingent or "material," determine a Spielraum for the elementary propositions; similarly the fully generalized logical propositions determine such a Spielraum as well, but in a degenerate sense: they characterize entire sub-manifolds of that space, by showing what the elements of that sub-manifold are, as well as how those elements might be related to each other. We should thus take quite seriously Wittgenstein's claim that the logical sentences show us features of the world: if

$$(\exists x, y): \varphi x \vee \varphi x. \varphi y \vee \varphi y$$

is a tautology, but

$$(\exists x, y, z): \varphi x \vee \varphi x. \varphi y \vee \varphi y. \varphi z \vee \varphi z$$

is senseless, then there are only two individuals in our world (however many other things of other types there may be), and that is certainly a truth with real consequences; it is just that this truth turns out empty when we try to say it.

b) Geometry and the Helmholtzian Manifold

This distinction between the a priori topological characteristics of our perceptual manifold, and the a posteriori facts appearing in the manifold is a fundamental feature of Hertz's picture-theory, which he had from Helmholtz. Many authors have considered the relation between Hertz's The Principles of Mechanics and the Tractatus, not least because Wittgenstein made that connection explicitly.

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These readings have focused almost exclusively on the "Introduction" to that book, and, to a lesser extent, on the passage concerning dynamical models which I discussed in the last chapter. But there is a more systematic connection, the root of which lies in a fundamental division within *The Principles of Mechanics*. Hertz divides the book into two parts (Books), prefacing them with the following remarks, respectively:


**Remark.** Experience is completely foreign to the considerations of the first book. All the stated assertions are *a priori* judgments in the sense of KANT. They rest on the laws of our internal intuition and the forms of the


logic of the subject making the assertions, and they have no other connection to the external experience of the latter, than that of these intuitions and forms.

Remark. In this second book we will understand Time, Space and Mass to be signs for objects of external experience, whose properties do not contradict those properties which we gave the homonymous magnitudes as forms of our inner intuition, or through definition. Our assertions concerning the relations between times, spaces and masses should therefore satisfy not only the demands of our minds, but should also correspond simultaneously to possible, above all future experiences. Therefore these assertions no longer rest alone on the laws of our intuition and thought, but in addition on past experience. However the contribution of the latter, to the extent that it is not already contained in the fundamental notions, we will condense in a single generalized expression, which we will set up as a fundamental law. There will then be no subsequent appeals to experience. The question of the correctness of our assertions thus coincides with that as to the correctness or general validity of this single assertion.

The first section of the *Principles of Mechanics* (Book I) assumes an inner spatial manifold which is "the space of Euclidean geometry, with all the qualities, that this geometry ascribes to it."13 This section describes the kinematic properties of systems on a *purely geometrical basis*. Thus the truths of Book I are truths good for any manifold with the characteristics of three-dimensional Euclidean space: to the extent that our experiences are represented in such a manifold, the geometric properties of the latter will appear as laws of the former. Whether or not all our experiences can be so resolved is the sort of question to which Hertz alludes in his article on Helmholtz. Both men were open to the possibility--soon realized by Einstein--that Euclidean manifolds of three dimensions *not* be up to the task. At the same time they believed that there had to be some level at which our experience exhibited such *a priori* structures, if only because of the brute fact that our sense-organs have pre-determined potentialities: we can see only a finite range of colours, hear a finite range of sounds, *et cetera*.14 It may be that we will never be able to determine the structure of those manifolds precisely; but that is a matter which needs to be investigated empirically.


14I return to this question in my discussion of prediction in the Conclusion.
The propositions of the first book are thus geometrical propositions that are binding on any manifold with the geometrical properties of Euclidean three-space. The choice of this space sets the limit on what Hertz calls the “thinkable” states of a given system: if we are considering a system of two points in three dimensions, then the thinkable states of the system are possible values of the six coordinates for these two points. If we assume a “connection” between the points in the system, such that some of the coordinates are functions of others, then the multiplicity of the system is reduced: some thinkable positions are excluded from consideration, and the remaining ones are then called possible positions of the system. The notions of “thinkable position,” “possible positions” and “connection” depend on one another: connections are what reduce thinkable possibilities to actual possibilities; conversely, the fact that certain thinkable positions are impossible implies a connection. Hertz’s Fundamental Law expresses such a connection as well: it is a single generalized proposition which excludes, from the class of the possible, all thinkable systems that do not conform to it. It puts a constraint on the class of thinkable systems, and thereby defines the class of what Hertz terms “natural systems.”\textsuperscript{15}

The beauty of this picture is that it leaves open the question of what constitutes a complete characterization of a physical system. Suppose that the two objects in our system are not mass-points but rigid bodies with an unknown number of components. Each rigid body is in fact a complex system with a huge number of thinkable positions: the rigid connections reduce this multiplicity to that of a system with two material points. The possible positions of the rigid bodies are coextensive with the thinkable positions of the points: by abstracting from the inner complexity of the bodies in our two-body system, we can regard it as a two-point system in three-space.

\textsuperscript{15}Hertz’s Fundamental Law quantifies over “material systems,” which are already a sub-class of thinkable systems: the definitions of the first Book have already given purely geometric criteria according to which such sub-classes can be defined. This does not however affect the general point made above.
And if these two points are themselves connected, such that some of their thinkable positions have been excluded as impossible, then the remaining positions are the possible ones of the two-point system. What we call "thinkable" and "possible" positions depends in other words on the space and the number of points that serve as our fundamental system of representation: a world of actually complex objects, which, however, were stable for long periods of time might appear to us as the world of thinkable experience. But from another point of view (that in which the internal connections of the complexes were merely temporary) it would be a world of—temporarily—possible experience, with a perhaps unsuspected thinkable multiplicity. That is the relation between the dynamical model and the modeled system in Hertz’s dynamics: the actual number of entities in system and model is not decisive—only that “the thinkably necessary (denknotwendige) consequences of the pictures always themselves be pictures of the naturally necessary (naturnotwendige) consequences of the depicted objects.”16 If one system is a dynamical mode of another, then every possible position of the model maps onto a possible position of the system it depicts; however, the actual complexity of the modeled system remains unknown and perhaps unknowable. In choosing a given frame of reference for our mechanics (e.g. three-dimensional Euclidean space) we impose a grid of Denknotwendigkeit on our representations. So long as that structure is consistent with the relations among the objects in our experience, the axioms of geometry will appear to us as conditions of our experience.

This notion of an imposed geometry, which creates a framework within which physical laws can be enunciated, fits the 6.3x sequence on the relation between physics and logic (the "net" metaphor) nicely. Although those sections have been much discussed in the literature, and despite the fact that Wittgenstein refers the

reader to Hertz when introducing the Herztian terms "gesetzmäßig" and "denkbar" in those passages, these connections have not been drawn. Since the topic is subsidiary to my immediate concerns (being more or less a direct consequence of the Helmholtz/Hertz/Wittgenstein epistemological view) I will not develop this point in great detail in what follows. I will however notice the following points. Most attempts to make sense of the physical and geometrical language of the Tractatus (Sachlage, Mannigfaltigkeit, logischer Raum, Spielraum, Bewegungsfreiheit, and so on) have been hindered by two facts: The English translations generally suppress the primary significance of the words in question, thus obscuring their interconnections, i.e. you cannot connect "state of affairs" (one translation for Sachlage) with "multiplicity" (Mannigfaltigkeit) unless you look at the German, and you have some understanding of the technical meanings of those terms. More importantly, however, one must resist the temptation to look for analogies between the mechanics of mass points and the changing properties of objects in a space of properties. The term "logical space," and the opening passages to the book would seem, taken together with the 6.3x sequence, to licence this connection. But what is common to Wittgenstein's and Hertz's work is not mechanics, but the epistemology originating with Helmholtz—that based on the notion of an a priori perceptual manifold. So when Hertz defines a mass-particle (Massenteilchen) in the first book as "a characteristic by means of which we correlate uniquely a definite point in space at a given time with another definite point in space at every other given time," he is not positing the existence of physical particles in external space. He is saying that the concept "particle" in the geometric part of the book will subsume anything which allows us to pick out particular classes of event-points: a particle is a regularity in a perceptual manifold; without such regularities in the manifold of experience, this definition, though logically conceivable, will not find

17Hertz, Heinrich. Die Prinzipien der Mechanik. p. 54.
a correlate in our experience (think of the snowy television screen of Chapter 2—here there would be no particles to identify). The definitions of Hertz's mechanics rest on Helmholtzian foundations, and those foundations can support other systems of mechanics, as both Hertz and Wittgenstein acknowledged. Wittgenstein sees his task as defining the properties of any such foundation: physics is a method of determining all true propositions, but logic concerns itself with defining all significant propositions; physics describes the Bau of the world, logic its Struktur.

c) Symbol and Sign

For Hertz and Helmholtz, the actual multiplicity of the outside world cannot be determined, for the multiplicity of the no-man's-land confines our possible experience to the network of possibilities that boundary defines: if our visual apparatus has a finite resolution, then we will not be able to experience alterations that lie beyond that threshold. If the visual field is not a continuous manifold, then it will consist of minima visibilia (pixels), each of which can take a colour value; and if the colour manifold is discontinuous, then the range of colours that each pixel can take will be finite. So the limited multiplicity of our perceptual apparatus puts an implicit constraint on the multiplicity of the external world. Perhaps the radiation impinging on my eye contains more information than my eye can resolve; nevertheless my eye will ignore such fine distinctions—they will not be wirk-lich for me. The relation between the hypothetical manifold of external reality and the manifold of my experience is precisely analogous to that between two systems, one of which has greater freedom of motion than the other: certain thinkable positions of the one are not thinkable in the other, i.e. if I perceive a small but composite particle as an atom, I will not be able to imagine (without, that is, using analytical methods) all the positions that its components might take. The reduced number of coordinates involved in my representation will not be able to represent all the thinkable positions of the more
complicated system, and the difference in the multiplicities will amount to an implicit constraint.

The fundamental propositions of mechanics constrain the multiplicity of the internal manifold yet further: they assume enough regularity in our experience to allow us to identify objects over time. We must for instance be able to make statements such as, "Point P moved from location x to location y in time t." Even that statement, as we saw in Chapter 2, will depend on our being able to see the same thing--P--at two places at different times. But the propositions of Hertz's mechanics go much further than that, for they assert regularities such as: all motions are continuously differentiable; only time-independent connections (that being Hertz's definition of *gesetzmäßige Verbindungen*) hold between points in a system; a system always follows the shortest and straightest path consistent with its internal constraint. All these propositions exclude thinkable occurrences in the *internal* manifold. For I *can* conceive of a discontinuous motion. To exclude such a process is to say that certain thinkable displacements are impossible, and that is the definition of a *connection*.

We just saw how the fully generalized propositions of the *Tractatus* determine a *Spielraum* for the elementary propositions. All propositions do this, of course, but the fully generalized ones cut across ranges of logical space. If $a$, $b$, $c$, $d$ are spatial coordinates, and $t$ a time coordinate, then the proposition:

$$(\varphi,a,b,c,d,t): \varphi(a,b,t).a \equiv c.b = d : \supset \varphi(c,d,t)$$

might mean that, e.g. a property $\varphi$ cannot be at two distinct spatial locations at the same time. As always for Wittgenstein, such a proposition only makes *sense*, if indeed I *can* imagine that it be false, *i.e.* only if the restriction on the free play of the world is a restriction of real (thinkable) possibilities. The above proposition, unlike,

$$(\varphi,a,b,t). \varphi(a,b,t) \lor \neg \varphi(a,b,t)$$
expresses a (perhaps true) contingent fact, in that it puts a constraint on the field of elementary propositions; whereas the latter tautology points to the limit of that field:

6.343 Die Mechanik ist ein Versuch, alle *wahren* Sätze, die wir zur Weltbeschreibung brauchen, nach Einem Plane zu konstruieren.\(^\text{18}\)

6.3432 Wir dürfen nicht vergessen, daß die Weltbeschreibung durch die Mechanik immer die ganz allgemeine ist. Es ist in ihr, z.B. nie von *bestimmten* materiellen Punkten die Rede, sondern immer nur von *irgend welchen*.\(^\text{19}\)

The relation of the reduced experiential manifold of physics to the manifold of all thinkable experience is analogous to that holding between that latter manifold and the external manifold of reality. As we may recall from Helmholtz, it is characteristic of this theory that the sequence of such pictures or mappings is open-ended:

Each picture is a depiction or mapping of the picture to its left: the lower arrows represent the transitive relation of truth-dependency (the truth value of any expression in a right-hand system is defined in terms of states or *Lagen* of the system on the left); the upper ones the depictive relation (each expression in a right-hand system makes a

\(^{18}\)6.343 Mechanics is one attempt to construct all *true* propositions which we need to describe the world according to a *single* scheme.

\(^{19}\)6.3432 We mustn’t forget that the description of the world by means of mechanics is always entirely general. One never talks in it of *e.g. specific* material points, but always only of some ones or others.
statement—conditions—the state of the region in the system on the left to which it points). But the mappings generally involve a loss of multiplicity: certain thinkable possibilities in the left-hand system may not be thinkable in the right-hand one. Since the relation is transitive, even the high-level concepts of physics depict circumstances in the perceptual manifold, and, consequently, in the real world; however they are necessarily imprecise, for at every level of abstraction, multiplicity is lost. The propositions of mechanics (remembering that it is only one sort of macroscopic language) refer to the world first by picking out physical propositions which they assert to be true; and, by means of a further translation into, or mapping onto, the core manifold, they make statements about external reality as well:

6.3431 Durch den ganzen logischen Apparat hindurch sprechen die physikalischen Gesetze doch von den Gegenständen der Welt.20

Logic specifies the essential characteristics of the perceptual manifold, just as mechanics specifies, using expressions quantified over the whole of the latter, the essential characteristics of the manifold of gesetzliche experience, e.g.

6.36 Wenn es ein Kausalitätsgesetz gäbe, so könnte es lauten: “Es gibt Naturgesetze.”

Aber freilich kann man das nicht sagen: es zeigt sich.21

6.361 In der Ausdrucksweise Hertz’s könnte man sagen: Nur gesetzmaßige Zusammenhänge sind denkbar.22

We do have a language in which we can state the laws of physics: we have a manifold of thinkable situations within which we define the possible ones by means of true, physical axioms, which assert the existence of law-like connections between these situations. In logic we have no background language; however we can conceive the

206.3431 Through the whole of the logical apparatus, the physical laws do in fact speak of the objects of the world.

216.36 If there were a Law of Causality, it might go: “There are natural laws.” But obviously one can’t say that: it shows itself.

226.361 In Hertz’s terminology one might say: Only normal connections are thinkable.
world beyond our logic, beyond the limits of our language, just as Hertz and Helmholtz did—as a manifold of even greater complexity, the inner workings of which do not make themselves manifest in our experience (readers familiar with Hertz will think, as did Wittgenstein, of Hertz's hidden masses). So in logic we use the tautologies to show the logical laws, just as in physics we use fully generalized propositions to say the physical ones. Both describe the geometries of a possibility space; the latter from without, the former from within.

d) Substances

We can see in the preceding diagram that the actual boundary between inner and outer world has no special significance, at least from the point of view of the outside observer (clearly for the consciousness being described, the matter is otherwise). This is a natural consequence, and a desired one, of Helmholtz's epistemology: since the *signs* which form the basis of our perceptions are not part of our conscious experience (we are never aware of the immediate effects of the world on our sense organs), and since the systems leading from external event (the vibrating string of a violin) to sensory medium (the pressure waves in the air) to sensory organ (the action of the tympanum, the movement of fluid in the cochlea) to psychical representation (the note Middle C) all stand in a depictive relationship to one another, it is obvious that the substrates of those systems (string, air, bone, liquid, nerve fibre) are not essential to their pictorial nature. That which we believe to be invariant in our experience—what we call substances—is largely a function of our historical circumstances, and of our skill in consciously undoing the low-level perceptual processes that aggregate sensations into concepts. Both Wittgenstein and Helmholtz make this point:

Früher galten Licht und Wärme als Substanzen, bis sich später herausstellte, daß sie vergängliche Bewegungsformen seien, und wir müssen
iminer noch auf neue Zerlegungen der jetzt bekannten chemischen Elemente gefaßt sein.\textsuperscript{23}

Heat and Light were earlier considered to be substances, until it turned out that they are transitory forms of motion, and we must always be prepared for new analyses of the chemical elements now known to us.

\textit{23.12.14 Charakteristisches Beispiel zu meiner Theorie der Bedeutung der physikalischen Naturbeschreibung: die beiden Wärmetheorien, einmal die Wärme als ein Stoff, ein andermal als eine Bewegung aufgefaßt.}\textsuperscript{24}

A characteristic example of my theory of the meaning of the physical description of nature: the two theories of heat, on the one hand heat as a substance, on the other, heat considered as a motion.

Seen from the point of view of a naive consciousness, heat and light are fundamental constituents of our perceptual manifold, and thus they get confused with substances: more complicated perceptions get combined out of these basic data, and those perceptions picture aggregates of the latter. A more practiced consciousness can, however, learn to undo such aggregates (not for all time, but, for instance, under laboratory conditions), and thereby arrive at the conclusion that there is a deeper manifold within which the pseudo-substances are merely regularities.

For Wittgenstein, the objects named in our everyday language are aggregates, and the expressions of the everyday language picture those aggregates by means of definitions. We have seen this several times already: $\Phi (A)$ may be analyzed by means of the definitions,

\[
\Phi (x) = Fx \cdot Gx \quad \text{and,}
\]

\[
q (A) = q a \cdot q b \cdot a R b \quad \text{so as to read:}
\]

\[
a R b \cdot F a \cdot F b \cdot G a \cdot G b.
\]

Under the constraints that

\[
a R b, \quad \text{and}
\]


\textsuperscript{24}\textit{Notebooks}. p. 37.
\( (x). Fx = Gx \)

\( \Phi(A) \) will appear to be a predicative proposition (it will exhibit the syntax of such a proposition). However \( \Phi(A) \) occludes the actual multiplicity of the fact that it describes: it seems to assert the existence of a Sachsverhalt but in fact it refers to a Tatsache. "Tatsache is," Wittgenstein writes Russell in 1919, "what corresponds to the logical product of elementary props when this product is true." And, he continues, "The reason why I introduce Tatsache before introducing Sachverhalt would want a long explanation."\(^{25}\) There is certainly no distinct logical role for Tatsachen to be found in the Tractatus: they are not anything more than collections of Sachverhalte, as Wittgenstein’s definition makes clear. But the expressions of our natural language, and the representations of which our conscious thought is composed do not show the full of complexity of their reference, any more than our representation of heat captures the multiplicity of the motions producing the heat perceptions. The relation between Sachverhalt and Tatsache is thus analogous to that holding for Helmholtz between simple aggregates of Empfindungsweisen and the complex aggregates (of the simple aggregates) which correspond to "finished pictures of objects." It is the stability and regularity in the groupings of the elementary facts which permit us to define complex objects, and, in consequence, to construct a language which functions as if the latter were simples, \( i.e. \) which treats their names syntactically as if they referred to indestructible entities. And, as a consequence, for both authors the boundary of our conscious mental life is not the boundary between mind and world.\(^{26}\) The entities which we experience as substances, for which we have everyday names, are those of which we are conscious, but they are far removed from the Empfindungsweisen or the elementary objects. Indeed Helmholtz thinks that the threshold of

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\(^{25}\) Notebooks, p. 130.

\(^{26}\) cf. Leibniz’s remarks on perception and the error of the Cartesians and the beginning of the Monadology.
consciousness is movable: we may be able to make fine distinctions among our perceptions in controlled conditions, or for brief periods of time, by concentrating our attention. But when we leave the lab, we stop doing so. And this phenomenon is part of our daily life: I see a sailboat on the water, I concentrate my attention on the sails, the colour of the hull and of the water. Having satisfied myself of those details, I return to the first perception, which is of a sailboat and a lake, and no longer of sails and water, of white triangles and blue surface.

Under differing circumstances, I am conscious of facts of different complexity, of a different resolution in the manifold of my experience. For both logician and physiologist, the depth of this resolution is not decisive, for the process of aggregating elementary sensations goes on continuously, largely beyond our conscious thought. This means that we may never be able to determine, by simple introspection, where the limits of our perceptions—thus of world and language—actually lie. Of course the empirical investigation of both the physical world and of our sensory physiology will lead to increasingly detailed accounts of the operation of both; however we cannot say in advance how deep the analysis may go, nor can we have any certainty that it has bottomed out. That did not matter to Wittgenstein, because he thought that he could say *a priori* what the essential properties of *any* manifold on which his Logic could be built would be. That is one half of the *Tractatus*, and it can be accomplished Wittgenstein believed, without our knowing what the specific elements of our language might be (predicates, two-place relations, and so forth). The other half is the description of the languages that get built on that foundation: the languages we talk, the artificial languages we construct. *How* are they connected to that primitive language of symbols, and *what* secures that connection?
e) The interpretation of the symbolic language

The difficulty can be most easily explained by means of an analogy to computer languages and hardware: a high-level computer language has a formal syntax, this being a necessary characteristic of any language that can be translated, or compiled, into machine language; at the same time it has been constructed with the aim of being easily understood by human programmers. A programmer uses the language without thinking of, indeed without even knowing, the manner in which the expressions of his program are mapped onto the machine-language. And even the simple dichotomy between high-level and machine-languages has no basis in reality: between compilation and run-time, the program will be translated several times (one need only think of a multi-pass compiler). Even at run-time the machine-language may be interpreted into microcode. That is the first thing you learn about structured computer architecture: the line between software and hardware is fuzzy (is the microcode interpreter a logical program, or a physical device?), and it is also arbitrary. For I might design a language or an operating system on the assumption that it will be run on dedicated hardware, i.e. that its instructions will be machine-executable, only to have that dedicated hardware replaced, at a later date, with an interpreter running on some hardware whose characteristics are unknown to me now.27 When we look at a simple program in such a language, say,

```c
main()
int a;
a = 1;
if (a == 1) printf ("Hello");
end;
```

we see (if we know the syntax of the language) what it will do at run-time. Indeed we are inclined to say that we know exactly what it will do: if you know the syntax, then you know what this program must do. But this “must” requires justification: the

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27The VM/CMS mainframe operating system for the IBM 3700 series is a classic example.
program is just a string of symbols; even loaded in memory, it is not yet a program, but merely data. What we really mean to say is: “If you understand the syntax of this programming language, then you will see what this program must do, when it is correctly compiled and run on appropriate hardware in proper working order.” The program mustn’t do anything, and the hardware on which it runs can always fail. If however we ignore such sources of error, and assume that, at some level, the hardware acts deterministically, then we can ground the logical necessity of the program’s functioning in a given manner in the physical necessity that the hardware execute the compiled code in a pre-determined way.

The sign/symbol distinction is intended to capture just such a relationship between the signs of the languages we speak (which correspond to the high-level languages) and the symbols that actually ground the logical syntax (which are the machine-language):

3.32 Das Zeichen ist das sinnlich Wahrnehmbare am Symbol.

3.325 Um diesen Irrtümern zu entgehen, müssen wir eine Zeichensprache verwenden, welche sie ausschließt, indem sie nicht das gleiche Zeichen in verschiedenen Symbolen, und Zeichen, welche auf verschiedene Art bezeichnen, nicht äußerlich auf die gleiche Art verwendet. Eine Zeichensprache also, die der logischen Grammatik—der logischen Syntax—gehorcht.

(Die Begriffsschrift Frege’s und Russell’s ist eine solche Sprache, die allerdings noch nicht alle Fehler ausschließt).

3.326 Um das Symbol am Zeichen zu erkennen, muß man auf den sinnvollen Gebrauch achten.29

28 Chips exploit quantum effects, and so they are inherently prone to random behaviours. Indeed the hardware fails constantly in a modern computer, but these errors are usually caught and corrected by low-level routines.

29 3.32 The sign is the sensibly perceptible part of the symbol. 1 3.325 In order to avoid these errors, we must use a sign-language that excludes them, in that it does not externally apply the same sign in different symbols, or signs which signify in different ways, in the same way. A sign-language, that is, that obeys logical grammar, logical syntax. 1 (The concept-script of Frege and Russell is such a language, which does not however exclude all mistakes). 1 3.326 In order to recognize the symbol in the sign, one must attend to the significant use.
In the absence of their use, the sensibly perceptible signs—the letters on this page, the sounds that I utter when reading them—are not language. They are merely those parts of the symbols that we consciously perceive; and if they were not connected to something that determined their use, then they would be nothing more than some facts. Of course one might take a position typical of the later Wittgenstein, and argue that the use is not something that is added to the sign, that it cannot be grounded beyond appealing to rules which can themselves be misunderstood. That would agree poorly with the language of the 3.32x series; moreover it would vitiate the entire argument of the book. For the slogan was always: The signs must ground the logical properties! The proto-theory was rejected because it could only be made plausible by assuming the existence of logical entities. That assumption was found untenable, because we can easily imagine circumstances in which they would not exist, meaning that the significance of some, supposedly primitive, propositions would depend on the truth of others. And so Wittgenstein concluded that the exempla on whose model we understand a given proposition were contained in the symbols themselves: names would fit one another only in certain ways; symbols for the logical functions would "show" their significance by means of their structural (internal) properties alone (without making mention of their referents, cf. 3.33, 6.126). Lastly the multitude of languages that do not literally embody these structural characteristics were to be defined as functioning in accordance with these:

3.341 Das Wesentliche am Satz ist also das, was allen Sätzen, welche den gleichen Sinn ausdrücken können, gemeinsam ist.

Und ebenso ist allgemein das Wesentliche am Symbol das, was alle Symbole, die denselben Zweck erfüllen können, gemeinsam haben.

3.343 Definitionen sind Regeln der Übersetzung von einer Sprache in eine andere. Jede richtige Zeichensprache muß sich in jede andere nach solchen Regeln übersetzen lassen: Dies ist, was sie alle gemeinsam haben.30

303.341 That which is essential to a proposition is therefore that which is common to all propositions capable of expressing the same sense. And similarly that which is essential to the symbol is in general what is common to all symbols that serve the same purpose. 3.343 Definitions are rules
Here Wittgenstein writes as if there need not be a language that guaranteed the usage of its expressions: there are simply classes of inter-translatable languages. But he wants to appeal simultaneously to some sort of logical syntax which is not a matter of convention. He has to do so, for otherwise the necessity of logic would be entirely conventional: you cannot displace the necessity of syntax from world to sign, and then deny that there are signs that embody that necessity. That would be as if one said: "Computer languages form a class of systems such that any program in the one is translatable into one in the other. This is what they have in common." That works only if you add that one of these languages is machine executable, *i.e.* that one of these *actually* functions in the manner that the others are supposed to. And once you have done that, you will have assumed something beyond the class of languages with which you began, namely the machine.

Wittgenstein feels justified in suppressing this claim because he believes that that which is not sensibly perceptible in the sign—the symbols—may well be excluded from our experience. In Helmholtz's epistemology, the subject's conscious perception of his mental processes stops at some threshold. The perceptions of which he is aware, which he aggregates and analyzes consciously, are only the tip of the mental iceberg. That does not mean that the subliminal operations are not part of his mind, are not psychical. He is just not aware of them. Conversely one cannot conclude that the processes of which we *are* aware do not have a physical substrate. Helmholtz dissociates the notion of consciousness from that of the mental, in that he maintains that the boundary between conscious and unconscious does not necessarily coincide with that between nervous system and world. This dissociation is indeed at the essence of the picture theory, for it is the functional characteristics of a system that
make it a picture of another, and that is independent of the medium of the picture, the substrate of the system:

Ich hebe hervor, daß über die Natur der Bedingungen, unter denen Vorstellungen entstehen, hier gar keine Voraussetzungen gemacht werden sollen. Ebenso gut, wie die realistische Ansicht, deren Sprache wir bisher gebraucht haben, wäre zulässig die Hypothese des subjektiven Idealismus. Wir könnten annehmen, daß all unser Wahrnehmen nur ein Traum sei, wenn auch ein in sich höchst konsequenter Traum, in dem sich Vorstellung aus Vorstellung nach festen Gesetzen entwickelte. In diesem Falle würde der Grund, daß eine neue scheinbare Wahrnehmung eintritt, nur darin zu suchen sein, daß in der Seele des Träumenden die Vorstellungen bestimmter anderer Wahrnehmungen und etwa auch Vorstellungen von eigenen Willensimpulsen bestimmter Art vorausgegangen sind. Was wir in der realistischen Hypothese Naturgesetze nennen, würden in der idealistischen Gesetze sein, welche die Folge der mit dem Charakter der Wahrnehmung auf einander folgenden Vorstellungen regeln.31

I emphasize that no demands ought be made here as to the conditions under which representations come to be. The hypothesis of subjective idealism would be just as good as the realistic point of view whose language we have used up until now. We could assume that all our perceptions are just a dream, even if a most consistent one, in which representation would develop out of representation according to fixed laws. In this case the ground for the occurrence of a new pseudo-representation could only be found in the fact that, in the soul of the dreamer, the representation of certain other perceptions, and perhaps also representations of his own will-impulses, had already occurred. That which we call natural laws under the realistic hypothesis would under the idealist one be laws regulating the sequence of those sequentially occurring representations having the character of perception.

The "language of realism" to which Helmholtz refers ascribes perceptions to external physical causes, and at least some portion of our internal representations, thus of our mental life, to the operations of the sensory organs and the nervous system. However, since the regularities among both perceptions and representations of (imagined) situations are describable without making any appeal to some sort of ontological ground, we can just as well dispense with such a ground. What appears on the realistic hypothesis as a pre-established harmony between external and internal systems can be seen on the idealistic one as a harmony between two chains of representations in a single consciousness. That is to say: the consciousness that

Helmholtz’s realistic hypothesis describes would have no right to infer the existence of external systems and natural laws, if it were to confine itself to the internal evidence of its experience; but it would certainly have no right to deny their existence either. Similarly it would not be able to say whether the regularities in its internal representations (which include will-impulses, but also representations of perceptions not present to it) were the consequence of some deeper physical or psychical order, or just brute facts of our mental life. The existence of regularities in our psychical life need not be explained, in other words. It might simply be the case that I always use certain symbols in certain ways, indeed that I find myself incapable of doing otherwise. My inability to use them otherwise will appear to me as a natural law of my thought. And if the laws of my inner representations mirror those of my outer ones, then I will experience these parallel regularities as parallel necessities, although they might well, from the point of view of a realistically inclined observer, appear as contingent facts, as consequences of the construction of the external world and the human organism. We know that Wittgenstein thought this way because, on the day that he notes down the remark that became 5.63—the proposition asserting that solipsism and realism converge—he makes the following series of remarks, which refer directly to the doctrine of psychophysical parallelism:

Ist es denn wahr, daß sich mein Charakter nach der psychophysischen Auffassung nur im Bau meines Körpers oder meines Gehirns und nicht ebenso im Bau der ganzen übrigen Welt ausdrückt?
Hier liegt ein springender Punkt.

Dieser Parallelismus besteht also eigentlich zwischen meinem Geist, i.e. dem Geist, und der Welt.

... 

Und die Antwort hierauf kann nur im psychophysischen Parallelismus liegen: Wenn ich so aussähe wie die Schlange und das täte, was sie tut, so wäre ich so und so.32

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32*Notebooks*, p. 85. Is it then true that my character expresses itself, as in the psycho-physical conception, only in the construction of my body or my brain, and not equally well in the construction of the entire remaining world? Here lies a decisive point. This parallelism thus inheres between my soul,
Fechner defined psychophysics as “an exact science of the functional relations or relations of dependence between body and soul, more generally between bodily and spiritual, physical and psychical worlds.”33 The way the world appears to a given organism is a function both of the nature of the world and of the nature of the organism engaged with it. The scientist describing that organism can treat that relationship as one holding between two natural systems. She need not ask after the ontological nature of world and soul, for she seeks only to correlate measurable changes in the one with measurable changes in the other. She will be able to view the experiential manifolds of snake and human as contingent consequences of their physical construction (the external world being the same for both). But the snake and the human subject interpret the world of their experience not as their world, but as the world: its structure will appear to them as necessary.

Wittgenstein’s remarks play on just this duality: the thinking subject can be seen realistically as a system embedded in a larger one which it mirrors; or solipsistically as two parallel streams of consciousness, the one outward, the other inward, such that the possibilities of the one are paralleled in the other. We can assume the existence of a perfect logical language, if we are willing to concede that the necessity which that language embodies is ultimately a phenomenon. On the other hand we can adopt the position that the necessity in question is grounded—in some sort of psycho-physical substrate beyond the threshold of my awareness (e.g. nerve tissue). This ambiguity is built into the picture-theory, for the latter denies the importance of the substrate, arguing that everything essential in our mental signs lies in their function (Helmholtz) or their structure (Hertz and Wittgenstein). We think by means of concrete symbols, but we prescind from their physical characteristics when we do so.

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And this means that we cannot, in principle, ever know the ontological laws that secure these functional and structural properties. In sum, the laws of our pure logical language are necessary for us, because we are built in the way we are, and there is no objection to our appealing to this psychical necessity (and psycho-physical parallelism) when explaining the kind of necessity that we ascribe to other signs, to those of the "macroscopic" languages.
Chapter 8 - Conclusion

When I introduced Helmholtz’s manifold-theory in Chapter 2, I mentioned that Wittgenstein, on his return to Cambridge in 1929, focused his attention on what he called “phenomenology:”

Es scheint viel dafür zu sprechen daß die Abbildung des Gesichttraumes durch die Physik wirklich die einfachste ist. D.h. Daß die Physik die wahre Phänomenologie wäre.

Aber dagegen läßt sich etwas einwenden: Die Physik strebt nämlich Wahrheit d.h. richtige Voraussagungen der Ereignisse an während das die Phänomenologie nicht tut sie strebt Sinn nicht Wahrheit an.

Aber man kann sagen: Die Physik hat eine Sprache und in dieser Sprache sagt sie Sätze. Diese Sätze können wahr oder falsch sein. Diese Sätze bilden die Physik und die Grammatik die Phänomenologie. [sic]

In these paragraphs Wittgenstein pulls together several of the themes I discussed in the last few chapters: Physics is identified as a body of true propositions; the semantics of Physics rests on Phenomenology—physical propositions describe regularities among phenomena; and the language of Physics rests on the “grammar” of Phenomenology. We saw this connection in the section on Hertz in Chapter 7: Physics consists of empirical propositions in that it limits the set of thinkable occurrences within the internal manifold—all conceivable constellations, and sequences of such constellations, of our primitive perceptions—to a class of physically possible ones.

The language in which physical laws are expressed must therefore be such as to allow the description of both classes: the ones which could thinkably happen, but physically (in fact) do not, and those which are both thinkable and physically possible.

Wittgenstein continues:

1Wittgenstein, Ludwig. Wiener Ausgabe, Band I, Philosophische Bemerkungen. ed. Michael Nedo. Vienna: Springer-Verlag, 1994. p. 4. There seem to be good reasons for saying that the depiction [Abbildung] of visual space by means of Physics really is the simplest. That is, that Physics is the true Phenomenology. 1 But against that one might object: Physics strives after truth, that is, after right predictions, while Phenomenology does not do this—it strives after sense not truth. 1 But one can say: Physics has a language and in this language it says propositions. These propositions can be true or false. These propositions make up Physics, and Grammar [makes up] Phenomenology.
Es gibt eine bestimmte Mannigfaltigkeit des Sinnes und eine andere Mannigfaltigkeit der Gesetze.

Die Physik unterscheidet sich von der Phänomenologie dadurch, daß sie Gesetze feststellen will. Die Phänomenologie stellt nur die Möglichkeiten fest.\(^\footnote{Wittgenstein, Ludwig. \textit{Wiener Ausgabe, Band I.} p. 4. There is a determinate manifold of sense, and another manifold of laws. Physics differs from Phenomenology in that it wants to determine laws. Phenomenology determines only the possibilities.}}\)

In the \textit{Tractatus} the actual nature and fundamental constituents of logical space could not be specified. Collectively, they played the role assigned to Phenomenology in the above passages: they were to determine the possibility space within which facts might appear. Alongside the possibility space and its geometry, there was to be a language whose expressive possibilities were of the same multiplicity as the core logical space. Those “expressive possibilities” depended on the (Theory of) Types at the atomic level, and on the number and kind of truth-functions and definitions at the molecular one.

In these notebooks Wittgenstein rejects one fundamental tenet of his earlier work: that the \textit{actual} structure of the manifold of experience is on the one hand unknowable, and, on the other, irrelevant to Logic. For Wittgenstein in 1929, Logic no longer takes care of itself, for the geometric properties of the various perceptual spaces are now a matter for empirical, indeed psychological investigations:

... Mein \textit{Beschreibung} [des Gesichtsraumes] muß also unbedingt den ganzen Gesichtsraum ja selbst seine Färbigkeit [\textit{sic}] enthalten auch wenn sie nicht sagt welche Farbe an jedem Ort ist.

D.h. Sie muß doch sagen daß eine Farbe an jedem Ort ist.

Heißt das nicht, daß die \textit{Beschreibung} den Raum soweit sie ihn nicht mit Constanten erfüllt, mit Variablen erfüllen muß.\(^\footnote{Wittgenstein, Ludwig. \textit{Wiener Ausgabe, Band I.} p. 54. ... My description [of visual space] must therefore necessarily contain the entire visual field, indeed its colouration, even if it does not say which colour is at which location. That is, it must at least say that there is a colour at each position. Does that not mean that the description, in so far as it does not fill space with constants, must fill it with variables.} \)
The relation between spatial location and colour is like that between function and argument ("Space and colour saturate one another. And the way in which they penetrate one another makes the visual field."\(^4\)), and the language that describes the possible connections of the elements of the visual field must, so Wittgenstein continues, share in this thinkable multiplicity:

Die Mannigfaltigkeit der räumlichen Beschreibung ist dadurch von vornherein gegeben, daß die Beschreibung die richtige Mannigfaltigkeit hat wenn sie vermag alle denkbaren Konfigurationen zu beschreiben.\(^5\)

That is, only if the language of description is so constituted that it can refer to each possible configuration of primitive visual sensations, will it be possible to talk about the full multiplicity of possible visual experiences.

The results of this work are peppered throughout the manuscripts making up Wittgenstein und der Wiener Kreis, in particular in Waismann’s notes on Wittgenstein’s concept of space:

Die Raumpunkte bilden in einem ganz andern Sinn eine "Menge" als etwa die Bücher oder die Hüte. ... Dieser Unterschied hängt zusammen mit dem Unterschied der Worte "sinnvoll" und "wahr". Die Menge von Hüten in diesem Zimmer wird gegeben durch eine Eigenschaft (Aussagefunktion). Kennen wir die Eigenschaft, so wissen wir damit noch nicht, ob etwas unter die Eigenschaft fällt und, falls ja, wieviel Dinge unter dieser Eigenschaft fallen. Nur die Erfahrung kann dies lehren. Der Extension der Eigenschaft entspricht hier eine Klasse von wahren Sätzen.


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\(^5\)Wittgenstein, Ludwig. *Wiener Ausgabe. Band I.* p. 55. The multiplicity of the spatial relationships is given in advance by means of the fact that the description has the right multiplicity when it can describe all thinkable configurations.
Eine Klasse von wahren Sätzen wird in ganz anderer Weise begrenzt als eine Klasse von sinnvollen Sätzen. Im ersten Fall wird die Grenze durch die Erfahrung gezogen, im zweiten Fall durch die Syntax der Sprache. Die Erfahrung begrenzt die Sätze von außen, die Syntax von innen. Der *Sinnbereich* einer Funktion (d.h. die Gesamtheit der x-Werte, für welche fx sinnvoll ist) ist von innen begrenzt durch die Natur der Funktion. Und so ist auch die Klasse der Raumpunkte von innen her—durch die Syntax der räumlichen Aussagen—begrenzt. [emphasis in original]

The points in space make up a “class” in a sense quite different from books or hats. ... This difference is connected to the difference between the words “significant” and “true”. The class of hats in this room is given by means of a quality (propositional function). If we know the quality, we still do not know whether something falls under this quality, and, if so, how many things fall under it. Only experience can teach us this. To the extension of the concept corresponds here a *class of true propositions*.

What is a spatial point? ... A spatial point occurs in our propositions in a completely different manner than does an object of reality, namely only ever as part of a description that concerns objects in reality. I can describe the position of a body by specifying its distance from other definite bodies. This description corresponds to a possible situation [*Sachverhalt*], whether or not the description is true or false. Thus a spatial point represents a possibility, namely the possibility of the situation of a body relative to other bodies. The expression of this possibility is: that the proposition that describes this situation has sense. To the totality of spatial points corresponds a totality of possibilities, thus a *class of significant propositions*.

A class of true propositions will be bounded in an entirely different way than a class of significant propositions. In the first case the boundary will be drawn in experience, in the second case through the syntax of the language. The significance-range of a function (that is, the totality of x-values for which fx is significant) is bounded from within through the nature of the function. And so too is the class of spatial points limited from within by the syntax of spatial statements.6 [emphasis in original]

Since this passage is not Wittgenstein’s (these are Waismann’s notes of conversations that took place in December, 1929), I can offer it only as secondary evidence for the reading I have advanced. It is, however, perfectly consonant with that reading, allowing of course for the break I just mentioned: the space under consideration is no longer a hypothetical “logical space,” but the space of our experience—the spatial portion of the visual field, for instance. Connected to this renewed interest in psychological spaces we find a new and pragmatic conception of space and multiplicity:

Einen Raumpunkt kennen wir, wenn wir den Weg kennen, der zu diesem Raumpunkt führt. Dieser Weg wird gegeben durch eine Satzform

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In 1913, the proposition and its corresponding fact were conceived as aggregates of things (names on the one hand, and objects on the other); however this conception was soon modified on the space/coordinate model. In the Tractatus the notions of object and name equivocate between the two conceptions, in exactly the same way as they do in Helmholtz's perceptual manifold: an elementary fact or perception is an aggregate of primitive entities, and it is at the same time an element of the manifold of isomorphic facts. The primitive entities are on the one hand things, but on the other hand the notion "thing in itself" fits them poorly—they are more like coordinates of the aggregates in which they occur. The proposition is thus already in the Tractatus a kind of "address" for a location in space. Here we see the next stage in this development: the proposition's multiplicity is no longer purely pictorial (so that a fact is statically represented by means of one with an isomorphic structure), but has become a multiplicity of instructions or actions:

10. Wenn man die Sätze als Vorschriften auffaßt, um Modelle zu bilden, wird ihre Bildhaftigkeit noch deutlicher.

Denn, damit das Wort meine Hand lenken kann, muß es die Mannigfaltigkeit der gewünschten Tätigkeit haben.

Und das muß das Wesen des negativen Satzes erklären. So könnte einer zum Beispiel das Verständnis des Satzes "Das Buch ist nicht rot" dadurch zeigen, daß er bei der Anfertigung eines Modells die rote Farbe Wegwirft.8

7Waismann, Friedrich. Ludwig Wittgenstein und der Wiener Kreis, p. 217. We know a spatial point when we know the way leading to this spatial point. This way will be given by means of a propositional form (e.g. 10 steps forward, then 5 steps to the right). To the totality of spatial points corresponds the totality of possible ways, thus the totality of propositional forms.

8Wittgenstein, Ludwig. Philosophische Bemerkungen. Schriften 2. ed. R. Rhee. Frankfurt am Main: Suhrkamp Verlag. 1964. p. 57. I 10. When one conceives propositions as instructions, their pictoriality becomes still clearer. For if a word [an expression] is able to guide my hand it must have the multiplicity of the desired action. And that must explain the essence of the negative proposition. Thus someone might show their understanding of the proposition "The book is not red" in the fact that they throw away the red colour [paint, crayon] while preparing a model.
The expression for a spatial location has three elements—the three arbitrarily defined coordinates—which map onto instructions telling me the actions I must perform to find or arrive at that location. Similarly a proposition such as “There is a red square at locations \([0 - 2, 0 - 2]\)” will describe the model:

And the sequence of actions I must carry out in constructing this model will themselves constitute a mapping of the multiplicity of the description onto a multiplicity of possible *Tätigkeiten*. This shift in the role of multiplicity within Wittgenstein’s philosophy is of importance to our understanding of both the earlier and the later work, for it translates the pictorial multiplicity of the *Tractatus* into the *grammar* of the middle and later work. Two passages from these notebooks put this watershed in stark relief:

Die Grammatik gibt der Sprache den nötigen Freiheitsgrad.\(^9\)

Die Grammatik ist eine “Theory of Logical Types/der logischen Typen”.

Ich nenne die Regel der Darstellung keine Convention, die sich durch Sätze rechtfertigen läßt, Sätze welche das Dargestellte beschreiben und zeigen, daß die Darstellung adäquat ist. Die Conventionen der Grammatik lassen sich nicht durch eine Beschreibung des Dargestellten rechtfertigen. Jede solche Beschreibung setzt schon die Regeln der Grammatik voraus. D.h.

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"Grammar" is a term we associate with the middle and later Wittgenstein, "degree of freedom" and its connection to the Theory of Types belong to the theory of the Tractatus and its predecessors. In equating the first of these with the last two, Wittgenstein explicitly connects all three notions in a manner paralleling the shift in his understanding of multiplicity: multiplicity was a static, structural property of facts and their pictures; however it became a multiplicity of the actions required to make or locate (thus to verify) the fact corresponding to the proposition. The correlate of the multiplicity of the possible facts was the "degree of freedom" of the symbolism used to indicate these facts, where this degree of freedom was a function of the number of types (the dimensionality of the logical space) and the numbers of their elements (its absolute extension): logical propositions had an unbounded degree of freedom, in that they placed no constraint on the fundamental manifold, and the contradictions had a degree of freedom of zero. The degree of freedom is equivalent to the possibilities of syntactic variation determined by the internal properties (the structural properties) of the symbolism, thus by the built-in type restrictions: it is the fundamental syntax of the language. Since in these theories of the late 1920's the connection between picture and intended fact is a temporal one, the static notion of syntax as a pictorial multiplicity must reflect this development as well, and the notion that replaces it is Grammar.

In the Tractatus theory, those pseudo-propositions that were to describe the fundamental properties of things ("There are two individuals," "Fx is a predicate,"
etc.) were seen as propositions whose truth was the condition for the significance of the elementary propositions, and they were consequently rejected as *sinnlos*. Here Wittgenstein makes the same claim for the grammatical rules and conventions that "every description presupposes." The first section of the *Philosophical Investigations* describes a language-game to which this modified manifold-theory is adequate:

Denke nun an diese Verwendung der Sprache: Ich schicke jemand einkaufen. Ich gebe ihm einen Zettel, auf diesem stehen die Zeichen: "fünf rote Äpfel". Er trägt den Zettel zum Kaufmann; der öffnet die Lade, auf welcher das Zeichen "Äpfel" steht; dann sucht er in einer Tabelle das Wort "rot" auf und findet ihm gegenüber ein Farbmuster; nun sagt er die Reihe der Grundzahlwörter—ich nehme an, er weiß sie auswendig—bis zum Worte "fünf" und bei jedem Zahlwort nimmt er einen Apfel aus der Lade, der die Farbe des Musters hat.11

Here we have a language-game in which the proposition is an instruction having "the multiplicity of a desired action," even though it is still a picture whose elements correspond to aspects of the fact it represents. For what are "five", "red" and "apple"? Each is, as Baker and Hacker observe,12 an instance of a type of word, of a grammatical category. And the bins, charts and lists specify the "location" of these signs within those grammatical categories: the row of fruit-boxes, the sequence of counting-words, and the colour-chart all map elements of three categories of words onto semantic elements by means of a sequence of conventional actions on the part of the shopkeeper. His charts and lists define a manifold of possible actions, which result in a manifold of possible orders: two green pears, seven red oranges, *etc*. If we ignored the shopkeeper and his lists, we might rightly say that the multiplicity of the signs on the slip reflects a multiplicity in the elements of the customer's order: the signs on the

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11Wittgenstein, Ludwig. *Philosophische Untersuchungen/Philosophical Investigations*. 2nd edition. eds. R. Rhees, G.E.M. Anscombe. Oxford: Basil Blackwell. 1958. I now think of the following use of language: I send someone shopping. I give him a slip on which stand the signs: "five red apples". He takes the slip to the shopkeeper, who opens the bin marked with the sign "apples"; then he looks up the word "red" in a table and finds a colour sample beside it; now he recites the sequence of cardinal numbers—I assume he knows them by heart—up to the word "five," and with each number he takes from the bin an apple having the colour of the sample.

slip all correlate with definite properties of the order (the order is made up of *five red apples*); but when we factor in the shopkeeper, we see that this pictorial relationship is mediated by a sequence of possible actions within an "action-space," within a manifold of possible actions: "... in order that a word may guide my hand it *must* have the multiplicity of the desired action."

I am in no way suggesting that this is the mature doctrine of the *Investigations*. On the contrary, this example is almost certainly a deliberate invocation of Wittgenstein's views from the twenties and thirties, and we would expect such references at the opening of a book that was intended to be read against the background of those earlier thoughts. Indeed the most obvious difference between the *Investigations* and the texts I have cited so far is that the insistence on *internal* connections between utterances, actions and results vanishes in the later work, and internal connections are of course the essence of pictoriality (*Bildhaftigkeit*). We saw in Chapter 5 how Wittgenstein criticized the theory Russell advanced in his *Analysis of Mind* because it "take[s] recognition to be an external relation," whereas the *Bildauflassung* "sees recognition as the cognition of an internal relation."13 On Russell's view of judging or understanding, says Wittgenstein, we cannot know unambiguously whether or not that which fulfills our expectation is in fact that which we intended. Only an internal relation between intention and intended can secure this fundamental property of language:

20. ... *Wenn man das Element der Intention aus der Sprache entfernt, so bricht damit ihre ganze Funktion zusammen.*

21. *Das Wesentliche an der Intention, an der Absicht, ist das Bild. Das Bild des Beabsichtigten.*14

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13 *Philosophische Bemerkungen.* p. 63.

14 *Philosophische Bemerkungen.* p. 63. 20. ... If one removes the element of intention from language its entire functions collapses. 21. The essential in an intention, in an expectation, is the picture. The picture of the intended.
The critical failing in Russell’s judgment-theory was that the decomposed propositional sign could not uniquely specify the fact it described: that was the theory Frege laughed out of court because it gave the jury nothing on which to judge but “thought-rubble.” The sticking point was that language must connect the speaker or hearer to things which are absent, and it must do so unambiguously, internally. Russell’s and Wittgenstein’s early theories did not distinguish between facts that are absent because we are not in their spatial proximity (they are not “present” to us, in Helmholtz’ language) and those which are absent because they have already, or have not yet, occurred. The proposition projects onto its corresponding fact atemporally—indeed the question of whether I can in fact verify a given proposition is not relevant: the fact that I cannot directly observe the state of affairs in the next room is not fundamentally different from my being unable to observe what happened yesterday. In these middle writings, however, it is not reference to the absent which is the defining function of language, but its ability to project forward in time. The concern that arose with Russell’s theory is, however, very much alive: if my utterance predicts or points to some future fact or event, then it must somehow capture here and now that which is supposed to be there and then. Just as in the *Tractatus*, this cannot mean that the utterance actually gives us its reference (Mont Blanc is not part of our minds): it must give us a way of identifying the referent if and when it occurs, and, conversely, of identifying that it has not occurred when and where we said it would. The proposition must give us a way to the fact or event it describes, and this way must be *determinate*. It is with this topic that I shall conclude my discussion, because it links the earliest roots of this conception—Helmholtz’s definition of *anschaulich vorstellbar*—with doctrines of both the middle and late Wittgenstein. As I said in the Introduction, the

15I believe, though I do not argue the point in this thesis, that there is a temporal manifold in the *Tractatus* theory, thus time-indices, or time-objects. How Wittgenstein’s conception of time changed from 1913-1930 would of course make for an interesting discussion; however it would be of little value until the topics I treat in the preceding chapters were well understood.
entrenchment of this conception in Wittgenstein's later work most probably has a double cause: on the one hand Wittgenstein had inherited an essentially Helmholtzian semiotic theory through Hertz; on the other hand the members of the Vienna Circle would have received Helmholtz's ideas as well (if only, as with Reichenbach, as a result of their interest in relativity theory). Wittgenstein may thus have had his position reinforced through contact with Schlick and Waismann.

Helmholtz introduced his manifold theory in conjunction with a definition of conceivable, a definition which depends in good measure on that theory. The definition says that something is conceivable when we can describe the sense-impressions it would bring about in us in all situations and from all points of view:

Wenn nun die Frage diskutiert werden soll, ob die Raumverhältnisse in metamathematischen Räumen anschaulich vorstellbar seien, so werden wir zunächst feststellen müssen, nach welcher Norm wir die Anschaulichkeit eines Objektes zu beurteilen haben, welches wir nie angeschaut haben.

Ich habe zu dem Ende eine Definition dessen aufgestellt, was wir als anschaulich vorstellbar anerkennen müßten, die dahin lautet, daß dazu erforderlich sei die vollständige Vorstellbarkeit derjenigen Sinnesindrücke, welche das betreffende Objekt in uns nach den bekannten Gesetzen unserer Sinnesorgane unter allen denkbaren Bedingungen der Beobachtung erregen, und wodurch es sich von anderen ähnlichen Objekten unterscheiden würde.

So if we should discuss the question of whether the spatial relations in metamathematical spaces are _anschaulich vorstellbar_, we will first have to determine the standard according to which we will judge the intuitability [_die Anschaulichkeit_] of an object which we have not yet seen/intuited [ _angeschaut haben_].

I have to this end prepared a definition of that which we must recognize as _anschaulich vorstellbar_. It says that what is required is the complete conceivability of those sense-impressions which the object in question would excite in us—in accordance with the known laws of our sense-organs—under all thinkable conditions of observation, and by means of which [ _i.e. the sense-impressions_] it would be distinguished from other similar objects. 16

The definition contains quantifiers: it refers to "all thinkable conditions of observation" and, in consequence, to all sense-impressions that could arise under those

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various conditions; indeed, the definition is only of value if it takes account of situations that are not, in Helmholtz’s language, present to us. Put otherwise, a definition of “conceivable experience” must make room for a class of experiences which we have not had, and perhaps never will have, but which are nonetheless demonstrably possible. These demands can only be met if we have means at our disposal for defining the class of possible experiences in advance, and that is what the manifold-theory of perception does: it provides us with an elementary sign-vocabulary—the elements of the manifolds, which are signs for Helmholtz—and postulates 1) that any experience we can have will consist in an aggregate of these elements, and 2) that all possible aggregates of these elements are possible primitive experiences. Furthermore, the theory remains open to the possibility that the manifolds in question not be the fundamental ones. Since the higher-order manifolds are defined in terms of the lower-order ones (as, for instance, a one-dimensional colour spectrum simplifies a three-dimensional one), this does not introduce the possibility of experiences that lie, so to speak, outside the original manifold, because all representations in the higher-order manifolds refer, even if imprecisely (with free play) to aggregates in the lower-order ones. “Blue” may turn out to have a more complicated extension that I originally thought, but the assertion that “Blue lies between Violet and Red in the colour-spectrum” is not overturned by the discovery of a three-dimensional spectrum. The statement will now appear to be true but imprecise, just as a description of a complex mechanism by means of a model with a reduced degree of freedom will fail to describe micro-mechanical motions of the mechanism’s components, but not, if the model is well chosen, possible macroscopic motions of consequence. Helmholtz’s manifold theory was, from the beginning, a theory postulating a primitive semantics (the primitive appearances) and sign-language (the primitive representations) for the purpose of defining a space of possible experience, thus of significant language. Moreover this language is structured by categories: each element of a category is
necessarily an element, and the category is simultaneously nothing beyond the set of its members. This relationship holds both for the primitive elements themselves and for the aggregates. Red is necessarily a determination of the concept colour, just as a spatial location is necessarily a determination of the concept space; and \([\text{Blue},2,3]\) is necessarily an element of the visual-space manifold, which consists of elements of the type \([\text{Colour},x\text{-value},y\text{-value}]\).

In the *Tractatus* the primitive manifold is also the space of possible occurrences. The types are the dimensions, their combinations the *Sachverhalte*, themselves determination of *Urbilder*, which are completely generalized concepts. The problem Wittgenstein had to solve was: Given that there are facts not present to us, how can we distinguish between those which *could be* present, and those which could not, between those which are possible and thinkable, and those which are neither? Without that account in place, we would have no way of distinguishing between false propositions and nonsense, between conceivable but absent experiences, and mere fantasy. We must have some unambiguous way of projecting our current thoughts onto absent states of affairs, and this meant for the young Wittgenstein, as it did for Helmholtz, that we must already be in possession of the elements by means of which we can describe these situations: we must already know of everything that could happen to us, and we must also be able to describe it. In so far as we cannot (and, quite obviously, we cannot) this failing is to be explained by a lack of multiplicity either in the language we speak, or in the sensory manifold of which we are conscious.

The sensory manifold, logical space, and the rules and conventions of language-games all play this role for Wittgenstein. They provide a space within which perceptions, elementary facts, or moves may play out—a space of meaning, without which linguistic expressions lose their determinate significance—a *Spielraum*. The
relation between the utterances and that space of meaning lies at the core of any such account: How do the utterances map onto the space determinately? How does a picture determine its subject, an intention its fulfillment, an instruction its execution? An explanation of this mapping relation will explain the sense of these utterances: how it is that they are able to convey determinate meaning, this being “the whole function” of language. In the Tractatus the answer was: the propositional signs are internally related to the locations they depict by means of a) shared internal, structural properties, and b) arbitrarily established denotations. In the account of the Philosophische Bemerkungen, the critical relation is that of intention: what is essential to intention is a shared multiplicity between picture, instructions, actions and finished model—this is grammar. In the Investigations, Wittgenstein breaks with the pictorial relation between expression and referent for good: only the network of conventions determined by a community of users can ground the connection between sense and meaning, picture and fact, or intention and intended. In an Investigations language-game an utterance still determines some future event or state of affairs (think of our shopkeeper); but this connection is no longer underwritten by a metaphysical relation between utterance and fulfillment (an internal connection). There is no fact or event simultaneous to my utterance (e.g. the structure of the utterance itself, or an intention accompanying it) that secures this relation. Thus an investigation of the immediate circumstances of a speech-act (the psychological states accompanying it) cannot provide an answer to questions concerning the nature of language and meaning.

Wittgenstein had to arrive at this extreme, not least because the project of psychophysical analysis that Müller, Fechner and Helmholtz inaugurated is still alive and well, and the physiological presuppositions of the latter’s manifold-theory—that our sense-organs have only certain potentialities, and that our sensations all result from permutations of and within those potentialities—is not easily doubted. In some sense we do have advance knowledge of possible experience, so something like the
manifold theory, thus the *Tractatus* theory, must be true. Husserl's Phenomenology, Wittgenstein's work in the thirties, and the positivists' project of constructing a language of unified science can all be seen as attempts to put the Helmholtz/Wittgenstein theory into epistemological practice. Only by denying that the content and structure of our thoughts are the primary determinants of what we call "meaning" can we avoid this conclusion. And even if we do draw it, it does not follow that analysis of the Helmholtzian stripe is bankrupt. The conclusion is a far more modest one: if our concern is to elucidate notions such as "language," "meaning," "intention" or "truth," then we must consider the full complex of activities that surround instances of language use (the grey rags and dust); but if it is not, then we are perfectly well entitled to follow the psychophysical project wherever it might lead.

That was Wittgenstein's fundamental error--an error that the positivists repeated. He thought he could intuit the essential properties of the fundamental manifolds, and that he could draw conclusions, even if only negative ones, from that logical intuition. Helmholtz, in contrast, contrived his manifold theory on the one hand because of mathematical and epistemological concerns, and on the other because it agreed well with the results of his careful psychophysical research. And he used the theory to *expand* the space of the thinkable, of the significant: he claimed that there were things yet unseen which were nonetheless conceivable, and in doing so he paved the way for a new physical theory--Einstein's--that rested on the possibility of similarly "inconceivable" occurrences, which indeed asserted that these were not only possible but actual. Wittgenstein and his compatriots also drew on the resources of Helmholtz's empirical research (whether knowingly or not) when positing a manifold of conceivability and significance. But they did not justify their postulates in the laboratory, and they used their meaning-space to *exclude* from the conceivable and the utterable great swaths of language they disliked. They based a negative conclusion on
a theoretical prejudice whose claim to empirical validity was certainly questionable. That was bad philosophy, and it gave rise to more; for many of the ideas in the *Tractatus* which seemed so revolutionary (especially to its British and American readers) were in fact fairly widespread in German circles at the turn of the century (although there too they did not survive the war). These shop-worn notions of 19th century German psychology became part of the *Fachsprache* of analytic philosophy, while their relation to the empirical sciences that spawned them faded into the background. Philosophers, as so often happens, were left quarreling amidst the ruins of an antique theory, the original architecture of which was unknown to them. This tendency to misidentify the conclusions of slightly outmoded scientific theories as common-sense data for philosophical analysis is probably the most regrettable in our discipline. If I am not mistaken, a good portion of "philosophy of mind" is firmly in this tradition. Perhaps the preceding treatment may serve as a cautionary tale.
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