Partition Calculus
for Topological spaces

by

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for the degree of Ph.D.
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Abstract

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The subject matter of this exposition is the study of partition relations for particular topological spaces. For topological spaces $X$ and $Y$, and for a positive integer $n$ and a cardinal $k$, the notation $X \rightarrow (Y)^n_k$ means that for any partition $f$ of $[X]^n$ into $k$ pieces, there is a subspace $H$ of $X$ homeomorphic to $Y$ and $f$ is constant on $[H]^n$. In contrast with the ordinary partition calculus in this area very few positive results are known. For example, it is consistent that for $n > 3$ and for any space $X$, we have $X \not\rightarrow (Y)^2_2$ for every topological spaces $Y$ in which not all the points are isolated. Even in the case of $n = 3$ if the space $Y$ is complicated enough the relation fails to be positive. It seems that the cases $n = 1$ and 2 are the cases where there exists some positive relations for non-trivial space $Y$. The spaces considered in this exposition are the followings: Ordinal spaces, Reals, Rationals, Cantor Cubes, $\Sigma$-products, Čech-Stone compactification of the integers, Alexandroff compactification of discrete spaces and the countable sequences converging to a point in the sense of an arbitrary ultrafilter of $\mathbb{N}$, denoted by $\omega \cup \{p\}$. We will present and review many techniques for proving topological relations when $n = 1$. Then we shall use a stepping-up procedure to obtain results for higher dimensions $n > 1$ when possible. Using ordinary axioms of set theory (ZFC), we will establish some positive relations for the space $2^\kappa$ such as $2^\kappa \rightarrow (2^\kappa, \mathbb{Q})^1$ and $2^\omega \rightarrow (\omega \cup \{p\})^1_\omega$, and $\Sigma_{\mathbb{N}} \not\rightarrow (\mathbb{N}_2)^3_3$, where $\Sigma_{\mathbb{N}}$ denotes $\Sigma_{\mathbb{N}} \{0, 1\}$, the sigma-product of $\mathbb{N}_2$ copies of the two point discrete space $\{0, 1\}$. Beyond ZFC, we introduce and apply the method of elementary submodels and use the SPFA and some of its consequences to establish the consistency of $2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^1$ as well as the consistency of $2^{\omega_1} \rightarrow (\omega_1 + 1, \omega_1)^1$. The additional axiom SPFA has a considerable large cardinal consistency strength, but we shall establish that the large cardinal assumption is necessary for the consistency of the positive relation $\Sigma_{\mathbb{N}} \rightarrow (\omega_1)^1_2$. The techniques of forcing and iterated forcing are used to define generic partitions for some spaces leading to the independence of the partition of the form $\Sigma_{\mathbb{N}} \rightarrow (\omega_1)^1_2$. 

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0.1 Introduction

Ramsey's theorem of 1930 is a two dimensional version of the standard pigeon-hole principle and states that whenever the edges of an infinite complete graph are partitioned into two pieces there is an infinite complete graph all edges of which belong to the same piece of partition. Later on, generalized in various ways, this theorem was expounded to the subjects that are today known as partition calculus and as Ramsey theory (see [EHRM] and [GRS]). In these subjects as well as in the subject matter of this thesis the Erdős-Rado arrow notation is very useful:

Notation: Given cardinal numbers $\kappa$, $\lambda$, $\mu$ and a positive integer $n$, the notation

$$\kappa \rightarrow (\lambda)^n_\mu$$

stands for the following statement:

*Given any partition of the $n$-tuples of a set $A$ of size $\kappa$ into $\mu$ pieces, $f : [A]^n \rightarrow \mu$, there is a set $H \subseteq A$ of size $\lambda$ which is homogeneous for $f$, that is, $f$ is constant on $[H]^n$. In this case it is understood that for some $\alpha < \mu$ we have $[H]^n \subseteq f^{-1}({\alpha})$. Such a set is said to be $\alpha$-homogeneous for $f$. The negation of this statement is denoted by*

$$\kappa \not\rightarrow (\lambda)^n_\mu.$$

Variations of this symbol will be introduced as the occasion arises.

Using this notation Ramsey's theorem becomes $\omega \rightarrow (\omega)^2_2$, where $\omega$ stands for the first infinite cardinal number. The finite Ramsey's theorem states that for any triple of positive integers $m,n$ and $k$ there is a positive integer $r$ such that

$$r \rightarrow (n)^m_k.$$

As a natural extension, the partition relations for order types such as ordinals and in general for partial orderings were introduced (see [EHMR] and [To1]). The homogeneous set in these cases must be some ordinal (type). One of the first and the
most important such result was \( \omega_1 \rightarrow (\alpha)^2_k \) due to J.Baumgartner and A.Hajnal [BH] which has been extended to all partially ordered sets by S.Todorcevic [To1] as follows: for a partially ordered set \( P \) that satisfies \( P \rightarrow (\omega)_m^1 \) we have the positive relation \( P \rightarrow (\alpha)^2_k \) for every countable ordinal \( \alpha \) and a positive integer \( k \).

Besides as order types the ordinals can be thought of as topological ordinal spaces (limit points are required to be limits in the topological sense.) These results have led to first question in the area of partition calculus for topological spaces: Galvin and Laver asked whether one of the partition relations \( \omega_1 \rightarrow (\alpha)^2_k \) or \( \mathbb{R} \rightarrow (\alpha)^2_k \) can be extended to topological ones. In other words can we always get the homogeneous set not only of order type \( \alpha \) but topologically homeomorphic to \( \alpha \). (incidentally, this question remains open to-date.)

The combinatorial nature of many problems in topology is well known and understood (see for example [To2] and [To3]). More recently, people have realized that the arrow notation can also be used to characterize certain families of topological spaces (see [To4] and [GS]). So, let us introduce the arrow notation for topological spaces.

For topological spaces \( X \) and \( Y \) and the cardinal number \( \kappa \)

\[
X \rightarrow (Y)_k^n
\]

means: For any partition \( f : [X]^n \rightarrow k \) there is a homogeneous subset \( H \) of \( X \) homeomorphic to the space \( Y \); we often identify \( H \) with \( Y \). From now on, all the sets \( X \) and \( Y \) are considered as topological spaces unless otherwise stated. If any risk of confusion is predicted the prefix \( \text{top} \) in front of \( Y \) may appear to emphasize that we are interested in a topological copy of \( Y \).

The arrow notation has the following frequently used variations:

\[
X \rightarrow (Y,Z)^n
\]

to mean that for spaces \( X, Y \) and \( Z \) and for each partition \( f : [X]^n \rightarrow \{0,1\} \) there is either a copy of \( Y \) in \( X \) such that \( f''([Y]^n) = \{0\} \), or a copy of \( Z \) in \( X \) such that \( f''([Z]^n) = \{1\} \).
The following convention will be used from time to time. The notation
\[ X \rightarrow (Y, (Z)_k)^n \]
means that for any partition of \([X]^n\) there is either a 0-homogeneous subspace homeomorphic to \(Y\) or for \(l \leq l < k\) there is an \(l\)-homogeneous subspace homeomorphic to \(Z\). A weakening of the positive arrow relation is the square bracket relation:
\[ X \rightarrow [Y]^n_k, \]
which means that for every \(f : [X]^n \rightarrow k\) there is \(H \subset X\) homeomorphic to \(Y\) and \(i < k\) such that \(i \notin f''[H]^n\).

It goes without saying that if \(X \rightarrow (Y)_k^n\) and \(X \leftrightarrow Z\) then \(Z \rightarrow (Y)_k^n\) as well. Similarly, if \(W \leftrightarrow Y\) then it follows that \(X \rightarrow (W)_k^n\). One has similar monotonicity properties regarding the negative relations.

Finally, the subscript \(k\) in the arrow notation is presently chosen to be finite but the subscript in general need not be a finite cardinal. We discuss some infinite partitions into \(\omega\) or \(\omega_1\) pieces but as we shall soon see, in partition calculus for topological spaces we are often concerned simply with the case \(k = 2\). Also notice that in this case the square and the round bracket notations will coincide.

**Simple facts about the topological arrow notation**

1. In Partition Calculus a superscript \(n = 1\) signals an easy problem the solution to which is a pigeon-hole argument together with some cofinality considerations. In partitioning topological spaces, however, \(n = 1\) is often a problem in topology that will be, as we shall soon see, often a difficult one.

2. A negative result can be extended to one with a larger superscript. Suppose for example that \(X\) and \(Y\) are arbitrary sets with \(|Y| > 2\). Then \(X \not\rightarrow (Y)_2^1\) implies \(X \not\rightarrow (Y, 3)^2\). To see this we take a partition \(f : X \rightarrow \{0, 1\}\) witnessing the negative relation \(X \not\rightarrow (Y)_2^1\), and define the partition \(g : [X]^2 \rightarrow \{0, 1\}\) by...
$g(x, y) = 0$ iff $f(x) = f(y)$ witnesses the negative relation $X \not
rightarrow (Y, 3)^2$. This argument is generalized (see lemma 1.2) to extend negative relations of any dimension to any higher dimension.

3. There are negative partition relations from the ordinary partition calculus for $n > 1$ that one can use to draw conclusions in the area of partition relations for topological spaces. One such result is the following:(originally due to K.Gödel, (see [Ka]))

$$2^\omega \not
rightarrow (3)^\omega.$$  

The proof for this relation is simple: Consider $X = \omega^2$, as the set of all branches of the binary tree of height $\omega$, and let $D$ denote the collection of all the nodes in such a tree. For any two elements $f, g \in X$ let $d(f, g)$ denote the separating node of $f$ and $g$. Of course size of $X$ is $2^{\aleph_0}$ and size of $D$ is $\omega$. Therefore $d$ is a partition of $X$ into $\omega$ many pieces, and since no three elements of $X$ can share the same branching node, $d$-homogeneous sets are of size at most 2. This result clearly restricts the possible positive partition relation for "small" topological space $X$ and infinite subscripts, in the case of $n = 2$. Such negative results are expounded, in partition calculus, to hold for $n > 2$ by using some stepping-up lemma, as discussed before (for more on these see section 1.1).

4. The ordinary Stepping-up lemma were extended by Komjath and Weiss to other parameters of the arrow formula. They proved: If a first countable space $X$ satisfies $X \rightarrow (\omega + 1)_\omega^1$ then for any countable ordinal $\alpha$ we have $X \rightarrow (\alpha)_\omega^1$, and in the case when character of space $X$ is $\aleph_1$ the relation $X \rightarrow (\omega^2 + 1)_\omega^1$, is independent [KW].

The main purpose of this exposition is to discuss the topological arrow relation $X \rightarrow (Y)^k$, where $X$ and $Y$ belong to the following list:

- the ordinal spaces,

- the set of rationals with the usual topology $\mathbb{Q}$,
- the Cantor Cubes $2^\kappa$,
- the $\Sigma$-products $\Sigma_\kappa\{0,1\}$ hereafter denoted by $\Sigma_\kappa$,
- the Baire space $B(\omega_1)$,
- the Čech-Stone compactification of the natural numbers $\beta\omega$,
- the sequences convergent in the sense of some arbitrary ultrafilter $\omega \cup \{p\}$, and
- the one point compactification of discrete spaces, $A(\kappa)$.

The simplicity or the universality of these spaces are the reasons for studying these spaces. There are consistency results that assure a negative relation for all topological spaces $X$, which shed light on the project we are just about to undertake. One such result announces that for $n > 2$ the positive relation $X \rightarrow (Y)_2^n$ implies that the space $Y$ must be scattered [HJW]. Another result, for the case $n > 3$ maintains that the positive relation $X \rightarrow (Y)_2^n$ implies that the space $Y$ must be discrete [Wel]. On the other hand there is a consistency result that ensures the existence of a topological space $X$ of size $\aleph_3$ that satisfies $X \rightarrow (\text{non-discrete})_\omega^3$ [JS]. The impact of these results on our project is obvious; here is a list of negative relations for any topological space $X$ and any $n > 3$ that we consistently have

1. $X \not\rightarrow (\omega + 1)_2^n$,
2. $X \not\rightarrow (\omega \cup \{p\})_2^n$,
3. $X \not\rightarrow (\mathbb{Q})_2^3$,
4. $X \not\rightarrow (\beta\omega)_2^3$, Also, lemma 2.1 of [HJW] implies
5. $X \not\rightarrow (\mathbb{Q})_2^3$.

And of course for the spaces $Y$ that embed $\omega + 1$, $\omega \cup \{p\}$, $\mathbb{Q}$ and $\beta\omega$ we obviously have $X \not\rightarrow (Y)_2^n$.
Existing methods in topological partition calculus for \( n = 1 \)

The earliest result in this area is, perhaps a result of Bernstein (1908):

There is a partition of the reals into two pieces neither piece contains a copy of the Cantor set.

The proof of this result is based on the following general scheme. Given topological spaces \( X \) and \( Y \), let \( \{ Y_\alpha : \alpha \in \kappa \} \) be an enumeration of all the possible homeomorphic copies of \( Y \) inside \( X \). If for all \( \alpha \in \kappa \) we have \( |Y_\alpha| \geq \kappa \) then \( X \not\rightarrow (Y)_2 \) follows easily. To construct a partition \( f : X \rightarrow \{0, 1\} \) witnessing this relation one chooses a pair \( \{x^0_\alpha, x^1_\alpha\} \) from \( Y_\alpha \setminus \bigcup_{\beta < \alpha} \{x^0_\beta, x^1_\beta\} \) and let \( f(x^i_\alpha) = i \). Then, one extends \( f \) arbitrarily over \( X \). To adopt this scheme to \( X = 2^\omega \) and \( Y = 2^\omega \) (see section 2.1 for definitions) we observe that since \( 2^\omega \) is separable every copy of \( Y \) in \( X \) is determined by a countable set. There are continuum many countable subsets of \( X = 2^\omega \) hence there are at most continuum many homeomorphic copies of the Cantor set. The cardinality condition is now satisfied. This argument was generalized in different ways (see [Ma], [We3] and [BSS]) to establish the consistency of \( X \not\rightarrow (2^\omega)_2 \) with certain size restrictions on the topological space \( X \).

In studying the relation \( \kappa \rightarrow (\omega_1)_2 \) one has access to some parts of each copy of \( \omega_1 \) in \( \kappa \) by using a combinatorial principle like \( \diamondsuit \) or \( \square_\kappa \), (see chapter 5 for an example of such a situation.) These principles guarantee the existence of sequences that approximate some subsets of \( \kappa \) that could be a copy of \( \omega_1 \). To define a partition of \( \kappa \) that admits no homogeneous sets of type \( \omega_1 \) one will have to construct the partition based on the information given at that stage by the combinatorial principle. One of the first instances of such a construction is due to K. Prikry and R. Solovay ([PS]) who proved \( \kappa \not\rightarrow (\omega_1)_2 \) for every ordinal \( \kappa \) using a variation of the combinatorial principle \( \square_\kappa \).

Another technique for proving a negative partition relation \( X \not\rightarrow (Y)_2 \) is to stratify the topological space \( X \) according to the specifications of the space \( Y \) so that any copy of the space \( Y \) has a known place in the stratified \( X \). Then we define \( f : X \rightarrow \{0, 1\} \), in a "geometric way" to preclude the possibility of any homogeneous copy of \( Y \) in \( X \) (for a simple example, see section 1.2.) This technique was used
in [CKS] to establish $\omega^* \not\rightarrow (\omega \cup \{p\})$ (for definition of $\omega \cup \{p\}$ and $\omega^*$ see section 1.6.) Also, this technique was used in [KW] to establish, assuming $\Diamond$, that for any topological space $X$ of size $\aleph_1$, we have $X \not\rightarrow (\omega_1)^2$.

Another technique, used in [HJW], is to inductively define a partition of a space $X$ starting from a very small subset $X_0$ of $X$ for which a negative relation exists, and trying to carefully extend the existing partition to larger and larger subspaces in such a way that no copy of the space $Y$ can be homogeneous for the partition constructed up to that stage of the iteration. Again to manage the limit stages of the iteration some assumption about the cardinal arithmetic has to be made.

Perhaps the fastest way to produce a negative partition relation is to use a *generic set* given by a forcing extension. If a generic set is used in defining a partition of certain space $X$, with some care this partition will split all the copies of a certain specified space $Y$ inside $X$. This technique is demonstrated in chapter 4 in proving the consistency of $\Sigma_\kappa \not\rightarrow (\omega_1)^2$ and $2^{\omega_1} \not\rightarrow (\omega_1)^2$. This method was used by W. Weiss [We4] to prove the consistency of $X \not\rightarrow (2^{\omega_1})^2$ for a metric space $X$ of a fixed size.

In the positive direction, the sequences given by the combinatorial principle $\Diamond$ can be used to approximate all the the partitions of the space $X$, and then to try to assign to each partition a homogeneous copy of the space $Y$ (for a sample of this construction see 5.1).

Another collection of axioms have been useful too. For example, the *Semi-Proper Forcing Axiom* (SPFA) (see 3.4) guarantees the existence of elementary submodels with very useful properties. Similar axioms such as *Reflection Principle*, (RP) (see 3.2) and Chang’s Conjecture, (see 3.5) guarantee the existence of continuous chains of elementary submodels extending a given elementary submodel. Such chains are often used to create a continuous collection of ordinals (hence homeomorphic copy of some ordinals) or to “smooth” a given collection of sets or functions (see 3.5). These techniques are used in chapter 3 to establish $2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^1$, as well as $2^{\omega_1} \rightarrow (\omega_1 + 1, \omega_1)^1$.

In chapter 1 we introduce the spaces for which we intend to study the topological
arrow notation. In this chapter we also give a survey of the existing topological relations, and use some well-known results from ordinary partition calculus to draw conclusions in the case \( n > 1 \). We also quote several classical results from general topology and use them to establish relations for \( n = 1 \).

Chapter 2 introduces the Cantor Cubes and establishes the positive relations \( 2^\kappa \rightarrow (2^\kappa, \mathbb{Q})^{1}_{\frac{1}{2}} \), and \( 2^{\omega_1} \rightarrow (2^{\omega_1}, (\mathbb{Q})_{\omega_1})^{1} \).

Chapter 3 introduces the method of elementary submodels and discusses the reflection principle, Chang's Conjecture and SPFA. A lemma of S. Todorcevic's appears and is used to improve a result of S. Shelah to \( 2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^{1} \). The strength of a positive relation for \( \Sigma \)-products is examined and is seen to have a large cardinal nature.

Chapter 4 contains a forcing argument establishing some negative partition relation about certain spaces \( X \) of size \( \aleph_2 \). This forcing argument is iterated to prove similar result about the space \( 2^{\omega_1} \).

Chapter 5 demonstrate a number of different principles and assumptions to bring three different arguments in establishing a positive relation \( 2^c \rightarrow (\omega \cup \{p\})^{1}_\omega \).

A chart of the partition relations for topological spaces summerizes some existing results together with the new ones, (see page 11.)
0.1.1 A list of the topological partition relations

1. \( \beta \omega \not\rightarrow (\omega + 1)^1 \): see 1.6.
2. \( \omega^* \not\rightarrow (\omega \cup \{p\})^1 \): see 1.6.
3. \( \beta \omega \not\rightarrow (2^\lambda)^1 \): see 1.
4. \( \beta \omega \not\rightarrow (\mathbb{Q})^1 \): see 1.
5. \( \beta \omega \not\rightarrow (A(\lambda))^1 \): see 1.7.
6. \( \omega_1 \rightarrow (\omega_1, (\alpha)\omega)^1, \alpha < \omega_1 \): see 1.2.
7. \( \kappa \not\rightarrow (\omega_1 + 1, \omega + 1) \): see 1.2.
8. \( \square \kappa : \kappa \not\rightarrow (\omega_1)^{1/2} \): see 1.2.
9. SPFA: \( \omega_2 \rightarrow (\omega_1)^{1/2} \): see 1.2.
10. \( \kappa \not\rightarrow (\mathbb{Q})^1 \): see 1.3.
11. \( \kappa \not\rightarrow (2^\omega)^{1/2} \): see 1.3.
12. \( \kappa \not\rightarrow (\omega \cup \{p\})^1 \): see 1.6.
13. \( \kappa \not\rightarrow (\beta \omega)^{1/2} \): see 12.
14. \( \kappa \not\rightarrow (A(\omega_1))^1 \): see 1.7.
15. \( 2^\kappa \not\rightarrow (\text{top } \kappa^+)^{1/2} \): see 2.1.
16. \( 2^\kappa \rightarrow (2^\kappa, \mathbb{Q})^1 \): see 2.3.
17. \( 2^\kappa \rightarrow (\alpha, \mathbb{Q})^1, \alpha < \kappa^+ \): see 16 and 2.1.
18. \( 2^\kappa \rightarrow (\alpha, \beta)^1, \alpha < \kappa^+, \beta < \omega_1 \): see 16 and the fact that any countable ordinal is embeddable in \( \mathbb{Q} \).
19. SPFA: \( 2^{\omega_1} \rightarrow (\omega_1 + 1, \omega_1)^1 \): see 3.4.
20. SPFA: \( 2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^{1/2} \): see 3.4.
21. CON \( 2^{\omega_1} \not\rightarrow (\omega_1)^{1/2} \): see 4.2.
22. \( 2^\omega \rightarrow (2^\omega, (\mathbb{Q})\omega)^1 \): see 1.3.
23. \( 2^\omega \not\rightarrow (2^\omega)^{1/2} \): see Introduction.
24. \( 2^{\omega_1} \rightarrow (2^{\omega_1}, \mathbb{Q}_{\omega_1})^1 \): see 2.4.
25. \( 2^\omega \rightarrow (\omega \cup \{p\})^{1/2} \): see 5.4.
26. \( 2^\omega \not\rightarrow (\omega \cup \{p\})^1 \): see 1.6.
27. $2^\kappa \not\rightarrow (\beta \omega[^1_2])$: see 1.6.
28. $2^{2^\kappa} \rightarrow (\beta \kappa, Q)^1$: see 2.3.
29. $2^\kappa \rightarrow (A(\kappa))[^1_2]_{cf(\kappa)}$: see 1.7.

30. SPFA: $\Sigma_{\omega_2} \rightarrow (\omega_1[^1_2])$: see 3.0.
31. $\Sigma_{\omega_2} \not\rightarrow (\omega_1 + 1)^1$: see 1.5.
32. CON $\Sigma_\kappa \not\rightarrow (\omega_1[^1_2])$: see 4.1.
33. $\Sigma_{\omega_2} \rightarrow (\omega_1, (\alpha)_\omega)^1 \alpha < \omega_1$: see 1.5.
34. $\Sigma_\kappa \not\rightarrow (\omega_1[^1_2])$: see 3.6.
35. $\Sigma_{\omega_2} \rightarrow (\omega_1[^1_2] \Rightarrow CC)$: see 3.5.
36. $\Sigma_{\omega_2} \rightarrow (2^\omega, Q_\omega)^1$: see 1.5.
37. CON $\Sigma_\kappa \not\rightarrow (2^\omega[^1_2])$: see 4.1.
38. CON $\Sigma_\kappa \not\rightarrow (2^\omega, \omega_1)^1$: see 4.1.
39. $\Sigma_{\omega_2} \rightarrow (2^\omega, (Q)_\omega)^1$: see 1.5.
40. $\Sigma_\kappa \not\rightarrow (\omega \cup \{p\})[^1_1]$: see 1.5.
41. $\Sigma_\kappa \not\rightarrow (\beta \omega[^1_1])$: see 40.
42. $\Sigma_\kappa \not\rightarrow (A(\omega_1))[^1_1]$: see 1.5.
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\frac{1}{1^2}((\eta)^2(\eta)\forall) & \frac{1}{1}(m_3) & \frac{1}{1}(m) & \frac{1}{1}(\varphi) & \frac{1}{1}(m_2) & \frac{1}{1}(m) \\
\hline
\frac{1}{1}(m_2) & \frac{1}{1}(m_2) & \frac{1}{1}(m_2) & \frac{1}{1}(m_2) & \frac{1}{1}(m_2) & \frac{1}{1}(m_2) \\
\hline
\frac{1}{1}(m) & \frac{1}{1}(m) & \frac{1}{1}(m) & \frac{1}{1}(m) & \frac{1}{1}(m) & \frac{1}{1}(m) \\
\hline
\frac{1}{1}(\psi) & \frac{1}{1}(\varphi) & \frac{1}{1}(\varphi) & \frac{1}{1}(\varphi) & \frac{1}{1}(\varphi) & \frac{1}{1}(\varphi) \\
\hline
\frac{1}{1}(\eta) & \frac{1}{1}(\eta) & \frac{1}{1}(\eta) & \frac{1}{1}(\eta) & \frac{1}{1}(\eta) & \frac{1}{1}(\eta) \\
\hline
\hline
m \leq \eta & \eta_1 & \eta_1 & \eta_1 & \eta_1 & \eta_1 \\
\hline
\end{array}
\]

Note: The table represents a chart of the topological partition relations \( \eta^\top \) of the ordinal \( (\Lambda \setminus 1)^\top \).
Chapter 1

Definitions and simple corollaries

In this chapter we will introduce some basic definitions and discuss some partition relations involving the ordinal spaces and scattered spaces. We will also introduce the spaces under consideration and extend some of the earlier results to these spaces. The relations in this chapter are all simple corollaries of definitions, existing relations in ordinary partition calculus and some well known theorems in general topology.

1.1 Stepping-up methods

In this section we prove a basic lemma which enables us to extend any negative partition relation to a negative relation for larger superscripts. Thereby many trivial and/or existing negative partition relations from partition calculus or topological partition calculus will find their topological counterpart for larger superscripts. Such a lemma is called a stepping-up lemma. We first quote one such lemma from ordinary partition calculus.

Lemma 1 (lemma 24.1 (5) [EHMR]) Let $n \geq 2$ be an integer and $\mu$ be an infinite cardinal and $\lambda$ be an arbitrary cardinal, infinite or finite. Then $\kappa \not\rightarrow (\lambda)^n_\mu$ implies $2^\kappa \not\rightarrow (\lambda+1)^{n+1}_\mu$. 
Using this stepping up lemma, one can, for example, step-up the relation $2^\omega \not\rightarrow (3)_\omega^3$ given above as follows: for each space $X$ of cardinality at most $2^{\aleph_0}$

$$X \not\rightarrow (4)_\omega^3.$$  

We will prove another lemma that allows us also to step-up from $n = 1$ to any larger $n$ but keeping the space $X$ and $Y$ fixed.

**Lemma 2** Let $X$ and $Y$ be infinite spaces, and let $m, n$ and $k$ be positive integers such that $m < n$. Then $X \not\rightarrow (Y)_k^m$ implies $X \not\rightarrow (Y,r)^n$, where $r = r(m,n,k)$ is the Ramsey number associated with the triple $(m,n,k)$, that is the least positive integer that satisfies $r \rightarrow (n)_k^m$.

**proof:** Choose a partition $f : [X]^m \rightarrow k$ witnessing $X \not\rightarrow (Y)_k^m$. We define a partition $g : [X]^n \rightarrow \{0,1\}$ as follows:

$$g(\{x_1, x_2, \ldots x_n\}) = 0 \iff |f''([\{x_1, x_2, \ldots, x_n\}]^m)| = 1.$$  

Observe that there can not exist a 0-homogeneous subset of $X$ for $g$, homeomorphic to $Y$, as such a space must be homogeneous for $f$. To show there is no 1-homogeneous subset of $X$ for $g$ of size $r$, we assume on the contrary that there is a subset $A$ of $X$ of size $r$ with $g''([A]^n) = \{1\}$. By the finite Ramsey’s theorem there is a set $\{x_1, x_2, \ldots, x_n\} \subset A$ which is homogeneous for $f$. This would imply that $g(\{x_1, x_2, \ldots, x_n\}) = 0$; a contradiction that proves that lemma. 

The instance of lemma 2 that is used very frequently is for $n = 2$ and $m = 1$:

$$X \not\rightarrow (Y)_k^1 \text{ implies } X \not\rightarrow (Y,k + 1)_k^2.$$  

The following variation of lemma 2 allows the mixed relations to step up:

**Lemma 3** For topological spaces $X, Y$ and $Z$ we have that

$$X \not\rightarrow (Y,Z)_2^1 \text{ implies } X \not\rightarrow (Y,Z,3)_3^2.$$
proof: The proof is similar to the proof of lemma 2. Given a partition \( f \) of \( X \) into two pieces, we define a partition \( g \) of \([X]^2\) into three pieces as follows:
for \( i = 0, 1 \) \( g(\{x, y\}) = i \) iff \( f(x) = f(y) = i \), and \( g(\{x, y\}) = 2 \) otherwise.

To extend the negative partition relations for infinitely many colors the following argument is used in [Wel]:

**Lemma 4** For infinite topological spaces \( X \) and \( Y \), the negative relation \( X \notightarrow (Y)_{\omega}^n \) can be stepped-up to \( X \notightarrow (Y)_{\omega}^n \) for any \( n > m \).

proof: Let \( k = C_m^n \), the number of the \( m \)-element subsets of a set of size \( n \). and let \( \{p_1, p_2, ... p_k\} \) be the first \( k \) prime numbers. Define a partition \( g : [X]^n \rightarrow \omega \) as follows: for a given set \( \{x_1, x_2, ... x_n\} \) let \( \{a_1, a_2, ..., a_k\} \) denote the \( n \)-element subsets of \( \{x_1, x_2, ... x_n\} \),

\[
g(\{x_1, x_2, ... x_n\}) = \prod_{1 \leq i \leq k} p_i^{f(a_i)}.
\]

\(\square\)

### 1.2 Partition relations for uncountable ordinal spaces

Since a considerable amount of work in studying the the partition relations for countable ordinal spaces has been done (see [Bal]), here we concentrate on the ordinal space \( \omega_1 \), the first uncountable ordinal.

An unbounded set closed in the ordered topology, (hereafter a *club*) subset of \( \omega_1 \) with the subspace topology is homeomorphic to \( \omega_1 \); any strictly increasing continuous function mapping \( \omega_1 \) onto a *club* would be a homeomorphism. A *stationary* subset of \( \omega_1 \) is a subset that intersects every *club* subset of \( \omega_1 \). Hence a *non-stationary* subset of \( \omega_1 \) is a set which is contained in the complement of a club. As a consequence we immediately have:

\[
\omega_1 \rightarrow (\text{closed and unbounded, stationary}).
\]
A countable intersection of clubs in $\omega_1$ is a club itself, hence a countable union of non-stationary subsets of $\omega_1$ is non-stationary. This gives an improvement:

$$\omega_1 \rightarrow (\text{closed and unbounded, (stationary)}_\omega)^1.$$ 

Considering a result of Friedman [Fr], which says a stationary subset of $\omega_1$ contains a topological copy of any countable ordinals, we arrive at:

$$\omega_1 \rightarrow (\omega_1, (\alpha)_\omega)^1 \text{ for every countable ordinal } \alpha \in \omega_1.$$ 

It follows from the ordinary Partition Calculus that

$$\omega_1 \nrightarrow (3)^2_\omega.$$ 

If we reduce the number of pieces in the partitions, although in the ordinary partition calculus for ordinals we have

$$\omega_1 \rightarrow (\omega_1, \alpha)^2$$

(see [To5]), we still don't get a comparable topological result. Indeed a partition $\omega_1 = S_0 \cup S_1$ into stationary sets establishes

$$\omega_1 \nrightarrow (\omega_1)^1_2,$$

which, using lemma (2), gives

$$\omega_1 \nrightarrow (\omega_1, 3)^2.$$ 

This gives us another striking difference between the topological and the ordinal version of partition calculus. Indeed, it is still open if $\omega_1 \rightarrow (\alpha)^2_2$ for topological copy of the arbitrary countable ordinal $\alpha$. 

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To find homogeneous copies of $\omega_1 + 1$ is much harder. Indeed for each ordinal $\kappa$

$$\kappa \not\rightarrow (\omega_1 + 1, \omega + 1)^1.$$  

This is witnessed by the following partition $f : \kappa \rightarrow 2$, defined by

$$f(\alpha) = 0 \text{ iff } \alpha \text{ is limit of countable cofinality.}$$

Clearly $\omega_1 + 1 \not\rightarrow f^{-1}(\{0\})$, and no continuous function can map $\omega + 1$ into $f^{-1}(\{1\})$ with the topology induced from $\kappa$. This is extended by lemma 3 to

$$\kappa \not\rightarrow (\omega_1 + 1, \omega + 1, 3)^2.$$  

Beyond this point we enter into the realm of independence results.

A result of Solovay and Prikry [SPI] uses a version of $\Box_\kappa$ to provide a partition of any ordinal $\kappa$ witnessing

$$\kappa \not\rightarrow (\omega_1)^1.$$  

On the other hand S.Shelah [Sh1] uses SPFA to establish

$$\omega_2 \rightarrow (\omega_1)^1.$$  

Applying lemma 2 to Prikry-Solovay result gives us the consistency of

$$\kappa \not\rightarrow (\omega_1, 3)^2,$$

for all ordinals $\kappa$. Recall that here there is no point in considering infinitely many colors unless we assume CH, since if CH fails, we have $\omega_2 \not\rightarrow (3)_\omega^2$. Also, in general we have $\omega_2 \not\rightarrow (4)_\omega^3$.  

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1.3 Baire Category arguments

Let's first remind ourselves of the notions dense in itself and scattered: A subset $A$ of a topological space $X$ is said to be dense in itself if every point of it is a limit point. It is said to be scattered if it contains no non-empty dense-in-itself subset. Equivalently $A$ is scattered in $X$ if the complement of $\overline{A}$, denoted by $(\overline{A})^c$, is dense in $X$. (see [En])

It is a basic consequence of these definitions that the intersection of two open dense subsets of a topological space is open dense. This implies that a dense in itself space cannot be the union of two scattered subspace. In terms of the topological arrow notation, this leads to the positive relation

$$\text{dense in itself} \rightarrow (\text{dense in itself})_k.$$

To extend this relation for infinite subscript we recall the

Baire Category Theorem: In a complete metric space a countable union of no-where dense sets is a co-dense set. (Of course this theorem has a more general statement, but we intend to use it here only in this special form, for more on this see [En].) It follows that

$$\text{dense in itself complete metric} \rightarrow (\text{dense in itself})_\omega.$$

An example of a complete metric space in our work is the space $2^\omega$, the Cantor Cube of countable weight, also known as the Cantor Set. (See 2.1 for definitions and the equivalence of the the Cantor Set and $2^\omega$.)

Another well known theorem that has a flavor of partition calculus and is used in the subsequent arguments is

Cantor-Bendixson theorem: Any topological space can be represented as the union of two disjoint sets, of which one is dense in itself and closed while the other is scattered.

We shall also use the following well-known scheme in constructing copies of the Cantor set in an uncountable $G_\delta$ subset of $2^\omega$:
**Lemma 5** Every uncountable $G_\delta$ subset of a complete separable metric space contains a copy of the Cantor set $2^\omega$.

**proof:** Notice that such a space $X$ is hereditarily Lindeloff. Assume $G = \bigcap_{n \in \omega} V_n$ is an uncountable $G_\delta$ set in $X$. Consider the collection of open sets

$$\mathcal{W} = \{W : |W \cap G| \leq \aleph_0\}.$$  

By the Hereditary Lindeloff property of $X$ one can find a countable $\mathcal{W}_0 \subseteq \mathcal{W}$ such that $\bigcup \mathcal{W}_0 = \bigcup \mathcal{W}$. Let $W = \bigcup \mathcal{W}$, and observe that $W \cap G$ is countable since $W \cap G = (\bigcup \mathcal{W}_0) \cap G$. Consider $F = G \setminus W$. Observe that $F$ is uncountable, and for any open nbd $U$ of a point $x \in F$, $U \cap G$ is uncountable.

Choose points $x_{<0>_0}, x_{<1>_0} \in F \cap V_1$ and in $V_1$ separate the points with open $U_{<0>}$ and $U_{<1>}$ of radius smaller than 1, with $\overline{U_{<0>}} \cap \overline{U_{<1>}} = \emptyset$. Next we choose new points $x_{<0>_1}, x_{<0>_1} \in U_{<0>} \cap F$ and separate them with balls of radius smaller than $1/2$. Continuing this process we construct a collection of open sets $\{U_s : s \in \omega^2\}$ to satisfy:

1. $U_s \subseteq V_n$ and $\text{diam}(U_s) < 1/n$ whenever $\text{dom}(s) \supseteq n$.

2. $\overline{U_s} \subseteq U_t$ whenever $s$ extends $t$, and

3. for all $s$ in $\omega^2$, $\overline{U_{s-0}} \cap \overline{U_{s-1}} = \emptyset$.

Notice that for each $f \in 2^\omega$, the intersection $\bigcap_{n \in \omega} \overline{U_{f|n}}$ is a singleton in $G$, and denote it by $\{x_f\}$.

**claim:** $X = \{x_f : f \in \omega^2\}$ is homeomorphic to $2^\omega$.

Let $H$ be the assignment $f \mapsto x_f$ clearly a bijection. We show that $H$ is open and continuous. To show this note that for $s \in \omega^2$, $H''\{f \in \omega^2 : f \supseteq s\} = U_s \cap X$. \(\square\)

And the following lemma will be used in constructing copies of the rationals $\mathbb{Q}$.

**Lemma 6** Any countable, first countable, dense in itself subspace of a regular space is homeomorphic to $\mathbb{Q}$.  

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proof: The proof follows from the following well-known results (see [En]):

1. a countable first countable space is second countable,

2. a second countable space is metrizable if and only if it is regular (Urysohn), and

3. any countable dense in itself metric space is homeomorphic to \( \mathbb{Q} \) (Sierpinski).

The following is a list of corollaries of the results mentioned above.

**Corollary 1**

1. If \( X \) is dense in itself, then \( X \rightarrow (\text{dense in itself})^1_2 \). Moreover, if \( X \) is dense in itself and Baire, then

\[
X \rightarrow (a \text{ dense in itself } G_\delta \text{ set}, (\text{dense in itself})^1_\omega).
\]

2. \( \mathbb{Q} \rightarrow (\mathbb{Q})^1_2 \).

3. If \( X \) is uncountable separable complete metric space,

then \( X \rightarrow (a \text{ dense in itself } G_\delta \text{ set }, (\mathbb{Q})^1_\omega) \).

4. Any uncountable \( G_\delta \) subset of \( 2^\omega \) contains a copy of \( 2^\omega \). Hence,

\[
2^\omega \rightarrow (\text{top } 2^\omega, \text{size } \aleph_1)^1_1.
\]

5. \( 2^\omega \rightarrow (2^\omega, (\mathbb{Q})^1_\omega) \).

proof: To prove the second part of (1) suppose \( \{ A_n : n \in \omega \} \) is a partition of the space \( X \). By Cantor-Bendixson theorem we partition each \( A_n \) into a scattered and a dense in itself. If no \( A_n \) has a dense in itself component, then \( (A_n)^c \) is open dense, and \( \bigcap_{1 \leq n < \omega} (A_n)^c \) is a \( G_\delta \) set and dense in itself. (2) and (3) follow from (1) and lemma (6), (4) follows from lemma (5), and (5) follows from (3) and lemma (5).
1.4 $\Sigma$-products

The $\Sigma_\kappa$-product of the space $\{0, 1\}$ is a subset of the space $\{0, 1\}^\kappa$ defined as follows:

$$\Sigma_\kappa = \Sigma_\kappa\{0, 1\} = \{ f \in \{0, 1\}^\kappa : |\text{supp}(f)| \leq \aleph_0 \},$$

where, $\text{supp}(f) = \{ \alpha < \kappa : f(\alpha) \neq 0 \}$.

**Remark:** The space $\Sigma_\kappa\{0, 1\}$ is Frechet space [En; 3.10.D], that is, any subset contains a sequence converging to any of its limit points. This implies that the ordinal space $\omega_1 + 1$ does not embed in $\Sigma_\kappa$, that is:

$$\Sigma_\kappa \not\rightarrow (\omega_1 + 1)_1^1.$$ Notice also that $2^\omega$ naturally embeds into $\Sigma_\kappa$ by restricting the product to $\omega$. Similarly $\omega_1$ embeds into $\Sigma_\kappa$ if $\kappa$ is uncountable. Therefore, the following positive relations hold :

(see sections 1.2 and 1.4)

$$\Sigma_\kappa \rightarrow (2^\omega, (Q)_\omega)_1^1, \text{ and } \Sigma_\kappa \rightarrow (\omega_1, (\alpha)_\omega)_1^1, \alpha < \omega_1.$$ To improve these results to $\Sigma_\kappa \rightarrow (\omega_1)_2^1$, $\Sigma_\kappa \rightarrow (2^\omega)_2^1$ or $\Sigma_\kappa \rightarrow (2^\omega, \omega_1)_1^1$ would not be possible in ZFC as the corollary 4.1 announces the consistency with ZFC of the following relations:

$$\Sigma_\kappa \not\rightarrow (\omega_1)_2^1, \quad \Sigma_\kappa \not\rightarrow (2^\omega)_2^1 \text{ and } \Sigma_\kappa \not\rightarrow (2^\omega, \omega_1)_1^1.$$ On the other hand we shall see (section 3-6) that in ZFC, we do have the restriction

$$\Sigma_{\omega_2} \not\rightarrow (\omega_1)_3^1.$$ Using lemma 2 this result extends to

$$\Sigma_{\omega_2} \not\rightarrow (\omega_1, 4)_2^1.$$
We shall also establish that

\[ \Sigma_{\omega_2} \rightarrow (\omega_1)^1_2 \] implies Chang's Conjecture,

thereby showing that the positive partition relation has a large cardinal consistency strength.

1.5 \( \beta \omega \)

The Čech-Stone compactification of the discrete space \( \omega \), is characterized as the collection of all the ultrafilters on \( \omega \). The space \( \omega \) is embedded in \( \beta \omega \) via the principal ultrafilters. For properties of the space \( \beta \omega \) see the exposition by W. Rudin [Ru]. on each ordinal \( \kappa \) one can show (see [JHS]) that:

\[ 2^\kappa \not\rightarrow (\beta \omega)^1_2. \]

Other observation about the space \( \beta \omega \) is that it embeds no copy of a convergent sequence, \( \omega + 1 \), that is

\[ \beta \omega \not\rightarrow (\omega + 1)^1_1. \]

If we remove the subspace \( \omega \) from \( \beta \omega \) we would be left with Čech-Stone remainder \( \omega^* = \beta \omega \setminus \omega \). It is known (see [Ru]) that elements of the space \( \omega^* \) have uncountable character less than or equal to the continuum, \( c \).

To motivate our next definition, let us recall the definition of the "Frechet filter " on \( \mathbb{N} \):

\[ \mathcal{F}_N = \{ A \subseteq \mathbb{N} : A \text{ is cofinite} \}. \]

The convergence of a sequence \( S \) to a point \( s \) can be rephrased this way: \( S \) converges to \( s \) if for any neighborhood \( U \) of \( s \) we have \( U \cap S \in \mathcal{F}_S \). In the sequel we identify a sequence with its index set. Rather imprecise, the above statement would then become: \( S \) converges to \( s \) if for any neighborhood \( U \) of \( s \) we have \( U \cap S \in \mathcal{F} \), taking
$\mathcal{F}$ to be the Frechet filter on $\mathbb{N}$. This idea can be generalized for an ultrafilter of $\omega$ that is an element of $\omega^*$. For a $p \in \omega^*$ we define $\omega \cup \{p\}$ to be a countable discrete set which converges to a point in the sense of an ultrafilter $p$. That is for any neighborhood $U$ of the limit point we have $U \cap \omega \in p$.

If we choose a point $p \in \omega^*$ since the topology of $\omega \cup \{p\}$ is inherited from $\beta\omega$ then also in the space $\omega \cup \{p\}$ the point $p$ has uncountable character. This fact implies that $\omega \cup \{p\}$ does not embed in the space $2^\omega$. Hence,

$$2^\omega \not\hookrightarrow (\omega \cup \{p\})^1_1.$$ 

Since the closure of every countable subset of a $\Sigma$-product $\Sigma_\kappa$ is second countable we can not have a copy of $\omega \cup \{p\}$ embedded in $\Sigma_\kappa$. Hence:

$$\Sigma_\kappa \not\hookrightarrow (\omega \cup \{p\})^1_1.$$ 

Neither could we have $\omega \cup \{p\}$ embedded in an ordinal space $\kappa$. This is because in ordinal spaces a countable set can have limit point of countable cofinality (hence countable character.) It is established in [CKS] that

$$\omega^* \not\hookrightarrow (\omega \cup \{p\})^2_2.$$ 

From this result one concludes that for any countably compact space $Y$,

$$\omega^* \not\hookrightarrow (Y)^1_2.$$ 

This is so because such a $Y$ must contain a copy of $\omega \cup \{p\}$.

It is also established in [CKS] that for any space $Y$ of size $2^\omega$:

$$\omega^* \not\hookrightarrow (Y)^1_2.$$
1.6 \( A(\kappa) \) and \( B(\omega_1) \)

Recall that \( A(\kappa) \), the one-point compactification of a discrete space of size \( \kappa \) is defined by adding one extra point to the discrete space of size \( \kappa \) and as neighborhood basis at that extra point we consider all the co-finite subsets of that set of size \( \kappa \). As such, \( A(\omega) \) is just a converging sequence. Also, for each \( \lambda \leq \kappa \) we have \( A(\lambda) \leftrightarrow A(\kappa) \). Therefore, \( A(\kappa) \not\leftrightarrow \beta\omega \) because \( \omega + 1 \not\leftrightarrow \beta\omega \).

It follows from the definition of \( A(\kappa) \) that the extra point of \( A(\kappa) \) has character \( \kappa \). Notice that \( A(\kappa) \not\leftrightarrow \lambda \) for any \( \kappa \) and \( \lambda \). Similarly, \( A(\omega_1) \not\leftrightarrow \Sigma_\kappa \). Finally, \( A(\kappa) \not\leftrightarrow 2^\lambda \) for each \( \lambda \leq \kappa \). However, the space \( A(\kappa) \) not only embeds in the cantor cube \( 2^\kappa \) but also we have positive partition relation about these spaces. There is a result (of Elkes, Erdos, Hajnal, based on a result of J. Baumgartner, for regular \( \kappa \) and of S.Todorcevic for singular \( \kappa \), see [We2]) as follows:

\[
2^\kappa \longrightarrow (A(\kappa))_{cf(\kappa)}^1.
\]

The space \( A(\kappa) \) is one of the simplest among the non-trivial spaces of size \( \kappa \) or of character \( \kappa \) which makes it a good candidate for positive partition relation regarding large target spaces. In this regard there is a result from [HJW] which shows that for any uncountable cardinal \( \kappa \) there is a topological space \( X \) such that

\[
X \longrightarrow (A(\kappa))_{\omega}^2.
\]

The Baire space of uncountable weight, \( B(\omega_1) \), is the countable power of the set \( \omega_1 \) with the discrete topology. We will establish in 3-4 that under SPFA we have

\[
2^{\omega_1} \longrightarrow (\omega_1 + 1, B(\omega_1))^1.
\]

This space is known to be universal for strongly zero-dimensional metrizable spaces of weight \( \omega_1 \). As such, this result can be interpreted as a result about this class of metrizable spaces.
Chapter 2

Partitioning Cantor cubes

In this section we study Cantor sets of uncountable weight and concentrate on positive results about the cube $2^{\omega_1}$. In section 2.1 basic properties of $2^{\kappa}$ are summarized, and in section 2.3 we consider partitions of $2^{\kappa}$ into finitely many pieces to establish

$$2^{\kappa} \rightarrow (2^{\kappa}, \mathbb{Q})^1.$$ 

In section 2.2 we introduce collections of subsets of $\omega_1$, and we investigate to which subspaces of $2^{\omega_1}$ they will translate. These families are then used in 2.4 to prove

$$2^{\omega_1} \rightarrow (2^{\omega_1}, (\mathbb{Q})_{\omega_1})^1$$

where $2^{\omega_1}$ is the refinement of the topology of $2^{\omega_1}$ by adding all the $G_\delta$-sets. (for more on this space see [CN].)

2.1 Cantor cubes

Consider the discrete topology on the set $\{0, 1\}$. The product topology on $\Pi_{i \in I}\{0, 1\}$, where $I$ is an index set of size $\kappa$, is what we call the Cantor cube of weight $\kappa$ and will denote by $2^{\kappa}$. Clearly if $X$ and $Y$ are two index sets of equal size, then $X^2$ and $Y^2$ with the Tychonoff product topology are homeomorphic.
We reserve $2^{\aleph_0}$ and $2^{\aleph_1}$ to denote the cardinality of the continuum and the cardinality of the power set of $\omega_1$. Instead, $2^\omega$ and $2^{\aleph_1}$ will be the Cantor cubes.

As usual, the point set of $2^\kappa$ is considered to be functions with domain $\kappa$ and range $\{0,1\}$, in other words $^\kappa 2$.

**Some facts and observations about $2^\kappa$**

1. The Cantor middle third set $C$ is homeomorphic to $2^\omega$. To see this one can identify elements of $C$ with sequences of 0's and 1's that is the branches of the binary tree $\omega 2$, This implies that $2^\omega \rightarrow \mathbb{R}$.

2. The weight and the character of $2^\kappa$ is $\kappa$. (see [Ho] for definitions and properties of cardinal functions .)

3. For each $\alpha < \kappa$ and $i \in 2$, define $B_\alpha(i) = \{ x \in 2 : x(\alpha) = i \}$. Then, the collection $\{ B_\alpha(i) : \alpha \in \omega_1, i \in 2 \}$ forms a subbase for $2^\kappa$. Since the $B_\alpha$'s are closed and open (clopen,) this topology is zero-dimensional.

4. $2^\kappa$ is universal for the zero dimensional spaces of weight $\kappa$, that is, any zero-dimensional space of weight at most $\kappa$ is embeddable in $2^\kappa$. In particular for all $\nu < \kappa$, $2^\nu \hookrightarrow 2^\kappa$.

5. Since weight of the space $\kappa^+$ is $\kappa^+$ and weight of the space $2^\kappa$ is $\kappa$ we conclude that $\kappa^+$ does not embed in $2^\kappa$. However, any ordinal $\alpha < \kappa^+$ being 0-dimensional of weight $\kappa$ is embeddable in $2^\kappa$.

6. For a regular $\kappa$, any non-empty subset $F$ of $2^\kappa$ which is an intersection of less than $\kappa$-many open sets of $2^\kappa$ contains a homeomorphic copy of $2^\kappa$: Pick a point $x \in F$ and let $\{ U_\alpha : \alpha \in \lambda < \kappa \}$ be open sets such that $F = \bigcap_{\alpha<\lambda} U_\alpha$. For each $\alpha$ choose $F_\alpha \in [\kappa]^{<\omega}$ such that $x \in \bigcap_{\gamma \in F_\alpha} B_\gamma(x(\gamma)) \subseteq U_\alpha$. Since $I = \kappa \setminus \bigcup_{\alpha<\lambda} F_\alpha$ has size $\kappa$, we have that $2^I \approx 2^\kappa$. Therefore,

$$2^\kappa \approx \bigcap_{\alpha<\lambda} \bigcap_{\gamma \in F_\alpha} B_\gamma(x(\gamma)) \subseteq \bigcup_{\alpha \in \lambda} U_\alpha.$$
7. As singletons of $2^\kappa$ are closed, item 6 above implies that

$$2^\kappa \rightarrow (\text{top } 2^\kappa, \text{ size } \kappa)^1.$$  

8. $2^{\kappa^+}$ can be partitioned into $2^\kappa$ pieces such that each piece is a $\kappa$-intersection of open sets. An example of such a partition is

$$\kappa^+ 2 = \bigcup \{A_f : f \in {}^2\kappa\}, \text{ where } A_f = \{h \in {}^{\kappa^+}2 : h|\kappa = f\}.$$  

By (2) each piece contains a copy of the $2^{\kappa^+}$. This implies

$$2^{\kappa^+} \rightarrow (\text{top } 2^{\kappa^+}, \text{ size } 2^\kappa)^1.$$  

A simple case of this is $\omega^\omega \rightarrow (\omega^\omega, 2^{\aleph_0})^1$.

2.2 Special subsets of $\mathcal{P}(\omega_1)$

We define certain collections of subsets of $\omega_1$ and use the map $\Psi : \mathcal{P}(\kappa) \rightarrow 2^\kappa$ defined by $\Psi(A) = \chi_A$, to translate them to some well known topological spaces. These collections will be used in later sections.

**Definition 1:** A $\mathcal{D}$-collection of subsets of $\kappa$ is a collection $\mathcal{D} = \{d_s : s \in {}^{<\omega}\omega\}$ of subsets of $\kappa$ such that:

1. $s \nsubseteq t$ implies $d_s \nsubseteq d_t$, and
2. $u = s \wedge t$ implies $d_s \cap d_t = d_u$.

**Definition 2:** A $\mathcal{C}$-collection is the family of all unions $\bigcup_{\alpha < \omega_1} c_{f|\alpha} \ (f \in {}^{<\omega_2})$, for some collection $T = \{c_s : s \in {}^{<\omega_2}\}$ of subsets of $\kappa$ such that:

1. $s \nsubseteq t$ implies $c_s \nsubseteq c_t$,
2. $u = s \wedge t$ implies $c_s \cap c_t = c_u$. 

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Lemma 1 The image under the map $\Psi$ of a $D$-family is homeomorphic to $\mathbb{Q}$.

proof: Let $D = \{d_s : s \in <\omega \omega \}$ and $Z = \Psi''(D)$. Clearly, $Z$ is countable and for each $s \in <\omega \omega$, $z_s = \Psi(d_s)$ is a limit point of $\{z_{s-n} : n \in \omega\}$ so that $Z$ is dense in itself. By lemma 1.4 it is enough to show that each point $z \in Z$ has a countable base.

To see this let $I = \bigcup_{s \in <\omega \omega} d_s$, and note that for every $z \in Z$ the basic open nbds of $z$ whose supports are included in the countable set $I$ form a neighborhood base of $z$ in $Z$. □

Lemma 2 The image under the map $\Psi$ of a $C$-collection is homeomorphic to $2^\omega$.

proof: Let $T = \{t_s : s \in <\omega \omega \}$ be the tree underlying the definition of a given $C$-collection, $C$, and we will prove that the map $\Phi$ defined by

$$f \mapsto \Psi\left( \bigcup_{\xi < \omega_1} t_{f|\xi} \right)$$

is a homeomorphism between $2^\omega$ and $\Psi''C$. Clearly, $\Phi$ is one-to-one. Consider a subbasic clopen set $B_\alpha(1)$. Then, $\Phi^{-1}(B_\alpha(1))$ is the set of all $g \in 2^\omega$ such that for some $\xi$, $\alpha \in t_{f|\xi}$. Note that by definition 2 this is equal to the set of all $g \in 2^\omega$ which extend $s$, where $s$ is the minimal such that $\alpha \in t_s$. Clearly, this is a clopen subset of $2^\omega$. To show that $\Phi$ is open consider an $s \in <\omega \omega 2$, of successor height and pick $\alpha \in t_s \setminus \bigcup_{r \subset s} t_r$. Then using definition 2 it is easily checked that

$$\Phi(\{f \in 2^\omega : s \subset f\}) = B_\alpha(1) \cap \Psi''C.$$

This completes the proof. □

2.3 Finite partitions of Cantor cubes

The following proposition is an scheme for producing a sequence converging to a fixed point, thereby producing a simple first countable subspace in the Cantor Cube $2^\omega$. 27
Then, we will iterate this scheme to produce a countable, dense in itself first countable subspace of the space $2^\omega$.

**Proposition 1**  
$2^\omega \to (2^\omega, \omega + 1)^1$.

**proof:** Consider a partition $\{A_n : n \in \omega\}$ of $\omega_1$ into uncountable sets. Given a partition $\{X_0, X_1\}$ of $2^\omega$, we fix $z \in X_1$ and for each $n \in \omega$ we define $Z_n$ to be the set of all $x \in 2^\omega$ such that

$$
\begin{cases}
  x(\alpha) = z(\alpha) & \text{for } \alpha \in \bigcup_{k<n} A_k \\
  x(\alpha) \neq z(\alpha) & \text{for } \alpha \in A_n \\
  x(\alpha) \text{ arbitrary} & \text{when } \alpha \in \bigcup_{k>n} A_k
\end{cases}
$$

As $|\bigcup_{k>n} A_k| = \aleph_1$, we have $Z_n \approx 2^\omega$. Assume $2^\omega \not\subseteq X_0$, hence $Z_n \cap X_1 \neq \emptyset$ for all $n$. For each $n$, we choose $z_n \in Z_n \cap X_1$, define $Z^n = \{z_k : k > n\}$ and let $Z = \{z_n : n \in \omega\}$.

Consider $B_\alpha(z) = \{x \in 2^\omega : x(\alpha) = z(\alpha)\}$. Arbitrarily fix $\alpha_i \in A_i$ for each $i \in \omega$, and define the open sets $U_n = B_{\alpha_0}(z) \cap \ldots \cap B_{\alpha_n}(z)$, and $V_n = U_{n-1} \cap B_{\alpha_n}^c(z)$.

Observe that:

1. For all $n \in \omega$, $(\forall m > n) Z_m \subseteq U_n$ and $(\forall m \leq n) Z_m \cap U_n = \emptyset$.

2. For all $n \in \omega$, $Z_n \subseteq V_n$ and $(\forall m \neq n) Z_m \cap V_n = \emptyset$. As the $V_n$'s are disjoint, $Z$ is a discrete subset of $\bigcup \{Z_n : n \in \omega\}$ in the induced topology from $2^\omega$.

3. For each $n$ and $\beta \in A_n$, $B_{\alpha_n}(z) \cap Z^n = B_{\beta}(z) \cap Z^n$. It follows that $(\forall \beta \in \omega_1)(\exists n \in \omega) U_n \cap Z^n \subseteq B_{\beta}(z) \cap Z^n$. (Indeed such $n$ is the integer with $\beta \in A_n$.) Hence, $z_n$'s converge to $z$ in the usual sense of a convergent sequence in the induced topology from $2^\omega$.

It is now apparent from the observations that $\{U_n \cap Z \cup \{z\} : n \in \omega\}$ forms a local basis at $z$, making $Z \cup \{z\}$ into a homeomorphic copy of $\omega + 1$ embedded in $X_1$.  \[\square\]

**Proposition 2**  
$2^\kappa \to (2^\kappa, \mathbb{Q})^1$.  

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proof: Let \( \{X_0, X_1\} \) be a partition of \( 2^\kappa \), and assume \( 2^\kappa \not\to X_0 \). Fix a \( z \in X_1 \) and by the method of the last proof construct a discrete collection \( \langle Z_{<n}> : n \in \omega \rangle \) of subspaces of \( 2^\kappa \) “converging” to \( z \). Of course, each \( Z_{<n>} \) is a homeomorphic copy of the space \( 2^\kappa \). As before, choose \( z_{<n>} \in Z_{<n>} \cap X_1 \), and in the subspace topology of \( Z_{<n>} \) repeat the process to construct a collection \( \{Z_{<nk>} : k \in \omega \} \), and we choose \( z_{<nk>} \in Z_{<nk>} \cap X_1 \). Similarly we construct open sets (of \( 2^\kappa \)) \( \{U_{<nk>} : k \in \omega \} \) such that \( \{U_{<nk>} \cap Z_{<n>} : k \in \omega \} \) forms a local base at \( z_{<n>} \), in the subspace topology of \( Z_{<n>} \). Observe that in the subspace topology of \( Z_{<n>} \) the point \( z_{<n>} \) is a limit point of the sequence \( \langle z_{<nk>} : k \in \omega \rangle \).

We repeat this process countably many times, to construct a collection \( \{Z_s : s \in ^{<\omega} \omega \} \) of homeomorphic copies of \( 2^\kappa \), a collection of points \( Z = \{z_s : s \in ^{<\omega} \omega \} \), where \( z_s \in Z_s \cap X_1 \), and a collection of open sets (of \( 2^\kappa \)) \( \{U_s : s \in ^{<\omega} \omega \} \) with the following properties:

1. each \( z_s \) is the a limit point of \( \{z_{s-k} : k \in \omega \} \),

2. each collection \( \{U_{s-k} \cap Z_s : k \in \omega \} \) is a local basis at \( z_s \), in the subspace \( Z_s \), witnessing that \( z_s \) is of countable character in \( Z_s \).

Observe that the subspace \( Z \) is countable, first countable and dense in itself. We apply lemma 4 to conclude that \( Z \) is homeomorphic to \( \mathbb{Q} \).

Corollary 1 For each cardinal \( \kappa \)

\[ 2^{2^\kappa} \to (\beta \kappa, \mathbb{Q})^1. \]

proof: Notice that \( \beta \kappa \) is zero-dimensional space of weight \( 2^\kappa \) hence it is embeddable in \( 2^{2^\kappa} \). Now the positive relation \( 2^{2^\kappa} \to (2^{2^\kappa}, \mathbb{Q})^1 \) implies the corollary. \( \square \)
2.4 Infinite partitions of $2^{\omega_1}$

The main book-keeping tool in the proof of the following result is a collection of 
$\sigma$-ideals on $\mathcal{P}(\omega_1)$, defined as follows:

Let $\{S_{\alpha} : \alpha < \omega_1\}$ be a partition of $\omega_1$ into uncountable sets. For each $\zeta < \omega_1$
define a proper $\sigma$-ideal $I_\zeta$ on $\mathcal{P}(\omega_1)$ as follows: $X \in I_\zeta$ iff

1. $(\forall \alpha \in \omega_1) |X \cap S_\alpha| < \aleph_1$, and

2. $(\forall \alpha < \zeta) X \cap S_\alpha = \emptyset$.

Hence, we have $I_0 \supseteq \ldots \supseteq I_\eta \supseteq I_\zeta \supseteq \ldots$ ($\eta < \zeta$).

Notice also that for each $\eta \in \omega_1$, $I_\eta$ is $\sigma$-complete, and that if we choose $A_\alpha \in I_\alpha$
for $\alpha \geq \eta$, then $\bigcup_{\eta \leq \alpha < \omega_1} A_\alpha \in I_\eta$.

**Lemma 3** Suppose $X, Y \in I_0$, and $X \cap Y = \emptyset$. Then,

$$[X, Y^c]_\zeta = \{ Z \subseteq \omega_1 : X \subseteq Z \subseteq Y^c \text{ and } Z \setminus X \in I_\zeta \}.$$ contains a lattice isomorphic to $(\mathcal{P}(\omega_1), \subseteq)$.

**proof:** Choose a $Z \subseteq Y^c$ such that $Z \supseteq X$ and $Z \setminus X \in I_{\zeta}$ and $|Z \setminus X| = \aleph_1$. For
example, let $W = \{ w_\mu : \zeta \leq \alpha < \omega_1 \}$ where $w_\mu \in S_\alpha \setminus (X \cup Y)$, then let $Z = X \cup W$.
Then, $\{ X \cup V : V \in \mathcal{P}(Z \setminus X) \}$ is isomorphic to the structure $(\mathcal{P}(\omega_1), \subseteq)$, and is
contained in $[X, Y^c]_{\zeta}$.

**Proposition 3 (CH)** Given a partition $\{ P_\alpha : \alpha \in \omega_1 \}$ of $\mathcal{P}(\omega_1)$, either there is
a 0-homogeneous $C$-collection or for some $\alpha \geq 1$, there is an $\alpha$-homogeneous $D$-
collection.

It follows from this and lemma (1) and (2) that we have the following fact.

**Corollary 2** [CH] \(2^{\omega_1} \rightarrow (2^{\omega_1}_\delta, (\mathbb{Q})_{\omega_1})^1\).
**Proof of proposition:**

Given $\zeta < \omega_1$ and $s \in \leq 2$, we construct a triple $(X_s, Y_s, i(s))$ of sets $X_s \subseteq \omega_1$, and $Y_s \subseteq \omega_1$ and $i(s) \in \omega_1$ that satisfy

1. $\{X_s : s \in \leq 2\}$ forms a binary tree under $\subseteq$, and for each $s \in \leq 2$ we have:
   - $X_s \cap Y_s = \emptyset$.
   - $E = \bigcup\{X_t : s$ extends $t\} \subsetneq X_s$ and $X_s \setminus E \in I_\zeta$.
   - $Y_s \in I_0$, and $\bigcup\{Y_t : s$ extends $t\} \subseteq Y_s$.
   - $X_s \in P_{i(s)}$, and for all $\gamma < \omega_1$, $0 < \gamma < i(s)$ implies that no $X \in P_\gamma$ exists such that $(1)-(4)$ hold for some $Y_s$ and $X$ replacing $X_s$.

   In addition, if $\zeta$ is a limit, or if $\zeta = \eta + 1$ for some $\eta$, and $s = t^\zeta$ for some $t \in \leq 2$, we require that:

   - if $\emptyset \neq Z \in I_\zeta$ and $Z \cap (Y_s \cup X_s) = \emptyset$, then $(Z \cup X_s) \notin P_{i(s)}$.

   Assume that $P_0$ does not contain a $C$-collection.

   The proof follows from the following three observations.

**Observation 1 [CH]** If at some level $\zeta = \eta + 1$ and for some $t \in \leq 2$, $X_{t^\zeta}$ and $Y_{t^\zeta}$ and $i(t^\zeta)$ can be defined, then $(X_{t^\zeta}, Y_{t^\zeta}, i(t^\zeta))$ is also possible to define.

**Proof:** Let $\leq 2 = \{s_\alpha : \alpha < \omega_1\}$ and assume that for some $\beta < \omega_1$, and all $\gamma < \beta$ the $X_{s^\gamma}$'s are constructed as well as $X_{s^\beta}$. We will choose a set $X_{s^\beta}$ that avoids

$$Y = Y_{s^\beta} \cup \bigcup_{\gamma < \beta} X_{s^\gamma} \cup (X_{s^\beta} \setminus X_s).$$

Note that $Y \in I_0$ as $I_0$ is a $\sigma$-ideal. Define

$$A = \{\delta > 0 : (\exists Z \in I_\zeta) \ Z \neq \emptyset \ \text{and} \ Z \cap Y = \emptyset \ \text{and} \ Z \cup X_{s_{\delta}} \in P_{\delta}\}.$$
Notice that \( A \neq \emptyset \), as otherwise for each non-empty \( Z \in I_{\zeta} \) if \( Z \cap Y = \emptyset \), then \( Z \cup X_{s^\alpha} \in P_0 \). This means that \( [X_{s^\alpha}, Y^\zeta]_{\zeta} \subseteq P_0 \), and lemma (3) gives us a contradiction with the assumption that \( P_0 \) does not contain a \( C \)-collection, (which can easily built inside \( \mathcal{P}(\omega_1) \) if CH holds.) Choose \( i(s^\alpha) = \min(A) \) and choose \( Z \) to witness that fact. Define \( X_{s^\alpha} = X_{s^\alpha} \cup Z \), and \( Y_{s^\alpha} = Y_{s^\alpha} \). Now \( X_{s^\alpha} \) satisfies (1)-(5).

\[ \square \]

**Observation 2:** The construction must be impossible for some \( \zeta \) and some \( s \in \xi^2 \).

*proof:* Otherwise, for each \( f \in \omega_2 \) let \( X_f = \bigcup_{s \leq f} X_s \). We will show that \( \mathcal{X} = \{X_f : f \in \omega_2\} \subseteq P_0 \), which is a contradiction as \( \mathcal{X} \) is a \( C \)-collection.

To this end, we fix a \( f \) and assume \( X_f \in P_\gamma \) for some \( \gamma > 0 \). For some limit \( \zeta > \gamma \) and \( s = f|\zeta \), by (3) we have \( A_{\zeta} = X_s \setminus \bigcup_{\eta < \zeta} \{X_t : t = f|\eta\} \in I_{\zeta} \). Let

\[
A = X_f \setminus \bigcup_{\eta < \zeta} \{X_t : t = f|\eta\} = \bigcup_{\zeta < \xi < \omega_1} A_{\xi},
\]

where \( A_{\xi} = X_{f|\xi} \setminus \bigcup_{\alpha < \xi} \{X_t : t = f|\alpha\} \). \( A \cap S_\alpha = \emptyset \) for all \( \alpha < \zeta \) as \( A_{\xi} \in I_{\xi} \) for all \( \xi \geq \zeta \) and

\[
|A \cap S_\alpha| = |\bigcup_{\zeta < \xi < \alpha} (A_{\xi} \cap S_\alpha)| < \aleph_1.
\]

Hence, \( A \in I_{\zeta} \).

Also, \( X_f \cap Y_s = \emptyset \). Then, (5) implies \( \gamma \geq i(s) \). Now, if \( s' = f|\xi \) for some other limit \( \xi > \zeta \), then \( \gamma \geq i(s') \geq i(s) \). Furthermore, from (6) we have \( \gamma \geq i(s') > i(s) \).

Since this is true for arbitrary limit \( \zeta \), it follows that \( \gamma > \delta \) for all \( \delta \in \omega_1 \); a contradiction that implies \( \gamma = 0 \).

\[ \square \]

**Observation 3:** When the construction stops, there is some \( \gamma > 0 \) such that \( P_\gamma \) contains a \( D \)-collection.

To prove the observation suppose we stopped at a level \( \zeta \) with \( s \in \xi^2 \). Consider \( A = \bigcup\{X_t : s \text{ extends } t\} \) and \( B = \bigcup\{Y_t : s \text{ extends } t\} \).

By our assumption \( [A, B^\zeta]_{\zeta} \not\subseteq P_0 \). Choose \( D \in I_{\zeta} \) with \( D \subseteq (A \cup B)^\zeta \) and \( \gamma > 0 \) such that \( X_s = D \cup A \in P_\gamma \). Consider \( Y_s = B \) and notice that \( (X_s, Y_s, \gamma) \) satisfies (1)-
(4) with \( \gamma = i(s) \). By repeated applications of (5), if needed, we find a set \( Y^1_s \) such that \( (X_s, Y^1_s, i(s)) \) satisfies (1)-(5). Now by the assumption (that the construction stops) (6) must fail for this triple. Choose \( Z_0 \in I_c \) witnessing this fact, i.e. \( Z_0 \cup X_s \in P_{i(s)} \).

We, recursively, build \( \{Z_n : n < \omega \} \) by, at stage \( n \) considering \( (X_s, Y^n_s, i(s)) \) where \( Y^n_s = Y_s \cup \bigcup \{Z_m : m < n\} \). Of course for each \( n \), \( X^n_s = X_s \cup Z^n \in P_{i(s)} \) and satisfies (1)-(5).

For each \( n \) we continue to find a set \( Z_{<n,0} \) witnessing that (6) fails for the triple \( (X^n_s, Y^n_s, i(s)) \). We will continue to construct a collection \( \{Z^{(n,m)} : n < \omega \} \) such that \( X^n_s \cup Z^{(n,m)} \in P_{i(s)} \), and a collection \( \{Y^{<n,m} : m \in \omega \} \) such that for all \( m < \omega \), \( Y^{<n,m} = \bigcup \{Z^{<n,k} : k < m\} \cup Y^n_s \), and that the triple \( (X^n_s, Y^{<n,m}_s, i(s)) \) satisfies (1)-(5). Notice that the collection \( \{X^n_s \cup Z^{<n,m} : m \in \omega \} \) forms a \( \Delta \)-system with root \( X^n_s \). Recursively, we obtain \( \{Z^r : r \in <\omega \omega \} \) and \( \{Y^r : r \in <\omega \omega \} \) and \( \{X^r : r \in <\omega \omega \} \) such that \( \{X^r \cup Z^r : m \in \omega \} \) forms a \( \Delta \)-system with root \( X^r \), \( X^r \cup Z^r \in P_{i(s)} \), and that the triple \( (X^r, Y^r, i(s)) \) satisfies (1)-(5). Together with \( X_s \), the family \( \{Z^r : r \in <\omega \omega \} \) forms a \( \mathcal{D} \)-collection. \( \square \)
Chapter 3

The method of elementary submodels

In this chapter the $\Sigma$-products of topological spaces are further studied and the consistency strength of $\Sigma_{\aleph_2}\{0,1\} \rightarrow (\omega_1)_2^1$ is examined. Semi-proper forcing axiom (SPFA) and the Chang's Conjecture (CC) are introduced and is shown that

$$\Sigma_{\aleph_2} \rightarrow (\omega_1)_2^1$$ implies $CC$.

Also, the reflection principle is introduced and is used in the construction of a structure that will be translated into $B(\omega_1)$, the Baire space of weight $\omega_1$, and it is shown that

$$SPFA \text{ implies } 2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^1.$$  

Section 1 introduces elementary submodels and some arguments about them. In section 2 the Reflection Principle and a consequence of it (a key lemma due to S. Todorcevic) are introduced. In section 3 a structure is constructed which is translated into a Baire space. In section 4 we use this to modify an argument of S. Shelah's and prove that under SPFA,

$$2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^1, \quad \text{and} \quad 2^{\omega_1} \rightarrow (\omega_1 + 1, \omega_1)^1.$$
Section 5 establishes that $\Sigma_\omega \rightarrow (\omega_1)^1$ implies $CC$, and section 6 uses the technique used in section 5 to establish the negative relation $\Sigma_\omega \not\rightarrow (\omega_1)^1$.

### 3.1 Elementary submodels

The proofs and the constructions in this chapter are based heavily on the notion of elementary submodels of some structure of the form $(H_\theta, \in)$ (see the notations below.)

**Definition 1** A submodel $M$ of a model $N$ is elementary, denoted by $M \prec N$, if for any formula $\phi$ with parameters from $M$, we have that $M \models \phi$ iff $N \models \phi$.

The following result is used in establishing the elementarity of a submodel of $H_\theta$.

**Tarski-Vaught test**: For a submodel $M$ of $N$, we have $M \prec N$ iff for each formula $\phi(y, x_1, ..., x_n)$ with parameters $x_1, ..., x_n$ from $M$

$$N \models \exists y \phi(y, x_1, ..., x_n) \Rightarrow (\exists y \in M) N \models \phi(y, x_1, ..., x_n).$$

**Notations**

1. For a cardinal $\theta$, $H_\theta$ denotes the collection of all sets whose transitive closure has size less than $\theta$. It is known that for a regular $\theta$, $(H_\theta, \in)$ is a transitive model of axioms of ZFC other than the power set axiom (see [Ku]).

2. For any formula $\phi$, $\phi^{H_\theta}$ denotes the relativization of the formula $\phi$ to $H_\theta$, and it means $H_\theta \models \phi$. It is known that $\phi^{H_\theta}$ is absolute for transitive models that contain $H_\theta$, in particular, if $\lambda > \theta$, then $(H_\lambda \models \phi^{H_\theta})$ iff $\phi^{H_\theta}$. (see [Ku])

3. We shall frequently and implicitly add a well ordering $\leq_w$ of $H_\theta$ as a predicate and we consider $(H_\theta, \in, \leq_w)$ instead of $(H_\theta, \in)$, so that for a given set $A$, we can take $Sk_\theta(A)$, the skolem closure of $A$ in $(H_\theta, \in, \leq_w)$.

4. For elementary submodels $M$ and $N$ with $M \prec N$, we say $N$ end-extends $M$ if

$$M \cap \omega_1 = N \cap \omega_1.$$
As a first elementary submodel argument we will show that

**Lemma 1** Suppose $M < H_\lambda$, $\theta < \lambda$ and $\theta \in M$. Then, $H_\theta \in M$ and $M \cap H_\theta < H_\theta$.

*proof:*

$H_\theta \in M$:

Since $\theta \in H_\lambda$, we have $H_\theta \in H_\lambda$, so that $H_\lambda \models \exists y \ (y = H_\theta)$. And as $M < H_\lambda$, we have $(\exists y \in M) H_\lambda \models y = H_\theta$, which means that $(\exists y \in M) y = H_\theta$. 

$M \cap H_\theta < H_\theta$:

Fix a formula $\phi(y, x_1, ..., x_n)$ with $x_1, ..., x_n \in M \cap H_\theta$ such that $H_\theta \models \exists y \phi(y, x_1, ..., x_n)$, then follow the chain of implications:

\[
H_\theta \models \exists y \ \phi(y, x_1, ..., x_n) \implies \exists y \in H_\theta \ \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies H_\lambda \models \exists y \in H_\theta \ \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies (\exists y \in M) H_\lambda \models y \in H_\theta \text{ and } \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies (\exists y \in M) M \models (y \in H_\theta \text{ and } \phi^{H_\theta}(y, x_1, ..., x_n))
\]

\[
\implies (\exists y \in M \cap H_\theta) M \models \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies (\exists y \in M \cap H_\theta) H_\lambda \models \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies (\exists y \in M \cap H_\theta) \ \phi^{H_\theta}(y, x_1, ..., x_n)
\]

\[
\implies (\exists y \in M \cap H_\theta) H_\theta \models \phi(y, x_1, ..., x_n).
\]

We, now, apply the Tarski-Vaught test to deduce $M \cap H_\theta < H_\theta$. □

We shall also need some standard facts about the stationary sets and closed and unbounded sets (clubs) in the structures of the form $[A]^{\aleph_0}$. The following facts will be occasionally used (see [Je].)
Facts about clubs and stationary sets

1. The collection of all countable elementary submodels of a $H_\theta$ forms a club in $[H_\theta]^{\aleph_0}$.

2. a) If $C$ is club in $[A]^{\aleph_0}$ then $C^* = \{ X \in [B]^{\aleph_0} : X \cap A \in C \}$, contains a club in $[B]^{\aleph_0}$, and if $C$ is club in $[B]^{\aleph_0}$ then $C|_A = \{ x \cap A : x \in C \}$, contains a club of $[A]^{\aleph_0}$.

   b) If $A \subseteq B$ and $S \subseteq [A]^{\aleph_0}$ is stationary, then $S^*$ is stationary in $[B]^{\aleph_0}$.
   Conversely, if $S$ is stationary in $[B]^{\aleph_0}$ then $S|_A$ is stationary in $[A]^{\aleph_0}$.

3. For any set $A$ we have:
   a) If $F : [A]^{<\omega} \rightarrow [A]^{\omega}$, then the set $C_F = \{ x \in [A]^{\omega} : (\forall e \in [x]^{<\omega}) F(e) \subseteq x \}$ is a club.
   b) (Kueker's lemma) For every club $C$ in $[A]^{\omega}$ there is a function $F : [A]^{<\omega} \rightarrow [A]^{\omega}$ such that $C_F \subseteq C$. (see [De])

3.2 Reflection Principle

The Reflection Principle (RP) is the following statement: (see [Be])

For any cardinal $\kappa$, any set $A$ of size $\kappa$, any stationary $S \subseteq [A]^{\aleph_0}$, any cardinal $\lambda > \kappa$ and any countable $M_0 < H_\lambda$, there is a continuous $\in$-chain $\{ M_\xi : \xi < \omega_1 \}$ of countable elementary submodels of $H_\lambda$ which starts from $M_0$, and an stationary subset $E \subseteq \omega_1$ such that $(\forall \xi \in E) M_\xi \cap A \in S$.

Given a countable elementary submodel $M$, and some ordinal $\alpha \not\in M$, we often extend $M$ to an elementary submodel that extends $M$ and contains $\alpha$. In doing so, many new objects that may not be desirable are also added. We use RP to provide an abundance of some carefully chosen ordinals so that extending $M$ using these ordinals does not include undesirable ordinals.
In the following discussions we assume $\theta = (2^{\aleph_2})^+$ and $\lambda = (2^{\aleph_2})^+$. And for any countable elementary submodel $M$ of $H_\lambda$ we let

$$D_M = \{ \alpha < \omega_2 : (\forall f \in M \cap \omega^2 \omega_2)(f \text{ is regressive} \implies f(\alpha) \in M) \}.$$

Lemma 2 (RP) The collection $T = \{ M \prec H_\theta : D_M \text{ is unbounded in } \omega \}$ contains a club subset of $[H_\theta]^{\aleph_0}$.

**proof:** Suppose $S = [H_\theta]^{\aleph_0} \setminus T$ is stationary. By RP find a continuous $\in$-chain $\{ M_\xi : \xi < \omega_1 \}$ of countable elementary submodels of $H_\lambda$ with $\theta \in M_0$, and a stationary $E \subseteq \omega_1$ such that $(\forall \xi \in E) M_\xi \cap H_\theta \in S$.

Let $\delta = \bigcup_{\xi < \omega_1} (M_\xi \cap \omega_2)$. Since the chain of submodels is an $\in$-chain, we have for each $\xi$, that $M_\xi \cap \omega_1 < M_{\xi+1} \cap \omega_1$, so that $\omega_1 \subseteq \bigcup_{\xi < \omega_1} M_\xi$. Hence, any $\alpha < \delta$ will fall in $\bar{M} = \bigcup_{\xi \in \omega_1} M_\xi$ as $\bar{M}$ is an elementary submodel and $\omega_1 \subseteq \bar{M}$. Therefore, $\delta$ is an ordinal.

Take a countable $N \prec H_\lambda$ such that $\{ M_\xi : \xi < \omega_1 \} \in N$ and $\delta, \theta \in N$ and $\eta = N \cap \omega_1 \in E$. This is possible because there are club many $N \prec H_\theta$ which contain $\{ M_\xi : \xi < \omega_1 \}$, by fact (1), and from fact (2), $E^* = \{ X \in [H_\lambda]^{\aleph_0} : X \cap \omega_1 \in E \}$ is stationary in $[H_\lambda]^{\aleph_0}$.

Observation: For each $\xi \in E$ we have $M_\xi \cap H_\theta \in S$, hence by the definition of $S$, $D_{M_\xi \cap H_\theta}$ is bounded in $\omega_2$. Let $\delta_\xi = \sup D_{M_\xi \cap H_\theta}$, hence $\delta_\xi < \omega_2$. Now, $M_\xi \in M_{\xi+1}$ and $H_\theta \in M_{\xi+1}$, so that $D_{M_\xi \cap H_\theta} \in M_{\xi+1}$. It follows that $\delta_\xi \in M_{\xi+1} \cap \omega_2$, hence $\delta_\xi < \delta$.

Claim 1: $N \cap \bigcup_{\xi < \omega_1} M_\xi = M_\eta$.

**proof:** Let $\eta = \omega_1^N$. Then, $\eta$ is a limit ordinal and $\bigcup_{\xi < \eta} M_\xi = M_\eta$. To see this, observe that if $\xi < \eta$ then $M_\eta \subseteq N$, so that $M_\eta \subseteq N \cap \bigcup_{\xi < \eta} M_\xi$. On the other hand, if $x \in N \cap \bigcup_{\xi < \eta} M_\xi$ there is $\xi < \omega_1$ such that $x \in M_\xi$ as follows:

$$H_\lambda \models "(\exists \xi < \omega_1) x \in M_\xi" \implies N \models "(\exists \xi < \omega_1) x \in M_\xi", \implies (\exists \xi < \omega_1 \cap N) N \models "x \in M_\xi" \implies (\exists \xi < \eta) H_\lambda \models "x \in M_\xi"$$

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\[(\exists \xi < \eta) \ x \in M_\xi \subseteq M_\eta. \]

\[\Box\]

**Claim 2:** \( \delta = \min (N \cap \omega_2 \setminus M_\eta) \).

**Proof:** We need to show that if \( \alpha < \delta \) and \( \alpha \in N \cap \omega_2 \), then \( \alpha \in M_\eta \).

If \( \alpha \in N \cap \delta \), then \( N \models (\exists \xi < \omega_1) \ \alpha \in M_\xi \cap \omega_2. \) Hence, \( (\exists \xi < \omega_1 \cap N) \ \alpha \in M_\xi \cap \omega_2, \) that is \( \alpha \in M_\xi \cap \omega_2 \) for some \( \xi < \eta \). \[\Box\]

**Claim 3:** \( \delta \in D_{M_\eta \cap H_\theta} \).

**Proof:** Let \( f \in M_\eta \cap H_\theta \cap \omega \omega_2 \) be regressive then, \( f \in N \). As \( \delta \in N \), we have \( f(\delta) \in N \cap \delta \), hence by claim (2) we have \( f(\delta) \in M_\eta \cap \omega_2 \). It follows that \( \delta \in D_{M_\eta \cap H_\theta} \). \[\Box\]

Claim (3) implies that \( \delta < \delta_\eta \), contradicting the observation which maintains \( \delta_\eta < \delta \). \[\Box\]

**Corollary 1** For any countable \( M \prec H_\lambda \), with \( \theta \in M \), \( D_M \) is unbounded in \( \omega_2 \).

**Proof:** Let \( C \subseteq \mathcal{T} \) be a club. From fact (3) we have

\[ H_\lambda \models "(\exists F : [H_\theta]^\omega \rightarrow [H_\theta]^\omega) \ C_F \subseteq C." \]

By elementarity of \( M \) and the fact that \( \theta \in M \) we have

\[ (\exists F \in M) \ (F : [H_\theta]^\omega \rightarrow [H_\theta]^\omega \text{ and } H_\lambda \models "C_F \subseteq C." ) \]

Being closed under \( F \), \( M \cap H_\theta \in C_F \), hence it is in \( C \), and \( D_{M \cap H_\theta} \) is unbounded in \( \omega_2 \).

The following argument shows that \( D_M \) must also be unbounded in \( \omega_2 \).

We observe that any \( f \in M \cap \omega \omega_2 \) is already in \( H_\theta \) because \( |tc(f)| < (2^{\omega_2})^+ = \theta \).

It follows that

\[ \forall \alpha \in D_{M \cap H_\theta} \ f(\alpha) \in M \cap H_\theta \subseteq M. \]

This, of course, is true for all \( f \) as described in the previous paragraph; so that
$D_{M \cap H_0} \subseteq D_M$. This implies unboundedness of $D_M$ in $\omega_2$. \hfill \Box

Fix a countable elementary submodel $M$ of $H_\lambda$, and let $M^{(e)}$ denote $sk_\lambda(M \cup \{e\})$.

**Remark:** $(\forall e \in D_M \setminus M) \ e = \min(M^{(e)} \cap \omega_2 \setminus M)$.

**proof:** If $\delta < e$ and $\delta \in M^{(e)}$, then there is a formula $\phi$ with free variables in $M \cup \{e\}$ that uniquely defines $\delta$. We define a regressive function $h \in M \cap \omega^2 \omega_2$ such that $h(e) = \delta$, which is indeed a regressive Skolem function, as follows:

$$h(z) = \begin{cases} 
\min\{\alpha < z : \phi(\alpha, a_1, \ldots, a_n, z)\} & \text{if not empty} \\
0 & \text{otherwise}
\end{cases}$$

Clearly, $h \in M \cap \omega^2 \omega_2$, and $h(e) = \delta$. It follows that $\delta \in M$. \hfill \Box

Similarly, one can prove that for each $e \in D_M$

$$M \cap \omega_1 = M^{(e)} \cap \omega_1.$$

### 3.3 A Correspondence Between $[\omega_2]^{\aleph_0}$ and $\mathcal{P}(\omega_1)$

In translating the structures between $[\omega_2]^{\aleph_0}$ and $\mathcal{P}(\omega_1)$ we use a mapping $\Phi$ defined based on an almost disjoint family of subsets of $\omega_1$, of size $\aleph_2$. That is, a collection $A = \{A_\alpha : \alpha < \omega_2\} \subset \mathcal{P}(\omega_1)$ such that for all distinct $\alpha$ and $\beta$ in $\omega_2$, $|A_\alpha \cap A_\beta| \leq \aleph_0$. The existence of such a family is a fact of ZFC.

The map $\Phi : ([\omega_2]^{\aleph_0}, \subseteq) \rightarrow (\mathcal{P}(\omega_1), \subseteq)$, first considered by B. Velickovic (see [We2]), is defined as follows:

$$\Phi(X) = \bigcup_{\alpha \in X} A_\alpha.$$

Notice that almost disjointness of $A$ implies that

$$(\forall X \in [\omega_2]^{\aleph_0})(\forall \alpha \notin X) \ |A_\alpha \setminus \bigcup_{\gamma \in X} A_\gamma | = \aleph_1.$$
And, this implies that $\Phi$ is an order-preserving injection, hence an isomorphism onto its image. The range of $\Phi$, however, is not all of $P(\omega_1)$, for example $\omega_1 \notin \Phi''([\omega_2]^\omega_0)$.

Clearly, the image under $\Phi$ of any continuous strictly increasing chain of length $\omega_1$ in $[\omega_2]^\omega_0$ is a copy of $\omega_1$ in $P(\omega_1)$. The union of such a chain, however, does not belong to $[\omega_2]^\omega_0$. For the purpose of finding converging $\omega_1$ sequences in $P(\omega_1)$ we will have to arrange for a copy of $\omega_1$ in $[\omega_2]^\omega_0$ whose image under $\Phi$ is destined to converge to a fixed point in $P(\omega_1)$. If we plan to have a copy of $\omega_1$ converging to an uncountable $Y \subseteq \omega_1$ we start with an almost disjoint family of subsets of $Y$.

In the next section we will be interested in finding homogeneous copies of $\omega_1$ and of a Baire space of weight $\aleph_1$. The image under $\Phi$ of the following structure $\mathcal{Y}$ contains such spaces.

**Construction of $\mathcal{Y}$ (RP)**

We start with a countable elementary submodel $M$. It is a consequence of $SPFA$ that $2^{\aleph_0} = \aleph_2$, (see[Be]). Therefore, $|^{<\omega_1}\omega_1| = \aleph_2$. Also, $|D_M| = \aleph_2$. We choose an ordering $\preceq$ of $^{<\omega_1}\omega_1$ in type $\omega_2$ that respects the original partial ordering of $^{<\omega_1}\omega_1$.

Then, using corollary 1, we construct $M = \{M^s : s \in ^{<\omega_1}\omega_1\}$ as follows:

1. $M^\emptyset = M$,

2. $M^s = \bigcup_{t \subseteq s} M^t$, for $s \in ^{<\omega_1}\omega_1$, for any limit $\alpha$.

3. $M^{s^{\gamma}} = sk_\lambda(M^s \cup \{e\})$, where

$$e = \min D_{M^s} \setminus \max\{\sup\{(M^t \cap \omega_2) : t \preceq s\}, \sup\{(M^{s^{\beta}} \cap \omega_2) : \beta < \gamma\}\}.$$

For each $s \in ^{<\omega_1}\omega_1$ let $X^s = M^s \cap \omega_2$, and $Y^s = \Phi(X^s)$, and define $\mathcal{Y} = \{Y^s : s \in ^{<\omega_1}\omega_1\}$. Notice that each branch of $\mathcal{Y}$ is isomorphic to $\omega_1$. Also, define $\mathcal{Y}' = \{Y^f : f \in \omega_1\}$, and $Z = \Psi''(\mathcal{Y}')$, where $\Psi$ is the characteristic function assignment map defined in section 2.2. As such, $Z$ is a subset of $2^{\omega_1}$ and with the subspace topology we have

**Lemma 3** $Z$ is homeomorphic to $B(\omega_1)$.
proof: Recall that $B(\omega_1)$ is $\omega_1$ with the product topology, where $\omega_1$ is given the discrete topology. Define $F : \omega_1 \to Z$, by $F(f) = \chi_{Y'f}$.

claim 1: $F$ is a bijection.

proof: If $f \neq g$ then there is $n \in \omega$ such that $f(n) \neq g(n)$. Let $s = f|_n = g|_n$. Suppose $f(n) = \gamma$, $g(n) = \delta$, $s^{-\gamma} \leq s^{-\delta}$, $M^{s^{-\gamma}} = sk_{\lambda}(M^s \cup \{\alpha\})$ and $M^{s^{-\delta}} = sk_{\lambda}(M^s \cup \{\beta\})$, for some $\alpha$ and $\beta$. By the construction, $M^s$ will never contain $\alpha$. Choose $\xi \in A_\alpha \setminus \Phi(X^s)$, and notice that $F(f)(\xi) \neq F(g)(\xi)$, so that $F(f) \neq F(g)$. □

claim 2: $F$ is open.

proof: Choose a subbasic open set of $\omega_1$, say $D(n, \alpha) = \{f \in \omega_1 : f(n) = \alpha\}$. Observe that

$$F''(D(n, \alpha)) = \{F(f) : f(n) = \alpha\} = \{\chi_{Y'f} : f(n) = \alpha\}$$

$$= \{\chi_{Y'f} : f|_{n+1}(n) = \alpha\} = \{\chi_Y : Y^s \subseteq Y \text{ with } s(n) = \alpha\}$$

$$= \{g \in Z : g(\gamma) = 1 \text{ where } \gamma \in Y^s, s(n) = \alpha\}$$

$$= \bigcup_{s(n) = \alpha} \bigcup_{\gamma \in Y^s} \{g \in Z : g(\gamma) = 1\},$$

which is open in the induced topology of $Z$ from $2^{\omega_1}$. □

claim 3: $F$ is continuous.

proof: Choose a subbasic open set in $Z$, say for some $i \in \{0, 1\}$ and $\alpha < \omega_1$, $B_\alpha(i) = \{f \in Z : f(\alpha) = i\}$.

case 1: $i = 1$. $B_\alpha(1) = \{h \in Z : h(\alpha) = 1\} = \{\chi_Y : Y \in Y' \text{ and } \alpha \in Y\}$. Choose $s \in \omega_1$ such that $\alpha \in Y^s \setminus Y^s$ for some $\beta \in \omega_1$. If no such $s$ exists then either $\alpha \in Y^s$ for all $s$ or $\alpha \notin Y^s$ for all $s$. Hence the set $\{Y \in Y' : \alpha \in Y\}$ is either empty or it is the entire $Z$; so that $B_\alpha(1) \cap Z$, is empty, or equals to $Z$. If such $s$ exists then $B_\alpha(1) = \{\chi_{Y'f} : s \subseteq f\}$. Then, $F^{-1}(B_\alpha(1)) = \{f \in \omega_1 : s \subseteq f\}$ which is open is $\omega_1$.

case 2: $i = 0$.

$$B_\alpha(0) = \{f \in Z : f(\alpha) = 0\} = \{\chi_Y : Y \in Y' \text{ and } \alpha \notin Y\}.$$
Consider $s$ as before. If no such $s$ exists then either $\alpha \notin Y^s$ for all $s$ or $\alpha \in Y^s$ for all $s$. In the former case $F^{-1}(B_\alpha(0)) = Z$, and in the latter $F^{-1}(B_\alpha(0)) = \emptyset$, both open sets. If such a $s$ exists then for some $\beta$

$$B_\alpha(0) = \{ \chi_Y : s \subset f \text{ and } s \beta \notin f \} \cup \{ \chi_Y : s \subset f \}^c,$$

which is open because each component of the union is clopen. This establishes the continuity of $F$, and finishes the proof.

\[ \Box \]

\section{3.4 SPFA}

A semi-proper partial order as a forcing notion does not destroy stationary subsets of $\omega_1$. The Semi Proper Forcing Axiom (SPFA) is the following statement:

For any collection $\mathcal{D}$ of dense open subsets of a semi-proper partial ordering $\mathbb{P}$ there exists a $\mathcal{D}$-generic filter $G$, i.e. a $G \subseteq \mathbb{P}$ such that:

1. $G$ is a filter, and

2. for all $D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

A typical use of SPFA that we shall have here is as follows. Suppose the elements of some partial order $\mathbb{P}$ are structures similar to countable ordinals, and a suitable $\mathcal{D}$ guarantees the extension of each element of $\mathbb{P}$ into arbitrary large countable structures. Then, the the $\mathcal{D}$-generic filter $G$ will contain large enough "compatible" structures, so that $\bigcup G$ becomes a structure similar to $\omega_1$.

It is known (see [Be]) that SPFA implies the RP.

\textbf{Proposition 1 (SPFA)} \textit{The following topological partition relations hold,}

\begin{itemize}
  \item[a)] $2^{\omega_1} \rightarrow (\omega_1 + 1, \omega_1)^1$,
  \item[b)] $2^{\omega_1} \rightarrow (\omega_1 + 1, B(\omega_1))^1$.
\end{itemize}
proof: It would be sufficient to prove that

$$P(\omega_1) \rightarrow (\omega_1 + 1, \mathcal{Y})^1,$$

where, \( \mathcal{Y} \) is as constructed in section 3.

Given a partition \( \{X_0, X_1\} \) of \( P(\omega_1) \) we assume that \( \mathcal{Y} \not\rightarrow X_1 \). Observe first that \( X_0 \) contains an uncountable subset of \( \omega_1 \), since otherwise for any uncountable co-uncountable subset \( Y \) of \( \omega_1 \), we would have

$$[Y, \omega_1] = \{ Z : Y \subseteq Z \subseteq \omega_1 \} \subseteq X_1.$$

Note that \([Y, \omega_1]\) as a lattice is isomorphic to \( P(\omega_1) \), which implies \( \mathcal{Y} \not\rightarrow X_1 \); a contradiction.

Choose an uncountable \( Y \in X_0 \), and let \( \mathcal{A} = \{ A_\alpha : \alpha < \omega_2 \} \) be an almost disjoint family of subsets of \( Y \) such that \( Y = \bigcup \mathcal{A} \). As in section 3.2 we define a map \( \Phi \) and we let \( K_i = \Phi^{-1}(X_i) \), for \( i = 0, 1 \).

Now, we define \( P \) as follows:

\[
p \in P \text{ iff } p = (X_\alpha : \alpha \leq \gamma) \subseteq K_0 \text{ is a countable continuous increasing closed chain. (Notice that each condition } p \text{ is considered to be a function with domain } \gamma + 1 \text{ for some countable ordinal } \gamma. \text{ Similarly, } \text{range}(p) \text{ is the collection } \{ X_\alpha : \alpha \leq \gamma \} \text{ of countable subsets of } \omega_2. \text{ ) The ordering of } P \text{ is end-extension.}
\]

**Case 1:** \( P \) is semi-proper.

Define for each \( \alpha \in \omega_1 \)

\[
D_\alpha = \{ p \in P : \alpha \in \text{dom } p \},
\]

and for each \( \alpha \in Y \), let

\[
E_\alpha = \{ p \in P : (\exists B \in \text{range}(p)) (\exists \beta \in B) \alpha \in A_\beta \}.
\]
We will argue that \( D_\alpha \)'s and \( E_\alpha \)'s are dense in \( P \).

\( D_\alpha \)'s are dense, as otherwise any extension of some \( \{X_\xi : \xi \leq \gamma \} \subseteq [\omega_2]^{\omega_0} \) is in \( K_1 \). This would mean that \( [\omega_2]^{\omega_0} \hookrightarrow K_1 \), which implies \( \mathcal{U} \hookrightarrow X_1 \).

Similarly, \( E_\alpha \)'s are dense because

\[
(\forall \alpha \in Y)(\exists \beta \in \omega_2) \alpha \in A_\beta,
\]

and for each \( \beta \in \omega_2 \), the collection \( B = \{B \in [\omega_2]^{\omega_0} : \beta \in B\} \) is isomorphic to \( [\omega_2]^{\omega_0} \) and therefore must contain element from \( K_0 \) extending any given chain \( p = \{X_\alpha : \alpha \leq \gamma \} \) from \( \mathcal{P} \).

Now let

\[
\mathcal{D} = \{D_\alpha : \alpha \in \omega_1\}
\]

and

\[
\mathcal{E} = \{E_\alpha : \alpha \in Y\},
\]

and by semi-properness of \( P \) we choose a \( \mathcal{D} \cup \mathcal{E} \)-generic subset \( G \) of \( P \).

Observe that \( g = \bigcup G \) satisfies:

1. \( g : \omega_1 \rightarrow K_0 \) is a continuous, increasing map. Therefore,
   \( \{\Phi(g(\alpha)) : \alpha < \omega_1\} \approx \omega_1 \), and is contained in \( X_0 \).

2. From the \( \mathcal{E} \)-genericity of \( G \) it follows that
   \[
   Y = \bigcup_{\alpha \in \omega_1} \Phi(g(\alpha)).
   \]

Of course, \( (2) \) implies that the copy of \( \omega_1 \) created by \( (1) \) will converge to \( Y \) (i.e. its union is \( Y_1 \)) giving rise to a copy of \( \omega_1 + 1 \) in \( X_0 \).

**Case 2:** \( P \) is not semi-proper. Then, there must exist an stationary subset \( S \) of \( \omega_1 \) such that,

\[\vdash_P \text{"} \hat{S} \text{ is not stationary in } \hat{\omega}_1 \text{"}\]
Fix a name $\tau$, a stationary $S$ as above, and a condition $p$ such that:

$$p \Vdash "\tau \text{ is a club in } \omega_1, \text{ and } \dot{S} \cap \tau = \phi."$$

Let $\lambda = (2^{2^{<\omega}})^+$, and notice that $H_\lambda$ contains all the information discussed this far. Take a countable elementary $M \prec H_\lambda$ which contains the above information as well as $M \cap \omega_1 = \delta \in S$.

**Subcase 1:** For all the elementary end-extensions $N$ of $M$ we have $N \cap \omega_2 \in K_1$. Recall that in section 3-3 we constructed a tree $M = \{M^s : s \in <\omega, \omega\}$ of elementary end-extensions of a given elementary submodels and a tree $U$ of the traces of elements of $M$ on $\omega_2$. Each branch of this tree translates to a copy of $\omega_1$ and an initial part of this tree translates to $B(\omega_1)$ (see section 3.3). Indeed the assumption that for any end-extension $N$ of $M$ we have $N \cap \omega_2 \in K_1$, implies that $U \subset K_1$ which is more than needed in both cases (a) and (b) of proposition.

**Subcase 2:** For some end-extension $N$ of $M$, we have $N \cap \omega_2 \in K_0$.

**Claim:** There is a descending chain of conditions $\{p_n : n < \omega\}$ below $p$, and an increasing sequence of countable ordinals $\{\gamma_n : n < \omega\}$ cofinal in $\delta$ such that

1. $p_n \Vdash \"\gamma_n \in \tau\"$,
2. $\bigcup_{n \in \omega} \text{dom } p_n = \delta$, and
3. $\bigcup_{n \in \omega} (\bigcup \text{range } p_n) = N \cap \omega_2$.

**Proof:** Pick a sequence $\langle \delta_n : n \in \omega \rangle$ of ordinals less than $\delta$ converging to $\delta$, let $N \cap \omega_2 = \{\zeta_n : n \in \omega\}$, and define

$$D_n = \{q \in P : \zeta_n \in \bigcup \text{range } q, \text{ and } \delta_n \in \text{dom } q\}.$$ 

Clearly, for each $n \in \omega$, $D_n \in N$ and $D_n$ is dense in $P$. Since $N \models p \Vdash \"\tau \text{ is a club in } \omega_1\\"$,

$$N \models (\exists \gamma \geq \delta_0)(\exists p_0 \in D_0) p_0 \leq p \ p_0 \Vdash \"\gamma_0 \in \tau\".$$
Next,
\[ N \models (\exists \gamma_1 \geq \max(\gamma_0, \delta_1) \exists p_1 \in D_1 \ p_1 \leq p_0 \text{ and } p_1 \vdash "\tilde{\gamma}_1 \in \tau", \]
and so on. Inductively, a descending sequence \( \{p_n : n < \omega\} \) is constructed, such that (1) holds and for each \( n \in \omega \ p_n \in D_n \cap N \). Then, \( \bigcup_{n<\omega} \text{dom} \ p_n = \delta \) and \( \bigcup_{n \in \omega} \bigcup \text{range} \ p_n = N \cap \omega_2. \)

Finally,
\[ q = \bigcup_{n \in \omega} p_n \cup \{(\delta, M \cap \omega_2)\}, \]
is a condition, as it is continuous and closed. Notice that,
\[ q \vdash \{\tilde{\gamma}_n : n < \omega\} \subseteq \tau \text{ and is unbounded below } \delta. \]

Therefore, \( q \models "\tilde{\delta} \in \tau". \)

But, \( q \leq p \) and \( p \models "\tilde{\delta} \in \tilde{\mathcal{S}}". \) It follows that \( q \models "\tilde{\delta} \in \tau \cap \tilde{\mathcal{S}}" \), which contradicts \( p \models "\tau \cap \tilde{\mathcal{S}} = \phi." \)

3.5 Chang's conjecture

The SPFA used in the last section is a forcing axiom whose consistency proof uses a strong large cardinal assumption. Would the consistency proof of \([\omega_2]^\aleph_0 \rightarrow (\omega_1)^1_2\) also need a large cardinal hypothesis? In this section we will demonstrate that this is indeed the case, (see corollary of proposition 2.)

Chang’s Conjecture, hereafter CC, is the statement that every structure of the form \( \langle \omega_2, \omega_1, <, ... \rangle \) for a countable language has an uncountable elementary submodel \( B \) such that \( B \cap \omega_1 \) is countable.

It is known (see [Ka]) that CC implies the consistency of ZFC hence it is is a large cardinal hypothesis.
Assume the negation of CC and choose a counter-example $\mathfrak{A} = \langle \omega_2, \omega_1, h, <, \ldots \rangle$ to CC that contains as a predicate a function $h : [\omega_2]^2 \rightarrow \omega_1$ with the property that $h(\alpha, \gamma) \neq h(\beta, \gamma)$ $\alpha < \beta < \gamma$, and that $h(\alpha, \gamma) \in \omega$ for $\alpha, \gamma \in \omega_1$. To each $\{\alpha, \beta\}$, we associate

$$B_{\alpha\beta} = \text{sk}\mathfrak{A}(\{\alpha, \beta\})$$

where, of course, $\text{sk}\mathfrak{A}(\{\alpha, \beta\})$ is Skolem closure of $\{a, b\}$ in the structure $\mathfrak{A}$, the smallest elementary model of $\mathfrak{A}$ that contains $\alpha$ and $\beta$. Notice that $B_{\alpha\beta} \cap \omega_1$ is a countable ordinal, so we can define the function $e : [\omega_2]^2 \rightarrow \omega_1$ as follows:

$$e(\alpha, \beta) = B_{\alpha\beta} \cap \omega_1.$$ 

It is easily seen that (see, for example, [To6])

**Lemma 4** For every uncountable $A \subseteq \omega_2$, the image $e''([A]^2)$ is uncountable.

**Notation**: For a topological space $X$ and a point $x_0 \in X$ let

$$\sum_{x_0} X = \{f \in \omega^\omega X : |\text{supp}(f)| \leq \aleph_0\},$$

where, $\text{supp}(f) = \{\alpha \in \omega_2 : f(\alpha) \neq x_0\}$. This is the $\Sigma$-product of the topological space $X$ around the point $x_0$. (see [En])

**Proposition 2** Suppose $X$ is a topological space such that $X^{\aleph_0}$ contains no copy of $\omega_1$. Then $\sum_{x_0} X \rightarrow (\omega_1)_{<\omega}^1$ implies CC.

**proof**: Assume the negation of CC and consider the function $e$ as above. Consider a partition $\{E_0, E_1\}$ of $\omega_1$ into stationary co-stationary subsets, and define $\Phi : \sum_{x_0} X \rightarrow \{0, 1\}$ as follows : for a given $g \in \sum_{x_0} X$, let

$$\Phi(g) = i \text{ iff } \sup(e''[\text{supp}(g)]^2) \in E_i, \quad i = 0, 1.$$ 

**Notations:**
1. \( \mathcal{F} = \{ f_\alpha : \alpha < \omega_1 \} \) denotes any copy of \( \omega_1 \) in \( \sum_{\omega_2} X \).

2. \( S_\alpha = \text{supp}(f_\alpha), \ S = \{ S_\alpha : \alpha < \omega_1 \} \) and for each \( \alpha < \omega_1, \ \lambda_\alpha = \sup(e''[S_\alpha]^2) \).

3. For each \( \alpha \in \omega_2, \ I_\alpha = \{ \gamma \in \omega_1 : \alpha \in S_\gamma \} \).

4. \( A = \{ \alpha \in \omega_2 : I_\alpha \text{ is uncountable} \} \).

5. \( \langle M_\xi : \xi < \omega_1 \rangle \) is an \( \varepsilon \)-chain of elementary submodels of \( H_\lambda \) for some large enough \( \lambda \) such that \( H_\lambda \) contains all the information discussed so far. Also assume \( e, \mathcal{F}, X, \Phi, S, A \in M_0 \).

6. \( C \) is a club in \( \omega_1 \) such that \( \forall \delta \in C \ M_\delta \cap \omega_1 = \delta \).

**Claim 1:** \( (\forall \delta \in C) \ S_\delta \subseteq M_\delta \).

**Proof:** For all \( \alpha < \delta, S_\alpha \in M_\delta \); since \( \{ f_\xi : \xi < \omega_1 \} \in M \) hence \( S_\alpha \subseteq M_\delta \) as \( S_\alpha \) is countable. By continuity at \( \delta \), every \( \beta \in S_\delta \) must belong to \( S_\alpha \) for cofinally many \( \alpha \) below \( \delta \); which implies that \( \beta \in M_\delta \).

**Claim 2:** \( (\forall \alpha < \omega_1) (\exists \beta > \alpha) \ S_\beta \setminus D_\alpha \neq \emptyset \).

**Proof:** Otherwise, the support of \( \mathcal{F}_\alpha = \{ f_\beta : \beta > \alpha \} \) is contained in \( S_\alpha \), which is a countable set, while \( \mathcal{F}_\alpha \) is homeomorphic to \( \omega_1 \). This means that the countable product \( X^{S_\alpha} \) contains a copy of \( \omega_1 \), which contradicts our assumption about \( X \). \( \square \)

**Remark:** Notice that if \( C \) is a club in \( \omega_1 \), then \( \{ f_\alpha : \alpha \in C \} \) is also homeomorphic to \( \omega_1 \) so that the above argument produces an ordinal \( \beta \in C \) as described above.

**Lemma 5** \( (\forall \delta \in C) \ S_\delta = A \cap M_\delta \).

**Proof:** \( \subseteq \):

Choose \( \delta \in C \) and \( \alpha \in S_\delta \). Then, \( \alpha \in M_\delta \) as \( S_\delta \subseteq M_\delta \), by remark (1). Assume for a moment that \( \alpha \not\in A \). Then, as \( A \in M_0 \subseteq M_\delta \),

\[ M_\delta \models \alpha \notin A. \]
Hence,

\[ M_\delta \models I_\alpha \text{ is countable.} \]

It follows that \( I_\alpha \subseteq M_\delta \cap \omega_1 = \delta \), so \( \delta \notin I_\alpha \), a contradiction.

\[ \exists : \]

Choose \( \alpha \in A \cap M_\delta \), then \( M_\delta \models \alpha \in A \), so that

\[ M_\delta \models \text{"} I_\alpha \text{ is uncountable."} \]

This transfers to:

\[ (\forall \beta < \delta) \ (\exists \eta < \delta) \ \beta < \eta \text{ and } \alpha \in S_\eta. \]

This together with the fact that \( \mathcal{F} \text{ is continuous at } \delta \) gives that \( \alpha \in S_\delta \). This finishes the proof. \( \square \)

**Claim 1:** \( A \text{ is uncountable.} \)

**proof:** Fix a \( \delta \in C \). Since \( A \in M_\delta \), then if it were countable it had to be a subset of \( M_\delta \). Hence, to prove \( A \) is uncountable it is enough to prove that \( A \setminus M_\delta \neq \emptyset \). To this end, choose a \( \gamma > \delta \) in \( C \) such that \( S_\gamma \setminus S_\delta \neq \emptyset \). The existence of such a \( \gamma \in C \) is guaranteed by claim (2) and the comment following it. Choose \( \alpha \in S_\gamma \setminus S_\delta \). Because \( S_\gamma = A \cap M_\gamma \) and \( S_\delta = A \cap M_\delta \), we have \( \alpha \in A \setminus M_\delta \). This finishes the proof. \( \square \)

Recall that \( \lambda_\alpha = \text{sup}(e''[S_\alpha]^2) \). We are planning to prove that \( \Lambda = \{ \lambda_\alpha : \alpha \in C \} \) contains a club subset of \( \omega_1 \); so that

\[ \Lambda \cap E_i \neq \emptyset \text{ for every } i = 0, 1, \]

which implies

\[ \Phi''(\mathcal{F}) = \{0, 1\}. \]

Since \( \mathcal{F} \) has an arbitrary copy of \( \omega_1 \) this means that \( \Phi \) witness \( \Sigma^1_2 X \rightarrow (\omega_1)_2 \)
finishing the proof of proposition 2.

**Claim 2**: For all $\delta \in C$, $\lambda_\delta = \delta$.

*Proof:* \[ \leq: \]
For all $\alpha$ and $\beta$ in $S_\delta$ we have $e(\alpha, \beta) \in M_\delta$ because $S_\delta \subseteq M_\delta$. Hence, $e(\alpha, \beta) \leq \delta$.
Therefore, $\lambda_\delta \leq \delta$.

\[ \geq: \]
suppose $\lambda_\delta < \delta$, hence $\lambda_\delta \in M_\delta$. Now it follows from lemma 4 that

\[ M_\delta \models "(\exists \alpha, \beta \in A) \ e(\alpha, \beta) > \lambda_\delta," \]
that is,

\[ (\exists \alpha, \beta \in M_\delta \cap A) \ e(\alpha, \beta) > \lambda_\delta. \]

By lemma (5) we have that $\alpha, \beta \in S_\alpha$ which contradicts $\lambda_\delta = \sup (e''[S_\delta])$. \[ \square \]

It follows from claim (2) that $C \subseteq \Lambda$ and this finishes the proof. \[ \square \]

**Corollary 2** $\Sigma^0 \rightarrow (\omega_1)^1_2$ implies CC. Or more precisely,

$[\omega_2]^\omega_0 \rightarrow (\omega_1)^1_2$ implies CC.

### 3.6 Partitioning the $\Sigma$-product into three pieces

In this section we modify the proof of section 3.5 to establish the following:

**Proposition 3** Suppose $X$ is a topological space such that $X^{\omega_0}$ contains no copy of $\omega_1$. Then

$$\mathop{\sum}_{\omega_2} X \notightarrow (\omega_1)^1_3.$$
Before we start the proof with the notation as in the previous section we will define
for each $g \in \sum_{\omega_2}^{x_0} X$,

$$D(g) = sk_{\alpha}(supp(g)).$$

The following observation about $D(g)$ are usefull:

1. For each $g$, $D(g) \cap \omega_1$ is an ordinal.

2. If $G = \{g_\alpha : \alpha < \omega_1\}$ is a sequence of elements of $\sum_{\omega_2}^{x_0} X$ such that
   $(supp(g_\alpha) : \alpha < \omega_1)$ is increasing and continuous (that is for $\alpha < \beta$, $supp(g_\alpha) \subseteq supp(g_\beta)$ and $supp(g_\alpha) = \bigcup_{\gamma < \alpha} supp(g_\gamma)$ for limit $\alpha$,) then
   \{ $D(g_\alpha) \cap \omega_1 : \alpha < \omega_1$ \} is an increasing continuous collection of ordinals in $\omega_1$.

3. For each $g$ and $h$ with $supp(g) \subseteq supp(h)$ if $D(g) \cap \omega_1 = D(h) \cap \omega_1$ then
   $min(D(h) \setminus D(g)) \cap \omega_2 \geq sup(D(g) \cap \omega_2)$.

   proof of observation 3: Let $\alpha \in (D(h) \cap supp(D(g))) \cap \omega_2$. We claim that $\alpha \in D(g)$.
   Choose $\gamma \in D(g) \cap \omega_2$ such that $\gamma > \alpha$ and consider the ordinal
   $$\xi = f(\alpha, \gamma) \in D(h) \cap \omega_1 = D(g) \cap \omega_1.$$ 

   Since $D(g)$ is an elementary submodel of $(\omega_2, \omega_1, f, ...)$ which contains $\xi$ and $\gamma$
   and since $\alpha$ is the unique solution to the equation $\xi = f(\alpha, \gamma)$ if follows that
   $\alpha \in D(g)$. $\square$

proof of proposition: We define maps

$$\Phi : \sum_{\omega_2}^{x_0} X \rightarrow \omega_1, \quad \Phi(g) = D(g) \cap \omega_1, \quad \text{and}$$

$$\Psi : \sum_{\omega_2}^{x_0} X \rightarrow \omega_1, \quad \Psi(g) = otp(D(g) \cap \omega_2).$$

Claim: Suppose the supports of the elements in $G = \{g_\alpha : \alpha < \omega_1\}$ form an
increasing and continuous chain of sets, then at least one of the $\Phi^{\prime} G$ or $\Psi^{\prime} G$ contains
a club.
proof of claim: It follows from observation (2) that $\Phi''\mathcal{F}$ is increasing and continuous, as the supports are increasing and continuous. If it is unbounded, $\Phi''\mathcal{F}$ will be a club in $\omega_1$, and if it is bounded then there must be some $\alpha_0$ such that

$$\Phi(g_\beta) = \Phi(g_{\alpha_0}) \ \forall \beta \geq \alpha_0.$$ 

In this case, by the observation (3) we have $D(g_\beta)$ end-extends $D(g_\alpha)$ whenever $\beta > \alpha \geq \alpha_0$. This property of the $D(g_\alpha)$'s will guarantee that $\Psi(g_\alpha)$ ($\alpha_0 \leq \alpha < \omega_1$), form an increasing continuous collection of ordinals in $\omega_1$ of length $\omega_1$. It follows that in this case $\Psi''\mathcal{G}$ contains a club subset of $\omega_1$. \qed

Now we define a coloring $c : \sum_{\omega_2}^{\omega_0}(X) \to \{0, 1, 2\}$ as follows. Let $\{S_0, S_1, S_2\}$ be a partition of $\omega_1$ into stationary sets, then

$$c(g) = \min\{i : \Phi(g) \notin S_i \text{ and } \Psi(g) \notin S_i\}.$$ 

To show that this partition does not allow a homogeneous copy of $\omega_1$ we assume $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$ is any copy of $\omega_1$ in $\sum_{\omega_2}^{\omega_0}X$. Using the arguments in the previous section we can select a subcollection $\mathcal{G} = \{g_\alpha : \alpha < \omega_1\}$ of $\mathcal{F}$ such that the supports \{supp$(g_\alpha) : \alpha < \omega_1$\} form an increasing continuous chain of subsets of $\omega_2$.

It follows from the claim that one of the $\Phi''\mathcal{G}$ or $\Psi''\mathcal{G}$ will be a club, which intersects all the $S_i$'s, that is, either for some $g_0, g_1$ and $g_2$ in $\mathcal{G}$, we have $\Phi(g_i) \in S_i$ for $i = 0, 1, 2$, or that for some $g_1, g_2$ and $g_2$ in $\mathcal{G}$, $\Psi(g_i) \in S_i$ for $i = 0, 1, 2$. In any case we have

$$c(g_0) \neq 0, \ c(g_1) \neq 1 \ \text{and} \ c(g_2) \neq 2.$$ 

This means that $\mathcal{F}$ is colored by $c$ in more than one color. \qed

Remark: It follows from proposition 3.6 that the negative result above is strongest possible, in the sense that the number of colors cannot be decreased to two.
Chapter 4

Negative Partition relations in forcing extensions

In this chapter we establish the consistency of some negative relations. In the first section we use a \(\sigma\)-closed forcing to partition any ground model topological space \(X\) in such a way that for any uncountable first countable space \(Y\) we have \(X \not
\hookrightarrow (Y)^1\). In the second section we use an iteration of \(\sigma\)-closed forcings to establish the consistency of the relation \(2^{\omega_1} \not
\hookrightarrow (\omega_1)^1\).

4.1 Generic filter as a partition

First we bring a definition and a lemma from general topology.

**Definition 1** A topological space \(X\) is said to concentrate around \(A \subseteq X\) whenever any open set about \(A\) contains all but at most countably many points of \(X\).

**Lemma 1**Uncountable first countable countably compact Hausdorff spaces cannot be concentrated around their countable subsets.

**proof:** First observe that:
1. Any topological space concentrated about a countable set is Lindelöf, hence it is enough to prove: "no uncountable, first countable compact space is concentrated about countable subsets."

2. No uncountable, first countable space can be concentrated about one point.

3. Let $B$ be the set of all $x \in X$ such that every open neighborhood of $x$ contains uncountably many points of $X$. Then, $B$ is closed. Also, if $X$ is compact first countable then $B$ is dense in itself.

This can be justified as follows. Let $x_0 \in B$ be isolated and find open set $U$ such that $U \cap B = \{x_0\}$. Fix a countable decreasing collection of open sets $\{U_n : n \in \omega\}$ with $\overline{U_0} \subseteq U$ and $\bigcap_{n \in \omega} U_n = \{x_0\}$. Since $U_0 \setminus \{x_0\} = \bigcup_{n \in \omega} (U_n \setminus U_{n+1})$ is uncountable for some $n_0$, $U_{n_0} \setminus U_{n_0+1}$ must be uncountable. Let $y$ be a complete accumulation point of $U_{n_0} \setminus U_{n_0+1}$. Then, $y \in \overline{U_0} \setminus U_{n_0+1} \subseteq U$. But, by definition, $y \in B$; a contradiction.

4. If $B \neq \emptyset$ and $X$ is compact first countable, then $B$ is uncountable.

This is an immediate consequence of Čech-Pospíšil's lemma (see [En]) and of the previous observation on $B$.

5. If $X$ is uncountable and concentrated about $A = \{a_n : n \in \omega\}$, then $B \cap A \neq \emptyset$, (in particular $B \neq \emptyset$.) To see this, suppose $A \cap B = \emptyset$ and choose countable open neighborhoods $U_n$ of $a_n$. But, $U$ is countable, $A \subseteq U = \bigcup_{n \in \omega} U_n$, and as $U$ is open, $X \setminus U$ is countable; a contradiction.

To prove the lemma, we assume $X$ is an uncountable, compact, first countable space concentrated about a set $A = \{a_n : n \in \omega\}$. Consider $B$ as in observation (3). Notice that by observations (4) and (5), $B$ is first countable, compact dense in itself. Hence, we may assume without loss of generality that $X$ has these properties.

Aiming towards a contradiction, we will find an open set $U \supset A$ whose complement is uncountable.

We use (2) to find an open set $U_0 \ni a_0$ whose complement in uncountable. Then, we choose an uncountable open set $V(a_0)$ whose closure is disjoint from closure of $U_0$.  

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Next, we apply (2) again to find an open set $U_1 \ni a_1$ together with uncountable open sets $V_{(0,i)}$, $i = 0, 1$ such that:

1. $\overline{V_{(0,i)}} \subset V_{(0)}$,
2. $\overline{V_{(0,0)}} \cap \overline{V_{(0,1)}} = \emptyset$, and
3. $\overline{V_{(0,i)}} \cap U_1 = \emptyset$.

Having constructed open sets $U_n \ni a_n$, $n \in \omega$, and $\mathcal{V} = \{V_s : s \in \omega^2\}$ such that

1. $\overline{V_s} \subset V_t$, whenever $s$ extends $t$,
2. $\overline{V_s} \cap U_n = \emptyset$, for all $n$ and all $s \in \omega^2$, and
3. $\overline{V_s \sim 0} \cap \overline{V_s \sim 1} = \emptyset$,

we define

$$Z = \bigcup_{f \in \omega^2} \bigcap_{n \in \omega} V_{f|n}.$$ 

Let $U = \bigcup_{n \in \omega} U_n$ and notice that we have:

1. $Z \cap U = \emptyset$,
2. $A \subset U$,
3. $Z$ is uncountable.

Thus, $U$ is the desired open set. \hfill \Box

Given a cardinal $\kappa$ and a topological space $X = (\kappa, \tau)$, over a model of $ZFC + CH$ we force with $\mathbb{P} = Fn(\kappa, 2, \omega_1)$. For any generic filter $G$, $\bigcup G : \kappa \rightarrow 2$ is a partition of $\kappa$, hence a partition of $X$ into two pieces.

**Proposition 1** With respect to the above generic partition, there is no homogeneous copy of any countably compact uncountable first countable space inside $X$. 

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Before proving the proposition we mention a basic consequence of countable closedness of $\mathbb{P}$:

**Remark 1** With the above notation, if $A$ is countable, and $\zeta$ is a name such that $p \models "\zeta$ is a subspace of $\tilde{X}$ with $\tilde{A} \subseteq \zeta$, and $(\forall a \in \tilde{A}) \chi(a, \zeta) = \aleph_0"$, then:

1. there is a condition $q \leq p$ and there is a countable collection of open sets $\mathcal{W}_A$ such that for any name $\nu$:

   $$ q \models "\nu \text{ is open and } (\forall a \in \tilde{A})a \in \nu \Rightarrow (\exists W \in \mathcal{W}_A) a \in W \cap \zeta \subseteq \nu \cap \zeta." $$

2. There is a condition $q \leq p$ such that for any name $\nu$,

   $$ q \models "\nu \text{ is open in } \zeta \text{ and } \tilde{A} \subseteq \nu \Rightarrow (\exists W \in \mathcal{T}) \tilde{A} \subseteq W \cap \zeta \subseteq \nu." $$

**proof:**

1. Let $A = \{a_n : n \in \omega\}$. There are names $\nu_n$ for $n \in \omega$, such that,

   $$ p \models "\nu_n \text{ is a countable local basis at } \tilde{a}_n \text{ with respect to } \zeta." $$

Since $\mathcal{T}$ is a basis for the topology of $X$ in the extension, one can find a condition $p_0 \leq p$ and a countable collection, $\mathcal{W}_0$, of open sets such that

$$ p_0 \models "(\forall \text{ open } \nu \exists \tilde{a}_0)(\exists W \in \mathcal{W}_0) \tilde{a} \in W \cap \zeta \subseteq \nu." $$

We call this process **deciding a local basis for $a_0$, below $p$**.

*Next, we decide a local basis for $a_1$, below $p_0$, and continue to construct a descending chain of conditions $p_n$, $n \in \omega$, along with countable collections of open sets $\mathcal{W}_n$ which are decided by the $p_n$'s to be the ground model local basis for the $a_n$'s.*
The desired collection $\mathcal{W}_A$ is the union of $\mathcal{W}_n$'s and the desired condition is $q = \bigcup_{n<\omega} p_n$.

2. In this part we use the condition $q$ and $\mathcal{W}_A$ as above. Given a name $\nu$, such that $q \models "A \subseteq \nu$ and $\nu$ is open," we know from (1) that $q \models "(\forall n \in \omega)(\exists W_n \in \mathcal{W}_n) a_n \in W_n \cap \zeta \subseteq \nu."$ i.e.

$$q \models "\exists W = \bigcup_{n \in \omega} W_n \ A \subseteq W \cap \zeta \subseteq \nu."$$

But $W \in \mathcal{T}$ as it is a countable union of elements of $\mathcal{T}$ and our partial order is $\sigma$-complete.

proof of proposition: Fix names $\xi$ and $\tau$, and a condition $p \in \mathbb{P}$ such that $p \models \xi$ is an uncountable first countable countably compact, and $\tau : \xi \rightarrow \bigcup \Gamma^{-1}(0)$.

Define for $r \leq p$,

$$A^r = \{ x \in \text{dom}(r) : (\exists q \leq r) \ q \models "(\exists \alpha \in \xi) \tau(\alpha) = \hat{x}" \}$$

and

$$A_r = \{ x \in X : r \models "(\exists \alpha \in \xi) \tau(\alpha) = \hat{x}" \}.$$  

Claim 1: With the notation as above, for any condition $r$ there exists $q \leq r$ such that $q \models "A^q \subseteq r"(\xi)."$

proof of claim: Let $q_0 = r$. For each positive integer $n$, having chosen $q_n$ as $\text{dom} \ q_n$ is countable, by countable closedness we choose a condition $q_{n+1} \leq q_n$ such that $q_{n+1} \models "A^{q_n} \subseteq r"(\xi)"$. Define $q = q_\omega = \inf \{ q_n : n \in \omega \}$ and notice that $A^q = \bigcup_{n \in \omega} A^{q_n}$, and $q \models "A^q \subseteq r"(\xi).$  

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We are planning to find a $z \in A$ and a condition below $p$ which contains $(z, 1)$. This, of course, implies that the image of $\tau$ meets the color 1 as well; hence the desired contradiction is reached.

Notice that whenever, in the following construction, for some $r \leq p$ it happens that $A_r \setminus \text{dom } r \neq \emptyset$, then we may choose $z \in A_r \setminus \text{dom } r$ and extend $r$ to $r \cup \{(z, 1)\}$ to get the required condition. Thus, we assume throughout, that for any $r \leq p$ we have $A_r \subseteq \text{dom } r$.

**Claim 2:** Given a condition $q \leq p$ there exists $r \leq q$ and a $\mathcal{W}_r \subset T$ such that

$$r \models \text{"$\nu$ is open in } \tau^n(\xi) \text{ and } \check{A}^r \subseteq \nu\text{"} \implies (\exists W \in \mathcal{W}_r) \check{A}^r \subseteq W \cap \tau^n(\xi) \subseteq \nu."$$

**proof:** By claim 1 extending $q$ we may assume that $q \models \text{"$\check{A}^q \subseteq \tau(\xi)\text{"}.}$ We use remark 2 to find a condition $q_0 \leq q$ and a ground model local basis $\mathcal{W}_0$ for $A^q$ in the subspace $\tau^n(\xi)$. We repeat to find $q_{n+1} \leq q_n$ and a ground model local basis $\mathcal{W}_{n+1}$ for $A^{q_n}$. We let $r = \bigcup_{n<\omega} q_n$ and $\mathcal{W}_r = \bigcup_{n<\omega} \mathcal{W}_n$. \(\square\)

**Claim 3:** There exist:

1. a sequence $\{\eta_n : n < \omega\}$ of elements of $\xi$ and a countable descending chain of conditions $p_n \leq p \ n < \omega$,

2. a sequence of open sets $\{U_n : n < \omega\}$ such that $A^{p_n} \subset U_n$, and

3. a countable collection of points $\{x_n : n < \omega\} \subset X$ such that $x_{n+1} \notin U_n$ for $n < m$ and $p_n \models \text{"$\tau(\eta_n) = \check{x}_n\text{"}"}$.

**proof:** Fix a name $\eta_0$, and find $q_0 \leq p$ and $x_0 \in X$ to satisfy

$$q_0 \models \text{"$\eta \in \tau^n(\xi) \text{ and } \tau(\eta_0) = \check{x}_0 \text{ and } A^{q_0} \subset \tau^n(\xi)\"}.$$  

Since $q_0 \models \text{"$\tau^n(\xi)$ is first countable countably compact,"}$ then by proposition 1, $q_0 \models (\exists U \text{ open in } \tau^n(\xi)) \check{A}^q \subset U \text{ and } \tau^n(\xi) \setminus U \text{ is uncountable."}$
Claim 2 guarantees that there is a condition \( p_0 \leq q_0 \) and an open set \( U_0 \) with \( A^{p_0} \subset U_0 \) and \( p_0 \vdash \overline{U_0} \subseteq U \). It follows that \( p_0 \vdash \text{"\( \tau "(\xi) \setminus U_0 \) is uncountable."} \)

Choose another name \( \eta_1 \), another point \( x_1 \notin A \setminus U_0 \) and \( q_1 \leq p_0 \) such that \( q_1 \vdash \text{"\( \tau (\eta_1) = x_1 " \).} \) Then, we apply claim 1 to find \( p_1 \) such that \( p_1 \vdash \text{"\( A^{p_1} \subseteq \tau "(\xi) " \),} \) and an open set \( U_1 \supset A^{p_1} \) such that \( p_1 \vdash \text{"\( \tau "(\xi) \setminus (U_0 \cup U_1) \) is uncountable.} \)

This describes the inductive steps of the construction, and finishes the proof of claim 3. \( \square \)

Define \( p_\omega := \bigcup_{n < \omega} p_n \). We have

\[
p_\omega \vdash \text{"\( \xi \) is first countable and countably compact"},\]

so that,

\[
p_\omega \vdash \{ \eta_n : n \in \omega \}^d \neq \emptyset. \quad (*)\]

**Claim 4**: There is a point \( z \in X \) and a point \( \eta \in \xi \) such that the sequence \( \langle x_n : n < \omega \rangle \) accumulates to \( z \), and such that \( p_\omega \vdash \text{"\( \tau (\eta) = z " \).} \)

**proof**: It follows from \((*)\) and the maximality principle that there is a name \( \eta \) and a sequence of names \( \{ \eta_{n_k} : k \in \omega \} \subseteq \{ \eta_n : n \in \omega \} \) such that

\[
p_\omega \vdash \eta \in \xi \text{ and } \eta_{n_k} \text{ converge to } \eta.\]

Since \( X \) is countably compact, there must be a subspace of the \( \{ x_n : n \in \omega \} \) say \( \{ x_{n_k} : k \in \omega \} \) that converges in \( X \) to a point \( z \). Then \( z \) and \( \eta \) are as required. \( \square \)

It follows, on the one hand, that \( z \in A_{p_\omega} \), and on the other hand, \( z \notin A^{p_n} \) for all \( n \in \omega \). This is because \( A^{p_n} \subset U_n \), \( U_n \) is open, and \( x_m \notin U_n \) \( \forall m > n \), while some infinite subsequence of \( \{ x_n \} \) converges to \( z \). By Claim 4 and the definition of \( A^{p_n} \) we must have that \( z \notin \text{dom}(p_n) \) for all \( n \). Therefore, \( z \notin \text{dom} p_\omega \). Now, we are in the situation where \( z \in A_{p_\omega} \setminus \text{dom} p_\omega \). The sought after condition is \( p_\omega \cup \{(z, 1)\} \), and this proves the proposition. \( \square \)
Corollary 1 For every topological space $X$ there is a forcing notion $\mathbb{P}$ which forces that $X \notightarrow (Y)_2^1$ for any first countable uncountable countably compact space $Y$. Indeed, one has: $X \notightarrow (Y_1, Y_2)^1$ for first countable uncountable countably compact spaces $Y_1$ and $Y_2$.

The problem with the above proof is that we can only guarantee that for the ground model topological space $X$ the above relation holds. Since our forcing notion was a $\sigma$-closed notion, we know that certain spaces will remain unchanged in the extension. An example of such spaces is $\Sigma$-products of any topological space. Hence we have

Corollary 2 there is a forcing extension in which we have

1. $\Sigma \notightarrow (\omega_1)_2^1$.
2. $\Sigma \notightarrow (2^\omega)_2^1$.
3. $\Sigma \notightarrow (2^\omega, \omega_1)^1$.

4.2 Iteration of countably closed forcings

In the above proposition we have shown how to generically partition each copy of a topological space $X$ that appears in the ground model. In general, new copies of such spaces can appear in the generic extension. In this case in order to produce a result of the sort $X \notightarrow (Y)_2^1$ in some generic extension, we need to iterate the process of the last section until it is guaranteed there are no new copies of such space produced and the existing ones are all generically partitioned at some stage of the iteration. We will demonstrate this process in establishing the following result.

Theorem 1 There is a forcing extension in which $2^{\omega_1} \notightarrow (\omega_1)_2^1$. 

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proof: Clearly we may assume GCH. We define the countable support iteration

\[ \langle \langle P_\alpha : \alpha \leq \omega_2 \rangle, \langle \pi_\alpha : \alpha < \omega_2 \rangle \rangle, \]

and a collection \( \{ \xi_\alpha : \alpha < \omega_2 \} \), of names as follows:

1. \( P_0 = 0 \) and \( \xi_0 = (2^{\omega_1})^V \).

2. For all \( \alpha \in \omega_2 \), \( \xi_\alpha \) is a \( P_\alpha \)-name for \( (2^{\omega_1})^{V[G_\alpha]} \setminus \bigcup_{\beta < \alpha} (2^{\omega_1})^{V[G_\beta]} \), and \( \pi_\alpha \) will be a \( P_\alpha \)-name such that \( \|P_\alpha \|^\pi_\alpha = F_n(\xi_\alpha, 2, \omega_1) \).

We will show that forcing with \( P \) produces a generic partition of \( 2^{\omega_1} \) which admits no homogeneous copies of \( \omega_1 \).

Remarks:

1. In defining the iteration we adopt Baumgartner's approach [Ba2]. We use the following convention in switching back and forth between \( P_\alpha \) and \( P_{\omega_2} \) for each \( \alpha < \omega_2 \): A condition \( p \in P_\alpha \) is a function with domain \( \omega_2 \) with support in \( \alpha \). Hence a \( P_\beta \)-name \( \tau \) would also be considered a \( P_\alpha \)-name for \( \beta < \alpha \).

2. For simplicity of notation we let \( P = P_{\omega_2} \), we use \( \|P_\omega \| \) to mean \( \|P_{\omega_2} \| \), and by name we mean \( P \)-name. For each \( \gamma \) with \( \alpha < \gamma \leq \omega_2 \) we use \( P_{[\alpha, \gamma]} \) to denote the remainder of \( P_\gamma \) after forcing with \( P_\alpha \), that is a partial ordering such that \( P_\gamma \) is isomorphic to \( P_\alpha \ast P_{[\alpha, \gamma]} \) where \( \ast \) stands for iteration of the partially ordered sets.

3. We will prove with the help of lemma 2 that each \( P_\alpha \) is forcing equivalent to \( F_n(\omega_2, 2, \omega_1) \), and as a corollary we will have that each \( P_\alpha \) is \( \sigma \)-closed and has the \( \aleph_2 \)-chain condition, and hence preserves cardinals and the GCH. Similarly \( P_{[\alpha, \gamma]} \) has \( \sigma \)-closed and has the \( \aleph_2 \)-cc.

4. In establishing the next lemma we use the partial order \( \prod_{\beta < \alpha} F_n(\omega_2, 2, \omega_1) \), that is the product of the partially ordered sets \( F_n(\omega_2, 2, \omega_1) \) with countable supports. Elements of this partial order are as usual the functions \( f \) with domain
Lemma 2 Let \((Q_\alpha; \mu_\alpha; \alpha \leq \omega_2)\) be a countable support iteration such that 

\[ \text{If } \mu_\alpha = F_n(\omega_2, 2, \omega_1), \text{ then each } Q_\alpha \text{ is forcing equivalent to } \prod_{\beta < \alpha} F_n(\omega_2, 2, \omega_1) \text{ and hence equivalent to } F_n(\omega_2, 2, \omega_1). \]

**proof:** First notice that \(\prod_{\beta < \alpha} F_n(\omega_2, 2, \omega_1)\) is isomorphic to \(\prod_{\beta < \alpha} F_n(\omega_2 \times \{\alpha\}, 2, \omega_1)\) which in turn is isomorphic to \(\prod_{\beta < \alpha} F_n(\omega_2 \times \alpha, 2, \omega_1)\) via the isomorphism

\[ i(p) = \bigcup_{\beta < \alpha} p(\beta). \]

Finally, \(\prod_{\beta < \alpha} F_n(\omega_2 \times \alpha, 2, \omega_1)\) is isomorphic with \(F_n(\omega_2, 2, \omega_1)\).

To prove the first equivalence, we conveniently let \(\Pi_\beta\) stands for \(\prod_{\xi < \beta} F_n(\omega_2, 2, \omega_1)\). We will show that \(\Pi_\beta\) is densely embedded into \(Q_\beta\) for each \(\beta \leq \alpha\). Each element of \(\Pi_\beta\) is a function with domain \(\beta\) and range \(F_n(\omega_2, 2, \omega_1)\); such function belongs to the ground model because its support is countable. Elements of \(Q_\beta\) are functions with countable support but their values are names of functions rather than the functions themselves. Of course the elements of \(\Pi_\beta\) can also be found among those of \(Q_\beta\), and we call such elements determined conditions. Hence, we may consider \(\Pi_\beta\) as a subordering of \(Q_\beta\) via identification with the set of all determined conditions.

We will show that the determined conditions are dense in \(Q_\beta\), that is

\[ (\forall p \in Q_{\gamma+1})(\exists s \in \Pi_\beta) s \leq p. \]

Let \(p \in Q_\beta\) be given. First assume \(\beta = \gamma + 1\). By definition of \(p\),

\[ p|_\gamma, \text{ } p(\gamma) \in F_n(\omega_2, 2, \omega_1). \]

Notice that \(p(\gamma)\) is a \(Q_\gamma\)-name for a countable function from \(\omega_1\) to \(\{0, 1\}\), and by \(\sigma\)-closedness, one can decide such a function. So that, \((\exists r \in Q_\gamma) r \leq p|_\gamma\) and \(\exists f \in [\omega_2]^{\leq \omega_0} \text{ such that } r \Vdash "f = p(\gamma)." \) Now, by induction, choose \(r' \leq r, \text{ } r' \in \Pi_\beta\).
and define $s$ as follows: $s|_{\gamma} = r'$ and $s(\gamma) = f$. Clearly, $s \in \Pi_{\beta+1}$ and $s \leq p$. If $\text{cf}(\beta) = \omega$, then we choose an strictly increasing sequence of ordinals $\langle \beta_n : n \in \omega \rangle$ cofinal in $\beta$. Define a descending sequence $\{r_n : n \in \omega\}$ of determined conditions such that $r_n \in \Pi_{\beta_n}$ and $r_n \leq p|_{\beta_n}$. Let $p_\omega \in \Pi_\beta$ be an extension of $\{p_n : n \in \omega\}$. Clearly $p_\omega \leq p$.

If $\text{cf}(\beta)$ is uncountable then, as usual, there is some $\gamma < \beta$ such that

$$\forall \delta > \gamma \models "p(\delta) = \emptyset."$$

We choose $r < p|_\gamma$ and extend $r$ to $s$ by $s(\delta) = \emptyset$, $\forall \delta > \gamma$.

In order to establish lemma 2 it remains to show that each $P_\alpha$ is forcing equivalent to $Q_\alpha$. This is done inductively and by considering the fact that at stage $\alpha$ we have

$$\models_{P_\alpha} \langle |\xi_\alpha| = \aleph_2, $$

that implies

$$\models_{P_\alpha} F_n(\xi_\alpha, 2, \omega_1) \approx F_n(\omega_2, \omega_1)."$$

\[\square\]

At stage $\alpha$ of the iteration, of course, $\bigcup_{\beta \leq \alpha} \xi_\beta$ is $2^{\omega_1}$, but in the final extension $\bigcup_{\beta \leq \alpha} \xi_\beta$ is just some subspace of $2^{\omega_1}$. This subspace, because of $\sigma$-closedness of the partial orders, is countably compact.

**Lemma 3** ($\forall \alpha < \omega_2$) $\models \langle \bigcup_{\beta \leq \alpha} \xi_\beta$ is countably compact.$$)

**proof:** Let $G$ be a $P$-generic set and $\tau$ and $\mu$ be names such that

$$\models \tau = \bigcup_{\beta < \alpha} \xi_\beta \text{and } V[G] \models \langle \mu_G \text{ is a countably infinite subset of } \tau_G $$.)

By $\sigma$-closedness of $P_{[\beta, \alpha]}$ we know that $\mu_G \in V[G_\alpha]$. But, $V[G_\alpha] \models \tau_G = \{0, 1\}^{\omega_1}$, so that

$$V[G_\alpha] \models \exists \zeta \in \tau_G \text{ } \zeta \text{ is an accumulation point of } \mu_G.$$
Hence there is a $\mathbb{P}_\alpha$-name $\sigma$ such that

$$V[G_\alpha] \models \sigma_G \in \tau_G \text{ and } \sigma_G \text{ is an accumulation point of } \tau_G.$$  \hspace{1cm} (†)

We now claim that

$$V[G] \models \sigma_G \in \tau_G \text{ and } \sigma_G \text{ is an accumulation point of } \tau_G.$$  

Otherwise,

$$V[G] \models (\exists \gamma < \omega_1)(\exists f \in \gamma 2) \text{ such that } \sigma_G \in U_f \cap \tau_G = \emptyset.$$  

As $f \in V[G_\alpha]$, we have $V[G_\alpha] \models "\sigma_G \in U_f \text{ and } U_f \cap \tau_G = \emptyset"$ which contradicts (†). This proves that $V[G_\alpha] \models "\tau_G \text{ is countably compact}."$ \hfill \Box

for all $\alpha \in \omega_2$ we let $\psi_\alpha$ be the a $\mathbb{P}_\alpha$-name for the generic partition of $\xi_\alpha$ and choose a name $\Psi$ such that $\models "\Psi = \bigcup_{\alpha < \omega_2} \psi_\alpha"$. Then,

**Remark 4:** $\models "\Psi : 2^{\omega_1} \to \{0, 1\} \text{ is a partition}"$.

**Claim:** $\models "\omega_1 \not\leftrightarrow \Psi^{-1}\{0\}"$.

**Proof of claim:** Assume, for a contradiction, that for some $p \in \mathbb{P}$ and some name $\tau$ we have

$$p \models "\tau : \omega_1 \leftrightarrow \Psi^{-1}\{0\}."

We will try to find some condition $q \leq p$, some ordinal $\alpha \in \omega_2$ and $\mathbb{P}_\alpha$-names $t$ and $\lambda$ such that $q|_\alpha \models "\lambda \in \xi_\alpha \cap \tau''(\omega_1) \text{ and } \langle \lambda, 1 \rangle \in q(\alpha)".$ Then we have

$$q \models \Psi(\lambda) = 1 \text{ and } \lambda \in \tau(\omega_1)$$

which is, of course, a contradiction to the assumption of $p \models "\tau''(\omega_1) \subset \Psi^{-1}\{0\}"$. To construct such a condition $q$ we proceed as follows.
**Remark 6:** The Čech-Stone Compactification of the space $\omega_1$ with the order topology is the space $\omega_1 + 1$.

*Proof of Remark:* Recall that if every continuous function from a Tychonoff space $X$ to the closed interval $I$ is continuously extendible to a compactification $\alpha(X)$ of $X$, then $\alpha(X)$ is equivalent to the Čech-Stone compactification of $X$, (see [Enl].) Also notice that every continuous real-valued function from $\omega_1$ is eventually constant, (for example see [SS].) Clearly, such function can be trivially extended to $\omega_1 + 1$. It follows that the space $\omega_1 + 1$ is the Čech-Stone compactification of the space $\omega_1$ with the order topology.

Therefore we can extend $\tau$ to the (unique) function $\tilde{\tau}$ defined on $\omega_1 + 1$. We choose a name $\theta$ for $\tilde{\tau}(\omega_1)$, and we have $p \models "\theta = \tau(\omega_1) \in 2^{\omega_1}"$.

Working in the ground model, we will construct the following collections:

1. a strictly increasing collection of ordinals $\langle \gamma_n : n \in \omega \rangle \subset \omega_1$,
2. an increasing collection of ordinals $\langle \alpha_n : n < \omega \rangle \subset \omega_2$, and $P_{\alpha_n}$-names $\lambda_n$,
3. a descending chain of conditions $\langle p_n : n < \omega \rangle$, with $p_0 \leq p$,
4. An extending collection $\{f_n : n < \omega\}$ of countable functions with domain countable ordinals and range $\{0, 1\}$,

such that the following properties hold:

(for simplicity $U_n$ will denote $U_{f_n} = \{g \in 2^\omega : f_n \subset g\}$ and notice that $U_n$ is closed $G_\delta$)

a) $p_n \models "\theta \in U_n, \tau(\gamma_n) = \lambda_n \in U_n \text{ and } \lambda_n \in \xi_{\alpha_n}"$, and
b) $\forall \beta < \omega_2 \ p_{n+1} \models (\text{dom } p_n(\beta) \cap \tau''(\omega_1)) \subset (2^\omega \setminus U_n)$.

Assume such sequences are constructed. We let $\alpha = \text{lim } \alpha_n$, $\gamma = \sup_{n<\omega} \gamma_n$, $p_\omega = \inf \{ p_n : n < \omega \}$ and $\lambda$ a $P_\alpha$-name such that $p_\omega \models "\lambda = \tau(\gamma) = \text{lim}_{n<\omega} \lambda_n"$.

This limit exists because $\tau$ is continuous at $\gamma$. It follows that:
c) \( p_\omega \models_\alpha \models_\beta \lambda \in \xi_\alpha \).

**Remark 7:** \((\forall \beta < \omega_2)\) \( p_\omega \models \lambda \notin dom p_\omega(\beta) \).

Proof of remark: If for some \( \beta \in \omega_2 \), \( p_\omega \models \lambda \in dom p_\omega(\beta) \), then there is an \( n \in \omega \) such that \( p_\omega \models \lambda \in dom p_n(\beta) \), which implies that \( p_\omega \models \lambda \notin U_n \), (see (b)). On the other hand,

\[
p_\omega \models \lambda = \lim \lambda_n, \text{ and } (\forall n \in \omega ) \lambda_n \in U_{f_n} \subseteq U_{f_{n-1}}.
\]

This implies that \( p_\omega \models \lambda \in \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} U_n \), that is \((\forall m \in \omega) p_\omega \models \lambda \in U_m \), which contradicts \( p_\omega \models \lambda \notin U_n \). \(\square\)

Next we define a condition \( q \in \mathbb{P} \) as follows:

\[
\begin{cases}
q|_\alpha = p_\omega|_\alpha, \\
q|_\alpha \models \"q(\alpha) = p_\omega(\alpha) \cup \{(\lambda, 1)\}\" \\
(\forall \beta > \alpha) q(\beta) = p_\omega(\beta).
\end{cases}
\]

On the one hand, \( q \models \"\lambda \in \tau''(\omega_1) \text{ and } \psi_{\alpha+1}(\lambda) = 1.\" \) On the other hand \( q \leq p \), which implies \( q \models \"\tau''(\omega_1) \subseteq \Psi^{-1}\{0\},\" \) and this is a contradiction. \(\square\)

**Construction of the sequences** \( \{\alpha_n : n \in \omega\}, \{\gamma_n : n \in \omega\} \) and \( \{p_n : n \in \omega\} \)

Recall that \( \theta \) stands for the \( \bar{\tau}(\omega_1) \) in the previous discussions.

**Lemma 4** Given a countable function \( f \) and a condition \( p \in \mathbb{P} \) with \( p \models \"\theta \in U_f,\" \) then, there is a countable function \( g \) with \( g \supseteq f \) and a condition \( q \leq p \) such that

\[
(\forall \beta < \omega_2) q \models \"\theta \in U_g \text{ and } dom p(\beta) \cap U_g \cap \tau''(\omega_1) = \emptyset.\"
\]

Proof of lemma: As the iteration has countable support, the support of \( p \), that is \( \{\gamma < \omega_2 : p(\gamma) \neq \emptyset\} \), is a countable set. Also, each coordinate of a condition is a countable function, so that we can enumerate \( \bigcup_{\beta < \alpha} dom p(\beta) = \bigcup_{\beta \in A} dom p(\beta) \) as
\{b_n : n < \omega\}. Let \( p_0 = p \) and \( f_0 = f \). For each \( n \geq 1 \) choose a \( p_n \leq p_{n-1} \) such that \( p_n \) decides whether \( b_n \) is in \( \tau''(\omega_1) \) or not, and choose \( f_n \supseteq f_{n-1} \) such that \( b_n \not\in U_{f_n} \) while \( p_n \models \"\theta \in U_{f_n}\" \). Finally, we let \( q = \inf \{ p_n : n < \omega \} \) and \( g = \bigcup_{n \in \omega} U_n \). Clearly, 

\((\forall \beta < \omega_2) \; q \models \"\theta \in U_g \text{ and } \text{dom} \; p(\beta) \cap \tau''(\omega_1) \cap U_g = \emptyset\". \)

Now we construct the sequences as follows:

Choose a \( p_0 \leq p \), a countable function \( f_0 \) and \( \alpha_0 < \omega_2 \) such that \( \alpha_0 \) is the least \( \alpha \) such that there is a \( \gamma < \omega_1 \) with \( p_0 \models \"\tau(\gamma) \in \xi_\alpha \cap U_0\" \), and let \( \gamma_0 \) be the least such \( \gamma \). Also let \( \lambda_0 \) be an \( \alpha_0 \)-name such that \( p_0 \models \"\tau(\lambda_0) = \lambda_0\" \).

For each \( n \geq 1 \), having chosen \( p_n, f_n, \gamma_n \) and \( \alpha_n \) such that \( \gamma_n > \gamma_{n-1} \), \( \alpha_n \geq \alpha_{n-1} \), \( p_n \leq p_{n-1} \) and \( f_n \supseteq f_{n-1} \) satisfying

\[ p_n \models \"\theta \in U_n, \; \tau(\gamma_n) \in U_n \cap \xi_{\alpha_n} \" \quad \text{and} \quad \]

\[(\forall \beta < \omega_2) \; p_n \models \"\text{dom} \; p_{n-1}(\beta) \cap \tau''(\omega_1) \cap U_n = \emptyset\", \]

we proceed as follows:

We apply lemma 4 to \( p_n \) and \( f_n \) to obtain \( q \leq p_n \) and \( f_{n+1} \supseteq f_n \) such that \( q \models \"\theta \in U_{f_{n+1}} \) and for all \( \beta < \omega_2 \) \( q \models \"\text{dom}(p(\beta)) \cap U_{f_{n+1}} \cap \tau''(\omega_1) = \theta, \" \) and consider the set \( N = \{ \gamma \geq \gamma_n : p_n \models \tau(\gamma) \in \bigcup_{\beta \leq \alpha_n} \xi_\beta \} \).

\textbf{Case 1} : \( N \) is countable. We choose \( \gamma_{n+1} \geq \sup N \), and \( p_{n+1} \leq q \) and \( \alpha_{n+1} \) to be the least \( \alpha > \alpha_n \) such that

\[ p_{n+1} \models \"\theta \in U_{n+1} \text{ and } \tau(\gamma_{n+1}) = \lambda_n \in U_{n+1} \cap \xi_{\alpha_{n+1}} \". \]

\textbf{Case 2} : \( N \) is uncountable. In this case we let \( \alpha_m = \alpha_n \) and choose \( \gamma_m > \gamma_{m-1} \) for all \( m > n \), with \( \gamma_m \in N \). Clearly, the condition (a) is satisfied. Similarly, condition (b) is satisfied since by the lemma,

\[(\forall \beta < \alpha_n) \; p_{n+1} \models \"\text{dom} \; p_{n-1}(\beta) \cap U_n \cap \tau''(\omega_1) = \emptyset, \; \theta \in U_n \text{ and } \tau(\gamma_n) \in U_n \". \]
Notice that by countable compactness of the $\bigcup_{\beta \leq \alpha} \xi_\beta$, $N$ must be closed. It follows that $p_\omega \vdash \text{"} \lambda \in N \text{"}$. To check the condition (c), observe that since $p_\omega \vdash \text{"} \lambda \in \bigcup_{\beta \leq \alpha} \xi_\beta \text{"}$, we need only to prove that $p_\omega \vdash \text{"} \lambda \not\in \bigcup_{\beta < \alpha} \xi_\beta \text{"}$. Assume the contrary. If case 1 happened all the time, then $p_\omega \vdash \text{"} \lambda \not\in \xi_\beta \text{"}$ for all $\beta < \alpha$ by the the choice of the $\gamma_n$'s and the fact that $(\forall n) \gamma > \gamma_n$, which implies that $\gamma > \sup N$. Therefore $\lambda \not\in \xi_\beta$ for any $\beta < \alpha$.

If case 2 happened at some point in the construction, then for some $n$ we have $\alpha = \alpha_n$, and $p_\omega \vdash \text{"} \lambda \in \bigcup_{\beta \leq \eta < \alpha} \xi_\beta \text{"}$ for some ordinal $\eta$. Of course we have $\eta < \alpha_n = \alpha$, and $\tau(\gamma_m) \in \bigcup_{\beta \leq \eta} \xi_\beta$ eventually for all $m$ which contradicts the minimality of $\alpha_n$. This finishes the construction of the sequences and the proof of the theorem. $\Box$
Chapter 5

Homogeneous copies of $\omega \cup \{p\}$

For a fixed non-principal ultrafilter $p$ on $\omega$,

**Definition 1** A point $x$ in a topological space $X$ is the $p$-limit of a sequence

$\{x_n : n < \omega\}$ whenever: $U$ is an open neighborhood of $x$ iff $\{n : x_n \in U\} \in p$.

$\omega \cup \{p\}$ denotes a sequence converging to a point in the sense of a $p$-limit.

In [CKS] it is established that for each $p \in \omega = \beta \omega \setminus \omega$, $\omega \not\rightarrow (\omega \cup \{p\})^1_2$. There are points $p \in \omega$ whose character is the continuum, so that for such $p$ the character of the limit $p$ in the space $\omega \cup \{p\}$ would also be the continuum, $c$. Hence the smallest Cantor Cube that embeds $\omega \cup \{p\}$ is $2^c$.

In this chapter we present some techniques for proving

$$2^c \rightarrow (\omega \cup \{p\})^1_\omega.$$  

We shall give two different proofs of this result, each using different assumption and technique. Section 1 demonstrates the basic construction used in other sections and establishes $2^c \rightarrow (\omega \cup \{p\})^1_2$. The theme of other sections is as follows. To prove a positive partition relation for a space $A$ we will list all the possible partitions of the set $A$ into countably many pieces, say $\{f_\alpha : \alpha < \kappa\}$, then recursively we define topologies $\tau_\alpha$ on initial pieces of $A$ in such a way that in $\tau_\alpha$ it is guaranteed that $f_\alpha$ will have a homogeneous copy of the space $\omega \cup \{p\}$. In section 2 we use a combinatorial principle for club subsets of $\omega_2$, as a means of “guessing” or “predicting” any possible countable
partition of a given set. Together with the negation of CH this would establish establish $2^c \rightarrow (\omega \cup \{p\})^1_\omega$. In section 3 we will use the hereditarily separability of the reals to give a ZFC argument establishing the same relation $2^c \rightarrow (\omega \cup \{p\})^1_\omega$.

5.1 The basic construction

Proposition 1 For a fixed $p \in \omega^*$, we have $2^c \rightarrow (\omega \cup \{p\})^1_\omega$.

proof: We define a 0-dimensional topology $\tau_p$, of weight at most continuum on $\omega^2 + 1$ which refines the usual topology and $(\omega^2 + 1, \tau_p) \rightarrow (\omega \cup \{p\})^1_\omega$. Then, since of course $(\omega^2 + 1, \tau_p)$ embeds into $2^c$, the proposition follows.

Recursively we construct $(\tau_\alpha : \alpha \leq \omega^2 + 1)$ in such a way that:

1. $\tau_\alpha$ is a 0-dimensional topology on $\alpha$ refining the usual topology;
2. $\forall \gamma < \beta < \alpha, \quad \tau_\gamma = \tau_\beta \cap \mathcal{P}(\gamma)$;
3. each limit point is a $p$-limit point.

Suppose for each $\beta < \alpha$ $\tau_\beta$ is defined. If $\alpha$ is a limit ordinal, then we define $\tau_\alpha = \langle \bigcup \{\tau_\beta : \beta < \alpha\} \rangle$. The notation $\langle X \rangle$ stand for the topology generated by $X$, that is the subbasic open sets are taken to be elements of $X$. If $\alpha$ is the successor of a successor ordinal, say $\alpha = \beta + 2$, then we declare $\{\beta + 1\}$ to be open: that is $\tau_\alpha = \langle \tau_{\beta+1} \cup \{\beta + 1\} \rangle$. For $\alpha = \beta + 1$ where $\beta$ is a limit ordinal, the basic neighborhoods at $\beta$ will be defined as follows: We fix a sequence of ordinals $\beta_n$ increasing to $\beta$ (if $\beta = \omega^2$ we require that the $\beta$'s be limit ordinals.) Then we define $u_{n,\beta} = (\beta_{n-1}, \beta_n]$, and for a fixed element $A \in p$ we let $W_{A,\beta} = \{\beta\} \cup \bigcup \{u_{n,\beta} : n \in A\}$. Finally, we declare $\mathcal{W}_\beta = \{W_{A,\beta} : A \in p\}$ to be a neighborhood basis at $\beta$, and define $\tau_{\beta + 1} = \langle \tau_\beta \cup \mathcal{W}_\beta \rangle$.

Notice that $\tau_p = \tau_{\omega^2+1}$ is a 0-dimensional topology: this follows from the fact that for all limit $\beta$ and for all $A \in p$, the $W_{A,\beta}$ is closed as it is a union of closed $u_n$'s together with the limit $\beta$ of the collection of $u_n$'s.

Also, $|p| \leq c$, which implies that at each limit point we have a basic neighborhood of size at most $c$. Therefore $\omega(\langle \tau_p, \omega^2 + 1\rangle) \leq c$. 

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It remains to show that \((\omega^2 + 1, \tau_p) \to (\omega \cup \{p\})_2^1\). Let \(\omega^2 + 1 = K_0 \cup K_1\), be a partition. Consider a sequence \(\{\beta_n : n < \omega\}\) of limit ordinals converging to \(\omega^2\), and consider \((\beta_{n-1}, \beta_n) = u_n\). If for some \(n\), \(u_n \subseteq K_i\), we have a copy of \(\omega \cup \{p\}\) in \(K_i\). This is because \(\beta_n\) is a limit point, hence there is a sequence from \((\beta_{n-1}, \beta_n)\) converging to \(\beta_n\) in the sense of \(p\). Of course, by the construction, any sequence converges in the sense of \(p\). On the other hand, if for all \(n\) \(u_n \cap K_i \neq \emptyset\), for \(i = 0, 1\), then we can choose pairs \(\{\gamma_0^n, \gamma_1^n\} \subseteq u_n\) such that \(\gamma_1^n \in K_i\). Both sequences \(\{\gamma_i^n : n < \omega\}, i = 0, 1\), have \(\omega^2\) as their \(p\)-limit. Therefore, if \(\omega^2 \in K_i\), then we have \(\{\gamma_i^n : n < \omega\} \subseteq K_i\), a copy of \(\omega \cup \{p\}\). \(\square\)

### 5.2 Club guessing principle

We introduce a combinatorial principle and we reconstruct a proof of its existence due to S. Shelah (presented by U. Abraham at the Toronto set theory seminar.) Then we use this principle and \(-CH\) in order to show that \(2^c \to (\omega \cup \{p\})_2^1\).

**Definition 2** Recall the notation \(S_0^2 = \{\lambda \in \omega_2 : cf(\lambda) = \omega\}\).

1. A Ladder System on \(S\) is a sequence \(\langle \eta_\delta : \delta \in S \rangle = \eta\), where each \(\eta_\delta\) is an \(\omega\)-sequence converging to \(\delta\).

2. A Ladder system \(\eta\) is club guessing if for each club \(C \subseteq S\), there is some \(\delta\) such that the range(\(\eta_\delta\)) \(\subseteq C\).

**Theorem 1 (Shelah) [Sh2]** There exists a club guessing ladder system on \(S_0^2\).

**proof:** Fix any ladder system \(\eta = \{\eta_\delta : \delta \in S_0^2\}\).

**Notation:** Given a club \(C \subseteq S_0^2\), we define \(CR(\eta, C)\) to be ladder system \(\mu = \{\mu_\delta : \delta \in S_0^2\}\), as follows:

\[(\forall \delta \in S_0^2)(\forall i \in \omega) \mu_\delta(i) = sup(C \cap \eta_\delta(i)).\]

Observe that:
1. \((\forall \delta \in S_0^2) \ (\forall i \in \omega) \ \mu_\delta(i) \leq \eta_\delta(i),\) and

2. if for some \(\delta, \ (\forall i \in \omega) \ \sup(C \cap \eta_\delta(i)) = \eta_\delta(i),\) then \(\text{range}(\eta_\delta) \subseteq C.\)

Define a decreasing sequence \(<C^\xi : \xi \in \omega_1>\) of clubs in \(\omega_2\) and \(<\eta^\xi : \xi \in \omega_1>\) of ladder systems as follows:

- \(C^0 = \omega_2,\)
- \(\eta^0 = \eta,\)
- \(\eta^{\xi+1} = CR(\eta^\xi, C^\xi),\)
- \(C^{\xi+1}\) is a club witnessing that \(\eta^{\xi+1}\) is not a club guessing ladder system,
- for a limit \(\xi, \eta^\xi\) is defined as follows: \(\eta^\xi(i) = \min\{\eta_\delta(i) : \gamma < \xi\}.\) (It follows from observation 1 that each \(\{\eta_\delta(i) : \gamma < \xi\}\) is a non-increasing sequence of ordinals.)
- \(C^\xi = \bigcap_{i<\xi} C^i, \ \xi\ a\ \text{limit ordinal};\)

We claim that this process must end at some \(\xi < \omega_1\) and hence, \(\eta^{\xi+1}\) is a club guessing ladder system. Assume for a contradiction that the process of choosing \(C^\xi\)'s is not interrupted and let \(C = \bigcap_{\xi < \omega_1} C^\xi.\) Clearly \(C\) is a club in \(\omega_2.\)

**Claim:** \(\forall \delta \in C \ \exists \epsilon_\delta \in \omega_1 \ \forall i < \omega \ \forall \epsilon > \epsilon_\delta \ \eta^\delta(i) = \eta^{\delta+\epsilon}(i).\) **proof of claim:** Since from observation 1 we know that for all \(\delta \in S_0^2\) and for each \(i \in \omega, \ \{\eta_\delta(i) : \gamma < \omega_1\}\) is a non-increasing sequence of ordinals we can inductively choose \(\xi_j \in \omega_1\) such that \((\forall \xi > \xi_j) \ \eta^\xi(j) = \eta^{\xi_j}(j),\) and let \(\epsilon = \sup \{\xi_j : j < \omega\}.\)

It follows from the claim that \(\eta^\xi = \eta^{\xi+\epsilon},\) which means by observation 2 that \(\text{range}(\eta^\xi) \subseteq C^{\xi+\epsilon},\) so that \(C^{\xi+\epsilon}\) would not witness that \(\eta^{\xi+\epsilon}\) is club guessing. This means that the process of our construction was interrupted. This proves the theorem.

\(\Box\)

We are ready to prove:
Proposition 2  There is a 0-dimensional topology $\tau_p$ on $S_0^2$ such that
\[
(S_0^2, \tau_p) \longrightarrow (\omega \cup \{p\})^1_{\omega}.
\]

Let $\eta = \{\eta_\delta : \delta \in A\}$ be a club guessing ladder system on $S_0^2$.

**proof of proposition:** Define a topology $\tau_p$ on $S_0^2$ which is 0-dimensional and is of weight $\leq \aleph_2$ that refines the usual topology.

for each $\lambda \in S_0^2$ and each $A \in p$ we define the following open neighborhood of $\lambda$ as follows:
\[
W_{A,\lambda} = \{[\eta_\lambda(n), \eta_\lambda(n + 1)) : n \in A\} \cup \{\lambda\}.
\]

Notice that each $W_{A,\lambda}$ is the union of a discrete collection of closed sets together with a singleton. In the usual topology, such a set is closed. By declaring every such set open in the new topology we are defining a clopen basis at each point $\lambda$.

Then by adding open sets of this form at each point of $S_0^2$, we get a refinement $\tau_p$ of the usual topology. Clearly, $\tau_p$ is 0-dimensional. In this topology, each point has a basis of cardinality $|p| \leq c$. It follows that $w((S, \tau_p)) = c.\aleph_2$.

To show $(S_0^2, \tau_p) \longrightarrow (\omega \cup \{p\})^1_{\omega}$, we fix a partition of $S_0^2$, say $f : S_0^2 \longrightarrow \omega$. We consider $A_f = \{n \in \omega : f^{-1}\{n\} \text{ is unbounded in } S_0^2\}$, and for each $n \in A_f$ we let
\[
C_n^f = \{\beta \in S_0^2 : (f|_{\beta})^{-1}\{n\} \text{ is unbounded below } \beta\},
\]
and define the club $C_f = \bigcap_{n \in A_f} C_n^f \setminus \sup(f^{-1}(\omega \setminus n))$. We then choose $\lambda$ such that $\eta_\lambda \subseteq C_f$. Let $n = f(\lambda)$. Clearly, $n \in A_f$, because otherwise $\lambda \not\in C_f$, whereas $\eta_\lambda \subseteq C_f$ implies that $\lambda \in C_f$.

Now, for all $k$, $\eta_\lambda(k) \in \text{range}(\eta_\lambda) \subset C_f$, which means that $f|_{\eta_\lambda(k)}^{-1}(n)$ is unbounded below $\eta_\lambda(k)$. We choose $\gamma_k \in (f|_{\eta_\lambda(k)})^{-1}(n) \cap [\eta_\lambda(k - 1), \eta_\lambda(k))$, for all $k$.

Now we have $\{\gamma_k : k \in \omega\} \longrightarrow_p \lambda$, which finishes the proof. $\square$

**Corollary 1** ($\neg CH$) $2^c \longrightarrow (\omega \cup \{p\})^1_{\omega}$. 74
proof: \( \neg \text{CH} \) implies that \( c \geq \aleph_2 \), so that \( (S_0^2, \tau_p) \) being 0-dimensional of weight at most \( c \) is embeddable in \( 2^c \).

\[ \square \]

5.3 Argument using hereditary separability of \( \mathbb{R} \)

Proposition 3 There is a 0-dimensional topology \( \tau_p \) of weight at most continuum on \( I \) such that

\[ \langle I, \tau_p \rangle \rightarrow (\omega \cup \{p\})^1_\omega. \]

Since such a topology is embeddable in \( 2^c \), we have:

Corollary 2 \( 2^c \rightarrow (\omega \cup \{p\})^1_\omega. \)

proof of proposition: Consider an enumeration of the irrationals \( I = \{a_\alpha : \alpha < c\} \), and let \( \{f_\alpha : \alpha < c\} \) be an enumeration of all the \( \omega \)-partitions of the countable sets of irrationals in such a way that each \( f \) appears cofinally often indexed by a limit ordinal.

Recursively, we define topologies \( \langle \tau_\alpha : \alpha < \omega_1 \rangle \), each \( \tau_\alpha \) a 0-dim topology of weight at most continuum on \( \{a_\beta : \beta < \alpha\} \) refining the Euclidean topology on \( I \). In defining \( \tau_\alpha \) we take care to assign a homogeneous copy of \( \omega \cup \{p\} \) to \( f_\alpha \), if possible. Then of course, \( \tau_p = \Sigma_{\alpha < c} \tau_\alpha \) would be a 0-dim topology on \( I \) of weight at most continuum.

At stage \( \alpha \), suppose \( \tau_\beta \) is constructed for all \( \beta < \alpha \), as required. If \( \alpha \) is limit, let \( \tau_\alpha = \Sigma_{\beta < \alpha} \tau_\beta \), and if \( \alpha \) is the successor of a successor ordinal, say \( \alpha = \beta + 1 \), let \( \tau_\alpha \) be the topology generated by \( \tau_\beta \cup \{\{a_\beta\}\} \).

The main case would be when \( \alpha = \beta + 1 \) for a limit \( \beta \). We start with a countable subset \( E \) of the reals such that \( f_\beta \) is an \( \omega \)-partition of \( E \) and \( a_\alpha \in \overline{E^c} \). If no such \( E \) is found we declare \( \{a_\alpha\} \) open. If \( a_\alpha \notin \text{dom} \ f_\alpha \) or \( a_\alpha \notin f_\alpha^{-1}(\{n\})^E \) for any \( n \), we let \( \{a_\alpha\} \) to be open, otherwise we proceed as follows. Let \( M \) denote the collection of all \( n \in \omega \) such that \( a_\alpha \in f_\alpha^{-1}(\{n\})^E \) and for each \( n \in M \) let \( S^n = \{a_\alpha^n : n \leq m < \omega\} \) witness that \( a_\alpha \notin f_\alpha^{-1}(\{n\})^E \). Also choose two sequences of the rationals: \( \langle q_k^l : k \in \omega \rangle \) strictly increasing and converging to \( a_\beta \) from left and \( \langle q_k^r : k \in \omega \rangle \) strictly decreasing
and converging to \( a_\beta \) from right. We reduce the \( S^n \)'s to subsequences such that in the usual ordering of the reals, \( a_n^m < a_{m+1}^n < a_\beta \) or \( a_\beta < a_{m+1}^n < a_m^n \). At this point we introduce a partition of \( M = M^r \cup M^l \), where \( M^r \) denotes the set of all the \( n \)'s in \( M \) with elements of the \( S^n \) approaching \( a_\beta \) from right in the usual ordering of the reals. Similarly \( M^l \) stands for approaching \( a_\beta \) from left. If \( n \in M^l \) we would like the \( a_n^m \)'s to belong to \( [q_m^l, q_{m+1}^l] \) and similarly, if \( n \in M^r \) we require that \( a_n^m \in [q_m^r, q_{m+1}^r] \).

Now, to \( a_\beta \) we assign neighborhoods that make it the \( p \)-limit of all these sequences. To define the subbasis at \( a_\beta \), let \( u_m^l = [q_m^l, q_{m+1}^l] \cup [a_\beta, \infty) \) and for each \( n \in M^r \) let \( u_m^r = (-\infty, a_\beta] \cup (q_{m+1}^r, q_m^r] \). Then to each \( t \in \{r, l\} \) and to each \( A \in p \) assign

\[
W_A^t = \bigcup \{u_m^t : m \in A\},
\]

and let \( \mathcal{W} = \{W_A^t : t \in \{r, l\}, A \in p\} \). Finally we define \( \tau_\alpha \) to be the topology generated by \( \tau_\beta \cup \mathcal{W} \).

Clearly, \( \tau_\alpha \) refines the usual topology on \( \tau_\alpha \) and since each \( u_m^t \) is cloed in the induced topology of the irrationals, in \( \tau_\alpha \) they are also open, making the \( \tau_\alpha \) zero-dimensional. This topology is of weight at most continuum as \( |\mathcal{W}| \leq \omega_1 \).

To show that such a topology will satisfy \((\mathcal{I}, \tau_p) \rightarrow (\omega \cup \{p\})_0^\dagger\), we proceed as follows. Given a coloring \( f : \mathcal{I} \rightarrow \omega \), we consider for each \( n < \omega \), a countable dense \( D_n \subseteq f^{-1}(\{n\}) \) and \( D = \bigcup_{n < \omega} D_n \). Then, \( D \) is countable. Choose a large enough limit ordinal \( \beta \) and a countable \( E \supseteq D \) such that \( f_\beta = f|_E \) and \( a_\beta \notin D \). If \( f(a_\beta) = n_0 \), then \( a_\beta \notin D \) implies that \( a_\beta \) is a cluster point of \( (f^{-1}(\{n_0\})) \). Take the sequence \( S^{n_0} = \{a_n : n \in \omega\} \) as in the construction at the stage \( \beta + 1 \) of the construction, and of course \( \{a_n : n \in \omega\} \cup \{a_\beta\} \) is homeomorphic to \( \omega \cup \{p\} \).
Bibliography


[En1] ———— cor 3.6.3 .

[En2] ———— 3.10.D


[SS2] ——— (p. 66-68)

[SS3] ——— (p. 70)


