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Derivation of general relativistic gravitational potential energy using principle of equivalence and gravitational time dilation

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Abstract

Einstein's theory of general relativity which has been experimentally proved to be true theory of gravity doesn't need gravitational potential energy to predict trajectory of particles in space. This is because general relativity is a purely geometric theory. Objects move along the geodesics in the curved space-time. The energy-momentum tensor that warps the space-time as per Einstein's field equations takes into account only the energy/momentum of matter and radiation. Thus, gravitational potential energy doesn't come into picture in Einstein's theory of gravity and its role is taken over by curvature of space-time. However, general relativistically correct expression for gravitational potential energy is required for energy conservation and some energy-based approaches in physics. Conventionally, correct form of gravitational potential energy is derived by using full mathematical formality of general relativity. In this paper, we describe an event by which we derive the same general relativistic expression for gravitational potential energy simply by using the principle of equivalence and gravitational time dilation.

Keywords: Gravitational potential energy, Principle of equivalence, General relativity

1. Introduction

As per Newton's law, gravitational force between two material bodies is calculated by taking derivative of the gravitational potential energy which is given by, $\phi = -GMm/r$. The gravitational force finally decides the motion of a particle in space populated with arbitrary distribution of mass. However, Einstein's theory of general relativity is a geometric theory of gravity in which dynamics of the objects are governed by curvature of space-time

created by distribution of energy-momentum in the universe. Every object follows the geodesic path in curved space-time and we don't need to calculate gravitational potential energy to predict the trajectory. General relativity has established itself as the true theory of gravity by numerous experimental confirmations such as deflection of light rays coming from distant star by sun, precession of perihelion of mercury and gravitational time dilation [1-2]. Metric of the space-time in general relativity is found out by solving the Einstein field equation given by [1],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (1)$$

Where $R_{\mu\nu}$ is Ricci tensor, R is Ricci scalar, $g_{\mu\nu}$ is metric tensor and $T_{\mu\nu}$ is energy-momentum tensor. It is important to note that the energy-momentum tensor on right hand side of Eq. (1) contains only the matter/radiation terms, not the gravitational potential energy. This doesn't mean, gravitational potential energy doesn't exist. It just means, in Einstein's theory of gravity, we deal with curvature of space-time to calculate the dynamics of material bodies. Role of gravitational potential energy has been taken over by curvature of space-time. That's why it is stated that *covariant* divergence (not the "ordinary divergence") of energy-momentum tensor is zero. Covariant divergence includes a term related to curvature. So, we can say, gravitational potential energy exists in the form of curvature of space-time.

In some energy-based branches of physics like Hamiltonian approaches of quantum mechanics, we need the exactly correct form of gravitational potential energy. Newton's law gives only the approximate value of this potential energy just like it predicts the approximate trajectory or dynamics of material objects. We have to apply the general theory of relativity to exactly calculate the gravitational potential energy between two masses, one large mass M and other infinitesimal mass m at distance R from it. It has been done previously [3] by using the full formality of General relativity and the potential energy ($P.E.$) is given by,

$$P.E. = mc^2 \left[\left(1 - \frac{2GM}{c^2 R} \right)^{1/2} - 1 \right] \quad (2)$$

We can observe that if Schwarzschild radius ($2GM/c^2$) is too small as compared to distance R , above expression reduces to,

$$P.E. = mc^2 \left[\left(1 - \frac{1}{2} \frac{2GM}{c^2 R} \right) - 1 \right] = -\frac{GMm}{R}, \text{ which is Newton's law.}$$

In this paper, we will derive Eq. (2) in a simpler method by means of a thought experiment without using the full formality of General relativity. We will however use the principle of equivalence which is the foundation on which the whole of general relativity stands and expressions of gravitational time dilation and special relativity. The advantage of our method is that the expression of general relativistic gravitational potential energy is derived by physical arguments and interesting consequence of principle of equivalence becomes evident.

2. Derivation of gravitational potential energy from principle of equivalence and gravitational time dilation

Let us consider a spherical mass M (a planet or star) as shown in Fig. 1. There are two stationary clocks, A and C. Clock A is at infinite distance from M and stationary clock C is at a distance of R from M . Let a particle with initial velocity u_0 and small rest mass m_0 with clock B attached to it falls freely from infinity. It crosses past the clock C with velocity u .

Fig. 1 Positions of stationary clocks A and C and freely falling particle with attached clock B used for derivation of general relativistic gravitational potential energy

As the particle falls freely in the gravitational field, by principle of equivalence [1, 4-5], space-time looks flat to a tiny observer standing on this particle. In other words, it is in a *local inertial frame* of reference and it doesn't experience any gravitational field in its local region of space during its free motion. Hence, the tiny observer on the particle is entitled to apply special theory of relativity to events happening in its *local region* at any instant of time. Let dT_A is the time interval (proper time) between two consecutive ticks of clock A. When the clock **B** crosses past clock A at infinity, the same time interval as measured by clock B will be given by (applying special theory of relativity),

$$(dT_A)_{\text{measured by B at } \infty} = \frac{dT_A}{\sqrt{1 - \frac{u_0^2}{c^2}}} \quad (3)$$

Due to gravitational time dilation [1], time interval in clock A as measured by clock C is given by,

$$(dT_A)_{\text{measured by C}} = dT_A \sqrt{1 - \frac{2GM}{c^2 R}} \quad (4)$$

Note that we can take the expression for gravitational time dilation from General relativity or Schwarzschild solution as aim of this paper is just to demonstrate how a modification of time coordinate leads to change in gravitational potential energy.

When the clock **B** crosses past clock C at radial distance R from M , since it is in a local inertial frame due to free fall, it can apply special relativity to translate the time $(dT_A)_{\text{measured by C}}$ in terms of its own time by relation,

$$(dT_A)_{\text{measured by B at R}} = \frac{(dT_A)_{\text{measured by C}}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Putting Eq. (4) in above expression,

$$(dT_A)_{\text{measured by B at R}} = \frac{dT_A \sqrt{1 - \frac{2GM}{c^2 R}}}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (5)$$

Since clock B is always in a locally inertial frame of reference due to free fall, its rate of ticking remains same and its measured value for the same parameter will also be same irrespective of where it is in space. So,

$$(dT_A)_{\text{measured by } B \text{ at } \infty} = (dT_A)_{\text{measured by } B \text{ at } R}$$

Using Eq. (3) and (5) in above,

$$\begin{aligned} \frac{dT_A}{\sqrt{1 - \frac{u_0^2}{c^2}}} &= \frac{dT_A \sqrt{1 - \frac{2GM}{c^2 R}}}{\sqrt{1 - \frac{u^2}{c^2}}} \\ \Rightarrow \frac{\sqrt{1 - \frac{u^2}{c^2}}}{\sqrt{1 - \frac{u_0^2}{c^2}}} &= \left(1 - \frac{2GM}{c^2 R}\right)^{1/2} \end{aligned} \quad (6)$$

Taking the gravitational potential energy as zero at ∞ and applying law of conservation of energy to the particle with rest mass m_0 described in the beginning of this section,

$$\begin{aligned} &(\text{Sum total of rest and kinetic Energy})_{At \infty} \\ &= (\text{Sum total of rest and kinetic Energy})_{At R} + (\text{Gravitational Potential Energy})_{At R} \end{aligned}$$

$$\Rightarrow (P.E.)_{At R} = \frac{m_0 c^2}{\sqrt{1 - \frac{u_0^2}{c^2}}} - \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\Rightarrow (P.E.)_{At R} = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \left(\frac{\sqrt{1 - \frac{u^2}{c^2}}}{\sqrt{1 - \frac{u_0^2}{c^2}}} - 1 \right)$$

$$\text{Or } (P.E.)_{At R} = mc^2 \left(\frac{\sqrt{1 - \frac{u^2}{c^2}}}{\sqrt{1 - \frac{u_0^2}{c^2}}} - 1 \right) \quad (7)$$

Where $m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$ = Relativistic mass of particle.

Using Eq. (6) in (7), we get,

$$(P.E.)_{At R} = mc^2 \left[\left(1 - \frac{2GM}{c^2 R}\right)^{1/2} - 1 \right] \quad (8)$$

Thus, we have derived the exact expression for the relativistic gravitation potential energy which is same as Eq. (2) using the principle of equivalence, gravitational time dilation and special theory of relativity.

3. Conclusion

General theory of relativity is the exact theory of gravity which has seen no violation of its predictions in any of the experimental observations conducted so far. In this theory, dynamics of the material bodies are determined by curvature of space-time given by Einstein's field equation which takes into account only the distribution of energy/momentum due to matter and radiation excluding gravitational potential energy. However, as covariant divergence of energy-momentum tensor for law of energy conservation includes a term related to curvature in space-time, this curvature can be interpreted as representing gravitational potential energy. In some branches of physics such as Hamiltonian approach, exact (general relativistically correct) form of this gravitational potential energy is required. In this paper, we have presented a sequence of events through which we could derive the general relativistic expression for gravitational potential energy simply by using the principle of equivalence, gravitational time dilation and expressions of special theory of relativity. Our expression exactly matches with the expression conventionally derived using full formality of general relativity. In the special case when the distance is much more than Schwarzschild radius, general relativistic expression for gravitational potential energy reduces to Newtonian expression of gravitational potential energy.

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