ADAPTIVE SWITCHING CONTROL APPLIED TO MULTIVARIABLE SYSTEMS

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

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In loving memory of my father

and

to our family
Therefore, the art of employing troops is that when the enemy occupies high ground, do not confront him: with his back resting on hills, do not oppose him. When he pretends to flee, do not pursue. Do not attack his elite troops. Do not gobble preferred baits. Do not thwart an enemy returning homewards. To a surrounded enemy you must leave a way of escape. Do not press an enemy at bay.

*Sun Tzu Ping Fa*
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Abstract

In this thesis, a family of adaptive control problems is examined and solved using robust self-tuning switching controllers. The motivation for using this type of controller is that, often in practise, no suitable mathematical model of the system to be controlled is available: conventional methods of adaptive controller design generally require specific \textit{a priori} plant information (e.g. it may have to be known if the plant is minimum phase), and thus cannot be implemented if such a knowledge is not known.

In contrast, this thesis shall generally assume that very little \textit{a priori} plant information is known - the main assumption being that the plant can be modelled by a finite dimensional linear time invariant (LTI) system. More specifically, for the adaptive control problem of a family of not necessarily strictly proper multi-input multi-output (MIMO) plants, a switching mechanism which requires less \textit{a priori} system information than previously considered is proposed. Utilizing this framework, various new self-tuning controllers then are presented, which solve the adaptive stabilization problem and the robust servomechanism problem for potentially unknown MIMO systems.

The proposed controllers appear to be quite attractive in their overall improved tuning transient response when compared with earlier results. Real-time experimental results of one particular class of switching controllers when applied to a multivariable hydraulic apparatus are presented, and illustrate the feasibility of applying such adaptive controllers to industrial process control problems.
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Chapter 1

Introduction

In the conventional design of controllers for multivariable systems, the general approach often adopted is to find a suitable nominal model for the plant, which is often a difficult task, and then to design a controller based upon this nominal model. In this case, if the nominal model captures sufficient dynamical aspects of the plant, effective control of the system may be possible to attain. However, if large unexpected structural changes subsequently occur in the system, severe limitations in practical performance will, in general, arise since conventional control schemes usually do not have the ability to control systems which are subject to unplanned extreme changes.

Currently, one method to deal effectively with the specific problem of parametric plant uncertainty is adaptive control; typically, the controllers employed in such schemes are nonlinear and time-varying, and consist of a compensator augmented with a tuning mechanism which adjusts the compensator gains to match a prespecified desired plant model. Although recent attempts to enhance the robustness properties [42], [43], improve the closed loop tuning transient response [80], [81], [10], weaken the restrictive sufficient a priori assumptions [69], [74], [70], [99], and eliminate [77] the unwanted “bursting” phenomena [3] have been successful for this class of controllers, these results generally are limited in scope due to the nature of the a priori assumptions still required.

During the past several years, however, switching control in both theory and practice has been applied successfully to multivariable systems to accomplish a wide variety of tasks [13], [101], [34], [59], [61], [64], [62], [65], [66], [72], [79], [12], [15], [80], [16], [19], [81], [10], [23], [17], [73], [97]; while many controllers of this type which use very little a priori plant
information have also traditionally enjoyed the extra benefit of being very robust to large plant uncertainties, one particular disadvantage of some of these schemes commonly has been an unpleasant closed loop susceptibility to substantial output transient responses. A brief review of some of the important contributions made in this area can be found in [57] and [58].

In this thesis, the primary focus therefore will be on designing simple robust adaptive switching control algorithms which attempt to use as little a priori system information as possible, and which attempt to provide a reasonable closed loop transient response. For instance, by now allowing for the possibility of cyclic switching to occur in the adaptive control problem for a finite family of MIMO plants considered by Miller and Davison [65], decreased closed loop tuning transients and a priori system assumptions, as well as reduced switching controller complexity, usually can be attained. Similarly, by utilizing this framework to solve the adaptive stabilization problem and the robust servomechanism problem for possibly unknown MIMO systems, corresponding desired transient improvements for these problems can also generally be achieved by selecting non-pathological controller and tuning parameters.

In order to determine the feasibility of applying such controllers to an industrial system, real-time application studies using one such class of controllers with almost no a priori plant information, when applied to an experimental multivariable hydraulic system, are carried out; these studies show the feasibility of using the obtained controllers in an industrial-type setting and, in addition, show that the controllers applied display desirable integrity features.

Before beginning, however, the following preliminary information will be required.

1.1 Notation

The following mathematical notation will be used in a fairly consistent manner throughout this thesis.

Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{C}$ denote respectively the set of real, positive real, natural, and complex numbers; $\mathbb{R}^n$ ($\mathbb{C}^n$) will be the n-dimensional real (complex) vector space, $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) the set of $m \times n$ real (complex) matrices, $\mathbb{C}^-$ ($\mathbb{C}^+$) the set of complex numbers with strictly negative (positive) real parts, and $\mathbb{C}^0$ the set of complex numbers lying strictly on
the imaginary axis. For any \(x, y \in \mathbb{N}\),

\[
x \mod y := x - \text{floor} \left( \frac{x}{y} \right) y
\]

where floor(*) rounds the expression * down to the nearest integer.

With \(x = [x_1 \ x_2 \ldots \ x_n]^T \in \mathbb{R}^n\), denote its \(p\)-norm as

\[
\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}, \quad p \in \mathbb{N}.
\]

and its \(\infty\)-norm as

\[
\|x\|_\infty = \|x\| := \max_{1 \leq i \leq n} |x_i|.
\]

For any arbitrary \(A \in \mathbb{C}^{n \times n}\), let \(\lambda(A)\) and \(\text{eig}(A)\) denote the eigenvalues of \(A\). Matrix \(A \in \mathbb{R}^{n \times n}\) will be said to be stable if \(\lambda(A) \subset \mathbb{C}^{-}\) and unstable otherwise.

For the general case when \(A \in \mathbb{R}^{m \times n}\), \(A^T\) will denote its matrix transpose, \(\text{rank}(A)\) its rank, and, if \(A\) also has full row rank, \(A^\dagger = A^T(AA^T)^{-1}\) its pseudo-inverse. In addition, the corresponding induced norms of \(A\) will be denoted as \(\|A\|_p\), with the \(\infty\)-norm calculated as

\[
\|A\|_\infty = \|A\| := \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|
\]

and the Frobenius norm given as

\[
\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}.
\]

where \(a_{ij}\) denotes the \((i, j)\) element of \(A\).

As a final point, let \(C^\infty(\mathbb{R}^n)\) denote the set of \(\mathbb{R}^n\)-valued functions defined on \(\mathbb{R}^+\) which are infinitely differentiable. A function \(f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^n\) will be said to lie in \(\mathcal{L}_\infty\) (\(f \in \mathcal{L}_\infty\)) if

\[
\|f\|_\infty = \|f\|
\]
1.2 Some Motivation

As mentioned earlier, during the past several years, there has been a considerable amount of interest and effort made towards developing controller design methods which require as little a priori plant information as possible [54], [34], [64], [62], [65], [72], [23]. The motivation for this interest stems from the fact that it is generally difficult and often impossible to obtain an accurate model representation of an actual industrial plant. As well, while conventional adaptive controllers currently have the ability to deal effectively with the problem of parametric plant uncertainty, important a priori plant information still is required; for example, in conventional model reference adaptive control for a single-input single-output (SISO) system, the four classical assumptions typically made are that [68], [82]:

(i) the plant is minimum phase;
(ii) an upper bound on the plant order exists and is known;
(iii) the relative degree is known; and
(iv) the sign of the high frequency gain is known.

Although recent developments have been able to remove condition (iv) [69], [74], and to weaken conditions (ii) [70] and (iii) [71], [98], given above, specific plant information (e.g. any plant zeros which lie in the open right half plane must be known to lie in a finite set [63]) still is needed. For an excellent historical overview of some of the major advancements in this area, see [75] and [76].

In this thesis, the primary focus will be on simple prerouted adaptive switching control algorithms\(^1\) which attempt to use as little a priori system information as possible; as well, due to potential implementation constraints, this work will also be concerned with robust

\[ := \sup_{t \geq 0} \| f(t) \| \]

exists.

\(^1\)A switching algorithm is said to be prerouted if the potential sequence of applied controllers is determined off-line, and fixed prior to the application of the switching mechanism.
adaptive schemes (cf. [90], for example, and the results contained therein) which are tolerant to bounded immeasurable noise disturbances, and which attempt to provide a reasonable closed loop transient response.

As an illustration of the type of improvement that may be obtained by using the proposed controllers, consider the problem in [85] of finding a stabilizing controller (in the sense that \( x(t) \to 0 \) as \( t \to \infty \) and \( [x \; \eta]^T \in L_\infty \), where \( x(t) \) and \( \eta(t) \) are, respectively, the state of the plant and controller) of the form

\[
\dot{\eta}(t) = f(y(t), \eta(t)),
\]
\[
u(t) = g(y(t), \eta(t))
\]

for the one-dimensional SISO plant

\[
\dot{x}(t) = ax(t) + bu(t),
\]
\[
y(t) = x(t)
\]

with both \( b \neq 0 \) and \( a > 0 \) unknown. In this instance, using the Mårtensson-type [54] controller [34]

\[
\dot{\eta}(t) = y^2(t),
\]
\[
h(\eta) = \sqrt{\log \eta},
\]

and
\[
u(t) = y(t) \cdot h(\eta(t))^{0.25} \cdot (\sin(\sqrt{h(\eta(t))}) + 1) \cdot \cos(h(\eta(t)))
\]

with \( a := 1, \ b := -1, \ x(0) := 1, \) and \( \eta(0) := 1, \) the undesirably large transient response (with a peak overshoot greater than 311000 in magnitude) presented in Figure 1.1 is obtained. Similarly, using the adaptive Nussbaum-type stabilizer [85], [74, pg. 549]

\[
\dot{\eta}(t) = y^2(t),
\]
\[
u(t) = y(t)\eta^2(t) \cos(\eta(t))
\]

with

\[
y(t) = x(t) + \epsilon(t),
\]
one can see that an arbitrarily small persistent measurement disturbance may destabilize the system if $y^2(t)$ is nonintegrable due to variations in $\epsilon(t)$. This effect can be seen in Figure 1.2, where the controller given by (1.3) is applied to (1.1) with $a := 1, b := 1, x(0) := 1, \eta(0) := 0,$ and $\epsilon(t) := 0.25 \sin(100t)$.

In contrast, using one of the new controllers proposed later in this thesis (Controller S1), the more desirable responses shown in Figures 3.1 and 3.2 are obtained, which correspond to the respective identical examples given for Figures 1.1 and 1.2.

### 1.3 Thesis Outline

The remainder of this thesis is organized as follows.

In Chapter 2, the adaptive control problem for a finite family of MIMO LTI plants is considered from the point of view of either stabilization or servomechanism control, and new theoretical results for a generalized class of switching controllers are obtained. Chapter 3 then re-examines the adaptive stabilization problem for the case when the MIMO LTI plant is almost entirely unknown (i.e. it is assumed only that the plant is stabilizable and detectable). In Chapter 4, the case of potentially unknown open loop stable MIMO plants is considered, and self-tuning PI and PID controllers which solve the robust servomechanism problem for constant reference and constant disturbance inputs are proposed: the corresponding PI case with control input constraints is next examined in Chapter 5. These servomechanism controller results are generalized in Chapter 6 for the case of possibly unknown MIMO plants, which may be open loop unstable, by applying an alternative method for resolving the adaptive servomechanism problem. Real-time experimental results of one class of controllers when applied to a multivariable hydraulic system then are presented in Chapter 7, showing the successful implementation of the PI and PID controllers developed in Chapters 4 and 5.
1.3. Thesis Outline

Figure 1.1: Simulated results of $y(t)$ with (1.2) applied to (1.1).

Figure 1.2: $(e(t) := 0.25 \sin(100t))$ Simulated results with (1.3) applied to (1.1) with $x$ (solid) and $\eta$ (dashed).
Chapter 2

Adaptive Switching Control of LTI MIMO Systems

In this chapter, we assume that the plant to be controlled can be modelled by a finite dimensional LTI system which is contained in a specified finite family of plant models. For each of these plant models, it is assumed that there is an associated servomechanism controller (which has been separately designed), and it is desired to obtain a switching controller which has the property that it will select the correct stabilizing controller from this family of controllers using as little structural information as possible. Moreover, a switching control mechanism which is robust in nature to all bounded piecewise continuous references and disturbances, and which requires less a priori system information than previously assumed in [65], is presented. As well, plant detectability and the construction of a bounding function $f$ are shown to be sufficient to ensure that switching eventually stops. Simulation results using this new controller are also presented, and compared with the corresponding output responses obtained using the schemes given in [65] and [73].

2.1 Switching Control for General Controller Structures

In this section, switching control for a finite set of $s$ plants, subject to the general control law

$$
\mathcal{K}_i : \begin{cases} 
\dot{\eta} &= G_i \eta + H_i y + J_i y_{ref} \\
u &= K_i \eta + L_i y + M_i y_{ref} 
\end{cases} , \quad i \in \{1, 2, \ldots, s\}, \quad (2.1)
$$
will be considered with $s \in \mathbb{N}$, $\eta \in \mathbb{R}^{\eta_1}$, $G_i \in \mathbb{R}^{\eta_1 \times \eta_i}$, $H_i \in \mathbb{R}^{\eta_i \times r}$, $J_i \in \mathbb{R}^{r \times r}$. $K_i \in \mathbb{R}^{m \times \eta_i}$, $L_i \in \mathbb{R}^{m \times r}$, and $M_i \in \mathbb{R}^{m \times r}$.

### 2.1.1 Preliminary Definitions and Results

Let each element

$$P_i := (A_i, B_i, C_i, D_i, E_i, F_i), \quad i \in \{1, 2, \ldots, s\}, \quad s \in \mathbb{N}$$

belonging to the finite set\(^1\) of possible plants to be controlled

$$P := \bigcup_{i=1}^{s} P_i$$

be of the finite dimensional form

\[
\begin{align*}
\dot{x} &= A_i x + B_i u + E_i w. \tag{2.2a} \\
y &= C_i x + D_i u + F_i w. \tag{2.2b} \\
e &= y_{\text{ref}} - y \tag{2.2c}
\end{align*}
\]

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^r$ is the plant output to be regulated, $w \in \mathbb{R}^q$ is the disturbance, and $e \in \mathbb{R}^r$ is the difference between the specified reference input $y_{\text{ref}}$ and the output $y$. In the discussions which will follow, we do not necessarily assume that $\eta_i, A_i, B_i, C_i, D_i, E_i,$ or $F_i$ are known, and we do not restrict $\lambda(A_i) \subset \mathbb{C}^-$, $i \in \{1, 2, \ldots, s\}$.

Observe that upon applying Controller $K_i$ to control plant model $P_i$, the resulting closed loop system is

\[
\begin{align*}
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} &= \begin{bmatrix}
A_i + B_i L_i \tilde{I}_i C_i & B_i K_i + B_i L_i \tilde{I}_i D_i K_i \\
H_i \tilde{I}_i C_i & G_i + H_i \tilde{I}_i D_i K_i
\end{bmatrix} \begin{bmatrix}
x \\
\eta
\end{bmatrix} + \begin{bmatrix}
B_i \\
0
\end{bmatrix} \begin{bmatrix}
y_{\text{ref}} \\
w
\end{bmatrix}, \tag{2.3a}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
y \\
\eta
\end{bmatrix} &= \begin{bmatrix}
\tilde{I}_i C_i & \tilde{I}_i D_i K_i \\
0 & I
\end{bmatrix} \begin{bmatrix}
x \\
\eta
\end{bmatrix} + \begin{bmatrix}
\tilde{I}_i D_i M_i & \tilde{I}_i F_i
\end{bmatrix} \begin{bmatrix}
y_{\text{ref}} \\
w
\end{bmatrix}. \tag{2.3b}
\end{align*}
\]

---

\(^1\)Since $P_i$ is a 6-tuple, a slight abuse of notation is used to define $P$ in an effort to maintain notational simplicity.
where

\[
\tilde{I}_i := (I - D_i L_i)^{-1} \quad (2.3c)
\]

and

\[
B_i := \begin{bmatrix}
B_i M_i + B_i L_i \tilde{I}_i D_i M_i & E_i + B_i L_i \tilde{I}_i F_i \\
J_i + H_i \tilde{I}_i D_i M_i & H_i \tilde{I}_i F_i
\end{bmatrix} \quad (2.3d)
\]

Preliminary definitions and results which are needed before proceeding are given as follows.

**Definition 2.1:** Consider matrices \((C, A, B) \in \mathbb{R}^{r \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\). Then \((C, A)\) is said to be *detectable* if there exists a matrix \(K \in \mathbb{R}^{n \times r}\) such that \(\lambda(A + KC) \subseteq \mathbb{C}^-\), and \((A, B)\) is said to be *stabilizable* if there exists a matrix \(L \in \mathbb{R}^{m \times n}\) such that \(\lambda(A + BL) \subseteq \mathbb{C}^-\).

**Definition 2.2:** ([31, pg. 17]) The *transmission zeros* of \((C, A, B, D) \in \mathbb{R}^{r \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{r \times m}\) are defined to be the set of complex numbers \(\lambda\) which satisfy the following inequality:

\[
\text{rank} \begin{bmatrix}
A - \lambda I & B \\
C & D
\end{bmatrix} < n + \min(m, r).
\]

**Definition 2.3:** A plant \(P_i\) (2.2) is said to be *minimum phase* if all of its transmission zeros lie in \(\mathbb{C}^-\); otherwise, plant \(P_i\) is said to be *non-minimum phase*.

**Definition 2.4:** ([26]) Consider the triple \((C, A, B) \in \mathbb{R}^{r \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\); then the set of *centralized fixed modes* of \((C, A, B)\), denoted by \(\Lambda(C, A, B)\), is defined as follows:

\[
\Lambda(C, A, B) := \bigcap_{K \in \mathbb{R}^{m \times r}} \lambda(A + BK C).
\]

**Definition 2.5:** ([24]) Consider \((C, A, B, D) \in \mathbb{R}^{r \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{r \times m}\); then the set of *decentralized fixed modes* of \((C, A, B, D)\) with respect to \(K\) denoted by \(\Lambda(C, A, B, D)\), is defined as follows:

\[
\Lambda(C, A, B, D) := \bigcap_{K \in \mathbb{K}} \lambda(A + BK (I - DK)^{-1} C).
\]
where $K \in \mathbb{K}$ is also chosen such that $(I - DK)^{-1}$ exists.

**Remark 2.1:** Given $D \in \mathbb{R}^{r \times m}$, then for almost all [30] $L \in \mathbb{R}^{n \times r}$, $(I - DL) \in \mathbb{R}^{r \times r}$ is invertible (i.e. given a fixed matrix $D$, then $(I - DL)$ is invertible for generic $L$).

**Remark 2.2:** From Lemma 2.3.3 of [37, pg. 59], if $F \in \mathbb{R}^{n \times n}$ and $\|F\| < 1$, then $(I - F)$ is nonsingular (i.e. $(I - F)^{-1}$ exists).

**Definition 2.6:** ([64]) A function $f : \mathbb{N} \to \mathbb{R}^+$ is said to be a *strong bounding function* ($f \in \text{SBF}$) if it is strictly increasing and if

$$\frac{f(i + 1)}{f(i)} \to \infty$$

as $i \to \infty$.

**Definition 2.7:** A function $f : \mathbb{N} \to \mathbb{R}^+$ is said to be a *modified strong bounding function* ($f \in \text{MSBF}$) if it is strictly increasing and if, for all constants $(c_0, c_1, c_2) \in \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^+$,

$$\frac{f(i)}{c_0 + c_1(i - 1) + c_2 \sum_{j=1}^{i-1} f(j)} \to \infty$$

as $i \to \infty$.

**Proposition 2.1:** There exists a MSBF (e.g. $f(i) = i \exp(i^2)$).

**Proof:** The proof follows upon first observing that

$$\sum_{j=1}^{i-1} f(j) \leq \int_1^{i} \tau \exp(\tau^2) d\tau \leq \exp(i^2)$$

for $i \geq 2$. In addition, since

$$\frac{i \exp(i^2)}{c_0 + c_1(i - 1) + c_2 \sum_{j=1}^{i-1} j \exp(j^2)} \geq \frac{i \exp(i^2)}{c_0 + c_1(i - 1) + c_2 \exp(i^2)} =: \Lambda(i)$$
for \((c_0, c_1, c_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\), and since

\[
\lim_{i \to \infty} \lambda(i) \to \infty.
\]

the result immediately follows.

**Proposition 2.2:** Assume that \(\bar{A}_i\) given in (2.3) is stable for a given choice of \((G_i, H_i, K_i, L_i)\); then

\[
\lambda(\bar{A}_i) \not\subset \mathbb{C}^0 \text{ for almost all } (G_i, H_i, K_i, L_i).
\]

**Proof:** The proof follows upon observing that \(\bar{A}_i\) can alternatively be written as

\[
\begin{bmatrix}
A_i & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
B_i & 0 & B_i & 0 \\
0 & I & 0 & I
\end{bmatrix}
\begin{bmatrix}
L_i \tilde{I}_i & K_i & 0 & 0 \\
H_i \tilde{I}_i & G_i & 0 & 0 \\
0 & 0 & 0 & L_i \tilde{I}_i D_i K_i \\
0 & 0 & 0 & H_i \tilde{I}_i D_i K_i
\end{bmatrix}
\begin{bmatrix}
C_i & 0 \\
0 & I \\
0 & I
\end{bmatrix}:
\]

(2.4)

using Definition 2.4 and the fact that \(\bar{A}_i\) is stable for a given choice of controller parameters \((G_i, H_i, K_i, L_i)\). It therefore follows that \(\bar{A}_i\) has no fixed modes lying in \(\mathbb{C}^0\) for all feedback matrices

\[
\bar{K}_i := \begin{bmatrix}
L_i \tilde{I}_i & K_i & 0 & 0 \\
H_i \tilde{I}_i & G_i & 0 & 0 \\
0 & 0 & 0 & L_i \tilde{I}_i D_i K_i \\
0 & 0 & 0 & H_i \tilde{I}_i D_i K_i
\end{bmatrix}
\]

satisfying the structure

\[
\begin{bmatrix}
* (I - D_i*)^{-1} & * & 0 & 0 \\
* (I - D_i*)^{-1} & * & 0 & 0 \\
0 & 0 & 0 & * (I - D_i*)^{-1} D_i * \\
0 & 0 & 0 & * (I - D_i*)^{-1} D_i *
\end{bmatrix}
\]

where \(*\) is an arbitrary matrix having appropriate dimensions. Hence, it immediately follows [30] that for almost all controller parameters \((G_i, H_i, K_i, L_i)\), \(\lambda(\bar{A}_i) \not\subset \mathbb{C}^0\).
Remark 2.3: When $D_i = 0$ for the proof of Proposition 2.2, one can alternatively express $\hat{A}_i$ as

$$
\begin{bmatrix}
A_i + B_i L_i C_i & B_i K_i \\
H_i C_i & G_i
\end{bmatrix} = 
\begin{bmatrix}
A_i & 0 \\
0 & I
\end{bmatrix} + 
\begin{bmatrix}
B_i & 0 \\
0 & I
\end{bmatrix} 
\begin{bmatrix}
L_i & K_i \\
H_i & G_i
\end{bmatrix} 
\begin{bmatrix}
C_i & 0 \\
0 & I
\end{bmatrix}.
$$

\[ \diamond \]

Lemma 2.1: Consider the system given in (2.2) with $u(t)$ given in (2.1) applied at time $t = 0$. Assume that in (2.3), the matrix

$$
\hat{A}_i := 
\begin{bmatrix}
A_i + B_i L_i \dot{I}_i C_i & B_i K_i + B_i L_i \dot{I}_i D_i K_i \\
H_i \dot{I}_i C_i & G_i + H_i \dot{I}_i D_i K_i
\end{bmatrix}
$$

is stable, and that $y_{ref}(t)$ and $w(t)$ are bounded piecewise continuous signals having respective $\mathcal{L}_\infty$ norms of $\tilde{y}_{ref}$ and $\tilde{w}$. Then there exist constants $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ independent of $z(0) = z_0 := [x(0)^T \eta(0)^T]^T$ such that

$$
\|z(t)\| \leq \zeta_1 \|z(0)\| + \zeta_2,
$$

and

$$
\|e(t)\| \leq \zeta_3 \|z(0)\| + \zeta_4
$$

for all $t \in [0, \infty)$.

Proof: Since $\hat{A}_i$ is stable, there exist constants $(\lambda, \zeta_1) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\|e^{\hat{A}_i t}\| \leq \zeta_1 e^{-\lambda t}$ for $t \geq 0$; using the additional fact that

$$
z(t) = e^{\hat{A}_i t} z_0 + \int_0^t e^{\hat{A}_i (t-\tau)} (\hat{B}_i^1 y_{ref}(\tau) + \hat{B}_i^2 w(\tau)) d\tau.
$$

where

$$
\begin{bmatrix}
\hat{B}_i^1 \\
\hat{B}_i^2
\end{bmatrix} :=
\begin{bmatrix}
B_i M_i + B_i L_i \dot{I}_i D_i M_i & E_i + B_i L_i \dot{I}_i F_i \\
J_i + H_i \dot{I}_i D_i M_i & H_i \dot{I}_i F_i
\end{bmatrix}.
$$

therefore

$$
\|z(t)\| \leq \zeta_1 \|z_0\| + \underbrace{\frac{\zeta_1}{\lambda} (\|\hat{B}_i^1\| \cdot \tilde{y}_{ref} + \|\hat{B}_i^2\| \cdot \tilde{w})}_{\zeta_2}.
$$
Likewise, since

\[ e = y_{\text{ref}} - y \]

\[ = [-\ddot{I}_i C_i - \ddot{I}_i D_i K_i] x + (I - \ddot{I}_i D_i M_i) y_{\text{ref}} - \ddot{I}_i F_i w, \]

it therefore follows that

\[
\|e(t)\| \leq \|[\ddot{I}_i C_i - \ddot{I}_i D_i K_i]\| \cdot (\zeta_1 \|z_0\| + \zeta_2) + \|I - \ddot{I}_i D_i M_i\| \cdot \dot{y}_{\text{ref}} + \|[\ddot{I}_i F_i]\| \cdot \dot{w}
\]

\[
= \zeta_3 \|z_0\| + \zeta_4
\]

where

\[
\zeta_3 := \|[\ddot{I}_i C_i - \ddot{I}_i D_i K_i]\| \cdot \zeta_1,
\]

\[
\zeta_4 := \|[\ddot{I}_i C_i - \ddot{I}_i D_i K_i]\| \cdot \zeta_2 + \|I - \ddot{I}_i D_i M_i\| \cdot \dot{y}_{\text{ref}} + \|[\ddot{I}_i F_i]\| \cdot \dot{w}
\]

for all \( t \in [0, \infty) \).

In order to consider the situation when \( y_{\text{ref}}(t) \) and \( w(t) \) are bounded continuous signals, and when \( D_i = 0 \) for all \( i \in \{1, 2, \ldots, s\} \), label **Controller F1** as

\[
\dot{\eta}(t) = G(t) \eta(t) + H(t) y(t) + J(t) y_{\text{ref}}(t), \quad \eta(t_k^-) \equiv 0, \quad t \in (t_k, t_{k+1}]
\]

\[
u(t) = K(t) \eta(t) + L(t) y(t) + M(t) y_{\text{ref}}(t)
\]

where \( k \in \{1, 2, 3, \ldots\}, \ i := ((k - 1) \mod s) + 1, \)

\[
(G(t), H(t), J(t), K(t), L(t), M(t)) := (G_i, H_i, J_i, K_i, L_i, M_i), \quad t \in (t_k, t_{k+1}]
\]

\( t_1 := 0 \), and where, for each \( k \geq 2 \) such that \( t_{k-1} \neq \infty \), the switching time \( t_k \) is defined by

\[
t_k := \begin{cases}
\text{min } t & \exists \\
i) t > t_{k-1}, \text{ and} & \text{if this minimum exists} \\
ii) \|[(\eta(t)^T e(t)^T)^T] = f(k - 1) & \text{otherwise}
\end{cases}
\]
with $f \in \text{MSBF}$. In addition, let **Assumption F1** be the following:

1. $\|\eta(0)\| < f(1)$;
2. $\|e(0)\| < f(1)^2$;
3. for each plant $P_i$ and each corresponding applied Controller $\mathcal{K}_i$, $i \in \{1, 2, \ldots, s\}$, the closed loop system is stable (and controller parameters ($G_i$, $H_i$, $J_i$, $K_i$, $L_i$, $M_i$) provide acceptable error regulation/disturbance rejection when the plant $P_i$ is subject to bounded piecewise constant reference and disturbance inputs):
   1. for each plant $P_i$, ($C_i$, $A_i$) is detectable; and
   2. for each $i, j \in \{1, 2, \ldots, s\}$, ($I - D_iL_j$) is invertible (see Remark 2.1).

The switching mechanism described by Controller F1 is schematically shown in Figure 2.1.

![Figure 2.1: A schematic setup of Controller F1.](image)

In Controller F1, norm bounds on $\eta(t)$ and $e(t)$ are used in an attempt to detect closed loop instability which might be caused if Controller $\mathcal{K}_i$ is applied to plant $P_j$. $i \neq j$. If this upper bound is met at any time during the tuning process, then a controller switch occurs.

---

$^2$This condition is required so that switching time $t_k$ is well defined for Controller F1; given $e(0)$, it can be met easily by appropriately scaling $f(i)$. 

---
and $\eta$ is reset to zero immediately following this switch. This reset action is performed since all candidate feedback controllers need not necessarily be of the same order, and since past experimental investigations [14] have indicated that reduced tuning transient responses generally can be attained via such a scheme. However, for the case when all candidate controllers have the same order, i.e. when $g_i = g_j$ for all $i, j \in \{1, 2, \ldots, s\}$, $\eta(t_k^+)$ need not necessarily be reset to zero after each switch; one can choose to continue to form $\eta(t)$ using the set of piecewise LTI systems given by $(G_i, H_i, J_i)$ with $\eta(t_k^+) = \eta(t_k)$.

### 2.1.2 Main Results

#### Continuous Signals

For the situation when $D_i = 0$ for all $i \in \{1, 2, \ldots, s\}$, and when $y_{ref}(t)$ and $w(t)$ are bounded continuous signals, the following result can be obtained:

**Theorem 2.1**: Consider a plant $P \in P$ with $D_i = 0$, $i \in \{1, 2, \ldots, s\}$, and with Controller F1 applied at time $t = 0$; then for every $f \in \text{MSBF}$, for every bounded continuous reference and disturbance signal, and for every initial condition $z(0) := [x(0) \, \eta(0)]^T$ for which Assumption F1 holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$ and constant matrices $(G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ such that $(G(t), H(t), J(t), K(t), L(t), M(t)) = (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ for all $t \geq t_{ss}$;

ii) the controller states $\eta \in L_{\infty}$ and the plant states $x \in L_{\infty}$; and

iii) if the reference and disturbance inputs are constant signals, then for almost all controller parameters $(G_i, H_i, K_i, L_i)$, asymptotic error regulation occurs, i.e. $e(t) \to 0$ as $t \to \infty$.

#### Piecewise Continuous Signals

For the situation when $D_i \neq 0$ for some $i \in \{1, 2, \ldots, s\}$ and/or when $y_{ref}(t)$ or $w(t)$ are bounded piecewise continuous signals, the switching criterion given for time $t_k$ in Controller F1 may not be well-defined. In order to circumvent such potential problems, Controller F1
can be simply modified by filtering the error signal $e(t)$, and defining $e_f(t)$ as

$$
\dot{e}_f = -\lambda e_f + \lambda e, \; \lambda \in \mathbb{R}^+.
$$

(2.5)

Hence, label Assumption F2 and Controller F2 to be, respectively, Assumption F1 with $e(t)$ replaced by $e_f(t)$, and Controller F1 with $e(t)$ replaced by $e_f(t)$ in the definition of switching time $t_k$.

**Lemma 2.2:** Consider the closed loop system formed by augmenting (2.3) together with (2.5). Assume that $\lambda(A_i) \subset \mathbb{C}^-$ and that $y_{ref}(t)$ and $u(t)$ are bounded piecewise continuous signals. Then there exist constants $(\zeta_1, \zeta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ independent of $\hat{z}(0) := [x(0)^T \eta(0)^T e_f(0)^T]^T$ such that

$$
\|\hat{z}(t)\| \leq \zeta_1 \|\hat{z}(0)\| + \zeta_2
$$

for all $t \in [0, \infty)$.

The following result can now be obtained.

**Theorem 2.2:** Consider a plant $P \in P$ with Controller F2 applied at time $t = 0$: then for every $f \in \text{MSBF}$ and $\lambda \in \mathbb{R}^+$, for every bounded piecewise continuous reference and disturbance signal, and for every initial condition $\hat{z}(0) := [x(0)^T \eta(0)^T e_f(0)^T]^T$ for which Assumption F2 holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$ and constant matrices $(G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ such that $(G(t), H(t), J(t), K(t), L(t), M(t)) = (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ for all $t \geq t_{ss}$;

ii) the controller states $\eta \in L_\infty$, the plant states $x \in L_\infty$, and the filtered error signal $e_f \in L_\infty$; and

iii) if the reference and disturbance inputs are constant signals, then for almost all controller parameters $(G_i, H_i, K_i, L_i)$, asymptotic error regulation occurs, i.e. $e(t) \to 0$ as $t \to \infty$.

Controller F2 provides a great deal of generality and versatility since:
all finite dimensional MIMO LTI plant models $P_i$ are assumed to have the general form given by (2.2), with $A_i$ not necessarily stable, $D_i$ not necessarily equal to zero, and $P_i$ not necessarily minimum phase;

- the plant models $P_i$ need not be controllable and/or observable;

- the set of all admissible controllers need only satisfy the structure presented in (2.1), and thus the candidate controllers need not have the same dimension;

- the class of piecewise continuous reference and disturbance signals allowable for the servomechanism controller design [27] of $K_i$ (and the implementation of Controller F2) is quite large provided only that $y_{ref} \in L_\infty$ and $w \in L_\infty$ (e.g. the class of sinusoidal references and disturbances is allowed);

- a priori bounds on either $y_{ref}(t)$ or $w(t)$ are neither needed nor estimated for the proposed controller;

- no extensive a priori on-line calculations are needed in order to implement Controller F2 (cf. the schemes given in [34] and [65]);

- the controller switching mechanism is very simple to implement in real-time, and is therefore attractive from a practical point of view;

- the switching mechanism does not depend directly on any explicit knowledge of the matrices associated with plant $P_i$ or candidate Controller $K_i$;

- no on-line estimation period is needed; and

- the switching mechanism is robust and will not suffer from chattering in the steady-state (cf. [39], for instance) for all bounded piecewise continuous reference and disturbance inputs.

As well, Theorems 2.1 and 2.2 will clearly also hold even if the finite number of candidate controllers is greater than or equal to the number of possible plants\(^3\).

In addition, in Theorem 2.2, the requirement that $y_{ref}(t)$ and $w(t)$ be bounded piecewise continuous functions and the restriction that switching cannot occur infinitely fast guarantees the existence and uniqueness of a solution [41] to the set of differential equations given

\(^3\)In fact, Theorems 2.1 and 2.2 will hold for an infinite number of plants (e.g. see Section 2.2.4) so long as there exist a finite number of candidate controllers which satisfy Assumptions F1 and F2 respectively.
by (2.3) and (2.5). Furthermore, without any loss of properties i) to iii) given in Theorem 2.2, filtered error signal $e_f(t)$ could also have been defined as

$$
\dot{x}_f = A_f x_f + B_f e,
$$

(2.6a)

$$
e_f = C_f x_f
$$

(2.6b)

where $x_f \in \mathbb{R}^r$, $\lambda(A_f) \subset \mathbb{C}^-$, and $C_f$ and $B_f$ are both invertible.

In fact, for the general situation when plant $P_i$ is described by

$$
\begin{align*}
\dot{x} &= A_i x + B_i u + E_i w + N_i \mu_1, \\
y &= C_i x + D_i u + F_i w + Q_i \mu_2,
\end{align*}
$$

(2.7a)

(2.7b)

properties i) and ii) of Theorem 2.2 will also hold for all bounded piecewise continuous noise signals $(\mu_1, \mu_2) \in \mathbb{R}^{\mu_1} \times \mathbb{R}^{\mu_2}$ with $(N_i, Q_i) \in \mathbb{R}^{n_x \times \mu_1} \times \mathbb{R}^{r \times \mu_2}$. This follows since the closed loop system with Controller $K_i$ applied may be expressed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{\eta} \\
\dot{e}_f \\
\dot{z}
\end{bmatrix} = A^{cl} \begin{bmatrix}
x \\
\eta \\
e_f \\
z
\end{bmatrix} + B^{cl} \begin{bmatrix}
y_{ref} \\
w \\
\mu_1 \\
\mu_2
\end{bmatrix},
$$

where

$$
I := (I - D_i L_i)^{-1},
$$

$$
A^{cl} := \begin{bmatrix}
A_i + B_i L_i \tilde{I} C_i & B_i K_i + B_i L_i \tilde{I} D_i K_i & 0 \\
H_i \tilde{I} C_i & G_i + H_i \tilde{I} D_i K_i & 0 \\
-\lambda \tilde{I} C_i & -\lambda \tilde{I} D_i K_i & -\lambda I
\end{bmatrix},
$$

and

$$
B^{cl} := \begin{bmatrix}
B_i M_i + B_i L_i \tilde{I} D_i M_i & E_i + B_i L_i \tilde{I} F_i & N_i & B_i L_i \tilde{I} Q_i \\
J_i + H_i \tilde{I} D_i M_i & H_i \tilde{I} F_i & 0 & H_i \tilde{I} Q_i \\
\lambda I - \lambda \tilde{I} D_i M_i & -\lambda \tilde{I} F_i & 0 & -\lambda \tilde{I} Q_i
\end{bmatrix}.
$$

Moreover, if $\|[\mu_1^T \mu_2^T]^T\| \to 0$, property iii) of Theorem 2.2 will once again be recovered. Since corresponding comments similar to those given here can also be made for Theorems
3.1, 3.2, 3.3, 4.2, 4.3, 4.4, 4.5, 6.1, and 6.2 (provided that the output signals are filtered accordingly) by using an identical argument, the details of these additional extensions will be omitted for brevity.

**Remark 2.4:** Let the switching time $t_k$ be defined as


t_k := \begin{cases} 
\min t \ni & \text{if this minimum exists} \\
\infty & \text{otherwise,}
\end{cases}

\text{if } i \ni t > t_{k-1}, \text{ and } \|\eta(t)\| = f(k - 1)

and define **Assumption A** to be Assumption F1 with the following additional condition:

vi) $H_j$ is left invertible for all $j \in \{1, 2, \ldots, s\}$ (i.e. $H_j$ has full column rank, and hence, there exists $H_j^T := (H_j^T H_j)^{-1} H_j^T$ such that $H_j^T H_j = I$).

Then properties i) to iii) of Theorem 2.1 will also hold for all bounded piecewise continuous reference and disturbance signals with $D_j$ not necessarily equal to zero for all $j$. This follows since, for any plant $P_m \in \mathcal{P}$, the pair

$$(H_j(I - D_m L_j)^{-1} C_m, A_m + B_m L_j(I - D_m L_j)^{-1} C_m)$$

is detectable. Furthermore, under Assumption A, if switching based upon filtered error signal $e_f(t)$ is still desired, then Theorem 2.2 will also hold for any $(C_f, A_f, B_f) \in \mathbb{R}^{r_f \times n_f} \times \mathbb{R}^{n_f \times n_f} \times \mathbb{R}^{n_f \times r}$ with $\lambda(A_f) \subset \mathbb{C}^-$.

**Remark 2.5:** Define

$$\dot{y}_f := -\lambda y_f + \lambda y,$$

and $$\dot{u}_f := -\lambda u_f + \lambda u,$$

where $\lambda \in \mathbb{R}^+$. Assume that

i) $\|y_f(0)\| < f(1)$; and

ii) $\|u_f(0)\| < f(1)$
2.1. Switching Control for General Controller Structures

both additionally hold. Then Theorem 2.2 will also hold using the switching criterion defined by

\[
t_k := \begin{cases} 
\min t \in & \\
\text{i) } t > t_{k-1} \text{, and} & \\
\text{ii) } \| \Xi(t) \| = f(k - 1) & \\
\infty & \text{otherwise.}
\end{cases}
\]

where

\[
\Xi(t) := [\eta(t)^T \ y_f(t)^T]^T.
\]

or \[
\Xi(t) := [\eta(t)^T \ e_f(t)^T \ y_f(t)^T]^T.
\]

or \[
\Xi(t) := [\eta(t)^T \ e_f(t)^T \ u_f(t)^T]^T.
\]

or \[
\Xi(t) := [\eta(t)^T \ y_f(t)^T \ u_f(t)^T]^T.
\]

or \[
\Xi(t) := [\eta(t)^T \ e_f(t)^T \ y_f(t)^T \ u_f(t)^T]^T.
\]

Given the general nature of Controller F2, one may also want to bound individually \( \eta(t) \) and \( e_f(t) \) by different bounding functions. This can be done as shown in the following definition and existence results.

**Definition 2.8:** Functions \( f_1 : \mathbb{N} \to \mathbb{R}^+ \) and \( f_2 : \mathbb{N} \to \mathbb{R}^+ \) are said to be *choosable modified strong bounding functions* (\( (f_1, f_2) \in \text{CMSBF} \)) if, for \( k \in \{1, 2\} \), \( f_k \) is strictly increasing and if, for all constants \( (c_0, c_1, c_2, c_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \),

\[
f_k(i) \rightarrow \infty \quad \text{as } i \rightarrow \infty.
\]

**Proposition 2.3:** There exist functions \( f_1 \) and \( f_2 \) such that \( (f_1, f_2) \in \text{CMSBF} \) (e.g., \( f_1(i) = \alpha i \exp(i^2) \) and \( f_2(i) = \beta i \exp(i^2) \)) where \( (\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ \).
Remark 2.6: Consider the switching criterion for Controller F2 given by

\[
\begin{align*}
    t_k := \begin{cases} 
        \min t \ni \\
        \text{i) } t > t_{k-1}, \text{ and } & \text{if this minimum exists} \\
        \text{ii) } ||\eta(t)|| = f_1(k-1) \text{ and/or } & ||e_f(t)|| = f_2(k-1) \\
        \infty & \text{otherwise}
    \end{cases}
\end{align*}
\]

where \((f_1, f_2) \in \text{CMSBF}\). Then with

i) \(||\eta(0)|| < f_1(1)\); and

ii) \(||e_f(0)|| < f_2(1)\):

and with conditions iii) to v) of Assumption F2 assumed to be true. Theorem 2.2 will also hold true in this situation.

Remark 2.7: The modifications given in Remark 2.6 can also be made to the switching mechanisms given in Theorems 3.2, 3.3, 4.2, 4.3, 4.4, 4.5, 6.1, and 6.2.

Remark 2.8: For simplicity, consider the case when \(D_i = 0, i \in \{1, 2, \ldots, s\}\); then the robust servomechanism problem can be solved (if possible) using the servocompensator design method given in [27]. For example, maintaining the structure, notation, and assumptions given in [27], with

\[
e := y - y_{\text{ref}}.
\]

a full order Luenberger observer of the form

\[
\dot{x} = (A_i + L_i C_i)x + B_i u - L_i y
\]

can be constructed to yield

\[
\dot{e} = (A_i + L_i C_i)e - L_i F_i w - E_i w
\]
where $\dot{e} := \dot{x} - x$. Since the closed loop system may be expressed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{\xi} \\
\dot{\hat{e}}
\end{bmatrix} =
\begin{bmatrix}
A_i + B_i P_0 & B_i P_i & B_i P_0 \\
B^* C_i & C^* & 0 \\
0 & 0 & A_i + \mathcal{L}_i C_i
\end{bmatrix}
\begin{bmatrix}
x \\
\xi \\
\hat{e}
\end{bmatrix} +
\begin{bmatrix}
0 & E_i \\
-B^* & B^* F_i \\
0 & -\mathcal{L}_i F_i - E_i
\end{bmatrix}
\begin{bmatrix}
y_{ref} \\
w
\end{bmatrix}.
$$

where

$$u = P_0 \dot{x} + P_i \xi,$$

one can therefore choose matrices $P_0$ and $P_i$ such that

$$
\begin{bmatrix}
A_i + B_i P_0 & B_i P_i \\
B^* C_i & C^*
\end{bmatrix} =
\begin{bmatrix}
A_i & 0 \\
B^* C_i & C^*
\end{bmatrix} +
\begin{bmatrix}
B_i \\
0
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_i
\end{bmatrix}
$$

is stable. Hence, the final controller may now be expressed in the form

$$
\begin{bmatrix}
\dot{\xi} \\
\dot{x} \\
\dot{\hat{e}}
\end{bmatrix} =
\begin{bmatrix}
C^* & 0 \\
B_i P_i & A_i + \mathcal{L}_i C_i + B_i P_0 \\
B^* \mathcal{L}_i 
\end{bmatrix}
\begin{bmatrix}
\xi \\
\dot{x} \\
\hat{e}
\end{bmatrix} +
\begin{bmatrix}
B^* \\
-\mathcal{L}_i \\
0
\end{bmatrix} y +
\begin{bmatrix}
-B^* \\
0
\end{bmatrix} y_{ref}.
$$

One can therefore apply switching Controller $\mathcal{F}^2$ to regulate and reject adaptively this particular class of bounded reference and disturbance signals.

Remark 2.9: For simulation purposes, with plant $P$ having system matrices given by

$$(A, B, C, 0, E, F),$$

and with Controller $\mathcal{K}_i$ applied, the closed loop system may be expressed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{\xi} \\
\dot{\hat{e}}
\end{bmatrix} =
\begin{bmatrix}
A & B P_i & B P_0 \\
B^* C & C^* & 0 \\
-\mathcal{L}_i C & B_i P_i & A_i + \mathcal{L}_i C_i + B_i P_0,
\end{bmatrix}
\begin{bmatrix}
x \\
\xi \\
\hat{e}
\end{bmatrix} +
\begin{bmatrix}
0 & E \\
-B^* & B^* F \\
0 & -\mathcal{L}_i F
\end{bmatrix}
\begin{bmatrix}
y_{ref} \\
w
\end{bmatrix},
$$

where $\hat{e} := \dot{x} - x$. Since the closed loop system may be expressed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{\xi} \\
\dot{\hat{e}}
\end{bmatrix} =
\begin{bmatrix}
A_i + B_i P_0 & B_i P_i & B_i P_0 \\
B^* C_i & C^* & 0 \\
0 & 0 & A_i + \mathcal{L}_i C_i
\end{bmatrix}
\begin{bmatrix}
x \\
\xi \\
\hat{e}
\end{bmatrix} +
\begin{bmatrix}
0 & E_i \\
-B^* & B^* F_i \\
0 & -\mathcal{L}_i F_i - E_i
\end{bmatrix}
\begin{bmatrix}
y_{ref} \\
w
\end{bmatrix}.
$$

where

$$u = P_0 \dot{x} + P_i \xi,$$
where the same notation as given in Remark 2.8 is maintained.

2.2 Simulation Results

In this section, the output response obtained by applying Controller F1 to a family of strictly proper plants will be considered. In the first example, results from the simulation will be compared with other output responses formed using the schemes given in [65] and [73]. For the second and third example, the unstable batch reactor model [25] will be used. This section is concluded with results obtained by using the family of plants given in [34].

2.2.1 A Family of Three SISO Plants

Consider the following family of three SISO LTI plant models taken from [65]:

Model \( P_1 \):

\[
\begin{aligned}
\dot{x} &= \begin{bmatrix} -0.75 & 7.25 \\ -2.25 & -8.25 \end{bmatrix} x + \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w, \\
y &= [ -1.2 \ 4.1 ] x
\end{aligned}
\]

with open loop eigenvalues of \(-4.5 \pm 1.5j\);

Model \( P_2 \):

\[
\begin{aligned}
\dot{x} &= \begin{bmatrix} 1.4 & 21.1 \\ -1.65 & -25.4 \end{bmatrix} x + \begin{bmatrix} -0.18 \\ 0.33 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w, \\
y &= [ -0.45 \ 7.0 ] x
\end{aligned}
\]

with open loop eigenvalues of 0.031 and \(-24.031\); and

Model \( P_3 \):

\[
\begin{aligned}
\dot{x} &= \begin{bmatrix} -1.9 & 10.9 \\ -1.3 & -12.2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w.
\end{aligned}
\]
2.2. Simulation Results

\[ y = \begin{bmatrix} 2 & -5 \end{bmatrix} x \]

with open loop eigenvalues of \(-3.535\) and \(-10.565\). As one can verify, using

**Controller \( K_1 \):** \( \dot{\eta} = e, \ u = -2.75\eta; \)

**Controller \( K_2 \):** \( \dot{\eta} = e, \ u = -2\eta + 7y; \) and

**Controller \( K_3 \):** \( \dot{\eta} = e, \ u = 25\eta - 6y \)

conditions iii) and iv) of Assumption F1 are both satisfied, and all controller-plant mismatches result in a closed loop unstable system\(^4\).

In Figures 2.2 and 2.3, the output response of the closed loop system with Controller F1 applied to plant \( P_3 \) is given for the case when \( f(k) = f_1(k) \) and \( f(k) = f_2(k) \) respectively, where

\[
(f_1(k), f_2(k)) := \begin{cases}
(20k, 50k), & 1 \leq k \leq 10 \\
\left(\left(\frac{k}{6}\right)^2 \exp\left(\frac{k}{6}\right)^3, \left(\frac{k}{6}\right)^2 \exp\left(\frac{k}{6}\right)^3\right), & k > 10;
\end{cases}
\]

for each figure.

\[ x_0 = x(0) := [1 \ 2]^T. \]

\( \eta(0) := 0, \ w(t) := 2, \) and \( y_{ref}(t) \) is a square wave (beginning at time \( t = 0 \)) having zero DC offset, a peak magnitude of 10, and a period of 20 seconds. Furthermore, in both instances, switches occur due to bounds on \( \eta(t_2) \) and \( \eta(t_3) \) being met or exceeded.

As can be seen, in each case, the transient response might be considered to be quite reasonable taking into account the fact that there are three different potential controller candidates which must be considered. In contrast, the transient peaks obtained using Controllers 1 and 2 given in [65] are substantially larger for the same set of controller candidates\(^5\). This occurrence is related in large part to the fact that Controller F1 has a potential cyclic switching action which may occur, whereas the switching mechanisms given in [65] will, at most, try each possible controller only once.

\(^4\)A controller-plant mismatch is said to occur if Controller \( K_i \) is applied to plant \( P_j \), where \( i \neq j \).

\(^5\)For Controllers 1 and 2, the peak magnitude of the output transients are approximately 2600 and 110 respectively.
2.2. Simulation Results

Figure 2.2: Simulated results with Controller F1 (having three candidate controllers) applied to plant $P_3$ for the case when $f(k) = f_1(k)$.

Figure 2.3: Simulated results with Controller F1 (having three candidate controllers) applied to plant $P_3$ for the case when $f(k) = f_2(k)$.
2.2. Simulation Results

Figure 2.4: Simulated results of plant $P_3$ with supervisory controller [73] applied.

For further comparison\footnote{The author acknowledges the gracious help and assistance of Wen-Chung Chang and A. S. Morse for providing the simulation code used to generate Figure 2.4.}. Figure 2.4 shows the output response of plant $P_3$ obtained when using the SISO supervisory control switching mechanism given in [73]. Here, dwell time $\tau_D$ is set to be equal to 0.1 seconds. with

$$\dot{\pi}_p := -10\pi_p + \epsilon_p^2, \quad p \in \{1, 2, 3\}.$$  

and Controller $\mathcal{K}_1$ is applied initially at time $t = 0$. Similar to the simulations shown in Figures 2.2 and 2.3,

$$w(t) := 2.$$

$$\eta(0) := 0.$$

and $x(0) := [1 \ 2]^T$.

Remark 2.10: The switching mechanisms for the controllers given in [65] and [73] generally necessitate more system information or \textit{a priori} computation than required for proposed Controllers $F_1$ and $F_2$. For instance, explicit use of the family of SISO transfer functions
2.2. Simulation Results

is needed in [73], while knowledge of each candidate

\[(A, B, C, 0, E, F)\]

system matrix is required in [65]. As such, it is conjectured that the switching mechanisms given by Controllers F1 and F2 will be more robust with respect to unmodelled plant perturbations and immeasurable noise disturbances than those mechanisms given in [65] and [73].

\[\text{Remark 2.11:} \] In Figure 2.4, dwell time \(\tau_D\) can be chosen to be 0.1 seconds since \(\tau_D\) can be made arbitrarily small without sacrificing performance in the absence of unmodelled dynamics and measurement errors [73].

In Figures 2.5 and 2.6, the system output response obtained (corresponding to Figures 2.2 and 2.3 respectively) for the case when \(P_2\) and \(K_2\) are replaced by Model \(P_2\):

\[
\begin{aligned}
\dot{x} &= \begin{bmatrix} -31 & -259 & -229 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w. \\
y &= \begin{bmatrix} 0 & 0 & 458 \end{bmatrix} x
\end{aligned}
\]

(with open loop eigenvalues of \(-1\) and \(-15 \pm 2j\) and Controller \(K_2\): \(\dot{\eta} = e. u = 2\eta + 3e\)

is shown. Using this new controller (\(K_2\)), one can verify that controller-plant mismatch \(K_2-P_3\) results in a stable closed loop system. This fact therefore accounts for the results shown in Figure 2.6, where \(K_2\) is chosen as the final steady state controller.\(^7\)

2.2.2 A Family of Ten MIMO Plants

In this example, we illustrate the implementation of Controller F1 when applied to the following (unstable batch reactor) MIMO plant taken from [25]:

\(^7\)In both Figures 2.5 and 2.6, all switches are due to the bound on \(\eta(t)\) being met or exceeded.
2.2. Simulation Results

Figure 2.5: (New $P_2$ and $K_2$) Simulated results with Controller F1 (having three candidate controllers) applied to plant $P_3$ for the case when $f(k) = f_1(k)$.

Figure 2.6: (New $P_2$ and $K_2$) Simulated results with Controller F1 (having three candidate controllers) applied to plant $P_3$ for the case when $f(k) = f_2(k)$.
Model $P_{10}$:

$$
\begin{pmatrix}
1.3800 & -0.2077 & 6.7150 & -5.6760 \\
-0.5814 & -4.2900 & 0 & 0.6750 \\
1.0670 & 4.2730 & -6.6540 & 5.8930 \\
0.0480 & 4.2730 & 1.3430 & -2.1040
\end{pmatrix}
\begin{pmatrix}
x \\
x + 5.6790 \\
x + 1.1360 \\
x + 1.1360
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
1.1360 \\
-3.1460
\end{pmatrix} u + E w,
$$

with open loop eigenvalues of 1.99, 0.0635, -5.0566, -8.6659. Let the corresponding controller ($K_{10}$) be

**Controller $K_{10}$:** $\dot{\eta} = e$, $u = \begin{pmatrix}
0.95 & -58.4 \\
79.3 & -0.72
\end{pmatrix} y + \begin{pmatrix}
-100 & 10000 \\
10000 & 100
\end{pmatrix} \eta$.

which has been designed to stabilize and regulate model $P_{10}$ subject to constant references and constant disturbances [25].

For simplicity, assume in the simulations that

$$
\begin{align*}
E &= [0.1, 0.1, 0.1, 0.1]^T, \\
F &= [0.1, 0.1]^T, \\
w(t) &= 2,
\end{align*}
$$

and set $x(0) := [1, 2, 3, 4]^T$, with all other initial conditions defined to be equal to zero at time $t = 0$. In addition, let $y_{ref}(t)$ (with $y_{ref}^1(t) = -y_{ref}^2(t)$ and $y_{ref}^1(t) := 10$ for $t \in [0, 10]$) be a square wave having zero DC offset, a peak magnitude of 10, and a period of 20 seconds.

The other potential controller candidates, obtained by using conventional controller design methods, are listed below:

**Controller $K_1$:** $\dot{\eta} = 5e$, $u = K(\eta + 3 \cdot 5e)$, $K := \begin{pmatrix}
0.0510 & -0.0198 \\
-0.0061 & 0.0239
\end{pmatrix}$;

**Controller $K_2$:** $\dot{\eta} = 2e$, $u = K(\eta + 2 \cdot 2e)$, $K := \begin{pmatrix}
-0.1316 & 1.3158 \\
1.3158 & -3.1579
\end{pmatrix}$;

**Controller $K_3$:** $\dot{\eta} = 2e$, $u = K(\eta + 2 \cdot 2e)$, $K := \begin{pmatrix}
-0.7000 & 1.3000 \\
-2.0000 & -2.0000
\end{pmatrix}$.
Controller $\mathcal{K}_4$: \[
\dot{\eta} = e, \ u = K (\eta + 2e), \ K := \begin{bmatrix} 0.3800 & -5.4000 \\ -0.6000 & 2.0000 \end{bmatrix};
\]

Controller $\mathcal{K}_5$: \[
\dot{\eta} = 0.15e, \ u = K (\eta + 0.25 \cdot 0.15e), \ K := \begin{bmatrix} 0.8520 & -3.4887 \\ 0.8824 & 0.3321 \end{bmatrix};
\]

Controller $\mathcal{K}_6$: \[
\dot{\eta} = e, \ u = K (\eta + e), \ K := \begin{bmatrix} 0.0945 & -0.4249 \\ 0.9272 & 1.2503 \end{bmatrix};
\]

Controller $\mathcal{K}_7$: \[
\dot{\eta} = e, \ u = K (\eta + e), \ K := \begin{bmatrix} -0.2846 & -0.9915 \\ -0.3421 & 0.0371 \end{bmatrix};
\]

Controller $\mathcal{K}_8$: \[
\dot{\eta} = e, \ u = K (\eta + e), \ K := \begin{bmatrix} -0.8241 & -0.1249 \\ 0.5050 & -0.6089 \end{bmatrix}; \text{ and}
\]

Controller $\mathcal{K}_9$: \[
\dot{\eta} = e, \ u = K (\eta + e), \ K := \begin{bmatrix} -0.9764 & -1.5901 \\ -3.4294 & 1.0485 \end{bmatrix}.
\]

Using the above controllers, one can verify that conditions iii) and iv) of Assumption F1 are both satisfied, and that all controller-plant ($P_{10}$) mismatches result in a closed loop unstable system except for Controller $\mathcal{K}_3$.

In Figure 2.7, simulation results with Controller F1 applied to plant $P_{10}$ and

\[f(k) := 20k \exp \left(\frac{k}{10}\right)^2\]

are given. In this instance, two switches occur due to the bounds on $\eta(t)$ and $e(t)$ which are met or exceeded at the respective times of 0.245 and 0.31 seconds, and switching stops once Controller $\mathcal{K}_3$ is applied. Similar to the output response shown in Figure 2.6, the above results again emphasize and illustrate the fact that the final steady-state controller gains may not necessarily correspond to the controller parameters associated with actual plant $P_m$. It is to be noted, however, that if the closed loop system consisting of plant $P_i$ with controller parameters $(G_j, H_j, J_j, K_j, L_j, M_j)$ is stable if and only if $i = j$, then the final controller gains applied will almost always be ensured to be $(G_i, H_i, J_i, K_i, L_i, M_i)$.

\footnote{For brevity, the system matrices of the other nine plants will not be given.}
2.2. Simulation Results

Figure 2.7: Simulated results with Controller F1 (having ten candidate controllers) applied to plant $P_{10}$ with $y_1$ (solid) and $y_2$ (dashed).

In contrast to these results, Figures 2.8 and 2.9 illustrate the output response of plant $P_{10}$ using the identical situation given for Figure 2.7, but with Controller $K_3$ now defined as follows:

Controller $K_3$: \[ \dot{\eta} = e, \quad u = \begin{bmatrix} -1.3776 & -0.0131 \\ -0.0131 & -1.3753 \end{bmatrix} y + \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} \eta. \]

In this instance, one can verify that Assumption F1 is satisfied, and that the closed loop system will indeed be stable if and only if Controller $K_{10}$ is applied.

2.2.3 A Family of Five MIMO Plants

In this example, Controller F1 will again be applied to (2.8)$^9$. As before, assume that

\[
E = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}^T, \\
F = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T, \\
w(t) = 2,
\]

$^9$In this example, however, the unstable batch reactor will be labelled as plant $P_3$. 


2.2. Simulation Results

Figure 2.8: Simulated results with Controller F1 (having ten candidate controllers) applied to plant $P_{10}$ with $y_1$ (solid) and $y_2$ (dashed).

Figure 2.9: Switching time instants with Controller F1 applied to plant $P_{10}$. (Controllers which are applied due to a previous bound on $\eta(t)$ or $e(t)$ being met or exceeded are marked by a ‘N’ or an ‘E’ respectively.)
and define

\[
x(0) := [1 \ 2 \ 3 \ 4]^T, \\
B^* := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
C^* := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
f(k) := 40k \exp\left(\frac{k}{5}\right)^2.
\]

Let all other initial conditions be equal to zero at time \( t = 0 \), and let \( y_{ref}(t) \) be a periodic triangular wave of the form shown in Figure 2.10. For completeness, the parameters of all candidate controllers are listed in Section B.1.

As one can verify, Assumption F1 is satisfied and all controller-plant \( (P_3) \) mismatches result in a closed loop unstable system. In Figures 2.10 and 2.11, the output response as well as the switching time instants of the closed loop system are given: here, all switches are due to bounds on \( e(t) \) being met or exceeded. Similar to the predicted theoretical results, Controller \( K_5 \) is also selected correctly (after approximately 0.6 seconds).

### 2.2.4 A Family of Unstable SISO Plants

In this last example, consider the respective set of unstable plants and controllers given by [34, pg. 1102]

\[
\begin{align*}
\dot{x}_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*}
\] (2.9a) (2.9b)
2.2. Simulation Results

Figure 2.10: Reference signals $y_{1\text{ref}}^1(t)$ (solid) and $y_{2\text{ref}}^2(t)$ (dash-dotted) (top plot); and switching time instants with Controller F1 applied to plant $P_3$ (bottom plot).

Figure 2.11: Simulated output response with Controller F1 (having five candidate controllers) applied to plant $P_3$ with $y_1$ (solid) and $y_2$ (dashed).
and

\[
\mathbf{K}(q) : \begin{cases} 
\dot{\eta} = G(q)\eta + H(q)y, \\
u = K(q)\eta + L(q)y 
\end{cases}
\]

where \( q \in [-0.5, 0.5] \), and

\[
G(q) := -30q^2 + 31q - 9, \\
H(q) := -150q^2 + 185q - 56, \\
K(q) := -5 + 6q, \\
L(q) := -31 + 30q;
\]

as one can verify, (2.9) is controllable and observable for all values of \( q \in \mathbb{R} \), and the closed loop system can be expressed as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{\eta}
\end{bmatrix} = 
\begin{bmatrix}
1 - 31q + 30q^2 & 1 & -5q + 6q^2 \\
-31 + 30q & 1 & -5 + 6q \\
-150q^2 + 185q - 56 & 0 & -30q^2 + 31q - 9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\eta
\end{bmatrix}
\]

where \( \lambda(\mathcal{A}) = \{-1, -2, -4\} \). The control objective here will be to stabilize the closed loop system.

With

\[
[x_1(0) \ x_2(0) \ \eta(0)] := [1 \quad 2 \quad 0],
\]

\[
f(k) := \begin{cases} 
25k, & 1 \leq k \leq 5 \\
60(k - 5)\exp((k - 5)^2), & k > 5.
\end{cases}
\]

\[
\begin{bmatrix}
\mathcal{K}_1 \ \mathcal{K}_2 \ \mathcal{K}_3 \ \mathcal{K}_4 \ \mathcal{K}_5
\end{bmatrix} := 
\begin{bmatrix}
\mathbf{K}(-0.5) \ \mathbf{K}(-0.25) \ \mathbf{K}(0) \ \mathbf{K}(0.25) \ \mathbf{K}(0.5)
\end{bmatrix}.
\]

(it is known from [34] that at least one Controller \( \mathcal{K}_i \), \( i \in \{1, 2, \ldots, 5\} \), will stabilize the system given by (2.9) for a fixed value of \( q \), \( q \in [-0.5, 0.5] \)) the output results shown in Figures 2.12, 2.13, and 2.14 are obtained\(^\text{10}\) for the indicated values of \( q \). Although the final

\(^{10}\)In these simulations, \( \eta(t^*_i) \) is not reset to zero immediately following any controller switch.
2.2. Simulation Results

Figure 2.12: \( q = -0.5 \) Simulated output response with Controller F1 (having five candidate controllers) applied to (2.9) with \( x_1 \) (solid) and \( x_2 \) (dashed).

steady-state controllers do not necessarily correspond in all instances to those given in [34], the output transient responses are still comparable\(^{11}\) in nature to those shown in Figures 3, 4, and 5 of [34, pg. 1102]\(^{12}\).

\(^{11}\)In Figure 2.14, the output response is improved noticeably over the transients shown in Figure 5 of [34]. This occurrence is due, in part, to the fact that when using the controller of [34], there exists a positive time interval immediately following each controller switch during which no further switching may occur.

\(^{12}\)These latter results are, however, obtained by using a much more computationally intensive on-line switching mechanism; in addition, a considerable amount of \textit{a priori} calculations must also be done in order to implement the scheme given by [34].
Figure 2.13: \((q = 0.125)\) Simulated output response with Controller F1 (having five candidate controllers) applied to (2.9) with \(x_1\) (solid) and \(x_2\) (dashed).

Figure 2.14: \((q = 0.5)\) Simulated output response with Controller F1 (having five candidate controllers) applied to (2.9) with \(x_1\) (solid) and \(x_2\) (dashed).
Chapter 3

Adaptive Stabilization of LTI MIMO Systems

Using the methods and techniques developed in Chapter 2, we now propose a new stabilizing adaptive controller for the class of first order strictly proper SISO LTI systems, and for the general class of finite dimensional strictly proper MIMO LTI systems considered in [54], [61], and [62]. As in Chapter 2, the controller is potentially cyclic in nature, and the emphasis will be to provide a robust switching mechanism which is insensitive to bounded piecewise continuous disturbances $w(t)$ and which attempts to provide an acceptable transient response; the switching mechanism proposed here, however, is simpler in nature than that given in [61], and it does not require a preliminary identification period as given in [62].

3.1 Adaptive Stabilization of First Order LTI SISO Systems

In this section, we again utilize the switching mechanism given in Section 2.1 to stabilize adaptively (in the sense that $x(t) \to 0$ as $t \to \infty$ and $[x \ u]^T \in L_\infty$ with $w(t) = 0$, and $[x \ u]^T \in L_\infty$ with $w(t) \neq 0$ and $w \in L_\infty$) the following first order SISO system:

\begin{align*}
\dot{x} &= ax + bu + ew, \quad (3.1a) \\
y &= cx + fw \quad (3.1b)
\end{align*}
where \( x \in \mathbb{R} \) is the state, \( u \in \mathbb{R} \) is the control input, \( y \in \mathbb{R} \) is the plant output, \( w \in \mathbb{R}^q \) is a bounded piecewise continuous disturbance, \((a, b, c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, b \neq 0, c \neq 0, \) and \((e^T, f^T) \in \mathbb{R}^q \times \mathbb{R}^q;\) by maintaining the assumption that \( b \neq 0 \) and \( c \neq 0, \) the system given by (3.1) is both stabilizable and detectable for all \( a \in \mathbb{R}. \)

In the past, one class of non-linear (one-dimensional) smooth adaptive stabilizing controllers for this type of system has been considered [85], and a particular stabilizing noise sensitive controller (which potentially gives a very large transient response, as observed in Figure 1.1) is given in the form [34]

\[
\begin{align*}
\dot{\eta}(t) &= y^2(t), \\
h(\eta) &= \sqrt{\log \eta}, \\
and \ u(t) &= y(t) \cdot h(\eta(t))^{0.25} \cdot (\sin(\sqrt{h(\eta(t))}) + 1) \cdot \cos(h(\eta(t))).
\end{align*}
\]

However, due to the nature of the problem considered in this section, and for brevity, the simulations presented here will exclude any comparison with other conventional adaptive control methods [38], [8], [78], [95] which are also known to be able to solve this problem.

**Definition 3.1:** A function \( f : \mathbb{N} \to \mathbb{R}^+ \) is said to be a SI bounding function \((f \in \text{SI BF})\) if it is strictly increasing and if, for all constants \((c_0, c_1, c_2, m_1, m_2, \tau, \bar{w}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \)

\[
\frac{f(i)}{c_0 + c_1 \sum_{j=1}^{i-1} (m_1 + m_2 \tau^j)(f(j) + \bar{w}) + c_2 \tau^i} \to \infty
\]

as \( i \to \infty. \)

**Proposition 3.1:** There exists a SI BF (e.g. \( f(i) = i^2 \exp(i^3) \)).

Maintaining the notation used earlier, let **Controller S1** be given as follows:

\[
u(t) = \varepsilon(t)K(t)y(t), \quad t \in (t_k, t_{k+1}]
\]

where \( k \in \{1, 2, 3, \ldots \}, \)

\[
S := \{ (\varepsilon_0, \tau) : \varepsilon_0 > 0, \tau > 1 \},
\]
3.1. Adaptive Stabilization of First Order LTI SISO Systems

\[ K_1 := 1, \]
\[ K_2 := -1, \]
\[ (\varepsilon_0, \tau) \in \mathcal{S}, \]
\[ \varepsilon(t) = \frac{\tau^k}{\varepsilon_0}, \quad t \in (t_k, t_{k+1}], \]
\[ K(t) = K_i, \quad i \in \{1, 2\}, \quad i = ((k - 1) \mod 2) + 1. \quad t \in (t_k, t_{k+1}], \]

\[ t_1 := 0, \text{ and where, for each } k \geq 2 \text{ such that } t_{k-1} \neq \infty, \text{ the switching time } t_k \text{ is defined by} \]

\[ t_k := \begin{cases} 
\min t \ni & \text{if this minimum exists} \\
\begin{array}{ll}
  \text{i) } t > t_{k-1}, \text{ and} \\
  \text{ii) } |y(t)| = f(k - 1) \\
\end{array} & \\
\infty & \text{otherwise}
\end{cases} \]

with \( f \in \text{S1BF}. \) Label \textbf{Assumption S1} to be \[ |y(0)| < f(1). \]

Here, the restriction that \( |y(0)| < f(1) \) is required in order to ensure once again that the switching time \( t_k \) is well defined for Controller S1. As well, Controller S1 works by monitoring plant output \( y(t) \) in an attempt to detect instability. Following each controller switch, and similar to the results presented in [101], the sign of \( K \) is changed, and gain \( \varepsilon \) is increased with the goal of using high gain output feedback to stabilize (3.1).

**Remark 3.1:** Consider the SISO system (3.1), where \((a, b, c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, b \neq 0, c \neq 0, \) and \( K := \{K_1, K_2\}; \) then for almost all \((\varepsilon_0, \tau) \in \mathcal{S}. \)

\[ \left( \bigcup_{j=1}^{2} \bigcup_{h=1}^{\infty} \left( a + b \frac{h}{\varepsilon_0} K_j c \right) \right) \cap \{0\} = \emptyset. \]

**Lemma 3.1:** Consider the first order SISO plant (3.1) with Controller S1 applied at time \( t = 0, \) and assume that the controller never stops switching; let \( w(t) \) be a piecewise continuous signal with \( w \in \mathcal{L}_\infty, \) and let \( \text{sign}(bK_jc) = -1 \) for one \( j \in \{1, 2\}. \) Then with \( k \)
sufficiently large such that \(((k - 1) \mod 2) + 1 = j\) and

\[
\left( a + b \frac{\tau^k}{\varepsilon_0} K_j c \right) < 0.
\]

the following properties hold for all \(t \in (t_i, t_{i+1}]\) (with \(i \geq k\), \(((l - 1) \mod 2) + 1 = j\):

\[
|x(t)| \leq \zeta_1 |x(t_i)| + \zeta_2 + \zeta_3 \tau^l.
\]

and

\[
|y(t)| \leq \zeta_4 |x(t_i)| + \zeta_5 + \zeta_6 \tau^l.
\]

where \((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\) are constants independent of \(l\) and \(x(t_i)\).

**Proof:** The proof follows upon first observing that

\[
\left( a + b \frac{\tau^l}{\varepsilon_0} K_j c \right) < 0
\]

for all \(l \geq k\), \(((l - 1) \mod 2) + 1 = j\); hence, there exists a constant \(\lambda \in \mathbb{R}^+\) such that

\[
\exp \left( \left( a + b \frac{\tau^l}{\varepsilon_0} K_j c \right) t \right) \leq \exp(-\lambda t)
\]

for \(l \geq k\), \(((l - 1) \mod 2) + 1 = j\), \(t \geq 0\).

On defining

\[
\begin{align*}
\bar{a} & := a + b \frac{\tau^l}{\varepsilon_0} K_j c, \\
\bar{e} & := e + b \frac{\tau^l}{\varepsilon_0} K_j f, \\
\text{and } \bar{w} & := \|w\|
\end{align*}
\]

it therefore now follows that

\[
x(t) = e^{\bar{a}(t-t_i)} x(t_i) + \int_{t_i}^{t} e^{\bar{a}(t-\tau)} \bar{e} w(\tau) d\tau. \quad t \in (t_i, t_{i+1}].
\]

and thus that

\[
|x(t)| \leq |x(t_i)| + \frac{1}{\lambda} \left( |e| + \frac{|f|}{\varepsilon_0} \tau^l \right) \bar{w}.
\]
The result therefore follows upon defining

\[
\begin{align*}
\zeta_1 & := 1, \\
\zeta_2 & := \frac{|e|\dot{w}}{\lambda}, \\
\zeta_3 & := \frac{b|f|\dot{w}}{\lambda\varepsilon_0}, \\
\zeta_4 & := |c|\zeta_1, \\
\zeta_5 & := |c|\zeta_2 + |f|\dot{w}, \\
\zeta_6 & := |c|\zeta_3.
\end{align*}
\]

and

\[
\zeta_6 := |c|\zeta_3.
\]

3.1.1 Main Results

**Theorem 3.1:** Consider the system given by (3.1) with Controller S1 applied at time \( t = 0 \); then for every \( f \in S1BF \) and \((\varepsilon_0, \tau) \in S\), for every bounded continuous disturbance signal, and for every initial condition \( x(0) \) for which Assumption S1 holds, the closed loop system has the properties that:

i) there exist a finite time \( t_{ss} \geq 0 \), a finite constant \( \varepsilon_{ss} > 0 \), and a constant \( K_{ss} \) such that \( \varepsilon(t) = \varepsilon_{ss} \) and \( K(t) = K_{ss} \) for all \( t \geq t_{ss} \);

ii) the plant state \( x \in L_\infty \); and

iii) if the disturbance inputs \( u(t) = 0 \), then for almost all \((\varepsilon_0, \tau) \in S\), \( x(t) \to 0 \) as \( t \to \infty \).

**Remark 3.2:** As in Theorem 2.2, Controller S1 will also work for bounded piecewise continuous disturbance signals upon filtering \( y(t) \) as

\[
\dot{y}_f = -\lambda y_f + \lambda y, \quad \lambda \in \mathbb{R}^+.
\]

where Assumption S1 and Controller S1 now are defined to be, respectively. Assumption S1 with \( y(t) \) replaced by \( y_f(t) \), and Controller S1 with \( y(t) \) replaced by \( y_f(t) \) in the given definition of switching time \( t_k \).
3.1.2 Simulation Results

To demonstrate the potential transient improvement that might be attained by using Theorem 3.1, consider the system (3.1) with

\[(c.a.b.e.f) := (1.1,-1.0,0),\]
\[x(0) := 1,\]
\[\varepsilon(k) := \frac{2^k}{5},\]
\[and f(k) := \begin{cases} 4k, & 1 \leq k \leq 5 \\ 10(k - 5)^2 \exp((k - 5)^3), & k > 5. \end{cases}\]

In Figure 3.1, the output response obtained using Controller S1 is shown. Similarly, using identical controller parameters with

\[(c.a.b.e.f) := (1.1,1.0,1),\]
\[x(0) := 1,\]
\[and w(t) := 0.25 \sin(100t),\]

the output illustrated in Figure 3.2 is obtained. In both instances, these results compare favourably when contrasted with the respective outputs shown in Figures 1.1 and 1.2, where a peak overshoot greater than 311000 in magnitude, and closed loop instability, respectively result.

3.2 Adaptive Stabilization of LTI MIMO Systems

In this section, the general problem of adaptively stabilizing the finite dimensional strictly proper (stabilizable and detectable) MIMO LTI system given by

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ew, \quad (3.2a) \\
y &= Cx + Fw \quad (3.2b)
\end{align*}
\]

is examined, where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) is the control input, \(y \in \mathbb{R}^r\) is the plant output, and \(w \in \mathbb{R}^q\) is the disturbance. The candidate feedback controllers which will be
3.2. Adaptive Stabilization of LTI MIMO Systems

Figure 3.1: \( w(t) := 0 \) Simulated results of \( y(t) \) with Controller S1 applied to (3.1).

Figure 3.2: \( w(t) := 0.25 \sin(100t) \) Simulated results of \( y(t) \) with Controller S1 applied to (3.1).
considered are of the form

\[ \mathcal{K}_i : \begin{cases} \dot{\eta} = G_i \eta + H_i y, \\ u = K_i \eta + L_i y \end{cases} \]  \hspace{1cm} (3.3)

where \( \eta \in \mathbb{R}^n \), \( G_i \in \mathbb{R}^{n \times n} \), \( H_i \in \mathbb{R}^{n \times r} \), \( K_i \in \mathbb{R}^{m \times n} \), \( L_i \in \mathbb{R}^{m \times r} \), and

\[ \tilde{\mathcal{K}}_i := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+n) \times (r+g_i)}. \]

Similar to Section 2.1, in the discussions which will follow, we do not necessarily assume that \( n, A, B, C, E, \) or \( F \) are known. and do not restrict \( \lambda(A) \subset \mathbb{C}^- \).

Preliminary definitions and results which are needed before proceeding are given as follows.

**Definition 3.2:** A function \( f : \mathbb{N} \to \mathbb{R}^+ \) is said to be a \( S^2 \) bounding function (\( f \in S^2BF \)) if it is strictly increasing and for all constants \( (c_0, c_1, m_1, m_2, \tau, \bar{w}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \),

\[ f(i) \\
\quad = \frac{c_0 + c_1 \sum_{j=1}^{i-1} (m_1 + m_2 \tau^j) (f(j) + \bar{w})}{i^{l_i}} \rightarrow \infty \]

as \( i \to \infty \).

**Proposition 3.2:** There exists a \( S^2BF \) (e.g. \( f(i) = i^2 \exp(i^3) \)).

**Definition 3.3:** ([92, pg. 44]) A set \( D \) is dense in \( \mathbb{R} \) if \( \bar{D} = \mathbb{R} \), where \( \bar{D} \) is the closure of \( D \).

**Definition 3.4:** A function \( h : \mathbb{N} \to \mathbb{R}^{(m+g_i) \times (r+g_i)} \) is a controller tuning function (\( h \in CTF \)) if

(i) \( \{h(i) : i \in \mathbb{N}\} \) is dense in \( \mathbb{R}^{(m+g_i) \times (r+g_i)} \) for a fixed value of \( g_i \in \mathbb{N} \cup \{0\} \); and

(ii) for \( h(i) := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+g_i) \times (r+g_i)} \),

\[ \|K_i\| \leq \tau_1^i \text{ and } \|L_i\| \leq \tau_2^i \]
for constants \((\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+\).

**Proposition 3.3:** Given \(g_i \in \mathbb{N} \cup \{0\}\), there exists a \(h \in \text{CTF} \).

**Proof:** Consider the situation where each element of \(h(i)\) successively examines, in a nested fashion, each interval \([-n, n], n \in \mathbb{N}\) and tries points \(\frac{1}{2n}\) apart. Then since \(\|h(1)\| = (r + g_i)\), and since each element of \(h(i)\) will increase in magnitude at most by one following any given switch, therefore

\[
\|h(i)\| \leq (r + g_i + 1)^i.
\]

Since

\[
\|K_i\| \leq \|h(i)\|
\]

and \(\|L_i\| \leq \|h(i)\|\),

the result follows. \(\square\)

As an example of how one might construct \(h(i)\), consider the situation when, for instance, \(g_i = 1, \ i \in \mathbb{N}\) and \(m = r = 1\); then \(h(i)\) can be defined as follows:

\[
h(1) = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(6) = \begin{bmatrix} 1.0 & 0.5 \\ 1.0 & 1.0 \end{bmatrix}, \quad \vdots
\]
\[
h(2) = \begin{bmatrix} 0.5 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(7) = \begin{bmatrix} 0.5 & 0.5 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(625) = \begin{bmatrix} -1.0 & -1.0 \\ -1.0 & -1.0 \end{bmatrix}
\]
\[
h(3) = \begin{bmatrix} 0.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(8) = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(626) = \begin{bmatrix} 2.0 & 2.0 \\ 2.0 & 2.0 \end{bmatrix}
\]
\[
h(4) = \begin{bmatrix} -0.5 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(9) = \begin{bmatrix} -0.5 & 0.5 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(627) = \begin{bmatrix} 1.75 & 2.0 \\ 2.0 & 2.0 \end{bmatrix}
\]
\[
h(5) = \begin{bmatrix} -1.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \quad h(10) = \begin{bmatrix} -1.0 & 0.5 \\ 1.0 & 1.0 \end{bmatrix}, \quad \vdots
\]
For the current problem under consideration, define

\[
\begin{align*}
\dot{x} &:= [x^T \eta^T]^T, \\
\tilde{u} &:= [u^T \eta^T]^T, \\
\tilde{y} &:= [y^T \eta^T]^T, \\
\begin{bmatrix}
\hat{A} & \hat{B} & \hat{E} \\
\tilde{C} & \tilde{F}
\end{bmatrix} &:= 
\begin{bmatrix}
A & 0 & B & 0 & E \\
0 & 0 & 0 & I & 0
\end{bmatrix}, \\
\end{align*}
\]

Since the closed loop system may therefore be expressed as

\[
\begin{align*}
\dot{x} &= \hat{A}\dot{x} + \hat{B}\tilde{u} + \hat{E}w, \quad (3.4a) \\
\dot{y} &= \tilde{C}\dot{x} + \tilde{F}w, \quad (3.4b) \\
\tilde{u} &= \hat{K}_i\tilde{y}, \quad (3.4c)
\end{align*}
\]

the stabilization of (3.2) can be seen to be equivalent to the problem of finding a feedback matrix \(\hat{K}_i \in \mathbb{R}^{(m+p_i)\times(r+q_i)}\) such that \(\lambda(\hat{A} + \hat{B}\hat{K}_i\hat{C}) \subset \mathbb{C}^-\). In fact, with \(l \in \mathbb{N} \cup \{0\}\),

\[
\Sigma_l := \{(C, A, B) : \text{there exists an } l\text{'th order LTI stabilizing compensator (3.3) which stabilizes (3.2)}\},
\]

and

\[
\Sigma := \bigcup_{l=0}^{\infty} \Sigma_l,
\]

then \(\Sigma_l \subset \Sigma_{l+1}\), and \((C, A, B) \in \Sigma\) if and only if \((A, B)\) is stabilizable and \((C, A)\) is detectable. As well, using this particular framework, the following result is obtained.

**Lemma 3.2:** Consider the plant (3.2), and assume that there exists a LTI controller (3.3) of known order \(g_i \in \mathbb{N} \cup \{0\}\) that stabilizes the closed loop system (3.4) (i.e. it is known
that with \( \hat{u} = \tilde{K}_i \tilde{y}, \lambda(\hat{A} + \tilde{B} \tilde{K}_i \tilde{C}) \subset \mathbb{C}^- \) for some value of \( \tilde{K}_i \). Then with

\[ \hat{u} = (\tilde{K}_i + \Delta \tilde{K}_i) \tilde{y} \]

applied at time \( t = 0 \), there exist constants \( (\gamma, \zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) independent of \( \tilde{x}(0) \) such that

\[
\| \tilde{x}(t) \| \leq \zeta_1 \| \tilde{x}(0) \| + \zeta_2,
\]

and

\[
\| \tilde{y}(t) \| \leq \zeta_3 \| \tilde{x}(0) \| + \zeta_4
\]

for all \( \| \Delta \tilde{K}_i \| \leq \gamma, \ t \geq 0 \).

**Proof:** By assumption, there exists a matrix \( \tilde{K}_i \in \mathbb{R}^{(m+q_1) \times (r+q_1)} \) such that \( \lambda(\hat{A} + \tilde{B} \tilde{K}_i \tilde{C}) \subset \mathbb{C}^- \). Hence, by the continuity of eigenvalues [96], there exists a constant \( \gamma \in \mathbb{R}^+ \) such that

\[
\lambda(\hat{A} + \tilde{B} \tilde{K}_i \tilde{C} + \tilde{B} \Delta \tilde{K}_i \tilde{C}) \subset \mathbb{C}^-
\]

for all \( \| \Delta \tilde{K}_i \| \leq \gamma \). In addition, there therefore also exist constants \( (\alpha, \bar{\lambda}) \in \mathbb{R}^+ \times \mathbb{R}^+ \) such that

\[
\| \exp((\hat{A} + \tilde{B} \tilde{K}_i \tilde{C} + \tilde{B} \Delta \tilde{K}_i \tilde{C})t) \| \leq \alpha \exp(-\bar{\lambda}t)
\]

for all \( \| \Delta \tilde{K}_i \| \leq \gamma, \ t \geq 0 \).

Define

\[
\hat{u} := (\tilde{K}_i + \Delta \tilde{K}_i) \tilde{y},
\]

\[
\hat{A} := \hat{A} + \tilde{B} \tilde{K}_i \tilde{C} + \tilde{B} \Delta \tilde{K}_i \tilde{C},
\]

and

\[
\hat{E} := \hat{E} + \tilde{B} \tilde{K}_i \tilde{F} + \tilde{B} \Delta \tilde{K}_i \tilde{F},
\]

and note that

\[
\tilde{x}(t) = e^{\hat{A}t} \tilde{x}(0) + \int_0^t e^{\hat{A}(t-\tau)} \hat{E}w(\tau) d\tau.
\]
With

\[ \beta := \sup_{\|\Delta \hat{K}\| \leq \gamma} \|\hat{E}\|, \]

and \( \tilde{w} := \|w\|, \)

therefore

\[ \|\hat{x}(t)\| \leq \alpha \|\hat{x}(0)\| + \frac{\alpha}{\lambda} \beta \tilde{w}, \]

and \( \|\hat{y}(t)\| \leq \|\hat{C}\| \cdot \left( \alpha \|\hat{x}(0)\| + \frac{\alpha}{\lambda} \beta \tilde{w} \right) + \|\hat{F}\| \cdot \tilde{w}; \)

hence, the result follows upon defining

\[ \zeta_1 := \alpha, \]
\[ \zeta_2 := \frac{\alpha \beta \tilde{w}}{\lambda}, \]
\[ \zeta_3 := \|\hat{C}\| \cdot \zeta_1. \]

and

\[ \zeta_4 := \|\hat{C}\| \cdot \zeta_2 + \|\hat{F}\| \cdot \tilde{w}. \quad \square \]

### 3.2.1 Using a Known Value of the Compensator Order

For the case when it is known that compensator (3.3) will stabilize plant (3.2) with \( y_k = p, \)
\( p \in \mathbb{N} \cup \{0\}, \)
using some appropriate choice of controller matrices \( G_i, H_i, K_i, L_i, \)
label **Controller S2** as

\[
\begin{align*}
\dot{\eta}(t) & = \ G(t)\eta(t) + \ H(t)y(t), \quad \eta(t) \equiv 0, \quad t \in (t_k, t_{k+1}] \\
u(t) & = \ K(t)\eta(t) + \ L(t)y(t)
\end{align*}
\]

where \( k \in \{1, 2, 3, \ldots\}, \ h \in \text{CTF}, \)

\[
\begin{bmatrix}
L(t) & K(t) \\
H(t) & G(t)
\end{bmatrix} := h(k), \quad t \in (t_k, t_{k+1}].
\]
3.2. Adaptive Stabilization of LTI MIMO Systems

$t_1 := 0$, and where, for each $k \geq 2$ such that $t_{k-1} \neq \infty$, the switching time $t_k$ is defined by

$$t_k := \begin{cases} \min t \ni & i) \ t > t_{k-1}, \text{ and} \\
& \text{if this minimum exists} \\
& \|\eta(t)^T y(t)^T\| = f(k-1) \\
\infty & \text{otherwise} \end{cases}$$

with $f \in S2BF$. In addition, let Assumption S2 be

i) $\|\eta(0)\| < f(1)$; and

ii) $\|y(0)\| < f(1)$.

**Theorem 3.2**: Consider the system given by (3.2) with Controller S2 applied at time $t = 0$; then for every $f \in S2BF$ and $h \in CTF$, for every bounded continuous disturbance signal, and for every initial condition $z(0) := [x(0)^T \eta(0)^T]^T$ for which Assumption S2 holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$ and constant matrices $(G_{ss}, H_{ss}, K_{ss}, L_{ss})$ such that

$$(G(t), H(t), K(t), L(t)) = (G_{ss}, H_{ss}, K_{ss}, L_{ss}) \text{ for all } t \geq t_{ss};$$

ii) the controller states $\eta \in L_{\infty}$ and the plant states $x \in L_{\infty}$; and

iii) if the disturbance inputs $w(t) = 0$, then for almost all controller parameters $(G, H, K, L)$, $x(t) \to 0$ as $t \to \infty$.

3.2.2 Using no Known Value of the Compensator Order

For the case when the order $g_i$ of a stabilizing compensator is unknown, but a lower bound $\alpha \in \mathbb{N} \cup \{0\}$ and an upper bound $\gamma \in \mathbb{N} \cup \{0\}$ is known such that $\alpha \leq g_i \leq \gamma$ with $\alpha < \gamma$, Controller S2 can be modified such that, starting with $\omega := \alpha$, one can search the $\mathbb{R}^{(m+\omega) \times (r+\omega)}$ parameter space near zero with a certain degree of fineness, and then increase (if necessary) the order of $\omega$ by one and repeat the process with an increased degree of fineness. This idea is formalized in the following definition.

**Definition 3.5**: Given that $\alpha \in \mathbb{N} \cup \{0\}$ is a lower bound and that $\gamma \in \mathbb{N} \cup \{0\}$ is an upper bound on $g_i$ such that $\alpha \leq g_i \leq \gamma$ with $\alpha < \gamma$, a function $h : \mathbb{N} \to \mathbb{R}^{(m+g_i) \times (r+g_i)}$ is
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a *modified controller tuning function* \((h \in \text{CTF}')\) if, with \((g_{i+1} - g_i) \in \{0, 1\}\), \(g_1 := \alpha\), and \(h(i) := \tilde{K}_i\), the following properties hold:

(i) \(\lim_{i \to \infty} g_i \to \gamma\);

(ii) \(\{h(i) : i \in \mathbb{N}\}\) is dense in \(\mathbb{R}^{(m+\gamma) \times (r+\gamma)}\); and

(iii) for \(h(i) := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+g_i) \times (r+g_i)}\),

\[\|K_i\| \leq \tau_i^1\] and \(\|L_i\| \leq \tau_i^2\)

for constants \((\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+\).

**Proposition 3.4:** There exists a \(h \in \text{CTF}'\).

**Proof:** Consider the situation where each element of \(h(i)\) successively examines, in a nested fashion, each interval \([-n, n]\), \(n \in \mathbb{N}\), and tries points \(\frac{1}{2^n}\) apart, and where, upon completing each nested search accordingly, \(g_i\) is increased (if necessary) by one and the nested search is restarted; then

\[\|h(i)\| \leq (r + \gamma + 1)^i\]

and hence the result immediately follows. \(\square\)

**Remark 3.3:** In an attempt to clarify the terse statements given in the proof of Proposition 3.4, consider the situation when \((\alpha, \gamma, m, r) = (0, 1, 1, 1)\); then \(h(i)\) can be defined as follows:

\[
h(1) = 1.0 \quad h(6) = \begin{bmatrix} 2.0 & 2.0 \\ 2.0 & 2.0 \end{bmatrix} \quad h(83526) = \begin{bmatrix} -2.0 & -2.0 \\ -2.0 & -2.0 \end{bmatrix}
\]

\[
h(2) = 0.5 \quad h(7) = \begin{bmatrix} 1.75 & 2.0 \\ 2.0 & 2.0 \end{bmatrix} \quad h(83527) = \begin{bmatrix} 3.0 & 3.0 \\ 3.0 & 3.0 \end{bmatrix}
\]

\[
h(3) = 0.0 \quad h(8) = \begin{bmatrix} 1.5 & 2.0 \\ 2.0 & 2.0 \end{bmatrix} \quad h(83528) = \begin{bmatrix} 2.875 & 3.0 \\ 3.0 & 3.0 \end{bmatrix}
\]

\[
h(4) = -0.5 \quad h(9) = \begin{bmatrix} 1.25 & 2.0 \\ 2.0 & 2.0 \end{bmatrix} \quad h(83529) = \begin{bmatrix} 2.75 & 3.0 \\ 3.0 & 3.0 \end{bmatrix}
\]
For the case when it is known that compensator (3.3) will stabilize plant (3.2) with $g_i = p$ using some $\alpha \leq p \leq \gamma$ with $\alpha, \gamma \in \mathbb{N} \cup \{0\}$, $\alpha < \gamma$, and using some appropriate choice of controller parameters $G_i$, $H_i$, $K_i$, $L_i$, label **Controller S2'** to be Controller S2, but with $h \in CTF'$.

**Theorem 3.3**: Consider the system given by (3.2) with Controller S2' applied at time $t = 0$; then for every $f \in S2BF$ and $h \in CTF'$, for every bounded continuous disturbance signal, and for every initial condition $z(0) := [x(0)^T \eta(0)^T]^T$ for which Assumption S2 holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$ and constant matrices $(G_{ss}, H_{ss}, K_{ss}, L_{ss})$ such that 

$$(G(t), H(t), K(t), L(t)) = (G_{ss}, H_{ss}, K_{ss}, L_{ss}) \text{ for all } t \geq t_{ss};$$

ii) the controller states $\eta \in \mathcal{L}_{\infty}$ and the plant states $x \in \mathcal{L}_{\infty}$; and

iii) if the disturbance inputs $w(t) = 0$, then for almost all controller parameters $(G, H, K, L)$, $\eta(t) \to 0$ as $t \to \infty$.

Here, the relative computational simplicity of Controllers S2 and S2' compare favourably when contrasted, for instance, with the controllers given in [74], [61], [62], and [57]. In essence, by properly constructing $f$ to have certain known a priori properties, the adaptive stabilization problem for stabilizable and detectable MIMO LTI systems can be solved by monitoring norm bounds on $\eta(t)$ and $y(t)$.

**Remark 3.4**: As noted in [62], if it is known a priori that there exists a gain $\hat{K}_i$ in a known set $S \subset \mathbb{R}^{(m+g_i)(r+g_i)}$ such that the closed loop system is stable, then one can restrict the search of the $\mathbb{R}^{(m+g_i)(r+g_i)}$ parameter space to $S$. For example, if

$$S = \bigcup_{i=1}^{p} \hat{K}_i, \ p \in \mathbb{N},$$
then one can define

\[
h(i) := \begin{cases} 
\hat{K}_1, & \text{if } ((i - 1) \mod p) + 1 = 1 \\
\hat{K}_2, & \text{if } ((i - 1) \mod p) + 1 = 2 \\
\vdots & \vdots \\
\hat{K}_p, & \text{if } ((i - 1) \mod p) + 1 = p 
\end{cases}
\]

as a controller tuning function.

Hence, if one can restrict the search of the \( \mathbb{R}^{(m-g_1) \times (r-g_1)} \) parameter space to \( S \) for stabilizing controller parameters \( \hat{K}_i \), where

\[
S := \bigcup_{i=1}^{p} \hat{K}_i, \quad p \in \mathbb{N}.
\]

then Theorem 3.2 reduces to Theorem 2.1 with \( y_{ref}(t) := 0 \). In this instance, \( f(k) \) need only satisfy the property that

\[
\frac{f(i)}{c_0 + c_1(i - 1) + c_2 \sum_{j=1}^{i-1} f(j)} \rightarrow \infty
\]

as \( i \rightarrow \infty \) for all constants \( (c_0, c_1, c_2) \in \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^- \).

**Remark 3.5:** If \( w(t) \) is a bounded piecewise continuous signal, then corresponding comments equivalent to the ones given in Remark 3.2 are also applicable to Theorems 3.2 and 3.3 provided that

(i) \( \|G_i\| \leq \tau_i^3 \); and

(ii) \( \|H_i\| \leq \tau_i^3 \)

are additionally satisfied for \( (\tau_3, \tau_i) \in \mathbb{R}^+ \times \mathbb{R}^+ \). (This additional restriction can always be met by using the construction methods given in the proof of Proposition 3.3 and Remark 3.3.)
3.2.3 Simulation Results

Example 1: SISO Unstable Nonminimum Phase Plant

Consider the following (controllable and observable) unstable nonminimum phase SISO plant taken from [62, pg. 604]:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2.5 & 0.75 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u + Ew. \tag{3.5a}
\]

\[
y =
\begin{bmatrix}
0.75 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + Fw \tag{3.5b}
\]

with poles given by \(-2\) and \(0.5 \pm j\), and zeros given by \(-1.5\) and \(0.5\). Assume that the plant is unknown, but that it is known that there exists a zero'\textsuperscript{th}-order stabilizing compensator for the system (i.e. it is known that for some value of \(L \in \mathbb{R}\) the closed loop system will be stable with \(u = L(y)\)). In addition, define \(h(i)\) as

\[
\begin{align*}
h(1) &= 1.0 \\
h(2) &= 0.5 \\
h(3) &= 0.0 \\
h(4) &= -0.5 \\
h(5) &= -1.0 \\
h(6) &= 2.0 \\
h(7) &= 1.75 \\
h(8) &= 1.5 \\
\vdots
\end{align*}
\tag{3.6}
\]

so that each successive interval \([-n, n], n \in \mathbb{N}\) is examined, and points \(\frac{1}{2^n}\) apart are chosen. Let

\[
E := 0, \quad F := 1, \quad x(0) := [1 \ 0 \ 0]^T, \quad \text{and } f(k) := \begin{cases} 2k, & 1 \leq k \leq 15 \\ 20(k - 15)^2 \exp((k - 15)^3), & k > 15. \end{cases}
\]
and observe that the closed loop system is stable if and only if

$$L_i \in \left(-1 + \frac{\sqrt{14}}{2}, \frac{10}{3}\right).$$

In Figures 3.3 and 3.4, output results obtained using Controller S2 are shown for \(w(t) := 0\) and \(w(t) := \sin(2t)\) respectively. In both instances, these results are comparable to those given in [61] and [62], and, in accordance with Theorem 3.2. \(L(t)\) remains constant after a finite number of switches \((L_{ss} = 2)\).

For comparison, using Controller S2 and the same initial conditions and parameters as given for Figure 3.3. but with \(h(i)\) defined as

\[
\begin{align*}
    h(1) &= 1.0 & h(6) &= -1.0 & h(11) &= 1.5 & h(16) &= 2.25 \\
    h(2) &= 0.0 & h(7) &= -0.5 & h(12) &= 2.0 & h(17) &= 2.0 \\
    h(3) &= -1.0 & h(8) &= 0.0 & h(13) &= 3.0 & h(18) &= 1.75 \\
    h(4) &= -2.0 & h(9) &= 0.5 & h(14) &= 2.75 & h(19) &= 1.5 \\
    h(5) &= -1.5 & h(10) &= 1.0 & h(15) &= 2.5 & : \\
\end{align*}
\]

the response shown in Figure 3.5 is obtained. In this instance, \(L_{ss} = 3\). and, unlike the results presented in Figure 1 and the highly oscillatory response shown in Figure 2 of [62], the state transient response suffers from only an approximate four-fold increase in peak magnitude when compared with Figure 3.3\(^1\).

**Example 2: Two Input-One Output Unstable Minimum Phase Plant**

As another example, consider the (stabilizable and observable) unstable minimum phase MISO plant [62. pg. 605]

\[
\begin{align*}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
    0.1 & 0.1 & 1 \\
    0 & 0.1 & 1 \\
    0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
+ \begin{bmatrix}
    0.1 & 0 \\
    0 & 0.4 \\
    0 & 0
\end{bmatrix} u + Ew. \\
\end{align*}
\]

\(^1\)In [62], Figure 2 suffers from an approximate seven-fold increase in peak magnitude when compared with Figure 1.
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Figure 3.3: \((w(t) := 0)\) Simulated results with Controller S2 applied to (3.5) using (3.6) with \(x_1\) (dotted), \(x_2\) (dashed), \(x_3\) (dash-dotted), and \(y\) (solid).

Figure 3.4: \((w(t) := \sin(2t))\) Simulated results with Controller S2 applied to (3.5) using (3.6) with \(x_1\) (dotted), \(x_2\) (dashed), \(x_3\) (dash-dotted), and \(y\) (solid).
3.2. Adaptive Stabilization of LTI MIMO Systems

Figure 3.5: \( w(t) = 0 \) Simulated results with Controller S2 applied to (3.5) using (3.7) with \( x_1 \) (dotted), \( x_2 \) (dashed), \( x_3 \) (dash-dotted), and \( y \) (solid).

\[
y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + Fw; \tag{3.8b}
\]

Assume once again that the plant is unknown, but that it is known that there exists a zero'th-order stabilizing compensator (i.e. it is known that for some value of \( L_t \in \mathbb{R}^2 \), the closed loop system will be stable with \( u = L_t y \)). For

\[
u := \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix} y,
\]

the closed loop system will be stable if and only if \((l_{11}, l_{21})\) lies in the region defined by

\[
-l_{11} - 4l_{21} - 2 > 0.
\]

and \( l_{11} + 1 > 0 \).
Define \( h(i) \) by

\[
\begin{align*}
    h(1) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & h(4) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & h(7) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & h(10) &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
    h(2) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & h(5) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & h(8) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & h(11) &= \begin{bmatrix} 2 \\ 1.5 \end{bmatrix} \\
    h(3) &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & h(6) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, & h(9) &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\end{align*}
\]

and let

\[
E := 0, \quad F := 1, \\
x(0) := \begin{bmatrix} 1 & 1 & 3 \end{bmatrix}^T, \\
\text{and } f(k) := \begin{cases} 6 + 2k, & 1 \leq k \leq 10 \\ 10(k - 10)^2 \exp((k - 10)^3). & k > 10. \end{cases}
\]

In Figures 3.6 and 3.7, the output response is shown with \( w(t) := 0 \) and \( u(t) := \sin(2t) \) respectively: once again, in accordance with Theorem 3.2, \( L(t) \) remains constant after a finite number of switches \( (L_{ss} = [1 \ -1]^T) \) even with \( w(t) \neq 0 \).

**Example 3: Simultaneous Stabilization Problem with Three Plants**

Alternatively, consider the simultaneous stabilization problem [94], [100], [11] of the following family of three unstable non-minimum phase SISO plant models taken from [36, pg. 1106]:

\[
P_1(s) := \frac{s - 7}{s - 4.6}, \quad P_2(s) := \frac{s - 2}{2s - 2.6}, \quad P_3(s) := \frac{s - 6}{4.8s - 24.6};
\]

let the corresponding

\[
\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]
\]
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Figure 3.6: \((w(t) := 0)\) Simulated results with Controller S2 applied to (3.8) using (3.9) with \(x_1\) (dotted), \(x_2\) (dashed), \(x_3\) (dash-dotted), and \(y\) (solid).

Figure 3.7: \((w(t) := \sin(2t))\) Simulated results with Controller S2 applied to (3.8) using (3.9) with \(x_1\) (dotted), \(x_2\) (dashed), \(x_3\) (dash-dotted), and \(y\) (solid).
state space representation for each plant be given as

\[
\begin{bmatrix}
4.6 & -2.4 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1.3 & -0.35 \\
1 & 0.5
\end{bmatrix}
\begin{bmatrix}
5.125 & -1.15 \\
1 & \frac{5}{24}
\end{bmatrix}
\]

and define their respective controller parameters to be

\[
\begin{bmatrix}
G_i & H_i \\
K_i & L_i
\end{bmatrix}
\]

controller parameters to be

\[
\begin{bmatrix}
-19.2 & -3.4 \\
-3.5 & 0
\end{bmatrix}
\begin{bmatrix}
-10.5 & -1.17 \\
-10 & 0
\end{bmatrix}
\begin{bmatrix}
-42.3 & -12.39 \\
-14 & 0
\end{bmatrix}
\]

One can verify that each respective controller-plant pair yields closed loop poles at \{-1, -1\}, \{-1, -1\}, and \{-0.5, -0.5\}. Furthermore, as shown in [36] and [11, pg. 81], there does not exist a fixed finite dimensional LTI controller which can simultaneously stabilize plants \(P_1\), \(P_2\), and \(P_3\).

Now using Controller S2 and the comments given in Remarks 2.6 and 3.4, with

\[x(0) := 1,\]
\[\eta(0) := 0,\]
\[E := 0,\]
\[F := 0,\]
\[\lambda := 10,\]
\[f_1(k) := \begin{cases} 5k \exp\left(\frac{k}{2}\right)^3, & 1 \leq k \leq 3 \\ k \exp(k^2), & k > 3. \end{cases} \]
\[f_2(k) := \begin{cases} 4 + 3 \exp\left(\frac{k}{1.5}\right)^3, & 1 \leq k \leq 3 \\ k \exp(k^2), & k > 3. \end{cases} \]

the output response for plants \(P_1\), \(P_2\), and \(P_3\) is shown in Figures 3.8, 3.9, and 3.10 respectively. Although the transient response in Figure 3.10 is significantly larger when compared to the results presented in Figures 3.8 and 3.9, this fact may be attributed to the actual
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Figure 3.8: Simulated results with Controller S2 applied to plant $P_1(s)$ (3.10) with $x$ (dashed) and $y$ (solid).

Figure 3.9: Simulated results with Controller S2 applied to plant $P_2(s)$ (3.10) with $x$ (dashed) and $y$ (solid).
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Figure 3.10: Simulated results with Controller S2 applied to plant $P_3(s)$ (3.10) with $x$ (dashed) and $y$ (solid).

Figure 3.11: Schematic set-up of the $\mathcal{H}_\infty$ design synthesis used for plant $P_3(s)$ (3.10).
3.2. Adaptive Stabilization of LTI MIMO Systems

controller designed for plant $P_3(s)$. Indeed, using Figure 3.11 and the $\mathcal{H}_\infty$ design technique \cite{33} with

$$W(s) := \left( \frac{a}{s/w_0 + 1} \right),$$

$a := 20$, $w_0 := 0.25$, $\epsilon_1 := 1 \times 10^{-6}$, and $\epsilon_2 := 1 \times 10^5$.

one can obtain

$$\begin{bmatrix} G_3 & H_3 \\ K_3 & L_3 \end{bmatrix} = \begin{bmatrix} -1.4663 \times 10^2 & -1.1459 \times 10^{-3} & -2.5016 \times 10^1 \\ -1.1459 \times 10^{-3} & -1.2919 \times 10^{-3} & -9.7748 \times 10^{-5} \\ -2.5016 \times 10^1 & -9.7745 \times 10^{-5} & 0 \end{bmatrix}.$$  

and the noticeably improved closed loop transient response shown in Figure 3.12.

![Diagram](image-url)

Figure 3.12: Simulated results with Controller S2 (cf. Figure 3.10) applied to plant $P_3(s)$ (3.10) with $x$ (dashed) and $y$ (solid).
Chapter 4

The Self-Tuning Robust Servomechanism

In this chapter, the robust self-tuning controllers presented in [64], [15], and [16] for cases involving both a known and an unknown estimate of the steady-state DC gain matrix $T$ will be reconsidered; as well, two new self-tuning proportional-integral-derivative (PID) controllers for similar corresponding cases will also be developed. In an attempt to improve the output transient response, the proposed controllers now switch based upon norm bounds on $\eta(t)$ and $e(t)$. The results presented here therefore indicate that a more general controller structure than originally given by Miller and Davison in [64] can be implemented on an open loop stable system using a switching criterion with very little a priori system information.

4.1 Self-Tuning Proportional-Integral Control

Consider the finite dimensional LTI system given by

\begin{align}
\dot{x} &= Ax + Bu + Ew, \\
y &= Cx + Du + Fw, \\
e &= y_{\text{ref}} - y
\end{align}

(4.1a) \quad (4.1b) \quad (4.1c)

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^r$ is the plant output to be regulated, $w \in \mathbb{R}^q$ is the disturbance, and $e \in \mathbb{R}^r$ is the difference between the specified reference input $y_{\text{ref}}$ and the output $y$. Assume that $m \geq r$, that $A$ is stable, that $n, A, B,
4.1. Self-Tuning Proportional-Integral Control

$C$, $D$, $E$, or $F$ are not necessarily known, and restrict $y_{ref}$ and $w$ to be bounded piecewise constant signals; let $f \in \text{MSBF}$, and define $\mathcal{T} := D - CA^{-1}B$. For the case when the estimate of $\mathcal{T}$, $\hat{\mathcal{T}}$, has full row rank, let $K := \hat{\mathcal{T}}^+$. (An estimate $(\hat{\mathcal{T}})$ of $\mathcal{T}$ can be obtained via $m$ steady state experiments outlined in [21].)

In this section, the self-tuning robust servomechanism controllers considered in [64], [15], and [16] for cases involving both a known and an unknown estimate of the steady-state DC gain matrix $\mathcal{T}$ will be re-examined. As in these earlier results, assume throughout that $\mathcal{T}$ has full row rank in order to form a tractable problem, and maintain the definitions (e.g. $f \in \text{MSBF}$) and notation given in previous chapters.

**Definition 4.1:** A function $g : \mathbb{N} \to \mathbb{R}^+$ is a tuning function ($g \in \text{TF}$) if $\lim_{k \to \infty} g(k) = 0$. If $g \in \text{TF}$ and if there exist finite constants $\epsilon_0 > 0$ and $\tau > 1$ so that $g(k) = \frac{\epsilon_0}{\tau^k}$ for $k \in \mathbb{N}$, then define $g \in \text{TF}'$ to be a modified tuning function.

**4.1.1 Using an Estimate of the DC Gain**

When an estimate of $\mathcal{T}$, given by $\hat{\mathcal{T}}$, is available, define the set of admissible controller parameters as

$$\hat{\Omega} := \{(f, g, \rho) : f \in \text{MSBF}, g \in \text{TF}', \rho \geq 0\}$$

and let $\mathcal{S} := \{(\epsilon_0, \tau) : \epsilon_0 > 0, \tau > 1\}$.

Label **Assumption PI1** to be

i) $\|\eta(0)\| < f(1)$;

ii) $\|e(0)\| < f(1)$;

iii) $-\mathcal{T}\hat{T}^+$ is stable; and

iv) $(I + \rho \frac{\epsilon_0}{\tau^i}DK)$, $i \in \{1, 2, 3, \ldots\}$, is invertible for fixed $\rho \geq 0$ (see Remark 4.1):

then with $\tilde{\sigma} = (f, g, \rho) \in \hat{\Omega}$, define **Controller PI1** as

$$\eta(t) = \int_{t_k}^{t} e(\tau)e(\tau)d\tau, \quad \eta(t^+_k) \equiv 0, \quad t \in (t_k, t_{k+1}]$$

$$u(t) = K(\eta(t) + \rho e(t)e(t))$$
where $k \in \{1, 2, 3, \ldots \}$, $K := \hat{T}^t$.

\[ \epsilon(t) = g(k), \quad t \in (t_k, t_{k+1}], \]

$t_1 := 0$, and where, for each $k \geq 2$ such that $t_{k-1} \neq \infty$, the switching time $t_k$ is defined by

\[
t_k := \begin{cases} 
\min t \ni & 
i) t > t_{k-1}, \text{ and} \\
\infty & \text{ii) } \|\eta(t)^T e(t)^T\| = f(k - 1) \ni \infty \text{ otherwise.}
\end{cases}
\]

**Remark 4.1**: Given $\rho \geq 0$ and $D \in \mathbb{R}^{r \times m}$, then for almost all $\epsilon \in \mathbb{R}^+$ and for almost all matrices $K \in \mathbb{R}^{m \times r}$, $(I + \rho \epsilon DK)$ is invertible.

The following results will also be needed.

**Theorem 4.1**: ([48, pg. 57]) Consider the singularly perturbed system given by

\[
\begin{align*}
\dot{x}_1 & = A_{11}x_1 + A_{12}x_2, \quad (4.2a) \\
\epsilon \dot{x}_2 & = A_{21}x_1 + A_{22}x_2 \quad (4.2b)
\end{align*}
\]

where $(x_1, x_2, \epsilon) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^-$. If $A_{22}^{-1}$ exists, then as $\epsilon \to 0$, $n_1$ eigenvalues of (4.2) tend to

\[ \lambda(A_{11} - A_{12} A_{22}^{-1} A_{21}), \]

while the remaining $n_2$ eigenvalues of (4.2) tend to infinity, with the rate of $\frac{1}{\epsilon}$, along the asymptotes defined by $\frac{1}{\epsilon} \lambda(A_{22})$.

**Corollary 4.1**: ([48, pg. 58]) Consider the singularly perturbed system given by (4.2). If $A_{22}^{-1}$ exists, and if $A_0 := A_{11} - A_{12} A_{22}^{-1} A_{21}$ and $A_{22}$ are asymptotically stable matrices, then there exists an $\epsilon^* \in \mathbb{R}^+$ such that for all $\epsilon \in (0, \epsilon^*)$ the system (4.2) is asymptotically stable.
Lemma 4.1: Consider the closed loop system

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
\hat{A} & BK\hat{I} \\
-\epsilon IC & -\epsilon DK
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix} +
\begin{bmatrix}
\rho e BK\hat{I} \\
\epsilon I
\end{bmatrix} y_{ref} +
\begin{bmatrix}
E - \rho e BK\hat{I}F \\
-\epsilon IF
\end{bmatrix} w
\] (4.3a)

with

\[
\hat{A} := A - \rho e BK\hat{I}C.
\] (4.3b)

and \( \hat{I} := (I + \rho e DK)^{-1} \). (4.3c)

Assume that \( \hat{I} \) exists (see Remark 4.1), and that \( A \) and \(-TK\) are both stable. Then there exist constants \((\alpha, \beta, \epsilon^*) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\) with the property that for every initial condition and for every pair of bounded piecewise continuous reference and disturbance signals.

\[
\|z(t)\| \leq \alpha\|z(0)\| + \beta \sup_{\tau \geq 0} (\|y_{ref}(\tau)\| + \|w(\tau)\|)
\]

for \( \epsilon \in (0, \epsilon^*) \). \( t \geq 0 \).

Proof: To prove Lemma 4.1, observe from (4.3) that

\[
eig(\hat{A}(\rho, \epsilon)) = \epsilon \cdot \eig(A(\rho, \epsilon))
\]

where

\[
A(\rho, \epsilon) :=
\begin{bmatrix}
-\hat{I}DK & -\hat{I}C \\
BK\hat{I} & \hat{A}
\end{bmatrix}
\frac{\epsilon}{\epsilon}
\]

From the comments given in [48, pg. 48] and Corollary 4.1, there then exists an \( \epsilon^* \in \mathbb{R}^+ \) such that \( \eig(A(\rho, \epsilon)) \subset \mathbb{C}^- \) for all \( \epsilon \in (0, \epsilon^*) \). Hence, by the continuity of eigenvalues, for fixed \( \rho \geq 0 \), there exist constants \((\alpha, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+\) such that

\[
\|e^{\hat{A}(\rho, \epsilon)t}\| \leq \alpha e^{-\lambda t}
\]

for all \( \epsilon \in (0, \epsilon^*) \), \( t \geq 0 \).
Define

\[ \beta_1 := \sup_{\epsilon \in (0, \epsilon^*)} \| B_1 \|, \]
and \[ \beta_2 := \sup_{\epsilon \in (0, \epsilon^*)} \| B_2 \|. \]

Since

\[ z(t) = e^{\tilde{A}(\rho, \epsilon) t} z(0) + \int_0^t e^{\tilde{A}(\rho, \epsilon)(t-\tau)} (B_1 y_{ref}(\tau) + B_2 w(\tau)) d\tau. \]

it therefore follows upon taking norms that

\[ \| z(t) \| \leq \alpha \| z(0) \| + \frac{\alpha}{\beta} (\beta_1 + \beta_2) \sup_{\tau \geq 0} \left( \| y_{ref}(\tau) \| + \| w(\tau) \| \right) \]

for all \( \epsilon \in (0, \epsilon^*), t \geq 0. \]

**Lemma 4.2:** Consider the matrix

\[ \tilde{A}(\rho, \frac{\epsilon_0}{\tau_1}, K) := \begin{bmatrix} \tilde{A} & BK \tilde{I} \\ -\frac{\epsilon_0}{\tau_1} \tilde{I}C & -\frac{\epsilon_0}{\tau_1} \tilde{I}DK \end{bmatrix} \] (4.4a)

where \( i \in \{1, 2, 3, \ldots \} \) and

\[ \tilde{I} := (I + \rho \frac{\epsilon_0}{\tau_1} DK)^{-1}, \] (4.4b)

and \( \tilde{A} := A - \rho \frac{\epsilon_0}{\tau_1} BK \tilde{I}C. \) (4.4c)

Assume that both \( A \) and \( -TK \) are stable; then for almost all \((\epsilon_0, \tau) \in S,\)

\[ \left( \bigcup_{i=1}^{\infty} \lambda(\tilde{A}(\rho, \frac{\epsilon_0}{\tau_i}, K)) \right) \cap \mathbb{C}^0 = \emptyset. \]

**Proof:** Since this proof closely follows the proof of Lemma 4 given in [64], only the major necessary modifications will be provided. To prove Lemma 4.2, observe that

\[ \tilde{I} := (I + \rho \epsilon DK)^{-1} \]
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(which is assumed to exist (see Remark 4.1)) can alternatively be rewritten as

\[ \dot{\hat{I}} := \frac{\dot{\hat{I}}}{\Omega}. \]

where \( \dot{\hat{I}} \) is a matrix whose elements are polynomials in \( \varepsilon \) and the elements of \( K \), and where \( \Omega \) is a polynomial in \( \varepsilon \) and the elements of \( K \) with the property that

\[ \lim_{\varepsilon \to 0} \Omega = 1. \]

Since

\[
\det \begin{bmatrix} sI - A + \rho \varepsilon BK \hat{I}C & -BK\hat{I} \\ \varepsilon \hat{I}C & sI + \varepsilon \hat{I}DK \end{bmatrix} = \frac{1}{\Omega^{n-r}} \det \hat{A}
\]

where

\[
\hat{A} := \begin{bmatrix} s\Omega I - \Omega A + \rho \varepsilon BK \hat{I}C & -BK\hat{I} \\ \varepsilon \hat{I}C & s\Omega I + \varepsilon \hat{I}DK \end{bmatrix}.
\]

and since \( \hat{A} \) is a matrix whose elements are polynomials of \( \varepsilon \) and the elements of \( K \), the proof of Lemma 4.2 can now continue using the identical method given in [64, pg. 521].

Lemmas 4.1 and 4.2 enable us to obtain the following.

**Theorem 4.2**: Consider the stable plant (4.1) with \( D = 0 \) and with Controller PI1 applied at time \( t = 0 \): then for every \( \varepsilon(0) \in \hat{\Omega} \), for every bounded constant reference and disturbance signal, and for every initial condition \( z(0) := [x(0)^T \quad \eta(0)^T]^T \) for which Assumption PI1 holds, the closed loop system has the properties that:

i) there exist a finite time \( t_{ss} \geq 0 \) and a finite constant \( \varepsilon_{ss} > 0 \) such that \( \varepsilon(t) = \varepsilon_{ss} \) for all \( t \geq t_{ss} \);

ii) the controller states \( \eta \in \mathcal{L}_\infty \) and the plant states \( x \in \mathcal{L}_\infty \); and

iii) if the reference and disturbance inputs are constant signals and \( g \in TF' \), then for almost all \( (\varepsilon_0, \tau) \in S \), \( e(t) \to 0 \) as \( t \to \infty \).
4.1.2 Using no Estimate of the DC Gain

For the situation when no estimate of $T$ is available, define the set of admissible controller parameters as

$$\tilde{\Omega} := \{(f,g,\rho,U) : f \in \text{MSBF}, g \in \mathcal{T}_F, \rho \geq 0, U \in \mathbb{R}^{m \times m} \text{ and is nonsingular}\}$$

and let $\mathcal{S} := \{ (\epsilon_0, \tau, U) : \epsilon_0 > 0, \tau > 1, U \in \mathbb{R}^{m \times m} \text{ and is nonsingular}\}$.

with $\tilde{\sigma} = (f,g,\rho,U) \in \tilde{\Omega}$ and Assumption PI1' defined to be

i) $\|\eta(0)\| < f(1)$;

ii) $\|e(0)\| < f(1)$; and

iii) $(I + \frac{\epsilon_0}{\tau}DK), i \in \{1,2,3,\ldots\}$, is invertible for fixed $\rho \geq 0$ (see Remark 4.1)

label Controller PI1' as

$$\eta(t) = \int_{t_k}^{t} \epsilon(\tau)e(\tau)d\tau, \quad \eta(t_k^+) \equiv 0, \quad t \in (t_k, t_{k+1}]$$

$$u(t) = K(t)(\eta(t) + \rho e(t)e(t))$$

where $k \in \{1,2,3,\ldots\}$,

$$e(t) = g(k), \quad t \in (t_k, t_{k+1}]$$

$\tau_1 := 0$, and where, for each $k \geq 2$ such that $t_{k-1} \neq \infty$, the switching time $t_k$ is defined by

$$t_k := \begin{cases} \min t \ni & \text{if this minimum exists} \\ i) \ t > t_{k-1}, \text{ and} \\ ii) \ |[\eta(t)^T e(t)^T]^T| = f(k-1) \infty \end{cases}$$

with

$$K(t) = U\mathcal{W}_i, \quad i \in \{1,2,\ldots,s\}, \quad i = ((k-1) \mod s) + 1, \quad t \in (t_k, t_{k+1}]$$
An explicit method for constructing $K_j := UW_j$ for the case when $m \geq r$ is given in [55] which ensures that $-TUW_j$ is stable for at least one $j \in \{1, 2, \ldots, s\}$, $s \in \mathbb{N}$.

Using the same method as shown in the outline of the proof of Lemma 4.2, and following exactly the proof given in [64, pp. 521-522], the following result can also be obtained.

**Lemma 4.3:** Consider the matrix $\tilde{A}(\rho, \epsilon, K)$ defined in (4.4) where $A$ is stable and $\rho \geq 0$ is fixed; then with $K := \{K_j : j \in \{1, 2, \ldots, s\}\}$ (an explicit method for calculating $K_j$ is given in [55]), for almost all $(\epsilon_0, \tau, U) \in \mathcal{S}'$,

$$\left(\bigcup_{j=1}^{s} \bigcup_{i=1}^{\infty} \lambda(\tilde{A}(\rho, \frac{\epsilon_0}{\tau^i}, K_j))\right) \cap \mathbb{C}^0 = \emptyset.$$

**Theorem 4.3:** Consider the stable plant (4.1) with $D = 0$ and with Controller PI1' applied at time $t = 0$; then for every $\tilde{\delta} \in \bar{\Omega}$, for every bounded constant reference and disturbance signal, and for every initial condition $z(0) := [x(0)^T \eta(0)^T]^T$ for which Assumption PI1' holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$, a finite constant $\epsilon_{ss} > 0$, and a matrix $K_{ss}$ such that $e(t) = \epsilon_{ss}$ and $K(t) = K_{ss}$ for all $t \geq t_{ss}$;

ii) the controller states $\eta \in \mathcal{L}_\infty$ and the plant states $x \in \mathcal{L}_\infty$; and

iii) if the reference and disturbance inputs are constant signals and $g \in \mathcal{T}F'$, then for almost all $(\epsilon_0, \tau, U) \in \mathcal{S}'$, $e(t) \to 0$ as $t \to \infty$.

### 4.1.3 Simulation Results

To illustrate the effect of this new switching mechanism on a system's potential closed loop transient response, consider the following MIMO plant taken from [64, pg. 517]:

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1/6 & 0 \\ 2/3 & 1 \\ 0 & 1/2 \end{bmatrix} u + Ew, \quad (4.5a)$$

$$y = \begin{bmatrix} 3 & -3/4 & -1/2 \\ 2 & -1 & 0 \end{bmatrix} x, \quad (4.5b)$$
which has a DC gain given by

\[ \mathcal{T} = \begin{bmatrix} 1 & 2 \\ 1/3 & 1 \end{bmatrix} \]

and a nominal transfer function matrix (i.e. with \( E = 0 \)) given by

\[ \frac{1}{(s + 1)^2} \begin{bmatrix} 1 - s & 2 - s \\ \left( \frac{1 - 3s}{3} \right) & 1 - s \end{bmatrix} . \]

Assume that the following estimate of \( \mathcal{T} \) is known:

\[ \hat{\mathcal{T}} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} ; \]

let

\[ \rho := 10. \]
\[ y_{\text{ref}}(t) := [-2 -2]^T. \]
\[ w(t) := 0. \]
\[ g(i) := \frac{10}{2^i} ; \]
\[ f(i) := \begin{cases} 4i, & 1 \leq i \leq 10 \\ 20(i - 10)^2 \exp((i - 10)^3), & i > 10. \end{cases} \tag{4.6} \]

and set all controller-plant initial conditions to be equal to zero at time \( t = 0 \). For this example, do not reset controller states \( \eta(t) \) to be equal to zero immediately following any controller switch.

In Figures 4.1 and 4.2, the output response of the closed loop system is shown with Controllers P2 [14] and PI1 applied respectively. As can be seen, in Figure 4.2, all controller switches are due to norm bounds on \( e(t) \) being met or exceeded; however, although the actual switching time instants shown in Figures 4.1 and 4.2 are relatively close to each other, a substantially improved transient response occurs in Figure 4.2. This result can be attributed to the fact that Controller PI1 now uses an additional norm bound (\( \|e(t)\| \)) in an attempt to detect instability.
4.1. Self-Tuning Proportional-Integral Control

Figure 4.1: Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller P2 [14] applied to (4.5).

Figure 4.2: Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller PI1 applied to (4.5).
4.2 Self-Tuning Proportional-Integral-Derivative Control

In this section, practical self-tuning proportional-integral-derivative (PID) controllers of the form [6]

\[ u = K(\eta + \rho \epsilon_1 e + \epsilon_2 d), \]
\[ \dot{\eta} = e \epsilon. \]
\[ d(s) = \frac{-s}{(s N + 1)} y(s) \]

will be considered when applied to the system given in (4.1) for situations involving both a known and an unknown estimate of the steady-state DC gain matrix \( \mathcal{T} \). As in Section 4.1, assume throughout that \( \mathcal{T} \) has full row rank in order to form a tractable problem.

Note that for constant parameters \( \rho, \epsilon, \epsilon_1, \epsilon_2, N, \) and \( K \), the closed loop system formed by augmenting (4.1) together with (4.7) can be expressed as

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta} \\
\dot{a}
\end{bmatrix} =
\begin{bmatrix}
A + BW\dot{I}C & BK + BW\dot{I}DK & \xi \\
-\epsilon\dot{I}C & -\epsilon\dot{I}DK & \epsilon\epsilon_2 N^2\dot{I}DK \\
-\dot{I}C & -\dot{I}DK & -NI + \epsilon_2 N^2\dot{I}DK
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
a
\end{bmatrix}
+ B
\begin{bmatrix}
y_{\text{ref}} \\
w
\end{bmatrix}
\]

where (see Remark 4.2)

\[
\dot{I} := (I + \rho \epsilon_1 DK + \epsilon_2 NK)^{-1},
\]
\[
\xi := -\epsilon_2 N^2 B(K + W\dot{I}DK),
\]
\[
W := -\rho \epsilon_1 K - \epsilon_2 NK,
\]
\[
\begin{bmatrix}
\rho \epsilon_1 BK + \rho \epsilon_1 BW\dot{I}DK & E + BW\dot{I}F \\
\epsilon I - \rho \epsilon_1 \dot{I}DK & -\epsilon\dot{I}F \\
-\rho \epsilon_1 \dot{I}DK & -\dot{I}F
\end{bmatrix}
\]
Furthermore, here, we consider "derivative" terms of the form

\[ d(s) = \frac{-s}{s/N + 1}y(s) \]

and \textit{not} of the form

\[ d(s) = \frac{s}{s/N + 1}(b_1 \cdot y_{ref}(s) - y(s)), \quad b_1 \in \mathbb{R}^+ \]

since, as noted in [6, pg. 7], the reference signal \( y_{ref}(t) \) is normally \textit{piecewise} constant in nature. As well, the proposed first order filter \((d(s)/y(s))\) is used in order to limit the noise sensitivity produced by the derivative action in any practical situation.

\textbf{4.2.1 Using an Estimate of the DC Gain}

For the self-tuning PID controller using a known estimate of \( T \), given by \( \hat{T} \), define the set of admissible controller parameters as

\[ \Omega_{PID} := \{(f,g,g_1,g_2,\rho,N) : f \in \text{MSBF}, g \in \text{TF}', g_1 \in \text{TF}', g_2 \in \text{TF}', \rho \geq 0, N > 0\}, \]

and label \textbf{Assumption PID1} to be

i) \( \|[[\eta(0)^T \ a(0)^T]]^T \| < f(1); \)

ii) \( \|e(0)\| < f(1); \)

iii) \( -\hat{T} \hat{T}^+ \) is stable; and

iv) \( (I + \rho \frac{\epsilon_0_1}{\tau_1} DK + \frac{\epsilon_0_2}{\tau_2} NDK), \ i \in \{1, 2, 3, \ldots\}, \) is invertible for \((\epsilon_0_1, \tau_1, \epsilon_0_2, \tau_2) \in S \times S \)

and for fixed \( \rho \geq 0, N > 0 \) (see Remark 4.2).

With \( \sigma_{PID} = (f,g,g_1,g_2,\rho,N) \in \Omega_{PID}, \) define \textbf{Controller PID1} as follows:

\[ \eta(t) = \int_{t_k}^{t} \epsilon(\tau)e(\tau)d\tau, \quad \eta(t_k^+) \equiv 0, \quad t \in (t_k, t_{k+1}] \]
\[ \dot{a}(t) = -Na(t) - y(t), \]
\[ d(t) = -N^2a(t) - Ny(t), \]
\[ u(t) = K (\eta(t) + \rho \epsilon_1(t)e(t) + \epsilon_2(t)d(t)) \]
where \( k \in \{1, 2, 3, \ldots \} \), \( K := T^t \),

\[
(\varepsilon(t), \varepsilon_1(t), \varepsilon_2(t)) = (g(k), g_1(k), g_2(k)), \quad t \in (t_k, t_{k+1}],
\]
t_1 := 0, and where, for each \( k \geq 2 \) such that \( t_{k-1} \neq \infty \), the switching time \( t_k \) is defined by

\[
t_k := \begin{cases} \min t \ni \\
\text{i) } t > t_{k-1}, \text{ and} \\
\text{ii) } \|[(\eta(t)^T a(t)^T e(t)^T)^T] = f(k-1) \\
\infty \quad \text{otherwise.}
\end{cases}
\]

**Remark 4.2:** Given \( \rho \geq 0 \) and \( D \in \mathbb{R}^{r \times m} \), then for almost all \((\varepsilon_1, \varepsilon_2, N) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) and for almost all matrices \( K \in \mathbb{R}^{m \times r} \), \((I + \rho \varepsilon_1 DK + \varepsilon_2 NDK) \) is invertible. \( \diamond \)

Once again, the following two lemmas can be obtained via methods similar to those used for Lemmas 4.1 and 4.2. In particular, to prove Lemma 4.5, one can again rewrite \( \hat{I} \) as

\[
\hat{I} := \frac{\hat{i}}{\Omega},
\]

where \( \hat{I} \) is a matrix whose elements are polynomials in \( \varepsilon_1, \varepsilon_2, \) and the elements of \( K \), and where \( \Omega \) is a polynomial in \( \varepsilon_1, \varepsilon_2, \) and the elements of \( K \) with the property that

\[
\lim_{(\varepsilon_1, \varepsilon_2) \to (0,0)} \Omega = 1.
\]

**Lemma 4.4:** Consider the closed loop system (4.8) where \( A \) and \(-TK\) are both stable. Then there exist constants \((\alpha, \beta, \varepsilon^*) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) with the property that for every initial condition and for every pair of bounded piecewise continuous reference and disturbance signals,

\[
||z(t)|| \leq \alpha ||z(0)|| + \beta \sup_{\tau \geq 0} (||y_{ref}(\tau)|| + ||w(\tau)||)
\]

for \( \varepsilon \in (0, \varepsilon^*), \varepsilon_1 \in (0, \varepsilon^*), \varepsilon_2 \in (0, \varepsilon^*), t \geq 0. \)
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Proof: Consider the situation when \((\epsilon_1, \epsilon_2) = (0, 0)\), and observe that

\[
eig(\bar{A}_{PID}) = \epsilon \cdot \eig(\bar{A}_{PID})
\]

where

\[
\bar{A}_{PID} := \begin{bmatrix}
    -DK & C & 0 \\
    BK & -\tfrac{A}{\epsilon} & 0 \\
    -DK & -\tfrac{C}{\epsilon} & \frac{NI}{\epsilon}
\end{bmatrix}.
\]

From Corollary 4.1, there then exists an \(\bar{\epsilon} \in \mathbb{R}^+\) such that \(\eig(\bar{A}_{PID}) \subseteq \mathbb{C}^-\) for all \(\epsilon \in (0, \bar{\epsilon})\). Hence, by the continuity of eigenvalues, there exist constants \((\alpha, \epsilon^*, \lambda) \in \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^+\) such that, with \(\mathcal{I} := (0, \epsilon^*)\),

\[
\eig(\bar{A}_{PID}) \subseteq \mathbb{C}^-
\]

and \(\|e^{\bar{A}_{PID}t}\| \leq \alpha e^{-\lambda t}\)

for all \((\epsilon, \epsilon_1, \epsilon_2) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I}, t \geq 0\).

Define

\[
J_1 := \sup_{(\epsilon, \epsilon_1, \epsilon_2) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I}} \|\mathcal{B}\|.
\]

Since

\[
z(t) = e^{\bar{A}_{PID}t}z(0) + \int_0^t e^{\bar{A}_{PID}(t-\tau)} \left( B \cdot [y_{ref}(\tau)^T w(\tau)^T] \right) d\tau.
\]

it therefore follows upon taking norms that

\[
\|z(t)\| \leq \alpha \|z(0)\| + \left( \frac{\alpha \beta_1}{\lambda} \right) \sup_{\tau \geq 0} \|y_{ref}(\tau)\| + \|w(\tau)\|
\]

for all \((\epsilon, \epsilon_1, \epsilon_2) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I}, t \geq 0\). \(\square\)

Lemma 4.5: Consider the matrix \(\bar{A}_{PID}(\rho, \epsilon, \epsilon_1, \epsilon_2, K, N)\) given in (4.8) where \(A\) is stable and \(\rho \geq 0\), \(N > 0\) are fixed; then if \(-TK\) is stable, for almost all \((\epsilon_0, \tau, \epsilon_0_1, \tau_1, \epsilon_0_2,\)
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\[ \tau_2 \in S \times S \times S, \]
\[ \left( \bigcup_{h=1}^{\infty} \lambda(\tilde{A}_{PID}(\rho, \frac{\epsilon_0}{\tau_1}, \frac{\epsilon_0}{\tau_2}, K, N)) \right) \cap \mathcal{C}^0 = \emptyset. \]

These results enable us to obtain the following.

**Theorem 4.4:** Consider the stable plant (4.1) with \( D = 0 \) and with Controller \( PID_1 \) applied at time \( t = 0 \); then for every \( \sigma_{PID} \in \Omega_{PID} \), for every bounded constant reference and disturbance signal, and for every initial condition \( z(0) := [x(0)^T \eta(0)^T a(0)^T]^T \) for which Assumption PID1 holds, the closed loop system has the properties that:

i) there exist a finite time \( t_{ss} \geq 0 \) and constants \( (\epsilon_{ss}, \epsilon_{ss1}, \epsilon_{ss2}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) such that \( \epsilon(t) = \epsilon_{ss}, \epsilon_1(t) = \epsilon_{ss1}, \epsilon_2(t) = \epsilon_{ss2} \) for all \( t \geq t_{ss} \);

ii) the controller states \( \eta, a \in \mathcal{L}_{\infty} \), and the plant states \( x \in \mathcal{L}_{\infty} \); and

iii) if the reference and disturbance inputs are constant signals and \( g \in \text{TF}' \), \( g_1 \in \text{TF}' \), \( g_2 \in \text{TF}' \), then for almost all \( (\epsilon_0, \tau, \epsilon_0_1, \tau_1, \epsilon_0_2, \tau_2) \in S \times S \times S. e(t) \to 0 \) as \( t \to \infty \).

4.2.2 Using no Estimate of the DC Gain

For the self-tuning PID controller using no known estimate of \( T \), define the set of admissible controller parameters as

\[ \Omega_{PID} := \{ (f, g, g_1, g_2, \rho, N, U) : f \in \text{MSBF}, g \in \text{TF}', g_1 \in \text{TF}', g_2 \in \text{TF}', \rho \geq 0, N > 0, U \in \mathbb{R}^{m \times m} \text{ and is nonsingular} \}; \]

let **Assumption PID1'** be Assumption PID1 with condition (iii) removed.

With \( \sigma_{PID}' = (f, g, g_1, g_2, \rho, N, U) \in \Omega_{PID} \), define **Controller PID1'** as

\[ \eta(t) = \int_{t_k}^{t} e(\tau)e(\tau)d\tau. \quad \eta(t_k^+) \equiv 0. \quad t \in (t_k, t_{k+1}] \]
\[ \dot{a}(t) = -Na(t) - y(t), \]
\[ d(t) = -N^2 a(t) - Ny(t), \]
\[ u(t) = K(t) (\eta(t) + \rho e_1(t)e(t) + e_2(t)d(t) \)
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where \( k \in \{1, 2, 3, \ldots\} \).

\[
(e(t), e_1(t), e_2(t)) = (g(k), g_1(k), g_2(k)), \quad t \in (t_k, t_{k+1}].
\]

\( t_1 := 0 \), and where, for each \( k \geq 2 \) such that \( t_{k-1} \neq \infty \), the switching time \( t_k \) is defined by

\[
t_k := \begin{cases}
\min t \ni & i) t > t_{k-1}, \quad \text{and} \\
\infty & \text{if this minimum exists}
\end{cases}
\]

\[
\quad \quad \text{ii) } \|[[\eta(t)^T a(t)^T e(t)^T]^T]\| = f(k - 1)
\]

with

\[
K(t) = U W_i, \quad i \in \{1, 2, \ldots, s\}. \quad i = ((k - 1) \mod s) + 1. \quad t \in (t_k, t_{k-1}].
\]

An explicit method for constructing the \( K_j := U W_j \) for the case when \( m \geq r \) is given in [55] which ensures that \( -TK_j \) is stable for at least one \( j \in \{1, 2, \ldots, s\} \). \( s \in \mathbb{N} \).

**Lemma 4.6:** Consider the matrix \( \tilde{A}_{PID}(\rho, \epsilon, \epsilon_1, \epsilon_2, K, N) \) given in (4.8) where \( A \) is stable and \( \rho \geq 0, N > 0 \) are fixed; then with \( K := \{K_j : j \in \{1, 2, \ldots, s\}\} \). for almost all \( (\epsilon_0, \tau, \epsilon_0_1, \epsilon_0_2, \tau_1, \tau_2, U) \in S \times S \times S' \),

\[
\left( \bigcup_{i=1}^{s} \bigcup_{\lambda=1}^{\infty} \lambda(\tilde{A}_{PID}(\rho, \epsilon_0, \epsilon_0_1, \epsilon_0_2, \tau_1, \tau_2, K_i, N)) \right) \cap \mathbb{C}^0 = \emptyset.
\]

**Theorem 4.5:** Consider the stable plant (4.1) with \( D = 0 \) and with Controller PID1' applied at time \( t = 0 \): then for every \( \sigma_{PID}' \in \Omega_{PID}' \). for every bounded constant reference and disturbance signal, and for every initial condition \( z(0) := [x(0)^T \eta(0)^T a(0)^T]^T \), for which Assumption PID1' holds. the closed loop system has the properties that:

i) there exist a finite time \( t_{ss} \geq 0 \), a matrix \( K_{ss} \), and constants \( (\epsilon_{ss}, \epsilon_{ss_1}, \epsilon_{ss_2}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) such that \( K(t) = K_{ss}, \epsilon(t) = \epsilon_{ss}, \epsilon_1(t) = \epsilon_{ss_1}, \epsilon_2(t) = \epsilon_{ss_2} \) for all \( t \geq t_{ss} \);

ii) the controller states \( \eta, a \in L_\infty \), and the plant states \( x \in L_\infty \); and
iii) if the reference and disturbance inputs are constant signals and \( g \in \text{TF}' \), \( g_1 \in \text{TF}' \), \( g_2 \in \text{TF}' \), then for almost all \((\varepsilon_0, \tau, \varepsilon_0_1, \tau_1, \varepsilon_0_2, \tau_2, U) \in S \times S \times S'\), \( e(t) \to 0 \) as \( t \to \infty \).

**Remark 4.3:** If \( D \neq 0 \) and/or if \( y_{ref}(t) \) or \( w(t) \) are bounded piecewise constant reference and disturbance signals, then Theorems 4.2, 4.3, 4.4, and 4.5 can be modified, as in Theorem 2.2, by letting \( \lambda \in \mathbb{R}^+ \), by defining

\[
\dot{e}_f := -\lambda e_f + \lambda e \\
\dot{u}_f := -\lambda u_f + \lambda u
\]

and by switching based upon norm bounds on \([\eta(t)^T a(t)^T e_f(t)^T]_T^T \) or \([\eta(t)^T a(t)^T e_f(t)^T u_f(t)^T]_T^T \) (assuming that \( e_f(0) < f(1) \) and \( u_f(0) < f(1) \)).

In Theorems 4.4 and 4.5, relatively little a priori plant information and on-line computation is required in order to successfully apply either Controller PID1 or Controller PID1' to a potentially unknown (not necessarily strictly proper) MIMO open loop stable system. In essence, with \( n, A, B, C, D, E, \) or \( F \) potentially unknown, but with \( T \) having full row rank and \( \lambda(A) \subset \mathbb{C}^- \), all proposed controllers will almost always provide asymptotic error regulation and disturbance rejection with \( y_{ref} \) and \( w \) bounded constant signals. Furthermore, in an attempt to reduce the transient tuning response, the switching criterion now is based partially upon the norm bound of \( e(t) \).

These facts therefore compare favourably when contrasted, for instance, with the SISO results outlined in [102], [84], [5], [35], [6], [40], [7], and with the (strictly proper) MIMO results given in [83], [88], [89], [53], [45], [52]. More specifically, it is assumed that \( m = r \) (with each decentralized control agent measuring only one plant output and manipulating only one plant input) and that an accurate model representation is available in [83]; that on-line manual tuning of the proportional-integral controllers occurs in [88]; that \( m = r, m \leq n \), and that input-output decoupling (in the sense of [32, pg. 652]) occurs in [89] and [45]; that \( m = r \) and that diagonal decentralized control occurs in [53]; and that diagonal dominance occurs in [52]. Moreover, the scheme of [83] admits the possibility of having to first design \( m! \) controllers, and "is based on a theorem and two heuristics".

**Remark 4.4:** Consider the situation when
4.2. Self-Tuning Proportional-Integral-Derivative Control

i) \((f_1, f_2) \in \text{CMSBF}\):

ii) \(\|\eta(0)\| < f_1(1)\): and

iii) \(\|a(0)\| < f_2(1)\).

Then, similar to Remark 2.4, Theorems 4.2, 4.3, 4.4, and 4.5 will also hold for all bounded piecewise constant reference and disturbance signals with \(D\) not necessarily equal to zero, and with the switching time \(t_k\) given by

\[
t_k := \begin{cases} 
\min t \geq i) \quad t > t_{k-1} \text{ and} \\
\quad \text{if this minimum exists} \\
\quad \quad \quad \text{ii) } \|\eta(t)\| = f_1(k - 1) \text{ and/or} \\
\quad \quad \quad \quad \|a(t)\| = f_2(k - 1) \\
\infty \quad \text{otherwise.}
\end{cases}
\]

This occurs since, with fixed admissible controller parameters. Controller PID1 can be expressed as

\[
\begin{bmatrix}
\dot{\eta} \\
\dot{a}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & -NI
\end{bmatrix} \begin{bmatrix}
\eta \\
a
\end{bmatrix} + \begin{bmatrix}
-\epsilon I \\
-I
\end{bmatrix} y + \begin{bmatrix}
\epsilon I \\
0
\end{bmatrix} \|y\|_\infty \;
\]

\[
u = \begin{bmatrix}
K \\
-\epsilon_2 N^2 K
\end{bmatrix} \begin{bmatrix}
\eta \\
a
\end{bmatrix} + [-\rho \epsilon_1 K - \epsilon_2 N K] y + [\rho \epsilon_1 b K] y_{\text{ref}}
\]

in its most general form (4.10). As well, in this instance, with \(\rho := 0\) and \(\epsilon_2 := 0\). (4.9) reduces essentially to the original definition of switching time \(t_k\) given for Controllers 2 and 2' in [64].

\[\Diamond\]

**Remark 4.5:** Theorems 4.4 and 4.5 are both equally valid for the (PI) case when \(\epsilon_2 := 0\) (see Theorems 4.2 and 4.3) and for the (ID) case when \(\rho := 0\).

\[\Diamond\]

Theorems 4.4 and 4.5 are also valid for self-tuning PID controllers of the form [9, pg. 222]

\[
u = K(\eta + \rho \epsilon_1 (b \cdot y_{\text{ref}} - y) + \epsilon_2 d),
\]

\[
\dot{\eta} = \epsilon (y_{\text{ref}} - y).
\]
where \((b, N) \in \mathbb{R}^+ \times \mathbb{R}^+\). This follows since the closed loop system may be expressed as (4.8), where

\[
B := \begin{bmatrix}
b(\rho e_1 BK + \rho e_1 BW \hat{D} K) & E + BW \hat{F} \\
\epsilon I - bp e_1 \hat{D} K & -\epsilon \hat{F} \\
-bp e_1 \hat{D} K & -\hat{F}
\end{bmatrix}
\]

and

\[
D := \begin{bmatrix}
bp e_1 \hat{D} K & \hat{F} \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Furthermore, in this particular case, the comments given in Remarks 4.3, 4.4, and 4.5 also hold true.

### 4.2.3 Simulation Results

**Example 1: Two Input-Two Output System**

Consider once again the system given by (4.5). As before, assume that the following estimate of \(\mathcal{T}\) is known:

\[
\hat{\mathcal{T}} = \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}.
\]

let \(f(i)\) be given by (4.6), with

\[
g(i) := \frac{10}{2^i},
\]

\[
y_{ref}(t) := [-2 \quad -2]^T.
\]

\[
w(t) := 0,
\]

and set all controller-plant initial conditions to be equal to zero at time \(t = 0\).
4.2. Self-Tuning Proportional-Integral-Derivative Control

Using Controller 2 defined in [64], the output response shown in Figure 4.3 is obtained. For comparison, the results obtained by applying Controller PID1 with

\[ g_1(i) = g_2(i) := g(i), \]

\( \rho := 1, \) and \( N := 1 \) are given in Figure 4.4; in this case, all initial conditions are also set to be equal to zero at time \( t = 0, \) the states of \( \eta(t) \) are not reset to zero after each controller switch, and the same modified strong bounding function \( f(i) \) as defined in (4.6) is used.

As can be seen, in this instance, the output transient response of Figure 4.4 is noticeably improved over that shown in Figure 4.3. (The one switch which occurs at 0.475 seconds in Figure 4.4 is due to the bound on \( [\eta(t)^T \ a(t)^T]^T. \) For further comparison, using similar initial conditions and parameter functions/values (unless otherwise noted) as in Figure 4.4, the plant output obtained using Controller PID1' and Controller PID1 is shown in Figures 4.5\(^1\) and 4.6 respectively. In Figure 4.5,

\[ f(i) := \begin{cases} 4\alpha(i) + \beta(i), & 1 \leq i \leq 50 \\ 20(i - 50)^2 \exp((i - 50)^3), & i > 50. \end{cases} \]

\[ g(i) := \frac{10}{2\alpha(i)}, \]

\[ \alpha(i) := \text{floor} \left( \frac{i + 5}{6} \right), \]

\[ \beta(i) := 0.2 \cdot ((k - 1) \mod 6), \]

and the cyclic switching action as summarized in Table 4.1 is implemented\(^2\); for Figure

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( (t_1, t_2) )</td>
<td>( (t_2, t_3) )</td>
<td>( (t_6, t_7) )</td>
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<tr>
<td>( K )</td>
<td>( W_1 )</td>
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<td>( W_1 )</td>
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<tr>
<td>( k )</td>
<td>8</td>
<td>9</td>
<td>10</td>
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<tr>
<td>( t )</td>
<td>( (t_8, t_9) )</td>
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<td>( (t_{10}, t_{11}) )</td>
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<td>( K )</td>
<td>( W_2 )</td>
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<td>( W_3 )</td>
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</table>

Table 4.1: Summary of the cyclic switching behaviour used for Figure 4.5.

---

1. Here, no estimate of \( T \) is assumed to be known.
2. In this case, five cycles through each of the six possible feedback matrices are required before switching stops; once again, the states of \( \eta(t) \) are not reset to zero after each controller switch.
Figure 4.3: Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller 2 given by Miller and Davison [64] applied to (4.5).

Figure 4.4: Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1 applied to (4.5).
4.2. Self-Tuning Proportional-Integral-Derivative Control

4.6. \( \rho \) is set to be equal to zero (with \( \eta(t) \) not reset to zero immediately following the one controller switch). Note that in Figure 4.5, switches which are due to the bound on \( e(t) \) or \([\eta(t)^T a(t)^T]^T\) are denoted by an 'E' or a 'X' respectively, while in Figure 4.6, the one switch is again due to the bound on \([\eta(t)^T a(t)^T]^T\).

**Remark 4.6:** For a two input-two output system, the six possible feedback matrices are

\[
W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_2 = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}, \quad W_3 = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}.
\]

and

\[
W_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W_5 = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}, \quad W_6 = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.
\]

Example 2: Four Input-Four Output Building Temperature System

As another example, consider the (four input-four output) partial decentralized temperature control problem of a multi-zone building [29] described by

\[
\begin{align*}
\dot{x} &= Ax + Bu + E\delta T_{out}, \\
y &= Cx.
\end{align*}
\]

(4.11a) (4.11b)

For completeness, matrices \( A, B, C, \) and \( E \) are given in Section B.2. As one can verify, \( \lambda(A) \subset \mathbb{C}^- \) [29, pg. 12] and

\[
\mathcal{T} = \begin{bmatrix} 24.8739 & 2.0509 & 1.3565 & 0.1552 \\ 2.0509 & 30.0662 & 0.1796 & 1.0794 \\ 1.3565 & 0.1796 & 26.4671 & 1.6125 \\ 0.1552 & 1.0794 & 1.6125 & 25.2580 \end{bmatrix}
\]

has full rank.
4.2. Self-Tuning Proportional-Integral-Derivative Control

Figure 4.5: Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to (4.5).

Figure 4.6: ($\rho = 0$) Simulated results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1 applied to (4.5).
4.2. Self-Tuning Proportional-Integral-Derivative Control

Assume that an estimate of \( \mathcal{T} \), given by

\[
\hat{T} = \begin{bmatrix}
24.40 & 5.21 & 4.03 & 1.44 \\
5.21 & 29.20 & 1.55 & 4.58 \\
4.03 & 1.55 & 27.90 & 4.84 \\
1.44 & 4.58 & 4.84 & 28.70
\end{bmatrix},
\]

is available, and let

\[
\begin{align*}
\rho & := 7, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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4.2. Self-Tuning Proportional-Integral-Derivative Control

Theoretical continuous time switching controller output response.

Figure 4.7: Simulated results of $y_1$ (solid), $y_2$ (dotted), $y_3$ (dash-dotted), and $y_4$ (dashed) with Controller PID1 applied to (4.11).

Fixed controller output response.

Figure 4.8: Simulated results of $y_1$ (solid), $y_2$ (dotted), $y_3$ (dash-dotted), and $y_4$ (dashed) with fixed integral controller (4.13) applied to (4.11).
Example 3: Four Input-Four Output Furnace System

For a final example, consider the situation when an estimate of $\mathcal{T}$ is assumed to be known, and when (filtered) Controller PID1 (using (4.10)) is applied to the following model-reduced (four input-four output) furnace system taken originally from [91, pg. 199]$^3$:

\[
\begin{align*}
\dot{x} &= A_R x + B_R u + E_R w, \\
y &= C_R x + D_R u + F_R w
\end{align*}
\]

with $\lambda(A_R) \subset \mathbb{C}^-$ and $D_R \neq 0$. Since

\[
\mathcal{T} := D_R - C_R A_R^{-1} B_R
\]

\[
= \begin{bmatrix}
1.00 & 0.70 & 0.30 & 0.20 \\
0.60 & 1.00 & 0.40 & 0.30 \\
0.35 & 0.40 & 1.00 & 0.60 \\
0.20 & 0.30 & 0.70 & 1.00
\end{bmatrix}
\]

one can therefore verify that

\[
\mathcal{T} := I
\]

(i.e. $K := I$) satisfies condition iii) of Assumption PID1.

In Figure 4.9, the output response of the closed loop system using Controller PID1 is shown with

\[
\begin{align*}
\dot{e_f} &= -10e_f + 10e, \quad e_f(0) \equiv 0. \\
\rho &= 1, \\
N &= 1, \\
b &= 2, \\
w(t) &= 2, \\
y_{ref}(t) &= [-2 -2 -2 -2]^T,
\end{align*}
\]

$^3$For brevity, the system matrices used are given in Section B.3.
4.2. Self-Tuning Proportional-Integral-Derivative Control

\[ f(i) := \begin{cases} 
\begin{align*}
15i, & \quad 1 \leq i \leq 3 \\
20(i-3)\exp((i-3)^2), & \quad i > 3.
\end{align*}
\end{cases} \]

\[ (\epsilon(i), \epsilon_1(i), \epsilon_2(i)) := \left( \frac{10}{2^i}, \frac{10}{2^i}, \frac{10}{2^i} \right). \]

and all initial conditions of the controller and plant defined to be equal to zero at time \( t = 0 \). Here, one controller switch occurs at 0.352 seconds (due to the norm bound on \([\eta(t)^T a(t)^T]^T\) being met or exceeded), and, similar to the earlier results, \( \eta(t) \) is not reset to zero immediately following this switch. Moreover, due to the relatively poor estimate of \( \mathcal{T} \) which is used for this particular class of reference and disturbance signals (\( \|\mathcal{T} - \mathcal{T}\| = 1.35 \)), and due to the fact that

\[ y(0) = \begin{bmatrix} 5.6914 & -0.1111 & -7.4865 & 2.1858 \end{bmatrix}^T. \]

one can therefore also conclude that Controller PID1 is indeed robust in nature.

Figure 4.9: \((D_R \neq 0)\) Simulated results of \( y_1 \) (solid), \( y_2 \) (dotted), \( y_3 \) (dash-dotted), and \( y_4 \) (dashed) with (filtered) Controller PID1 applied to (4.14).

Finally, we include for completeness Figure 4.10, which shows the output response\(^4\)

\(^4\)In this case, no further controller switches occur for \( t > 0 \).
obtained using the same controller, initial conditions, and parameter functions/values as given for Figure 4.9 for the nominal (unreduced) input-output transfer function matrix given by

\[
\begin{bmatrix}
1 & 0.7 & 0.3 & 0.2 \\
1 + 4s & 1 + 5s & 1 + 5s & 1 + 5s \\
0.6 & 1 & 0.4 & 0.3 \\
1 + 5s & 1 + 4s & 1 + 5s & 1 + 5s \\
0.35 & 0.4 & 1 & 0.6 \\
1 + 5s & 1 + 5s & 1 + 4s & 1 + 5s \\
0.2 & 0.3 & 0.7 & 1 \\
1 + 5s & 1 + 5s & 1 + 5s & 1 + 4s
\end{bmatrix}
\]

(4.15)

and Figure 4.11, which shows the corresponding (integral control) output results\(^5\) produced using \(\rho := 0\) and \(\epsilon_2 := 0\).

---

\(^5\)No further controller switches occur for \(t > 0\) in Figure 4.11.
4.2. Self-Tuning Proportional-Integral-Derivative Control

Theoretical continuous time switching controller output response.

Figure 4.10: Simulated results of $y_1$ (solid), $y_2$ (dotted), $y_3$ (dash-dotted), and $y_4$ (dashed) with Controller PID1 applied to (4.15).

Theoretical continuous time switching controller output response.

Figure 4.11: $((\rho, \epsilon_2) = (0, 0))$ Simulated results of $y_1$ (solid), $y_2$ (dotted), $y_3$ (dash-dotted), and $y_4$ (dashed) with Controller PID1 applied to (4.15).
Chapter 5

The Self-Tuning Servomechanism with Control Input Constraints

Similar to the adaptive tracking problem with control input constraints considered originally in [60] and [67], in this chapter, the general structure of the previously proposed class of proportional-integral (PI) controllers (Controllers P11 and P11') is further modified to incorporate control signal saturation constraints. As in Chapter 4, the controller presented here attempts to improve the tuning output transient response over that obtained by using conventional integral (I) control. Unlike, however, the continuous time results given in [46], [93] and the discrete time SISO settings considered in [86], [87], [2], [1], and [4], for example, where certain structural information typically is assumed to be known in advance, the results presented here are given for continuous time, finite dimensional multivariable systems and once again require very little a priori system information. Simulation results as well as initial experimental studies (see Section 7.4.3) obtained when using this new controller tend to indicate that desirable improvements in the closed loop tuning response generally can be achieved.

5.1 Constrained Self-Tuning Proportional-Integral Control

Consider the finite dimensional LTI system given by

\[
\dot{x} = Ax + Bu + Ew, \quad (5.1a)
\]

\[
y = Cx + Du + Fw, \quad (5.1b)
\]
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^r \) is the plant output to be regulated, \( w \in \mathbb{R}^q \) is the disturbance, and \( e \in \mathbb{R}^r \) is the difference between the specified reference input \( y_{ref} \) and the output \( y \). Assume that \( m \geq r \), that \( A \) is stable, that \( n \), \( A, B, C, D, E, \) or \( F \) are not necessarily known, and restrict \( y_{ref} \) and \( w \) to be bounded constant signals; let \( T := D - CA^{-1}B \). In addition, in order to form a tractable problem, assume throughout that \( T \) has full row rank, and that, if available, the estimate of \( T, \hat{T} \) has full row rank.

With

\[
\begin{bmatrix}
  u \\
  y
\end{bmatrix} = [u_1 \ u_2 \ldots \ u_m]^T
\]

and \( u_{i}^{\text{min}} < u_{i}^{\text{max}}, \ i \in \{1,2,\ldots, m\} \), define \( U := \{ u \in \mathbb{R}^m : u_i^{\text{min}} \leq u_i \leq u_i^{\text{max}} \} \). \( \partial U \) to be the boundary of \( U \), \( U^0 \) to be the interior of \( U \), and \( u^c \) to be the center of \( U \). Let \( (b, \epsilon, \lambda, \rho) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \), \( K \in \mathbb{R}^{m \times r} \) with \( \text{rank}(K) = r \), and define

\[
\begin{align*}
\dot{\eta} & := \epsilon(y_{ref} - y), \\
\dot{e}_b & := -\epsilon\lambda e_b + \epsilon\lambda(b y_{ref} - y), \\
u & := K(\eta + \rho e_b) + u^c.
\end{align*}
\]

On augmenting (5.1) together with (5.2), the closed loop system and equilibrium points can be expressed respectively as

\[
\begin{pmatrix}
  \dot{x} \\
  \dot{\eta} \\
  \dot{e}_b \\
  \dot{z}
\end{pmatrix} = \begin{bmatrix}
  A & BK & \rho BK \\
  -\epsilon C & -\epsilon DK & -\rho\epsilon DK \\
  -\epsilon\lambda C & -\epsilon\lambda DK & -\epsilon\lambda I - \rho\epsilon\lambda DK
\end{bmatrix} \begin{bmatrix}
  x \\
  \eta \\
  e_b \\
  z
\end{bmatrix} + \begin{bmatrix}
  0 & E & B \\
  \epsilon I & -\epsilon F & -\epsilon D \\
  \epsilon\lambda b I & -\epsilon\lambda F & -\epsilon\lambda D
\end{bmatrix} \begin{bmatrix}
  0 \\
  F \\
  D
\end{bmatrix} \begin{bmatrix}
  y_{ref} \\
  w \\
  u^c
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \dot{y} \\
  \dot{u}
\end{bmatrix} = \begin{bmatrix}
  C & DK & \rho DK \\
  0 & K & \rho K
\end{bmatrix} \begin{bmatrix}
  x \\
  \eta \\
  e_b
\end{bmatrix} + \begin{bmatrix}
  0 & F & D \\
  0 & 0 & I
\end{bmatrix} \begin{bmatrix}
  y_{ref} \\
  w \\
  u^c
\end{bmatrix},
\]
and

\[
\begin{bmatrix}
  x_{ss} \\
  \eta_{ss} \\
  e_{bss}
\end{bmatrix}
= -\tilde{A}^{-1}(\rho, \epsilon, \lambda, K) \cdot
\begin{bmatrix}
  0 & E & B \\
  \epsilon I & -\epsilon F & -\epsilon D \\
  \epsilon \lambda b I & -\epsilon \lambda F & -\epsilon \lambda D
\end{bmatrix}
\begin{bmatrix}
  y_{ref} \\
  w \\
  u^c
\end{bmatrix}.
\]

where \( \tilde{A}^{-1}(\rho, \epsilon, \lambda, K) \) is well defined, and is given by

\[
\tilde{A}^{-1}(\rho, \epsilon, \lambda, K) :=
\begin{bmatrix}
  A^{-1} + A^{-1}BK(TK)^{-1}CA^{-1} & A^{-1}BK(TK)^{-1}/\epsilon & 0 \\
  -(TK)^{-1}CA^{-1} & -(TK)^{-1} - \rho I)/\epsilon & \rho I/(\epsilon \lambda) \\
  0 & I/\epsilon & -I/(\epsilon \lambda)
\end{bmatrix}.
\]

Before proceeding, the following preliminary definitions and results are required.

**Proposition 5.1:** ([67, pg. 878]) With \( y_{ref} \in \mathbb{R}^r \) and \( w \in \mathbb{R}^q \) bounded constant signals, there exists a control input signal \( u : [0, \infty) \to U \) so that \( \lim_{t \to \infty} e(t) = 0 \) if and only if

\[
y_{ref} + (CA^{-1}E - F)w \in \{ Tu : u \in U \}.
\]

**Definition 5.1:** ([67, pg. 880]) A function \( \tilde{k} : \mathbb{N} \to \{ K \in \mathbb{R}^{m \times r} : \|K\| = 1, \ \text{rank}(K) = r \} \) is a \( K \) tuning function \( (\tilde{k} \in \text{KTF}) \) if

(i) \( \{ \tilde{k}(i) : i \in \mathbb{N} \} \) is dense in \( \{ K \in \mathbb{R}^{m \times r} : \|K\| = 1, \ \text{rank}(K) = r \} \); and

(ii) \( \{ \tilde{k}(i) : i \geq n \} = \{ \tilde{k}(i) : i \in \mathbb{N} \} \) for all \( n \in \mathbb{N} \).

**Remark 5.1:** If \( m = r = 1 \), then \( \tilde{k}(i) := (-1)^i + 1 \) is a \( K \) tuning function.

**Proposition 5.2:** Consider a complex matrix \( M \in \mathbb{C}^{4 \times 4} \) defined by

\[
M :=
\begin{bmatrix}
  -\lambda - \rho \lambda \Lambda & -\lambda \Lambda & 0 & 0 \\
  -\rho \Lambda & -\Lambda & 0 & 0 \\
  0 & 0 & -\lambda - \rho \lambda \Lambda^* & -\lambda \Lambda^* \\
  0 & 0 & -\rho \Lambda^* & -\Lambda^*
\end{bmatrix}.
\]
5.1. Constrained Self-Tuning Proportional-Integral Control

where

\( (\lambda, \rho) \in \mathbb{R}^+ \times \mathbb{R}^+ \).

\( j := \sqrt{-1}. \)

\( a \in \mathbb{R}^+. \)

\( b \in \mathbb{R}, b \neq 0. \)

\( \Lambda := a + bj, \)

\( \text{and} \) \( \Lambda^* := a - bj. \)

Then

\( \text{eig}(M) \subseteq \mathbb{C}^- \)

for all admissible parameter values.

**Proof:** Observe that

\[
\det(\Gamma I - M) = \Gamma^4 + \Gamma^3 \gamma_3 + \Gamma^2 \gamma_2 + \Gamma \gamma_1 + \gamma_0. \tag{5.4a}
\]

where

\[
\gamma_3 = 2a + 2\lambda + 2\rho \lambda a. \tag{5.4b}
\]

\[
\gamma_2 = 2\lambda a (2 + \rho \lambda) + \lambda^2 + (1 + \rho \lambda)^2 (a^2 + b^2). \tag{5.4c}
\]

\[
\gamma_1 = 2\lambda^2 a + 2\lambda(1 + \rho \lambda)(a^2 + b^2). \tag{5.4d}
\]

\[
\text{and} \gamma_0 = \lambda^2 (a^2 + b^2). \tag{5.4e}
\]

Hence, the result immediately follows upon constructing the Routh table [49. pp. 164-168] for (5.4). \( \square \)

**Remark 5.2:** For the case when

\[
M := \begin{bmatrix}
-\lambda - \rho \lambda a & -\lambda a \\
-\rho a & -a
\end{bmatrix}
\]
with \((a, \lambda, \rho) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\). it immediately follows that

\[
eig(M) \subseteq \mathbb{C}^-
\]

since

\[
det(\Gamma I - M) = \Gamma^2 - \Gamma \cdot \text{trace}(M) + \det(M).
\]

Furthermore, since

\[
(a + \lambda + \rho \lambda a)^2 - 4a\lambda = (\lambda - a)^2 + 2a^2 \lambda \rho + 2a^2 \lambda^2 \rho + a^2 \lambda^2 \rho^2.
\]

and hence

\[
0 < (a + \lambda + \rho \lambda a)^2 - 4a\lambda < (a + \lambda + \rho \lambda a)^2.
\]

it therefore follows that

\[
eig(-M) \subseteq \mathbb{R}^+.
\]

**Lemma 5.1:** With \((\lambda, \rho) \in \mathbb{R}^- \times \mathbb{R}^+\) and \(\eig(-TK) \subseteq \mathbb{C}^-\). \(TK \in \mathbb{R}^{r \times r}\). consider the matrix \(M \in \mathbb{R}^{2r \times 2r}\) given by

\[
M := \begin{bmatrix}
-\lambda I - \rho \lambda TK & -\lambda TK \\
-\rho TK & -TK
\end{bmatrix}.
\]

Then

\[
eig(M) \subseteq \mathbb{C}^-
\]

for all admissible parameter values.

**Proof:** Let

\[
J := \Delta TK \Delta^{-1}
\]
be the block diagonal Jordan decomposition [37. pg. 339] of $TK$, and let
\[ \text{eig}(TK) = \{ \Lambda_1, \Lambda_2, \ldots, \Lambda_r \}. \]

Observe that
\[
\text{eig}(M) = \text{eig} \left( \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} -\lambda I - \rho \lambda TK & -\lambda TK \\ -\rho TK & -TK \end{bmatrix} \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{bmatrix} \right) \\
= \text{eig} \left( \begin{bmatrix} -\lambda I - \rho \lambda J & -\lambda J \\ -\rho J & -J \end{bmatrix} \right) \\
= \text{eig}(S_1) \cup \text{eig}(S_2) \cup \ldots \cup \text{eig}(S_r)
\]
where
\[
S_i := \begin{bmatrix} -\lambda - \rho \lambda \Lambda_i & -\lambda \Lambda_i \\ -\rho \Lambda_i & -\Lambda_i \end{bmatrix}, \quad i \in \{1, 2, \ldots, r\};
\]
hence, since

(i) \: \text{eig}(-TK) \subset \mathbb{C}^-; \: \text{and}

(ii) \: \text{all eigenvalues of } TK \text{ must occur in complex conjugate pairs}

the result follows upon applying Proposition 5.2 and noting Remark 5.2. \hfill \Box

Theorem 4.1 and Lemma 5.1 enable one to obtain the following result.

**Proposition 5.3:** Consider the closed loop system given by (5.3): then with $(\lambda, \rho) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\text{eig}(-TK) \subset \mathbb{C}^-$, $TK \in \mathbb{R}^{r \times r}$, there exists a constant $\epsilon^* \in \mathbb{R}^+$ such that
\[
\text{eig}(\tilde{A}(\rho, \epsilon, \lambda, K)) \subset \mathbb{C}^-
\]
for all $\epsilon \in (0, \epsilon^*)$.

**Definition 5.2:** ([67]) Bounded constant input signals $y_{ref} \in \mathbb{R}^r$ and $w \in \mathbb{R}^q$ are said to be feasible with respect to $(A, B, C, D, E, F)$ if
\[
\mathcal{T}^\dagger [y_{ref} + (CA^{-1}E - F)w] + [I - \mathcal{T}^\dagger \mathcal{T}] u^c \in U^0.
\]
For the case when no estimate of $\mathcal{T}$ is available, define **Controller C1** as

$$
\eta(t) = \int_{t_k}^{t} \epsilon(\tau) \epsilon(\tau) d\tau, \quad \eta(t_k^+) = 0, \quad t \in (t_k, t_{k+1}]
$$

$$
e_b(t) = \int_{t_k}^{t} e^{-\epsilon(t) \lambda(t-\tau)} \epsilon(\tau) \lambda (y_{ref}(\tau) - y(\tau)) d\tau, \quad e_b(t_k^+) = 0, \quad t \in (t_k, t_{k+1}]
$$

$$
u(t) = K(t) (\eta(t) + \rho(t) e_b(t)) + u^c
$$

where $k \in \{1, 2, 3, \ldots \}$,

$$(\epsilon(t), \rho(t)) = (g(k), \hat{\rho}(k)), \quad t \in (t_k, t_{k+1}],$$

$t_1 := 0$, and where, for each $k \geq 2$ such that $t_{k-1} \neq \infty$, the switching time $t_k$ is defined by

$$
t_k := \begin{cases} 
\min t \ni 
\text{i) } t > t_{k-1}, \text{ and } & \text{if this minimum exists} \\
\text{ii) } \nu(t) \in \partial U \\
\infty & \text{otherwise}
\end{cases}
$$

with

$$
K(t) = \tilde{k}(k), \quad t \in (t_k, t_{k+1}],
$$

and $g \in TF', \rho \in TF$, $\tilde{k} \in KTF$, and $(b, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+$.

**Remark 5.3:** Equation (5.3) can alternatively be written as

$$
\begin{bmatrix}
x' \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
A & BK \\
-\epsilon C & -\epsilon DK
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix} +
\begin{bmatrix}
0 \\
\epsilon I
\end{bmatrix}
y_{ref} +
\begin{bmatrix}
E & B \\
-\epsilon F & -\epsilon D
\end{bmatrix}
\begin{bmatrix}
w \\
\bar{u}
\end{bmatrix} +
\begin{bmatrix}
B \\
-\epsilon D
\end{bmatrix}
u^c,
$$

$$
\begin{bmatrix}
y \\
u
\end{bmatrix} =
\begin{bmatrix}
C & DK \\
0 & K
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix} +
\begin{bmatrix}
F & D \\
0 & I
\end{bmatrix}
\begin{bmatrix}
w \\
\bar{u}
\end{bmatrix} +
\begin{bmatrix}
D \\
I
\end{bmatrix}
u^c,
$$

where $\bar{u} := \rho Ke_b$.

A direct consequence of Remark 5.3 and Theorem 2 of [67] is the following.

**Theorem 5.1:** Consider the stable plant (5.1) with $\text{rank}(\mathcal{T}) = r$, and with Controller C1
applied at time $t = 0$; then for every $(b, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+$, for every $g \in \mathcal{T}_F$, $\tilde{p} \in \mathcal{T}_F$, $\tilde{k} \in \mathcal{K}_F$.

for every bounded constant reference and disturbance signal $(y_{\text{ref}}, w) \in \mathbb{R}^r \times \mathbb{R}^q$ which is feasible with respect to $(A, B, C, D, E, F)$, and for every initial condition $x(0) \in \mathbb{R}^n$, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$, a matrix $K_{ss}$, and constants $(\epsilon_{ss}, \rho_{ss}) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $K(t) = K_{ss}$, $\epsilon(t) = \epsilon_{ss}$, $\rho(t) = \rho_{ss}$ for all $t \geq t_{ss}$;

ii) the controller states $\eta$, $e_b \in \mathcal{L}_\infty$, and the plant states $x \in \mathcal{L}_\infty$; and

iii) if the final closed loop system has no eigenvalues lying on $\mathbb{C}^0$, then $e(t) \to 0$ as $t \to \infty$.

As in [67], Controller C1 essentially works by detuning the control system via parameter $\epsilon(t)$ each time any input control signal saturates. Since

$$
\dot{e}_b(t) = -\epsilon(t)\lambda e_b(t) + \epsilon(t)\lambda(b y_{\text{ref}}(t) - y(t)),
$$

each controller switch lessens the filter "speed" attributed to $e_b(t)$. As well, for property iii) of Theorem 5.1, one can also show that the final closed loop system generically will have no eigenvalues lying in $\mathbb{C}^0$.

**Remark 5.4:** ([67, pg. 881]) If a good estimate of $\mathcal{T}$, $\hat{T}$, is available by, for instance, conducting $m$ steady state experiments [21] on the open loop system, then one can set $K(t) := \hat{T}^+$ assuming that $-\mathcal{T} \hat{T}^+$ is stable. Except in the SISO case, however, now the set of admissible $y_{\text{ref}} \in \mathbb{R}^r$ and $w \in \mathbb{R}^q$ does not include the set of all feasible pairs; however, if $(y_{\text{ref}}, w)$ are feasible, then it follows that $(y_{\text{ref}}, w)$ are admissible provided that $\|K - \hat{T}^+\|$ is sufficiently small.

### 5.2 Simulation Results

**Example 1: SISO Nonminimum Phase System**

Consider the SISO nonminimum phase stable plant [67, pg. 883]

$$
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1.01 & -1.2 & -1.2
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u + \begin{bmatrix}
0 \\
0.3 \\
0
\end{bmatrix} w,
$$

(5.5a)
5.2. Simulation Results

\[ y = [1 -2 0] x. \]  \hspace{1cm} (5.5b)

whose nominal transfer function is given by

\[ \frac{100(1 - 2s)}{100s^3 + 120s^2 + 120s + 101} \]

where

\[ (\mathcal{T}, u_{ss}) = \left( \frac{100}{101}, \frac{83}{50} \right). \]

Assume that \((u^{\min}, u^{\max}) = (-5.5)\), and set

\[ (g(k), \bar{k}(k)) := \left( \frac{4}{2^k}, (-1)^{k-1} \right), \]

\[ \bar{\rho}(k) := \frac{10 \cdot 2}{2^{(k-1) \mod 20} + 1}, \]

\[ (b, \lambda) := (1, 10), \]

\[ (y_{\text{ref}}, w) := (2, 1), \]

and \(x(0) = 0\).

In Figures 5.1 and 5.2, the output responses respectively obtained upon applying Controllers 2 [67] and C1 to (5.5) are shown for the above parameter functions and values. Observe that in both instances, the final closed loop systems are stable \((\epsilon_{ss} = \frac{4}{25}\) and \(\epsilon_{ss} = \frac{4}{211}\) respectively), reference tracking and disturbance rejection occurs, and the control input constraint placed on \(u(t)\) is satisfied for all time. Furthermore, despite the relatively smaller value of \(\epsilon_{ss}\) in Figure 5.2 when compared with that of Figure 5.1, the transient responses of both figures are roughly comparable in nature.

**Example 2: Three Input-Three Output Distillation Column**

As another example, consider a MIMO minimum phase binary distillation tower with pressure variation [20], whose model is stable and is obtained by a linearization of the system about a standard operating point. Let the three control signal inputs be the reboiler steam temperature \(u_1 (^\circ\text{F})\), the condenser coolant temperature \(u_2 (^\circ\text{F})\), and the reflux ratio \(u_3\); the three outputs to be regulated are the bottom product composition \(y_1\) (mole fraction of
5.2. Simulation Results

Figure 5.1: Simulated results with Controller 2 [67] applied to (5.5).

Figure 5.2: Simulated results with Controller C1 applied to (5.5).
5.2. Simulation Results

the more volatile component in liquid phase), the top product composition \(y_2\) (mole fraction of the more volatile component in liquid phase), and the pressure \(y_3\) (atmosphere) [67, pg. 884]. With primary disturbance input \(w\) (mole fraction of the more volatile component in liquid phase) associated with changes in the input feed composition stream of the system, restrict

\[-1^\circ\text{F} \leq u_1(t) \leq 1^\circ\text{F}. \quad -1^\circ\text{F} \leq u_2(t) \leq 1^\circ\text{F}. \quad -0.1 \leq u_3(t) \leq 0.1.\]

and assume, after using the methods outlined in [21], that the following estimate of \(T\) to two significant digits is available:

\[
\hat{T} = \begin{bmatrix}
-0.0049 & 0.0062 & 7.9 \\
-0.00099 & 0.00066 & 4.4 \\
0.017 & 0.034 & 17
\end{bmatrix}
\]

Now with system matrices \((A, B, C, D, E, F)\) given in Section B.4.

\[
K(t) := \hat{T}^\top \text{ (see Remark 5.4).}
\]

\[
g(k) := \frac{0.5}{2^k},
\]

\[
\tilde{p}(k) := \frac{3 \cdot 2^{(k-1) \mod 20}-1}{2^{(k-1) \mod 20}-1}.
\]

\[
(b, \lambda) := (2, 100),
\]

\[
y_{ref} := [0 \quad 0 \quad 0]^T.
\]

\[
w := 0.2.
\]

and \(z(0) \equiv 0.\)

the results shown in Figures 5.3 and 5.4 are respectively obtained upon applying Controller 2 [67] and Controller C1 to the distillation tower. In these figures, \(y_1(t)\) and \(u_1(t)\), \(y_2(t)\) and \(u_2(t)\), and \(y_3(t)\) and \(u_3(t)\) are represented as solid, dashed, and dash-dotted lines.

Observe that in this instance, the transient response shown in Figure 5.4 is discernibly improved over that shown in Figure 5.3. In addition, in both of these cases, the final closed loop systems are once again stable (\(\epsilon_{ss} = \frac{0.5}{2^7}\) and \(\epsilon_{ss} = \frac{0.5}{2^5}\) respectively), reference tracking and disturbance rejection occurs, and the control input constraint imposed on \(u(t)\)
is successfully maintained for all time.

**Example 3: Four Input-Four Output Multi-Zone Building Control**

As a final example, consider the MIMO minimum phase partial decentralized temperature control problem of a multi-zone building whose stable model was given earlier in (4.11). Let

\[
K(t) := \begin{bmatrix}
0.0436 & -0.0075 & -0.0059 & 0 \\
-0.0075 & 0.0364 & 0 & -0.0054 \\
-0.0059 & 0 & 0.0377 & -0.0060 \\
0 & -0.0054 & -0.0060 & 0.0367
\end{bmatrix},
\]

\[
g(k) := \frac{10}{2^k},
\]

\[
\tilde{g}(k) := \frac{5 \cdot 2}{2^{((k-1) \mod 20)+1}},
\]

\[
(b, \lambda) := (1.5, 10),
\]

\[
y_{ref}(t) := [-2 2 3 2.5]^T,
\]

\[
\delta T_{out}(t) := -5.
\]

and restrict

\[
-1 \leq u_1(t) \leq 1, \quad -1 \leq u_2(t) \leq 1, \quad -1 \leq u_3(t) \leq 1, \quad -1 \leq u_4(t) \leq 1.
\]

Upon using these values and on applying Controllers 2 [67] and C1 respectively to (4.11), the output results shown in Figures 5.5 and 5.6 are obtained, where \( y_1(t) \) and \( u_1(t) \), \( y_2(t) \) and \( u_2(t) \), \( y_3(t) \) and \( u_3(t) \), and \( y_4(t) \) and \( u_4(t) \) are represented as solid, dashed, dash-dotted, and dotted lines. Again, as expected, the final closed loop systems are stable \( \varepsilon_{ss} = \frac{10}{12} \) in both instances), reference tracking and disturbance rejection occurs, and the control input constraint imposed on \( u(t) \) is successfully maintained for all time.
5.2. Simulation Results

Figure 5.3: Simulated results with Controller 2 [67] applied to a distillation tower.

Figure 5.4: Simulated results with Controller C1 applied to a distillation tower.
5.2. Simulation Results

Theoretical continuous time switching controller output response.

![Graph of Plant Output versus Time](image)

Control signal $u$ versus time.

![Graph of Control Signal $u$ versus Time](image)

Figure 5.5: Simulated results with Controller 2 [67] applied to (4.11).

Theoretical continuous time switching controller output response.

![Graph of Plant Output versus Time](image)

Control signal $u$ versus time.

![Graph of Control Signal $u$ versus Time](image)

Figure 5.6: Simulated results with Controller C1 applied to (4.11).
Chapter 6

Adaptive Tracking of LTI MIMO Systems

Using the methods and results presented in Chapter 3, we consider now the finite dimensional strictly proper MIMO adaptive tracking problem for the general class of reference and disturbance signals whose behaviour can be described by a linear combination of bounded piecewise continuous sinusoidal and polynomial functions. Once again, by monitoring plant output \( y(t) \) and/or error signal \( e(t) \), the emphasis will be to provide a robust controller which is insensitive to bounded piecewise continuous disturbances \( w(t) \) and which attempts to provide an acceptable transient response. Initial simulation studies using these new controllers tend to indicate that such desirable improvements in the tuning response usually can be achieved when compared with, for instance, the outputs obtained using the computationally more intensive algorithms presented in [59] and [66].

6.1 Preliminary Definitions and Results

Similar to Section 3.2, consider the finite dimensional strictly proper (stabilizable and detectable) MIMO LTI system given by

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ew, \\
y &= Cx + Fw, \\
e &= y_{\text{ref}} - y
\end{align*}
\]

(6.1a)  
(6.1b)  
(6.1c)
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^r \) is the plant output to be regulated, \( w \in \mathbb{R}^q \) is the disturbance, and \( e \in \mathbb{R}^r \) is the difference between the specified reference input \( y_{ref} \) and the output \( y \). In addition, assume that \( n, A, B, C, E, \) or \( F \) are not necessarily known.

Let \( \alpha(s) := \sum_{i=0}^{p} \alpha_i s^i \) with \( p \in \mathbb{N} \) and \( \alpha_p := 1 \), and let the roots of \( \alpha(s) \) lie in \( \mathbb{C}^+ \cup \mathbb{C}^0 \); as well, restrict

\[
y_{ref}(t) \in \left\{ f \in C^{\infty}(\mathbb{R}^r) : \sum_{i=0}^{p} \alpha_i f^{(i)}(t) = 0, f \in L_{\infty} \right\} \quad (6.2a)
\]

and \( w(t) \in \left\{ f \in C^{\infty}(\mathbb{R}^q) : \sum_{i=0}^{p} \alpha_i f^{(i)}(t) = 0, f \in L_{\infty} \right\} \). \( (6.2b) \)

The control objective in this chapter will be to stabilize the system given by (6.1) such that asymptotic reference tracking and disturbance rejection occurs for the class of \( y_{ref}(t) \) and \( w(t) \) defined by (6.2). Thus, in order to form a tractable problem [27], assume too that

(i) \( m \geq r \); and

(ii) the transmission zeros of \( (C, A, B, 0) \) do not coincide with the zeros of \( \alpha(s) \).

To form the servocompensator, consider the situation where matrices \( \hat{A} \in \mathbb{R}^{p \times p} \) and \( \hat{B} \in \mathbb{R}^p \) are chosen so that \( \text{det}(sI - \hat{A}) = \alpha(s) \) and \( (\hat{A}, \hat{B}) \) is controllable. Using [27], define

\[
\hat{A}^* := \text{block diagonal} \ (\hat{A}, \ldots, \hat{A}) \in \mathbb{R}^{p \times pr}.
\]

\[
\hat{B}^* := \text{block diagonal} \ (\hat{B}, \ldots, \hat{B}) \in \mathbb{R}^{p \times r}.
\]

and let the servocompensator \( \xi \) be given by

\[
\dot{\xi} := \hat{A}^* \xi + \hat{B}^* e. \quad (6.3)
\]

Combining plant (6.1) together with servocompensator (6.3), the augmented system can therefore be written as

\[
\dot{x} = \tilde{A} \bar{x} + \tilde{B} u + \tilde{E} \bar{w}. \quad (6.4a)
\]

\[
\bar{y} = \tilde{C} \bar{x} + \tilde{F} \bar{w}. \quad (6.4b)
\]
where

$$\dot{x} := \begin{bmatrix} x^T & \xi^T \end{bmatrix}^T.$$  \hspace{1cm} (6.4c)

$$\dot{y} := \begin{bmatrix} y^T & \xi^T \end{bmatrix}^T.$$  \hspace{1cm} (6.4d)

$$\dot{w} := \begin{bmatrix} y_{ref}^T & w^T \end{bmatrix}^T.$$  \hspace{1cm} (6.4e)

$$\begin{bmatrix} \hat{A} & \hat{B} & \hat{E} \end{bmatrix} := \begin{bmatrix} A & 0 & B & 0 & E \\ -\hat{B}^* & \hat{A}^* & 0 & \hat{B}^* & -\hat{B}^*F \end{bmatrix}.$$  \hspace{1cm} (6.4f)

and $$\begin{bmatrix} \hat{C} & \hat{F} \end{bmatrix} := \begin{bmatrix} C & 0 & 0 & F \\ 0 & I & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (6.4g)

**Lemma 6.1:** ([28]) Given the plant (6.1), the servocompensator (6.3), and the augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -\hat{B}^*C & \hat{A}^* \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & E \\ \hat{B}^* & -\hat{B}^*F \end{bmatrix} \begin{bmatrix} y_{ref} \\ w \end{bmatrix}.$$  \hspace{1cm} (6.4h)

then \((\hat{A}, \hat{B})\) is stabilizable and \((\hat{C}, \hat{A})\) is detectable.

Due to the structure of the constructed servocompensator \(\xi\), any LTI controller which stabilizes (6.4) will provide asymptotic error regulation and disturbance rejection [27]. Therefore, using the results given in Lemma 6.1, let the adaptive stabilizing controller for (6.4) be of the form

$$\mathcal{K}_i : \begin{cases} \dot{\eta} = G_i \eta + H_i \tilde{y} \\ u = K_i \eta + L_i \tilde{y} \end{cases}$$  \hspace{1cm} (6.5)

where \(\eta \in \mathbb{R}^{g_i}, G_i \in \mathbb{R}^{g_i \times g_i}, H_i \in \mathbb{R}^{g_i \times (r+pr)}, K_i \in \mathbb{R}^{m \times g_i}, L_i \in \mathbb{R}^{m \times (r+pr)}, \) and

$$\mathcal{\bar{K}}_i := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+g_i) \times (r+pr+g_i)}.$$
In addition, note that upon applying controller (6.5) to system (6.4), there always exists a $g_i \leq n$ such that the final closed loop system is stable.

**Definition 6.1:** A function $h : \mathbb{N} \rightarrow \mathbb{R}^{(m+g_i) \times (r+p-r+g_i)}$ is a general controller tuning function ($h \in \text{GCTF}$) if

1. \( \{h(i) : i \in \mathbb{N}\} \) is dense in \( \mathbb{R}^{(m+g_i) \times (r+p-r+g_i)} \) for fixed values of $g_i \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N} \cup \{0\}$; and
2. for $h(i) := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+g_i) \times (r+p-r+g_i)}$,

   \[ \|K_i\| \leq \tau_1^i \text{ and } \|L_i\| \leq \tau_2^i \]

   for constants $(\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.

**Proposition 6.1:** Given $g_i \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N} \cup \{0\}$, there exists a $h \in \text{GCTF}$.

**Definition 6.2:** Given that $\alpha \in \mathbb{N} \cup \{0\}$ is a lower bound and that $\gamma \in \mathbb{N} \cup \{0\}$ is an upper bound on $g_i$ such that $\alpha \leq g_i \leq \gamma$ with $\alpha < \gamma$, a function $h : \mathbb{N} \rightarrow \mathbb{R}^{(m+g_i) \times (r+p-r+g_i)}$ is a modified general controller tuning function ($h \in \text{GCTF'}$) if, with $(g_{i+1} - g_i) \in \{0, 1\}$, $g_i := \alpha$, $h(i) := \hat{K}_i$, and $p \in \mathbb{N} \cup \{0\}$, the following properties hold:

1. \( \lim_{i \to \infty} g_i \rightarrow \gamma \);
2. \( \{h(i) : i \in \mathbb{N}\} \) is dense in \( \mathbb{R}^{(m-\gamma) \times (r+p-r+\gamma)} \); and
3. for $h(i) := \begin{bmatrix} L_i & K_i \\ H_i & G_i \end{bmatrix} \in \mathbb{R}^{(m+g_i) \times (r+p-r+g_i)}$,

   \[ \|K_i\| \leq \tau_1^i \text{ and } \|L_i\| \leq \tau_2^i \]

   for constants $(\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.

**Proposition 6.2:** There exists a $h \in \text{GCTF'}$.

Using the adaptive stabilization results of Chapter 3, the following results are obtained.
6.2 Using a Known Value of the Compensator Order

For the case when it is known that compensator (6.5) will stabilize the augmented system (6.4) with \( g_l = l, l \in \mathbb{N} \cup \{0\} \), using some appropriate choice of controller matrices \( G_i, H_i, K_i, L_i \), define **Controller T1** as

\[
\begin{align*}
\dot{\eta}(t) &= G(t)\eta(t) + H(t)\bar{y}(t), \quad \xi(t_k^+) \equiv 0, \quad \eta(t_k^+) \equiv 0, \quad t \in (t_k, t_{k+1}] \\
u(t) &= K(t)\eta(t) + L(t)\bar{y}(t)
\end{align*}
\]

where \( k \in \{1, 2, 3, \ldots \} \), \( h \in \text{GCTF} \).

\[
\begin{bmatrix}
L(t) & K(t) \\
H(t) & G(t)
\end{bmatrix} := h(k), \quad t \in (t_k, t_{k+1}],
\]

\( t_1 := 0 \), and where, for each \( k \geq 2 \) such that \( t_{k-1} \neq \infty \), the switching time \( t_k \) is defined by

\[
t_k := \begin{cases} 
\min t \ni & \text{if this minimum exists} \\
i) \ t > t_{k-1}, \text{ and} \\
\text{ii) } \|\eta(t)^T \bar{y}(t)^T\|^T = f(k - 1) \\
& \infty \quad \text{otherwise}
\end{cases}
\]

with \( f \in \text{S2BF} \). Label **Assumption T1** to be

i) \( \|\eta(0)\| < f(1) \); and

ii) \( \|\bar{y}(0)\| < f(1) \).

**Theorem 6.1:** Consider the system given by (6.1), and assume that there exists a solution to the robust servomechanism problem for the class of reference and disturbance signals defined by (6.2); then with the corresponding servocompensator (6.3) implemented, and with Controller T1 applied at time \( t = 0 \), for every \( f \in \text{S2BF} \) and \( h \in \text{GCTF} \), for every bounded reference and disturbance signal of the form given by (6.2), and for every initial condition \( [x(0)^T \xi(0)^T \eta(0)^T]^T \) for which Assumption T1 holds, the closed loop system has the properties that:

i) there exist a finite time \( t_{ss} \geq 0 \) and constant matrices \((G_{ss}, H_{ss}, K_{ss}, L_{ss}) \) such that

\[
(G(t), H(t), K(t), L(t)) = (G_{ss}, H_{ss}, K_{ss}, L_{ss}) \quad \text{for all } t \geq t_{ss};
\]
6.3. Using no Known Value of the Compensator Order

ii) the controller states $\xi, \eta \in L_\infty$, and the plant states $x \in L_\infty$; and

iii) for almost all controller parameters $(G, H, K, L)$, the final eigenvalues of the closed loop system will lie in $C^-$. 

6.3 Using no Known Value of the Compensator Order

For the case when it is known that compensator (6.5) will stabilize the augmented system (6.4) with $g_i = l$ using some $\alpha \leq l \leq \gamma$ with $\alpha, \gamma \in \mathbb{N} \cup \{0\}$, $\alpha < \gamma$, and using some appropriate choice of controller parameters $G_i, H_i, K_i, L_i$, label Controller $T_1'$ to be Controller $T_1$, but with $h \in \text{GCTF}'$.

Theorem 6.2: Consider the system given by (6.1), and assume that there exists a solution to the robust servomechanism problem for the class of reference and disturbance signals defined by (6.2); then with the corresponding servocompensator (6.3) implemented, and with Controller $T_1'$ applied at time $t = 0$, for every $f \in \text{S2BF}$ and $h \in \text{GCTF}'$, for every bounded reference and disturbance signal of the form given by (6.2), and for every initial condition $[x(0)^T \xi(0)^T \eta(0)^T]^T$ for which Assumption $T_1$ holds, the closed loop system has the properties that:

i) there exist a finite time $t_{ss} \geq 0$ and constant matrices $(G_{ss}, H_{ss}, K_{ss}, L_{ss})$ such that $(G(t), H(t), K(t), L(t)) = (G_{ss}, H_{ss}, K_{ss}, L_{ss})$ for all $t \geq t_{ss}$; 

ii) the controller states $\xi, \eta \in L_\infty$, and the plant states $x \in L_\infty$; and

iii) for almost all controller parameters $(G, H, K, L)$, the final eigenvalues of the closed loop system will lie in $C^-$. 

If $A$ is stable and if $\alpha(s) = s$, then the controllers proposed in Chapter 4 of this thesis can be used to achieve asymptotic reference tracking and disturbance rejection for this particular class of signals. As well, upon filtering $\gamma(t)$ as 

$$\dot{\gamma}_f = -\lambda \gamma_f + \lambda \gamma, \ \lambda \in \mathbb{R}^+, $$
and upon making corresponding changes in the assumption and controller definitions (e.g. see Theorem 2.2 and Remark 3.5), Theorems 6.1 and 6.2 will also work for those reference and disturbance signals whose behaviour can be described by a linear combination of bounded piecewise continuous sinusoidal and polynomial functions. Furthermore, the comments given in Remark 3.4 are equally valid for for Theorems 6.1 and 6.2. In this instance, by using more system information, and hence, by reducing the parameter space search for a stabilizing $h(i)$, one would therefore also expect an improved output transient response.

**Remark 6.1:** ([21]) If $A$ is stable in (6.1) and the roots of $\alpha(s)$ lie in $\mathbb{C}^0$, then there exists a gain $\bar{K} \in \mathbb{R}^{m \times pr}$ so that $\tilde{A} + \tilde{B}[0 \bar{K}]\tilde{C}$ is stable. Hence, if plant (6.1) has an $l$'th-order stabilizing compensator, then the augmented system given by (6.4) does as well.

One can also combine plant (6.1) together with servocompensator (6.3) to form the augmented system

$$
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-\bar{B}^*\bar{C} & \bar{A}^*
\end{bmatrix}
\begin{bmatrix}
x \\
\xi
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} u +
\begin{bmatrix}
0 & E \\
\bar{B}^* & -\bar{B}^*F
\end{bmatrix}
\begin{bmatrix}
y_{ref} \\
w
\end{bmatrix},
$$

$$
\begin{bmatrix}
e \\
\xi
\end{bmatrix} =
\begin{bmatrix}
-C & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
x \\
\xi
\end{bmatrix} +
\begin{bmatrix}
l & -F \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
y_{ref} \\
w
\end{bmatrix},
$$

where $(\tilde{A}^*, \tilde{B}^*)$ is stabilizable and $(\tilde{C}^*, \tilde{A}^*)$ is detectable. Hence, Controllers $T_1$ and $T_1'$ will also work using the switching mechanism defined by

$$
t_k := \begin{cases} 
\min t \ni \\
\text{i) } t > t_{k-1}, \text{ and } \\
\text{if this minimum exists} \\
\text{ii) } \|\Xi(t)\| = f(k - 1) \\
\infty \quad \text{otherwise},
\end{cases}
$$

where

$$
\Xi(t) := \begin{bmatrix} \eta(t)^T & \bar{y}^*(t)^T \end{bmatrix}^T
$$

or

$$
\Xi(t) := \begin{bmatrix} \eta(t)^T & y(t)^T & e(t)^T & \xi(t)^T \end{bmatrix}^T,
$$
assuming that \(\|e(0)\| < f(1)\) and \(\|\xi(0)\| < f(1)\) also hold at time \(t = 0\).

### 6.4 Simulation Results

**Example 1: SISO Unstable Minimum Phase Plant**

Consider the (controllable and observable) unstable minimum phase SISO plant [66. pg. 57]

\[
\begin{align*}
\dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 2 \\ 1 \end{bmatrix} w. \\
\dot{x}_2 &= 1.1 \\
y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-2]w \\
\end{align*}
\]

with open loop eigenvalues of \(-2\) and \(1\), and assume that there exists a solution to the robust servomechanism problem for constant references and constant disturbances. Let \(y_{ref}(t) := 10, w(t) := 1\).

\[
f(k) := \begin{cases} 
3.5k. & 1 \leq k \leq 15 \\
30(k - 15)^2 \exp((k - 15)^3). & k > 15.
\end{cases}
\]

and set all controller-plant initial conditions to be equal to zero at time \(t = 0\). Since \(m = r = 1\) and \(\alpha(s) = s\), a choice of \((\hat{A}^*, \hat{B}^*) = (0, 1)\) yields an appropriate corresponding servocompensator \(\xi\) for the particular class of reference and disturbance signals chosen.

Assume that the given plant (6.6) is unknown, but that it is known that there exists a zero'th-order stabilizing compensator for the open loop system. In addition, using the comments given in Remark 6.1 and the results given in [21], it suffices to search for a stabilizing feedback matrix \(L_i\) in the set \(\mathbb{R} \times [-1, 1]\); hence, define \(h(i)\) as

\[
\begin{align*}
h(1) &= [-1.0 - 1.0] & h(6) &= [1.0 1.0] & h(11) &= [-1.5 - 1.0] \\
h(2) &= [-1.0 1.0] & h(7) &= [-2.0 - 1.0] & h(12) &= [-1.5 - 0.5] \\
h(3) &= [0.0 - 1.0] & h(8) &= [-2.0 - 0.5] & h(13) &= [-1.5 0.5] \\
h(4) &= [0.0 1.0] & h(9) &= [-2.0 0.5] & h(14) &= [-1.5 1.0] \\
h(5) &= [1.0 - 1.0] & h(10) &= [-2.0 1.0] & \vdots
\end{align*}
\]
and note that for
\[ u := [l_{11}, l_{12}] \hat{y}, \]
the closed loop system is stable if and only if \((l_{11}, l_{12})\) lies in the regions defined by
\[
\begin{align*}
1 - l_{11} &> 0, \\
l_{12} &> 0, \\
\text{and } l_{11}^2 + l_{11}(1 - l_{12}) - 2 &> 0.
\end{align*}
\]

Figure 6.1: Simulated results of \(y(t)\) with Controller T1 applied to system (6.6) using (6.7). (Controllers which are applied due to a previous bound on \(\xi(t)\) or \(y(t)\) being met or exceeded are marked by a ‘X’ or ‘Y’ respectively.)

In Figure 6.1, simulated output results are shown using Controller T1 with the given initial conditions and parameter functions/values defined earlier. (Controllers which are applied due to a previous bound on \(y(t)\) or \(\xi(t)\) being met or exceeded are marked by a ‘Y’ or ‘X’ respectively.) As expected, Controller T1 eventually stops switching, and \(L_{ss} = [-2 1]\). In addition, in this instance, \(e(t) \to 0\) as \(t \to \infty\), and the response shown in Figure 6.1 is noticeably improved over that shown in Figure 1 of [66].
Example 2: SISO Stable Nonminimum Phase Plant

As a second example, consider the (controllable and observable) stable nonminimum phase SISO plant [59]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -3 & -3
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u. \quad (6.8a)
\]

\[
y = 
\begin{bmatrix}
0.25 & -1 & 1
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + Fw \quad (6.8b)
\]

(with open loop poles at \(-1\) and zeros at 0.5) where it is assumed that (6.8) is known only to be stable, and that a solution to the robust伺服 mechanism problem exists for constant references and constant disturbances. Using the results given in Remark 6.1 and [21], define \(h(i)\) as follows:

\[
\begin{align*}
h(1) &= [0.0 \quad 1.0] \\
h(2) &= [0.0 \quad 0.0] \\
h(3) &= [0.0 \quad -1.0] \\
h(4) &= [0.0 \quad 2.0] \\
h(5) &= [0.0 \quad 1.5] \\
h(6) &= [0.0 \quad 1.0] \\
h(7) &= [0.0 \quad 0.5] \\
h(8) &= [0.0 \quad 0.0] \\
h(9) &= [0.0 \quad -0.5] \\
h(10) &= [0.0 \quad -1.0] \\
h(11) &= [0.0 \quad -1.5] \\
h(12) &= [0.0 \quad -2.0] \\
h(13) &= [0.0 \quad 3.0] \\
h(14) &= [0.0 \quad 2.75]
\end{align*}
\]

With

\[
u := [0 \quad l_{12}] \bar{y},
\]

the closed loop system is stable if and only if

\[
l_{12} \in \left(0, \frac{-25 + 9\sqrt{33}}{32}\right).
\]
Upon defining \((\dot{A}^*, \dot{B}^*) := (0.1), y_{\text{ref}}(t) := 1, w(t) := \sin(2t), F := 1\), and

\[
  f(k) := \begin{cases} 
  \frac{k^2}{4.5} & 1 \leq k \leq 10 \\
  10(k - 10)^2 \exp((k - 10)^3) & k > 10 
  \end{cases}
\]

with all controller-plant initial conditions set to be equal to zero at time \(t = 0\), the output response shown in Figure 6.2 is obtained using Controller T1. In this instance, although the transient magnitude is comparable to that given in Figure 2 of [59], the response shown here is much more sluggish in nature; however, as expected, the controller is indeed robust and eventually stops switching \((L_{ss} = [0 0.5])\) even with a sinusoidal disturbance.

![Theoretical continuous time switching controller results.](image)

![Switching time instants of controller L=1.](image)

Figure 6.2: Simulated results of \(y(t)\) with Controller T1 applied to (6.8) using (6.9).

**Example 3: SISO Unstable Minimum Phase Plant**

As another example, consider the (controllable and observable) unstable minimum phase SISO plant [59]

\[
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  2 & -1
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} u.
\]

(6.10a)
6.4. Simulation Results

\[
y = \begin{bmatrix} \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

(6.10b)

with poles given by $-2$ and $1$: assume again that the plant (6.10) is unknown, that there exists a solution to the robust servomechanism problem for constant reference and constant disturbance signals, and that there exists a zero'th-order stabilizing compensator for the augmented system (6.4) of the form

\[
u = [0 \ l_{12}] \tilde{y}.
\]

With $u$ given by (6.11), the closed loop system is stable if and only if

\[l_{12} \in (3, \infty).
\]

Define $h(i)$ as

\[
\begin{align*}
h(1) &= [0.0 \ 1.0] & h(6) &= [0.0 \ 1.0] & h(11) &= [0.0 \ -1.5] \\
h(2) &= [0.0 \ 0.0] & h(7) &= [0.0 \ 0.5] & h(12) &= [0.0 \ -2.0] \\
h(3) &= [0.0 \ -1.0] & h(8) &= [0.0 \ 0.0] & h(13) &= [0.0 \ 4.0] \\
h(4) &= [0.0 \ 2.0] & h(9) &= [0.0 \ -0.5] & h(14) &= [0.0 \ 3.75] \\
h(5) &= [0.0 \ 1.5] & h(10) &= [0.0 \ -1.0] & \vdots
\end{align*}
\]

(6.12)

and let $(\hat{A}^*, \hat{B}^*) := (0, 1)$, $y_{ref}(t) := 1$.

\[
\text{and } f(k) := \begin{cases} \frac{k^2}{12} & 1 \leq k \leq 20 \\
15(k - 20)^2 \exp((k - 20)^3) & k > 20
\end{cases}
\]

with all controller-plant initial conditions set to be equal to zero at time $t = 0$. the output response given in Figure 6.3 is obtained upon applying Controller T1. As anticipated, Controller T1 eventually stops switching ($L_{ss} = [0 \ 4]$), and, in this case, the transient response is also improved over that shown in Figure 3 of [59].
6.4. Simulation Results

Figure 6.3: Simulated results of $y(t)$ with Controller T1 applied to (6.10) using (6.12).

Example 4: MISO Unstable Minimum Phase Plant

Finally, consider once again the (stabilizable and observable) unstable minimum phase MISO plant [62, pg. 605] given in (3.8). Assume that the plant (3.8) is unknown, that there exists a solution to the robust servomechanism problem for constant reference and constant disturbance signals, and that there exists a zeroth order stabilizing compensator for the augmented system (6.4) of the form

$$u = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} y \\ \xi \end{bmatrix}.$$  \hspace{1cm} (6.13)

One can verify that with $u$ given by (6.13), the closed loop system is stable if and only if

$$l_{12} < 0,$$

$$l_{11} + 4l_{21} < -2,$$

and

$$(0.1 + 0.1l_{11} + l_{12} + 4l_{22})(-2 - l_{11} - 4l_{21}) + l_{12} > 0.$$
Let

\[(\dot{\bar{A}}, \dot{\bar{B}}) := (0, 1),\]
\[y_{\text{ref}}(t) := 1,\]
\[w(t) := \sin(t),\]
\[E := 0,\]
\[F := 1.\]

\[f(k) := \begin{cases} 
10 + \exp((k/6)^2), & 1 \leq k \leq 16 \\
500(k - 16)^2 \exp((k - 16)^3), & k > 16.
\end{cases}\]

and set all controller-plant initial conditions to be equal to zero at time \(t = 0\). Define \(h(i)\) as

\[
\begin{align*}
\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} & \quad \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} & \quad \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} & \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} & \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & \quad \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} & \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\end{align*}
\]

(6.14)
One can verify that for the given listed values of $h(i)$,

$$h(2), h(10), \text{ and } h(20)$$

form stable closed loop systems.

In Figure 6.4, output results are shown using Controller T1 with the previously defined parameters and functions. Again, as anticipated, the controller eventually stops switching, with

$$L_{ss} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$ 

and the controller is indeed robust even with a persistent sinusoidal disturbance. For comparison, using $h(i)$ defined as

$$h(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad h(8) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \quad h(15) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$h(2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad h(9) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad h(16) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$h(3) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad h(10) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad h(17) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$h(4) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad h(11) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad h(18) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

$$h(5) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad h(12) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad h(19) = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$h(6) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad h(13) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \quad h(20) = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

$$h(7) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad h(14) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \vdots$$
the larger output transient response given in Figure 6.5 is obtained, and

\[ L_{ss} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}. \]
6.4. Simulation Results

Figure 6.4: Simulated results of $y(t)$ with Controller T1 applied to (3.8) using (6.14).

Figure 6.5: Simulated results of $y(t)$ with Controller T1 applied to (3.8) using (6.15).
Chapter 7

Experimental Results

In this chapter, the robust self-tuning PI and PID controllers presented in Chapters 4 and 5 (for cases involving both a known and an unknown estimate of the steady-state DC gain matrix $T$) will be implemented and examined on a multivariable industrial system called MARTS (Multivariable Apparatus for Real Time Control Studies), located at the Systems Control Group, Department of Electrical and Computer Engineering, University of Toronto. This experimental equipment consists of a collection of industrial actuators and sensors to produce a highly interacting multivariable system. Moreover, using MARS, it will be shown how these new controllers possess certain desirable integrity features in spite of gross changes which may occur unexpectedly to the MARS configuration.

7.1 Experimental Apparatus

The MARS facility consists of an interconnection of industrial commercial actuators and sensors which monitor and control a nonlinear hydraulic dynamic process. Although there exist many different control configurations for this apparatus, the system which we will primarily be concerned with is depicted in Figure 7.1. Here, the control objective using this particular arrangement will be to regulate the level of both column heights, if possible, for all initial conditions and disturbances applied.

In this system, the by-pass, drainage, and interconnecting valves are all adjustable manually to enable the selection of desired equilibrium column heights, and to control the degree of interaction existing between both columns. Actuator valves for both columns also enable one to individually apply positive constant disturbances $w_1$ and $w_2$. To measure
both column heights (through the use of a current-pressure transducer) and to control the actuator and control valves (by using a current-pneumatic actuator). A Texas Instruments (TM 990/101/MA-1) real time digital computer using the PDOS operating system with an additional 12 bit analog-digital (A/D) and digital-analog (D/A) board (not shown) is used. Valid inputs $u$ to the control valves are therefore restricted numerically to the interval

$$0 \text{ (shut)} \leq u \leq 4095 \text{ (fully open)},$$

while column height measurements $h$ generally fall in the range (due to the sensors' calibration) of

$$1200 \text{ (bottom)} \leq h \leq 4095 \text{ (top)}.$$ 

Physically, this later interval corresponds to a height range of about 1.2 meters, or, equivalently, to approximately 0.4 mm. per digital division of resolution.

In Table 7.1, a listing of some of the various major components of the MARS appa-
ratus is given. Note that as the interconnecting valve angle increases from $0^\circ$ to $45^\circ$, the plant becomes increasingly difficult to control, and that due to the valve transducers used, both control valves utilize a mechanical feedback linkage to give an equilibrium valve position that is proportional to the input current signal\(^1\). Additional information concerning the equipment used, the system startup procedure, as well as other possible experimental configurations can be found in [22].

Since many commonly used controller synthesis design techniques need an accurate representation of the actual plant, and since conventional adaptive controllers still typically require specific plant information (e.g. the order of a stabilizing controller, an upper bound on the order of the plant, the relative degree of the plant) in order to guarantee acceptable controller performance, the variable structure of MARS presents an ideal situation for one to implement and examine the robust self-tuning PI and PID controllers proposed in Chapters 4 and 5.

### 7.2 Linearized Model of MARS

In order to demonstrate the general difficulty and uncertainty in obtaining an accurate model representation of an unknown system, consider the interconnection structure of MARS shown in Figure 7.2.

Let $i \in \{1, 2\}$, and define

$$u_i := \text{input to control valve } i,$$

---

\(^1\)This occurs as opposed to using pneumatic feedback to give an equilibrium output pressure that is proportional to the input current signal.
7.2. Linearized Model of MARTS

Figure 7.2: Cross sectional view of the interconnected columns of MARTS (not to scale).

\[ y_i := \text{measured output (height) of column } i. \]
\[ A_j := \text{cross sectional area. } j \in \{1, 2, 3\}. \]
\[ g := \text{acceleration due to gravity.} \]
\[ h_i := \text{liquid height in column } i. \]
\[ Q_{in_i}(u_i) := \text{input flow rate to column } i. \]
\[ Q_i := \text{output flow rate from column } i. \]
\[ \theta := \text{interconnection valve angle } (0^\circ \leq \theta \leq 90^\circ). \]
\[ C_d := \text{coefficient of discharge.} \]
\[ C_d(\theta) := \text{coefficient of discharge with respect to } \theta. \]

and \[ C_d(0) := 0 \text{ (i.e. } \theta = 0^\circ \Leftrightarrow \text{interconnection valve is shut).} \]

Here, the time delay occurring between the input signal \( u_i(t) \) and the output flow rate \( Q_i(t) \) is ignored, and it is assumed that any actuator valve nonlinearities have been eliminated (by using, for instance, nonlinear compensation gains).

Defining

\[ h_i := h_{i3} + \delta h_i \]
\[ \text{and } u_i := u_{i3}^{in} + \delta u_{i}^{in}, \]
where \( h_{is} \) and \( u_{in} \) are, respectively, the steady state height of and input to column \( i \) and control valve \( i \). the following linearized model of MRTS can be obtained for the case when \( h_{is} > h_{2s} \) [14], [51]:

\[
\begin{bmatrix}
\delta h_1 \\
\delta h_2
\end{bmatrix} = A
\begin{bmatrix}
-\beta_1 - \gamma(\theta) \\
\gamma(\theta)
\end{bmatrix}
\begin{bmatrix}
\delta h_1 \\
\delta h_2
\end{bmatrix}
+ B
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\delta u_1 \\
\delta u_2
\end{bmatrix},
\]

(7.1a)

\[
\begin{bmatrix}
\delta y_1 \\
\delta y_2
\end{bmatrix} = C
\begin{bmatrix}
\delta h_1 \\
\delta h_2
\end{bmatrix}
\]

(7.1b)

where

\[
\beta_i := \frac{C_d A_2}{A_1} \sqrt{\frac{g}{2h_{is}}},
\]

(7.1c)

and

\[
\gamma(\theta) := \frac{C_d(\theta) A_3}{A_1} \sqrt{\frac{g}{2(h_{is} - h_{2s})}}.
\]

(7.1d)

In (7.1), \( \delta y_i \) is the perturbed liquid level of column \( i \) \((i \in \{1, 2\})\), and \( \delta u_i \) is the perturbed input liquid flow rate from control valve \( i \) entering column \( i \).

**Remark 7.1:** A similar derivation for the case when \( h_{is} < h_{2s} \) yields the same basic format for (7.1), with

\[
\gamma(\theta) := \frac{C_d(\theta) A_3}{A_1} \sqrt{\frac{g}{2(h_{2s} - h_{is})}}
\]

being the only slight modification needed. In addition, for the case when \( h_{is} \approx h_{2s} \) and \( \theta \neq 0^\circ \), one can also show [14] that the system behaves as a single column apparatus obeying the equations

\[
\dot{X} = -\beta X + \delta u_1 + \delta u_2,
\]

\[
\delta y_i = \frac{1}{2} X
\]

where \( X := \delta h_1 + \delta h_2 \), and where the (realistic) assumption is made that \( \beta := \beta_1 \approx \beta_2 \).

Hence, (7.1) may be seen to be equally valid under this particular condition.

Using the experimental apparatus with the interconnection valve angle \( \theta \) set at 30°, (\( \beta_1, \)}
In this case, one can also verify that $-0.0248$ and $-0.0519$ are the stable eigenvalues of $A$.

Alternatively, using singular value analysis and the model identification and reduction algorithm given in [50], a discrete time model given by

$$x(k+1) = \mathcal{F}x(k) + Gu(k). \tag{7.3a}$$
$$y(k) = Hx(k). \tag{7.3b}$$

corresponding to a sampling time period $T$ of 2 seconds, was experimentally identified [14] for $\theta = 30^\circ$, where:

$$
\begin{bmatrix}
\mathcal{F} & G & H
\end{bmatrix} = \begin{bmatrix}
0.9783 & -0.0032 & 0.1674 & 0.8759 & 0.5623 & -1.0041 \\
-0.0150 & 0.9636 & -0.4568 & 0.3873 & 1.6699 & 0.6109
\end{bmatrix}. \tag{7.3c}
$$

with the stable eigenvalues of $\mathcal{F}$ given by 0.9811 and 0.9609.

The above results show the general difficulty in obtaining consistent mathematical models for an unknown system; although both models $(C,A,B)$ in (7.1) and $(H,\mathcal{F},G)$ in (7.3) approximately describe the behaviour of the system, the models are, in fact, not consistent with each other (e.g. with $T = 2$, $\lambda(e^{TA}) = \{0.901, 0.952\}$).

**Remark 7.2:** Due to the applied nonlinear compensation gains (to remove any actuator valve nonlinearities) and the mechanical feedback nature of the control valves, any admissible value of $\delta u_i$ calculated by a control algorithm can be used directly to control the actuator valves [44, pp. 254-259].

In this chapter, unless otherwise stated, the models listed above will **not** be used in any manner to obtain the experimental results which will be presented in Section 7.4.
7.3 Conventional Controller Design Results

In this section, experimental results obtained by using the high performance controller design methods given in [28] and [25] (for constant reference and disturbance inputs), which both require that an accurate mathematical model of the system be available, are presented.

In particular, upon using the MARTS model given in (7.2) for \( \theta = 30^\circ \), and the performance index [28]

\[
J_e = \int_0^\infty (e(t)^T e(t) + \epsilon \dot{y}(t)^T \dot{y}(t)) \, dt
\]

(7.4)

with \( \epsilon \in \mathbb{R}^+ \), the optimal controller which minimizes (7.4) is given by

\[
\dot{u} = \begin{bmatrix} K_0^l & K_1^l \end{bmatrix} \begin{bmatrix} \dot{x} \\ e \end{bmatrix}
\]

where \( x := [\delta h_1 \, \delta h_2]^T \), \( e := (y_{\text{ref}} - x) \), and \( y_{\text{ref}} \) is the desired reference tracking signal; hence, with \( \epsilon = 1 \), the optimal controller obtained for the MARTS system is

\[
u(t) = -K_0^l e(t) + K_1^l \int_0^t e(\tau) \, d\tau
\]

(7.5a)

where

\[
\begin{bmatrix} K_0^l & K_1^l \end{bmatrix} := \begin{bmatrix} -1.3776 & -0.0131 & 1.0000 & 0.0000 \\ -0.0131 & -1.3753 & 0.0000 & 1.0000 \end{bmatrix}.
\]

(7.5b)

When controller (7.5) is digitally implemented on the MARTS system using a sampling time period \( T \) of 0.4 seconds, a reference input signal of

\[
(y_{\text{ref}}^1(t), y_{\text{ref}}^2(t)) := \begin{cases} 
(3000, 2500), & 0 \leq t < 500 \\
(3500, 2000), & 500 \leq t < 1000 \\
(3000, 2500), & 1000 \leq t < 1500 \\
(3500, 2000), & 1500 \leq t < 2000 \text{ seconds},
\end{cases}
\]

and an interconnection valve angle \( \theta \) of 30\(^\circ\), the experimental results presented in Figure 7.3 are obtained. As can be seen, in this instance, the controller (7.5) provides excellent
7.3. Conventional Controller Design Results

performance, and the desired control objective is achieved. (The sensitivity exhibited by column height \(y_2(t)\) when \(y_{ref}^2(t) = 2000\) can be attributed to the (relatively small) value of \(\epsilon\) used in (7.4), and hence, to an undesirably large closed loop system bandwidth.)

![Graph showing experimental results](image)

Figure 7.3: Experimental proportional-integral results of \(y_1\) (dotted) and \(y_2\) (dashed) with \(T = 0.4\) seconds, \(\theta = 30^\circ\), and conventional controller (7.5) applied to the MARTS system.

However, when an unexpected event occurs using the conventional controller (7.5), catastrophic and unacceptable results may, and almost certainly will, occur, as demonstrated in the sample experimental output response shown in Figure 7.4. Here, the following gross change in the MARTS configuration was made at \(t = 1000\) seconds:

With the controller (7.5) (which is designed for the case when \((\theta, \epsilon) = (30^\circ, 1)\)) implemented on the MARTS system, the plant's configuration was suddenly changed at \(t = 1000\) seconds by reversing output leads \(y_1(t)\) and \(y_2(t)\) with the reference input signal given by

\[
(y_{ref}^1(t), y_{ref}^2(t)) := \begin{cases} 
(3500, 2500), & 0 \leq t < 1000 \\
(2500, 3500), & t \geq 1000 \text{ seconds} 
\end{cases}
\]

(7.6) applied.
7.3. Conventional Controller Design Results

Figure 7.4: Experimental proportional-integral results of $y_1$ (dotted) and $y_2$ (dashed) with $T = 0.4$ seconds, $\theta = 30^\circ$, and with the outputs reversed at $t = 1000$ seconds, showing failure of conventional controller (7.5).

Figure 7.5: Experimental proportional-integral results of $y_1$ (dotted) and $y_2$ (dashed) with $T = 0.4$ seconds, $\theta = 0^\circ$, and with the outputs reversed at $t = 1000$ seconds, showing failure of conventional controller (7.5).
As can be seen, in this situation, the closed loop system fails to track (7.6) for such a severe configuration change: indeed, even with \( \theta = 0^\circ \). Figure 7.5 shows that a similar problem still occurs on applying controller (7.5) with \( T = 0.4 \) seconds to the MARTS system. Unfortunately, such failures of this type of controller are not unexpected, since drastic changes in the plant have occurred at \( t = 1000 \) seconds.

It will now be shown that one class of the self-tuning robust servomechanism controllers proposed previously does not have this limitation in the sense that these specified controllers can readily adjust to severe plant configuration changes\(^2\).

### 7.4 Switching Controller Output Results

As evidenced in Section 7.3, conventional controller (7.5) is unable to carry out its specified mandate when gross structural changes occur to the MARTS configuration. Hence, consider the following definition of intelligent control adopted from [23] and [17].

**Definition 7.1**: An intelligent controller for a plant is a controller which successfully carries out its mandate for the plant's nominal operating conditions, as well as for any unexpected (i.e. unplanned) events which may occur.

Although Definition 7.1 is loose in the sense that the wording might not be considered to be well defined, the flavour of the definition is clear. For example, if one designs a controller so that it successfully controls the nominal plant \( \mathcal{N} \), as well as when the plant is in some failure mode \( \mathcal{X} \), say, then the controller has the property that it displays certain integrity features (which is very desirable), but the controller is not intelligent; the controller is intelligent only if it successfully controls the plant, either in nominal mode \( \mathcal{N} \) or failure mode \( \mathcal{X} \), in spite of the fact that failure mode \( \mathcal{X} \) was not anticipated in the design of the controller.

In this section, the self-tuning proportional-integral (PI) and proportional-integral-derivative (PID) controllers presented earlier in Chapters 4 and 5 will be implemented and examined on the MARTS apparatus. Using almost no a priori system information, in Sections 7.4.2, 7.4.3, and Appendix C, Controllers PID1' (Theorem 4.5) and C1 (Theorem

\(^2\)Due to the nature of the controllers considered, plant configuration changes which maintain the assumptions that \( \text{eig}(A) \subset \mathbb{C}^- \) and \( \text{rank}(T) = r \) only will be considered.
7.4. Switching Controller Output Results

5.1) will be shown to possess intelligent-like properties. Furthermore, where applicable, the generalized switching criterion defined by

\[
\begin{align*}
t_k := 
\begin{cases} 
\min t \ni \\
\text{i) } t > t_{k-1}, \text{ and } \\
\text{ii) } \|\eta(t)^T a(t)^T\| = f_1(k-1) \text{ and/or } \\
\|e_f(t)\| = f_2(k-1) \\
\infty
\end{cases}
\end{align*}
\]

will be used in the experiments.

For brevity, the details concerning various important practical issues (e.g. saturation constraints and integrator “windup” of both \(\eta(t)\) and \(a(t)\) (but not \(e_f(t)\))) will not be repeated here, but can instead be found in [14]. As well, similar to [14], a zero order hold is also used to obtain

\[
\begin{bmatrix}
\exp(-NTI) & \left(\frac{\exp(-NTI) - I}{N}\right) \\
-N^2 & -N \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\exp(-\lambda TI) & I - \exp(-\lambda TI) \\
I & 0 \\
\end{bmatrix}
\]

as the discrete time \(\begin{bmatrix} A_d & B_d \\ C & D \end{bmatrix}\) equivalents to the continuous time systems given by

\[
\begin{align*}
\dot{a}(t) &= -Na(t) - y(t), \\
\dot{d}(t) &= -N^2a(t) - Ny(t)
\end{align*}
\]

and

\[
\dot{e}_f = -\lambda e_f + \lambda e, \quad \lambda \in \mathbb{R}^+
\]

respectively.
7.4. Switching Controller Output Results

7.4.1 Using a Known Estimate of $\mathcal{T}$

Since an estimate $\hat{\mathcal{T}}$ of the DC gain matrix $\mathcal{T}$ can be obtained by performing two steady-state experiments on the MARTS apparatus, with $\theta = 30^\circ$, consider the following empirical measurement of $\hat{\mathcal{T}}$ [14]:

$$\hat{\mathcal{T}} = \begin{bmatrix} 21.8 & 18.1 \\ 5.6 & 46.5 \end{bmatrix}.$$ (7.7)

observe that

$$\|\hat{\mathcal{T}}^{-1}\|_F = \left\| \begin{bmatrix} 0.0510 & -0.0198 \\ -0.0061 & 0.0239 \end{bmatrix} \right\|_F = 0.06.$$

and that, from Remark 4.6,

$$\|W_j\|_F = \sqrt{2}, \ j \in \{1, 2, \ldots, 6\}.$$

Hence, in an attempt to maintain some consistency in norm values with those candidate feedback matrices to be used later in Section 7.4.2, define

$$K := 23.5654 \cdot \hat{\mathcal{T}}^{-1} = \begin{bmatrix} 1.2018 & -0.4666 \\ -0.1437 & 0.5632 \end{bmatrix}$$

so that $\|K\|_F = \sqrt{2}$.

With Controller PID1 applied using

$$T := 0.75 \text{ seconds.}$$ (7.8a)

$$\theta := 40^\circ.$$ (7.8b)

$$f_1(k) := \begin{cases} 5k^k, & 1 \leq k \leq 5 \\ (k - 3) \exp(k - 3)^2, & k > 5. \end{cases}$$

$$f_2(k) := \begin{cases} 210k, & 1 \leq k \leq 5 \\ (k - 3) \exp(k - 3)^2, & k > 5. \end{cases}$$
Figure 7.6: \((N = 3)\) Experimental proportional-integral-derivative results of \(y_1\) (solid) and \(y_2\) (dashed) with \(\theta = 40^\circ\) and Controller PID1 applied to the MARTS system.

Figure 7.7: \((N = 5)\) Experimental proportional-integral-derivative results of \(y_1\) (solid) and \(y_2\) (dashed) with \(\theta = 40^\circ\) and Controller PID1 applied to the MARTS system.
7.4. Switching Controller Output Results

\[(g(k), g_1(k), g_2(k)) := \begin{pmatrix} \frac{10}{5k} & \frac{10}{5k} & \frac{10}{5k} \end{pmatrix},\]

\[\begin{array}{l}
(3000, 2500), & 0 \leq t < 300 \\
(2500, 2000), & 300 \leq t < 600 \\
(3000, 2500), & 600 \leq t < 900 \\
(2500, 2000), & 900 \leq t < 1200 \\
(3000, 2500), & 1200 \leq t < 1500 \\
(2500, 2000), & t \geq 1500 \text{ seconds.}
\end{array}\]

\[(y_{ref}^1(t), y_{ref}^2(t)) := \begin{pmatrix} \rho \ := \ 20. \\
(7.8d) \\
\lambda \ := \ 10. \] (7.8e)

the output responses shown in Figures 7.6 and 7.7 are obtained for the case when \(N := 3\) and \(N := 5\) respectively. In both examples, \(\eta(t)\) is not reset to be equal to zero immediately following any controller switch, and as expected, the controller eventually stops switching (a total of three switches occurs in each figure). In addition, although the interconnection valve angle \(\theta\) is now set to be equal to 40°, tracking of the given reference heights occurs, and the controller is indeed robust.

7.4.2 Using no Known Estimate of \(T\)

In Figure 7.8, the output response obtained using Controller PIDI' and (7.8) with

\[N := 10.\]

\[f_1(k) := \begin{cases} 5(\alpha(k))^\alpha(k), & 1 \leq k \leq 30 \\
(\kappa - 28)\exp(\kappa - 28)^2, & k > 30.
\end{cases}\]

\[f_2(k) := \begin{cases} 210\alpha(k) + \beta(k), & 1 \leq k \leq 30 \\
(\kappa - 28)\exp(\kappa - 28)^2, & k > 30.
\end{cases}\]

\[(g(k), g_1(k), g_2(k)) := \left(\frac{10}{5\alpha(k)}, \frac{10}{5\alpha(k)}, \frac{10}{5\alpha(k)}\right),\]

\[\alpha(k) := \text{floor}\left(\frac{k + 5}{6}\right),\]

\[\beta(k) := 35 \cdot ((k - 1 \mod 6) + 1).\]

and no a priori estimate of \(T\) is shown; in this instance, utilizing the cyclic switching action summarized in Table 4.1 (with \(a(t)\) and \(e_f(t)\) both additionally reset to be equal
to zero immediately following any controller switch), a total of 18 switches occurs within approximately one minute, and tracking of (7.8c) once again is achieved. As in Figures 7.6 and 7.7, the sluggish behaviour which is apparent in $y_1(t)$ during the initial transition in height from 3000 to 2500 can be attributed primarily to valve saturation constraints.

For comparison, in Figure 7.9, experimental results are presented using Controller PID1' and the identical system setup as given for Figure 7.8, but with output leads $y_1(t)$ and $y_2(t)$ initially reversed.

$$(h_{ref}^1(t), h_{ref}^2(t)) := (3000, 2500). \quad t \geq 0. \quad (7.10)$$

and with positive constant disturbance $w_2$ applied to column 2 at time $t = 0$. In this case, there is a total of 23 switches, and tracking of (7.10) occurs after approximately 800 seconds. Furthermore, from (7.7), these results are entirely consistent in the fact that for Figure 7.8.

$$\lambda(-\overline{F}W_j) \subset \mathbb{C}^- \iff j = 1. \quad (7.7)$$

while for Figure 7.9,

$$\lambda \left( - \begin{bmatrix} 5.6 & 46.5 \\ 21.8 & 18.1 \end{bmatrix} W_j \right) \subset \mathbb{C}^- \iff j = 6. \quad (7.7)$$

As a final example, in Figure 7.10, output results for Controller PID1' are shown using (7.9) with

$$T := 0.75 \text{ seconds}, \quad \theta := 20^\circ, \quad (7.11a)$$

$$(y_{ref}^1(t), y_{ref}^2(t)) := (2500, 2250). \quad t \geq 0. \quad (7.11b)$$

$$\rho := 20, \quad (7.11c)$$

and with both output leads reversed after 1200 seconds. Once again, as anticipated, tracking of the given reference heights occurs, and $k_{ss} = 24$. Alternatively, using (7.9) and (7.11)
7.4. Switching Controller Output Results

Figure 7.8: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 40^\circ$ and Controller PID1' applied to the MARTS system.

Figure 7.9: Experimental proportional-integral-derivative results of $h_1$ (dashed) and $h_2$ (solid) with $\theta = 40^\circ$ and Controller PID1' applied to the reversed MARTS system.
Figure 7.10: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 20^\circ$ and Controller PID1 applied to the MARTS system.

Figure 7.11: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1 applied to the MARTS system.
7.4. Switching Controller Output Results

with no output lead reversal and

\[
\theta(t) := \begin{cases} 
-40^\circ, & 0 \leq t < 750 \\
0^\circ, & t \geq 750 \text{ seconds.}
\end{cases}
\]

\[ (y_{ref}^1(t), y_{ref}^2(t)) := (2500, 2000), \quad t \geq 0. \]

the results shown in Figure 7.11 are obtained and \( k_{ss} = 19 \).

Additional experimental results obtained upon applying Controller PID1' to MRTS are summarized in Appendix C.

7.4.3 Using Controller C1

For a final set of experimental results, let

\[
\begin{bmatrix}
\exp(-\epsilon\lambda TI) & I - \exp(-\epsilon\lambda TI) \\
I & 0
\end{bmatrix}
\]

be the discrete time \( \begin{bmatrix} A_d & B_d \\ C & D \end{bmatrix} \) equivalent to the continuous time system given by

\[
\dot{e}_b = -\epsilon\lambda e_b + \epsilon\lambda (b_{ref} - y). \quad (b, \epsilon, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.
\]

On applying Controller C1 (in order to eliminate any potential reset windup issues caused by control signal saturation constraints) with no a priori estimate of \( \mathcal{T} \), and on using the cyclic switching action for \( K(t) \) summarized in Table 4.1, together with

\[
T := 0.5 \text{ seconds.} \quad \tag{7.12a}
\]

\[
g(k) := \frac{10}{2^{((k-1) \text{ mod } 6)+1}}. \quad \tag{7.12b}
\]

\[
g_2(k) := \frac{2}{2^{((k-1) \text{ mod } 120)+1}}. \quad \tag{7.12c}
\]

\[
\theta := 30^\circ. \quad \tag{7.12d}
\]

\[(b, \lambda) := (1, 10), \quad \tag{7.12e}
\]

and \( (y_{ref}^1(t), y_{ref}^2(t)) := (2750, 2500), \)
the output results shown in Figures 7.12 and 7.13 are respectively obtained for $\rho(k) := 5 \cdot g_2(k)$ and $\rho(k) := 0$. In both instances, switching eventually ceases ($\kappa_{ss} = 85$ and $\kappa_{ss} = 61$), and tracking of $y_{ref}(t)$ occurs after approximately 500 seconds.

Alternatively, for the case when (7.12) is applied with Controller C1 using

$$(b, \lambda) := (1.50),$$

$$(y_1^{ref}(t), y_2^{ref}(t)) := (2500, 2500),$$

and with output leads $y_1(t)$ and $y_2(t)$ suddenly reversed at $t = 1500$ seconds, the respective results shown in Figures 7.14 and 7.15 are obtained for $\rho(k) := 0.05 \cdot g_2(k)$ and $\rho(k) := 0$. Once again, switching eventually ceases ($\kappa_{ss} = 66$ and $\kappa_{ss} = 72$), and tracking of the given reference inputs still occurs despite the unanticipated drastic plant changes which take place at $t = 1500$ seconds. Moreover, in both of the situations presented here, Controller C1 does indeed successfully result in a noticeably improved tuning transient response when compared with that obtained by using Controller 2 given in [67].
Figure 7.12: Experimental proportional-integral results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 30^\circ$ and Controller C1 applied to the MARTS system.

Figure 7.13: Experimental integral results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 30^\circ$ and Controller 2 [67] applied to the MARTS system.
Figure 7.14: Experimental proportional-integral results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 30^\circ$ and Controller C1 applied to the MARTS system.

Figure 7.15: Experimental integral results of $y_1$ (solid) and $y_2$ (dashed) with $\theta = 30^\circ$ and Controller 2 [67] applied to the MARTS system.
Chapter 8

Conclusions

In this chapter, the main contributions of this work are listed, and possible future research directions are discussed.

8.1 Summary of Results

In this thesis, a new class of switching controllers using a variant of the switching mechanism originally presented by Miller and Davison in [64] has been constructed to solve a number of problems in which as little \emph{a priori} plant information as possible is assumed to be known. For example, the following classes of problems have be investigated and solved:

- a general adaptive switching problem for a family of not necessarily strictly proper MIMO plants (Chapter 2);
- an adaptive stabilization problem for possibly unknown MIMO systems (Chapter 3);
- a self-tuning proportional-integral-derivative (PID) robust servomechanism problem for constant reference and constant disturbance inputs for stable plants (Chapter 4);
- a self-tuning proportional-integral (PI) robust servomechanism problem with control input constraints for constant reference and constant disturbance inputs for stable plants (Chapter 5); and
- an adaptive servomechanism problem for potentially unknown MIMO plants (Chapter 6).
As well, all controllers developed here are robust with respect to bounded immeasurable additive noise disturbances applied to control signal \( u(t) \) and/or plant output \( y(t) \).

Since one of the main objectives of this work has been to reduce the potential closed loop tuning response through the use of as little \textit{a priori} plant information as possible, the controllers presented here therefore are significant in the sense that, with the exception of Controller C1, switching now is based partially upon direct norm bounds on the system output error. Moreover, initial simulation results obtained when using these new controllers tend to indicate that a desirable improvement in the system transient response generally can be achieved by selecting non-pathological controller and tuning parameters.

An experimental real-time application study of one such class of switching controllers, using almost no \textit{a priori} plant information when applied to a multivariable hydraulic system (MARTS), indicates that the proposed controllers are feasible to implement in an industrial environment; in fact, it has been shown that such controllers operate satisfactorily in the presence of extreme structural changes occurring unexpectedly in the plant.

### 8.2 Future Research Directions

While many of the results presented here are positive, and therefore give assurance as to the development of a full theory of general adaptive switching controllers, numerous outstanding points still remain open for future examination. For example, although switching now is based partially upon direct norm bounds on the system output error, an unacceptably large output transient response may still occur if there exists a large number of candidate feedback controllers. This situation can be seen in Figures 2.8 and 2.11, and hence, further investigations into the reduction of initial tuning transients are warranted.

However, as one example of the potential generality afforded by the family of switching controllers proposed in this thesis, consider the time-varying plant [56. pg. 41]

\[
\begin{align*}
\dot{x} &= \left(1 + (\sin(t))^2\right) A x + (\cos(t)) B u, \\
y &= (\cos(t)) C x, 
\end{align*}
\]

(8.1a) (8.1b)

which is outside the class of systems considered in this work. Assume that the plant is \textit{unknown}, but that it is known that there exists a zero'th order stabilizing compensator for
8.2. Future Research Directions

the system (i.e. it is known that for some value of \( L_t \in \mathbb{R} \), the closed loop system will be stable with \( u = L_t y \)). As one can verify, since the closed loop system may be expressed as

\[
x(t) = \exp \left[ \frac{(3 + L_t) t}{2} + \frac{(L_t - 1) \sin(2t)}{4} \right] x(0)
\]

for a fixed value of \( L_t \in \mathbb{R} \). \((A + BL_t C)\) is exponentially stable (in the sense made precise in [47, pp. 167-168]) if and only if \( L_t < -3 \).

Upon applying Controller S2 and on defining \( h(i) \) as

\[
\begin{align*}
&h(1) = -1 \\
h(2) = 0 \\
h(3) = 1 \\
h(4) = 2 \\
h(5) = 1.5 \\
h(6) = 1 \\
h(7) = 0.5 \\
h(8) = 0 \\
h(9) = -0.5 \\
h(10) = -1 \\
h(11) = -1.5 \\
h(12) = -2 \\
h(13) = -4 \\
h(14) = -3.75 \\
h(15) = -3.5 \\
h(16) = -3.25 \\
h(17) = -3 \\
h(18) = -2.75 \\
h(19) = -2.5 \\
\end{align*}
\]

and

\[
f(k) := \begin{cases} 
1.5, & k = 1 \\
2, & k = 2 \\
((k - 1)!)^{0.25} + 1, & 3 \leq k \leq 20 \\
2(k - 19)^2 \exp((k - 19)^3), & k > 20.
\end{cases}
\]

the sample output responses given in Figures 8.1 and 8.2 are obtained for the case when \( x(0) = 1 \) and \( x(0) = 0.001 \) respectively. Since successful control occurs, it therefore is conjectured that subsequent extensions of these controllers to various classes of linear time-varying systems may also be conceivable.

Indeed, using the many well known techniques that have been applied so successfully to conventional model reference adaptive control problems, it may likewise be advantageous to replace bounding function \( f(k) \) by a dynamic one, \( \hat{f} \), where \( \hat{f} \) would have certain resetting properties (during time periods when no further switches occur) and could possibly be governed by the general nonlinear differential equation

\[
\dot{\hat{f}} = \hat{g}(\hat{f}, k, \eta, t, u, y, y_{ref}), \quad \hat{f}(0) = \hat{f}_0.
\]
Figure 8.1: \(x(0) = 1\) Simulated results with Controller S2 applied to (8.1) using (8.2) with \(x\) (dashed) and \(y\) (solid).

Figure 8.2: \(x(0) = 0.001\) Simulated results with Controller S2 applied to (8.1) using (8.2) with \(x\) (dashed) and \(y\) (solid).
Furthermore, due to the simplistic switching mechanisms proposed in this thesis, additional improvements in the current transient tuning response (through, for instance, the use of a more complex switching structure and/or increased *a priori* system information) also are surmised to be viable in any future work.
Appendix A

Proofs of Main Results

In this appendix, remaining detailed proofs of the main results presented in Chapters 2, 3, and 4 are given.

A.1 Adaptive Switching Control of LTI MIMO Systems

A.1.1 Theorem 2.1

Proof: The proof is by contradiction, and essentially works by constructing a Luenberger observer to estimate the unknown plant state $x(t)$. Using norm bounds on controller state $\eta(t)$ as well as a priori properties of the bounding function $f$, a contradiction can then be shown to occur.

The proof proceeds in the following four phases:

- Phase 1 obtains a bound on the observer error;
- Phase 2 obtains a bound on the estimated plant states;
- Phase 3 obtains a bound on the augmented plant and controller states; and
- Phase 4 uses a priori properties of bounding function $f$ and Lemma 2.1 to show that a contradiction occurs.

**Phase 1:** To prove property i), assume that there exist a controller parameter $f \in \text{MSBF}$, a continuous reference $y_{\text{ref}}$ having norm $\tilde{y}_{\text{ref}}$, a continuous disturbance $w$ having norm $\tilde{w}$, and an initial condition $z(0) = [x(0)^T \eta(0)^T]^T$ for which Assumption F1 holds, but property
A.1. Adaptive Switching Control of LTI MIMO Systems

i) does not; it follows that \( t_i \) is defined for all \( i \in \mathbb{N} \). Furthermore,

\[
P \in \mathbb{P} \implies P = P_m
\]

for some constant \( m \in \{1, 2, \ldots, s\} \).

Since \( (C_m, A_m) \) is detectable, this implies that there exists a matrix \( \tilde{M} \) such that \( \lambda(A_m + \tilde{M}C_m) \subset \mathbb{C}^- \). Hence, one can (theoretically) construct a (full order) Luenberger observer of the form

\[
\dot{x} = (A_m + \tilde{M}C_m)x + B_m u - \tilde{M}y + (E_m + \tilde{M}F_m)w.
\]

with \( \tilde{x}(t_1) \) an arbitrary constant vector in \( \mathbb{R}^{n_m} \) and \( \lambda(A_m + \tilde{M}C_m) \subset \mathbb{C}^- \). Also, with

\[
\dot{e} := \dot{x} - x,
\]

therefore

\[
\begin{align*}
\dot{e} &= \dot{x} - x \\
&= (A_m + \tilde{M}C_m)e
\end{align*}
\]

and

\[
\dot{e}(t) = e^{(A_m + \tilde{M}C_m)(t-t_1)}\dot{e}(t_1)
\]

for \( t \in (t_1, t_p] \), \( p \in \mathbb{N}, p \geq 2 \). Upon recalling that the stability of matrix \( (A_m + \tilde{M}C_m) \) implies that there exist constants \( (\lambda_1, -\lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \) so that \( \|e^{(A_m + \tilde{M}C_m)t}\| \leq \lambda e^{\lambda t} \) for \( t \geq 0 \), it follows (after taking norms) that for \( t \in (t_1, t_p] \),

\[
\|\dot{e}(t)\| \leq \lambda_1 \cdot \|\dot{e}(t_1)\|.
\]

**Phase 2:** Similarly, since

\[
\dot{x} = (A_m + \tilde{M}C_m)x + B_m u - \tilde{M}y + (E_m + \tilde{M}F_m)\underbrace{w} \in \mathcal{E}
\]
therefore
\[
\dot{\mathbf{x}}(t) = e^{(A_m+\hat{M}C_m)(t-t_1)}\dot{\mathbf{x}}(t_1) + \int_{t_1}^{t} e^{(A_m+\hat{M}C_m)(t-\tau)}(B_m u(\tau) - \hat{M} y(\tau) + \mathcal{E} w(\tau))d\tau:
\]

upon defining
\[
\begin{align*}
m_1 & := \max_{j\in\{1,2,\ldots,s\}} ||K_j||, \\
m_2 & := \max_{j\in\{1,2,\ldots,s\}} ||L_j||, \\
m_3 & := \max_{j\in\{1,2,\ldots,s\}} ||M_j||, \\
m_4 & := ||B_m||, \\
m_5 & := ||\hat{M}||, \\
m_6 & := ||E_m + \hat{M}F_m||,
\end{align*}
\]

and on noting that
\[
||y(t)|| \leq f(p-1) + \bar{y}_{ref}
\]
and \[||u(t)|| \leq m_1 f(p-1) + m_2 (f(p-1) + \bar{y}_{ref}) + m_3 \bar{y}_{ref}\]

for \(t \in (t_1, t_p]\), it also follows that
\[
||\dot{\mathbf{x}}(t)|| \leq \bar{\alpha} \cdot ||\dot{\mathbf{x}}(t_1)|| + \frac{\bar{\alpha}}{||\lambda||} (m_2 m_4 \bar{y}_{ref} + m_3 m_4 \bar{y}_{ref} + m_5 \bar{y}_{ref} + m_6 \bar{w}) + \frac{\bar{\alpha}}{||\lambda||} (m_1 m_4 + m_2 m_4 + m_5) \cdot f(p-1).
\]

**Phase 3:** Now using the fact that
\[
\dot{\epsilon}(t) = \dot{x}(t) - x(t),
\]
and thus that
\[
||x(t)|| \leq ||\epsilon(t)|| + ||\dot{x}(t)||
\]
for $t \in (t_1, t_p]$, it therefore follows that

$$\|x(t)\| \leq \tilde{\alpha} \cdot \|\dot{e}(t_1)\| + \tilde{\alpha} \cdot \|\dot{x}(t_1)\| + \frac{\tilde{\alpha}}{|\lambda|} (m_2 m_4 \tilde{y}_{ref} + m_3 m_4 \tilde{y}_{ref} + m_5 \tilde{y}_{ref} + m_6 \tilde{w}) + \frac{\tilde{\alpha}}{|\lambda|} (m_1 m_4 + m_2 m_4 + m_5) \cdot f(p - 1)$$

$$\leq c_0 + c_1 f(p - 1) =: \Gamma(p)$$

for $t \in (t_1, t_p]$ and for finite constants $c_0 > 0, c_1 > 1$. In addition, because

$$\|z(t)\| = \max\{\|x(t)\|, \|\eta(t)\|\}.$$ 

therefore

$$\|z(t)\| \leq \Gamma(p) \quad (A.1)$$

for $t \in (t_1, t_p]$.

**Phase 4:** Since Lemma 2.1 holds for all $\hat{j}$ such that $(\hat{j} - 1) \mod s = m - 1$, and since $f \in \text{MSBF}$, there exists a finite $\tilde{j}$, $((\tilde{j} - 1) \mod s) = m - 1$, such that

(i) $\zeta_1 \Gamma(\hat{j}) + \zeta_2 < f(\hat{j})$; and

(ii) $\zeta_3 \Gamma(\tilde{j}) + \zeta_4 < f(\tilde{j})$

are both satisfied for $t \in (t_j, t_{j+1})$: if we now set $t = t_{j+1}$, then the inequalities

$$\|z(t_{j+1})\| < f(\hat{j}),$$

$$\|e(t_{j+1})\| < f(\tilde{j})$$

contradict our definition of $t_{j+1}$; hence property i) is true.

From property i) and the bound given in (A.1), property ii) follows: also, from i), there exist matrices $G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss}$, and a $t_{ss} \geq 0$ such that $(G(t), H(t), J(t), K(t), L(t), M(t)) = (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ for all $t \geq t_{ss}$; it therefore follows from Proposition 2.2 that for almost all $(G_i, H_i, K_i, L_i), \tilde{A}_i$ will have no eigenvalues in $\mathbb{C}$; hence, property iii) follows since for almost all $(G_i, H_i, K_i, L_i)$, the excited modes of the final closed loop system will be stable.
A.1.2 Theorem 2.2

Proof: To prove property i), assume that there exist controller parameters \( f \in \text{MSBF} \) and \( \lambda \in \mathbb{R}^+ \). A piecewise continuous reference \( y_{\text{ref}} \) having norm \( \tilde{y}_{\text{ref}} \), a piecewise continuous disturbance \( w \) having norm \( \tilde{w} \), and an initial condition \( \tilde{z}(0) = [x(0)^T \eta(0)^T e_f(0)^T]^T \) for which Assumption F2 holds, but property i) does not; it follows that \( t_i \) is defined for all \( i \in \mathbb{N} \). Furthermore,

\[
P \in \mathbb{P} \implies P = P_m
\]

for some constant \( m \in \{1, 2, \ldots, s\} \); as such, for \( t \in (t_{p-1}, t_{p}] \), \( p \in \mathbb{N}, p \geq 2 \), it also follows that

\[
\begin{bmatrix}
\dot{x} \\
\dot{\eta} \\
\dot{e}_f
\end{bmatrix} =
\begin{bmatrix}
A_{p-1}^{11} & A_{p-1}^{12} & A_{p-1}^{13} \\
A_{p-1}^{21} & A_{p-1}^{22} & A_{p-1}^{23} \\
A_{p-1}^{31} & A_{p-1}^{32} & A_{p-1}^{33}
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
e_f
\end{bmatrix} +
\begin{bmatrix}
B_{p-1}^{11} & B_{p-1}^{12} \\
B_{p-1}^{21} & B_{p-1}^{22} \\
B_{p-1}^{31} & B_{p-1}^{32}
\end{bmatrix}
\begin{bmatrix}
y_{\text{ref}} \\
w
\end{bmatrix}
\]

\[
\begin{bmatrix}
y \\
\eta \\
e_f
\end{bmatrix} =
\begin{bmatrix}
\tilde{I}_{p-1}C_m & \tilde{I}_{p-1}D_mK_{c_{p-1}} & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
x \\
\eta \\
e_f
\end{bmatrix} +
\begin{bmatrix}
\tilde{I}_{p-1}D_mM_{c_{p-1}} & \tilde{I}_{p-1}F_m
\end{bmatrix}
\tilde{u}
\]

where \( x(t_{p-1}^-) = x(t_{p-1}) \).

\[
\begin{bmatrix}
A_{p-1}^{11} & A_{p-1}^{12} & A_{p-1}^{13} \\
A_{p-1}^{21} & A_{p-1}^{22} & A_{p-1}^{23} \\
A_{p-1}^{31} & A_{p-1}^{32} & A_{p-1}^{33}
\end{bmatrix} =
\begin{bmatrix}
A_{m} + B_{m}L_{c_{p-1}}\tilde{I}_{p-1}C_m & B_{m}(I + L_{c_{p-1}}\tilde{I}_{p-1}D_m)K_{c_{p-1}} & 0 \\
H_{c_{p-1}}\tilde{I}_{p-1}C_m & G_{c_{p-1}} + H_{c_{p-1}}\tilde{I}_{p-1}D_mK_{c_{p-1}} & 0 \\
-\lambda\tilde{I}_{p-1}C_m & -\lambda\tilde{I}_{p-1}D_mK_{c_{p-1}} & -\lambda I
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
B_{p-1}^{11} & B_{p-1}^{12} \\
B_{p-1}^{21} & B_{p-1}^{22} \\
B_{p-1}^{31} & B_{p-1}^{32}
\end{bmatrix} =
\begin{bmatrix}
B_{m}(I + L_{c_{p-1}}\tilde{I}_{p-1}D_m)M_{c_{p-1}} & E_{m} + B_{m}L_{c_{p-1}}\tilde{I}_{p-1}F_m \\
J_{c_{p-1}} + H_{c_{p-1}}\tilde{I}_{p-1}D_mM_{c_{p-1}} & H_{c_{p-1}}\tilde{I}_{p-1}F_m \\
\lambda I - \lambda\tilde{I}_{p-1}D_mM_{c_{p-1}} & -\lambda\tilde{I}_{p-1}F_m
\end{bmatrix}
\]

Since \((C_m, A_m)\) is detectable, consider the partial state estimation problem [18, pg. 361]
given by

\[
\begin{align*}
\dot{\tilde{z}} &= A_{p-1}^{cl} \tilde{z} + B_{p-1}^{cl} \bar{u}, \\
\bar{y} &= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{z} \\
&= \begin{bmatrix} \eta \\ e_f \end{bmatrix}
\end{align*}
\]

for \( t \in (t_{p-1}, t_p] \). Upon setting

\[
P_{p-1} := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}.
\]

on noting that

\[
P_{p-1}^{-1} = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.
\]

and upon defining

\[
\tilde{z} := P_{p-1} \tilde{z}.
\]

it therefore follows that

\[
\begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11}^{p-1} & \tilde{A}_{12}^{p-1} \\ \tilde{A}_{21}^{p-1} & \tilde{A}_{22}^{p-1} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_{11}^{1p-1} & \tilde{B}_{12}^{1p-1} \\ \tilde{B}_{21}^{2p-1} & \tilde{B}_{22}^{2p-1} \end{bmatrix} \bar{u}. \tag{A.2}
\]

\[
\begin{align*}
\bar{y} &= \begin{bmatrix} I & 0 \end{bmatrix} \tilde{z} \\
&= \tilde{z}_1 \\
&= \begin{bmatrix} \eta \\ e_f \end{bmatrix}
\end{align*}
\]
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where

\[
\begin{bmatrix}
\bar{A}_{p-1}^{11} & \bar{A}_{p-1}^{12} \\
\bar{A}_{p-1}^{21} & \bar{A}_{p-1}^{22}
\end{bmatrix} :=
\begin{bmatrix}
A_{p-1}^{22} & A_{p-1}^{23} & A_{p-1}^{21} \\
A_{p-1}^{32} & A_{p-1}^{33} & A_{p-1}^{31} \\
A_{p-1}^{42} & A_{p-1}^{43} & A_{p-1}^{41}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\bar{B}_{p-1}^{11} & \bar{B}_{p-1}^{12} \\
\bar{B}_{p-1}^{21} & \bar{B}_{p-1}^{22}
\end{bmatrix} :=
\begin{bmatrix}
B_{p-1}^{21} & B_{p-1}^{22} \\
B_{p-1}^{31} & B_{p-1}^{32} \\
B_{p-1}^{41} & B_{p-1}^{42}
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} =
\begin{bmatrix}
\eta \\
e_f \\
x
\end{bmatrix}.
\]

In addition, by rewriting (A.2) as

\[
\dot{y} = \bar{A}_{p-1}^{11} \bar{y} + \bar{A}_{p-1}^{12} \bar{z}_2 + \bar{B}_{p-1}^{11} y_{ref} + \bar{B}_{p-1}^{12} w.
\]

\[
\dot{z}_2 = \bar{A}_{p-1}^{22} \bar{z}_2 + \bar{A}_{p-1}^{21} \bar{y} + \bar{B}_{p-1}^{21} y_{ref} + \bar{B}_{p-1}^{22} w.
\]

and on defining

\[
u := \bar{A}_{p-1}^{21} \bar{y} + \bar{B}_{p-1}^{21} y_{ref} + \bar{B}_{p-1}^{22} w.
\]

\[
\bar{w} := \dot{y} - \bar{A}_{p-1}^{11} \bar{y} - \bar{B}_{p-1}^{11} y_{ref} - \bar{B}_{p-1}^{12} w = \bar{A}_{p-1}^{12} \bar{z}_2.
\]

therefore

\[
\dot{\bar{z}}_2 = \dot{x} = A_{p-1}^{11} x + \nu.
\]

\[
\bar{w} = \bar{A}_{p-1}^{12} x.
\]

Now since \((C_m, A_m)\) is detectable, let matrix \(\bar{M}\) be chosen such that \(\lambda(A_m + \bar{M} C_m) \subset \mathbb{C}^-\). Observe that \((\bar{A}_{p-1}^{12}, A_{p-1}^{11})\) is detectable for all \(p \in \mathbb{N}, p \geq 2\), and consider

\[
\dot{x} = (A_{p-1}^{11} + G_{p-1} \bar{A}_{p-1}^{12}) \dot{x} + A_{p-1}^{21} \bar{y} + \bar{B}_{p-1}^{21} y_{ref} + \bar{B}_{p-1}^{22} w - G_{p-1} \bar{w}.
\]
where $\hat{x}(t^+_{p-1}) = \hat{x}(t_{p-1})$ and

$$G_{p-1} := \begin{bmatrix} 0 & B_m \Lambda_{p-1} - \bar{M}\bar{F}_{p-1}^T \end{bmatrix}$$

for $t \in (t_{p-1}, t_p]$ with $\hat{x}(t_1)$ an arbitrary constant vector in $\mathbb{R}^{n_m}$ and $\lambda(A_{p-1}^{11} + G_{p-1}\bar{A}_{p-1}^{12}) = \lambda(A_m + \bar{M}C_m) \subset \mathbb{C}^-$. Also, with

$$\hat{e} := \hat{x} - \tilde{x},$$

therefore

$$\dot{\hat{e}} = \dot{\hat{x}} - \dot{\tilde{x}} = (A_m + \bar{M}C_m)\hat{e}$$

and

$$\hat{e}(t) = e^{(A_m + \bar{M}C_m)(t-t_1)}\hat{e}(t_1)$$

for $t \in (t_1, t_p]$. Upon recalling that the stability of matrix $(A_m + \bar{M}C_m)$ implies that there exist constants $(\bar{\alpha}, -\bar{\lambda}) \in \mathbb{R}^+ \times \mathbb{R}^-$ so that $\|e^{(A_m + \bar{M}C_m)t}\| \leq \bar{\alpha}e^{\bar{\lambda}t}$ for $t \geq 0$, it follows (after taking norms) that for $t \in (t_1, t_p]$,

$$\|\hat{e}(t)\| \leq \bar{\alpha} \cdot \|\hat{e}(t_1)\|.$$

Similarly, upon defining

$$\tilde{x} := \hat{x} + G_{p-1}\tilde{y},$$

$$B_{p-1}^1 := \bar{A}_{p-1}^{21} + G_{p-1}\bar{A}_{p-1}^{11} - (A_m + \bar{M}C_m)G_{p-1},$$

$$B_{p-1}^2 := \bar{B}_{p-1}^{21} + G_{p-1}\bar{B}_{p-1}^{11},$$

and

$$B_{p-1}^3 := \bar{B}_{p-1}^{22} + G_{p-1}\bar{B}_{p-1}^{12}$$
for \( t \in (t_{p-1}, t_p) \), and on noting that

\[
\dot{x} = \begin{bmatrix} A_m + \hat{M}C_m \end{bmatrix} x + \begin{bmatrix} B_{p-1}^1 & B_{p-1}^2 & B_{p-1}^3 \end{bmatrix} \begin{bmatrix} y \ y_{ref} \ w \end{bmatrix}, \quad t \in (t_{p-1}, t_p),
\]

and \( \dot{x}(t_{p-1}^+) = \dot{x}(t_{p-1}^+) + G_{p-1}^0 \ddot{y}(t_{p-1}^+) \),

therefore

\[
\dot{x}(t) = e^{(A_m + \hat{M}C_m)(t-t_1)} \dot{x}(t_1) + \sum_{j=1}^{p-1} \left( \int_{t_{p-j}}^{t_{p-j+1}} e^{(A_m + \hat{M}C_m)(t-\tau)} B_{p-j} \ddot{u}(\tau) \, d\tau \right) + \\
\sum_{j=2}^{p-1} \left( e^{(A_m + \hat{M}C_m)(t-t_j)} (G_j \ddot{y}(t_j^+) - G_{j-1} \ddot{y}(t_j)) \right),
\]

however, since

\[
\|\ddot{y}(t)\| \leq f(p-1)
\]

and \( \|\ddot{u}(t)\| \leq f(p-1) + \ddot{y}_{ref} + \ddot{w} \)

for \( t \in (t_1, t_p) \), and with

\[
m_1 := \max_{j \in \{1, 2, \ldots, s\}} \|G_j\|,
\]

\[
m_2 := \max_{j \in \{1, 2, \ldots, s\}} \|B_j\|,
\]

it also follows that

\[
\|\dot{x}(t)\| \leq \ddot{\alpha} \cdot \|\dot{x}(t_1)\| + \sum_{j=1}^{p-1} \ddot{\alpha} m_2 (f(p-j) + \ddot{y}_{ref} + \ddot{w}) + \sum_{j=2}^{p-1} 2\ddot{\alpha} m_1 f(j-1).
\]

Now using the fact that

\[
\dot{e}(t) = \dot{x}(t) - x(t),
\]

\[
\ddot{e}(t) = \ddot{x}(t) - \ddot{G}_{p-1} \ddot{y}(t), \quad t \in (t_{p-1}, t_p),
\]
and thus that
\[ \|x(t)\| \leq \|\tilde{e}(t)\| + \|\tilde{x}(t)\| \]
\[ \leq \|\tilde{e}(t)\| + \|\tilde{x}(t)\| + m_1 f(p - 1) \]

for \( t \in (t_1, t_p) \), it therefore follows that
\[ \|x(t)\| \leq \tilde{\alpha} \cdot \|\tilde{e}(t_1)\| + \tilde{\alpha} \cdot \|\tilde{x}(t_1)\| + \sum_{j=1}^{p-1} \frac{\tilde{\alpha}}{|\lambda|} m_2(f(p - j) + \tilde{y}_{ref} + \tilde{w}) + \]
\[ \sum_{j=2}^{p-1} 2\tilde{\alpha}m_1 f(j - 1) + m_1 f(p - 1) \]
\[ \leq c_0 + c_1 (p - 1) + c_2 \sum_{j=1}^{p-1} f(j) \]
\[ =: \Gamma(p) \]

for \( t \in (t_1, t_p) \) and for finite constants \( c_0 > 0, c_1 > 0, c_2 > 1 \). In addition, because
\[ \|\tilde{z}(t)\| = \max\{\|x(t)\|, \|\eta(t)\|, \|e_f(t)\|\}, \]

therefore
\[ \|\tilde{z}(t)\| \leq \Gamma(p) \quad (A.3) \]

for \( t \in (t_1, t_p) \).

Since Lemma 2.2 holds for all \( \tilde{j} \) such that \( ((\tilde{j} - 1) \mod s) = m - 1 \), and since \( f \in \text{MSBF} \), there exists a finite \( \tilde{j} \), \( ((\tilde{j} - 1) \mod s) = m - 1 \), such that
\[ \zeta_1 \Gamma(\tilde{j}) + \zeta_2 < f(\tilde{j}) \]
is satisfied for \( t \in (t_{\tilde{j}}, t_{\tilde{j}+1}) \); if we now set \( t = t_{\tilde{j}+1} \), then the inequality
\[ \|\tilde{z}(t_{\tilde{j}+1})\| < f(\tilde{j}) \]
contradicts our definition of \( t_{\tilde{j}+1} \); hence property i) is true.
From property i) and the bound given in (A.3), property ii) follows: also, from i), there exist matrices $G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss}$, and a $t_{ss} \geq 0$ such that $(G(t), H(t), J(t), K(t), L(t), M(t)) = (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss})$ for all $t \geq t_{ss}$; it therefore follows from Proposition 2.2 that for almost all $(G_i, H_i, K_i, L_i)$, $\tilde{A}_i$ will have no eigenvalues in $\mathbb{C}^0$: hence, property iii) follows since for almost all $(G_i, H_i, K_i, L_i)$, the excited modes of the final closed loop system will be stable. 

\section*{A.2 Adaptive Stabilization of LTI Systems}

\subsection*{A.2.1 Theorem 3.1}

\textbf{Proof:} To prove property i), assume that there exist controller parameters $f \in S1BF$ and $(\epsilon_0, \tau) \in S$, a continuous disturbance $w$ having norm $\tilde{w}$, and an initial condition $x(0)$ for which Assumption S1 holds, but property i) does not: it follows that $t_i$ is defined for all $i \in \mathbb{N}$.

Since

\begin{align*}
\dot{x} &= ax + bu + ew, \\
y &= cx + fw
\end{align*}

and $c \neq 0$, therefore

\[|x(t)| \leq \left| \frac{y(t)}{c} \right| + \left| \frac{f}{c} \right| \tilde{w} \]

for all $t \geq 0$. As well, using the fact that

\[|y(t)| \leq f(p - 1)\]

for $t \in (t_1, t_p], p \in \mathbb{N}, p \geq 2$, it therefore follows that

\begin{align*}
|x(t)| &\leq \left( \frac{f(p - 1) + |f|\tilde{w}}{|c|} \right) \\
&\leq c_0 + c_1 f(p - 1) \\
&=: \Gamma(p)
\end{align*}

(A.4a) (A.4b) (A.4c)
A.2. Adaptive Stabilization of LTI Systems

for \( t \in (t_1, t_p] \) and for constants \((c_0, c_1) \in \mathbb{R}^- \times \mathbb{R}^-\).

Consider now a \( j \) sufficiently large so that Lemma 3.1 holds for all \( q \geq j, t \in (t_q, t_{q-1}]\), with \(((q - 1) \mod 2) = ((j - 1) \mod 2)\). Since \( f \in S1BF \), there exists a finite \( j \geq j \), \(((j - 1) \mod 2) = ((j - 1) \mod 2)\), such that Lemma 3.1 holds and

\[
\zeta_1 \Gamma(j) + \zeta_5 + \zeta_6 r^j < f(j)
\]

is satisfied for \( t \in (t_j, t_{j+1}] \); if we now set \( t = t_{j-1} \), then the inequality

\[
|y(t_{j-1})| < f(j)
\]

contradicts our definition of \( t_{j+1} \); hence property i) is true.

From property i) and the bound given in (A.4), property ii) follows: also, from i), there exist an \( \varepsilon_{ss} \in \mathbb{R}^+ \), \( K_{ss} \in \{1, -1\} \), and a \( t_{ss} \geq 0 \) such that \( \varepsilon(t) = \varepsilon_{ss} \) and \( K(t) = K_{ss} \) for all \( t \geq t_{ss} \). Since \( K_{ss} \in K \) and \( \varepsilon_{ss} = \frac{\tau^i}{\varepsilon_0} \) for some \( i \in \mathbb{N} \), it follows from Remark 3.1 that for almost all \( (\varepsilon_0, \tau) \in S \), \( (a + b \varepsilon_{ss} K_{ss} c) \neq 0 \); hence property iii) follows since for almost all \( (\varepsilon_0, \tau) \in S \), the excited modes of the final closed loop system will be stable. \( \square \)

A.2.2 Theorem 3.2

Proof: To prove property i), assume that there exist controller parameters \( f \in S2BF \) and \( h \in CTF \), a continuous disturbance \( w \) having norm \( \hat{w} \), and an initial condition \( z(0) = [x(0)^T \eta(0)^T]^T \) for which Assumption S2 holds, but property i) does not: it follows that \( t_i \) is defined for all \( i \in \mathbb{N} \).

Since \((C, A)\) is detectable, this implies that there exists a matrix \( M \) such that \( \lambda(A + MC) \subset \mathbb{C}^- \). Hence, one can (theoretically) construct a (full order) Luenberger observer of the form

\[
\dot{x} = (A + MC)\dot{x} + Bu - My + (E + MF)w
\]

with \( \dot{x}(t_i) \) an arbitrary constant vector in \( \mathbb{R}^n \) and \( \lambda(A + MC) \subset \mathbb{C}^- \). Also, with

\[
\dot{e} := \dot{x} - x,
\]
therefore

\[
\dot{e} = \dot{x} - \dot{\hat{x}} = (A + MC)\dot{e}
\]

and

\[
\dot{\hat{e}}(t) = e^{(A+MC)(t-t_1)}\dot{\hat{e}}(t_1)
\]

for \( t \in (t_1, t_p) \), \( p \in \mathbb{N}, p \geq 2 \). Upon recalling that the stability of matrix \((A + MC)\) implies that there exist constants \((\alpha, -\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+\) so that \(\|e^{(A+MC)t}\| \leq \alpha e^{\lambda t}\) for \( t \geq 0 \). it follows (after taking norms) that for \( t \in (t_1, t_p) \).

\[
\|\dot{\hat{e}}(t)\| \leq \alpha \cdot \|\dot{\hat{e}}(t_1)\|
\]

Similarly, since

\[
\dot{\hat{x}} = (A + MC)\dot{\hat{x}} + Bu - My + (E + MF)w.
\]

therefore

\[
\hat{x}(t) = e^{(A+MC)(t-t_1)}\hat{x}(t_1) + \int_{t_1}^{t} e^{(A+MC)(t-\tau)}(Bu(\tau) - My(\tau) + (E + MF)w(\tau))d\tau.
\]

upon defining

\[
\begin{align*}
m_1 &= \|B\|, \\
m_2 &= \|M\|, \\
m_3 &= \|E + MF\|,
\end{align*}
\]

and on noting that

\[
\|y(t)\| \leq f(p - 1)
\]

and \( \|u(t)\| \leq \tau_1^{p-1}f(p - 1) + \tau_2^{p-1}f(p - 1) \)

for constants \((\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+\) and for \(t \in (t_1, t_p]\), it also follows that

\[
\|\dot{x}(t)\| \leq \hat{\alpha} \cdot \|\dot{x}(t_1)\| + \frac{\hat{\alpha} \cdot m_3 \dot{w}}{\lambda} + \frac{\hat{\alpha}(m_1 \tau_1^{p-1} f(p-1) + m_2 \tau_2^{p-1} f(p-1)) + m_2 f(p-1) + m_3 \dot{w}}{\lambda}.
\]

Now using the fact that

\[
\dot{e}(t) = \dot{x}(t) - x(t).
\]

and thus that

\[
\|x(t)\| \leq \|\dot{e}(t)\| + \|\dot{x}(t)\|
\]

for \(t \in (t_1, t_p]\), it therefore follows that

\[
\|x(t)\| \leq \hat{\alpha} \cdot \|\dot{x}(t_1)\| + \hat{\alpha} \cdot \|\dot{x}(t_1)\| + \frac{\hat{\alpha} m_3 \dot{w}}{\lambda} + \frac{\hat{\alpha}(m_1 \tau_1^{p-1} + \tau_2^{p-1} f(p-1) + m_2 f(p-1) + m_3 \dot{w})}{\lambda} \leq c_0 + c_1 f(p-1) + c_2 \tau^{p-1} f(p-1) =: \Gamma(p)
\]

for \(t \in (t_1, t_p]\) and for finite constants \(c_0 > 0, c_1 > 1, c_2 > 0, \tau > 0\). In addition, because

\[
\|z(t)\| = \max\{\|x(t)\|, \|\eta(t)\|\}.
\]

therefore

\[
\|z(t)\| \leq \Gamma(p) \quad (A.5)
\]

for \(t \in (t_1, t_p]\).

Since \(h \in \text{CTF}\), and since it is known that \(g_i = q\) will stabilize (3.2), where \(q \in \mathbb{N} \cup \{0\}\), for some \(\tilde{K}_i \in \mathbb{R}^{(m+g_i) \times (r+g_i)}\), consider now a \(j\) sufficiently large so that Lemma 3.2 holds for \(t \in (t_j, t_{j+1})\): with \(f \in \text{S2BF}\) and \(h \in \text{CTF}\), there therefore exists a finite \(\tilde{j} \geq j\) such that Lemma 3.2 holds and

\[
\zeta_3 \Gamma(\tilde{j}) + \zeta_4 < f(\tilde{j})
\]
is satisfied for \( t \in (t_j, t_{j+1}] \); if we now set \( t = t_{j+1} \), then the inequality

\[
\| \tilde{y}(t_{j+1}) \| < f(j)
\]

contradicts our definition of \( t_{j+1} \); hence property i) is true.

From property i) and the bound given in (A.5), property ii) follows: also, from i), there exist matrices \( G_{ss}, H_{ss}, K_{ss}, L_{ss} \), and a \( t_{ss} \geq 0 \) such that \( (G(t), H(t), K(t), L(t)) = (G_{ss}, H_{ss}, K_{ss}, L_{ss}) \) for all \( t \geq t_{ss} \); it therefore follows from Proposition 2.2 that for almost all \( (G_i, H_i, K_i, L_i) \), \( \tilde{A}_i \) will have no eigenvalues in \( \mathbb{C}^0 \); hence, property iii) follows since for almost all matrices \( (G_i, H_i, K_i, L_i) \), the excited modes of the final closed loop system will be stable.

\[\square\]

A.3 The Self-Tuning Robust Servomechanism

A.3.1 Theorem 4.2

Proof: To prove property i), assume that there exist a controller parameter \( \hat{\sigma} \in \hat{\Omega} \), a constant reference \( y_{ref} \) having norm \( \|y_{ref}\| \), a constant disturbance \( w \) having norm \( \|w\| \), and an initial condition \( z(0) = [z(0)^T \eta(0)]^T \) for which Assumption P1 holds, but property i) does not: it follows that \( t_i \) is defined for all \( i \in \mathbb{N} \).

Since \( \lambda(A) \subset \mathbb{C}^+ \), there exist constants \((\bar{\rho}, -\bar{\lambda}) \in \mathbb{R}^+ \times \mathbb{R}^+ \) so that \( \|e^{\lambda t}\| \leq \bar{\rho}e^{\bar{\lambda}t} \) for \( t \geq 0 \). In addition, since

\[
\dot{x} = Ax + Bu + Ew.
\]

therefore

\[
x(t) = e^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)}(Bu(\tau) + Ew(\tau))d\tau.
\]

upon defining

\[
m_1 := \|B\|,
\]

\[
m_2 := \|E\|,
\]

\[
m_3 := \|K\| \left(1 + \frac{\epsilon_0 \rho}{\tau}\right),
\]
and on noting that

\[ \|e(t)\| \leq f(i - 1) \]

and \[ \|u(t)\| \leq m_3 f(i - 1) \]

for \( t \in (t_1, t_i) \), \( i \geq 2 \), it also follows that

\[ \|x(t)\| \leq \alpha \|x(t_1)\| + \frac{\alpha}{|\lambda|} (m_1 m_3 f(i - 1) + m_2 \tilde{w}) \]

\[ \leq c_0 + c_1 f(i - 1) \]

\[ =: \Gamma(i) \]

for \( t \in (t_1, t_i] \) and for finite constants \( c_0 > 0, c_1 > 1 \). In addition, because

\[ \|z(t)\| = \max\{\|x(t)\|, \|\eta(t)\|\} \]

therefore

\[ \|z(t)\| \leq \Gamma(i) \quad (A.6) \]

for \( t \in (t_1, t_i] \).

Since Lemma 4.1 holds for a sufficiently large \( j \), and since \( f \in \text{MSBF} \) and \( g \in \text{TF}' \), there exists a finite \( \tilde{j} \geq j \) such that

(i) \( \alpha \tilde{\Gamma}(\tilde{j}) + \beta (\tilde{y}_{ref} + \tilde{w}) < f(\tilde{j}) \); and

(ii) \[ \left\| [ -\tilde{I}_j C \quad -\tilde{I}_j D K ] \right\| \cdot \tilde{\Gamma}(\tilde{j}) + \| \tilde{I}_j \| \cdot \tilde{y}_{ref} + \| \tilde{I}_j F \| \cdot \tilde{w} < f(\tilde{j}) \]

are both satisfied for \( t \in (t_j, t_{j+1}] \); if we now set \( t = t_{j+1} \), then the inequalities

\[ \|z(t_{j+1})\| < f(\tilde{j}), \]

\[ \|e(t_{j+1})\| < f(\tilde{j}) \]

contradict our definition of \( t_{j+1} \); hence property i) is true.

From property i) and the bound given in (A.6), property ii) follows; also, from i), there
exist an \( \epsilon_{ss} \in \mathbb{R}^+ \) and a \( t_{ss} \geq 0 \) such that \( \epsilon(t) = \epsilon_{ss} \) for all \( t \geq t_{ss} \). Since \( \epsilon_{ss} = \frac{\epsilon_0}{\tau_i} \) for some \( i \in \mathbb{N} \), it follows from Lemma 4.2 that for almost all \((\epsilon_0, \tau) \in S\), \( \bar{A}(\rho, \epsilon_{ss}) \) will have no eigenvalues in \( \mathbb{C}^0 \); hence, property iii) follows since for almost all \((\epsilon_0, \tau) \in S\), the excited modes of the final closed loop system will be stable.

A.3.2 Theorem 4.4

Proof: Although the proof of Theorem 4.4 is very similar in nature to that provided for Theorem 4.2, it will nevertheless be given in complete detail. To prove property i), assume that there exist a controller parameter \( \sigma_{PID} \in \Omega_{PID} \), a constant reference \( \tilde{y}_{ref} \) having norm \( \gamma_{ref} \), a constant disturbance \( w \) having norm \( \gamma \), and an initial condition \( x(0) = [x(0)^T \eta(0)^T \alpha(0)^T]^T \) for which Assumption PID1 holds, but property i) does not: it follows that \( t_i \) is defined for all \( i \in \mathbb{N} \).

Since \( \lambda(A) \subset \mathbb{C}^- \), there exist constants \((\alpha, -\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ \) so that \( \|e^{At}\| \leq \alpha e^{\lambda t} \) for \( t \geq 0 \). In addition, since

\[
\dot{x} = Ax + Bu + Ew,
\]

therefore

\[
x(t) = e^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)}(Bu(\tau) + Ew(\tau))d\tau:
\]

upon defining

\[
m_1 := \|B\|,
\]
\[
m_2 := \|E\|,
\]
\[
m_3 := \|K\| \left( 1 + \frac{\epsilon_{01} \rho}{\tau_1} + \frac{\epsilon_{02} N^2}{\tau_2} + \frac{\epsilon_{03} N}{\tau_2} \right),
\]
\[
m_4 := \frac{\epsilon_{02} N \gamma_{ref} \|K\|}{\tau_2},
\]

and on noting that

\[
\|e(t)\| \leq f(i - 1)
\]
and \( \|u(t)\| \leq m_3 f(i - 1) + m_4 \)
for \( t \in (t_1, t_i] \), \( i \geq 2 \), it also follows that

\[
\|x(t)\| \leq \tilde{\alpha} \cdot \|x(t_1)\| + \frac{\tilde{\alpha}}{\lambda} \left( m_1m_3 f(i - 1) + m_1m_4 + m_2\tilde{\omega}\right)
\leq c_0 + c_1f(i - 1)
=: \Gamma(i)
\]

for \( t \in (t_1, t_i] \) and for finite constants \( c_0 > 0, c_1 > 1 \). In addition, because

\[
\|x(t)\| = \max\{\|x(t)\|, \|\eta(t)\|, \|a(t)\|\},
\]

therefore

\[
\|x(t)\| \leq \Gamma(i) \tag{A.7}
\]

for \( t \in (t_1, t_i] \).

Since Lemma 4.4 holds for a sufficiently large \( \tilde{j} \), and since \( f \in \text{MSBF} \) and \((g.g_1,g_2) \in TF' \times TF' \times TF' \), there exists a finite \( \tilde{j} \geq \tilde{j} \) such that

(i) \( \alpha\Gamma(\tilde{j}) + \beta(\tilde{g}_{\text{ref}} + \tilde{\omega}) < f(\tilde{j}); \) and

(ii) \( \|(-\tilde{I}_jC - \tilde{I}_jDK \ g_2(\tilde{j})N^2\tilde{I}_jDK)\cdot\tilde{\Gamma}(\tilde{j}) + \|I-\rho g_1(\tilde{j})\tilde{I}_jDK\cdot\tilde{g}_{\text{ref}} + \|\tilde{I}_jF\cdot\tilde{\omega} < f(\tilde{j}) \)

are both satisfied for \( t \in (t_j, t_{j+1}] \); if we now set \( t = t_{j+1} \), then the inequalities

\[
\|z(t_{j+1})\| < f(\tilde{j}),
\]

\[
\|e(t_{j+1})\| < f(\tilde{j})
\]

contradict our definition of \( t_{j+1} \); hence property i) is true.

From property i) and the bound given in (A.7), property ii) follows; also, from i), there exist \((\epsilon_{ss}, \epsilon_{ss1}, \epsilon_{ss2}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \) and a \( t_{ss} \geq 0 \) such that \( \epsilon(t) = \epsilon_{ss}, \epsilon_1(t) = \epsilon_{ss1}, \)

and \( \epsilon_2(t) = \epsilon_{ss2} \) for all \( t \geq t_{ss} \). Since \( \epsilon_{ss} = \frac{\epsilon_0}{\tau_1}, \epsilon_{ss1} = \frac{\epsilon_{s1}}{\tau_1}, \) and \( \epsilon_{ss2} = \frac{\epsilon_{s2}}{\tau_2} \) for some \( i \in \mathbb{N} \), it follows from Lemma 4.5 that for almost all \((\epsilon_0, \tau, \epsilon_{s1}, \tau_1, \epsilon_{s2}, \tau_2) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S}, \)

\( \tilde{A}_{PID}(\rho, \epsilon_{ss}, \epsilon_{ss1}, \epsilon_{ss2}, K, N) \) will have no eigenvalues in \( \mathbb{C}^0 \); hence, property iii) follows since for almost all \((\epsilon_0, \tau, \epsilon_{s1}, \tau_1, \epsilon_{s2}, \tau_2) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S}, \) the excited modes of the final closed loop
A.3. The Self-Tuning Robust Servomechanism

The system will be stable. □

A.3.3 Theorem 4.5

Proof: The proof of Theorem 4.5 is very similar in nature to the proof given for Theorem 4.4 upon making appropriate changes to accommodate the fact that there are now $s$ possible feedback matrices: hence, only the major modifications to the latter will be given. To prove property i), one can form (in a manner analogous to the proof given for Theorem 4.4) the inequality

\[
\|z(t)\| \leq c_0 + c_1 f(i - 1) =: \Gamma(i) \tag{A.8a}
\]

where finite constants $c_0 > 0$, $c_1 > 1$, and where $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}$, $i \geq 2$. If we now set $q \in \{1, 2, \ldots, s\}$ such that $-TK_q$ is stable, and find a $\hat{j}$ sufficiently large so that Lemma 4.4 holds for all $i \geq \hat{j}$ with $(i - 1) \mod s = q - 1$. then since $f \in \text{MSBF}$ and $(g, g_1, g_2) \in \text{TF'} \times \text{TF'} \times \text{TF'}$, there exists a finite $\hat{j} \geq \hat{j}$, with $(\hat{j} - 1) \mod s = q - 1$, such that

\[
\begin{align*}
(i) & \quad c\Gamma(\hat{j}) + \beta(\hat{g}_{\text{ref}} + \hat{w}) < f(\hat{j}); \\
(ii) & \quad \|[-I_jC - I_jDK_q g_2(\hat{j})N^2I_jDK_q] \|\cdot \Gamma(\hat{j}) + \|I - \rho g_1(\hat{j})I_jDK_q\| \cdot \hat{g}_{\text{ref}} + \|I_jF\| \cdot \hat{w} < f(\hat{j})
\end{align*}
\]

are both satisfied for $t \in (t_j, t_{j+1}]$; if we now set $t = t_{j+1}$, then the inequalities

\[
\begin{align*}
\|z(t_{j+1})\| & < f(\hat{j}), \\
\|e(t_{j+1})\| & < f(\hat{j})
\end{align*}
\]

contradict our definition of $t_{j+1}$; hence property i) is true.

From property i) and the bound given in (A.8), property ii) follows: also, from i), there exist a matrix $K_{ss}$, constants $(\epsilon_{ss}, \epsilon_{ss1}, \epsilon_{ss2}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. and a $t_{ss} \geq 0$ such that $K(t) = K_{ss}$, $\epsilon(t) = \epsilon_{ss}$, $\epsilon_1(t) = \epsilon_{ss1}$, and $\epsilon_2(t) = \epsilon_{ss2}$ for all $t \geq t_{ss}$. Since $K_{ss} \in K$, and $\epsilon_{ss} = \frac{\epsilon_0}{\tau_1}$, $\epsilon_{ss1} = \frac{\epsilon_{10}}{\tau_1^2}$, $\epsilon_{ss2} = \frac{\epsilon_{20}}{\tau_2}$ for some $i \in \mathbb{N}$, it follows from Lemma 4.6 that for almost all
\[(\varepsilon_0, \tau, \varepsilon_0, \tau_1, \varepsilon_0, \tau_2, U) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S}' \]. \(\mathcal{A}_{PID}(\rho, \varepsilon_{ss}, \varepsilon_{ss1}, \varepsilon_{ss2}, K_{ss}, \mathcal{N})\) will have no eigenvalues in \(\mathbb{C}^0\); hence, property iii) follows since for almost all \((\varepsilon_0, \tau, \varepsilon_0, \tau_1, \varepsilon_0, \tau_2, U) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S}',\) the excited modes of the final closed loop system will be stable. \(\square\)
Appendix B

Miscellaneous Data

System matrices used for various simulation examples are given in this appendix.

B.1 Controller Parameters for a Family of Five Plants

In this section, the controller parameters for the structure

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{x}
\end{pmatrix} = \begin{bmatrix}
C^* & 0 \\
B_i P_i & A_i + L_i C_i + B_i P_0
\end{bmatrix} \begin{pmatrix}
\xi \\
\dot{x}
\end{pmatrix} + \begin{bmatrix}
B^* \\
-C_i
\end{bmatrix} y + \begin{bmatrix}
-B^* \\
0
\end{bmatrix} y_{ref}.
\]

\[
u = \begin{bmatrix}
P_i \\
P_0_i \\
K_i
\end{bmatrix} \begin{pmatrix}
\xi \\
\dot{x}
\end{pmatrix}
\]

used for the simulation example provided in Section 2.2.3 (Figure 2.11) are listed below:

\[a1 = \begin{pmatrix}
-0.0110 \\
-0.0077
\end{pmatrix} \begin{pmatrix}
-0.0016 \\
-0.0185
\end{pmatrix}
\]

\[b1 = \begin{pmatrix}
0.0843 \\
-0.2320
\end{pmatrix} \begin{pmatrix}
0.4431 \\
0.2007
\end{pmatrix}
\]

\[c1 = \begin{pmatrix}
0.5623 \\
1.6699
\end{pmatrix} \begin{pmatrix}
-1.0041 \\
0.6109
\end{pmatrix}
\]

\[L1 = \begin{pmatrix}
-1.2075 \\
4.1199
\end{pmatrix} \begin{pmatrix}
-1.9822 \\
-1.3827
\end{pmatrix}
\]
B.1. Controller Parameters for a Family of Five Plants

\[
P_1 = \\
\begin{bmatrix}
-53.4731 & -82.0000 & 2.5265 & 4.1452 \\
0.8104 & 0.6962 & -55.7095 & -51.0607
\end{bmatrix}
\]

\[
P_{O1} = \\
\begin{bmatrix}
-15.2059 & 33.1999 \\
-23.1610 & -8.5880
\end{bmatrix}
\]

\[
a_2 = \\
\begin{bmatrix}
-3.4448 & 1.3718 & 2.6741 & 2.5732 & -1.6488 \\
-2.4608 & -2.1160 & 3.3258 & 0.1274 & 1.0232 \\
-3.4053 & -3.4634 & -3.3366 & -1.1018 & 0.2684 \\
-0.5795 & -2.0056 & 1.0996 & -1.6816 & 1.6565 \\
-0.6210 & 0.5293 & -2.3358 & 0.6782 & -2.8446
\end{bmatrix}
\]

\[
b_2 = \\
\begin{bmatrix}
0.7286 & -0.5112 \\
-2.3775 & -0.0020 \\
-0.2738 & 1.6065 \\
0 & 0.8476 \\
0 & 0.2681
\end{bmatrix}
\]

\[
c_2 = \\
\begin{bmatrix}
0 & 0.1479 & -0.3367 & 1.5578 & 0 \\
0 & -0.5571 & 0.4152 & 0 & 1.1226
\end{bmatrix}
\]

\[
L_2 = \\
\begin{bmatrix}
-11.6448 & 2.7962 \\
8.9375 & 6.5512 \\
-11.0019 & -9.0574 \\
-10.2933 & -5.3340 \\
13.6818 & 1.6416
\end{bmatrix}
\]

\[
P_2 = \\
\begin{bmatrix}
35.2311 & -17.0037 & -175.7748 & 16.7539 \\
-50.5557 & -70.7379 & -30.2269 & 38.1306
\end{bmatrix}
\]

\[
P_{O2} = \\
\begin{bmatrix}
2.7617 & 8.5847 & -16.1114 & 22.5226 & 42.6001 \\
0.7637 & 2.0877 & -5.9681 & -15.3160 & 20.0489
\end{bmatrix}
\]

\[
a_3 = \\
\begin{bmatrix}
-2.7904 & 0.2769 & -0.6891 & 0.8506 & -0.2145 \\
0.2111 & -2.4407 & -0.0134 & 3.2803 & -1.0485 \\
-0.9388 & -1.0797 & -0.8947 & 0.8379 & -0.0735 \\
-0.0662 & -0.7764 & 0.2955 & -1.0820 & 3.4812 \\
-0.6250 & -3.1716 & -1.4806 & -0.7923 & -2.4274
\end{bmatrix}
\]

\[
b_3 = \\
\begin{bmatrix}
0 & -1.6333 \\
0 & 0 \\
-0.1612 & 0 \\
0.5377 & -1.2175
\end{bmatrix}
\]
The true plant is given by \((A, B, C)\).

\[
A =
\]
B.2 Partial Decentralized Control of a Multi-Zone Building

The following system matrices were used to obtain Figure 4.7:

\[ A = \]

\[
\begin{bmatrix}
1.3800 & -0.2077 & 6.7150 & -5.6760 \\
-0.5814 & -4.2900 & 0 & 0.6750 \\
1.0670 & 4.2730 & -6.6540 & 5.8930 \\
0.0480 & 4.2730 & 1.3430 & -2.1040 \\
\end{bmatrix}
\]

\[ B = \]

\[
\begin{bmatrix}
0 \\
0 \\
5.6790 \\
1.1360 \\
1.1360 \\
\end{bmatrix}
\]

\[ C = \]

\[
\begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[ L = \]

\[
\begin{bmatrix}
-7.9087 & -3.4248 \\
-0.5715 & -3.0574 \\
-5.5657 & -20.0400 \\
-6.1998 & -18.3393 \\
\end{bmatrix}
\]

\[ P = \]

\[
\begin{bmatrix}
7.2353 & 4.1008 & -22.0228 & -14.1473 \\
20.6627 & 17.2250 & -10.9506 & 0.6496 \\
\end{bmatrix}
\]

\[ PQ = \]

\[
\begin{bmatrix}
0.5770 & -2.0548 & 0.4698 & -0.5874 \\
5.3283 & -0.1451 & 3.9826 & -4.2342 \\
\end{bmatrix}
\]
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B.2. Partial Decentralized Control of a Multi-Zone Building

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B.2. Partial Decentralized Control of a Multi-Zone Building

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\end{array}

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\begin{array}{cccccccc}
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\text{Columns 8 through 14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{Columns 15 through 21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
B.3 A Four Input-Four Output Furnace Model

The following matrices were used in the simulation to obtain Figure 4.9:

\[
\begin{align*}
A_R &= \\
&= \begin{bmatrix}
-0.2003 & -0.0029 & -0.0002 & 0.0003 & 0.0000 & 0.0000 & 0.0000 \\
-0.0044 & -0.2497 & 0.0000 & 0.0000 & -0.0004 & 0.0000 & 0.0000 \\
0.0002 & 0.0000 & -0.2500 & 0.0000 & -0.0010 & 0.0007 & 0.0000 \\
0.0008 & 0.0000 & 0.0000 & -0.2500 & -0.0004 & -0.0005 & 0.0000 \\
0.0000 & 0.0003 & -0.0004 & 0.0003 & -0.2000 & 0.0000 & 0.0001 \\
0.0000 & -0.0002 & 0.0003 & 0.0001 & 0.0000 & -0.2000 & 0.0002 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0001 & -0.2500
\end{bmatrix}
\]

\[
B_R = \\
= \begin{bmatrix}
-0.5606 & -0.5049 & -0.5340 & -0.4621 \\
-0.4696 & -0.9996 & -0.3625 & -0.4097 \\
-0.5579 & -1.1862 & 0.0228 & 0.6893 \\
0.7695 & -0.5779 & 0.8317 & -0.5298 \\
-0.1268 & 0.3153 & -0.5383 & 0.3288 \\
0.3292 & -0.3941 & -0.1948 & 0.1050 \\
0.0658 & -0.3228 & -0.2961 & 0.2897
\end{bmatrix}
\]

\[
C_R = \\
= \begin{bmatrix}
-0.1154 & -0.0594 & -0.1380 & 0.1364 & 0.0683 & -0.1450 & 0.4705 \\
-0.1498 & -0.0737 & -0.0594 & -0.0881 & -0.0582 & 0.1142 & -0.1340 \\
-0.1628 & -0.1014 & 0.1738 & 0.0882 & 0.1050 & 0.0092 & -0.4302 \\
-0.1303 & -0.1820 & 0.1418 & -0.0398 & -0.1070 & -0.1015 & 0.1630
\end{bmatrix}
\]

\[
D_R = \\
= \begin{bmatrix}
0.0042 & 0.0528 & 0.0717 & -0.0687 \\
-0.0002 & -0.0030 & -0.0041 & 0.0039 \\
-0.0059 & -0.0738 & -0.1004 & 0.0962 \\
0.0015 & 0.0191 & 0.0259 & -0.0249
\end{bmatrix}
\]

\[
E_R = \\
= \begin{bmatrix}
0.1000 \\
0.1000 \\
0.1000 \\
0.1000 \\
0.1000
\end{bmatrix}
\]
These matrices were obtained upon using the model reduction methods given in [26] on the original furnace model given in [91, pg. 199] and [26]. The system matrices used to obtain Figures 4.10 and 4.11 are listed below.

\[ A = \]  
Columns 1 through 7  
\[
\begin{array}{cccccccc}
-0.2323 & 0.0171 & 0.0110 & 0.0077 & 0.0009 & 0.0146 & -0.0106 \\
0.0023 & -0.2549 & 0.0032 & -0.0250 & -0.0224 & -0.0369 & 0.0013 \\
0.0067 & 0.0053 & -0.2565 & 0.0252 & 0.0286 & 0.0318 & -0.0039 \\
0.0235 & -0.0113 & -0.0215 & -0.2404 & 0.0258 & -0.0128 & -0.0050 \\
0.0055 & -0.0178 & -0.0326 & 0.0240 & -0.2003 & 0.0094 & 0.0035 \\
-0.0224 & 0.0305 & 0.0316 & 0.0167 & -0.0101 & -0.1877 & -0.0010 \\
-0.0572 & -0.0279 & -0.0134 & -0.0031 & -0.0021 & 0.0055 & -0.2240 \\
-0.0018 & -0.0127 & -0.0074 & 0.0061 & -0.0053 & -0.0014 & 0.0007 \\
\end{array}
\]  
Column 8  
\[
\begin{array}{cccccccc}
0.0045 \\
-0.0123 \\
0.0057 \\
0.0073 \\
-0.0055 \\
0.0020 \\
-0.0003 \\
-0.2038 \\
\end{array}
\]  

\[ B = \]  
\[
\begin{array}{cccccccc}
-0.3339 & -0.2226 & -0.4942 & -0.4155 \\
-0.1606 & -0.2472 & 0.1345 & 0.3298 \\
0.1476 & -0.3294 & 0.0593 & -0.4351 \\
0.1986 & -0.2699 & -0.2105 & -0.2577 \\
-0.1571 & 0.2450 & 0.0557 & 0.2807 \\
0.0764 & -0.0484 & -0.0740 & -0.0235 \\
-0.0196 & 0.0502 & 0.0288 & 0.0094 \\
-0.0376 & 0.0579 & 0.0084 & 0.0615 \\
\end{array}
\]  

\[ C = \]  
Columns 1 through 7  
\[
\begin{array}{cccccccc}
-0.4177 & -0.3583 & -0.3344 & 0.3396 & -0.2580 & -0.0917 & 0.0403 \\
-0.2547 & -0.4481 & -0.1575 & -0.1221 & -0.0002 & 0.0932 & 0.0006 \\
-0.3225 & 0.0137 & 0.3904 & -0.2739 & 0.2818 & 0.0959 & -0.0427 \\
-0.2585 & 0.2041 & -0.2276 & 0.0027 & -0.0905 & 0.0186 & 0.0195 \\
\end{array}
\]  

\footnote{In essence, the system given by (4.14) was formulated for didactic purposes with \( D_R \neq 0 \).}
B.4. Matrices used for a Binary Distillation Tower

The following matrices were used to obtain Figures 5.3 and 5.4:

\[ A = \]

<table>
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<tbody>
<tr>
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<tr>
<td>9.5000e-03</td>
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\[ B = \]

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<td>0.0003</td>
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\[ D = \]

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B.4 Matrices used for a Binary Distillation Tower

The following matrices were used to obtain Figures 5.3 and 5.4:

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</table>
### B.4. Matrices used for a Binary Distillation Tower

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2.2000e-02 & 0 & 0 & 0 & 0 & 0 \\
-4.2200e-02 & 2.8000e-02 & 0 & 0 & 0 & 0 \\
2.0200e-02 & -4.8200e-02 & 3.7000e-02 & 0 & 2.0000e-04 \\
0 & 2.0200e-02 & -5.7200e-02 & 4.2000e-02 & 5.0000e-04 \\
0 & 0 & 2.0200e-02 & -4.8300e-02 & 5.0000e-04 \\
0 & 0 & 0 & 2.5500e-02 & -1.8500e-02
\end{bmatrix}
\]

**B =**

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
5.0000e-06 & -4.0000e-05 & 2.5000e-03 \\
2.0000e-06 & -2.0000e-05 & 5.0000e-03 \\
1.0000e-06 & -1.0000e-05 & 5.0000e-03 \\
0 & 0 & 5.0000e-03 \\
0 & 0 & 5.0000e-03 \\
-5.0000e-06 & 1.0000e-05 & 5.0000e-03 \\
-1.0000e-05 & 3.0000e-05 & 5.0000e-03 \\
-4.0000e-05 & 5.0000e-05 & 2.5000e-03 \\
-2.0000e-05 & 2.0000e-05 & 2.5000e-03 \\
4.6000e-04 & 4.6000e-04 & 0
\end{bmatrix}
\]

**C =**

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Appendix C

Additional Experimental Results

In this appendix, additional experimental results, obtained upon applying Controller PID' (for the case when no estimate of \( T \) is available) to the MARTS apparatus, are presented for the class of piecewise constant reference and disturbance inputs. A listing of the controller parameters and the individual system setup used for each figure is summarized in Table C.1, where the reference heights referred to are the following:

\[
(y_{\text{ref}}^1(t), y_{\text{ref}}^2(t)) := \begin{cases} 
(3000, 2500), & 0 \leq t < 600 \\
(2500, 2000), & 600 \leq t < 1200 \\
(3000, 2500), & t \geq 1200 \text{ seconds.}
\end{cases} \tag{C.1}
\]

\[
(h_{\text{ref}}^1(t), h_{\text{ref}}^2(t)) := \begin{cases} 
(3000, 2500), & 0 \leq t < 2250 \\
(2500, 2000), & t \geq 2250 \text{ seconds.}
\end{cases} \tag{C.2}
\]

\[
(h_{\text{ref}}^1(t), h_{\text{ref}}^2(t)) := \begin{cases} 
(3250, 3000), & 0 \leq t < 2250 \\
(3000, 2500), & 2250 \leq t < 3000 \\
(3250, 3000), & t \geq 3000 \text{ seconds.}
\end{cases} \tag{C.3}
\]

\[
(h_{\text{ref}}^1(t), h_{\text{ref}}^2(t)) := (3500, 2500), \quad t \geq 0. \tag{C.4}
\]

\[
(h_{\text{ref}}^1(t), h_{\text{ref}}^2(t)) := (3000, 2500), \quad t \geq 0. \tag{C.5}
\]

In addition, for all instances, unless otherwise stated,

\[
(g_1(k), g_2(k)) := (g(k), g(k)),
\]

\[
\theta := 40^\circ.
\]
\[ \lambda := 10. \]

and \( T := 0.75 \text{ seconds}. \)

<table>
<thead>
<tr>
<th>Fig.</th>
<th>( f_1(k) )</th>
<th>( f_2(k) )</th>
<th>( g(k) )</th>
<th>( \rho )</th>
<th>( N )</th>
<th>( \alpha_{ss} )</th>
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<th>Rvsd.</th>
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<td>210( \alpha + 3 )</td>
<td>10/5( a )</td>
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<td>6 ( W_1 )</td>
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<td>( X/N )</td>
<td>( C.5 )</td>
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</table>

Table C.1: Summary of the parameters used for Figures C.1-C.12. where \( \alpha(k) := \text{floor}\left(\frac{k + 5}{6}\right) \) and \( \beta(k) := 35 \cdot (((k - 1) \mod 6) + 1) \).

**Additional notes and comments:**

(a) In Figures C.11 and C.12, output leads \( y_1(t) \) and \( y_2(t) \) are reversed at \( t = 2250 \) seconds.
C. Additional Experimental Results

Figure C.1: Experimental integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.

Figure C.2: Experimental integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.
C. Additional Experimental Results

Figure C.3: Experimental integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MXRTS system.

Figure C.4: Experimental integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.
C. Additional Experimental Results

Figure C.5: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.

Figure C.6: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.
C. Additional Experimental Results

Figure C.7: Experimental proportional-integral-derivative results of \( h_1 \) (dashed) and \( h_2 \) (solid) with Controller PID1' applied to the reversed MARTS system.

Figure C.8: Experimental proportional-integral-derivative results of \( h_1 \) (dashed) and \( h_2 \) (solid) with Controller PID1' applied to the reversed MARTS system.
C. Additional Experimental Results

Figure C.9: Experimental proportional-integral-derivative results of $h_1$ (dashed) and $h_2$ (solid) with Controller PID1' applied to the reversed MARTS system.

Figure C.10: Experimental proportional-integral-derivative results of $h_1$ (dashed) and $h_2$ (solid) with Controller PID1' applied to the reversed MARTS system.
C. Additional Experimental Results

Figure C.11: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.

Figure C.12: Experimental proportional-integral-derivative results of $y_1$ (solid) and $y_2$ (dashed) with Controller PID1' applied to the MARTS system.
Bibliography


