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Higher dimensional black hole solutions with scalar hair/dilaton field and Toroidal horizons and the interior solution

Younes Younesizadeh¹, Yahya Younesizadeh²

^{1,2}*Department of physics, South Tehran Branch, Islamic Azad University, Tehran, Iran.*

Abstract

In this paper we investigate the black hole solutions with toroidal horizons in scalar hair/dilaton gravity. First, we obtain the field equations in n -dimensions, then we propose some different models (Ansatz) and find the exact solutions for these type of ansatzs. These solutions are not asymptotically (anti-)de Sitter or flat, except in one special case. We also show that the BTZ and BTZ-like solutions will emerge from some of these solutions as a special case. We also show that when the event horizon radius gets bigger and bigger, the temperature will be the same in various dimensions. The only difference is noticeable near the origin (this statement is clear in diagrams). For these solutions, we obtained a new version of the Smarr formula as well. Also, we show that the presence of the scalar field makes the black holes to be more stable near the origin except for the BTZ case. We can say in general that the presence of scalar field is an important factor in black hole's stability investigations. In the critical behavior analysis we find that there is no evidence to show the existence of P-V criticality. We present here a class of interior solutions corresponding to the solution in scalar hair gravity exterior. The solution which is obtained in linear case is regular and well-behaved at the stellar interior.

1 Introduction

Gravity still is a bizarre concept in the universe. Einstein theory of gravity (Which is called General Relativity) is an elegant theory in describing the gravitational interaction over many years of experimental tests [1]. These experimental tests range from the short scale (mm) tests in the laboratory frame to vast Solar System tests and is in high consistency with the gravitational waves emitted by binary pulsars. More interesting and recent one was the detection of Advanced LIGO and Virgo interferometers On August 17, 2017 that made their first observation of a binary neutron star pulse, called GW170817 [2]. Also, this detection revealed the existence of merging stellar-mass binary black holes (BBHs). Using the signal of binary neutron star system and testing the GR related to the dynamics of the source, the polarization of gravitational waves and the propagation of these waves indicates the highly agreement with the predictions of General Relativity (GR).

From the standard model of cosmology, General Relativity (Einstein theory of gravity) is considered as a theory to describe gravitational interactions over different scales. Around the two decade ago, 1998 was a revolutionary year for lots of physicists. They discovered that the expansion of the Universe is speeding up, not slowing down [3, 4]. There are some important observational data for this accelerated expansion like observations of type Ia supernovae [5, 6, 7, 8, 9], Cosmic Microwave Background (CMB) measurements [10, 11, 12, 13, 14, 15, 16, 17] and

¹E-mail: younesizadeh88@gmail.com

²E-mail: younesizadeh@cfu.ac.ir

some important studies of the large-scale structure [18, 19, 20, 21, 22, 23]. All measurements from these evidence are in a highly agreement with the Λ CDM model [24] [25]. This acceleration phase of the universe becomes one of the most challenging problems in theoretical physics in few past decades. In the GR framework, the acceleration phase could originate from an unknown cosmological model known as the dark energy. The first one and simplest scenario in this case is the cosmological constant, introduced by Einstein when his field equations explained a dynamic universe not a static one. Although, for explaining the current acceleration of the Universe, the cosmological constant value required to be incredibly small. From another point of view, Particle physics in some sense predicts the existence of vacuum energy state and brings a value for the cosmological constant. This prediction is unbelievably larger (more than 50 orders of magnitude) than the observed values which come from GR context. For sure, the discovery of the acceleration phase of the Universe may require us to modify the theory of gravity on cosmological scales and also the standard model of cosmology which is based on the General Relativity.

It is not an exaggeration to say that the black hole concept is one of the bizarre concepts in all physics. This concept might be treated as the solution of the Einstein's theory of gravity where some new characteristics of space-time emerge. One of these characteristics is the event horizon. This place is where even the light can not escape from the surface gravity of massive objects like massive stars and experiences red shift³. Behind the event horizon everything crash on the singularity. The existence of singularity first predicted by the discovery of the Schwarzschild solution and after that by the Kerr solution. In this unpleasant place, the concept of space-time may doom and some novel ideas are needed.

For the low energy scales, the Einstein theory of gravity (General Relativity) has been tested widely [26] [27] [28] [29]. Despite this, it is required that the General Relativity should be modified at very high energy levels close to the Planckian scale. The great ideas like the existence of extra dimensions can be found in the context of String theory and brane cosmology [30] [31]. As a modification on the Einstein's theory of gravity, the first modified model is the Lovelock gravity [32] where the action in this model contains higher order curvature terms and it holds the Einstein's theory as a special case in itself. This model does not contain the kind of terms with more than second derivatives of the metric, and consequently free of ghosts [33]. Also, new models like the Gauss-Bonnet gravity are the special case of this theory (Lovelock gravity). The Gauss-Bonnet gravity is the second order Lovelock gravity and emerges naturally from the low energy limit of the string theory [34].

In recent years, some authors showed a lot interest in scalar field gravity. They have worked on different kind of scalar fields for different purposes. The first attempts were considered by Fisher, who obtained a static spherically symmetric solution for massless scalar fields [35]. Dilaton gravity is one of the important scalar fields in black hole investigations and More specifically it is a kind of scalar fields which occurs in the low limit of the string theory. The presence of dilaton field has a deep influence on the physical phenomenon and leaves such interesting results. These results are the influence of the dilaton field on the asymptotic behaviour of space-time and thermodynamical quantities like temperature, entropy(S) and the heat capacity. Here we introduce some attempts in the concept of dilaton gravity. The first uncharged solution were obtained by Mignemi and Wiltshire in these works [36] [37] [38]. Poletti and Wiltshire first obtained the charged black hole solutions in the Einstein-Maxwell-dilaton gravity with the use of Liouville-type potentials in [39]. For the sake of an exact non-asymptotically flat, non-AdS

³When the light wants to escape the surface gravity of the massive objects, it losses energy and its frequency diminished. Gravitational red shift has been measured for the sun and earth which can be consider as the low-gravity objects. For more massive and compact objects like the neutron stars, this red shift becomes tremendous.

black hole solutions with electric charged in higher dimensions in the Einstein-Maxwell-dilaton gravity, Chan, Horne and Mann presented these kind of solutions in [40]. Also, the static, charged, non-asymptotically flat, non-AdS BH⁴ solutions in arbitrary dimensions were obtained by Wang and Cai in [41].

One of the important investigations in this work is to obtain a class of interior solution corresponding to the exterior space-time in dilaton gravity. In the relativistic astrophysics, it is fascinating to investigate the static anisotropic objects. This investigation leads to describe the interior region of stellar objects [42] [43] [44].

The paper is organized as follow: In section 2, we introduce the new solutions when we have the electromagnetic field as well. Section 3 is devoted to the P-V criticality analysis and we show that the EOS does not admit any inflection point and does not admit critical behaviour. In section 4 we introduce the action and field equations in n dimensions. In section 5, we introduce the concept of quasilocal mass and its relation in details. In section 6, we propose three models and calculate the field equations to obtain the exact solutions for these models and also we analyze these solutions in different ways like the asymptotics behaviour, locations of the singularity and the event horizon and the special case of these solutions. In section 7, we first introduce thermodynamical quantities (Temperature, entropy(S) and the heat capacity) and then calculate their relations correspondence to our solutions. We also check their properties schematically on diagrams. In section 8, we obtain the interior solution in n dimensions when the corresponding exterior space-time is the solution in dilaton gravity. We also show that the solution which is obtained in linear case is regular and well-behaved at the stellar interior. Finally in last section, we give a summary of our work and then propose new points for future works.

2 Charged solution in dilaton gravity

Here we introduce the action of charged black holes in the Einstein-Maxwell gravity which is coupled to a dilaton field. The general form of this action is studied in literature [61, 62, 63, 64]:

$$S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left[\mathcal{R} - \frac{4}{n-2} \nabla^\mu \Phi \nabla_\mu \Phi - V(\Phi) - e^{-\frac{4\Phi}{n-2}} F \right] \quad (1)$$

Where $\mathcal{R}$ is the Ricci scalar, $V(\Phi)$ is the potential of dilaton field. The last term in this relation is actually the representation of coupling between electrodynamics and scalar field. This term is as $\mathcal{L}(F, \Phi) = -F e^{-\frac{4}{n-2}\Phi}$ where $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ (In electromagnetic context, $F_{\mu\nu}$ is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). Varying this action with respect to the metric $g_{\mu\nu}$, the gauge field A_μ and dilaton field Φ after some simplifications, yields:

$$\mathcal{R}_{\mu\nu} = \frac{4}{n-2} \left[\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right] + \left[\frac{1}{(n-2)} g_{\mu\nu} - \frac{2}{F} F_{\mu\gamma} F_\nu^\gamma \right] \mathcal{L}(F, \Phi) \quad (2)$$

$$\nabla_\mu (\sqrt{-g} \mathcal{L}_F(F, \Phi) F^{\mu\nu}) = 0 \quad (3)$$

$$\partial_\mu \partial^\mu \Phi = \frac{n-2}{8} \frac{\partial V(\Phi)}{\partial \Phi} + \frac{1}{2} \mathcal{L}(F, \Phi) \quad (4)$$

Here we assume the most general static metric ansatz in n -dimensions [47]:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + H(r)^2 \sum_{i=1}^{n-2} d\phi_i^2 \quad (5)$$

⁴black hole

The coordinate ranges are $-\infty < t < \infty$, $r \geq 0$, and $-\pi \leq \phi_i \leq \pi$. Also, the metric signature is $(-, +, +, +, \dots)$.

Now, we consider this gauge potential:

$$A_\mu = h(r)\delta_\mu^t \implies F_{tr} = -h'(r) \quad (6)$$

Integration of Maxwell equation Eq.(3) gives:

$$E(r) = F_{tr} = \frac{qe^{\frac{4\Phi}{n-2}}}{H^{n-2}} \quad (7)$$

For this model $H(r) = \alpha r^p$ and solving the equations of motion, we have:

$$f(r) = \xi_1 r^{2p} + \xi_2 r^{1-(n-2)p} - \frac{2q^2}{L} r^{2(1-(n-2)p-\sqrt{p(p-1)})} \quad (8)$$

where:

$$L = \left(3 + (4 - 3n)p\right)\sqrt{p(p-1)} + \left(n^2 - 3n + 4\right)p^2 + \left(1 - 2n\right)p + 1 \quad (9)$$

And q is the electric charge.

3 P-V critically

In this section, we are going to study the critical behavior of our solution by plotting and analyzing the P-V diagrams. Here the cosmological constant Λ is regarded as a thermodynamic pressure $P = -\frac{\Lambda}{8\pi}$. The corresponding conjugate quantity for the pressure is the thermodynamic volume V which can be obtained from the following relation:

$$V = \left(\frac{\partial H}{\partial P}\right)_{S,Q} = \left(\frac{\partial M}{\partial P}\right)_{S,Q} \quad (10)$$

Where, M is the total mass and H is the enthalpy of black hole.

Also, the critical points in P-V diagram can be found from the following relations as the inflection points:

$$\left(\frac{\partial P}{\partial v}\right)_{v=v_c, T=T_c} = \left(\frac{\partial^2 P}{\partial v^2}\right)_{v=v_c, T=T_c} = 0 \quad (11)$$

Where v is the specific volume and its relation with the horizon radius is as [65].

$$v = \frac{4r_+}{d-2} \quad (12)$$

the corresponding equation of state(EOS) reads:

$$P = \frac{2\pi L T r_+ + (1 - (n-2)p - 2\sqrt{p(p-1)})q^2 r_+^{2(1-(n-2)p-\sqrt{p(p-1)})}}{4\pi L(np-1)r_+^{2p}}; \quad v = 4r_+ \quad (13)$$

This equation is like such an equations that does not admit any inflection point and does not admit critical behaviour.

Figure 1: P versus r_+ when p changes. We have assumed $T=1$, $q=1$ and $n=4$.

4 Solution in dilaton gravity when $q=0$

In this paper we investigate the case of scalar hair gravity. The n-dimensional action in the presence of scalar potential reads:

$$S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left[\mathcal{R} - \frac{4}{n-2} \nabla^\mu \Phi \nabla_\mu \Phi - V(\Phi) \right] \quad (14)$$

$\mathcal{R}$ is the Ricci scalar and $V(\Phi)$ is the potential of scalar hair (we set here that $8G=c=1$). Varying this action with respect to metric and scalar field after some simplifications, yields [45] [46]:

$$R_{\mu\nu} = \frac{4}{n-2} \left(\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) \quad (15)$$

$$\partial_\mu \partial^\mu \Phi = \frac{n-2}{8} \frac{dV(\Phi)}{d\Phi} \quad (16)$$

The static metric ansatz in n-dimensions is [47]:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + H(r)^2 \sum_{i=1}^{n-2} d\phi_i^2 \quad (17)$$

The equations (14) and (15) can be simplified to the following equations

$$\frac{-f(r)}{2H(r)} (H(r)f''(r) + (n-2)H'(r)f'(r)) = \frac{1}{n-2} (-f(r)V(\Phi)) \quad (18)$$

$$\frac{1}{2H(r)f(r)} (2(n-2)f(r)H''(r) + H(r)f''(r) + (n-2)H'(r)f'(r)) = \frac{4}{n-2} \left((\Phi'(r))^2 + \frac{V(\Phi)}{4f(r)} \right) \quad (19)$$

$$f(r)H(r)H''(r) + (n-3)f(r)(H'(r))^2 + f'(r)H(r)H'(r) = \frac{1}{n-2} (H(r)^2 V(\Phi)) \quad (20)$$

$$\frac{1}{H(r)^{n-2}} (f(r)H(r)^{n-2} \Phi'(r))' = \frac{n-2}{8} \left(\frac{dV(\Phi)}{d\Phi} \right) \quad (21)$$

Here prime means derivative with respect to r . From equations (17) and (18) we obtain

$$\frac{H''(r)}{H(r)} = \frac{4}{n-2} (\Phi'(r))^2 \quad (22)$$

And also from equations (17), (19) the relation between $f(r)$ and $H(r)$ is

$$\frac{H(r)}{2}f''(r) + \frac{n-4}{2}H'(r)f'(r) - \left(H''(r) + (n-3)\frac{H'(r)^2}{H(r)}\right)f(r) = 0 \quad (23)$$

This is an important equation in this paper because it give us the crucial part of the metric function($f(r)$). In the next section we show how it is possible to find this unknown function with some assuming ansatzs for the other unknown function($H(r)$).

For the metric ansatz(Eq.(16)), the Ricci and Kretschmann scalars are

$$R = \frac{f''(r)H(r)^2 + 2(n-2)f(r)H(r)H''(r) + (n-2)(n-3)f(r)H'(r)^2}{H(r)^2} + \frac{2(n-2)f'(r)H(r)H'(r)}{H(r)^2} \quad (24)$$

$$R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} = \frac{1}{4}f''(r)^2 + (n-2)\frac{f(r)^2H''(r)^2}{H(r)^2} + \frac{(n-2)}{2}\frac{f'(r)^2H'(r)^2}{H(r)^2} + (n-2)\frac{f(r)f'(r)H'(r)H''(r)}{H(r)^2} + \frac{(n-2)(n-3)}{2}\frac{f(r)^2H'(r)^4}{H(r)^4} \quad (25)$$

5 Quasilocal mass

The metrics in this paper are not asymptotically flat. So, it is required to use quasilocal formalism to define the mass of the solutions(for details, see [51]). If a metric is written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{N(r)^2} + r^2d\phi^2 \quad (26)$$

then the quasilocal mass by an observer(static one) at r (the radius of the spacelike hypersurface boundary) is given by

$$\mathcal{M}(r) = 2\sqrt{f(r)}(N_0(r) - N(r)) \quad (27)$$

The two terms in the bracket is the thermodynamic internal energy. in this relation(27), $N_0(r)$ is a function which determines the zero of the energy for the background. When the spacetime is asymptotically flat, the ADM⁵ mass M is determined in the equation (27) in the limit of $r \rightarrow \infty$. It should be noted that the quasilocal mass is not a positive entity in all cases. The above equations can admit negative values for some ranges of r , too. Basically, for a physical solution, the ADM mass M is positive.

6 Solutions

Now, we assume some arbitrary ansatzs for $H(r)$ in this section. By these proposing models, we want to see their effects on our solutions and even more on the thermodynamical quantities like the temperature, entropy(S), etc.

⁵Arnowitt-Deser-Misner

6.1 Model.1

The first model is:

$$H(r) = \alpha r^p \quad (28)$$

By using this relation and solving the equation (22) one can reach the following relation for $\phi(r)$:

$$\Phi(r) = \pm \left(\sqrt{p(p-1)} \left(\frac{n}{2} - 1 \right) \ln(r) \right) + \Phi_0 \quad p \geq 1 \quad (29)$$

Here α , Φ_0 and p are constants. ($p=1$ produces the trivial constant scalar hair case and it is not important for us here). By replacing this ansatz (Eq(28)) in Eq(23), the following relation for $f(r)$ will obtain:

$$f(r) = \xi_1 r^{2p} + \xi_2 r^{1+(2-n)p} \quad (30)$$

ξ_1 and ξ_2 are integration constants. The space-time metric in n -dimensions is given by:

$$ds^2 = -(\xi_1 r^{2p} + \xi_2 r^{1+(2-n)p}) dt^2 + \frac{dr^2}{\xi_1 r^{2p} + \xi_2 r^{1+(2-n)p}} + \alpha^2 r^{2p} \sum_{i=1}^{n-2} d\phi_i^2 \quad (31)$$

For finding the horizons and singularities, one should solve these equations $f(r)=0$ and $H(r)=0$. From these two equations we obtain $r=0$ (in this case, g_{tt} and $g_{\phi_i \phi_i}$ vanish and g_{rr} diverges) and $r = \frac{1}{(\frac{-\xi_1}{\xi_2})^{\frac{1}{np-1}}}$ (here g_{tt} vanishes and g_{rr} diverges again). now, it is worthwhile to introduce

the Ricci scalar for this metric to understand the basic properties of this solution. Actually, from the Ricci scalar we can find out where the curvature singularity (physical singularity) is.

$$R = \frac{p \left(\xi_1 (n-1)(np+2p-2)r^{2p} + \xi_2 (p-1)(n-2)r^{1+(2-n)p} \right)}{r^2} \quad (32)$$

It is obvious that the curvature singularity is located at $r = 0$.

With some manipulations in the relations of this section, We can reach the BTZ metric solution [52] [53]. Basically, we just put $\alpha=1$, $p=1$ and $n=3$. So, we reach the following metric

$$ds^2 = -(\xi_1 r^2 + \xi_2) dt^2 + \frac{dr^2}{\xi_1 r^2 + \xi_2} + r^2 d\phi^2 \quad (33)$$

In this relation, ξ_1 and ξ_2 can be considered as cosmological constant and mass quantity, respectively. The metric Eq.(31) is a generalisation of the BTZ($n=3$) and BTZ-like solutions in higher dimensions (for more detail one can see [55]).

For producing an exact solution from the equations of motion, one should find an explicit function for the potential function $V(\Phi)$ which all equations (18,19,20,21) are consistent. After some calculation one can reach this relation for the potential function:

$$V(\Phi) = \xi_1 p (np-1)(n-2) e^{4\sqrt{\frac{p-1}{p}} \left(\frac{\Phi-\Phi_0}{n-2} \right)} \quad (34)$$

This potential function is indeed the Liouville-type potentials [49]. In the case of $p=1$ $n=3$, the potential will be $V(\Phi) = 2\xi_1$. This is actually for the BTZ case and it is clear that ξ_1 is related to the cosmological constant through $\xi_1 = -\frac{2\Lambda}{(n-1)(n-2)}$. We also check the BTZ-like ($p=1$) case where the potential is $V(\Phi) = (n-1)(n-2)\xi_1$. In general, we have:

$$V(\Phi) = -2\Lambda \frac{p(np-1)}{(n-1)(n-2)} e^{4\sqrt{\frac{p-1}{p}} \left(\frac{\Phi-\Phi_0}{n-2} \right)} \quad (35)$$

From these relations and concepts, it is obvious that the cosmological constant is coupled to the scalar field in a nontrivial way and in the absence of this field, the action of scalar field reduces to the action of Einstein gravity with cosmological constant (ordinary gravity which is called General Relativity).

In Fig.2, we want to see the effect of "p" parameter on the radius of the event horizon in three dimensions (n=3). In order to do this purpose, we first choose $\xi_1 = 2, \xi_2 = -5$ in Eq.(30) and then plot the lapse function (f(r)) for five different values of "p". The blue, red, yellow, green and black colors are for the p=1, p=2, p=3, p=4 and p=5 values, respectively. In this case, the event horizon radius decreases with the increases of the "p" parameter. We should note again that the p=1 is the BTZ case. (The event horizon radius is where the diagrams cross the r axis.)

Figure 2: Shows the plot of lapse function $f(r)$ versus the radius of event horizon r for five different parameters of p : $p=1$ for blue, $p=2$ for red, $p=3$ for yellow, $p=4$ for green, and $p=5$ for black colored curve.

By this coordinate transformation $\alpha^2 r^{2p} \rightarrow r^2$, metric (31) becomes

$$ds^2 = - \left(\xi_1 \left(\frac{\mathbf{r}}{\alpha} \right)^2 + \xi_2 \left(\frac{\mathbf{r}}{\alpha} \right)^{\frac{1+(2-n)p}{p}} \right) dt^2 + \frac{\left(\frac{1}{\alpha^2 p^2} \right) \left(\frac{\mathbf{r}}{\alpha} \right)^{\frac{2(1-p)}{p}} d\mathbf{r}^2}{\left(\xi_1 \left(\frac{\mathbf{r}}{\alpha} \right)^2 + \xi_2 \left(\frac{\mathbf{r}}{\alpha} \right)^{\frac{1+(2-n)p}{p}} \right)} + \mathbf{r}^2 \sum_{i=1}^{n-2} d\phi_i^2 \quad (36)$$

Which is (anti)de Sitter asymptotically when $p=1$. without the loss of generality, we put $\alpha=1$

$$ds^2 = - \left(\xi_1 \mathbf{r}^2 + \xi_2 \mathbf{r}^{\frac{1+(2-n)p}{p}} \right) dt^2 + \frac{\left(\frac{1}{p^2} \right) \mathbf{r}^{\frac{2(1-p)}{p}} d\mathbf{r}^2}{\left(\xi_1 \mathbf{r}^2 + \xi_2 \mathbf{r}^{\frac{1+(2-n)p}{p}} \right)} + \mathbf{r}^2 \sum_{i=1}^{n-2} d\phi_i^2 \quad (37)$$

The metrics that we present in this work are not asymptotically flat and so it is required to use the so called quasilocal formalism [48] to define the mass quantity. By using the quasilocal mass formula from the section 5 and the relation (36), one can reach the following relation for it by the choice of the background solution as $\xi_2 = 0 \Rightarrow N_0(\mathbf{r}) = \left(\xi_1 p^2 \mathbf{r}^{2\left(\frac{2p-1}{p}\right)} \right)^{\frac{1}{2}}$.

$$M = 2p \mathbf{r}^{\frac{p-1}{p}} \left(\xi_1 \mathbf{r}^2 + \xi_2 \mathbf{r}^{\frac{1-p}{p}} \right)^{\frac{1}{2}} \left[\left(\xi_1 \mathbf{r}^2 \right)^{\frac{1}{2}} - \left(\xi_1 \mathbf{r}^2 + \xi_2 \mathbf{r}^{\frac{1-p}{p}} \right)^{\frac{1}{2}} \right] \quad (38)$$

By taking the upper bound of the radius $\mathbf{r}$ which is infinity(∞), we find

$$M = -p\xi_2 \quad (39)$$

in this case, $\xi_2 = -\frac{M}{p}$. Here we set the mass M to be positive ($M > 0$), so $\xi_2 < 0$.

6.2 Model.2

$$H(r) = \frac{\beta}{r^{p-1}} \quad (40)$$

Putting this function in equation (22) yields

$$\Phi(r) = \pm \left(\sqrt{p(p-1)} \left(\frac{n}{2} - 1 \right) \ln(r) \right) + \Phi_0 \quad p \geq 1 \quad (41)$$

Also, β , Φ_0 and p are constant. When we put this ansatz in equation (23), one can find

$$f(r) = \eta_1 r^{2-2p} + \eta_2 r^{3-n-(2-n)p} \quad (42)$$

η_1 and η_2 are integration constants. The metric for this model ($H(r) = \frac{\beta}{r^{p-1}}$) is

$$ds^2 = -(\eta_1 r^{2-2p} + \eta_2 r^{3-n-(2-n)p}) dt^2 + \frac{dr^2}{\eta_1 r^{2-2p} + \eta_2 r^{3-n-(2-n)p}} + \frac{\beta^2}{r^{2(p-1)}} \sum_{i=1}^{n-2} d\phi_i^2 \quad (43)$$

Just like the previous section, By some manipulations the above metric(42) can be simplified to the BTZ metric. In this case, we just put $p=0$, $\beta=1$, and $n=3$ and one can reach

$$ds^2 = -(\eta_1 r^2 + \eta_2) dt^2 + \frac{dr^2}{\eta_1 r^2 + \eta_2} + r^2 d\phi^2 \quad (44)$$

As we said before in the previous section, here η_1 and η_2 are also related to the cosmological constant and mass, respectively.

As an exact solution, the potential is give by the following relation which is also the Liouville-type potentials like the previous section.

$$V(\Phi) = \eta_1(p-1)(np-n+1)(n-2)e^{-4\sqrt{\frac{p}{p-1}}\left(\frac{\Phi-\Phi_0}{n-2}\right)} \quad (45)$$

For the BTZ case($p=0$ $n=3$) the potential is $V(\Phi) = 2\eta_1$.

Like the previous section, by this coordinate transformation $\frac{\beta^2}{r^{2(p-1)}} \rightarrow \mathbf{r}^2$, we can generate the asymptotically (anti)de Sitter solutions in the case of $p=0$. As is clear without this coordinate transformation this metric (42) reduces to the asymptotically (anti)de Sitter case when $p=0$, but this coordinate transformation is needed for finding the quasilocal mass quantity.

6.3 Model.3

In this section we propose a relation between $H(r)$ and $\Phi(r)$ through an exponential term.

$$H(r) = e^{\Phi(r)} \quad (46)$$

From Eq.(22) we have

$$\Phi(r) = \frac{\ln(N(\gamma_1 r + \gamma_2))}{N} \quad (47)$$

N is $\frac{n(n-4)}{(n-2)^2}$ ($n \neq 4$) in this equation and γ_1, γ_2 are constant. So $H(r)$ is

$$H(r) = (N(\gamma_1 r + \gamma_2))^{\frac{1}{N}} \quad (48)$$

By putting this relation in equation (23), one can obtain

$$f(r) = \epsilon_1 \left(r + \frac{\gamma_2}{\gamma_1}\right)^{\frac{2}{N}} + \epsilon_2 \left(r + \frac{\gamma_2}{\gamma_1}\right)^{\frac{2+N-n}{N}} \quad (49)$$

ϵ_1 and ϵ_2 are integration constants. Without the lose of generality, we assume $\gamma_1 = 1$. In this model, the metric is:

$$ds^2 = -(\epsilon_1(r+\gamma_2)^{\frac{2}{N}} + \epsilon_2(r+\gamma_2)^{\frac{2+N-n}{N}})dt^2 + \frac{dr^2}{\epsilon_1(r+\gamma_2)^{\frac{2}{N}} + \epsilon_2(r+\gamma_2)^{\frac{2+N-n}{N}}} + (N(r+\gamma_2))^{\frac{2}{N}} \sum_{i=1}^{n-2} d\phi_i^2 \quad (50)$$

To find an exact solution, it is required to obtain an explicit function for $V(\Phi)$ which the equations (18), (20) and (21) work together and everything is consistent. So, by some algebraic calculations and manipulations, we could reach this relation for $V(\Phi)$

$$V(\Phi) = \epsilon_1 \frac{(n-N)(n-2)}{N^2} \left(\frac{e^{N\Phi}}{N} \right)^{\frac{2}{N}-2} \quad (51)$$

There is a point in the metric relation Eq.(49). The γ_2 can be absorbed in a coordinte transformation, because this parameter is not a curvature singularity and is just a coordinate singularity. By proposing this transformation $\mathbf{r} = r + \gamma_2$, this unphysical singularity will disappear and the metric would be

$$ds^2 = -(\epsilon_1 \mathbf{r}^{\frac{2}{N}} + \epsilon_2 \mathbf{r}^{\frac{2+N-n}{N}})dt^2 + \frac{d\mathbf{r}^2}{\epsilon_1 \mathbf{r}^{\frac{2}{N}} + \epsilon_2 \mathbf{r}^{\frac{2+N-n}{N}}} + (N\mathbf{r})^{\frac{2}{N}} \sum_{i=1}^{n-2} d\phi_i^2 \quad (52)$$

If someone check the Ricci and Kretschmann scalars from section 4, it would be clear that this metric has a curvature singularity at $\mathbf{r} = 0$ and the event horizon radius is at $\mathbf{r} = \left(-\frac{\epsilon_1}{\epsilon_2}\right)^{\frac{N}{N-n}}$.

By this special coordinate transformation $(N\mathbf{r})^{\frac{2}{N}} \rightarrow X^2$, one can find out that this solution has unusual asymptotics which is not asymptotically flat or (anti)de Sitter.

7 Thermodynamics

In this important section, we want to calculate and investigate the thermodynamic quantities in details. We also plot diagrams for these quantities to see their properties more clear, schematically.

7.1 Temperature and Entropy(S)

From the proposing formula for balak holes temperature via Hawking and Bekenstein, we can calculate this parameter for the previous black hole solutions. The Hawking-Bekenstein temperature formula is

$$T = \frac{|g'_{tt}(r)|}{4\pi} \sqrt{-g^{tt}(r)g^{rr}(r)}|_{r=r_+} \quad (53)$$

This formula means we first take the derivative of tt metric component with respect to r and also find the inverse of tt and rr metric components and finally put the r_+ (The outer event horizon) instead of r.

In this moment, we want to calculate the Hawking-Bekenstein temperature for the first model($H(r) = \alpha r^p$) in section (6). From Eq.(30), we have

$$T = \frac{2\xi_1 p r_+^{2p-1} + \xi_2(1 + (2-n)p)r_+^{(2-n)p}}{4\pi} \quad (54)$$

In three dimensions($n=3$), the temperature is

$$T = \frac{2\xi_1 p \left(\frac{-\xi_1}{\xi_2}\right)^{-\frac{2p-1}{3p-1}} + \xi_2(1-p) \left(\frac{-\xi_1}{\xi_2}\right)^{\frac{p}{3p-1}}}{4\pi} \quad (55)$$

In the case of BTZ solution, one can just put $p=1$

$$T = \frac{\xi_1}{2\pi \sqrt{-\frac{\xi_1}{\xi_2}}} \quad (56)$$

From equation (54) and interpretation of ξ_1 and ξ_2 in subsection (6.1) as the cosmological constant and mass, it would be interesting to plot the Hawking-Bekenstein temperature versus the event horizon. For this purpose, in Figures 3 and 4 we have showed the effect of "n" and "p" on the temperature, schematically. In Fig.3, when the event horizon get bigger and bigger, the temperature is the same in different dimensions. The only difference can be seen for the small values of the event horizon. In Fig.4, the blue diagram is for the $p=1$ case which is the trivial constant scalar hair(This is the BTZ case). The green and red diagrams are the nontrivial cases where the effect of dilaton field is clear on the temperature. The temperature value is high near the origin and grows fast far from the origin.

Figure 3: In this figure, we want to see the diagrams of the temperature in different dimensions versus the event horizon. The blue one is for these values $\alpha = 1, p=1, \xi_1 = 1, \xi_2 = -1$ and $n=3$. The green is for $\alpha = 1, p=1, \xi_1 = 1, \xi_2 = -1$ and $n=4$ and the red is for $\alpha = 1, p=1, \xi_1 = 1, \xi_2 = -1$ and $n=5$.

Figure 4: In this figure, the purpose is seeing the effect of our model on the temperature in three dimensions. The blue one is for these values $\alpha = 1, p=1, \xi_1 = 1, \xi_2 = -1$ and $n=3$. The green is for $\alpha = 1, p=2, \xi_1 = 1, \xi_2 = -1$ and $n=3$ and the red is for $\alpha = 1, p=3, \xi_1 = 1, \xi_2 = -1$ and $n=3$.

For the n -dimensional space-time, entropy(S) has a general form and is needed some details. The line element for the $(n-2)$ ⁶ dimensional horizon has the following form and obviously is the end part of metric ansatz (17)

$$d\sigma^{n-2} = H(r)^2 \sum_{i=1}^{n-2} d\phi_i^2 \quad (57)$$

In order to find the area of black hole horizon for the obtaining metric (31), one can calculate and reach the following relation for the event horizon area

$$A_+ = \prod_{i=1}^{n-2} \int_0^{2\pi} d\phi_i \sqrt{\gamma} = (2\pi\alpha r_+^p)^{n-2} \quad (58)$$

In this relation γ is the determinant of this line element. The entropy(S) of black hole can be calculated due to the Wald formula [54]. After some algebraic calculations one can reach

$$S = 2(2\pi\alpha r_+^p)^{n-2} \quad (59)$$

Now we want to introduce the entropy(S) diagrams for different values of " n " and " p " versus the event horizon. In Figures 5 and 6, the red diagram is for the trivial constant scalar hair case($p=1$). The other diagrams(blue and green) which have the effect of scalar field and higher dimensions in themselves, experience the fast growth far from the origin.

⁶In this paper we set $n=d+1$. n is the total dimension, not just the space part.

Figure 5: In this figure, we want to see the diagrams of the entropy in different dimensions. The red one is for these values $\alpha = 1, p=1$ and $n=3$. The blue is for $\alpha = 1, p=1$ and $n=4$ and the green is for $\alpha = 1, p=1$ and $n=5$.

Figure 6: In this figure, the purpose is seeing the effect of our model on the entropy in three dimensions. The red one is for these values $\alpha = 1, p=1$ and $n=3$. The blue is for $\alpha = 1, p=2$ and $n=3$ and the green is for $\alpha = 1, p=3$ and $n=3$.

By putting the event horizon radius in this relation from sub-section (6.1), we have

$$S = 2 \left(\frac{2\pi\alpha}{\left(\frac{-\xi_1}{\xi_2}\right)^{\frac{p}{np-1}}} \right)^{n-2} \quad (60)$$

In this paragraph we want to compare the entropy(S) of BTZ black hole with our solutions, graphically. These solutions are obtained from our proposing models. First, one can just put $\alpha=1, n=3$ (spacetime dimension) and $p=1$. By this values the BTZ entropy(S) will extract⁷. Here, we just set $\alpha = 1$, $n=3$ and let "p" to be a variable. so we have

$$S = 2 \left(\frac{2\pi}{\left(\frac{-\xi_1}{\xi_2}\right)^{\frac{p}{3p-1}}} \right) \quad (61)$$

In this relation we set $\xi_1=1$ (cosmological parameter) and let ξ_2 (mass parameter) to have different values. In this case, we want to see the effect of "p" as a variable and the ratio of $\frac{-\xi_1}{\xi_2}$ on entropy, graphically. If this ratio is "1", the entropy(S) is always " 4π " and does not depend on the "p" parameter. So, the cases $\frac{-\xi_1}{\xi_2} > 1$ and $\frac{-\xi_1}{\xi_2} < 1$ will be interesting. In the Fig.7, the red diagram is for the $\frac{-\xi_1}{\xi_2} > 1$ case and the green diagram is for the $\frac{-\xi_1}{\xi_2} < 1$ case.

⁷we note that in the BTZ entropy relation, ξ_1 and ξ_2 have been interpreted as cosmological constant and mass in the sub-section (6.1) © The Author(s) or their Institution(s)

Figure 7: In this diagram, p=1 is the BTZ case.

7.2 Smarr formula

In this section, it is worthwhile to introduce the Smarr formula for these types of black holes solutions [56] [57] [58]. In this case, we combine the Eqs.(38,54 ,59) and use the relation $r = \frac{1}{(\frac{-\xi_1}{\xi_2})^{\frac{1}{np-1}}}$ for the event horizon radius. After some calculations we reach this relation:

$$M = \frac{1}{(2\pi)^{n-3}} \frac{p}{np-1} TS \quad (62)$$

For the BTZ case(n=3, p=1), the Smarr formula reduces to this relation(for more details one can check [59]):

$$M = \frac{1}{2} TS \quad (63)$$

7.3 Heat capacity and stability condition

In order to investigate the stability of our solutions, one can find the heat capacity for these solutions and talk about their stability and even plot their diagrams which is more convenient in this case. The positive heat capacity will indicate that the black hole is thermodynamically stable. On the other hand, the negative heat capacity indicates that the black hole is not thermodynamically stable. Finally, when the heat capacity experiences a critical situation and blows up, it can be a signal where the black hole may go under phase transition.

We can calculate the thermal heat capacity from this relation

$$C_Q = T_+ \left(\frac{\partial S_+}{\partial T_+} \right) \quad (64)$$

Also, one can write the heat capacity relation in another useful way. As we have obtained the Temperature and Entropy(S) for different models, it would be simple to calculate the heat capacity from the following relation [60]

$$C_Q = T \left(\frac{\partial S}{\partial r_+} \right)_Q \left(\frac{\partial T}{\partial r_+} \right)_Q^{-1} = T \left(\frac{\partial S}{\partial T} \right)_Q = T \left(\frac{\partial^2 M}{\partial S^2} \right)_Q^{-1} \quad (65)$$

From the equations (54) and (59), one can compute the heat capacity for the model.1 as

$$C_Q = - \frac{2(n-2)(2\pi\alpha r_+^p)^{n-2} \left((np-2p-1)\xi_2 r_+^{1+(2-n)p} - 2p\xi_1 r_+^{2p} \right)}{(n-2)(np-2p-1)\xi_2 r_+^{1+(2-n)p} + 4(p-\frac{1}{2})\xi_1 r_+^{2p}} \quad (66)$$

It is interesting to check the effect of dimension(n) and our proposing models which is clear as "p" parameter on the heat capacity diagrams. This purpose can be seen in Figures 8 and 9.

Figure 8: In this figure, we want to see the effect of the scalar field on the diagrams of the heat capacity. The left diagrams are for these values $\xi_1 = 1$, $\xi_2 = -1$ and $n=3$ (BTZ case). The right diagrams are for $\xi_1 = 1$, $\xi_2 = -1$ and $n=4$.

Figure 9: In this figure, we want to see the effect of the scalar field on the diagrams of the heat capacity. The left diagrams are for these values $\xi_1 = 1$, $\xi_2 = -1$ and $n=3$ (BTZ case). The right diagrams are for $\xi_1 = 1$, $\xi_2 = -1$ and $n=4$.

Figure 10: In this figure, the purpose is seeing the effect of our model on the heat capacity in different dimensions. The left diagrams are for these values $p=1$, $\xi_1 = 1$, $\xi_2 = -1$. The right diagrams are for $p=2$, $\xi_1 = 1$, $\xi_2 = -1$.

Figure 11: In this figure, the purpose is seeing the effect of our model on the heat capacity in different dimensions. The left diagrams are for these values $p=1$, $\xi_1 = 1$, $\xi_2 = -1$. The right diagrams are for $p=2$, $\xi_1 = 1$, $\xi_2 = -1$.

In Fig.7, we want to see the effect of the scalar field on the local stability of the black hole solutions. As is clear in Eq.(29), $p=1$ is the no scalar field case($p=1 \rightarrow \Phi = 0$). When $p>1$, we can say that the effect of this field is considered. In the left diagrams for $n=3$ (BTZ case), it is obvious that the presence of the scalar field makes the black holes to be unstable near the origin. On the other hand, for higher dimensions($n>3$) the presence of this field makes the black holes to be more stable near the origin. In the next section, we will show that for the charged solutions the presence of the scalar field makes the BHs to be more stable near the origin. In general, the presence of scalar field is an important factor in black hole's stability investigations.

8 Interior solution

In this section, we want to find the exact interior solution in the Einstein gravity when its exterior counterpart is the charged solution in the Einstein-Maxwell gravity coupled to a dilaton field.

The general line element for toroidal space-time in n -dimensions can be written as follows:

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + \alpha^2 r^{2m} \sum_{i=1}^{n-2} d\phi_i^2 \quad (67)$$

$g(r)$ is an unknown function which is our goal to obtain and α, m are just two constants. We take an anisotropic matter distribution in nature with this energy momentum tensor represented by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \quad (68)$$

Where ρ is the energy density, p is the pressure and $(n+1)$ -velocity is as

$$u_\mu = -e^{A(r)}(1, 0, 0, 0, \dots, 0) \quad (69)$$

The exact form of the energy-momentum tensor components is $T^\mu_\nu = (-\rho, P_r, P_t, \dots, P_t)$, where P_r and P_t are the radial and tangential pressure for considering matter distribution, respectively. The trace of energy-momentum tensor is $tr(T_{\mu\nu}) = -\rho + P_r + (n-1)P_t$.

By solving the Einstein field equations(using the above line element)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\pi T_{\mu\nu} \quad (70)$$

in the presence of the negative cosmological constant for this matter distribution, we find the following equations:

$$\frac{m(n-2)}{e^{2B}} \left(\frac{B'}{r} - \frac{(n-1)m-2}{2r^2} \right) = \pi\rho + \Lambda \quad (71)$$

$$\frac{m(n-2)}{e^{2B}} \left(\frac{A'}{r} + \frac{(n-3)m}{2r^2} \right) = \pi P_r - \Lambda \quad (72)$$

$$\frac{r^2 A'' + r^2 A'^2 - r^2 A' B' + (n-3)mr(A' - B') + \frac{(n-2)(n-3)}{2}m^2 - (n-3)m}{r^2 e^{2B}} = \pi P_t - \Lambda \quad (73)$$

In all equations, prime denotes as the derivative with respect to r . The energy density is obtained from Eq.(71) as:

$$\rho = \frac{1}{\pi} \left[\frac{m(n-2)}{e^{2B}} \left(\frac{B'}{r} - \frac{(n-1)m-2}{2r^2} \right) - \Lambda \right] \quad (74)$$

And the radial pressure from Eq.(72) as:

$$P_r = \frac{1}{\pi} \left[\frac{m(n-2)}{e^{2B}} \left(\frac{A'}{r} + \frac{(n-3)m}{2r^2} \right) + \Lambda \right] \quad (75)$$

Also the tangential pressure from Eq.(73) as:

$$P_t = \frac{1}{\pi} \left[\frac{r^2 A'' + r^2 A'^2 - r^2 A' B' + (n-3)mr(A' - B') + \frac{(n-2)(n-3)}{2}m^2 - (n-3)m}{r^2 e^{2B}} + \Lambda \right] \quad (76)$$

Now, we consider a linear equation of state(EOS):

$$P_r = \alpha_1 \rho + \alpha_2 \quad (77)$$

Where α_1 and α_2 are just two constant parameters.

The mass parameter within a radius r is defined as:

$$M(r) = \int_0^r \rho dV_{n-1} \quad (78)$$

Where

$$dV_{n-1} = (2\pi)^{n-2} r^{n-2} dr \quad (79)$$

To ensure that the mass and energy density functions are regular at $r = 0$, we assume $2B(r) = ((n-1)m-2)\ln(r) + a\ln(r^{-m})$, where a is a positive constant. Using this assuming function, the energy density is as:

$$\rho = \frac{1}{\pi} \left[\frac{-am^2(n-2)r^{m(a-n+1)}}{2} - \Lambda \right] \quad (80)$$

Using this equation for the energy density and Eqs.(75,77), the unknown function $A(r)$ can be obtained as

$$A(r) = \frac{(a\alpha_1 + n - 3)\ln(\frac{1}{r^{m(a-n+1)}})}{2(a - n + 1)} + \frac{2(1 + \alpha_1)\Lambda - 2\pi\alpha_2}{2m^2(n - 2)(a - n + 1)} + C \quad (81)$$

Where C is just an integration constant.

The radial pressure is as

$$P_r = \frac{\alpha_1}{\pi} \left[\frac{-am^2(n - 2)r^{m(a-n+1)}}{2} - \Lambda \right] + \alpha_2 \quad (82)$$

And from the Eq.(76), we have

$$P_t = \frac{1}{4\pi} \left[a \left(a(\alpha_1^2 - \alpha_1) + (n - 1)\alpha_1 + n - 3 \right) p^2 r^{p(a-n+1)} + 4\Lambda \right] \quad (83)$$

In past few years, there have been different arguments that considered for the existence of the anisotropy of star models. This characteristic may be due to the presence of the solid core, phase transitions, two-fluid mixtures, slow rotation, etc [66].

The measure of anisotropy can be investigated through the following relation:

$$\Delta = P_t - P_r \quad (84)$$

This anisotropy brings a force which is directed outward when $\Delta > 0$ and inward when $\Delta < 0$. Also, $\Delta = 0$ corresponds to the special case of an isotropic pressure of the charged fluid stellar configuration.

Figure 12: Δ versus $r(km)$ when Λ changes. We have assumed $a=4$, $m=2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$ and $n=4$.

Also the mass function has the following form

$$M(r) = (2\pi)^{n-2} \left[-\frac{\Lambda r^{n-1}}{\pi(n-1)} - \frac{am^2(n-2)r^{(a-n+1)m+n-1}}{2\pi((a-n+1)m+n-1)} \right] \quad (85)$$

This mass equation shows that $M(r) = 0$ at $r = 0$ which ensures the regular behaviour of the mass function at the centre and of course when $a > n - 1$ (in this equation the cosmological constant is a negative quantity).

8.1 Compactness and redshift

The compactness of the stellar configuration can be obtained as follow:

$$u = \frac{M(r)}{r} \Big|_{r=R} = (2\pi)^{n-2} \left[-\frac{\Lambda R^{n-2}}{\pi(n-1)} - \frac{am^2(n-2)R^{(a-n+1)m+n-2}}{2\pi((a-n+1)m+n-1)} \right] \quad (86)$$

And the surface redshift Z_s can be obtained from this compactness as

$$Z_s = (1 - 2u)^{-\frac{1}{2}} - 1 \quad (87)$$

For anisotropic spheres in (3+1) dimensional gravity the surface redshift is may be larger than 2. The value 2 is found by Buchdahl [67]. For some fix values we obtain this value for the surface redshift as 2.00616. The fix values are $a=n=4$, $m=2$, $\Lambda = -55.4118$ and the stellar configuration with radius $R=2.4$ km.

Figure 13: u versus $r(km)$ when Λ changes. We have assumed $a=4$, $m=2$ and $n=4$.

8.2 Matching condition

We assume that the exterior space-time of our interior metric is described by the first model Eq.(31):

$$ds^2 = -(\xi_1 r^{2p} + \xi_2 r^{1+(2-n)p})dt^2 + \frac{dr^2}{\xi_1 r^{2p} + \xi_2 r^{1+(2-n)p}} + \alpha^2 r^{2p} \sum_{i=1}^{n-2} d\phi_i^2 \quad (88)$$

Where at the boundary $r=R$, continuity of the metric yields the following junction conditions:

$$e^{2A(R)} = \xi_1 R^{2p} + \xi_2 R^{1+(2-n)p} \quad (89)$$

$$e^{-2B(R)} = \xi_1 R^{2p} + \xi_2 R^{1+(2-n)p} \quad (90)$$

$$R^{2m} = R^{2p} \quad (91)$$

Moreover, the radial pressure should be vanished at the boundary, i.e., $P_r(r=R) = 0$. These conditions can be used to fix the constants C , m .

$$e^{2C} = \frac{\xi_1 R^{2p} + \xi_2 R^{1+(2-n)p}}{R^{m(\alpha_2+n-3)} e^{\frac{2(1+\alpha_2)\Lambda+2\pi\alpha_2}{m^2(n-2)(a-n+1)}}} \quad (92)$$

Also, from the third condition $m=p$. Finally, from the pressure condition we have:

$$R = \left(\frac{2(\alpha_1 \Lambda - \pi \alpha_2)}{a \alpha_1 m^2 (n-2)} \right)^{\frac{1}{m(a-n+1)}} \quad (93)$$

8.3 Figures and physical analysis

Figure 14: Energy density ρ versus $r(\text{km})$. Also $\Lambda = -10$, $a=4$ and $n=4$.

As is clear, the energy density is a decreasing functions of r .

$$\frac{d\rho}{dr} = \frac{-am^3(a-n+1)(n-2)r^{m(a-n+1)-1}}{2\pi} < 0 \quad (94)$$

This is one important condition for a physically acceptable model.

In Fig.14, one can see the effect of the p parameter on the energy density. When p increases, the energy density vanishes in a shorter radius.

Figure 15: Radial pressure P_r versus $r(\text{km})$. Also $\Lambda = -10$, $a=4$ and $n=4$ $\alpha_1 = 0.1$ and $\alpha_2 = 0.2$.

Pressure is also a decreasing functions of r .

$$\frac{dP_r}{dr} = \frac{-a\alpha_1 m^3(n-2)(a-n+1)r^{m(a-n+1)-1}}{2\pi} < 0 \quad (95)$$

This is one of the pillars for a physically acceptable model as well.

In the two following figures, we have showed the effect of p and n parameters on the mass behaviour. In the two figures, there is a special radius where M is maximum. In Fig.16, this

special radius decreases when p parameter increases and in Fig.17 decreases when dimensional parameter n increases, too.

We can also show that $\frac{d^2\rho}{dr^2} < 0$ and $\frac{d^2P_r}{dr^2} < 0$ which indicate the maximality of central density and pressure.

Figure 16: Mass M (solar mass) versus r (km). We have assumed $\Lambda = -5$, $a=4$ and $n=4$.

Figure 17: Mass M (solar mass) versus r (km). Also we have considered $\Lambda = -10$, $a=7$ and $p=2$.

The Ricci scalar for the interior solution can be obtained as the following form:

$$R_{Ricci-scalar} = \frac{a(a\alpha_1^2 + (-a + n - 3)\alpha_1 + n - 1)}{2} p^2 r^{p(a-n+1)} \quad (96)$$

One can check that the Ricci scalar is regular at the origin $r=0$ and well behaved in the stellar interior.

9 Concluding remarks

In this work, we obtained a new class of black hole solutions with toroidal horizons in dilaton gravity where cosmological constant has been emerged from these solutions naturally. For finding these solutions, we proposed three functional models for $H(r)$. After that, we obtained all unknown functions in the field equations which make these solutions to be the exact solutions in the dilaton gravity. We also showed that these solutions are not asymptotically (anti-)de Sitter or flat (The asymptotically (anti)de Sitter case obtained in one special case for first model when $p=1$). Even though, we investigated the thermodynamical properties of these solutions. one important point was that the temperature was the same in different dimensions

when the event horizon got bigger and bigger. Although, we have showed that the presence of the dilaton field makes the black holes to be more stable near the origin except for the BTZ case. We can say in general that the presence of dilaton field is an important factor in black hole's stability investigations. For these solutions, we obtained a new version of the Smarr formula which is interesting and in special case of $p=1$ and $n=3$, reduces to the BTZ case. In the P-V criticality analysis, we showed that the EOS does not admit any inflection point and does not admit critical behaviour. Finally, we obtain the interior solution in n dimensions when the corresponding exterior space-time is the solution in dilaton gravity. We also show that the solution which is obtained in linear case is regular and well-behaved at the stellar interior. Also, we showed that some physical conditions for the energy density and pressure are valid.

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