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The bosonic and fermionic propagators under an external electromagnetic field in a Euclidean manifold

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In this paper, we will calculate the bosonic as well as fermionic propagators under classical homogeneous and constant magnetic and electric fields in a Euclidean space. For this, we will reassess the Ritus' method for calculating the Feynman propagator.

Keywords: Feynman propagator; Ritus' method; Electromagnetic field.

I. INTRODUCTION

It is well known that the Feynman propagator is an amount very important in Quantum Field Theory (QFT) and that, therefore it must be calculated for the most diverse situations in which quantum fields are subject, for example in periodic boundary conditions, antiperiodic boundary conditions and in external fields.

Taking into account external electromagnetic fields, the propagator can be calculated using the technique developed by J. Schwinger in 1951, the proper time method [1]. The proper time method is a powerful way to calculate the propagator under external fields, but it uses one dimension additional to get the propagator: the proper time S in the range $[0, +\infty]$. Thus, the expressions to the propagators are given by integrals on proper time S . These integrals can be very difficult to solve [1, 2]. In the decade of 1970, V. Ritus calculated the fermionic field propagator under an external electromagnetic field of an innovative way [3–5], namely: by the diagonalization of the Dirac operator. In other words, the method consists in to find

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eigenfunctions of Dirac's operator such that the propagator is written as in free form. That is, it is found a kind of Fourier transform for the operator $\not{D}$, being D_μ the covariant derivative. A few years ago, the Ritus' method also has been used to calculate the propagator of a charged particle of spin 1 in the context of electroweak theory [6, 7] and of spin 1/2 in low dimensions [8]. Here, we will reassess the Ritus' method for calculating Feynman propagator under external, constant and homogeneous electric and magnetic fields in the z direction.

The paper is organized as follows: in section II, we present the Ritus' method. As we want to be didactic, let us divide it into four subsections. In subsection **A** we calculate the free scalar propagator using the eigenfunctions of ∂_μ operator: the plane waves. In the subsection **B** we calculate the scalar propagator under an external magnetic and electric classical fields along to z direction. Let us calculate the eigenfunctions of operator D^2 and show that they form a complete set of solutions. From this set, we will be able to expand the propagator in momenta space. In subsection **C**, we calculate the Feynman propagator of the free spinor field. Again, to be pedagogical, we uses the Ritus' method in the $\not{D}$ operator context, i.e., we write the free fermionic propagator by expansion over plane waves. In subsection **D**, we calculate the spinor propagator field under an electric and magnetic external background fields in the z direction, through the eigenfunctions associated to $\not{D}$ operator. We will make our final considerations in section III. Throughout the text, we will use a four-dimensional Euclidean space and the natural system of units in which $c = \hbar = 1$. Our definitions agree with Ref. [9].

II. RITUS' METHOD

The Ritus' method can be describe in few words as follows. The field propagator satisfy an differential equation. This differential equation has a specific operator. We solved the eigenvalue equation for this specific operator. Then, we expand the propagator in terms of operator's eigenfunctions.

To get an general view of Ritus' method, let us define the operator $\mathcal{O}$ that satisfies the field equation

$$\mathcal{O} \varphi = 0,$$

and the respective propagator that satisfy

$$\mathcal{O} \Delta(u, u') = \pm \delta^4(u - u').$$

Then, we solve the eigenvalue equation

$$\mathcal{O} f_p(u) = \lambda f_p(u).$$

If the eigenfunctions $f_p(u)$ form a complete set, we have

$$\int dp f_p(u) f_p^*(u) = \delta^4(u - u').$$

Thus, the propagator can be expand in terms of $f_p(u)$ in the momenta space:

$$\Delta(u, u') = \int dp f_p(u) \tilde{\Delta}(p) f_p^*(u'),$$

being $\tilde{\Delta}(p) = \pm 1/\lambda$.

Depending on the background involved, finding the eigenfunctions of the $\mathcal{O}$ operator can be a difficult task. But, notice that if we have an operator $\mathcal{O}'$ such that

$$[\mathcal{O}, \mathcal{O}'] = 0,$$

then the eigenfuctions $f_p(u)$ will be also eigenfunctions of $\mathcal{O}'$. Therefore, we can work with $\mathcal{O}'$ operator instead $\mathcal{O}$.

To facilitate understanding, let us show how the Ritus' method works in four cases: free scalar field; scalar field under external fields $\mathbf{E}$ and $\mathbf{B}$; free spinor field; and spinor field under external fields $\mathbf{E}$ and $\mathbf{B}$.

A. Free scalar field propagator

The free Klein-Gordon equation in four-dimensional Euclidean space is given by

$$(-\partial_\mu \partial_\mu + m^2) \phi(u) = 0,$$

where $\partial_\mu = (\partial_\tau, \partial_x, \partial_y, \partial_z)$ and $u = (\tau, x, y, z)$. The Feynman propagator associated to free scalar field satisfy [9]

$$(\partial_\mu \partial_\mu - m^2) G(u - u') = -\delta^4(u - u'). \quad (1)$$

Note that the plane wave in Euclidean space, $\exp(ik_\alpha u_\alpha)$, are eigenfunction of ∂_ν with ik_ν eigenvalue. Furthermore, we have $[\partial_\mu \partial_\mu, \partial_\nu] = 0$. For this, plane waves are also eigenfunctions of $\partial_\mu \partial_\mu$ operator, with $-(k_\mu)^2$ eigenvalue. Since plane waves form a complete set of functions, we can write

$$G(u - u') = \int \frac{d^4 k}{(2\pi)^4} [\exp(ik_\mu u_\mu)] g(k) \times [\exp(ik_\nu u'_\nu)]^*, \quad (2)$$

being $g(k)$ the propagator in the momenta space. It is found through application $(\partial_\mu \partial_\mu - m^2)$ at Eq. (2). After this, we use Eq. (1) and take into account that

$$\delta^4(u - u') = \int \frac{d^4 k}{(2\pi)^4} [\exp(ik_\mu u_\mu)] [\exp(ik_\nu u'_\nu)]^*. \quad (3)$$

The result is

$$g(k) = \frac{1}{k^2 + m^2}, \quad (4)$$

where $k^2 = k_\tau^2 + k_x^2 + k_y^2 + k_z^2$. Now, we can go back in Eq. (2) and replace $g(k)$ by Eq. (4).

B. Scalar field propagator under an electromagnetic external field

Now let us calculate the bosonic propagator under an electric external field, $\mathbf{E} = E_0 \hat{z}$ and a magnetic external field, $\mathbf{B} = B_0 \hat{z}$. For this, we will use the minimal coupling in the Euclidean space [10]: $\partial_\mu \rightarrow \partial_\mu + ieA_\mu^{ext}$. This is the covariant derivative in the Euclidean space, D_μ . We choose the gauge $A_\mu^{ext} = (-zE_0, 0, xB_0, 0)$. The propagator, in this case, satisfy the equation

$$(D_\mu D_\mu - m^2) G(u, u', A) = -\delta^4(u - u'). \quad (5)$$

Notice that $[D_\mu^2, \partial_\nu] \neq 0$. Therefore, the planes waves are not eigenfuntions of D_μ^2 operator.

Let $E_p(u)$ the eigenfunctions of operator D_μ^2 . If we get the complete set formed by $E_p(u)$, then we will can write

$$G(u, u', A) = \int dp E_p(u) \mathcal{G}(p) E_p^*(u'). \quad (6)$$

To find $\mathcal{G}(p)$, we have to apply the operator $(D_\mu^2 - m^2)$ over Eq. (6), to use the Eq. (5) and write the delta function in terms of $E_p(u)$.

Therefore, let us solve the eigenvalue equation

$$D_\mu^2 E_p = p^2 E_p. \quad (7)$$

According to our gauge, the operator D_μ^2 becomes

$$\begin{aligned} D_\mu^2 = & \partial_\tau^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 - 2i\omega_E z \partial_\tau \\ & + 2i\omega_B x \partial_y - \omega_E^2 z^2 - \omega_B^2 x^2. \end{aligned} \quad (8)$$

being $\omega_E \equiv eE_0$ and $\omega_B \equiv eB_0$.

From Eq. (8), we notice that variables (τ, z) and (x, y) are coupling. Thus, to solve Eq. (7), we make the ansatz [10–12],

$$E_p(u) = C \exp [i(\omega_E a_\tau \tau + \omega_B a_y y)] X(x) Z(z), \quad (9)$$

where C is a normalization constant.

Replacing the ansatz on the equation (7), we get

$$\begin{aligned} \left(\frac{Z''(z)}{Z} - \omega_E^2 (z - a_\tau)^2 \right) = \\ - \left(\frac{X''(x)}{X} - \omega_B^2 (x + a_y)^2 - p^2 \right) \equiv -const. \end{aligned} \quad (10)$$

To obtain finite solutions in z variable, we choose $const = \omega_E(2n + 1)$, $n = 0, 1, 2, \dots$. In x variable, the choose that produces finite solutions is $-p^2 - const = \omega_B(2\ell + 1)$, $\ell = 0, 1, 2, \dots$. Thus,

$$p^2 = -[\omega_E(2n + 1) + \omega_B(2\ell + 1)], \quad n, \ell = 0, 1, 2, \dots \quad (11)$$

The Eq. (11) gives the finite solutions in terms of Hermite polynomials, namely

$$\begin{aligned} X_\ell(x) = \frac{1}{\sqrt{2^\ell \ell!}} \left(\frac{\omega_B}{\pi} \right)^{\frac{1}{4}} \exp[-\omega_B(x + a_y)^2/2] \\ \times H_\ell[\sqrt{\omega_B}(x + a_y)], \end{aligned} \quad (12)$$

and

$$\begin{aligned} Z_n(z) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega_E}{\pi} \right)^{\frac{1}{4}} \exp[-\omega_E(z - a_\tau)^2/2] \\ \times H_n[\sqrt{\omega_E}(z - a_\tau)]. \end{aligned} \quad (13)$$

The Eqs. (12) and (13) can be written in terms of the Hermite functions [13]

$$h_m(s) \equiv \frac{1}{\sqrt{2^m m!} \sqrt{\pi}} \exp\left(-\frac{s^2}{2}\right) H_m(s).$$

These functions form a complete set, i.e.,

$$\int_{-\infty}^{+\infty} ds \, h_\ell(s) h_n(s) = \delta_{\ell,n} \quad (14)$$

and

$$\sum_{n=0}^{+\infty} h_n(s) h_n(s') = \delta(s - s'). \quad (15)$$

Therefore, we get

$$\begin{aligned} E_p(u) = (\sqrt[4]{\omega_E \omega_B} / 2\pi) \exp[i(\omega_E a_\tau \tau + \omega_B a_y y)] \\ \times h_\ell[\sqrt{\omega_B}(x + a_y)] h_n[\sqrt{\omega_E}(z - a_\tau)]. \end{aligned} \quad (16)$$

From Eqs. (16) and (15) we can show the completeness relation

$$\begin{aligned} \int dp \, E_p(u) E_p^*(u') &= \sum_{n=0}^{+\infty} \sum_{\ell=0}^{+\infty} \int \omega_E \omega_B da_\tau da_y E_p(u) E_p^*(u') \\ &= \delta^4(u - u'). \end{aligned} \quad (17)$$

and the orthogonality

$$\begin{aligned} \int d^4u E_p(u) E_{p'}^*(u) &= \delta[\omega_E(a_\tau - a'_\tau)] \delta[\omega_B(a_y - a'_y)] \\ &\times \delta_{n,n'} \delta_{\ell,\ell'}. \end{aligned} \quad (18)$$

Thus, the scalar field propagator under external electromagnetic field is given in terms of the Ritus' eigenfunctions, $E_p(u)$:

$$G(u, u', A) = \omega_E \omega_B \sum_{n=0}^{+\infty} \sum_{\ell=0}^{+\infty} \int da_\tau da_y E_p(u) \mathcal{G}(p) E_p^*(u'), \quad (19)$$

with $E_p(u)$ given by Eq. (16) and p^2 written in Eq. (11).

Applying $(D_\mu^2 - m^2)$ at relation given in the Eq. (19) and using the Eqs. (5), (7), (11) and (17), we found $\mathcal{G}(p)$ in the Euclidean space

$$\mathcal{G}(p) = \frac{1}{\omega_E(2n+1) + \omega_B(2\ell+1) + m^2}. \quad (20)$$

Note that $\mathcal{G}(p)$ actually does not depends of p , but only of electric and magnetic fields and the Landau levels, (n, ℓ) .

It is important to emphasize the role of external fields in the propagator. We have a kind of dimensional reduction by introduction these external fields. In fact, to see this more easily, let us just look at the kernel propagator written in the momenta space (in the limit $u \rightarrow u'$), without and with external fields:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \rightarrow \sum_{n,\ell=0}^{\infty} \frac{\omega_E \omega_B}{\omega_E(2n+1) + \omega_B(2\ell+1) + m^2}.$$

The magnetic field in the direction z makes the reduction $D \rightarrow (D-2)$ and k^2 becomes $(k_\tau^2 + k_z^2 + k_x^2 + k_y^2) \rightarrow (k_\tau^2 + k_z^2 + \omega_B(2\ell+1))$. However, the external electric field also causes the dimensional reduction in the two remain dimensions: $(D-2) \rightarrow (D-4)$. Thus, the effective result of including $\mathbf{E}$ and $\mathbf{B}$ on the system is the dimensional reduction $D \rightarrow (D-4)$. Besides to the dimensional reduction, the electric and magnetic fields generate a discrete spectrum for the propagator, since the Landau levels (n, ℓ) carry this characteristic.

C. Free spinor field propagator

Let us find the well know free Dirac field propagator by the Ritus' method. In the Euclidean space, we have

$$(\not{D} + im) \psi(u) = 0,$$

where

$$\not{\partial} = \gamma_\mu \partial_\mu ; \{ \gamma_\mu, \gamma_\nu \} = -2 \delta_{\mu\nu}.$$

Explicitly

$$\gamma_0 = -i \begin{pmatrix} 0_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0_2 \end{pmatrix} ; \gamma_j = \begin{pmatrix} 0_2 & -\sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad (21)$$

being $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$ and $\sigma_3 = \sigma_z$ the Pauli matrices.

The free propagator satisfy the equation

$$(\not{\partial} + im) S(u - u') = \delta^4(u - u'). \quad (22)$$

Since $[\not{\partial}, \partial_\nu] = 0$, eigenfunctions of operator ∂_ν , namely, planes waves, are also eigenfunctions of $\not{\partial}$, with $\not{k}$ eigenvalue:

$$\not{\partial}[\exp(i k_\mu u_\mu)] = [\exp(i k_\mu u_\mu)] i \not{k}, \quad (23)$$

in this order.

Thus, expanding the Dirac propagator in terms of planes waves, we obtain

$$S(u - u') = \int \frac{d^4 k}{(2\pi)^4} [\exp(i k_\mu u_\mu)] \tilde{s}(k) \times [\exp(i k_\nu u'_\nu)]^*. \quad (24)$$

The free propagator in the momenta space, $\tilde{s}(k)$, can be found by application of operator $(\not{\partial} + im)$ at equation (24), and by use of the Eqs. (23), (22) and Eq. (3), namely

$$\tilde{s}(k) = \frac{i(\not{k} - m)}{k^2 + m^2}. \quad (25)$$

D. Spinor field propagator under an electromagnetic external field

Take into account the minimal coupling, the Dirac equation under both electric and magnetic fields in the z direction is given by

$$(\not{D} + im) \Psi(u) = 0, \quad (26)$$

where again $D_\mu = \partial_\mu + ieA_\mu^{ext}$ and $A_\mu^{ext} = (-zE_0, 0, xB_0, 0)$. The Feynman propagator is

$$(\not{D} + im) S(u, u', A) = \delta^4(u - u'). \quad (27)$$

Note that $[\not{D}, \partial_\nu] = ie\gamma_\mu[A_\mu^{ext}, \partial_\nu] \neq 0$. Therefore, the wave planes are not eigenfunctions of operator $\not{D}$. However, $[\not{D}^2, \not{D}] = 0$. Thus, let us find the eigenfunctions of quadratic Dirac operator.

If we could find the eigenfunctions $\mathbb{E}_p$ of operator $\not{D}$, or equivalently $\not{D}^2$, we can write the fermionic propagator by expansion in terms of them.

We can show that

$$\not{D}^2 = (ie[\gamma_\mu, \gamma_\nu]F_{\mu\nu} - 4D_\mu^2) / 4,$$

where D_μ^2 , is given by Eq. (8) and the only no null values of $F_{\mu\nu}$ is $F_{03} = -F_{30} = E_0$ and $F_{12} = -F_{21} = B_0$. Using explicitly the γ matrices write in Eq. (21), we have

$$\not{D}^2 = \omega_E(\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B(\mathbb{I}_2 \otimes \sigma_z) - D_\mu^2, \quad (28)$$

where

$$\tilde{\mathbb{I}}_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (29)$$

From bosonic case, we know that the operator D_μ^2 has its eigenvalues given by $[-\omega_E(2n+1) - \omega_B(2\ell+1)]$ and its eigenfunctions given by $E_p(u)$ (see Eq. (16)). Therefore, we have just that find the spin related part eigenvalue of $\not{D}^2$, i.e., $[\omega_E(\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B(\mathbb{I}_2 \otimes \sigma_z)]$.

We need find the matrices Ω_{σ_E} and Ω_{σ_B} such that

$$\begin{aligned} (\tilde{\mathbb{I}}_2 \otimes \sigma_z) \Omega_{\sigma_E} &= \sigma_E \Omega_{\sigma_E}, \\ (\mathbb{I}_2 \otimes \sigma_z) \Omega_{\sigma_B} &= \sigma_B \Omega_{\sigma_B}, \end{aligned} \quad (30)$$

with $\sigma_E = \sigma_B = \pm 1$.

After we make the tensorial products in Eq. (30), we easily found the structure of Ω_{σ_E} and Ω_{σ_B} . Therefore, the eigenfunctions of $\not{D}^2$ are given by [7, 12]

$$\mathbb{E}_p(u) = \sum_{\sigma_E, \sigma_B = \pm 1} E_{p, \sigma_E, \sigma_B}(u) \Omega_{\sigma_E, \sigma_B}, \quad (31)$$

where

$$\begin{aligned} \Omega_{\sigma_E, \sigma_B} &= \text{diag} \left(\delta_{1, \sigma_E} \delta_{1, \sigma_B}; \delta_{-1, \sigma_E} \delta_{-1, \sigma_B}; \right. \\ &\quad \left. \delta_{-1, \sigma_E} \delta_{1, \sigma_B}; \delta_{1, \sigma_E} \delta_{-1, \sigma_B} \right), \end{aligned} \quad (32)$$

and $E_{p, \sigma_E, \sigma_B}(u)$ are given by Eq. (16). Thus,

$$\not{D}^2 \mathbb{E}_p = [\omega_E(2n+1 + \sigma_E) + \omega_B(2\ell+1 + \sigma_B)] \mathbb{E}_p. \quad (33)$$

Although we already know the eigenfunctions of operator $\not{D}$, i.e., the Ritus' eigenfunctions $\mathbb{E}_p(u)$ given by Eq. (31), we do not know what are its eigenvalues, and for this reason, we still cannot expand the fermionic propagator as we did in Eqs. (2), (19) and (24) in terms of $\mathbb{E}_p$. To solve this problem, Ritus, postulated the relation

$$\not{D} \mathbb{E}_p = \mathbb{E}_p i \not{p}, \quad (34)$$

in this order, for a specific four-vector $\bar{p}$.

In the **appendix A**, we calculate explicitly the four-vector $\bar{p}$. The result is

$$\begin{aligned} \bar{p}_\mu &= \left(\sqrt{\omega_E(2n+1+\sigma_E)}, 0, \sqrt{\omega_B(2\ell+1+\sigma_B)}, 0 \right), \\ n, \ell &= 0, 1, 2, \dots \end{aligned} \quad (35)$$

Take into account the Eq. (17) and that,

$$\sum_{\sigma_E, \sigma_B = \pm 1} \sum_{\sigma_{E'}, \sigma_{B'} = \pm 1} \Omega_{\sigma_E, \sigma_B} \Omega_{\sigma_{E'}, \sigma_{B'}} = \mathbb{I}_4,$$

is easy show that

$$\int dp \mathbb{E}_p(u) \mathbb{E}_p^\dagger(u') = \delta^4(u - u'), \quad (36)$$

and

$$\begin{aligned} \int d^4u \mathbb{E}_p(u) \mathbb{E}_{p'}^\dagger(u) &= \delta[\omega_E(a_\tau - a'_\tau)] \delta[\omega_B(a_y - a'_y)] \\ &\times \delta_{n,n'} \delta_{\ell,\ell'}. \end{aligned} \quad (37)$$

Now, we finally can expand the fermionic field propagator in terms $\mathbb{E}_p$, namely

$$S(u, u', A) = \int dp \mathbb{E}_p(u) \mathcal{S}(p) \mathbb{E}_p^\dagger(u'), \quad (38)$$

Replacing the expansion given in Eq. (38) within Eq. (27) and using the Eqs. (34) and (36), we obtain

$$\mathcal{S}(\bar{p}) = \frac{i(\not{\bar{p}} - m)}{\bar{p}^2 + m^2}, \quad (39)$$

with the $\bar{p}_\mu$ components given by Eq. (35). We also have used that $\not{\bar{p}}^2 = -\bar{p}^2$. Finally, the Dirac propagator under an external constant electromagnetic field is

$$S(u, u', A) = \omega_E \omega_B \sum_{n=0}^{+\infty} \sum_{\ell=0}^{+\infty} \int da_\tau da_y \mathbb{E}_p(u) \mathcal{S}(\bar{p}) \mathbb{E}_p^\dagger(u'), \quad (40)$$

with $\mathbb{E}_p(u)$ write in Eq. (31).

As in the scalar case, the external electromagnetic field in the Dirac propagator induces the dimensional reduction $D \rightarrow (D - 4)$ and provides a discrete spectrum for the propagator, through the Landau levels (n, ℓ) . But now we have spin degrees of freedom and matrix structure inherent to spinor.

III. CONCLUSIONS / COMMENTS

In this paper, we used the Ritus' method to calculate the Feynman propagator under magnetic and electric external fields. First, we used the Klein Gordon equation and the minimal coupling prescription to find the Ritus' eigenfunctions and its eigenvalues. After that, we apply the method at the Dirac equation. In this case, we have had an additional problem: we did not know the eigenvalues associated with the Dirac operator under a constant electromagnetic field. Thus, one more condition was necessary to find the fermionic propagator. We define that $\not{D} \mathbb{E}_p = \mathbb{E}_p i \not{\bar{p}}$ for some $\bar{p}_\mu$ to be found. In the **appendix A**, we calculate explicitly the components of $\bar{p}$. In the **appendix B**, we calculating the non-relativistic limit to scalar and spinor propagators. The Ritus' method is a powerful technique to find the Feynman propagator in an external field. Since the bosonic propagator, as well the fermionic propagator were written in a diagonal form, the expressions involved became simpler than those obtained by other methods, for example, the proper time method [1, 14]. With this manuscript, we hope that the Ritus' method becomes more widespread in the academic community.

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Appendix A: The components of $\bar{p}_\mu$

Now let us demonstrate the $\bar{p}_\mu$ components written in (35). For this, we will write explicitly the left side of Eq. (34). Take into account the Eq. (31), we have

$$\mathbb{E}_p = \begin{pmatrix} E_{1,1} & 0 & 0 & 0 \\ 0 & E_{-1,-1} & 0 & 0 \\ 0 & 0 & E_{-1,1} & 0 \\ 0 & 0 & 0 & E_{1,-1} \end{pmatrix}. \quad (41)$$

Using the γ matrices given in Eq. (21) we get

$$\not{D} \mathbb{E}_p = \begin{pmatrix} 0 & 0 & (-iD_0 - D_3)E_{-1,1} & (-D_1 + iD_2)E_{1,-1} \\ 0 & 0 & (-D_1 - iD_2)E_{-1,1} & (-iD_0 + D_3)E_{1,-1} \\ (-iD_0 + D_3)E_{1,1} & (D_1 - iD_2)E_{-1,-1} & 0 & 0 \\ (D_1 + iD_2)E_{1,1} & (-iD_0 - D_3)E_{-1,-1} & 0 & 0 \end{pmatrix}, \quad (42)$$

where $D_0 = \partial_\tau - i\omega_E z$, $D_1 = \partial_x$, $D_2 = \partial_y + i\omega_B x$ and $D_3 = \partial_z$.

The right side of Eq. (34) is

$$\mathbb{E}_p i\vec{p} = \begin{pmatrix} 0 & 0 & (\bar{p}_0 - i\bar{p}_3)E_{1,1} & (-i\bar{p}_1 - \bar{p}_2)E_{1,1} \\ 0 & 0 & (-i\bar{p}_1 + i\bar{p}_2)E_{-1,-1} & (\bar{p}_0 + i\bar{p}_3)E_{-1,-1} \\ (\bar{p}_0 + i\bar{p}_3)E_{-1,1} & (i\bar{p}_1 + \bar{p}_2)E_{-1,1} & 0 & 0 \\ (i\bar{p}_1 - \bar{p}_2)E_{1,-1} & (\bar{p}_0 - i\bar{p}_3)E_{1,-1} & 0 & 0 \end{pmatrix}. \quad (43)$$

After we compare the Eqs. (42) and (43) and solve four sets of differential coupling equations, we find that the components of $\bar{p}_\mu$ satisfy

$$\bar{p}_0^2 + \bar{p}_3^2 = \omega_E(2n + 1 + \sigma_E),$$

and

$$\bar{p}_1^2 + \bar{p}_2^2 = \omega_B(2\ell + 1 + \sigma_B),$$

for $n, \ell = 0, 1, 2, \dots$ and $\sigma_E = \sigma_B = \pm 1$. To an appropriate coordinate system, we can fix $\bar{p}_1 = \bar{p}_3 = 0$, and thus, the Eq. (35) is demonstrated.

Appendix B: The non-relativistic propagators

In this appendix, we will apply the Ritus' method to obtain the non-relativistic (NR) expressions for the propagators. In the Euclidean space, the non-relativistic limit is got by replace [15]

$$\varphi(u) \rightarrow \varphi_{NR}(u) \cdot \exp \left[- \left(\frac{mc^2}{\hbar} \right) \tau \right],$$

on the field equations.

Klein-Gordon propagator

Let us start from Klein-Gordon equation under the external fields **E** and **B**:

$$(-D_\mu D_\mu + m^2) \phi(u) = 0. \quad (44)$$

Taking into account the Eq. (8) and the non-relativistic limit in natural units, the equation (44) becomes

$$[-(\partial_\tau^2 - 2m \partial_\tau + m^2) - \nabla^2 + 2i\omega_E z(\partial_\tau - m) - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2 + m^2] \phi(u)_{NR} = 0.$$

Since that in non relativistic limit, the terms that involve the rest energy are bigger than the others ones in brackets, we can write

$$(2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2) \phi(u)_{NR} = 0. \quad (45)$$

From Eq. (45), we notice that propagator in non relativistic limit satisfy

$$(2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2) G(u, u', A)_{NR} = -\delta^4(u - u'). \quad (46)$$

Using the Ritus' method, we get

$$G(u, u', A)_{NR} = \int dp E_{pNR}(u) \mathcal{G}(p)_{NR} E_{pNR}^*(u'),$$

with

$$(2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2) E_{p_{NR}}(u) = p_{NR}^2 E_{p_{NR}}(u). \quad (47)$$

To solve Eq. (47), we use the same ansatz of Eq. (9), the result is

$$E_{p_{NR}}(u) = (\sqrt[4]{\omega_E \omega_B} / 2\pi) \exp[i(\omega_E a_\tau \tau + \omega_B a_y y)] \times h_\ell[\sqrt{\omega_B}(x + a_y)] h_n\left[\sqrt{\omega_E}\left(z - \frac{im}{\omega_E}\right)\right], \quad (48)$$

and

$$\mathcal{G}(p)_{NR} = 1/p_{NR}^2,$$

where $p_{NR}^2 = [\omega_E(2n+1) + \omega_B(2\ell+1) + m^2 + 2m\omega_E a_\tau i]$, $n, \ell = 0, 1, 2, \dots$ and $h_m(s)$ are the Hermite functions.

Note that the eigenfunctions dependent on the electric field have an imaginary argument. Furthermore, we have a continuous spectrum for the propagator with the presence of poles in the complex plane, due to term $2m\omega_E a_\tau i$ in p_{NR}^2 .

Dirac propagator

Applying the operator $(\not{D} - im)$ at Eq. (26), we obtain

$$\left(\omega_E(\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B(\mathbb{I}_2 \otimes \sigma_z) - D_\mu^2 + m^2\right) \Psi(u) = 0. \quad (49)$$

Using the non relativistic limit $\Psi(u) \rightarrow \Psi_{NR}(u) \cdot \exp(-m\tau)$ on Eq. (49), the Dirac equation becomes

$$\left[\omega_E(\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B(\mathbb{I}_2 \otimes \sigma_z) + 2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2\right] \Psi_{NR}(u) = 0.$$

Thus, the Dirac propagator satisfy

$$\begin{aligned} &\left[\omega_E(\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B(\mathbb{I}_2 \otimes \sigma_z) + 2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2\right] S_{NR}(u, u', A) \\ &= \delta^4(u - u'). \end{aligned} \quad (50)$$

Let us expand $S_{NR}(u, u', A)$ in terms of eigenfunctions $\mathbb{E}_{p_{NR}}(u)$

$$S_{NR}(u, u', A) = \int dp \mathbb{E}_{p_{NR}}(u) \mathcal{S}_{NR}(p) \mathbb{E}_{p_{NR}}^\dagger(u), \quad (51)$$

where,

$$\mathbb{E}_{p_{NR}}(u) = \sum_{\sigma_E, \sigma_B = \pm 1} E_{p_{NR}, \sigma_E, \sigma_B}(u) \Omega_{\sigma_E, \sigma_B}, \quad (52)$$

with $E_{p_{NR},\sigma_E,\sigma_B}(u)$ given by Eq. (48) and $\Omega_{\sigma_E,\sigma_B}$ is given by Eq. (32).

The eigenvalue equation

$$\left[\omega_E (\tilde{\mathbb{I}}_2 \otimes \sigma_z) + \omega_B (\mathbb{I}_2 \otimes \sigma_z) + 2m \partial_\tau - \nabla^2 - 2im\omega_E z - 2i\omega_B x \partial_y + \omega_E^2 z^2 + \omega_B^2 x^2 \right] \mathbb{E}_{p_{NR}}(u) = \bar{p}_{NR}^2 \mathbb{E}_{p_{NR}}(u),$$

gives

$$\bar{p}_{NR}^2 = [\omega_E(2n+1+\sigma_E) + \omega_B(2\ell+1+\sigma_B) + m^2 + 2m\omega_E a_\tau i], \quad n, \ell = 0, 1, 2, \dots,$$

Also we have

$$\mathcal{S}_{NR}(p) = 1/\bar{p}_{NR}^2.$$

Since $-\infty < a_\tau < +\infty$, the spectrum of non relativistic limit of the Dirac propagator becomes continuous.