SAMPLED-DATA REPETITIVE CONTROL SYSTEMS

by

Ali Langari

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

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0-612-27986-3
To Tadj Khadje-Masoudi and Akbar Langari,

my parents
Sampled-Data Repetitive Control Systems

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Abstract

Repetitive control is employed in numerous industrial applications to allow systems to track or reject unknown periodic signals of a known period. This thesis takes a novel approach to the design and analysis of such systems, by introducing a useful performance measure, referred to as the induced power-norm. This measure represents the maximum power-norm of the steady-state error vector in the system, for all periodic inputs of unit power-norm. The approach taken here is also new in that it is a sampled-data formulation. Hence, the intersample behavior is directly taken into account.

First, a methodology is developed for designing optimal sampled-data repetitive controllers, based on minimizing the power-norm of the steady-state error vector for a given periodic input. It is shown that such an optimal controller always exists. This methodology is then generalized to the case of an unknown periodic input by minimizing the induced power-norm. Fast discretization is verified to be a useful computational tool for obtaining suboptimal controllers in both methodologies. To demonstrate these methodologies, active suppression of fan noise present in an acoustic duct is discussed with promising results.

Also formulated and analyzed in this thesis is a robust tracking problem for sampled-data repetitive control systems in the presence of structured linear periodically time-varying perturbations. Specifically, we investigate whether the induced power-norm of the closed-loop system remains below a given bound for a class of such perturbations. The result is stated in terms of a necessary and sufficient condition that involves Dullerud’s generalized notion of structured singular values for operators. Computational aspects are addressed with a numerical example.
Acknowledgements

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As well, my warmest thanks to Benoit Boulet, Mark Lawford and Ryan Leduc for their pure friendship and encouragement throughout. As network system administrators, it is thanks to them that I always had my computer system up and running.

And lastly, there are no words that will suffice to thank my parents, my brothers Reza and Abdol, and my only sister Ladan. I would have never come so far without their rock solid support and unconditional love.
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<td>13</td>
<td>Real number set</td>
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<td>$\mathbb{C}$</td>
<td>15</td>
<td>Complex number set</td>
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<tr>
<td>$\mathbb{R}^n$</td>
<td>13</td>
<td>$n$-dimensional vector space over real numbers</td>
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<tr>
<td>$\mathbb{C}^n$</td>
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<td>$n$-dimensional vector space over complex numbers</td>
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<td>$E$</td>
<td>28</td>
<td>$\mathbb{R}^n$ for any $n$</td>
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<tr>
<td>$\mathbb{R}_+$</td>
<td>13</td>
<td>Set of non-negative real numbers</td>
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<td>Set of non-negative integers</td>
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<td>$G$</td>
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<td>$\hat{g}(s)$</td>
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<td>$F$</td>
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<td>$K_c$</td>
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<td>Continuous-time controller</td>
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<td>$K$</td>
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<td>FDLTI discrete-time controller</td>
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<td>$\hat{k}(\lambda)$</td>
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<td>Transfer matrix of $K$ ($\lambda = z^{-1}$)</td>
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<td>Periodic sampler of sampling period $h/N$ (fast sampler)</td>
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Periodic zero-order hold synchronized with $S_N$ (fast hold)

Ratio of the sampling rates of $S_N$ and $S$

$S_N$ restricted to $[0, h)$

$H_N$ restricted to $\{0, 1, \ldots, N - 1\}$

Usually, the Planck's constant! In this thesis, $h/N$

Set of all sequences from $\{0, 1, \ldots, N - 1\}$ to $C$

Period of reference commands/disturbance signals

Continuous-time periodic input of period $T$

Error vector in the system, controlled output

Periodic component of $z$

Transient component of $z$

Control input

Measured output

power-norm of periodic signal $w$ of period $T$, defined by $\left[\frac{1}{T} \int_0^T w(t)'w(t)dt\right]^{1/2}$

Set of all continuous-time periodic signals of period $T$ of finite power-norm

Discrete-time periodic input of period $m$

power-norm of discrete-time periodic inputs of period $m$, defined by $\left[\frac{1}{m} \sum_{n=0}^{m-1} \omega(n)'\omega(n)\right]^{1/2}$

Set of all discrete-time periodic signals of period $m$

$\lambda$-transform of discrete-time signal $\omega \in \Omega_m$

Discrete Fourier Transform (DFT) of $\omega$ in $\Omega_m$

$e^{-j\frac{2\pi}{m}}$ in the DFT formulas, (2.6) and (2.7)

Linear space of all functions from $[0, h)$ to $C$

Linear space of all square-integrable functions in $C$
\( \mathcal{K}_p \) 21 Linear space of all functions in \( \mathcal{K} \) of finite \( p \) - norm

\( \mathcal{K}_\infty \) 21 Linear space of all functions in \( \mathcal{K} \) of finite \( \infty \) - norm

\( \ell(\mathbb{Z}_+, \mathcal{K}) \) 21 Linear space of all functions from \( \mathbb{Z}_+ \) to \( \mathcal{K} \)

(function-valued discrete-time signals)

\( \ell(\mathbb{Z}_+, \mathcal{K}_2) \) 21 Linear space of all functions from \( \mathbb{Z}_+ \) to \( \mathcal{K}_2 \)

\( \ell_2(\mathcal{K}_2) \) 22 Linear space of all square-integrable functions in \( \ell(\mathbb{Z}_+, \mathcal{K}_2) \)

\( \Lambda \) 23 The mapping from a signal \( v \in \ell_2(\mathcal{K}_2) \) to its \( \lambda \) - transform \( \hat{v} \)

\( \Omega_m(\mathcal{K}_2) \) 21 Set of all function-valued discrete-time periodic signals

of period \( m \)

\( M \) 25 Number of sampling periods \( h \) in \( T \) (\( T/h \) is assumed to be an integer.)

\( L_c \) 24 Continuous-time lifting

\( L \) 30 Discrete-time lifting

\( L \) 97 \( L \) restricted to \( \{0, 1, \ldots, N - 1\} \)

\( \Phi_K \) 89 Operator from \( w \) to \( z \) in Figure 5.1

\( J_{zw}(K) \) 85 Induced power-norm from \( w \) to \( z \)

\( \Phi_K \) 89 Operator from \( w := L_c w \) to \( z := L_c z \)

\( \Psi_K \) 89 Operator from \( \omega \) to \( \zeta \) in Figure 5.4

\( J_{\zeta\omega}(K) \) 87 Induced power-norm from \( \omega \) to \( \zeta \)

\( \Psi_K \) 89 Operator from \( \omega \) to \( \zeta \) in Figure 5.5

\( J_{\zeta\omega}(K) \) 88 Induced power-norm from \( \omega \) to \( \zeta \)

\( D_h \) 25 Delay-by-\( h \) operator on \( \mathcal{L}_2 \)

\( U \) 25 Unilateral shift on \( \ell_2(\mathcal{K}_2) \)

\( \mathbb{D} \) 22 Open unit disc in \( \mathbb{C} \)

\( \mathbb{D} \) 22 Closed unit disc in \( \mathbb{C} \)

\( \partial \mathbb{D} \) 22 Unit circle in \( \mathbb{C} \)
\( \mathcal{H}_2(\mathbb{D}, \mathcal{K}_2) \) 22 Linear space of all functions \( \hat{\vartheta} \) from \( \hat{\mathbb{D}} \) to \( \mathcal{K}_2 \), analytic on \( \mathbb{D} \) with \( \| \hat{\vartheta} \|^2_{\mathcal{H}_2(\mathbb{D}, \mathcal{K}_2)} := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \| \hat{\vartheta}(re^{j\theta}) \|^2_{\mathcal{K}} \, d\theta < \infty \\
\mathbb{L}(\mathcal{X}) \) 25 Set of all bounded linear transformations on a normed space \( \mathcal{X} \) \\
\mathbb{L}(\mathcal{K}_2) \) 29 Set of all bounded linear transformations on \( \mathcal{K}_2 \) \\
\mathcal{H}_\infty(\mathbb{D}, \mathbb{L}(\mathcal{K}_2)) \) 29 Linear space of all functions \( \hat{\vartheta} \) from \( \hat{\mathbb{D}} \) to \( \mathbb{L}(\mathcal{K}_2) \), analytic on \( \mathbb{D} \) with \( \| \hat{\vartheta} \|^2_{\mathcal{H}_\infty(\mathbb{D}, \mathbb{L}(\mathcal{K}_2))} := \sup_{\|\lambda\| < 1} \| \hat{\vartheta}(\lambda) \| < \infty \\
\mathcal{A}(\mathbb{D}, \mathbb{L}(\mathcal{K})) \) 30 Linear space of all functions in \( \mathcal{H}_\infty(\mathbb{D}, \mathbb{L}(\mathcal{K}_2)) \), but continuous on \( \hat{\mathbb{D}} \) with \( \| \hat{\vartheta} \|^2_{\mathcal{A}(\mathbb{D}, \mathbb{L}(\mathcal{K}_2))} := \max_{\|\lambda\| = 1} \| \hat{\vartheta}(\lambda) \| < \infty \\
\mathbb{L}_\mathcal{A}(\mathbb{D}, \mathbb{L}(\mathcal{K}_2)) \) 30 Set of all \( h \)-periodic operators with the transfer function of their lifted associated continuous on \( \partial\hat{\mathbb{D}} \) \\
\Delta \) 115 Perturbation to the plant \\
\Delta \) 118 Continuously-lifted \( \Delta \), \( L_c^{-1}\Delta L_c \) \\
\hat{\Delta}(\lambda) \) 118 Operator-valued transfer function of \( \Delta \) \\
\check{\Delta} \) 118 Operator value of \( \hat{\Delta}(\lambda) \) at a given \( \lambda \) \\
\Delta_{PTV} \) 118 Robust stability uncertainty set \\
\Delta_{rt} \) 127 Robust tracking uncertainty set \\
\mathcal{X}_s \) 118 Set of all structured perturbations \\
\mathcal{X}_{PTV} \) 118 Set of all periodically time-varying structured perturbations \\
\mathcal{U}\mathcal{X}_{PTV} \) 118 Open unit ball in \( \mathcal{X}_{PTV} \) \\
\mu \) 122 Structured singular value \\
\mathbb{F}_u \) 123 Upper linear fractional transformation
Chapter 1

Introduction

Repetitive control is applied to a system to make it track or reject unknown periodic reference commands or disturbance signals of which only the period is known. Thus, repetitive control systems are servomechanisms with periodic exogenous signals. However, there are a few reasons that make these systems stand out.

One is the large number and wide range of applications that they have found ever since they were introduced to the control community by Inoue et al. [INK+81]. Among those applications are the following: rejection of power supply interferences [NH86], control of robotic manipulators which perform a repetitive task, such as painting or picking and placing [OHN87, TAT88, SHKT90], accurate placement of the read/write head on a selected track of a disk-drive system, where the eccentricity of the track causes major run-outs [CT90], restoration of periodic signals distorted by nonlinear measurement devices [HHS92], control of peristaltic pumps intended for periodic pumping of blood in dialysis machines [HS93], noncircular machining [TT88, TT94], attitude stabilization of satellites [BM92, LHP+94], precision control of compact disc mechanisms [DSVS95], rejection of unknown periodic load disturbances in continuous steel casting processes [MTBR96], active suppression of vibrations [Hil96], and active noise attenuation in finite-length ducts [Hu95, Hu96]. In all these applications, there are periodic signals to be tracked or rejected, and depending on the application, there are certain practical issues arising from the periodicity of the signals that need to be considered.
In addition, repetitive control systems have posed theoretical challenges to researchers studying them, especially in continuous-time formulations [HON85, HY85, HYON88, YH88, Yam93]. The most fundamental theoretical issue is that an arbitrary periodic signal may have an infinite number of harmonics. This makes the space of exogenous signals infinite-dimensional. Hence intuitively speaking, to have steady-state tracking of all those signals, or simply put, perfect tracking, an infinite-dimensional controller would be needed. Dealing with an infinite-dimensional system involves more effort and requires that many questions be re-investigated, the most important one being stabilizability of the system. In fact, we will see in Chapter 3 that there is an important set of plants, namely the set of all strictly proper plants, that cannot be stabilized if the requirement of perfect tracking is imposed on them.

Repetitive control can also be regarded as a learning scheme. This is because, in reaching the steady state, the error signal goes down in amplitude from one period to the other, which is interpreted as if the plant is learning, as periods go by. With this interpretation, repetitive control resembles what is called iterative learning control, where the error in tracking a finite-duration reference signal is reduced by means of correcting the input in each iteration, based on the error observed in the preceding iterations. For example, see [AKM85, AKMT85, MK85, BCG88]. However, there is a subtle difference between iterative learning and what we call repetitive control. In iterative learning, the plant is always reset to the same initial conditions after every iteration, whereas, in repetitive control, the control process continues from where it has finished at the end of the preceding period. This also means that, in iterative learning, the control action need not be causal. Keeping in mind the differences between the two techniques, we see that they can be applied interchangably, thus enabling a wider variety of applications.

In the next section, we give a brief description of the application of repetitive control to cancelling fan noise in acoustic ducts. This is then followed by a brief historical review of the previous work on repetitive control with a focus on the ideas motivating this thesis. Finally, we summarize the main contributions of the thesis, followed by a glance at the remaining chapters.
1.1 Active Noise Control in an Acoustic Duct

Acoustic noise suppression in the ducts or pipework used in Heating, Ventilation and Air Conditioning (HVAC) systems, or the exhaust systems in numerous industrial applications is of significant importance. For instance, concert halls and hospitals naturally require a low level of noise. Another example is industrial work environments. In these areas, it is very critical to reduce the noise from fans in air handlers and ventilation ducts, as the prolonged exposure to such noise has been shown to result in fatigue and loss of concentration in people. A minimum level of acoustic noise is also desirable in office buildings, classrooms, meeting rooms, and living rooms.

Traditionally, passive silencers are used for noise attenuation in ducts. These silencers, which are constructed from some energy absorbent materials, line the ducts, especially around the corners, to reduce the turbulence caused by the rapid movement of air. Passive silencers are used because they attenuate noise over a wide range of frequencies. However, they carry some disadvantages. They are ineffective in low frequencies, they are bulky, they cause flow restrictions and, more importantly, they are costly.

An alternative approach that is of current and increasing interest is active control [NE92, HRS93, Hu95, Hu96, HAV+96]. This approach is based on the concept that reducing the noise along the duct and at its opening to a room reduces the overall level of the noise in that room. In this respect, certain locations are picked in the duct and a secondary sound wave is injected into it through an array of secondary sources, with the goal of counterbalancing or minimizing the noise of the original source at those points. Because of their small size, active noise controllers reduce the noise without much physical modification to the duct. At the same time, low-frequency noise can be attenuated efficiently by means of a properly designed controller. Also, in contrast to passive silencers, active noise controllers can cost less. For a comprehensive review of the work on active noise control systems, see [KM96].

Now, since the fan noise, which is a main contributing factor to the overall duct noise, is a periodic disturbance, repetitive control makes a potentially useful candi-
date for its active cancellation. In fact, Hu shows in [Hu95, Hu96] that, by using repetitive control, periodic noise can be attenuated extensively. In this thesis, we will also demonstrate the suitability of repetitive control as a useful technique in minimizing periodic noise, but also take into account the intersample behavior that might arise from the A/D and D/A operations in the digital implementation of such a controller. In [Hu95] and [Hu96] a digital implementation of a continuous-time repetitive controller, originally proposed in [HYON88], and a discrete-time internal model are used, respectively, to cancel out periodic disturbances of a given period. The disadvantage of discretizing continuous-time controllers is that high sampling rates, of at least a few times the bandwidth of both the plant and the disturbance, are required to stay close to the original continuous-time performance. Moreover, discrete-time controllers require high sampling rates, because the intersample behavior is ignored in their design. As we will see, the frequency response of a duct can have quite a broadband nature. This means that, since the plant bandwidth is high to begin with, very high sampling rates would be required for control. By way of simulation, we will show that at a fixed, low sampling rate, the sampled-data technique yields a much better performance, thus relaxing the requirement for high sampling rates.

1.2 Motivation for this Thesis

The work on repetitive control was pioneered by Inoue, Nakano, and their co-workers [INI81, INK+81], in application to the highly accurate control of a proton synchrotron magnet power supply. This was followed by theoretical studies by Hara et al. [HON85] and by Hara and Yamamoto [HY85]. A more detailed analysis appeared later in [HYON88, YH88]. These works, which incorporate the internal model principle, imply that, in order to achieve steady-state tracking of periodic signals of a given period $T$, a compensator which can generate all such signals should be included in the closed loop. (Just as in the case of tracking steps, where an integrator is introduced in the loop.) A rigorous derivation of such a generator is presented in Chapter 3. However to understand some of the issues relating to these papers,
consider the delay system of Figure 1.1, where the delay equals $T$. As illustrated in
this figure, a desired periodic waveform of period $T$ can be generated at the output of
this system, by feeding a single cycle of that waveform into its input. Let us denote
this system by $K_g$ and let the transfer function for it be $\hat{k}_g(s)$. Then from the block
diagram we have that,

$$\hat{k}_g(s) = \frac{1}{1 - e^{-Ts}}. \quad (1.1)$$

It can be seen that $\hat{k}_g(s)$ has an infinite number of poles on the imaginary axis
at $\pm j2k\pi/T$, $k = 0, 1, \ldots$; hence, it is an infinite-dimensional system. Yamamoto
proved [Yam93], that, in order to have perfect tracking, it is in fact necessary to
include (1.1) in the closed loop.

Figure 1.1: Periodic signal generator.

However, it was shown in [HY85, HON85] that the class of systems which can
be stabilized with $\hat{k}_g(s)$ in the loop is very limited, namely the class of systems
whose transfer functions have a relative degree of zero. In other words, simultaneous
tracking and stability with the proposed infinite-dimensional controller is not possible
for strictly proper plants, which represent most practical systems. One may attribute
this to the unreasonable requirement that tracking should occur for arbitrary periodic
signals, including the high-frequency components that are not in the passband of the
plant.

To overcome this limitation, Inoue et al. in [INK+81] and Hara et al. in [HON85]
weighted the delay term in $\hat{k}_g(s)$ by a stable low-pass filter $\hat{q}(s)$ of infinity-norm less
than or equal to 1. That is, \( \hat{k}_q(s) \) is replaced by

\[
\hat{k}_{g_{\text{mod}}}(s) = \frac{1}{1 - \hat{q}(s)e^{-Ts}},
\]

with

\[
\sup_{-\infty < f < \infty} |\hat{q}(j2\pi f)| \leq 1.
\]

In this technique, called modified repetitive control, the stabilizability constraint on strictly proper plants is removed; however, high-frequency tones cannot be tracked anymore. A controller synthesis algorithm based on this modification was introduced in [HON85] for single-input, single output (SISO) minimum phase plants and it was extended in [HY85] to nonminimum phase plants. The generalization of the synthesis algorithm to the multivariable case appeared in [HYON88]. Later, using the same modified compensator (1.2), Peery and Özbay presented a design methodology in [PO93, PO96] with the added feature of robust performance in a standard \( \mathcal{H}_\infty \) framework [DFT92].

Now let us see if we can explain why this low-pass filtering is so helpful. By examining \( \hat{k}_{g_{\text{mod}}}(s) \), we realize that weighting the delay term by a low-pass filter removes the unstable high-frequency poles of the controller. Therefore, the controller adds only a finite number of unstable poles to the system, which now becomes easier to stabilize than when there is an infinite number of unstable poles present. At the same time, removing the high-frequency poles of the controller violates the internal model principle since the resulting model doesn’t account for the high order harmonics of the periodic signals. Thus, the modified-repetitive-control remedy trades tracking ability for stability by considering a finite number of unstable poles.

We can now raise two points. First, since perfect tracking is not possible, we need some sort of a tracking measure, that is, a tool that can be used to compare the performance of different controllers. Secondly, by low-pass filtering the delay term, one is basically limited to finite-dimensional controllers. Noting these two points, this thesis is motivated by the following question: why not seek from the beginning a finite-dimensional controller and require that it tracks only periodic inputs which
are restricted to have negligible high-frequency components? To demonstrate this, we consider the setup of Figure 1.2, where $G$ is a finite-dimensional linear time-invariant (FDLTI) generalized plant, $F$ is a low-pass filter, $K_c$ is a continuous-time controller to be designed, $w$ is a periodic input of period $T$, and $z$ is the tracking error that we want to be small. We propose the design of a finite-dimensional $K_c$ to minimize an appropriate tracking measure. Our criterion is that the power of the steady-state tracking error should be minimized for the worst periodic input $w$ of unit power. Note that, since there are high-frequency components of $w$ that can leak through the low-pass filter $F$ (since the filter is not ideal), it is not be possible to have tracking for every input $w$.

Figure 1.2: A continuous-time repetitive control system.

Lin and Ho [LH92] have proposed a finite-dimensional controller as well. However, their design does not consider a tracking measure. It simply includes an internal model of a finite number of single tones and then solves a sensitivity minimization problem.

In addition, it is well-known that the flexibility and convenience of digital implementation of controllers are accompanied by sampling rate limitations and repetitive controllers are no exceptions. Thus, one mandate of a design methodology for repetitive controllers should be to take possible intersample behavior into account. For this reason, we consider the sampled-data setup of Figure 1.3, where solid and dotted lines represent continuous-time and discrete-time signals, respectively. The filter $F$ is now absorbed into the generalized plant $G$ and $S$ and $H$ denote, respectively, the periodic sampler and zero-order hold. Our goal here is to design a finite-dimensional discrete-
time controller $K$ to make the continuous-time plant $G$ track/reject continuous-time periodic signals $w$ of a given period $T$. This is done by defining a performance measure, called the *induced power-norm*, which also considers the intersample behavior of the steady-state tracking error. In contrast, previous works are based either on an analog design with a digital implementation [NH86], which may require very high sampling rates to stay close to the analog performance, or a discretization of the analog system followed by a discrete-time design, which doesn’t take the intersample behavior into account [TTC89]. For a comprehensive review of discrete-time repetitive control systems, see [Hi96].

![Diagram](image)

Figure 1.3: Sampled-data repetitive control setup.

Finally, the work in this thesis is motivated by the fact that, a repetitive controller which offers a certain level of performance for the plant model might not perform as well when it is implemented. Hence, developing robustness analysis tools is greatly desired. This work aims at formulating a robustness analysis problem for repetitive control systems in a sampled-data framework. While robustness issues of repetitive control systems with respect to plant variations have been the subject of study of previous works [TT94, Hi94], the major effort in those has been in the context of digital control, using a discretized plant. Since in practice the discrete-time controller is connected to an analog plant, a sampled-data approach to this issue makes more sense.
1.3 Contributions of the Thesis

The contributions of this thesis can be summarized as follows:

- The introduction of a tracking measure for repetitive control systems [LF94a]. This measure, referred to as the induced power-norm, represents the power of the steady-state tracking error for the worst-case periodic input of unit power and is well-defined for periodically time-varying sampled-data systems, as well as LTI continuous- and discrete-time systems. An important feature of the induced power-norm of a sampled-data system is that it takes the intersample error into account.

- The formulation of a design problem for sampled-data repetitive control systems with known periodic inputs, where minimization of the steady-state tracking error is used to design the controller [LF95]. The main feature of this formulation is that it places a major emphasis on the intersample error or ripple, as is referred to in some of the previous works. In [FE86], Franklin et al. proposed a technique for ripple-free sampled-data servomechanisms. But they require that the continuous-time part of the system include an internal model for the periodic inputs. This is a strong condition and in the case that the continuous-time plant does not have the required internal model, some analog pre-compensation is necessary. Hara et al. [HTK90] proposed a different technique for ripple attenuation in repetitive control systems, which involves generalized hold functions and asynchronous sampling. Compared to the standard sampled-data design, where the hold is zero-order and is synchronous with the sampler, this technique is far more complex. For a general treatment of ripple-free sampled-data systems, see [Yam94].

- The development of a design methodology for sampled-data repetitive control systems with unknown periodic inputs, where minimization of the induced power-norm is used to design the controller [LF94a]. Fast discretization is verified to be a useful computational tool in both obtaining suboptimal controllers
and evaluating their tracking performance.

- The formulation and analysis of a robust tracking problem for sampled-data repetitive control systems [LF96]. Specifically, we give a necessary and sufficient condition that can be used to investigate whether the induced power-norm of a sampled-data repetitive control system remains below a given bound for a class of structured linear periodically time-varying perturbations. Our result is based on Dullerud's framework for treatment of robust stability and robust performance of sampled-data systems with respect to such perturbations.

- The demonstration of the suitability of repetitive control as a useful technique in minimizing periodic noise in acoustic ducts. In this example, we show that, for a fixed sampling rate, the methodology developed in this thesis, yields a much better performance than a pure discrete-time design.

1.4 The Remaining Chapters

The rest of this thesis is organized as follows.

Chapter 2 contains the fundamental building blocks that we will use for the analysis and design of sampled-data repetitive control systems. We begin with the main ingredient for repetitive control, continuous- and discrete-time periodic signals. Then we introduce the power-norm as a tool for measuring such signals. This is followed by studying the behavior of multi-input, multi-output (MIMO) discrete-time LTI systems which are excited by discrete-time periodic signals. Then, we define a new measure for these systems, called the induced power-norm, which helps identify how big the steady-state component of the output can be, provided we know how big the periodic input is. We close the chapter with the notion of lifting which allows us to convert our sampled-data system to a discrete-time LTI system.

In Chapter 3, we present a concise overview of continuous-time repetitive control systems. Specifically, we demonstrate in a rigorous manner that simultaneous stability and perfect tracking of arbitrary periodic signals is not possible for strictly proper
plants. This then leads us to a problem formulation that sets the base for the approach of this work to the design and analysis of repetitive control systems.

In Chapter 4 we formulate a design problem for sampled-data repetitive control systems when the periodic input is known. Since the input is fixed, it can not be arbitrarily high-frequency; thus, we will not lowpass the input in this formulation. Our approach is to minimize the power-norm of the steady-state tracking error; this way, the intersample behavior is taken into account. We prove that an optimal controller always exists but, due to the sampled-data nature of the problem, it is not easily computable. Therefore, an approximation technique called fast discretization is introduced, which gives us a suboptimal controller that can be easily computed. However, the introduction of this approximation brings up some convergence issues. Finally, we show that the performance of a given sampled-data repetitive controller can be computed to any degree of accuracy with fast discretization. In preparation for the next chapter, we show that under certain mild conditions in the SISO case, the resulting controller is also optimal for all periodic inputs that involve the same harmonics as the input on which the design is based.

In Chapter 5, the aim is at generalizing the methodology of Chapter 4 to the case where the periodic input is unknown. To this end, the definition of the induced power-norm from Chapters 2 and 3 is extended to sampled-data systems, however, now taking the intersample behavior into consideration. Minimization of this measure is then used as a criterion for designing sampled-data repetitive controllers. As in Chapter 4, we exploit fast discretization to find a suboptimal controller and verify that fast discretization can be used to compute the induced power-norm of a given sampled-data repetitive control system to any desired degree of accuracy. Finally, we apply the methodologies developed in this chapter and Chapter 4 to the suppression of fan noise in an acoustic duct.

In Chapter 6, we formulate and analyze a robust tracking problem for sampled-data repetitive control systems in the presence of structured linear periodically time-varying perturbations. We show that necessary and sufficient conditions can be obtained that guarantee tracking robustness of the system under such perturbations.
In preparation for this result, we will see an overview of Dullerud's framework for stability robustness of sampled-data systems.

Finally, in Chapter 7 we state our concluding remarks and some directions for future research.
Chapter 2

Fundamental Building Blocks

This chapter contains notation, preliminaries and some of the building blocks common to most of the coming chapters. We begin with the definitions of periodic signals and systems, which constitute the main ingredient of sampled-data repetitive control.

2.1 Periodic Signals

Repetitive control is concerned with the steady-state response of systems to periodic inputs. A well-known fact is that a periodic input to a stable LTI system produces a periodic output (same period) when the time interval is \((-\infty, \infty)\). If the time interval is \([0, \infty)\), the output is the sum of a transient response and a periodic signal. However, a sampled-data repetitive control system is time-varying and hence its response to periodic inputs is not so simple. To investigate the situation in this case, this thesis assumes that the time origin for signals is at zero and rigorously analyzes the response of a sampled-data system to periodic inputs.

Let \(\mathbb{R}, \mathbb{R}^+, \) and \(\mathbb{Z}^+\) denote the spaces of real, nonnegative real, and nonnegative integer numbers. Denote by \(\mathcal{L}(\mathbb{R}^+, \mathbb{R}^p)\) the linear space of all functions from \(\mathbb{R}^+\) to \(\mathbb{R}^p\), that is, continuous-time signals, and by \(\ell(\mathbb{Z}^+, \mathbb{R}^p)\) the space of all functions from \(\mathbb{Z}^+\) to \(\mathbb{R}^p\), discrete-time signals (sequences). A continuous-time signal \(w\) in \(\mathcal{L}(\mathbb{R}^+, \mathbb{R}^p)\) is periodic of period \(T\) (\(T > 0\)) if
Chapter 2. Fundamental Building Blocks

A familiar example is \( \cos \alpha t \) in which case \( T \) is equal to \( 2\pi/\alpha \). Let

\[
\mathcal{W}_T = \left\{ w : w \text{ is of period } T, \frac{1}{T} \int_0^T w(t)^* w(t) dt < \infty \right\},
\]

(2.2)

where prime denotes transpose (conjugate transpose is denoted by *). For \( w \) in \( \mathcal{W}_T \) define its power-norm by

\[
\|w\|_{\mathcal{W}_T} = \left[ \frac{1}{T} \int_0^T w(t)^* w(t) dt \right]^{1/2}.
\]

(2.3)

Analogously, a discrete-time signal \( \omega \) in \( \ell^2(\mathbb{Z}_+, \mathbb{R}^p) \) satisfying

\[
\omega(n + m) = \omega(n), \quad n \geq 0
\]

(2.4)

is periodic of period \( m \) \((m > 0)\). Define

\[
\Omega_m = \{ \omega : \omega \text{ is of period } m \}.
\]

For \( \omega \in \Omega_m \) define its power-norm by

\[
\|\omega\|_{\Omega_m} = \left[ \frac{1}{m} \sum_{n=0}^{m-1} \omega(n)^* \omega(n) \right]^{1/2}.
\]

(2.5)

For ease of notation, the norm subscripts are dropped hereafter.

For periodic discrete-time signals the following results are useful. First, recall the standard discrete Fourier transform (DFT) equations [OS89]:

\[
\tilde{\omega}(k) = \sum_{n=0}^{m-1} \omega(n) W^{nk}, \quad k \geq 0
\]

(2.6)

\[
\omega(n) = \frac{1}{m} \sum_{k=0}^{m-1} \tilde{\omega}(k) W^{-nk}, \quad n \geq 0,
\]

(2.7)
where \( W := e^{-j \frac{2\pi}{m}} \). Let \( \mathbb{C} \) denote the set of complex numbers. Hence, the DFT coefficients of a discrete-time periodic signal are in \( \mathbb{C} \) in general. Symbolically, we have

\[
\tilde{\omega}(k) \xrightarrow{\text{DFT}} \omega(n)
\]

\[
\omega(n) \xrightarrow{\text{DFT}^{-1}} \tilde{\omega}(k)
\].

Notice that \( \tilde{\omega}(k) \) is of period \( m \) too.

For a general (not necessarily periodic) discrete-time signal \( \phi(n) \), its \( \lambda \)-transform (where \( \lambda = z^{-1} \)) is defined by

\[
\hat{\phi}(\lambda) = \sum_{n=0}^{\infty} \phi(n) \lambda^n.
\] (2.8)

It follows from (2.7) and (2.8) that the \( \lambda \)-transform of a periodic signal \( \omega(n) \) can be written in the form

\[
\hat{\omega}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\tilde{\omega}(k)}{1 - \lambda W^{-k}}.
\] (2.9)

Conversely, one can easily show that any discrete-time signal \( \omega(n) \) whose \( \lambda \)-transform is or can be expressed in the form

\[
\hat{\omega}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\hat{\omega}(k)}{1 - \lambda W^{-k}}
\] (2.10)

for some \( \hat{\omega}(k) \in \mathbb{C} \), \( k = 0, 1, \ldots, m - 1 \), is periodic of period \( m \) and

\[
\tilde{\omega}(k) = \hat{\omega}(k), \quad k = 0, 1, \ldots, m - 1.
\] (2.11)

That is, \( \tilde{\omega}(k) \) is the DFT of \( \omega(n) \).

The following lemma shows that the power-norm of a periodic discrete-time signal equals the power-norm of its DFT, divided by \( m \).
Lemma 2.1 For $\omega(n)$ periodic of period $m$

$$
\|\omega\| = \frac{1}{m} \left[ \sum_{k=0}^{m-1} \tilde{\omega}(k)^* \tilde{\omega}(k) \right]^{1/2}.
$$

Proof Noting that

$$
\sum_{n=0}^{m-1} W^{qn} = \begin{cases} 
0, & q \neq 0 \\
m, & q = 0
\end{cases},
$$

we get

$$
\|\omega\|^2 = \frac{1}{m} \sum_{n=0}^{m-1} \omega(n)^* \omega(n)
= \frac{1}{m} \sum_{n=0}^{m-1} \left( \frac{1}{m} \sum_{k=0}^{m-1} \tilde{\omega}(k) W^{-kn} \right)^* \left( \frac{1}{m} \sum_{l=0}^{m-1} \tilde{\omega}(l) W^{-ln} \right)
= \frac{1}{m^3} \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \tilde{\omega}(k)^* \tilde{\omega}(l) \sum_{n=0}^{m-1} W^{(k-l)n}
= \frac{1}{m^2} \sum_{k=0}^{m-1} \tilde{\omega}(k)^* \tilde{\omega}(k).
$$


2.2 Input-Output Relationship for LTI Systems

Now, we will look at the behavior of stable multi-input, multi-output LTI discrete-time systems when they are excited by periodic inputs. We will verify that the response of such systems to periodic inputs converges to a steady-state component. Then, having the power-norm as a tool for measuring the size of periodic signals, we define a new measure called the induced power-norm for such systems which helps identify how big this steady-state component can be, if we know how big the periodic input is. But first a general definition.
Definition 2.1 Let $F$ be a linear transformation of a normed linear space $\mathcal{X}$ into a normed linear space $\mathcal{Y}$. The induced norm of $F$ is

$$\|F\| := \sup_{x \in \mathcal{X}, \|x\| \leq 1} \|Fx\|.$$ 

If $\|F\| < \infty$, then $F$ is called a bounded linear transformation.

Now, consider an LTI discrete-time system $P$ with input $v$, output $\psi$ and transfer matrix $\hat{p}(\lambda)$. It is assumed that $\hat{p}(\lambda)$ is defined and continuous on the closed unit disc and is analytic and bounded in its interior. Denote the set of all square-summable functions in $\ell(\mathbb{Z}^+, \mathbb{R}^p)$ by $\ell_2$. The assumption on $P$ then implies its stability (boundedness) on $\ell_2$. Examples of such systems are LTI systems with rational transfer functions with all the poles outside the closed unit disc. From (2.9), for $v$ periodic of period $m$ we have

$$\hat{v}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\hat{\nu}(k)}{1 - \lambda W^{-k}}.$$ 

So

$$\hat{\psi}(\lambda) = \hat{p}(\lambda) \hat{v}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\hat{p}(\lambda)}{1 - \lambda W^{-k}} \hat{\nu}(k).$$

From this, one can decompose $\hat{\psi}$ into steady-state and transient components, $\hat{\psi}_{ss}, \hat{\psi}_{tr}$:

$$\hat{\psi}(\lambda) = \hat{\psi}_{ss}(\lambda) + \hat{\psi}_{tr}(\lambda) \quad (2.13)$$

$$\hat{\psi}_{ss}(\lambda) := \frac{1}{m} \sum_{k=0}^{m-1} \frac{\hat{p}(W^k)}{1 - \lambda W^{-k}} \hat{\nu}(k) \quad (2.14)$$

$$\hat{\psi}_{tr}(\lambda) := \frac{1}{m} \sum_{k=0}^{m-1} \frac{\hat{p}(\lambda) - \hat{p}(W^k)}{1 - \lambda W^{-k}} \hat{\nu}(k). \quad (2.15)$$

The advantage of such a decomposition is the following. Since $\hat{p}$ is continuous, each term in (2.15) is analytic and bounded in $|\lambda| < 1$. Therefore $\hat{\psi}_{tr}(\lambda)$ is analytic and bounded in $|\lambda| < 1$. This implies that $\psi_{tr}(k) \to 0$. On the other hand, by
the discussion around (2.10), \( \hat{\psi}_{ss}(\lambda) \) represents a periodic signal of period \( m \). So eventually, \( \psi \) approaches its periodic steady-state component, \( \psi_{ss} \). Also, note that \( \nu \mapsto \psi_{ss} \) is a linear transformation on \( \Omega_m \).

**Definition 2.2** Suppose \( P \) is an \( \ell_2 \)-stable LTI discrete-time system. Its induced power-norm is

\[
J_{\psi v} := \sup_{\begin{array}{c}
u \in \Omega_m, \|\nu\| \leq 1 \\
\end{array}} \|\psi_{ss}\|.
\tag{2.16}
\]

**Remark 2.1** The induced power-norm depends on the input period, \( m \).

The next result gives an explicit formula for \( J_{\psi v} \). Let \( \bar{\sigma} \) denote the maximum singular value for a matrix [ZGD95].

**Lemma 2.2** For an \( \ell_2 \)-stable LTI discrete-time system \( P \),

\[
J_{\psi v} = \max_{0 \leq k \leq m-1} \bar{\sigma} [\hat{p}(W^k)].
\tag{2.17}
\]

**Proof** Lemma 2.1 gives

\[
\|\psi_{ss}\|^2 = \frac{1}{m^2} \sum_{k=0}^{m-1} \hat{\psi}_{ss}(k)^* \hat{\psi}_{ss}(k).
\]

From (2.9) and (2.14)

\[
\hat{\psi}_{ss}(k) = \hat{p}(W^k) \hat{\nu}(k).
\]

Thus

\[
\|\psi_{ss}\|^2 = \frac{1}{m^2} \sum_{k=0}^{m-1} [\hat{p}(W^k) \hat{\nu}(k)]^* [\hat{p}(W^k) \hat{\nu}(k)] \\
\leq \frac{1}{m^2} \sum_{k=0}^{m-1} \bar{\sigma} [\hat{p}(W^k)]^2 \|\hat{\nu}(k)\|^2 \\
\leq \max_k \bar{\sigma} [\hat{p}(W^k)]^2 \frac{1}{m^2} \sum_{k=0}^{m-1} \|\hat{\nu}(k)\|^2 \\
= \max_k \bar{\sigma} [\hat{p}(W^k)]^2 \|\nu\|^2.
\]
This proves that
\[ J_{\psi \nu} \leq \max_k \bar{\sigma} \left( \hat{p} (W^k) \right); \]
it remains to show that \( J_{\psi \nu} \) achieves this upper bound. In this respect, we construct a signal \( \nu \in \mathcal{W}_T \) which turns the above inequalities into equalities. In doing so, we note that \( \nu \) is restricted to be real by the definition of \( \Omega_m \) (page 14). Let \( k_{\text{max}} \) denote the index at which \( \bar{\sigma} \left( \hat{p} (W^k) \right) \) takes its maximum value and let \( \bar{\nu} \) be a vector where \( \| \hat{p} (W^{k_{\text{max}}}) \bar{\nu} \| \) achieves its maximum (Euclidean) norm. Now, \( W^{k_{\text{max}}} \) is either real or complex. If it is real, which will happen if \( k_{\text{max}} = 0 \) or \( k_{\text{max}} = m/2 \) for even \( m \), then \( \hat{p} (W^{k_{\text{max}}}) \) and, thus, \( \bar{\nu} \) are real. In this case, signal \( \nu \) with DFT coefficients
\[ \bar{\nu}(k) = \begin{cases} \bar{\nu}, & k = k_{\text{max}} \\ 0, & \text{else} \end{cases} \]
will be real and it can be easily verified that \( \nu \) clears all the above inequalities to equalities. For complex \( W^{k_{\text{max}}} \), \( \hat{p} (W^{k_{\text{max}}}) \) is complex and, thus, \( \bar{\nu} \) is complex. In this case, it suffices to set
\[ \bar{\nu}(k) = \begin{cases} \bar{\nu}, & k = k_{\text{max}} \\ \bar{\nu}, & k = m - k_{\text{max}} \\ 0, & \text{else} \end{cases} \]
where bar denotes the conjugate, and note that \( \hat{p} (W^{m-k_{\text{max}}}) = \bar{\hat{p}} (W^{k_{\text{max}}}) \) and \( \| \hat{p} (W^{m-k_{\text{max}}}) \bar{\nu} \| = \bar{\sigma} \left( \hat{p} (W^{m-k_{\text{max}}}) \right) \| \bar{\nu} \| \).

**Example:** Consider the multivariable system
\[
\hat{p}(\lambda) = \begin{bmatrix}
\lambda^2 + 1 \\
\lambda^2 + 0.25\lambda + 1.1 \\
\lambda^2 - 0.8\lambda + 1.2 \\
\lambda \\
\end{bmatrix}.
\]
This system is analytic for \( |\lambda| \leq 1 \) and hence is stable. So by the analysis performed in this section, the response of this system to a periodic input approaches a steady state that is periodic with the same period as that of the input. Let us compute the induced power-norm of this system for periodic inputs of period \( m = 5 \). So
$W = e^{-j \frac{2\pi}{5}}$. By Lemma 2.2, we only have to compute the maximum singular value of $\hat{p}(\lambda)$ for 5 values of $\lambda$, that is, for $W^0, W^1, W^2, W^3$ and $W^4$. These values are listed in Table 2.1. Thus by (2.17), $J_{\psi \nu} = 4.5795$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\sigma(\hat{p}(\lambda))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W^0$</td>
<td>1.7969</td>
</tr>
<tr>
<td>$W^1$</td>
<td>4.5795</td>
</tr>
<tr>
<td>$W^2$</td>
<td>1.7818</td>
</tr>
<tr>
<td>$W^3$</td>
<td>1.7818</td>
</tr>
<tr>
<td>$W^4$</td>
<td>4.5795</td>
</tr>
</tbody>
</table>

Table 2.1: Maximum singular values of $\hat{p}$.

We have also brought in Table 2.2 the induced power-norm of the given system for different input periods. From this table, we observe that the induced power-norm is a function of the input period, $m$, confirming Remark 2.1. Also, we see that for large values of $m$, $J_{\psi \nu}$ converges to a final value. This can be explained from the maximum singular value plot of $\hat{p}$ in Figure 2.1, as follows. As we consider larger periods, more points from the singular value plot participate in the induced power-norm computation, as is suggested by Lemma 2.2. Hence, for large periods, the induced power-norm of the system approaches the maximum of this plot, 5.4915 in this case. This maximum is in fact the infinity norm of $\hat{p}$, $\| \hat{p} \|_\infty$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{\psi \nu}$</td>
<td>1.80</td>
<td>1.80</td>
<td>4.58</td>
<td>4.58</td>
<td>4.58</td>
<td>4.58</td>
<td>5.00</td>
<td>5.26</td>
<td>5.40</td>
<td>5.47</td>
<td>5.49</td>
</tr>
</tbody>
</table>

Table 2.2: Maximum singular value of $\hat{p}$.

### 2.3 Function-Valued Periodic Signals

Discrete-time periodic signals that were introduced in Section 2.1 are vector-valued, that is, they take values in $\mathbb{R}^p$. In contrast to these finite-dimensional signals, we
will have discrete-time function-valued periodic signals that are infinite-dimensional. These signals are useful in sampled-data repetitive control.

Let $h > 0$; it will be the sampling period in later sections. Denote by $\mathcal{K}$ the space of functions from $[0, h)$ to $\mathbb{C}^r$. Elements of this space are vector-valued continuous-time functions defined over the interval $[0, h)$. Define

$$\mathcal{K}_p := \left\{ v : v \in \mathcal{K}, \|v\| := \left[ \int_0^h \|v(t)\|^p \right]^{1/p} < \infty \right\}$$

for $p < \infty$ and

$$\mathcal{K}_\infty := \left\{ v : v \in \mathcal{K}, \|v\| := \text{ess sup}_{t \in [0,h]} \|v(t)\| < \infty \right\}$$

for $p = \infty$. Specifically, we are interested in $\mathcal{K}_2$, that is, the space of all functions in $\mathcal{K}$ that are square-integrable. Then denote by $\ell(\mathbb{Z}_+, \mathcal{K})$ the space of all functions from $\mathbb{Z}_+$ to $\mathcal{K}$, function-valued discrete-time signals and by $\ell(\mathbb{Z}_+, \mathcal{K}_2)$ the space of all function-valued discrete-time signals that take values in $\mathcal{K}_2$. A signal $\omega$ in $\ell(\mathbb{Z}_+, \mathcal{K}_2)$ of period $m$ is then defined exactly in the same way that a vector-valued periodic signal is defined, that is, by (2.4). The difference is that now $\omega$ takes values in $\mathcal{K}_2$. Denote by $\Omega_m(\mathcal{K}_2)$ the space of all function-valued periodic signals of period $m$ and
define the power-norm for $\omega$ in $\Omega_m(\mathcal{K}_2)$ by

$$
\|\omega\|_{\Omega_m(\mathcal{K}_2)} := \left[ \frac{1}{m} \sum_{n=0}^{m-1} \|\omega(n)\|_{\mathcal{K}_2}^2 \right]^{1/2}.
$$

(2.18)

Observing that discrete Fourier transform equation pair (2.6, 2.7) do not depend on the space where the periodic signals take their values in, they will still be valid for function-valued periodic signals. However, the Fourier transform takes values in $\mathcal{K}_2$ now.

The next lemma gives a frequency-domain expression for the power-norm of function-valued periodic signals. The proof is similar to that of Lemma 2.1 and hence is omitted.

**Lemma 2.3** For $v(n) \in \Omega_m(\mathcal{K}_2)$

$$
\|v\| = \frac{1}{m} \left[ \sum_{k=0}^{m-1} \|\hat{v}(k)\|^2 \right]^{1/2}.
$$

(2.19)

The notion of $\lambda$-transform too can be extended to function-valued signals [Hil48, SNF70]. Let $\ell_2(\mathcal{K}_2)$ denote the space of functions from $\mathbb{Z}_+$ to $\mathcal{K}_2$ that are squaresummable:

$$
\ell_2(\mathcal{K}_2) = \left\{ v : \|v\|_{\ell_2(\mathcal{K}_2)}^2 := \sum_{n=0}^{\infty} \|v(n)\|_{\mathcal{K}_2}^2 < +\infty \right\}.
$$

Denote the open unit disk, the closed unit disk, and the unit circle by $\mathbb{D}$, $\overline{\mathbb{D}}$, and $\partial\mathbb{D}$, respectively. A function $\hat{v} : \mathbb{D} \rightarrow \mathcal{K}_2$ is analytic if for each $\lambda_0 \in \mathbb{D}$ the limit

$$
\lim_{\lambda \to \lambda_0} \frac{\hat{v}(\lambda) - \hat{v}(\lambda_0)}{\lambda - \lambda_0}
$$

exists in $\mathcal{K}_2$. Define $\mathcal{H}_2(\mathbb{D}, \mathcal{K}_2)$ to be the space of functions mapping $\mathbb{D}$ to $\mathcal{K}_2$ that are analytic on $\mathbb{D}$ and such that

$$
\|\hat{v}\|_{\mathcal{H}_2(\mathbb{D}, \mathcal{K}_2)}^2 := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|\hat{v}(r e^{i\theta})\|_{\mathcal{K}_2}^2 \, d\theta < \infty.
$$
For \( v \in \ell_2(\mathcal{K}_2) \), define the \( \lambda \)-transform

\[
\hat{v}(\lambda) = \sum_{n=0}^{\infty} v(n) \lambda^n.
\]

From [SNF70], \( \hat{v} \in \mathcal{H}_2(\mathbb{D}, \mathcal{K}_2) \). Moreover from the same reference, the following proposition holds, where \( \Lambda \) denotes the mapping from \( v \) to \( \hat{v} \).

**Proposition 2.1** The mapping \( \Lambda : \ell_2(\mathcal{K}_2) \rightarrow \mathcal{H}_2(\mathbb{D}, \mathcal{K}_2) \) defines an isometric isomorphism.

This simply means that this mapping is linear, one-to-one, onto and norm preserving.

Here too, as in (2.9), the \( \lambda \)-transform of a periodic signal \( v \in \Omega_m(\mathcal{K}_2) \) can be written in the form

\[
\hat{v}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\tilde{v}(k)}{1 - \lambda W^{-k}}.
\]  \hspace{1cm} (2.20)

As well, a discrete-time function-valued signal \( v \) with a \( \lambda \)-transform that can be expressed as

\[
\hat{v}(\lambda) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{\tilde{v}(k)}{1 - \lambda W^{-k}},
\]  \hspace{1cm} (2.21)

for some \( \tilde{v}(k) \in \mathcal{K}_2, \ k = 0, 1, \ldots, m - 1 \), is a periodic signal of period \( m \) and its DFT, \( \tilde{v}(k) \), equals \( \tilde{v}(k) \).

The next section will tell us how function-valued signals appear in our analysis of sampled-data systems.

### 2.4 Lifting and Periodic Systems

Lifting is a mathematical technique that is used to associate LTI systems to a certain kind of time-varying systems, namely *periodic systems*. We shall see the definition of lifting and periodic systems momentarily, but the advantage of lifting is already clear: By lifting we get LTI systems. The lifting technique will be a very important developmental tool in this thesis.
There are two types of lifting, continuous-time and discrete-time, which will be discussed in the sequel.

**Continuous-Time Lifting and Periodic Systems**

*Continuous lifting* $L_c$ is a linear transformation that allows converting a vector-valued continuous-time signal into a function-valued discrete-time, defined by

$$L_c : \mathcal{L}(\mathbb{R}_+, \mathbb{C}^p) \rightarrow \ell(\mathbb{Z}_+, \mathcal{K}), \quad v = L_c v$$

$$ [v(n)](\tau) = v(\tau + nh), \quad \tau \in [0, h), \quad n \in \mathbb{Z}_+. $$

This is depicted in Figure 2.2. Continuous lifting was introduced by Yamamoto [Yam90]. We see that the lifted signal is a discrete-time signal that takes values in $\mathcal{K}$; it is a function-valued signal. The block diagram of Figure 2.3 is used to represent lifting.

![Figure 2.2: Continuous lifting illustrated.](image)

![Figure 2.3: Block diagram for lifting.](image)
The inverse of lifting, $L_c^{-1}$, exists and is given by

$$L_c^{-1} : \ell(\mathbb{Z}_+, \mathcal{K}) \to \mathcal{L}(\mathbb{R}_+, \mathcal{C}^p), \quad v = L_c^{-1}u$$

$$v(\tau + nh) = [v(n)](\tau), \quad \tau \in [0, h), \quad n \in \mathbb{Z}_+.$$

We are particularly interested in the operation of lifting on $\mathcal{L}_2(\mathbb{R}_+, \mathcal{C}^p)$, abbreviated by $L_2$ from now on, and on $\mathcal{W}_T$. This allows us to have some nice properties for lifting including the following result [BPFT91].

**Proposition 2.2** $L_c$ and $L_c^{-1}$ are isometric isomorphisms between $L_2$ and $\ell_2(\mathcal{K}_2)$.

A similar result exists for periodic signals. Let $\mathbb{N}$ denote the set of positive integers.

If $T$ is an integer multiple of $h$, that is, $T = Mh$ for some $M \in \mathbb{N}$, then for $w \in \mathcal{W}_T$, $w := L_c w \in \Omega_M(\mathcal{K}_2)$ and

$$\|w\|_{\mathcal{W}_T} = \|w\|_{\Omega_M(\mathcal{K}_2)}.$$

The following result gives a restatement of this fact.

**Proposition 2.3** If $T = Mh$ for some $M \in \mathbb{N}$, then $L_c$ is an isometric isomorphism from $\mathcal{W}_T$ to $\Omega_M(\mathcal{K}_2)$.

For any normed space $\mathcal{X}$, denote by $L(\mathcal{X})$ the space of all bounded linear transformations on $\mathcal{X}$. Then it follows from Proposition 2.2 that for $P \in L(L_2)$, the lifted system, $P := L_cP L_c^{-1}$, is in $L(\ell_2(\mathcal{K}_2))$ and their induced norms are equal.

Now we will see how lifting helps to associate LTI systems to periodic systems. Let $D_h$ denote the delay-by-$h$ operator on $L_2$, that is, for $f \in L_2$,

$$(D_h f)(t) = \begin{cases} 0 & 0 \leq t < h \\ f(t - h) & t \geq h \end{cases}$$

and let us denote the unilateral shift on $\ell_2(\mathcal{K}_2)$ by $U$, that is, for $v \in \ell_2(\mathcal{K}_2)$, $(Uv)(k) = v(k - 1)$. One can then see that $UL_c = L_cD_h$ and $D_hL_c^{-1} = L_c^{-1}U$.

An operator $P : L_2 \to L_2$ is called $h$-periodic if it satisfies

$$PD_h = D_hP. \quad (2.22)$$
In other words, delaying the input to an $h$-periodic system by $h$ delays the output by the same amount. For $h$-periodic $P$, the lifted system, $P = L_cPL_c^{-1}$, satisfies

$$
UP = UL_cPL_c^{-1} = L_cD_hPL_c^{-1} = L_cPD_hL_c^{-1} = L_cPL_c^{-1}U = PU.
$$

This shows that the lifted system is LTI. In fact the following statement holds.

**Proposition 2.4** The mapping $P \to P$ is an isometric isomorphism from the space of bounded $h$-periodic operators on $\mathcal{L}_2$ to the space of bounded LTI operators on $\ell_2(K_2)$.

By the definition of $h$-periodic systems, all LTI operators on $\mathcal{L}_2$ are $h$-periodic for all $h > 0$. Another familiar example is the standard sampled-data system drawn in Figure 2.4. The generalized plant $G$ is continuous-time FDLTI with state-space representation

$$
\hat{g}(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & 0 & 0
\end{bmatrix},
$$

and the controller $K$ is discrete-time LTI. The sampling operator $S$ and the zero-order hold operator $H$ are of period $h$ and defined as follows:

$$
S : \mathcal{L}(\mathbb{R}_+, \mathbb{R}^p) \to \ell(\mathbb{Z}_+, \mathbb{R}^p), \quad \psi = Su \\
\psi(k) = u(kh) \\
H : \ell(\mathbb{Z}_+, \mathbb{R}^p) \to \mathcal{L}(\mathbb{R}_+, \mathbb{R}^p), \quad y = Hv \\
y(t) = v(k), \quad t \in [kh, (k+1)h).
$$
It is straightforward to check that the system from $w$ to $z$, $\Phi_K$, in Figure 2.4 too is $h$-periodic and hence by (2.23) the lifted system, $\Phi_K = L_c \Phi_K L_c^{-1}$, displayed in Figure 2.5, is LTI. First absorb the lifting operator together with the sample and hold devices into the plant $G$ to arrive at Figure 2.6. The lifted plant $G$ is

$$
G = \begin{bmatrix}
  L_c & 0 \\
  0 & S \\
\end{bmatrix}
\begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22} \\
\end{bmatrix}
\begin{bmatrix}
  L_c^{-1} & 0 \\
  0 & H \\
\end{bmatrix}.
$$

(2.24)

Figure 2.5: Continuous lifting to get a LTI system.

Given the state space representation for $G$, one can obtain a state space representation for $\hat{G}$ [BPFT91]. We have

$$
\hat{g}(\lambda) = \begin{bmatrix}
  A_d & B_1 & B_{2d} \\
  C_1 & D_{11} & D_{12} \\
  C_2 & 0 & 0
\end{bmatrix},
$$

(2.25)
where

\[ A_d = e^{hA}, \quad B_{2d} = \int_0^h e^{rA} B_2 d\tau, \]

and

\[
\begin{align*}
B_1 : \mathcal{K} \to \mathbb{E}, & \quad B_1 u = \int_0^h e^{(h-\eta)A} B_1 u(\eta) \, d\eta \\
C_1 : \mathbb{E} \to \mathcal{K}, & \quad (C_1 x)(\tau) = C_1 e^{\tau A} x \\
D_{11} : \mathcal{K} \to \mathcal{K}, & \quad (D_{11} u)(\tau) = C_1 \int_0^\tau e^{(\tau-\eta)A} B_1 u(\eta) \, d\eta \\
D_{12} : \mathbb{E} \to \mathcal{K}, & \quad (D_{12} v)(\tau) = D_{12} v + C_1 \int_0^\tau e^{(\tau-\eta)A} d\eta B_2 v;
\end{align*}
\]

here \( \mathbb{E} \) denotes \( \mathbb{R}^n \) for any \( n \). Assuming that the state space representation for the FDLTI controller \( K \) is given by

\[
\dot{k}(\lambda) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},
\]

that of \( \Phi_K \) will be

\[
\dot{\Phi}_K(\lambda) = \begin{bmatrix} A_{cd} & B_{cd} \\ C_{cd} & D_{cd} \end{bmatrix},
\]

where

\[
A_{cd} = \begin{bmatrix} A_d + B_{2d}D_KC_2 & B_{2d}C_K \\ B_KC_2 & A_K \end{bmatrix},
\]

\[
B_{cd} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},
\]

\[
C_{cd} = \begin{bmatrix} C_1 + D_{12}D_KC_2 & D_{12}C_K \end{bmatrix},
\]

\[
D_{cd} = D_{11}.
\]

The notion of transfer function can be generalized to operators on \( \ell_2(\mathcal{K}_2) \) such as
Figure 2.6: LTI lifted system.

lifted periodic systems. The function \( \hat{g} : \hat{\mathbb{D}} \rightarrow \mathbf{L}(\mathbb{K}_2) \) is analytic at \( \lambda_0 \in \mathbb{D} \) if

\[
\lim_{\lambda \to \lambda_0} \frac{\hat{g}(\lambda) - \hat{g}(\lambda_0)}{\lambda - \lambda_0}
\]

exists in \( \mathbf{L}(\mathbb{K}_2) \). Let \( \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \) be the normed space of functions \( \hat{g} \) from \( \hat{\mathbb{D}} \) to \( \mathbf{L}(\mathbb{K}_2) \) that are analytic on \( \mathbb{D} \), along with the norm

\[
\|\hat{g}\| := \sup_{|\lambda| < 1} \|\hat{g}(\lambda)\|.
\]  

(2.37)

For element \( \hat{g} \in \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \), define the multiplication operator, \( \hat{\Theta}_{\hat{g}} : \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \rightarrow \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \), by

\[
\left( \hat{\Theta}_{\hat{g}} \hat{\nu} \right)(\lambda) = \hat{g}(\lambda)\hat{\nu}(\lambda)
\]

for \( \hat{\nu} \in \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \) and \( \lambda \in \mathbb{D} \). The following connection holds between the linear operators on \( \ell_2(\mathbb{K}_2) \) and functions in \( \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \), [FF90].

**Proposition 2.5** (i) If \( G : \ell_2(\mathbb{K}_2) \rightarrow \ell_2(\mathbb{K}_2) \) is bounded, LTI and causal, then there exists \( \hat{g} \in \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \) so that \( G = \Lambda^{-1}\hat{g}\Lambda \). (ii) Every multiplication operator \( \hat{\Theta}_{\hat{g}} \), defined from a function \( \hat{g} \in \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \), defines a bounded operator on \( \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \).

Moreover, the \( \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \rightarrow \mathcal{H}_2(\mathbb{D}, \mathbb{K}_2) \) induced norm of the operator is equal to \( \|\hat{g}\|_\infty \).

With this result, one can see that \( \mathcal{H}_\infty(\mathbb{D}, \mathbf{L}(\mathbb{K}_2)) \) is isomorphic with the space of bounded LTI operators on \( \ell_2(\mathbb{K}_2) \), i.e., it is formed of the transfer functions for such operators. These transfer functions, which are operator-valued, could even have
discontinuity on the unit circle. It is technically more convenient to work with a subspace of $\mathcal{H}_\infty(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))$, denoted by $\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))$, which consists of members of $\mathcal{H}_\infty(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))$ that are continuous on the unit circle. For $\hat{g} \in \mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))$, the norm can be written as

$$||\hat{g}||_\infty = \max_{\lambda \in \hat{D}} ||\hat{g}(\lambda)||.$$ 

Now, define

$$\mathcal{L}_{\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))} := \left\{ L_c^{-1} \Lambda^{-1} \hat{\theta}_p \Lambda L_c : \hat{p} \in \mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2)) \right\}. \quad (2.38)$$

Thus, members of $\mathcal{L}_{\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}))}$ have the transfer function of their lifted associates in $\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))$. From Propositions 2.1, 2.2 and 2.5, it is immediate that $\mathcal{L}_{\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))}$ is a subspace of $\mathcal{L}(\mathcal{L}_2)$. Also, it is straightforward to check that members of $\mathcal{L}_{\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))}$ commute with $D_h$. That is, they are $h$-periodic. However, there are $h$-periodic operators that don't find themselves in $\mathcal{L}_{\mathcal{A}(\mathbb{D}, \mathcal{L}(\mathcal{K}_2))}$, those for which the transfer function for the associated lifted system is not continuous on $\hat{D}$.

**Discrete-Time Lifting and Periodic Systems**

Discrete-time lifting is a mathematical construction similar to continuous-time lifting. This type of lifting is commonly used (under the name "blocking") in multirate signal processing [Vai93] and is due to Friedland [Fri60]. By using lifting one can convert a multirate periodic system to a single-rate system. Define the *lifting operator* $L$ by

$$L : \ell(\mathbb{Z}_+, \mathbb{R}^p) \rightarrow \ell(\mathbb{Z}_+, \mathbb{R}^{pn}), \quad v = Lv$$

$$v(k) = \begin{bmatrix} v(kN) \\ v(kN + 1) \\ \vdots \\ v(kN + N - 1) \end{bmatrix}.$$ 

So, if $v$ is a signal which is referred to subperiod $h/N$, its *lifted* associate $v$ can be referred to period $h$. Note though that the dimension of the lifted signal is $N$ times that of $v$. 
The inverse of lifting, $L^{-1}$, is defined by

$$L^{-1}: \ell(\mathbb{Z}_+, \mathbb{R}^N) \rightarrow \ell(\mathbb{Z}_+, \mathbb{R}^p), \quad u = L^{-1}v$$

$$
\begin{bmatrix}
 v(kN) \\
 v(kN + 1) \\
 \vdots \\
 v(kN + N - 1)
\end{bmatrix}
= u(k).
$$

From these definitions, we see that $L$ ($L^{-1}$) maps periodic signals of period $MN$ ($M$) to periodic signals of period $M$ ($MN$). Moreover the power-norm is scaled up (down) under $L$ ($L^{-1}$) by the constant factor $\sqrt{N}$, since for $\nu = Lv$, where $v$ is periodic of period $MN$, we have

$$
\|\nu\|^2 = \frac{1}{M} \sum_{k=0}^{M-1} v(k)^t v(k)
$$

$$
= \frac{1}{M} \sum_{k=0}^{M-1} \begin{bmatrix}
 v(kN) \\
 v(kN + 1) \\
 \vdots \\
 v(kN + N - 1)
\end{bmatrix}^t \begin{bmatrix}
 v(kN) \\
 v(kN + 1) \\
 \vdots \\
 v(kN + N - 1)
\end{bmatrix}
$$

$$
= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} v(kN + l)^t v(kN + l)
= \frac{1}{M} \sum_{i=0}^{MN-1} v(i)^t v(i)
= N\|v\|^2.
$$

So

$$
\|\nu\| = \sqrt{N}\|v\|. 
\quad (2.39)
$$

Figure 2.7 displays a multirate system that will be used in later sections. Here,
the generalized plant $G$ is continuous-time FDLTI with

$$
\hat{g}(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix}
$$

and the controller $K$ is discrete-time FDLTI. Samplers $S$ and $S_N$ are periodic of periods $h$ and $h/N$, respectively, and synchronized with them correspondingly are hold devices $H$ and $H_N$. This is an example of $N$-periodic systems for which the output shifts by $N$ samples if the input does. Similar to the pure sampled-data case, discrete lifting can be used to associate an LTI system to this periodic system. First absorb the samplers and holds into the plant $G$ to get the setup in Figure 2.8. Here

$$
P := \begin{bmatrix} S_N & 0 \\ 0 & S \end{bmatrix} G \begin{bmatrix} H_N & 0 \\ 0 & H \end{bmatrix}.
$$

(2.40)

Then introduce the discrete-time lifting operator and its inverse in this setup to get the setup in Figure 2.9. The system from $\omega$ to $\zeta$ is single rate. Take the lifting and its inverse into $P$ as in Figure 2.10 where

$$
P := \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix}.
$$

(2.41)
It is straightforward to show that $P$ is LTI. A state space representation for $P$ from [CF95] is as follows. Define

$$A_d = e^{hA}, \quad B_{2d} = \int_0^h e^{\tau A} d\tau$$
$$A_f = e^{hA/N}, \quad [B_{1f}, B_{2f}] = \int_0^{h/N} e^{\tau A} d\tau [B_1, B_2]$$

(2.42)

$$B_{1d} = \begin{bmatrix}
A_{f}^{N-1}B_{1f} & A_{f}^{N-2}B_{1f} & \cdots & B_{1f}
\end{bmatrix}$$

(2.43)

$$C_{1d} = \begin{bmatrix}
C_1 \\
C_1A_f \\
\vdots \\
C_1A_f^{N-1}
\end{bmatrix}$$

(2.44)
We have now developed enough tools for our treatment of sampled-data repetitive control systems.

\[
D_{11d} = \begin{bmatrix}
D_{11} & 0 & \cdots & 0 \\
C_1B_{1f} & D_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_1A_f^{N-2}B_{1f} & C_1A_f^{N-3}B_{1f} & \cdots & D_{11}
\end{bmatrix}
\]

\[
D_{12d} = \begin{bmatrix}
D_{12} \\
C_1B_{2f} + D_{12} \\
\vdots \\
C_1A_f^{N-2}B_{2f} + \cdots + C_1B_{2f} + D_{12}
\end{bmatrix}
\]

\[
D_{21d} = \begin{bmatrix}
D_{21} & 0 & \cdots & 0
\end{bmatrix}
\]

Then

\[
\hat{p}(\lambda) = \begin{bmatrix}
A_d & B_{1d} & B_{2d} \\
C_{1d} & D_{11d} & D_{12d} \\
C_2 & D_{21d} & 0
\end{bmatrix}
\]

Figure 2.10: Single-rate lifted system.
Chapter 3

Repetitive Control: An Overview

In this chapter, we will have a quick overview of repetitive control systems in continuous time. Our purpose in doing so is to study a fundamental limitation of repetitive control systems. This will then lead us to a design methodology which forms the main idea of this thesis.

3.1 Repetitive Control and The Internal Model Principle

Consider the unity-feedback setup shown in Figure 3.1 where the plant $P$ is SISO FDLTI and the input $w$ is an arbitrary periodic input of period $T$. We intend to design the controller $K$ to make the output $y$ follow $w$, that is, our goal is to get the tracking error $z$ to go to zero as we proceed in time.

In tracking problems like this, the internal model principle proposed by Francis and Wonham [FW75] plays a key role. According to this principle, for the tracking error $z$ to go to zero in the steady state, it is necessary and sufficient that the generator for the reference command be included in the stable closed loop. By the generator for a reference command, we mean a linear system $K_g$ which for some initial conditions and no input generates that reference command at its output. For instance, the generator for step reference commands would be just a simple integrator. Inclusion of $K_g$ in the
Figure 3.1: A repetitive control system; \( w \) is periodic of period \( T \).

A stable closed loop would then mean a system such as the one in Figure 3.2, where \( K_s \) is a stabilizing controller. In other words, controller \( K \) of Figure 3.1 has two components: a generator of the reference command, \( K_g \), and a stabilizing controller, \( K_s \).

Figure 3.2: Generator \( K_g \) and stabilizing controller \( K_s \) bring tracking to the system.

Now, let's denote the transfer function of \( K_g \) by \( \hat{k}_g(s) \). Then in the case of step tracking, \( \hat{k}_g(s) = 1/s \), that is, the pole of \( \hat{k}_g(s) \) is the same as the pole of the Laplace transform of the reference command - in this case, a step signal. Analogously, if tracking of a sinusoid, say, \( \sin(2\pi t/T) \), were desirable, \( \hat{k}_g(s) \) would need two poles at \( \pm j2\pi/T \), i.e., the poles of the Laplace transform for \( \sin(2\pi t/T) \). In other words,

\[
\hat{k}_g(s) = \frac{1}{s^2 + (2\pi/T)^2}
\]  

(3.1)

would be a candidate to be taken in the loop.

An arbitrary periodic input \( w \in \mathcal{W}_T \), however, may have an infinite number of
harmonics [WB82], with a Fourier series representation given by

$$w(t) = \sum_{n=-\infty}^{\infty} \tilde{w}(n)e^{j2\pi nt}, \quad t \geq 0.$$  

(3.2)

Intuitively, the internal model principle then suggests that the loop should include an infinite product of the form

$$\hat{k}_g(s) = \cdots \times \frac{-j2\pi/T}{s - j2\pi/T} \times \frac{1}{s} \times \frac{j2\pi/T}{s - j2\pi/T} \times \cdots$$

$$= \frac{1}{s} \prod_{n=1}^{\infty} \left(\frac{(2n\pi/T)^2}{s^2 + (2n\pi/T)^2}\right).$$

Based on the identity

$$\sinh(\pi s) = \pi s \prod_{n=1}^{\infty} \left(1 + \frac{s^2}{n^2}\right),$$

Yamamoto shows in [Yam93] that

$$\hat{k}_g(s) = T e^{-T s/2} \frac{1}{1 - e^{-T s}}.$$  

Since $T e^{-T s/2}$ is just a delay term, we can simply take

$$\hat{k}_g(s) = \frac{1}{1 - e^{-T s}}$$  

(3.3)

to act as the generator of the reference command. In fact, by looking at the block diagram for $\hat{k}_g(s)$ in Figure 3.3, we observe that $\hat{k}_g(s)$ can produce any periodic waveform, by just being exposed to one period of that waveform at its input.

There is however a subtle point that needs to be emphasized here. The internal model principle in its original form assumes that the generator for the reference commands is finite-dimensional, which is certainly not the case with (3.3). So the legitimacy of using the internal model principle needs to be verified in the infinite-dimensional case. This was done by Yamamoto in [Yam93]. Specifically, he proved the need for the infinite-dimensional compensator (3.3) for perfect tracking.

Despite the applicability of the internal model principle in the infinite-dimensional
case, perfect tracking of periodic signals is not possible for all plants. This was shown by Hara and Yamamoto in [HY85] where they proved that with (3.3) in the loop, stabilization cannot be achieved for strictly proper plants, which represent most practical systems.

Denote by $\mathcal{H}_\infty$ the space of all scalar complex-valued functions of a complex variable that are analytic and bounded in the open right half-plane.

**Definition 3.1** Controller $K_s$ is *admissible* if its transfer function is proper and the ratio of two functions in $\mathcal{H}_\infty$.

The class of admissible controllers, hence, includes all the FDLTI controllers, which have real rational transfer functions, as well as controllers that involve delays, such as (3.3).

**Theorem 3.1** In Figure 3.2, suppose that $P$ is strictly proper and that $\hat{k}_y(s)$ is given by (3.3). Then there is no admissible controller $K_s$ that can stabilize the closed loop.

We present a proof here which compared to the original one in [HY85] is more concise\(^1\). First we need a definition.

**Definition 3.2** Functions $a, b \in \mathcal{H}_\infty$ are *strongly coprime* if there exist $x, y \in \mathcal{H}_\infty$ such that $ax + by = 1$.

\(^1\)I am grateful to Abie Feintuch for suggesting this proof.
The celebrated Corona Theorem from complex function theory can be called to check for the strong coprimeness of a pair of functions \( a, b \in \mathcal{H}_\infty \). Let \( \mathbb{C}_+ \) denote the closed right half-plane.

**Lemma 3.1** [Gar81] For a pair of functions \( a, b \in \mathcal{H}_\infty \) to be strongly coprime, it is necessary and sufficient that \( \inf_{s \in \mathbb{C}_+} [ |a(s)| + |b(s)| ] > 0 \).

Then, let \( G \) be a given LTI system with transfer function \( \hat{g}(s) \). Unlike \( P \) in Figure 3.1, \( G \) is not required to be finite-dimensional.

**Definition 3.3** Transfer function \( \hat{g} \) is said to have a *strong coprime factorization* over \( \mathcal{H}_\infty \) if there are strongly coprime \( a, b \in \mathcal{H}_\infty \) such that \( \hat{g} = a/b \).

We need one more result for the proof of Theorem 3.1.

**Theorem 3.2** [Smi89] In Figure 3.4, suppose that \( G \) is stabilizable, that is, there is an admissible controller \( K_s \) that stabilizes the closed loop. Then \( \hat{g} \) has a strong coprime factorization over \( \mathcal{H}_\infty \).

**Proof of Theorem 3.1** Let

\[
\hat{g}(s) = \hat{p}(s) \frac{1}{1 - e^{-Ts}}.
\]

This choice for \( \hat{g} \) makes the setup of Figure 3.4 to be the same as the setup of interest in Figure 3.2 with the repetitive generator (3.3) in the loop. Then, let

\[
\hat{p}(s) = \frac{\hat{n}(s)}{\hat{d}(s)}
\]

be a coprime factorization of \( \hat{p} \), where \( \hat{n} \) and \( \hat{d} \) are in \( \mathcal{H}_\infty \) [DFT92]. Since \( P \) is strictly proper, \( \hat{n} \) will have an additional property that

\[
\lim_{s \to \infty} \hat{n}(s) = 0.
\]

Also \( 1 - e^{-Ts} \) has arbitrarily large roots on the imaginary axis. Thus

\[
\inf_{s \in \mathbb{C}_+} \left[ |\hat{n}(s)| + \left| \hat{d}(s) (1 - e^{-Ts}) \right| \right] = 0,
\]
Figure 3.4: Plant $P$ and generator $K_g$ are now grouped into $G$.

So by Lemma 3.1 the numerator and the denominator in

$$
\hat{g}(s) = \frac{\hat{n}(s)}{d(s)(1 - e^{-Ts})}.
$$

are not strongly coprime. By Theorem 3.2 then, $G$ is not stabilizable, which implies that $P$ with (3.3) in the loop is not stabilizable.

This result in fact shows that simultaneous stabilization and tracking of unknown periodic signals is not achievable when the plant is strictly proper. There is an intuitive explanation for this fundamental limitation as well. A strictly proper plant attenuates the high frequency harmonics to a great extent. As a result, the controller will try to compensate for this by creating extremely large inputs to the plant which will consequently result in the instability of the system.

Now that perfect tracking of arbitrary periodic signals is not possible, a compromise has to be made. We will discuss in the next section an outline of the approach of this thesis in making this compromise, as well as the proposed design methodology in the next section.

### 3.2 The Induced Power-Norm Approach

In this section, we intend to formulate a design problem for continuous-time repetitive control systems. Our sole reason in doing this formulation is to motivate the approach
of this thesis; hence we will not solve this problem. It should be emphasized that this formulation is not restricted to the SISO case.

Let \( G \) be a stable multi-input, multi-output FDLTI continuous-time plant with input \( w \), output \( z \) and transfer matrix \( \hat{g}(s) \). A well-known fact is that for \( w \in \mathcal{W}_T \), the output \( z \) is the sum of a transient response and a periodic signal of the same period \( T \). In fact if we denote this periodic signal by \( z_{ss} \) and bring in the Fourier series representation (3.2) for \( w \), we get

\[
z_{ss}(t) = \sum_{n=-\infty}^{\infty} \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \tilde{w}(n) e^{j \frac{2\pi n t}{T}} , \quad t \geq 0.
\]  

(3.4)

It is also clear that the mapping from \( w \) to \( z_{ss} \) is linear on \( \mathcal{W}_T \).

**Definition 3.4** For a stable FDLTI plant \( G \), the *induced power-norm* is

\[
J_{zw} := \sup_{w \in \mathcal{W}_T, \|w\|=1} \|z_{ss}\|.
\]  

(3.5)

The induced power-norm, hence, is a measure of how big the steady-state output can be when the input ranges over all its possibilities in \( \mathcal{W}_T \). Since \( z \) usually represents the tracking error, the induced power-norm is in fact a tracking measure. The following result is obtained.

**Lemma 3.2**

\[
J_{zw} = \sup_{n \in \mathbb{Z}} \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] .
\]  

(3.6)

**Proof** First we note that for \( w \in \mathcal{W}_T \) with Fourier series representation (3.2),

\[
\|w\|^2 = \sum_{n=-\infty}^{\infty} \|\tilde{w}(n)\|^2.
\]

Then similar to the proof of Lemma 2.2, we can show that

\[
J_{zw} \leq \sup_{n \in \mathbb{Z}} \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] .
\]
To show that the upperbound is actually achieved, we note that given $\varepsilon > 0$, there exists $n_0 > 0$ such that

$$\left| \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] - \sup_{n \in \mathbb{Z}} \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] \right| \leq \varepsilon.$$  

Because of symmetry, it is also true that

$$\left| \bar{\sigma} \left[ \hat{g} \left( e^{-j \frac{2\pi n}{T}} \right) \right] - \sup_{n \in \mathbb{Z}} \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] \right| \leq \varepsilon.$$  

Let $\hat{w}$ be a unit-norm vector where $\hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \hat{w}$ achieves its maximum (Euclidean) norm. In (3.2) set

$$\hat{w}(n) = \begin{cases} \hat{w}, & n = n_0 \\ \overline{\hat{w}}, & n = -n_0 \\ 0, & \text{else.} \end{cases}$$

Then noting that symmetry implies $\| \hat{g} \left( e^{-j \frac{2\pi n}{T}} \right) \overline{\hat{w}} \| = \bar{\sigma} \left[ \hat{g} \left( e^{-j \frac{2\pi n}{T}} \right) \right] \| \overline{\hat{w}} \|$, we get from above

$$\left| \| z_{ss} \| - \sup_{n \in \mathbb{Z}} \bar{\sigma} \left[ \hat{g} \left( e^{j \frac{2\pi n}{T}} \right) \right] \right| < \varepsilon,$$

which shows that $\| z_{ss} \|$ can be made arbitrarily close to the upper bound.

Now, having the induced power-norm as a tracking measure and knowing that stabilization of the system in Figure 3.1 with the infinite-dimensional generator in the loop fails for strictly proper plants, we would like to find out what is the best tracking that is achievable with an FDLTI controller.

**Problem 3.1 (Optimal repetitive control for unknown periodic inputs)**

With respect to Figure 3.1,

$$\begin{align*}
\text{minimize} & \quad \sup_{K: \text{FDLTI and stabilizing}} \sup_{w \in \mathcal{W}_T, \|w\|=1} \| z_{ss} \|. \\
\end{align*}$$

(3.7)

By recalling Lemma 3.2 and letting $T_K$ denote the system from $w$ to $z$ in Figure 3.1,
this problem boils down to

\[
\text{minimize } \sup_{n \in \mathbb{Z}} \sigma \left( \hat{t}_K \left( e^{j2n\pi} \right) \right). 
\]

We will not need an extensive analysis on Figure 3.1 before we realize that the result of this minimization cannot be less than 1, regardless of what the strictly proper plant is. The reason for this is of course the direct path that connects \( w \) to \( z \).

Noting that \( \mathcal{W}_T \) is too large a class of periodic signals for tracking, it should be restricted. One way of imposing this restriction is to introduce a fictitious lowpass filter \( F \) at the input as in Figure 3.5 and let \( \mathcal{W}_T \) be the class of periodic inputs to the filter. In this way, what used to appear at \( r \) without the filter \( F \), will now have its high frequency harmonics attenuated, certainly a more realistic setup.

Figure 3.5: High-frequency tones are attenuated by the fictitious filter \( F \).

As was mentioned earlier, we will not try to solve (3.7) in this thesis. Instead, we design a sampled-data controller. The next two chapters deal with this.
Chapter 4

Sampled-Data Repetitive Control: A Known Periodic Input

In this chapter we formulate the sampled-data repetitive control problem for known periodic inputs. While the internal model principle makes it very simple to find a solution to this problem in continuous time, the solution in a sampled-data framework where intersample behavior can occur, is not straightforward. We then show that in the SISO case, the resulting controller is also optimal for all periodic inputs that are constituted by the same harmonics that are present in the periodic signal for which the controller was originally designed.

4.1 Problem Formulation

The setup of interest is shown in Figure 4.1. The plant $G$ is a continuous-time FDLTI system with a minimal realization of the form

$$
\hat{g}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{bmatrix}
$$ (4.1)

in which the matrices $D_{22}$ and $D_{21}$ are set to zero for well-posedness and because of
the presence of the sampler. Assume also that \((A, B_2)\) is stabilizable and that \((C_2, A)\) is detectable. Exogenous input \(w\) contains all reference commands or disturbance signals, which is assumed to be a known periodic signal of period \(T\), that is, \(w \in \mathcal{W}_T\), and the output \(z\) contains all the tracking errors to be minimized. Control input and measured outputs are denoted by \(u\) and \(y\), respectively.

![Sampled-data repetitive control system](image)

**Figure 4.1:** Sampled-data repetitive control system.

The controller \(K\) is a sampled-data controller with two assumptions regarding its sampling period \(h\): (1) There is an integer number of sampling periods within one period of the periodic input, i.e., \(T = Mh\) for some integer \(M\), and (2) the matrix \(A\) in the realization of \(G\) satisfies the nonpathological sampling condition that for each eigenvalue \(\lambda\) of \(A\) with \(\text{Re} \; \lambda \geq 0\), none of the points \(\lambda + j \frac{2\pi k}{h}\), \(k \neq 0\), is an eigenvalue of \(A\). This latter condition guarantees stabilizability of the system. Bring in Figure 4.2, the discretization of the system in Figure 4.1. The class of controllers over which the optimization is performed is \(\mathcal{C}\), the class of FDLTI causal controllers for which the \(A\)-matrix of the discretized system in Figure 4.2 is stable. As shown in [CF91], under mild technical conditions these controllers achieve internal stability for the sampled-data setup.

Due to the presence of the sampler, the system in Figure 4.1 from \(w\) to \(z\), denoted by \(\Phi_K\), is not an LTI system. However, since the sampler and hold are periodic of period \(h\) and synchronized, for \(K \in \mathcal{C}\), \(\Phi_K\) defines an \(h\)-periodic operator on \(\mathcal{L}_2\). This allows us to have some of the nice properties that appear in LTI signals and systems theory as the following theorem, originally proved in [LF94b], indicates.
Theorem 4.1 Suppose $K \in \mathcal{C}$. Then in the sampled-data setup of Figure 4.1, for each $w \in \mathcal{W}_T$, there exist unique signals $z_{ss}$ in $\mathcal{W}_T$ and $z_{tr}$ in $\mathcal{L}_2$ such that $z = z_{ss} + z_{tr}$. Moreover, the mapping from $w$ to $z_{ss}$ is linear on $\mathcal{W}_T$.

Proof The map $\Phi_K$ is $h$-periodic. Therefore, the lifted map from $w$ to $\tilde{z}$, $\tilde{\Phi}_K$, shown in Figure 4.3, is LTI. Since the controller $K$ is stabilizing the system, $\Phi_K$ is bounded on $\mathcal{L}_2$. By Proposition 2.2, $\Phi_K = L_c \Phi_K L_c^{-1}$, defines a bounded map on $\ell_2(\mathcal{K}_2)$. Thus, by Proposition 2.5, there exists $\hat{\Phi}_K$ in $\mathcal{H}_\infty(\mathcal{D}, \mathcal{L}(\mathcal{K}_2))$ so that $\Phi_K = \Lambda^{-1} \Theta_{\hat{\Phi}_K} \Lambda$. Also, $w \in \mathcal{W}_T$ and $T = Mh$ imply that $w \in \Omega_M(\mathcal{K}_2)$. Hence from (2.20) its $\lambda$-transform is given by

$$\tilde{w}(\lambda) = \frac{1}{M} \sum_{k=0}^{M-1} \frac{\tilde{w}(k)}{1 - \lambda W^{-k}},$$

where $\tilde{w}(k)$ is the DFT of $w(n)$. Therefore

$$\tilde{z}(\lambda) = \hat{\Phi}_K(\lambda) \frac{1}{M} \sum_{k=0}^{M-1} \frac{\tilde{w}(k)}{1 - \lambda W^{-k}}$$
Figure 4.4: LTI lifted system.

\[
\begin{align*}
\dot{z}(k) &= \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{\phi}_K(\lambda)\hat{w}(k)}{1 - \lambda W^{-k}} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \left[ \hat{\phi}_K(W^k) + \hat{\phi}_K(\lambda) - \hat{\phi}_K(W^k) \right] \hat{w}(k) \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{\phi}_K(W^k)\hat{w}(k)}{1 - \lambda W^{-k}} + \frac{1}{M} \sum_{k=0}^{M-1} \frac{[\hat{\phi}_K(\lambda) - \hat{\phi}_K(W^k)]\hat{w}(k)}{1 - \lambda W^{-k}} \\
&= \dot{z}_{ss}(\lambda) + \dot{z}_{tr}(\lambda),
\end{align*}
\]

where

\[
\dot{z}_{ss}(\lambda) := \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{\phi}_K(W^k)\hat{w}(k)}{1 - \lambda W^{-k}}
\]  

(4.2)

and

\[
\dot{z}_{tr}(\lambda) := \frac{1}{M} \sum_{k=0}^{M-1} \frac{[\hat{\phi}_K(\lambda) - \hat{\phi}_K(W^k)]\hat{w}(k)}{1 - \lambda W^{-k}}.
\]  

(4.3)

Noting that \(\dot{z}_{ss}\) is of the form in (2.21), it is a periodic signal of period \(M\). Also in the expression for \(\dot{z}_{tr}\),

\[
\frac{\hat{\phi}_K(\lambda) - \hat{\phi}_K(W^k)}{1 - \lambda W^{-k}}
\]

is in \(\mathcal{H}_\infty(\mathbb{D}, L(\mathcal{K}_2))\) because it is analytic and bounded on \(\mathbb{D}\). Therefore, \(\dot{z}_{tr} \in \mathcal{H}_2(\mathbb{D}, \mathcal{K}_2)\). By Proposition 2.1, \(\dot{z}_{tr} \in \ell_2(\mathcal{K}_2)\).

Now, define \(z_{ss} := L^{-1}_c\dot{z}_{ss}\) and \(z_{tr} := L^{-1}_c\dot{z}_{tr}\). Thus, \(z = z_{ss} + z_{tr}, z_{ss} \in \mathcal{W}_T\) and by Proposition 2.2, \(z_{tr} \in L_2\). So the first statement of the theorem is proved. Also, from above it is seen that the mapping from \(w\) to \(z_{ss}\) is linear.
Chapter 4. Sampled-Data Repetitive Control: A Known Periodic Input

Corollary 4.1 The DFT coefficients of $z_{ss}$ are

$$\tilde{z}_{ss}(k) = \hat{\omega}_K (W^k) \tilde{u}(k).$$  \hspace{1cm} (4.4)

In this decomposition, the steady-state component, $z_{ss}$, is in $\mathcal{W}_T$ and hence it is periodic of period $T$. The transient component, $z_{tr}$, is in $\mathcal{L}_2$ and therefore

$$\lim_{n \to \infty} \frac{1}{T} \int_{nT}^{(n+1)T} z_{tr}(t) z_{tr}(t) dt = 0$$

which shows that $z_{tr}$ will have less and less energy over each period as periods go by. So, in this sense the output $z$ approaches a unique steady-state periodic component, $z_{ss}$. Now, to have good tracking, emphasis has to be put on the power of $z_{ss}$. So we pose the following problem.

**Problem 4.1 (Optimal sampled-data repetitive control for a known periodic input)** With respect to Figure 4.1,

$$\min_{\kappa \in C} \|z_{ss}\|.$$ \hspace{1cm} (4.5)

**Remark 4.1** Since $z_{ss}$ is a continuous-time signal, Problem 4.1 takes the intersample error directly into the design.

We will solve this problem in the next section.

### 4.2 The Optimal Controller

To find the solution, we note that

$$\|z_{ss}\| = \|\tilde{z}_{ss}\|$$

from Proposition 2.3 and that
\[
\|z_{ss}\| = \frac{1}{M} \left[ \sum_{k=0}^{M-1} \|\tilde{z}_{ss}(k)\|^2 \right]^{1/2}
\]

from Lemma 2.1. These along with Corollary 4.1 allow us to write Problem 4.1 as

\[
\minimize_{K \in \mathcal{C}} \sum_{k=0}^{M-1} \left\| \hat{\phi}_K(W^k) \tilde{u}(k) \right\|^2.
\]  

(4.6)

The advantage to this new form is that it tells us of the dependence of the functional to be minimized on two elements: one that doesn’t depend on the controller \(K\), \(\tilde{u}(k)\), and one that does, \(\hat{\phi}_K(W^k)\). The next step is to use a parametrization technique to represent the dependence of \(\hat{\phi}_K(W^k)\) on \(K\) in a more explicit manner. With respect to the state-space representation of \(G\), given in (2.25), all the controllers in \(\mathcal{C}\) can be represented as the input-output system of the block diagram in Figure 4.5 [Doy84], where \(Q\) is stable FDLTI and \(J\) has the realization

\[
\hat{j}(\lambda) = \begin{bmatrix}
A_d + B_{2d}F + HC_2 & -H & B_{2d} \\
F & 0 & -I \\
-C_2 & I & 0
\end{bmatrix}
\]  

(4.7)

in which \(F\) and \(H\) are any matrices such that \(A_d + B_{2d}F\) and \(A_d + HC_2\) are stable. For a reference see [CF95]. By utilizing this characterization in Figure 4.4, the input-output transfer matrix from \(u\) to \(z\) will be an affine function of \(\tilde{q}(\lambda)\), that is,

\[
\hat{\phi}_K(\lambda) = \hat{\phi}_1(\lambda) + \hat{\phi}_2(\lambda) \tilde{q}(\lambda) \hat{\phi}_3(\lambda),
\]  

(4.8)

where

\[
\hat{\phi}_1(\lambda) = \begin{bmatrix}
A_d + B_{2d}F & B_{2d}F \\
0 & A_d + HC_2 \\
C_1 + D_{12}F & D_{12}F
\end{bmatrix}
\begin{bmatrix}
B_1 \\
-B_1 - HD_{12} \\
D_{11}
\end{bmatrix}
\]
Then (4.6) becomes

\[ \text{minimize } \sum_{k=0}^{M-1} \left\| \hat{\phi}_2 (W^k) \hat{w}(k) + \hat{\phi}_2 (W^k) \hat{q} (W^k) \hat{\phi}_3 (W^k) \hat{w}(k) \right\|^2. \]  

(4.10)

The optimization functional in (4.10) involves several elements. Let's pause here to remind ourselves about the spaces that these elements live in or operate on: \( \hat{w}(k) \) is the DFT of \( w(k) \), the (continuously) lifted \( w(t) \), and hence it is in \( \mathcal{K}_2 \). Also from (2.27), (2.28), (2.29) and (2.30) we see that \( \hat{\phi}_2 (W^k) \) is in \( \mathbf{L}(\mathcal{K}_2) \) and that \( \hat{\phi}_2 (W^k) \) and \( \hat{\phi}_3 (W^k) \) are bounded linear transformations from \( \mathcal{E} \) to \( \mathcal{K}_2 \) and from \( \mathcal{K}_2 \) to \( \mathcal{E} \), respectively. The remaining element, \( \hat{q}(W^k), \) is a complex-valued matrix.

The solution to (4.10) is then as follows. Note that the functional in (4.10) depends on \( \hat{q}(\lambda) \) only at the points \( W^k \); hence the solution is not unique.

**Lemma 4.1** A minimizing \( Q \)-parameter for (4.10) is

\[ \hat{q}(\lambda) = \sum_{n=0}^{M-1} \rho(n) \lambda^n \]  

(4.11)
where
\[ r(n) = \frac{1}{M} \sum_{k=0}^{M-1} \tilde{r}(k) W^{-nk}, \quad (4.12) \]

[that is, \( r(n) \overset{\text{DFT}^{-1}}{=} \tilde{r}(k) \)], and \( \tilde{r}(k) \) solves

\[
\minimize_{\tilde{r}(k)} \| \tilde{\phi}_1 (W^k) \hat{\omega}(k) + \tilde{\phi}_2 (W^k) \tilde{r}(k) \tilde{\phi}_3 (W^k) \hat{\omega}(k) \|, \quad k = 0, 1, \ldots, M - 1. \quad (4.13)
\]

**Proof** The solution in (4.10) is in general greater than or equal to

\[
\minimize_{\tilde{r}(0), \tilde{r}(1), \ldots, \tilde{r}(M - 1)} \sum_{k=0}^{M-1} \| \tilde{\phi}_1 (W^k) \hat{\omega}(k) + \tilde{\phi}_2 (W^k) \tilde{r}(k) \tilde{\phi}_3 (W^k) \hat{\omega}(k) \|^2. \quad (4.14)
\]

Specifically, they are equal if there is a stable \( \hat{q}(\lambda) \) that satisfies

\[
\hat{q}(W^k) = \tilde{r}(k) \quad k = 0, 1, \ldots, M - 1.
\]

On the other hand, since the optimization variables are independent, (4.14) is equivalent to (4.13). Hence if one interpolates the points \( \tilde{r}(k) \) that solve the minimization problem in (4.13) with a polynomial, which of course is stable, an optimal \( \hat{q}(\lambda) \) is obtained.

Polynomial (4.11) is the unique one of degree \( M - 1 \) that interpolates the points \( \tilde{r}(k) \). [Proof: From (4.11), \( \hat{q}(W^k) \overset{\text{DFT}}{=} r(n) \).] Of course, there are other polynomials of higher degree that interpolate, as well as, conceivably, lower order rational functions that interpolate. The minimizing \( Q \)-parameter obtained from this theorem can be plugged back in the general representation of all stabilizing controllers, Figure 4.5, to get the controller itself.

So, one only has to find the solution to \( M \) minimization problems in (4.13). The remainder of this section outlines how (4.13) might be solved numerically. These minimization problems all share the general form
minimize \( \|a + Bxc\| \), \( \quad (4.15) \)

where \( a \) is in \( K_2 \), \( c \) is in \( E \) and \( B \) and \( X \) are bounded operators from \( E \) to \( K_2 \) and from \( E \) to \( E \), respectively. Thus \( X \) is a matrix. The next step is to define

\[ x := Xc \quad (4.16) \]

which takes (4.15) to

minimize \( \|a + Bx\| \), \( \quad (4.17) \)

where \( x \) is in \( E \) again. (We can always backsolve for \( X \) such that \( x = Xc \) as long as \( c \neq 0 \).) To solve (4.17) we note from [Lue69] that \( K_2 \) can be expressed as the orthogonal direct sum of

\[ \Sigma := \text{Range}(B) \quad (4.18) \]

and

\[ \Sigma^\perp := \text{Null}(B^*) \quad (4.19) \]

Since \( E \) is finite dimensional, so is \( \Sigma \), hence it is closed. Denote by \( \Pi \) the orthoprojector of \( K_2 \) onto \( \Sigma \). Then

\[ a = a_1 + a_2 \quad (4.20) \]

where

\[ a_1 := \Pi a \quad \text{and} \quad a_2 := (I - \Pi)a. \quad (4.21) \]

Now equation

\[ a_1 + Bx = 0 \quad (4.22) \]

is guaranteed to have at least one solution. Any solution will be a solution to (4.17) as well and the corresponding minimum will be \( \|a_2\| \).

The solution, as can be seen, involves an operator equation and hence is not easy to find. This can be attributed to the fact that the sampled-data system of interest
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experiences both continuous- and discrete-time dynamics which makes a simple solution not directly available. In the next section, we take a different approach where an approximation of the input-output pair is used to find controllers that in performance can be as close as desired to the optimal performance.

4.3 Problem Reformulation via Fast Discretization

Problem 4.1 involves both continuous-time and discrete-time signals and hence is not amenable to a simple analysis. Instead, a problem close to the original problem is formulated here which, as we shall see, has a state-space solution that can be easily coded in MATLAB.

Before we start reformulating the problem, note that there are two time periods in Figure 4.1: $T$ is the period of the periodic input $w$; $h$ is the sampling period. By assumption, $T = Mh$, where $M$ is an integer.

The new formulation follows Keller and Anderson [KA92] and is based on approximating two signals: the periodic signal $w$ for which the design of optimal sampled-data repetitive controller is desired and the output $z$, which we would like to minimize. The way this approximation is done is the following. First, $w$ is sampled by a sampler $S_N$ of sampling period $h/N$, $N$ being an integer. Since $S_N$ clocks $N$-times faster than $S$, it is called a fast sampler. Samples of $w$ which are at a rate of $\frac{1}{h/N}$ are then held by a fast hold $H_N$ synchronized with $S_N$. This process is illustrated in Figure 4.6 for the input signal, $w$. Obviously, if $w$ is periodic of period $T$, $\omega$ will be periodic of period $MN$ and $\bar{w}$ will be periodic of period $T$, that is,

$$w \in \mathcal{W}_T \Rightarrow \omega \in \Omega_{MN} \Rightarrow \bar{w} \in \mathcal{W}_T.$$

By applying fast-discretization to the sampled-data setup of Figure 4.1, we arrive at Figure 4.7. Here, $N$ is a design parameter. In this way, the periodic input to the plant may not be exactly the one intended, but by choosing $N$ large enough it can be made as close as desired. Also, by choosing $N$ large, we will still be able to control
Figure 4.6: Approximating signals by fast-discretization.

the intersample behavior of the output. So, it is expected that the setup of Figure 4.7 will emulate the sampled-data setup of Figure 4.1 for large $N$.

Figure 4.7: Approximation by way of fast discretization.

We need the following result before we can state a design problem for the new setup.

**Theorem 4.2** For $K$ stabilizing in Figure 4.7, and for $w \in \mathcal{W}_T$, output $\hat{z}$ can be uniquely decomposed into a steady-state component $\hat{z}_{ss} \in \mathcal{W}_T$ and a transient component $\hat{z}_{tr} \in \mathcal{L}_2$.

**Proof** The system in Figure 4.7 is $h$-periodic. Use continuous-time lifting in the same way as in Theorem 4.1.

The new design problem can then be stated as

$$\min_{K \in \mathcal{C}} \|\hat{z}_{ss}\|.$$

To see the advantage of fast discretization, let's move to Figure 4.8 where all the signals are discrete-time. This setup has three time periods: $T$ and $h$ as before, and
$h/N$, the period of the fast sampler. Thus $T = Mh = MN(h/N)$. Then take the samplers and hold devices into the plant $G$ as shown in Figure 4.9 where

$$P := \begin{bmatrix} S_N & 0 \\ 0 & S \end{bmatrix} G \begin{bmatrix} H_N & 0 \\ 0 & H \end{bmatrix}.$$  \hspace{1cm} (4.24)

![Figure 4.8: Fast discretized system.](image)

![Figure 4.9: Two-rate discrete-time system.](image)

**Theorem 4.3** In Figure 4.8, suppose $K \in \mathcal{C}$. For each periodic $\omega$ of period $MN$, there exist unique signals $\zeta_{ss}$ in $\Omega_{MN}$ and $\zeta_{tr}$ in $\ell_2(\mathbb{Z}_+)$ such that $\zeta = \zeta_{ss} + \zeta_{tr}$. Moreover the mapping from $\omega$ to $\zeta_{ss}$ is linear on $\Omega_{MN}$.

Being similar to that of Theorem 4.1, the proof is omitted.

Now $\zeta_{tr} \in \ell_2(\mathbb{Z}_+)$ implies that it goes to zero in the limit as $n$ goes to infinity, that is,

$$\lim_{n \to \infty} \zeta_{tr}(n) = 0.$$
This means that $\zeta$ approaches a steady-state behavior, namely, $\zeta_{ss}$. The following result allows restating (4.23) in terms of $\zeta_{ss}$.

**Corollary 4.2** In Figure 4.7, suppose $K \in \mathcal{C}$. Then

$$\dot{\zeta}_{ss} = H_N \zeta_{ss}.$$  

The minimization problem (4.23) then takes the form

$$\min_{K \in \mathcal{C}} \|H_N \zeta_{ss}\|.$$  

But

$$\|H_N \zeta_{ss}\|^2 = \frac{1}{T} \int_0^T (H_N \zeta_{ss})(t)'(H_N \zeta_{ss})(t)dt$$

$$= \frac{1}{T} \sum_{k=0}^{MN-1} \int_{kh/N}^{(k+1)h/N} (H_N \zeta_{ss})(t)'(H_N \zeta_{ss})(t)dt$$

$$= \frac{1}{T} \sum_{k=0}^{MN-1} \int_{kh/N}^{(k+1)h/N} \zeta_{ss}(k)' \zeta_{ss}(k)dt$$

$$= \frac{1}{Mh} \sum_{k=0}^{MN-1} \zeta_{ss}(k)' \zeta_{ss}(k)(h/N)$$

$$= \frac{1}{MN} \sum_{k=0}^{MN-1} \zeta_{ss}(k)' \zeta_{ss}(k)$$

$$= \|\zeta_{ss}\|^2.$$  

So (4.25) reduces to

$$\min_{K \in \mathcal{C}} \|\zeta_{ss}\|.$$  

This problem relates to Figure 4.9 and involves discrete-time signals only. Also, note that the class of controllers over which the minimization is performed is the same as before.

Figure 4.9 is the two-rate discrete-time system of Section 2.4; some of the signals are referred to the base period $h$ on the real-time clock while the others are referred to
the subperiod $h/N$ of the base period. As we saw in Chapter 2, discrete-time lifting helps associate an LTI counterpart to this system. The lifted system is shown in Figure 4.10 with $P$ given by (2.40). For $K \in \mathcal{C}$, Theorem 4.3 allows a decomposition of $\zeta$ into two components for $\omega \in \Omega_{MN}$, its steady state component $\zeta_{ss}$ and its transient component $\zeta_{tr}$. Since $\zeta = L\zeta$ and $\omega = L\omega$, for a stabilizing controller, there is a natural decomposition for $\zeta$ when $\omega \in \Omega_M$, namely, $\zeta = \zeta_{ss} + \zeta_{tr}$, where $\zeta_{ss} := L\zeta_{ss} \in \Omega_M$ and $\zeta_{tr} := L\zeta_{tr} \in \ell_2(\mathbb{Z}_+)$. Also, from (2.39) we see that $\|\zeta_{ss}\| = \sqrt{N} \|\zeta_{ss}\|$. Therefore, (4.26) can be stated in terms of the lifted output $\zeta_{ss}$:

$$\min_{K \in \mathcal{C}} \|\zeta_{ss}\|. \quad (4.27)$$

Figure 4.10: Lifting gives a single-rate system.

Now the system from $\omega$ to $\zeta$, shown in Figure 4.11 with $P$ given by (2.41), is not only single-rate but also LTI. With a similar approach to what was taken in Section 4.2, we can find a suboptimal controller that minimizes the norm in (4.27) and hence the norm in (4.26).

### 4.4 Suboptimal Controllers

In this section we present the solution to (4.27). Let's denote by $\Psi_K$ the mapping from $\omega$ to $\zeta$ in Figure 4.9. Then the lifted map $\Psi_K = L\Psi_K L^{-1}$ in Figure 4.11 is LTI and thus by Lemma 2.1 and (2.15), (4.27) can be written as
Figure 4.11: Single-rate lifted system.

\[
\text{minimize } \sum_{k=0}^{M-1} \left\| \hat{y}_K (W^k) \tilde{w}(k) \right\|^2. 
\] (4.28)

To find the solution to this minimization problem, we follow the same path that we took in Section 4.2 for the system with the continuous lifting, namely, we parametrize all controllers in \( \mathcal{C} \). With respect to the state space representation of \( P \) from Section 2.4, all the stabilizing controllers can be represented as the input-output system of the block diagram in Figure 4.12 [Doy84], where \( Q \) is stable FDLTI and \( J \) has the realization

\[
\hat{j}(\lambda) = \begin{bmatrix}
A_d + B_{2d}F + HC_2 & -H & B_{2d} \\
F & 0 & -I \\
-C_2 & I & 0
\end{bmatrix}
\] (4.29)

in which \( F \) and \( H \) are matrices such that \( A_d + B_{2d}F \) and \( A_d + HC_2 \) are stable.

Figure 4.12: Stabilizing controllers.

We see here that characterization of the stabilizing controllers is the same as what
we had for the true sampled-data system of Figure 4.1. With this characterization, the input-output transfer matrix in Figure 4.11 from \( \omega \) to \( \zeta \) is an affine function of \( \hat{q}(\lambda) \), that is,

\[
\dot{\psi}_k(\lambda) = \dot{\psi}_1(\lambda) + \dot{\psi}_2(\lambda) \dot{q}(\lambda) \dot{\psi}_3(\lambda),
\]

(4.30)

where

\[
\begin{align*}
\dot{\psi}_1(\lambda) &= \begin{bmatrix} A_d + B_{2d}F & B_{2d}F & B_{1d} \\ 0 & A_d + HC_2 & -B_{1d} - HD_{12d} \\ C_{1d} + D_{12d}F & D_{12d}F & D_{11d} \end{bmatrix}, \\
\dot{\psi}_2(\lambda) &= \begin{bmatrix} A_d + B_{2d}F & B_{2d} \\ C_{1d} + D_{12d}F & D_{12d} \end{bmatrix}, \\
\dot{\psi}_3(\lambda) &= \begin{bmatrix} A_d + HC_2 & B_{1d} + HD_{12d} \\ C_2 & D_{21d} \end{bmatrix}.
\end{align*}
\]

(4.31)

Then (4.28) becomes

\[
\minimize_{\dot{q}(\lambda) \text{ stable}} \sum_{k=0}^{M-1} \left\| \dot{\psi}_1(W^k) \tilde{\omega}(k) + \dot{\psi}_2(W^k) \dot{q}(W^k) \dot{\psi}_3(W^k) \tilde{\omega}(k) \right\|^2.
\]

(4.32)

The main difference with the sampled-data case is that now the participating elements in the minimization are vector- and matrix-valued as opposed to function- and operator-valued.

For the solution, we have a finite-dimensional counterpart to Lemma 4.1.

**Lemma 4.2** A minimizing \( Q \)-parameter for problem (4.32) is

\[
\dot{q}(\lambda) = \sum_{n=0}^{M-1} r(n) \lambda^n
\]

(4.33)

where

\[
r(n) = \frac{1}{M} \sum_{k=0}^{M-1} \tilde{r}(k) W^{nk},
\]

(4.34)
[that is, \( r(n) \xrightarrow{\text{DFT}^{-1}} \tilde{r}(k) \)], and \( \tilde{r}(k) \) solves

\[
\text{minimize } \left\| \hat{\psi}_1(W^k) \hat{\omega}(k) + \hat{\psi}_2(W^k) \tilde{r}(k) \hat{\psi}_3(W^k) \hat{\omega}(k) \right\|, \quad k = 0, 1, \ldots, M - 1.
\]

(4.35)

The proof is omitted.

So, one only has to find the solution to \( M \) minimization problems in (4.35). These minimization problems share the general form

\[
\text{minimize } \| a + BXc \|, \quad x
\]

(4.36)

where \( a \) and \( c \) are complex vectors and \( B \) and \( X \) are complex matrices. This minimization problem always has a solution. If \( c = 0 \), then any \( X \) is a solution and the minimum is \( \| a \| \). If \( c \neq 0 \), then the solution can be obtained in two steps as follows. First find a solution to

\[
\text{minimize } \| a + Bx \|, \quad x
\]

(4.37)

Then find \( X \) by obtaining any solution to

\[
Xc = x.
\]

(4.38)

Noting that (4.38) can be formulated as a minimization problem, two least squares minimization problems need to be solved to find the solution to (4.36). This can be easily coded in say MATLAB. Since (4.37) and (4.38) might have more than one solution, depending on other possible performance specifications, a particular solution pair may prove to be better than others. But as long as the power-norm minimization is the sole performance criterion, all solution pairs to (4.37) and (4.38) are as good since they all solve (4.32).
4.5 Convergence Issues

In Section 4.1, Problem 4.1 we stated the optimal sampled-data repetitive control problem for a known periodic input. There, we used the power-norm of the steady-state component of the output, $z_{ss}$, to be the tracking measure which means that $\|z_{ss}\|$ is what is looked at when it comes to evaluating the performance of a given repetitive controller in a sampled-data setup. However, we learned in Section 4.2 that computing this norm might not be so easy due to the sampled-data nature of the problem. To deal with the computational complexity, a reformulation of Problem 4.1 was introduced in (4.23), as outlined in Figure 4.7, which relies on fast discretization. This approach uses the power-norm of the steady-state component of the approximate signal $\hat{z}_{ss}$ as the tracking measure. The fact that $\hat{z}_{ss}$ is an approximation to $z_{ss}$ brings up two problems. The first problem, which we call the analysis problem, is whether for a given controller $K$, one can compute $\|z_{ss}\|$ to any degree of accuracy by means of fast discretization. That is, for a given controller, whether one can get $\hat{z}_{ss}$ arbitrarily close to $z_{ss}$, or even $\|\hat{z}_{ss}\|$ to $\|z_{ss}\|$ for that matter, by choosing $N$ large enough. As we will show in this section, the answer here is affirmative, under a very mild condition on the periodic input $w$. The other problem is the design problem. When fast discretization is used, since the input-output pair is approximated, the controller may fall afar from optimality and we would then be interested to know whether controllers can be designed that in performance converge to the optimal controller. This remains as an open problem.

For the analysis problem, we concentrate on the setup of Figure 4.13. In this figure, outputs of the sampled-data setup in Figure 4.1 and the setup in Figure 4.7 are compared for the same periodic input $w$ and for the same controller. From Theorem 4.2, $\hat{z}_{ss}$ is in $\mathcal{W}_T$. The following convergence result is obtained.

**Theorem 4.4** In Figure 4.13, suppose that $K \in \mathcal{C}$ and that $w$ in $\mathcal{W}_T$ is such that

$$\lim_{N \to \infty} \|(I - H_N S_N)w\| = 0. \quad (4.39)$$
Moreover, the convergence rate in this limit is at least \(1/N\).

**Remark 4.2** Assumption (4.39) is merely a statement of the fact that for the fast-discretization technique to provide us with a good approximation of the controller performance, it should first give a good approximation of the periodic input signal. To understand this better, consider the following periodic input that is of period \(T = 1\),

\[
 w(t) = \begin{cases} 
 0 & \text{t rational} \\
 1 & \text{otherwise.} 
\end{cases}
\]

It is straightforward to see that

\[
 H_N S_N w(t) = 0
\]
for all $t$ regardless of what the value of $N$ is. Therefore $\hat{z}(t) = 0$ for all $t$ as well. So there will be no convergence in this case. This example is of course an extreme case and (4.39) holds for a fairly large class of signals. For example, all piecewise continuous signals satisfy (4.39), and hence (4.39) is no restriction in practice.

Theorem 4.4 can be proved relatively easily by continuous lifting and the help of some convergence results. First we will see what is the effect of lifting on $H_N S_N$, which is a main block in Figure 4.13. Let $\mathcal{E}_N$ denote the space of all finite sequences from $\{0, 1, \ldots, N - 1\}$ to $\mathbb{C}$. Also let $\tilde{h} := \frac{h}{N}$ be the sampling period of the fast sample and hold. So $\tilde{h} \to 0$ as $N \to \infty$. Define

$$S_N : \mathcal{K} \to \mathcal{E}_N \quad \psi = S_N u$$
$$\psi(l) = u(l\tilde{h}), \quad l = 0, 1, \ldots, N - 1,$$

and

$$H_N : \mathcal{E}_N \to \mathcal{K}, \quad y = H_N u$$
$$y(\tau) = \psi(l), \quad l\tilde{h} \leq \tau < (l + 1)\tilde{h}, \quad l = 0, 1, \ldots, N - 1.$$

The operators $S_N$ and $H_N$ would have been, respectively, restrictions of $S_N$ and $H_N$ to $[0, h)$ and $\{0, 1, \ldots, N - 1\}$ if they were acting on real-valued signals. Now for signal $\bar{x} \in \ell(\mathbb{Z}_+, \mathcal{K})$, $y = L_c H_N S_N L_{L_c}^{-1} \bar{x}$ is also a signal in $\ell(\mathbb{Z}_+, \mathcal{K})$ and for all $n \in \mathbb{Z}_+$

$$[y(n)](\tau) = [\bar{x}(n)](l\tilde{h}), \quad l\tilde{h} \leq \tau < (l + 1)\tilde{h}, \quad l = 0, 1, \ldots, N - 1.$$

That is, for each $n$

$$y(n) = H_N S_N \bar{x}(n).$$

Also, it is straightforward to verify that for $\bar{x} \in \Omega_M(\mathcal{K}_2)$, $y$ is also in $\Omega_M(\mathcal{K}_2)$ and

$$\tilde{y}(k) = H_N S_N \tilde{x}(k).$$

(4.41)
Next we will see a series of intermediate convergence results. For the first result, we need a special case of M. Riesz's general Convexity Theorem [SW71].

**Lemma 4.3** Suppose that $G$ is a bounded linear transformation on $\mathcal{K}_1$ and $\mathcal{K}_\infty$ with induced norms $M_1$ and $M_\infty$, respectively. Then for every $0 \leq p \leq \infty$, $G : \mathcal{K}_p \to \mathcal{K}_p$ is bounded, with induced norm $M_p \leq M_1^\frac{1}{p} M_\infty^\frac{1}{p}$.

Let $f$ be an element in $\mathcal{K}_1$ and let $F$ be a linear transformation on $\mathcal{K}$ given by

$$(Fu)(t) := \int_0^t f(t - \tau) u(\tau) d\tau.$$ 

Define

$$\tilde{f}_h(t) := \sup_{a \in [0, \min(t, h))] \|f(t) - f(t - a)\|, \quad t \in [0, h),$$

and

$$f_h := \sup_{t \in [0, h]} \|f(t)\|.$$

**Theorem 4.5** If $\lim_{t \to 0} \|\tilde{f}_h\|_1 = 0$ and $f_h$ is finite, then $(I - H_N S_N) F$ converges to zero in the induced norm for every $0 \leq p \leq \infty$.

This theorem is essentially the same as Theorem 9.3.3 in [CF95]. The difference is that there the time interval of interest is infinite whereas here it is of a finite duration.

**Proof** By Lemma 4.3, it suffices to show that $M_1$ and $M_\infty$ for $(I - H_N S_N) F$ tend to zero as $N$ tends to infinity. Take $u$ in $\mathcal{K}_1$ or $\mathcal{K}_\infty$ and set $y = Fu$. For $t \in [0, h)$, let $k$ be such that $kh \leq t < (k + 1)h$. Then

$$[(I - H_N S_N)y](t) = y(t) - y(kh) = \int_0^t f(t - \tau) u(\tau) d\tau - \int_0^{kh} f(\tau - \tau - kh) u(\tau) d\tau$$

$$= \int_0^{kh} [f(t - \tau) - f(\tau - kh)] u(\tau) d\tau + \int_0^t f(t - \tau) u(\tau) d\tau.$$ 

Therefore

$$\|[(I - H_N S_N)y](t)\| \leq \int_0^{kh} \|f(t - \tau) - f(\tau - kh)\| \|u(\tau)\| d\tau.$$
\[ + \int_{kh}^{t} \| f(kh - \tau) \| u(\tau) \| d\tau \]
\[ \leq \int_{0}^{kh} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau + f_h \int_{kh}^{t} \| u(\tau) \| d\tau \]
\[ \leq \int_{0}^{t} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau + f_h \int_{kh}^{(k+1)h} \| u(\tau) \| d\tau. \quad (4.42) \]

Now for \( u \in K_{\infty} \)

\[ \| (I - H_N S_N) y \|_{\infty} \leq (\| \bar{f}_h \|_1 + hf) \| u \|_{\infty}. \]

Hence \((\| \bar{f}_h \|_1 + hf)\), which by assumption vanishes as \( N \to \infty \), is an upper bound

for \( M_{\infty} \). Therefore \( M_{\infty} \to 0 \) as \( N \to \infty \).

For \( u \in K_1 \), integrate 4.42 to get

\[ \int_{kh}^{(k+1)h} \| (I - H_N S_N) y \| (t) \| dt \leq \int_{kh}^{(k+1)h} \int_{0}^{t} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau dt \]
\[ + f_h \int_{kh}^{(k+1)h} \int_{kh}^{t} \| u(\tau) \| d\tau dt \]
\[ = \int_{kh}^{(k+1)h} \int_{0}^{t} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau dt \]
\[ + hf_h \int_{kh}^{(k+1)h} \| u(\tau) \| d\tau dt. \]

If we add all the inequalities that we get for \( k = 0, 1, \ldots, N - 1 \),

\[ \| (I - H_N S_N) y \|_1 \leq \int_{0}^{h} \int_{0}^{t} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau dt + hf_h \| u \|_1. \quad (4.43) \]

But

\[ \int_{0}^{h} \int_{0}^{t} \bar{f}_h(t - \tau) \| u(\tau) \| d\tau dt = \int_{0}^{h} \int_{t}^{h} \bar{f}_h(t - \tau) \| u(\tau) \| dtd\tau \]
\[ = \int_{0}^{h} \| u(\tau) \| \left( \int_{\tau}^{h} \bar{f}_h(t - \tau) dt \right) d\tau \]
\[ = \int_{0}^{h} \| u(\tau) \| \left( \int_{0}^{h-\tau} \bar{f}_h(t) dt \right) d\tau \]
\[
\int_0^h \| u(\tau) \| \left( \int_0^h \tilde{f}_h(t) \, dt \right) \, d\tau \\
= \| \tilde{f}_h \|_1 \| u \|_1.
\]

So \( \| f_h \|_1 + h \| f_h \| \) is an upper bound for \( M_1 \) and hence for \( M_p \) as well. Thus \( M_p \) goes to zero as \( N \) goes to infinity.

We need the following standard lemma for some more convergence results.

**Lemma 4.4** Suppose that \( A \) is a square matrix. Then

\[
\| e^A \| \leq e^{\| A \|}.
\]  

**Proof** The result is immediate from the Taylor series for \( e^A \),

\[
e^A = I + A + \frac{A^2}{2!} + \cdots,
\]

which gives

\[
\| e^A \| \leq 1 + \| A \| + \frac{\| A \|^2}{2!} + \cdots
\]

\[
= e^{\| A \|}.
\]

**Lemma 4.5** In the state space representation for \( \phi_K \) given in (2.32), for every \( K_p \) norm

\[
\lim_{N \to \infty} \| (I - H_N S_N) D_{11} \| = 0.
\]  

**Proof** We use Lemma 4.4 to get

\[
\tilde{f}_h(t) \leq \| B_1 \| \cdot \| C_1 \| \cdot \| e^A \| \sup_{a \in [0, A]} \| I - e^{-aA} \|
\]

\[
\leq \| B_1 \| \cdot \| C_1 \| \| e^{\| A \|} \| (e^{\| A \|} - 1).
\]

From here it is immediate that \( \| \tilde{f}_h \|_1 \) goes to zero. Also it is obvious that \( f_h \) is finite. The statement of the lemma is then immediate from Theorem 4.5.
Corollary 4.3 The rate of convergence of \(|(I - H_N S_N) D_{11}|\) to zero as \(N \to 0\) is at least \(1/N\).

Lemma 4.6 In the state space representation for \(\Phi_K\) given in (2.32), the \(\mathcal{K}_2\)-to-\(\mathcal{E}\) induced norm of \(B_d\) satisfies

\[
\|B_d\| \leq \sqrt{h} \|A\| \|B_1\|. \tag{4.46}
\]

Proof For \(w \in \mathcal{K}_2\)

\[
\|B_d w\| = \left\| \int_0^h e^{(h-\tau)A} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(\tau) d\tau \right\|
\leq \int_0^h \|e^{(h-\tau)A}\| \left\| \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right\| \|w(\tau)\| d\tau
\leq \int_0^h e^{h\|A\|} \|B_1\| \|w(\tau)\| d\tau
\leq \int_0^h e^{h\|A\|} \|B_1\| \|w(\tau)\| d\tau.
\]

The result is obtained by bringing out the constants and using the Cauchy-Schwarz inequality. \(\blacksquare\)

Lemma 4.7 In the state space representation for \(\Phi_K\) given in (2.32),

\[
\lim_{N \to \infty} \|(I - H_N S_N) \mathcal{C}_d\| = 0 \tag{4.47}
\]

(\(\mathcal{E}\)-to-\(\mathcal{K}_2\) induced norm).

Proof From (2.35) we note that

\[
\mathcal{C}_d = [C_1 \quad D_{12}] \begin{bmatrix} I & 0 \\ D_K & C_K \end{bmatrix}. \tag{4.48}
\]
So it suffices to show that

\[
\lim_{N \to \infty} \|(I - H_N S_N)C_1\| = 0 \tag{4.49}
\]

and

\[
\lim_{N \to \infty} \|(I - H_N S_N)D_{12}\| = 0. \tag{4.50}
\]

To see (4.49), for \( t \in [0, h) \), we let \( k \) be such that \( k \leq t \leq (k + 1)h \). Then

\[
[(I - H_N S_N)C_1 x](t) = (C_1 x)(t) - (C_1 x)(kh) = C_1 e^{tA} x - C_1 e^{khA} x = C_1 e^{khA} [e^{(t-k)A} - I] x.
\]

Hence

\[
||(I - H_N S_N)C_1 x||(t) \leq ||C_1|| ||e^{khA}|| ||e^{(t-k)A} - I|| ||x|| \\
\leq ||C_1|| e^{kh\|A\|} ||e^{\|A\| - 1}|| ||x|| \\
\leq ||C_1|| e^{h\|A\|} (e^{\|A\| - 1}) ||x||.
\]

By a simple integration

\[
||(I - H_N S_N)C_1|| \leq \sqrt{h} ||C_1|| e^{h\|A\|} (e^{\|A\| - 1}). \tag{4.51}
\]

For the limit in (4.50), again we let for \( t \in [0, h) \), \( k \) be such that \( k \leq t \leq (k + 1)h \). Then

\[
[(I - H_N S_N)D_{12} x](t) = \left( D_{12} + C_1 \int_0^t e^{(t-t)A} d\tau B_2 \right) x \\
- \left( D_{12} + C_1 \int_0^{kh} e^{(kh-t)A} d\tau B_2 \right) x \\
= C_1 \left[ \int_0^t e^{(t-t)A} d\tau - \int_0^{kh} e^{(kh-t)A} d\tau \right] B_2 x
\]
Hence

\[ \|(I - H_N S_N)D_{12} x(t)\| \leq \|C_1\| \|B_2\| \|x(t)\| \int_0^t e^{(t-\tau)A} d\tau - \int_0^{kh} e^{(kh-\tau)A} d\tau \]. \quad (4.52) 

But

\[ \int_0^t e^{(t-\tau)A} d\tau - \int_0^{kh} e^{(kh-\tau)A} d\tau = \int_0^{kh} (e^{(t-\tau)A} - e^{(kh-\tau)A}) d\tau + \int_0^t e^{(t-\tau)A} d\tau \\
\quad = e^{khA} (e^{(t-kh)A} - I) \int_0^{kh} e^{-\tau A} d\tau + \int_0^t e^{(t-\tau)A} d\tau \]

which by Lemma 4.4 and Cauchy-Schwarz inequality gives

\[ \left\| \int_0^t e^{(t-\tau)A} d\tau - \int_0^{kh} e^{(kh-\tau)A} d\tau \right\| \leq \|e^{khA}\| \|e^{(t-kh)A} - I\| \int_0^{kh} \|e^{-\tau A}\| d\tau \\
\quad + \int_0^t \|e^{(t-\tau)A}\| d\tau \\
\quad \leq e^{h\|A\|} (e^{h\|A\|} - 1) \int_0^h e^{h\|A\|} d\tau + \int_0^t e^{h\|A\|} d\tau \\
\quad \leq [he^{h\|A\|} (e^{h\|A\|} - 1) + h] e^{h\|A\|}. \]

**Corollary 4.4** The rate of convergence of \( \|(I - H_N S_N)C_{cl}\| \) to zero as \( N \to 0 \) is at least \( 1/N \).

**Lemma 4.8** For a stabilizing controller \( K \) in Figure 4.1, the transfer function \( \hat{\phi}_K(\lambda) \) of the lifted system satisfies

\[ \lim_{N \to \infty} \|(I - H_N S_N) \hat{\phi}_K(\lambda)\| = 0 \quad \forall \lambda \in \bar{D}. \quad (4.53) \]

**Proof** From (2.32)

\[ \hat{\phi}(\lambda) = D_{11} + \lambda C_{cl} (\lambda I - A_{cl})^{-1} B_{cl}. \]

Since \( K \) is stabilizing, the inverse of \( \lambda I - A_{cl} \) exists for \( \lambda \in \bar{D} \) and

\[ \left\| (I - H_N S_N) \hat{\phi}_K(\lambda) \right\| \leq \|(I - H_N S_N)D_{11}\| \]
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\[ + |\lambda| \| (I - H_N S_N) C_d \| \cdot \| (\lambda I - A_d)^{-1} \| \cdot \| B_d \|. \]

Lemmas 4.5-4.7 then imply (4.53).

**Corollary 4.5** The rate of convergence of \( \| (I - H_N S_N) \hat{\Phi}_K (\lambda) \| \) to zero as \( N \to 0 \) is at least \( 1/N \) for all \( \lambda \in \bar{D} \).

**Corollary 4.6** For a stabilizing controller and \( N \) in \( \mathbb{N} \), \( H_N S_N \hat{\Phi}_K (\lambda) \) is bounded for all \( \lambda \) in \( \bar{D} \) and

\[ \| \hat{\Phi}_K (\lambda) \| = \lim_{N \to \infty} \| H_N S_N \hat{\Phi}_K (\lambda) \| \quad \forall \lambda \in \bar{D}. \quad (4.54) \]

**Proof of Theorem 4.4** The lifted setup for convergence analysis is shown in Figure 4.14 with a state space representation for \( \Phi_K \) given by (2.32). Since the steady-state error signal \( z_{ss} - \bar{z}_{ss} \) is in \( \mathcal{W}_T \), Proposition 2.3 allows us to write

\[ \| z_{ss} - \bar{z}_{ss} \| = \| z_{ss} - \bar{z}_{ss} \| . \quad (4.55) \]

Also from Lemma 2.3 we can express the power-norm of the lifted signal \( \bar{z}_{ss} - \bar{z}_{ss} \) in terms of the power-norm of its DFT, \( \bar{z}_{ss} (k) - \bar{z}_{ss} (k) \), that is,

\[ \| z_{ss} - \bar{z}_{ss} \| = \frac{1}{M} \left[ \sum_{k=0}^{M-1} \| \bar{z}_{ss} (k) - \bar{z}_{ss} (k) \|^2 \right]^{1/2}. \]

So it is enough to show that

\[ \lim_{N \to 0} \| \bar{z}_{ss} (k) - \bar{z}_{ss} (k) \| = 0, \ k = 0, 1, \ldots, M - 1. \quad (4.56) \]

Based on Corollary 4.1 and (4.41), the block diagram in Figure 4.15 shows how \( \bar{z}_{ss} (k) \) and \( \bar{z}_{ss} (k) \) are related to \( \bar{u} (k) \). From this diagram

\[ \bar{z}_{ss} (k) - \bar{z}_{ss} (k) = \hat{\phi} (W_k) \bar{u} (k) - H_N S_N \hat{\phi} (W_k) H_N S_N \bar{u} (k) \]

\[ = \hat{\phi} (W_k) \bar{u} (k) \]
Figure 4.14: Lifted setup for convergence analysis.

\[
- \quad [(I - H_N S_N) - I] \hat{\phi}(W^k)[(I - H_N S_N) - I] \bar{\omega}(k) \\
= \quad \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k) \\
+ \quad (I - H_N S_N) \hat{\phi}(W^k) \bar{\omega}(k) \\
- \quad (I - H_N S_N) \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k).
\]

This implies that

\[
\| \tilde{z}_{ss}(k) - \tilde{\tilde{z}}_{ss}(k) \| \leq \| \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k) \| \\
+ \quad \| (I - H_N S_N) \hat{\phi}(W^k) \bar{\omega}(k) \| \\
+ \quad \| (I - H_N S_N) \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k) \| .
\]

So we need to show that

\[
\lim_{N \to \infty} \| \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k) \| = 0 \quad (4.57)
\]

\[
\lim_{N \to \infty} \| (I - H_N S_N) \hat{\phi}(W^k) \bar{\omega}(k) \| = 0, \quad (4.58)
\]

and

\[
\lim_{N \to \infty} \| (I - H_N S_N) \hat{\phi}(W^k)(I - H_N S_N) \bar{\omega}(k) \| = 0. \quad (4.59)
\]
We start by proving (4.57). Proposition 2.3 implies that
\[ \| (I - H_N S_N) w \| = \| L_c (I - H_N S_N) w L_c^{-1} \| \]
\[ = \| L_c (I - H_N S_N) L_c^{-1} L_c w L_c^{-1} \| \]
\[ = \| (I - L_c H_N S_N L_c^{-1}) w \| . \]

By calling Lemma 2.3 and (4.41) we get
\[ \| (I - H_N S_N) w \|^2 = \frac{1}{M} \sum_{k=0}^{M-1} \| (I - H_N S_N) \tilde{w}(k) \|^2 . \quad (4.60) \]

The assumption made in (4.39) then implies that
\[ \lim_{N \to 0} \| (I - H_N S_N) \tilde{w}(k) \| = 0, \quad k = 0, 1, \ldots, M - 1. \quad (4.61) \]

From here (4.57) will be immediate if we note that
\[ \| \hat{\phi}_K (W^k) (I - H_N S_N) \tilde{w}(k) \| \leq \| \hat{\phi} (W^k) \| \| (I - H_N S_N) \tilde{w}(k) \| . \]

To see (4.58) and (4.59), we just need to note that
\[ \| (I - H_N S_N) \hat{\phi} (W^k) \tilde{w}(k) \| \leq \| (I - H_N S_N) \hat{\phi} (W^k) \| \| \tilde{w}(k) \| . \]
and
\[ \left\| (I - H_N S_N) \hat{\phi} (W^k) (I - H_N S_N) \bar{w}(k) \right\| \leq \left\| (I - H_N S_N) \hat{\phi} (W^k) \right\| \cdot \left\| (I - H_N S_N) \bar{w}(k) \right\|. \]

and call Lemma 4.8 and (4.61).

The statement about the convergence rate is a direct result of Corollary 4.5. ■

**Corollary 4.7** In Figure 4.13, suppose that $K$ is in $C$ and that $w \in \mathcal{W}_T$ is such that
\[ \lim_{N \to \infty} \|(I - H_N S_N)w\| = 0. \tag{4.62} \]

Then
\[ \lim_{N \to \infty} \|z_{ss}\| - \|\tilde{z}_{ss}\| = 0. \tag{4.63} \]

### 4.6 Design Example

In this section we present a simple but illustrative example to demonstrate the design technique developed in Chapter 4. Consider the feedback setup of Figure 4.16. The plant $P$ is a second order system with transfer function
\[ \hat{p}(s) = \frac{0.3s + 1}{(0.25s + 1)(s + 1)}. \]

The reference signal $w$ is the periodic signal
\[ w(t) = 0.6 \sin\left(\frac{2\pi}{10}t\right) + 0.3 \sin\left(\frac{2\pi}{10}t\right) + 0.3 \sin\left(\frac{2\pi}{10}t\right) \]
of period $T = 10$. The sampling period is chosen to be $h = 0.5$. This $h$ is intentionally comparable to the time constants of the plant, $\tau_1 = 1$ and $\tau_2 = 0.25$. From a practical viewpoint, this $h$ would be considered too large. Thus there are $M = 20$ sampling intervals in each period of the periodic reference signal. We design the controller...
for zero, one, and nine intersample points, that is, for $N = 1, 2,$ and 10. The case $N = 1$ represents the conventional discrete-time design, where the emphasis is put on the error signal only at the sampling instants, whereas for $N = 2$ and $N = 10,$ the power of the error signal is computed and minimized for one and nine extra points in between the sampling instants, respectively. Thus one expects to see better tracking in the latter cases.

Figure 4.16: The setup for the design example.

To do the design, we first put the setup of Figure 4.16 in the form of the setup in Figure 4.1 and incorporate the technique developed in Section 4.4. The steady-state tracking error is shown in Figure 4.17 for $N = 1, 2,$ and 10. As can be seen, the steady-state tracking error is zero at the sampling instants for $N = 1,$ but not for $N = 2$ or 10. However, we observe that by taking just one intersample point, i.e., $N = 2,$ the overall error reduces dramatically. The error is further decreased if we take an even larger number of intersample points, i.e., $N = 10.$ As we take more intersample points, not much improvement in the performance is observed and the error signals almost coincide with that for $N = 10.$ This convergence is not surprising as by taking more intersample points, the system in Figure 4.8 is expected to give a better emulation of the sampled-data setup in Figure 4.1. We have also listed the power-norm of the steady-state tracking error for different values of $N$ in Table 4.1 for a quantitative comparison. This table shows that the power of the error reduces almost 50% by taking one intersample point. Finally, in Figure 4.18 we have plotted the reference signal $w$ and the output $y$ of the system with the controller designed for $N = 10$ to show that the transient response of the system is acceptable.
Table 4.1: Power-norm of the steady-state error for different values of $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|z_{ss}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0256</td>
</tr>
<tr>
<td>2</td>
<td>0.0134</td>
</tr>
<tr>
<td>10</td>
<td>0.0118</td>
</tr>
</tbody>
</table>

Figure 4.17: Steady-state tracking error for different number of intersample points.
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4.7 Controller Optimality for Other Periodic Inputs

Our effort in this chapter so far has been on designing an optimal sampled-data repetitive controller for tracking a fixed known periodic input. In this regard, we formulated Problem 4.1 and showed in Section 4.2 that there is always a solution to this problem. That is, for a fixed known \( w \in \mathcal{W}_T \) in Figure 4.1, there is always a stabilizing controller \( K \) that would minimize the power-norm of the steady-state component of the tracking error \( z \). However, \( K \) is not necessarily optimal for other periodic inputs in \( \mathcal{W}_T \). Considering that the goal in repetitive control is likely not just tracking a single periodic input, we would like to know how \( K \) will perform for other periodic inputs in \( \mathcal{W}_T \). In this section, we will show that in fact in the case of SISO systems, \( K \) retains its optimality for some other signals in \( \mathcal{W}_T \). As we will see, these signals form quite a large class of periodic signals under practical assumptions.

Suppose that in our main setup in Figure 4.1, the known input is a one-dimensional periodic signal \( w \in \mathcal{W}_T \) that has a finite Fourier series representation.
where \( \tilde{w}(i) \in \mathbb{C} \), \( n = 0, \pm 1, \ldots, \pm l \). For this input, we know from Section 4.2 that there is an optimal controller that minimizes the power-norm of the steady-state component \( z_{ss} \) of the tracking error \( z \). This minimization is embodied in Problem 4.1.

Now consider another input, \( w_1 \), whose Fourier series representation involves the same sinusoids that \( w \) does, but its Fourier coefficients are not necessarily the same. Let

\[
\begin{align*}
  w_1(t) = \sum_{i=-l}^{l} \tilde{w}_1(i) e^{j \frac{2\pi it}{T}}
\end{align*}
\]  

be the corresponding Fourier series. A different design problem would be to find a controller that minimizes the power-norm of the steady-state component of the tracking error for \( w_1 \). Denote this tracking error by \( z_1 \). This minimization problem can then be stated as

**Problem 4.2**

\[
\text{minimize } \| z_{1ss} \|.
\]

As was stated earlier in this section, a solution to Problem 4.1 need not in general be a solution to Problem 4.2. However, as we will see shortly, this can indeed be the case under some mild conditions. First we need some preliminaries. We start by a standard result from algebra.

**Lemma 4.9** [BM65] For given integers \( i \in \mathbb{Z} \) and \( m \in \mathbb{N} \), there exist unique integers \( q(i, m) \) and \( r(i, m) \) such that

\[
i = mq(i, m) + r(i, m), \quad 0 \leq r < m.
\]

Next recall from Proposition 2.3 that since \( T = Mh \) for some \( M \in \mathbb{N} \), for periodic signal \( w \in \mathcal{W}_T \), the (continuously) lifted signal \( \mathcal{L}_w \), i.e., \( \mathcal{L}_w \), is a discrete-time, function-valued periodic signal of period \( M \). The following lemma shows that when
$w$ is a pure tone, the $M$ Fourier coefficients $\tilde{w}(k)$, $k = 0, 1, \ldots, M - 1$, of $w$, given from (2.6) by

$$\tilde{w}(k) = \sum_{n=0}^{M-1} W^k w(n), \tag{4.68}$$

can be found very simply. Recall that $\tilde{w}(k)$ are functions on the time interval $[0, h)$.

First define the function sequence $\xi_i : [0, h) \to \mathbb{C}, \ i \in \mathbb{Z}$ by

$$\xi_i(t) = e^{j2\pi it}. \tag{4.69}$$

**Lemma 4.10** For a fixed $i$ in $\mathbb{Z}$, let $w \in \mathcal{W}_T$ be given by

$$w(t) = e^{j2\pi it}, \ t \geq 0. \tag{4.70}$$

Then for $0 \leq k \leq M - 1$,

$$\tilde{w}(k) = \begin{cases} M\xi_i & k = r(i, M) \\ 0 & \text{else.} \end{cases} \tag{4.71}$$

**Proof** The lifted signal $w$ is a discrete-time, function-valued signal. Its $n$th sample, $n = 0, 1, 2, \ldots$, is a function piece defined on the time-interval $[0, h)$, described by

$$[w(n)](t) = [(L_c w)(n)](t) = w(t + nh) = e^{j2\pi t(t + nh)} = e^{j2\pi n} e^{j2\pi nh}.$$

Noting that $T = Mh$ and $W = e^{-j\frac{2\pi}{T}}$, we get

$$[w(n)](t) = W^{-in}e^{j\frac{2\pi it}{T}}. \tag{4.72}$$
By substituting \((4.72)\) in \((4.68)\), we obtain

\[
\begin{align*}
[\tilde{w}(k)](t) &= \sum_{n=0}^{M-1} W^{kn} \left[ W^{-in} e^{j2\pi t} \right] \\
&= e^{j2\pi t} \sum_{n=0}^{M-1} W^{(k-i)n} \\
&= \xi_i(t) \sum_{n=0}^{M-1} W^{(k-i)n}.
\end{align*}
\]

Now the above sum equals \(M\) whenever \(k-i\) is divisible by \(M\). By Lemma 4.9, there is one and only one such \(k\) in \([0, M-1]\). The sum equals zero otherwise.

Now since Lemma 4.9 states that \(r(i, M)\) is a unique integer in \([0, M-1]\), we can see that of the \(M\) Fourier coefficients of \(w\), only one is nonzero. For example, in the case that \(M = 5\), that is, \(T = 5h\), and \(i = 1\), that is, \(w(t) = e^{j2\pi t}\), the Fourier coefficients of \(w\) are

\[
\tilde{w}(k) = \begin{cases} 
0 & k = 0 \\
5\xi_1 & k = 1 \\
0 & 2 \leq k \leq 4,
\end{cases} \quad (4.73)
\]

since \(r(1, 5) = 1\). The corresponding coefficients when \(i = -1\) or \(w(t) = e^{-j2\pi t}\) are

\[
\tilde{w}(k) = \begin{cases} 
0 & 0 \leq k \leq 3 \\
5\xi_{-1} & k = 4.
\end{cases} \quad (4.74)
\]

This is because \(-1 = -1 \times 5 + 4\), which implies that \(r(-1, 5) = 4\). For \(w(t) = 1\), these coefficients are simply given by

\[
\tilde{w}(k) = \begin{cases} 
5 & k = 0 \\
0 & 1 \leq k \leq 4.
\end{cases} \quad (4.75)
\]

It is also interesting to see a case where \(w \in \mathcal{W}_T\) is not just a single harmonic. Since the DFT equation \((4.68)\) is linear, the corresponding Fourier coefficients for such \(w\) may be obtained by summing up those of the constituting terms, which are
in turn easily computable from Lemma 4.10. For instance, let \( l = 2 \) in (4.64), that is,

\[
w(t) = \sum_{i=-2}^{2} \tilde{w}(i) e^{j \frac{2\pi i t}{T}}, \quad t \geq 0.
\] (4.76)

For \( M = 5 \), the Fourier coefficients of the exponentials with \( i = -1, 0, 1 \) are given by (4.73-4.75) and those of the ones with \( i = -2, 2 \) can be computed similarly. Hence

\[
\tilde{w}(k) = \begin{cases} 
5\tilde{w}(k)\xi_k & k = 0, 1, 2 \\
5\tilde{w}(k - 5)\xi_{k-5} & k = 3, 4. 
\end{cases}
\] (4.77)

A point to observe here is that, for each \( k \) in \([0,4]\), \( \tilde{w}(k) \) involves only one of the coefficients in the Fourier series representation (4.76) for \( w \), namely, \( \tilde{w}(k) \). Clearly, with \( M = 5 \) this observation holds for all \( l \leq 2 \). However, this will not be the case when \( l \) is greater than 2. For example, for

\[
w(t) = \sum_{i=-3}^{3} \tilde{w}(i) e^{j \frac{2\pi i t}{T}}, \quad t \geq 0,
\] (4.78)

the Fourier coefficients of \( w \) are

\[
\tilde{w}(k) = \begin{cases} 
5\tilde{w}(k)\xi_k & k = 0, 1 \\
5[\tilde{w}(2)\xi_2 + \tilde{w}(-3)\xi_{-3}] & k = 2 \\
5[\tilde{w}(3)\xi_3 + \tilde{w}(-2)\xi_{-2}] & k = 3 \\
5\tilde{w}(-1)\xi_{-1} & k = 4. 
\end{cases}
\] (4.79)

In obtaining this relation, we note from Lemma 4.9 that \( r(2, 5) = r(-3, 5) = 2 \). From Lemma 4.10, this means that the nonzero Fourier coefficient for pure tones \( e^{j \frac{2\pi t}{T}} \) and \( e^{j \frac{2\pi(-3)t}{T}}, \quad t \geq 0 \), both happen at \( k = 2 \). This in turn implies that \( \tilde{w}(2) \) and \( \tilde{w}(-3) \) should both appear in the expression for \( \tilde{w}(2) \). The presence of \( \tilde{w}(3) \) and \( \tilde{w}(-2) \) in the expression for \( \tilde{w}(3) \) can be explained similarly.

This example brings our attention to the important fact that when the number of harmonics in (4.64), i.e., \( 2l + 1 \), is more than \( M \), aliasing can occur. Whether the periodic input for which the repetitive controller is designed has aliasing plays a
crucial role in the controller optimality for other periodic inputs. This will be shown in the main result of this section. But let's first summarize our observations in the following lemma.

**Lemma 4.11** Suppose that in (4.64)

\[ l \leq \frac{M - 1}{2}. \]  

(4.80)

Then the Fourier coefficients of \( w \) are given by

\[ \tilde{w}(k) = \begin{cases} 
M\tilde{w}(k)\xi_k & k = 0, 1, \ldots, l \\
M\tilde{w}(k - M)\xi_{k-M} & k = M - l, M - l + 1, \ldots, M - 1 \\
0 & \text{else.}
\end{cases} \]  

(4.81)

**Proof** From Lemma 4.9 we see that

\[ r(i, M) = \begin{cases} 
i & i = 0, 1, \ldots, l \\
M + i & i = -l, -l + 1, \ldots, -1.
\end{cases} \]  

(4.82)

Now constraint (4.83) on \( l \) implies that

\[ (M - l) - l = M - 2l \geq 1. \]

This inequality together with Lemma 4.10 and linearity of the DFT formula (4.68) gives (4.81).

**Theorem 4.6** Suppose that

\[ l \leq \frac{M - 1}{2} \]  

(4.83)

and that all the \( 2l + 1 \) coefficients present in (4.64) are nonzero. Then any \( K \in C \) that solves Problem 4.1 is a solution to Problem 4.2 also.

**Proof** In Section 4.2, we learned that by using controller parametrization, Prob-
Lemma 4.1 boils down to

\[
\min_{q(\lambda) \text{ stable}} \sum_{k=0}^{M-1} \left\| \hat{\phi}_1(W^k) \hat{\omega}(k) + \hat{\phi}_2(W^k) \hat{q}(W^k) \hat{\phi}_3(W^k) \hat{\omega}(k) \right\|^2. \tag{4.84}
\]

The assumption made on \( l \) allows us to bring in Lemma 4.11 to reduce the optimization functional in (4.84) to

\[
\sum_{k=0}^{l} |\hat{\omega}(k)|^2 \left\| \hat{\phi}_1(W^k) \xi_k + \hat{\phi}_2(W^k) \hat{q}(W^k) \hat{\phi}_3(W^k) \xi_k \right\|^2 + \sum_{k=M-l}^{M-1} |\hat{\omega}(k-M)|^2 \left\| \hat{\phi}_1(W^k) \xi_{k-M} + \hat{\phi}_2(W^k) \hat{q}(W^k) \hat{\phi}_3(W^k) \xi_{k-M} \right\|^2. \tag{4.85}
\]

Since all the \( \hat{\omega}(k) \) are nonzero by assumption, then \( \hat{q} \) is optimal iff it solves

\[
\left\| \hat{\phi}_1(W^k) \xi_k + \hat{\phi}_2(W^k) \hat{q}(W^k) \hat{\phi}_3(W^k) \xi_k \right\|^2 \tag{4.86}
\]

for each \( k = 0, \ldots, l \) and

\[
\left\| \hat{\phi}_1(W^k) \xi_{k-M} + \hat{\phi}_2(W^k) \hat{q}(W^k) \hat{\phi}_3(W^k) \xi_{k-M} \right\|^2 \tag{4.87}
\]

for each \( k = M - l, \ldots, M - 1 \).

From the proof, we note the importance of the assumptions made in the statement of this theorem. Without constraint (4.83) and with \( \hat{\omega}(k) \) nonzero, the input will have aliasing as it was illustrated by an example earlier. With aliasing present, some of the Fourier coefficients of \( \hat{w} \) will involve more than one of the coefficients \( \hat{w}(k) \) in (4.64). This means that the optimization functional in (4.84) will not be linear in \( |\hat{w}(k)|^2 \) as in (4.85) and hence cannot be broken down into independent pieces as in (4.86) and (4.87). Thus the optimal \( \hat{q} \) will not in general be independent of the coefficients of the exponentials in (4.64); it will be input-dependent.

Another point to note here is that constraint (4.83) need not necessarily limit \( l \) to be small. Indeed, \( l \) could be quite large since \( M \) could be large due to the fact that
there are typically many sampling periods in one period of the periodic input. This theorem, thus, suggests a strategy for designing a sampled-data repetitive controller that is optimal for a large class of periodic inputs: Pick any periodic input described by (4.64) in which \( l \) is the largest integer that is less than \( \frac{M-1}{2} \) and has all the exponentials with nonzero coefficients, and call the design algorithm developed in this chapter. However, one realizes from the proof that such a strategy stays limited to the SISO case. Our intention is to extend this strategy to MIMO systems by introducing the notion of induced power-norm for sampled-data systems.
Chapter 5

Sampled-Data Repetitive Control: Unknown Periodic Inputs

In the preceding chapter, we developed a methodology for designing a sampled-data repetitive controller for known periodic inputs. We also showed that, under certain mild conditions in the SISO case, the resulting controller is not only optimal for the input on which the design is based, but also for all other periodic inputs that share the same harmonics. This chapter aims at generalizing this methodology to the case where the periodic input is unknown. In this respect, a useful measure called the induced power-norm is introduced. This measure represents the power of the steady-state error vector in the system for the worst periodic input of unit power. The induced power-norm also takes the intersample behavior into consideration. Minimization of this measure is then used as a criterion for designing sampled-data repetitive controllers.

5.1 Induced Power-Norm

Consider the setup shown in Figure 5.1. This setup is the same as that in Figure 4.1 except that now the input w is not known; it can be any signal in \( \mathcal{W}_T \) normalized to \( \|w\| = 1 \). In the preceding chapter, one result of Theorem 4.1 was that the output \( z \) approaches a steady-state component \( z_{ss} \) for any periodic input \( w \in \mathcal{W}_T \). This led us to choosing the power-norm of this steady-state component, which is in \( \mathcal{W}_T \) itself, to
be the tracking measure. However, this measure directly depends on what the input is. For an unknown periodic input we need a tracking measure that considers all the possibilities that the input can assume. One such measure that treats all the periodic inputs in $\mathcal{W}_T$ equally is defined by

**Definition 5.1** The *induced power-norm* of the system in Figure 5.1 is

\[
J_{zw}(K) := \sup_{w \in \mathcal{W}_T, \|w\| \leq 1} \|z_{ss}\|.
\]  

That is, the induced power-norm equals the power of the error vector $z$ in the worst case for inputs of unit power.

![Sampled-data repetitive control system](image)

Figure 5.1: Sampled-data repetitive control system.

Now within the class $\mathcal{W}_T$ are all pure tones of arbitrarily large frequency, $\frac{2\pi f}{T}$. This will bound the induced-power norm from below if there is a feed-through path from the periodic input $w$ to the tracking error $z$, that is, if the $D_{11}$-matrix in the state-space representation for $G$ is nonzero. So $\mathcal{W}_T$ may be too large. In fact, $\mathcal{W}_T$ is too large for practical applications where the exogenous periodic input is rich only in a few harmonics. In this respect, in Definition 5.1 $\mathcal{W}_T$ perhaps should be replaced with the set of all *practically* possible periodic inputs. However, since this would be too application dependent, a better remedy is to introduce a fictitious low-pass filter $F$ at the input as in Figure 5.2 and let $\mathcal{W}_T$ be the class of periodic inputs to the filter. In this way, by choosing $F$ one can shape $\mathcal{W}_T$ to get the desired spectrum at the input to the plant $G$. 
Figure 5.2: Fictitious filter $F$ shapes the spectrum of the exogenous input $w$.

Absorbing $F$ in the plant $G$, we can equivalently assume that $G_{11}$ and $G_{21}$ are lowpass throughout.

5.2 Problem Formulation

The induced power-norm provides us with a tracking measure that can be used for designing repetitive controllers for unknown periodic inputs.

**Problem 5.1 (Optimal sampled-data repetitive control for unknown periodic inputs)**

$$\min_{K \in \mathcal{C}} J_{zw}(K).$$  \hspace{2cm} (5.2)

Problem 5.1 involves both continuous-time and discrete-time signals and hence is not amenable to a simple analysis. Following our approach in Chapter 4, we formulate a problem close to it instead which, as we shall see, has a state-space solution.

Note in Figure 5.1 that there are two time periods: $T$ is the period of the periodic input $w$; $h$ is the sampling period. We still assume that $T = Mh$, where $M$ is an integer.

To state the approximate formulation, we move to the setup of Figure 5.3. This setup is the same as that in Figure 4.8, that is, we model the input $w$ as the output of a fast zero-order hold $H_N$ of period $h/N$ and the output $z$ is looked at through a fast sampler $S_N$ that is synchronized with $H_N$. The integer $N$ is a design parameter.
In contrast with Section 4.3 where $\omega$ was fixed, here we let it span over the space of all discrete-time periodic signals that would make $w$ periodic of period $T$. Recalling that $T$, $h$ and $h/N$ are related by

$$T = Mh = MN(h/N),$$

this would mean that we take $\omega$ to be in $\Omega_{MN}$. In this way, we may not be able to cover all $\mathcal{W}_T$ at the output of $H_N$, but by choosing $N$ large enough, we expect that the setup of Figure 5.3 will emulate the sampled-data setup of Figure 5.1.

![Diagram]( attachment://diagram.png)

Figure 5.3: Fast discretization to emulate Figure 4.1.

Now, let's take the samplers and holds into the plant $G$ as we did in Section 4.3. This is shown in Figure 5.4, where

$$P := \begin{bmatrix} S_N & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} H_N & 0 \\ 0 & H \end{bmatrix}. \quad (5.3)$$

Recalling Theorem 4.3, we see that for $\omega \in \Omega_{MN}$ and a stabilizing controller $K$, the output $\zeta$ can be uniquely decomposed into two components, $\zeta_{ss}$ in $\Omega_{MN}$ and $\zeta_{tr}$ in $\ell_2(\mathbb{Z}^+)$, so that $\zeta = \zeta_{ss} + \zeta_{tr}$. Also, the map from $\omega$ to $\zeta_{ss}$ is linear on $\Omega_{MN}$. This allows us to define an induced power-norm for the system in Figure 5.4.

**Definition 5.2** The induced power-norm of the system in Figure 5.4 is defined by

$$J_{\zeta,\omega}(K) = \sup_{\omega \in \Omega_{MN}, \|\omega\| \leq 1} \|\zeta_{ss}\|. \quad (5.4)$$
And we have a counterpart to Problem 5.1:

\[
\min_{K \in \mathcal{C}} J_{\zeta}(K).
\]

Note that the class of controllers over which the minimization is performed is the same as before.

Next, introduce the discrete-time lifting operator and its inverse in the two-rate system of Figure 5.4 to arrive at the LTI system of Figure 5.5a and take them into the plant \( P \) to get Figure 5.5b, where \( P \) is given by (2.41). A state-space representation for \( P \) is obtained through (2.42-2.48). Since \( \zeta = L\zeta \) and \( \omega = L\omega \), there is a natural decomposition for \( \zeta \) when \( \omega \in \Omega_M \), namely, \( \zeta = \zeta_{ss} + \zeta_{tr} \) where \( \zeta_{ss} \in \Omega_M \) and \( \zeta_{tr} \in \ell_2(\mathbb{Z}_+) \). Moreover, the mapping from \( \omega \) to \( \zeta_{ss} \) is linear. Hence the following definition is allowed.

**Definition 5.3** In Figure 5.5b, the induced power-norm is

\[
J_{\zeta_{ss}}(K) = \sup_{\omega \in \Omega_M, \|\omega\| \leq 1} \|\zeta_{ss}\|.
\]

The corresponding minimization problem would then be

\[
\min_{K \in \mathcal{C}} J_{\zeta_{ss}}(K).
\]
Since $L$ scales the power-norm by a factor of $\sqrt{N}$ by the analysis on page 31, (5.5) and (5.7) are equivalent. The advantage of (5.7) is that it is stated on a system that is LTI. This allows us to employ frequency-domain formulas which help to obtain a closed-form solution for an optimal controller. We see this in the next section.

In what follows it will be convenient to use the following notation:

\[ \Phi_K : \text{ operator from } w \text{ to } z \text{ in Figure 5.1} \]
\[ \Phi_K : \text{ operator from } w := L_c w \text{ to } z := L_c z \]
\[ \Psi_K : \text{ operator from } \omega \text{ to } \zeta \text{ in Figure 5.4} \]
\[ \Psi_K : \text{ operator from } \omega \text{ to } \zeta \text{ in Figure 5.5}. \]

Thus

\[ \Phi_K = L_c \Phi_K L_c^{-1}, \quad \Psi_K = L \Psi_K L^{-1} \]

and $\Phi_K, \Psi_K$ are LTI with transfer functions $\hat{\Phi}_K(\lambda), \hat{\Psi}_K(\lambda)$; the former is operator-valued, the latter matrix-valued.

### 5.3 Design Procedure

In this section we present the solution to (5.7). From Lemma 2.2, (5.7) can be written as the minimax problem

\[
\min_{K \in \mathcal{C}} \max_{0 \leq k \leq M-1} \bar{\sigma} \left[ \hat{\psi}_K(W^k) \right], \quad (5.8)
\]
where we recall that $W = e^{-j\frac{2\pi}{M}}$. To find the solution to this problem, we parametrize all controllers in $C$ exactly in the same way as was done in Section 4.3. With respect to the state-space realization of $P$ in (2.48), namely

$$
\tilde{p}(\lambda) = \begin{bmatrix}
A_d & B_{1d} & B_{2d} \\
\frac{C_{1d}}{1} & \frac{D_{11d}}{1} & \frac{D_{12d}}{1} \\
C_2 & D_{21d} & 0
\end{bmatrix},
$$

(5.9)

all the stabilizing controllers can be represented as the input-output system of the block diagram in Figure 5.6, where $Q$ is stable FDLTI and $J$ has the realization

$$
\tilde{j}(\lambda) = \begin{bmatrix}
A_d + B_{2d}F + HC_2 & -H & B_{2d} \\
F & 0 & -I \\
-C_2 & I & 0
\end{bmatrix}
$$

(5.10)

in which $F$ and $H$ are matrices such that $A_d + B_{2d}F$ and $A_d + HC_2$ are stable.

The advantage of this characterization is that the input-output transfer matrix in Figure 5.5b from $\omega$ to $\zeta$ is an affine function of $\hat{q}(\lambda)$, that is,

$$
\hat{\psi}_K(\lambda) = \hat{\psi}_1(\lambda) + \hat{\psi}_2(\lambda) \hat{q}(\lambda) \hat{\psi}_3(\lambda),
$$

(5.11)

where $\hat{\psi}_1(\lambda)$, $\hat{\psi}_2(\lambda)$ and $\hat{\psi}_3(\lambda)$ are given in (4.31). Then (5.8) becomes

$$
\text{minimize} \max_{\hat{q}(\lambda) \text{ stable}} \sigma \left[ \hat{\psi}_1(W^k) + \hat{\psi}_2(W^k) \hat{q}(W^k) \hat{\psi}_3(W^k) \right].
$$

(5.12)

The solution to this problem is as follows.

**Lemma 5.1** A minimizing $Q$-parameter for (5.12) is

$$
\hat{q}(\lambda) = \sum_{n=0}^{M-1} r(n) \lambda^n
$$

(5.13)
Figure 5.6: Stabilizing controllers

where

$$r(n) = \frac{1}{M} \sum_{k=0}^{M-1} \tilde{r}(k) W^{-nk},$$

(5.14)

[that is, \(r(n) \xrightarrow{\text{DFT}^{-1}} \tilde{r}(k)\)], and \(\tilde{r}(k)\) solves

$$\min_{\tilde{r}(k)} \bar{\sigma} \left[ \hat{\psi}_1 (W^k) + \hat{\psi}_2 (W^k) \tilde{r}(k) \hat{\psi}_3 (W^k) \right]. \quad k = 0, 1, \ldots, M - 1. \quad (5.15)$$

**Proof** The minimum in (5.12) is in general greater than or equal to

$$\min_{\tilde{r}(0), \tilde{r}(1), \ldots, \tilde{r}(M-1)} \max_{0 \leq k \leq M-1} \bar{\sigma} \left[ \hat{\psi}_1 (W^k) + \hat{\psi}_2 (W^k) \tilde{r}(k) \hat{\psi}_3 (W^k) \right]. \quad (5.16)$$

Specifically, they are equal if there is a stable \(\hat{q}(\lambda)\) that satisfies

$$\hat{q}(W^k) = \tilde{r}(k) \quad k = 0, 1, \ldots, M - 1,$$

which there surely is: Formulas (5.13) and (5.14) give such \(\hat{q}(\lambda)\).

Furthermore, since the optimization variables are independent, (5.16) is equivalent to (5.15) \(\blacksquare\).

Polynomial (5.13) is the unique one of degree \(M - 1\) that interpolates the points \(\tilde{r}(k)\). Of course, there are other polynomials of higher degree that interpolate, as well as, conceivably, lower order rational functions that interpolate. The minimizing
Chapter 5. Sampled-Data Repetitive Control: Unknown Periodic Inputs

$Q$-parameter obtained from this theorem can be plugged back in the general representation of all stabilizing controllers, Figure 5.6, to get the controller itself.

So, one only has to find the solution to $M$ minimization problems in (5.15). These minimization problems share the general form

$$\min_X \sigma(A + BXC),$$

where $A, B, C, X$ are complex matrices. This problem has a closed-form solution. The next lemma takes care of this. Let $\dagger$ denote the Moore-Penrose generalized inverse.

**Lemma 5.2** Let

$$B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^*$$

(5.18)

and

$$C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^*$$

(5.19)

be the singular value decompositions of $B$ and $C$. Define

$$\bar{A} = U_B^* A V_C$$

(5.20)

$$\bar{X} = V_B^* X U_C$$

(5.21)

and let

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}$$

(5.22)

be natural partitions induced by the product

$$\begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \bar{X} \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix}.$$  

(5.23)

Then

$$\min_X \sigma(A + BXC) = \max \left( \bar{\sigma} \left[ \begin{array}{c} \bar{A}_2 \\ \bar{A}_4 \end{array} \right], \bar{\sigma} \left[ \begin{array}{cc} \bar{A}_3 & \bar{A}_4 \end{array} \right] \right) =: \alpha$$

(5.24)
and a minimizing solution is given by

\[ X = V_B \begin{bmatrix} \Sigma_B^{-1} \tilde{X} \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_C^* \]  \hspace{1cm} (5.25)

where

\[ \tilde{X} = -\bar{A}_1 - \bar{A}_2 \bar{A}_4^* (\alpha I - \bar{A}_4 \bar{A}_4^*)^\dagger \bar{A}_3. \]  \hspace{1cm} (5.26)

**Proof** Set \( Y := A + BXC \) and \( \tilde{Y} := U_B^* Y V_C \). Since matrices \( U_B \) and \( V_C \) are unitary, we have

\[ \bar{\sigma}(\tilde{Y}) = \bar{\sigma}(Y). \]  \hspace{1cm} (5.27)

On the other hand,

\[
\tilde{Y} = \bar{A} + \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \bar{X} \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix} + \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} \bar{X} \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} Z & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}
\]

where

\[
Z = \bar{A}_1 + [\Sigma_B \ 0] \bar{X} \begin{bmatrix} \Sigma_C \\ 0 \end{bmatrix}.
\]  \hspace{1cm} (5.28)

From the main result in [DKW82],

\[ \min_Z \bar{\sigma}(\tilde{Y}) = \alpha \]  \hspace{1cm} (5.29)

and a minimizing solution is

\[ Z = -\bar{A}_2 \bar{A}_4^* (\alpha I - \bar{A}_4 \bar{A}_4^*)^\dagger \bar{A}_3. \]  \hspace{1cm} (5.30)
So an optimal $\bar{X}$ for $\dot{Y}$ satisfies

$$
\begin{bmatrix}
\Sigma_B & 0
\end{bmatrix}
\bar{X}
\begin{bmatrix}
\Sigma_C \\
0
\end{bmatrix}
= Z - \bar{A}_1
$$

$$
= -\bar{A}_1 - \bar{A}_2 \bar{A}_4^\dagger (\alpha I - \bar{A}_4 \bar{A}_4^\dagger) \bar{A}_3
$$

$$
= : \bar{X}.
$$

This implies that one minimizing $\bar{X}$ is given by

$$
\bar{X} = \begin{bmatrix}
\Sigma_B^{-1} \bar{X} \Sigma_C^{-1} & 0 \\
0 & 0
\end{bmatrix}.
$$

(5.31)

Recalling (5.21), we arrive at (5.25).

### 5.4 Summary of Design Procedure

In this section, we summarize the controller design procedure. The procedure takes a minimal realization of $G$, the period $T$ of periodic inputs to be tracked or rejected, the sampling period $h$ and the number of intersample points $N$. By assumption, $T/h$ is an integer, $M$. The design steps are as follows:

**Step 1:** Starting with the realization of $G$, obtain a realization of $P$ in Figure 5.5b (formulas (2.42-2.48)).

**Step 2:** Parametrize all the stabilizing controllers as in Figure 5.6. Get matrices $\dot{\psi}_i (\lambda)$ ($i = 1, 2, 3$) in (4.31).

**Step 3:** Using Lemma 5.2, find the solutions $\bar{r}(k)$ to the following $M$ minimization problems:

$$
\text{minimize } \bar{c} \left[ \dot{\psi}_1 (W^k) + \dot{\psi}_2 (W^k) \bar{r}(k) \dot{\psi}_3 (W^k) \right], \quad k = 0, 1, \ldots, M - 1.
$$
The maximum of these minimums is the optimal induced power-norm in Figure 5.5b.

**Step 4:** Take the DFT\(^{-1}\) of \(\tilde{r}(k)\) to get \(r(k)\). A minimizing \(Q\)-parameter is given by
\[
q(\lambda) = \sum_{n=0}^{M-1} r(n) \lambda^n.
\]

**Step 5:** Insert \(Q\) in Figure 5.6 to arrive at the optimal controller.

This procedure has been coded in MATLAB.

### 5.5 Convergence Issues

As in the case of a fixed known periodic input, Section 4.5, the method of fast discretization gives rise to some convergence issues in the case of unknown periodic inputs as well. The first issue, the *analysis problem*, is if for a given repetitive controller, the induced power-norm of a sampled-data system can be computed to any desired degree of accuracy by fast discretization. The other issue, the *design problem*, is if a sequence of controllers can be designed by fast discretization which in performance converge to the optimal controller. In this section we will look only at the analysis problem.

To see the analysis problem, we focus on Figure 5.7. The following result shows that the induced power-norm from \(\omega\) to \(\zeta\) approaches that from \(\omega\) to \(z\), as larger and larger number of intersample points \(N\) are taken in the fast-discretization process.

**Theorem 5.1**

\[
J_{zw}(K) = \lim_{N \to \infty} J_{\zeta \omega}(K) \tag{5.32}
\]

**Remark 5.1** As we saw in Section 5.2, \(J_{\zeta \omega}(K)\) equals \(J_{\zeta \omega}(K)\). Since the system from \(\omega\) to \(\zeta, \Psi_K\), is an LTI discrete-time system, \(J_{\zeta \omega}(K)\) is readily computable from Lemma 2.2. So in fact this theorem provides us with an algorithm for computing \(J_{zw}(K)\).
The proof of Theorem 5.1 requires some preliminaries. The first result, which is the counterpart of Lemma 2.2, gives a frequency-domain expression for $J_{zw}(K)$.

**Lemma 5.3**

$$J_{zw}(K) = \max_{0 \leq k \leq M-1} \| \hat{\phi}_K (W^k) \|.$$  \hspace{1cm} (5.33)

**Proof** Exactly as in Lemma 2.2, we can derive that

$$J_{zw}(K) \leq \max_k \| \hat{\phi}_K (W^k) \|.$$

To show that $J_{zw}(K)$ achieves this upper bound, let $k_{\text{max}}$ denote the index at which $\| \hat{\phi}_K (W^k) \|$ takes its maximum value. For given $\epsilon > 0$, there exists $\tilde{w} \in K_2$ of unit norm so that

$$\| \| \hat{\phi}_K (W^{k_{\text{max}}}) \| - \| \hat{\phi}_K (W^{k_{\text{max}}}) \| < \epsilon.$$

Set

$$\tilde{w}(k) = \begin{cases} \tilde{w}, & k = k_{\text{max}} \\ 0, & \text{else.} \end{cases}$$

Then from above

$$\| \|z_{ss}\| - \| \hat{\phi}_K (W^{k_{\text{max}}}) \| \| < \epsilon,$$

which shows that $\|z_{ss}\|$ can be made arbitrarily close to the upper bound. \hfill \blacksquare

The next lemma is the core of the proof to Theorem 5.1.
Lemma 5.4

\[ \left\| \hat{\phi}_K(\lambda) \right\| = \lim_{N \to \infty} \left\| \hat{\psi}_K(\lambda) \right\| \quad \forall \lambda \in \mathbb{D}. \] (5.34)

Before seeing the proof, we note that in the statement of this lemma, \( \hat{\phi}_K(\lambda) \) is operator-valued whereas \( \hat{\psi}_K(\lambda) \) is matrix-valued. Nevertheless, these two transfer functions are related in an interesting way. First write

\[
\Psi_K = L \Psi_K L^{-1} = L S_N \Phi_K H N L^{-1} = L S_N L_c^{-1} L_c \Phi_K L_c^{-1} L_c H N L^{-1} = (L S_N L_c^{-1}) \Phi_K (L_c H N L^{-1}). \] (5.35)

Then define

\[ L : \mathcal{E}_N \to \mathbb{C}^N \quad y = L x, \]

\[
y = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}.
\]

This operator is basically the restriction of the discrete-time lifting operator to finite sequences of length \( N \). The inverse of \( L \) exists and is given by

\[ L^{-1} : \mathbb{C}^N \to \mathcal{E}_N \quad y = L^{-1} x, \]

\[
x = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n) \end{bmatrix}.
\]

It is straightforward to check that \( L \) and \( L^{-1} \) are norm-preserving.
Now for $L_cH_NL^{-1}$, from the block diagram

\[
\begin{array}{ccc}
\omega & L^{-1} & \omega \\
L^{-1} & H_N & w \\
H_N & w & L_c \\
\end{array}
\]

we can write

\[
[w(n)](\tau) = w(nh + \tau), \quad 0 \leq \tau < h, \quad n \in \mathbb{Z}_+.
\]

Also from the same diagram

\[
w(nh + \tau) = \omega(nN + l) \quad l\bar{h} \leq \tau < (l + 1)\bar{h}, \quad l = 0, 1, \ldots, N - 1
\]

and

\[
\omega(n) = \begin{bmatrix}
\omega(nN) \\
\omega(nN + 1) \\
\vdots \\
\omega(nN + N - 1)
\end{bmatrix}.
\]

Hence

\[
w(n) = H_N L^{-1} \omega(n).
\]

The $\lambda$-transforms of $w$ and $\omega$ are then related by

\[
\hat{w}(\lambda) = H_N L^{-1} \hat{\omega}(\lambda).
\] (5.36)

Similarly, we can derive a relation between the $\lambda$-transforms of the input $z$ and the output $\zeta$ of $LSNLc^{-1}$, shown in the following block diagram:

\[
\begin{array}{ccc}
\zeta & L\zeta^{-1} & z \\
L\zeta^{-1} & z & S_N \\
S_N & \zeta & L \\
\end{array}
\]
For $n$ in $\mathbb{Z}_+$ we can write

$$
\zeta(n) = \begin{bmatrix}
\zeta(nN) \\
\zeta(nN + 1) \\
\vdots \\
\zeta(nN + N - 1)
\end{bmatrix}.
$$

Also

$$
\zeta(nN + l) = z(nh + lh)
$$

and

$$
z(nN + lh) = [\tilde{z}(n)](lh).
$$

Therefore

$$
\zeta(n) = LS_N\tilde{z}(n).
$$

And from here

$$
\hat{\zeta}(\lambda) = LS_N\hat{\tilde{z}}(\lambda). \quad (5.37)
$$

The following lemma is then a direct result of combining (5.35), (5.36) and (5.37).

**Lemma 5.5**

$$
\hat{\psi}_K(\lambda) = LS_N\hat{\phi}_K(\lambda)H_NL^{-1} \quad (5.38)
$$

The relation between the norms of $\hat{\psi}_K(\lambda)$ and $\hat{\phi}_K(\lambda)$ is even simpler. By invoking the norm-preserving property of $L$ and $L^{-1}$ in (5.38), we get

$$
\|\hat{\psi}_K(\lambda)\| = \|S_N\hat{\phi}_K(\lambda)H_N\| \quad (5.39)
$$

We need one more result to prove Lemma 5.4.

**Lemma 5.6**

$$
\|S_N\hat{\phi}_K(\lambda)H_N\| = \|H_NS_N\hat{\phi}_K(\lambda)\|.
$$
Proof For $\mathbf{y}$ in $\mathcal{E}_N$

\[
\|H_N\mathbf{y}\|^2 = \int_0^h \| [H_N \mathbf{y}](t) \|^2 dt = \sum_{l=0}^{N-1} \int_{lh}^{(l+1)h} \| [H_N \mathbf{y}](t) \|^2 dt = h\|\mathbf{y}(l)\|^2 = h\|\mathbf{y}\|^2,
\]

that is,

\[
\|H_N\mathbf{y}\| = \sqrt{h}\|\mathbf{y}\|.
\]

Therefore

\[
\left\| S_N \tilde{\Phi}_K(\lambda) H_N \right\| = \sqrt{h} \left\| S_N \tilde{\Phi}_K(\lambda) \right\| = \left\| H_N S_N \tilde{\Phi}_K(\lambda) \right\|.
\]

Proof of Lemma 5.4 Immediate from (5.39), Lemma 5.6 and Corollary 4.6.

Proof of Theorem 5.1 Immediate from Lemmas 2.2, 5.3 and 5.4.

5.6 Fan Noise Suppression in an Acoustic Duct

For our design example, we will consider suppressing the periodic noise that is generated by a fan in an exhaust duct. The physical setup we study is an experimental setup at the University of Michigan [HAV+96]. In this first attempt in applying sampled-data repetitive control to noise attenuation in ducts, we use the same analytical model of an open-ended duct derived in [HAV+96]. Besides the derivation, that paper also considers the case of $H_2$ control applied to the experimental duct. Some other parameters, such as the speaker parameters, are borrowed from that paper as well.

We start by a brief description of the setup considered here and a derivation of its state-space model.
The Acoustic Duct

The experimental setup in [HAV+96] is essentially a rectangular acoustic duct as shown in Figure 5.8. The duct is open at both ends, with transversal dimensions $l_1$ and $l_2$ and length $L_D$. It is assumed that $l_1/L_D, l_2/L_D \ll 1$. With this assumption, the duct acts as a one-dimensional acoustic waveguide with waves traveling along its longitudinal axis $x$. The disturbance in the duct is a pressure signal that is injected into the duct through a disturbance speaker located at $x_D$. In an industrial setting, this disturbance might be actually originating from, say, the fan in an air conditioning system, in which case it would have a significant periodic component. The disturbance speaker is excited by a voltage $w$. A measuring microphone is installed on the duct wall at some point $x_M$. The output $y$ of this microphone is then used to compute a control signal $u$ that excites an actuating speaker placed at $x_A$. The acoustic signal $z$ at a point $x_o$ is the signal to be minimized. In an industrial setting, $x_o$ would be at the point where duct enters the workspace. One could take the feedback measurement exactly at the same point where minimization of the acoustic signal is desired, in which case $x_M$ and $x_o$ will be equal.

Now let's see a more detailed description. First denote by $p(x, t)$ the acoustic pressure at point $x$ along the $x$-axis at time $t$. In fact, the signals that we are interested in are $p(x_o, t)$ and $p(x_M, t)$. Also let's respectively denote by $a_A(t)$ and $a_D(t)$ the baffle accelerations of the actuating and the disturbance speakers. The linearized partial differential equation that governs the ideal dynamics of $p(x, t)$ along with the corresponding boundary conditions at the two open ends is then given by [HAV+96]

$$
\begin{align*}
  \frac{1}{c^2} p_{tt}(x, t) &= p_{xx}(x, t) + \rho_0 a_D(t) \delta(x - x_D) + \rho_0 a_A(t) \delta(x - x_A) \\
  p(0, t) = p(L_D, t) &= 0, \quad \forall t \geq 0,
\end{align*}
$$

with $c$ being the velocity of the acoustic wave (343 m/s in air under standard conditions of temperature and altitude) and $\rho_0$ being the equilibrium density (1.21 kg/m$^3$ for air under standard conditions). This setup along with a digital implementation
of the controller is summarized in the block diagram of Figure 5.9.

Next we shall find a state-space description for the system from \( w \) and \( u \) to \( z \) and \( y \) in this diagram. This system is denoted by \( G \). First for \( i \in \mathbb{N} \), define the spatial frequency
\[
k_i := \frac{i \pi}{L_D}
\]
and the time frequency
\[
\omega_i := c k_i.
\]
Let \( V_i : [0, L_D] \to \mathbb{R} \) be the function defined by
\[
V_i(x) = c \sqrt{2/L_D} \sin k_i x
\]
and let
\[
b^{P}_i := \rho_0 V_i(x_D)
\]
\[
b^{A}_i := \rho_0 V_i(x_A).
\]
Also let \( q_i : \mathbb{R}_+ \to \mathbb{R} \) be a function that satisfies

\[
\begin{aligned}
\dot{q}_i(t) + \omega_i^2 q_i(t) &= b_i^p a_D(t) + b_i^a a_A(t) \\
q(0) = \dot{q}_i(0) &= 0.
\end{aligned}
\] (5.46)

Then the solution to the differential equation (5.40) can be derived by the method of separation of variables as

\[
p(x, t) = \sum_{i=1}^{\infty} V_i(x) q_i(t).
\] (5.47)

Corresponding to the infinite set of second-order differential equations in (5.46), there is an infinite-dimensional state space representation. However, we limit ourselves here to finite dimensions. Preserving only \( n \) modes, define the state vector

\[ x_p := [q_1 \ \dot{q}_1 \ \ldots \ q_n \ \dot{q}_n]'. \]

We also use a pressure type microphone, whose output is proportional to the input pressure. Assuming a normalized gain of 1 for the microphone, we get

\[
\begin{aligned}
\dot{x}_p &= A_p x_p + B_{p1} a_D + B_{p2} a_A \\
z &= C_{p1} x_p \\
y &= C_{p2} x_p,
\end{aligned}
\] (5.48) (5.49) (5.50)
where

\[ A_p = \text{block-diagonal} \left( \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta_1\omega_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta_n\omega_n \end{bmatrix} \right) \] (5.51)

\[ B_{p1} = [0 \ b_1^p \ \ldots \ 0 \ b_n^p]' \] (5.52)

\[ B_{p2} = [0 \ b_1^d \ \ldots \ 0 \ b_n^d]' \] (5.53)

\[ C_{p1} = [V_1(x_0) \ 0 \ \ldots \ V_n(x_0) \ 0] \] (5.54)

\[ C_{p2} = [V_1(x_M) \ 0 \ \ldots \ V_n(x_M) \ 0]. \] (5.55)

Notice that damping coefficients \(\zeta_1, \ldots, \zeta_n\) have been introduced here for practical considerations. In practice, these coefficients are obtained via identification algorithms.

The dynamics of the control and the disturbance speakers should be included in the state space description as well. The transfer functions from \(w\) to \(a_D\) and \(u\) to \(a_A\) are assumed to be given by

\[ \frac{\dot{a}_D(s)}{\bar{w}(s)} = \frac{K_D s^2}{s^2 + 2\zeta_D \omega_D s + \omega_D^2} \] (5.56)

\[ \frac{\dot{a}_A(s)}{\bar{u}(s)} = \frac{K_A s^2}{s^2 + 2\zeta_A \omega_A s + \omega_A^2}, \] (5.57)

where \(K_A\) and \(K_D\) are the speaker gains, \(\omega_A\) and \(\omega_D\) are the speaker natural frequencies, and \(\zeta_A\) and \(\zeta_D\) are the damping ratios. Corresponding to these transfer functions are state-space descriptions given by

\[ \dot{x}_D = A_D x_D + B_D w \] (5.58)

\[ a_D = C_D x_D + D_D w \] (5.59)

\[ \dot{x}_A = A_A x_A + B_A u \] (5.60)

\[ a_A = C_A x_A + D_A u, \] (5.61)
where

\[
A_D = \begin{bmatrix}
0 & 1 \\
-\omega_D^2 & -2\zeta_D \omega_D
\end{bmatrix}
\]  

(5.62)

\[
B_D = \begin{bmatrix}
0 \\
K_D
\end{bmatrix}
\]  

(5.63)

\[
C_D = [-\omega_D^2 - 2\zeta_D \omega_D]
\]  

(5.64)

\[
D_D = K_D
\]  

(5.65)

\[
A_A = \begin{bmatrix}
0 & 1 \\
-\omega_A^2 & -2\zeta_A \omega_A
\end{bmatrix}
\]  

(5.66)

\[
B_A = \begin{bmatrix}
0 \\
K_A
\end{bmatrix}
\]  

(5.67)

\[
C_A = [-\omega_A^2 - 2\zeta_A \omega_A]
\]  

(5.68)

\[
D_A = K_A.
\]  

(5.69)

By combining the state vectors \( x_P, x_A \) and \( x_D \) into one vector \( x \), i.e., \( x = [x'_D \ x'_A \ x'_P]' \), and combining the state-space descriptions for the speakers and the duct, we arrive at the state-space description for \( G \):

\[
\dot{x} = Ax + B_1w + B_2u
\]  

(5.70)

\[
y = C_1x
\]  

(5.71)

\[
z = C_2x.
\]  

(5.72)

Here

\[
A = \begin{bmatrix}
A_D & 0 & 0 \\
0 & A_A & 0 \\
B_P, C_D & B_P, C_A & A_P
\end{bmatrix}
\]  

(5.73)
Chapter 5. Sampled-Data Repetitive Control: Unknown Periodic Inputs

\[
B_1 = \begin{bmatrix} B_D \\ 0 \\ B_{P_1}D_D \end{bmatrix} \quad (5.74)
\]
\[
B_2 = \begin{bmatrix} 0 \\ B_A \\ B_{P_2}D_A \end{bmatrix} \quad (5.75)
\]
\[
C_1 = [0 \quad 0 \quad C_{P_1}] \quad (5.76)
\]
\[
C_2 = [0 \quad 0 \quad C_{P_2}] \quad (5.77)
\]

With this state-space description, we arrive at Figure 5.10, where the sample and hold model, respectively, the A/D and D/A of Figure 5.9, and \(K\) is a FDLTI discrete-time controller to be designed. Denote the system from \(w\) to \(z\) by \(\Phi_K\), as before.

![Sampled-data setup for the duct control system.](image)

**Simulation**

We are interested in suppressing the noise that is generated by a 3600rpm fan, installed at one end of a duct of length \(L_D = 4.0195m\). Hence, the periodic noise is of period \(T = 1/f = 60Hz\). In our setup in Figure 5.8, we model this noise by a periodic signal that is injected into the duct through a properly positioned speaker. In doing so, we require all the modes that are preserved in the finite-dimensional model of the duct to be controllable from the disturbance input. This specification makes sure
that all the modes of interest are excited by the disturbance. As a result of this requirement, \( x_0 = 0 \) is immediately excluded from our choices since by examining (5.43), (5.44) and (5.52), we realize that this choice makes \( B_{p_1} = 0 \), which translates into having no disturbance at all. Also, the disturbance speaker cannot be at, say, \( x_D = L_D/8 = 0.502 \text{m} \), since from (5.44) it makes \( b_{q}^{D} = 0 \). A possible location not too far in the duct for the disturbance speaker is \( x_D = 0.114 \text{m} \), for which at least eight modes are controllable. For the disturbance speaker, we borrow the parameters \( K_D = 1 \), \( \omega_D = 2\pi \times 67 \text{rad/sec} \) and \( \zeta_D = 74\% \) from [HAV+96].

Our goal is to minimize the effect of the noise at one point near the end of the duct. The point picked for this simulation is \( x_o = 3.66 \text{m} \). We choose a co-located arrangement for the actuating speaker and measuring microphone, that is, \( x_A = x_M \).

In selecting this common location, we require that the system from the control input \( u \) to the measured output \( y \) in (5.48-5.49) be controllable and observable. By examining these equations, we note that this requirement is met if we set \( x_A \) and \( x_M \) to be the same as \( x_o \).

For simplicity, we consider only 8 modes in the duct model. The natural frequencies for these modes, which are obtained from (5.42), are brought in Table 5.1. The damping ratios are randomly generated in a range that covers those of the identified model in [HAV+96]. The Bode plot for the system from the disturbance \( w \) to the output \( z \) is shown in Figure 5.11. From the magnitude plot, the open-loop induced power-norm equals 2.34. We also consider the Bode plot of the system from \( w \) to \( z \) with many modes included in the duct model, to verify that the duct is of a broadband frequency response. This plot is shown in Figure 5.12 for 80 modes.

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_i % )</td>
<td>4.78</td>
<td>8.34</td>
<td>0.37</td>
<td>7.36</td>
<td>1.93</td>
<td>9.29</td>
<td>0.83</td>
<td>7.38</td>
</tr>
<tr>
<td>( \omega_i/(2\pi) )</td>
<td>42.67</td>
<td>85.33</td>
<td>128.00</td>
<td>170.67</td>
<td>213.33</td>
<td>256.00</td>
<td>298.67</td>
<td>341.33</td>
</tr>
</tbody>
</table>

Table 5.1: The damping ratios and natural frequencies of the modes preserved in the finite-dimensional model for the duct.

For designing the controller, we consider two methods, the induced power-norm
Figure 5.11: Bode plot for the system from $w$ to $z$ for 8 modes.

Figure 5.12: Bode plot for the system from $w$ to $z$ for 80 modes.
minimization of this chapter and the power-norm minimization of Chapter 4. In order to study the intersample behavior of the system, we choose to sample only \( M = 6 \) times during each period of the periodic input for the control operation. This gives \( h = T/M = 1/(fM) = 1/360 \text{sec} \), or a sampling rate of 360Hz for the controller. From the Bode plot of the model with 8 modes, this sampling rate compares with the bandwidth of the plant and hence is considered to be slow.

**Method 1: Induced Power-Norm Minimization** By following the design procedure of Section 5.4 for \( N = 1, 2, 4 \) and 10, we obtain four different controllers, which minimize the induced power-norm of the sampled-data system, fast-discretized respectively at \( N = 1, 2, 4 \) and 10 times the sampling rate of the controller. The case \( N = 1 \) represents the conventional discrete-time design, where the emphasis is put on the error signal only at the sampling instants, whereas for \( N = 2, N = 4 \), and \( N = 10 \), the error signal is computed and minimized for one, three and nine extra points in between the sampling instants, respectively. Thus one expects to see better tracking in the latter cases.

The resulting induced power-norms with these controllers in place are listed in Table 5.2. This table also includes the open-loop, or *uncompensated* induced power-norm, which is obtained from the magnitude bode plot in Figure 5.11. From this table, we see that the conventional discrete-time design, i.e., \( N = 1 \), yields a very poor performance, considering that the corresponding induced power-norm is even larger than that of the uncompensated system. However, we observe that by taking just one intersample point, i.e., \( N = 2 \), the induced power-norm reduces drastically to

\[
\frac{0.3623}{2.3401} \times 100 = 15.48\%
\]

of the induced power-norm of the uncompensated system.

**Method 2: Power-Norm Minimization** In this method, we pick a periodic disturbance \( w \) of fundamental frequency 60Hz and design the controller so that the power-norm of the corresponding steady-state output noise at \( x_o, z_{ss} \), is minimized. The disturbance \( w \) that is used in this example is shown in Figure 5.13. This signal
Table 5.2: Induced power-norm in Figure 5.10 for different controllers obtained by Method 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$J_{zw}(K)$</th>
<th>Open-loop induced power-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.1451</td>
<td>2.3401</td>
</tr>
<tr>
<td>2</td>
<td>0.3623</td>
<td>2.3401</td>
</tr>
<tr>
<td>4</td>
<td>0.3616</td>
<td>2.3401</td>
</tr>
<tr>
<td>10</td>
<td>0.3616</td>
<td>2.3401</td>
</tr>
</tbody>
</table>

was generated in MATLAB by a random choice of the Fourier coefficients in

$$w(t) = \sum_{n=1}^{5} w_s(n) \sin(2\pi n 60) + w_c(n) \cos(2\pi n 60).$$  \hspace{1cm} (5.78)

These coefficients are given in Table 5.3. We calculate the power-norm of $w$ to be $\|w\| = 1.0987$.

![Figure 5.13: Input to the duct at $x_D$.](image)

As in Method 1, we use fast discretization and design the controller for zero, one, three, and nine intersample points, that is, for $N = 1, 2, 4, \text{and } 10$. One period of $z_{ss}$ resulting from the disturbance $w$ for these controllers is shown in Figure 5.14. Also,
Table 5.3: Fourier coefficients for the disturbance $w$ in Method 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$w_s(n)$</th>
<th>$w_c(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1864</td>
<td>-0.5238</td>
</tr>
<tr>
<td>2</td>
<td>-0.8202</td>
<td>-0.9864</td>
</tr>
<tr>
<td>3</td>
<td>0.5993</td>
<td>0.0614</td>
</tr>
<tr>
<td>4</td>
<td>-0.2517</td>
<td>-0.1704</td>
</tr>
<tr>
<td>5</td>
<td>-0.0634</td>
<td>0.0095</td>
</tr>
</tbody>
</table>

Table 5.4: The power-norm of $z_{ss}$ for different values of $N$. 

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|z_{ss}|$</th>
<th>$|z_{uncomp}|$</th>
<th>$|z_{ss}|/|z_{uncomp}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1141</td>
<td>0.1950</td>
<td>0.585</td>
</tr>
<tr>
<td>2</td>
<td>0.0647</td>
<td>0.1950</td>
<td>0.332</td>
</tr>
<tr>
<td>4</td>
<td>0.0537</td>
<td>0.1950</td>
<td>0.275</td>
</tr>
<tr>
<td>10</td>
<td>0.0495</td>
<td>0.1950</td>
<td>0.254</td>
</tr>
</tbody>
</table>

Comparison and Discussion Now let's compare Methods 1 and 2. Table 5.5 lists the induced power-norm of the sampled-data setup for the two sets of controllers that were designed by these methods. We see that the discrete-time design, i.e., $N = 1$, yields the same induced power-norm in both of these methods; thus they are as poor in attenuating the noise at the output. From this table, we also observe that for
other values of $N$, the induced power-norms of the sampled-data system with the controllers designed by Method 2 are very close to those achieved by controllers that are designed by Method 1. This is very interesting because in Method 2, the design is not based on minimization of the induced power-norm. In other words, even though it is one arbitrary disturbance that we base the design on, the resulting controllers performs very close to the controllers that are based on all of $W_T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$J_{zw}(K)$ Method 1</th>
<th>$J_{zw}(K)$ Method 2</th>
<th>Open-loop induced power-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.1451</td>
<td>3.1451</td>
<td>2.3401</td>
</tr>
<tr>
<td>2</td>
<td>0.3623</td>
<td>0.4802</td>
<td>2.3401</td>
</tr>
<tr>
<td>4</td>
<td>0.3616</td>
<td>0.4690</td>
<td>2.3401</td>
</tr>
<tr>
<td>10</td>
<td>0.3616</td>
<td>0.4858</td>
<td>2.3401</td>
</tr>
</tbody>
</table>

Table 5.5: Comparison between the induced power-norms for the controllers obtained by Methods 1 and 2.

We have also applied the disturbance $w$ of Method 2 to the sampled-data setup, under controllers that are designed by Method 1. One period of the resulting steady-
state noise, $z_{ss}$, is plotted in Figure 5.15, along with one period of $z_{uncomp}$. The power-norms of these signals are listed in Table 5.6. Our first observation is that the steady-state output noise for $N = 1$ is the same as in Method 2. This can be attributed to the fact the resulting controllers for $N = 1$ are both discrete-time internal models of the periodic signals of interest. Finally, we observe that for all other values of $N$, the steady-state output noise is at a higher level compared to Figure 5.14. This can be explained by noting that we are now considering controllers that are not designed for the specific input considered here; they are worst-case based.

![Figure 5.15](image_url)

Figure 5.15: One period of the steady-state noise at $x_o$, with controllers designed by Method 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|z_{ss}|_{\text{Method 1}}$</th>
<th>$|z_{ss}|_{\text{Method 2}}$</th>
<th>$|z_{uncomp}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1141</td>
<td>0.1141</td>
<td>0.1950</td>
</tr>
<tr>
<td>2</td>
<td>0.0647</td>
<td>0.0666</td>
<td>0.1950</td>
</tr>
<tr>
<td>4</td>
<td>0.0537</td>
<td>0.0687</td>
<td>0.1950</td>
</tr>
<tr>
<td>10</td>
<td>0.0494</td>
<td>0.0696</td>
<td>0.1950</td>
</tr>
</tbody>
</table>

Table 5.6: Comparison between the $\|z_{ss}\|$ for the controllers obtained by Methods 1 and 2.
Chapter 6

Robustness Analysis of Sampled-Data Repetitive Control Systems

This chapter formulates and analyzes a robust tracking problem for sampled-data repetitive control systems in the presence of structured linear periodically time-varying perturbations. Two pertinent issues are addressed. One is stability robustness, that is, if stability will be retained under such perturbations. The other is robust tracking, that is, with stability robustness guaranteed, whether the tracking criterion will be met for all variations of the plant under the given class of perturbations. We take the tracking measure to be the induced power-norm of Chapter 5 and the tracking criterion is that only induced power-norms which are less than a prespecified bound are acceptable. Dullerud’s generalized notion of structured singular value is used to answer these questions in the form of a necessary and sufficient condition.

6.1 Stability and Tracking Robustness

Consider the sampled-data setup of Figure 6.1. The generalized plant $G$ is FDLTI and $w \in \mathcal{W}_T$. The controller $K$ is a discrete-time FDLTI repetitive controller that might have been obtained through different design methodologies, for example, by
doing a digital implementation of an analog design [NH86], by discretizing the plant and performing a discrete-time design [TTC89, HS93], or by a sampled-data approach as we have seen in Chapters 4 and 5. In any case, the assumption is that $K$ stabilizes the system in Figure 6.1 and offers a good level of tracking. This, however, might not be the case when $K$ is implemented in the actual system of which $G$ represents only a model. By implementing $K$ on the real system, we might end up having poor tracking or even instability. What we plan to do in this chapter is to create some tools that could assist us in expanding our knowledge about the performance of $K$ when the plant is subject to uncertainty. This issue has also been the subject of study of previous works, [TT94, Hil94], but the major effort has been in the context of digital control where the plant has been discretized. Since in practice, the discrete-time controller is connected to an analog plant, a sampled-data approach to this issue is more precise.

Figure 6.1: A sampled-data repetitive control system.

To state our robust tracking problem, we move to the setup of Figure 6.2, where with abuse of notation, $G$ is still allowed to denote the FDLTI generalized plant, which now accommodates perturbations. Inputs $p_1$ and $p_2$ are introduced for stability definition purposes, as we will see later. Perturbation $\Delta$ represents the uncertainty in the plant. Typically, $\Delta$ might be LTI. More generally, to discuss what type of $\Delta$ should be allowed, let us first note that obviously, for output $z$ to track periodic signals, it has to get to steady state first. Not all perturbations allow this. Therefore, to have a well-defined problem, one has to constrain perturbation $\Delta$ so that a steady state can be guaranteed for the output $z$. One class of such perturbations that makes
this possible is the class of stable linear $h$-periodic perturbations, as we will see in a later section. Recalling that all LTI perturbations are $h$-periodic, we see that they constitute another candidate class of perturbations. For simplicity, we formulate and solve the problem only for $h$-periodic perturbations. However, this would be too conservative if we don't need more than LTI perturbations to describe the uncertainty in the plant [Dul96]. Also, we would like to work with structured perturbations since the solution to the full-block perturbation case will then be immediate. As a by-product, necessary and sufficient conditions for robust stability and tracking are obtained. (The conditions would be only sufficient for LTI perturbations.)

![Sampled-data repetitive control system with uncertainty.](image)

We assume that $K$ stabilizes and meets the tracking criterion for the nominal system ($\Delta = 0$). The tracking measure is the induced power-norm of the preceding chapter. As we saw, this measure is the power of the steady-state tracking error $z$ for the worst periodic input $w$ of unit power. As well, this measure features the largest amount of power that can build up in the error in between the sampling points, that is, in the intersample behavior. The main question is robust stability, that is, if the system will retain its stability under such perturbations. Also it is important to have a way of testing if under perturbations of the given class the tracking criterion is met, that is, if the induced power-norm will stay less than a prespecified bound.

To answer these questions, we need some preliminaries.
6.2 Robustness Analysis Setup

We begin with the plant $G$. In dealing with $G$, we group $q_1$ and $w$ together as one input and $q_2$ and $z$ together as one output. Corresponding to this, we have the partition

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}.$$  \hfill (6.1)

We also assume that $G$ has a minimal realization of the form

$$\hat{g}(s) = \begin{bmatrix} A & [B_1^1 B_1^2] & B_2 \\ C_1^1 & 0 & 0 & D_{12} \\ C_1^2 & 0 & 0 & D_{12}^2 \\ C_2 & [0 0] & 0 \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix}.$$  \hfill (6.2)

The matrices $D_{11}$, $D_{12}$, and $D_{22}$ are set to zero for well-posedness and because $w$ is assumed to be lowpass filtered.

The controller $K$ is assumed to have minimal realization

$$\hat{k}(\lambda) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$  \hfill (6.3)

and is assumed to internally stabilize the nominal plant, that is, for no input present, i.e., $w = p_1 = p_2 = 0$, and for any initial conditions $x_o(0)$ and $x_K[0]$ of plant $G$ and controller $K$, $x_o(t) \to 0$ and $x_K[k] \to 0$ as $t \to \infty$, $k \to \infty$.

With regard to the uncertainty, we assume that perturbation $\Delta$ represents a stable linear structured $h$-periodic system. As we have seen in Section 2.4, by applying continuous lifting to an $h$-periodic system, one obtains an LTI associated system that is easier to work with. To be precise, we consider only those $h$-periodic systems that are in $L_A(\mathcal{B}_L(K_2))$. From (2.38) we recall that the operator-valued transfer functions for the lifted associates of members of $L_A(\mathcal{B}_L(K_2))$ are continuous on $\hat{D}$. This continuity brings some convenience to the analysis. To have $\Delta$ structured, take $\Delta$ in...
\[
X_s := \{ \Delta = \text{diag}(\Delta_1, \ldots, \Delta_l) : \Delta_k \in L(L_2), \ 1 \leq k \leq l \}.
\] (6.4)

As can be seen, this imposes a structure on \( \Delta \) with respect to its Euclidean dimensions. In summary, we take \( \Delta \) to be in

\[
X_{PTV} := X_s \cap L(A(D, L(K_2))).
\] (6.5)

Note that by setting \( l = 1 \) in \( X_s \), we find the solution for the full-block perturbation, i.e., unstructured case. Corresponding to each \( \Delta \in X_{PTV} \) there is \( \Delta = L_c^{-1} \Delta L_c \) with transfer function

\[
\hat{\Delta}(\lambda) = \begin{bmatrix}
\hat{\Delta}_1(\lambda) & 0 \\
\vdots & \ddots \\
0 & \hat{\Delta}_l(\lambda)
\end{bmatrix},
\]

where \( \hat{\Delta}_k \) is in \( A(D, L(K_2)) \). So for \( \Delta \in X_{PTV} \), \( \hat{\Delta} \) is a mapping from \( D \) to

\[
\Delta_{PTV} := \{ \bar{\Delta} \in L(K_2) : \bar{\Delta} = \text{diag}(\bar{\Delta}_1, \ldots, \bar{\Delta}_l), \ \bar{\Delta}_i \in L(K_2), \ i = 1, \ldots, l \}.
\] (6.6)

Finally, the input \( w \) is in \( \mathcal{W}_T \) and we retain the assumption that there is an integer number of sampling periods in \( T \), i.e., \( T = Mh \).

### 6.3 Problem Formulation

In this section we give a precise formulation of the robust tracking analysis problem. First, let's see if the problem is well-defined, that is, if for \( \Delta \in X_{PTV} \) in Figure 6.2 and a controller \( K \) that stabilizes the perturbed system, the output \( z \) gets to a steady state for \( w \in \mathcal{W}_T \). In Figure 6.2, set \( p_1 = p_2 = 0 \) and bring in \( L_c \) and \( L_c^{-1} \) to get the system in Figure 6.3. Then absorb them together with the sample and hold in the plant \( G \) and the perturbation \( \Delta \) to arrive at Figure 6.4.
Figure 6.3: Continuous-time lifting of the perturbed repetitive control system.

Figure 6.4: The lifted system is single-rate discrete-time.
The lifted plant $G$ is

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} L_c G_{11} L_c^{-1} & L_c G_{12} H \\ SG_{21} L_c^{-1} & SG_{22} H \end{bmatrix}. \tag{6.7}$$

As we saw in Section 2.4, $G$ is an LTI system with a state space representation given by (2.25-2.30). The lifted perturbation $\Delta$, $L_c \Delta L_c^{-1}$, is LTI too since $\Delta$ is $h$-periodic. So the lifted system is LTI. Now denote the mapping $w \mapsto z$ by $\Phi_\Delta$. Since the controller $K$ stabilizes the perturbed system, $\Phi_\Delta$ is bounded on $L_2$. By Proposition 2.2, the LTI lifted map from $w$ to $z$, $\Phi_\Delta := L_c \Phi_\Delta L_c^{-1}$, defines a bounded map on $\ell_2(K_2)$. Thus, by Proposition 2.5, there exists $\hat{\phi}_\Delta \in \mathcal{H}_\infty(\mathbb{D}, L(K_2))$ so that $\Phi_\Delta = \Lambda^{-1} \Theta \hat{\phi}_\Delta \Lambda$. Moreover, from the state space representation for $G$ we see that $\hat{\phi}_\Delta$ is continuous. So is $\hat{\Delta}$ since it is in $X_{PTV}$ and hence in $A(\mathbb{D}, L(K_2))$. Therefore $\hat{\phi}_\Delta$ is continuous on $\mathbb{D}$. Also, $w \in \mathcal{W}_T$ and $T = Mh$ imply that $w \in \Omega_M(K_2)$ and that its $\lambda$-transform is given by (2.20). Therefore

$$\hat{\xi}(\lambda) = \hat{\phi}_\Delta(\lambda) \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{w}(k)}{1 - \lambda W^{-k}}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{\phi}_\Delta(\lambda) \hat{w}(k)}{1 - \lambda W^{-k}}$$

$$= \hat{\xi}_{ss}(\lambda) + \hat{\xi}_{tr}(\lambda)$$

where

$$\hat{\xi}_{ss}(\lambda) := \frac{1}{M} \sum_{k=0}^{M-1} \frac{\hat{\phi}_\Delta(W^k) \hat{w}(k)}{1 - \lambda W^{-k}}. \tag{6.8}$$

and

$$\hat{\xi}_{tr}(\lambda) := \frac{1}{M} \sum_{k=0}^{M-1} \frac{[\hat{\phi}_\Delta(\lambda) - \hat{\phi}_\Delta(W^k)] \hat{w}(k)}{1 - \lambda W^{-k}}. \tag{6.9}$$

Noting that $\hat{\xi}_{ss}$ is of the form in (2.21), it is a periodic signal of period $M$ with DFT coefficients

$$\hat{\xi}_{ss}(k) = \hat{\phi}_\Delta(W^k) \hat{w}(k). \tag{6.10}$$

Also in the expression for $\hat{\xi}_{tr}$,
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\[
\left[ \frac{\hat{\phi}_\Delta(\lambda) - \hat{\phi}_\Delta(W_k)}{1 - \lambda W^{-k}} \right]
\]

is in \( \mathcal{H}_\infty(\mathbb{D}, L_2(\mathcal{K}_2)) \) because it is analytic and bounded on \( \mathbb{D} \). Therefore, \( \hat{z}_{tr} \in \mathcal{H}_2(\mathbb{D}, \mathcal{K}_2) \). By Proposition 2.1, \( \hat{z}_{tr} \in \ell_2(\mathcal{K}_2) \).

Now, define \( z_{ss} := L_c^{-1} \hat{z}_{ss} \) and \( z_{tr} = L_c^{-1} \hat{z}_{tr} \). Thus, \( z = z_{ss} + z_{tr}, z_{ss} \in \mathcal{W}_T \) and by Proposition 2.2, \( z_{tr} \in \mathcal{L}_2 \). This shows for \( \Delta \in \mathcal{X}_{PTV} \) and a controller that stabilizes the perturbed system, the output \( z \) approaches a steady-state component, \( z_{ss} \). Also, from above, it is seen that the mapping from \( w \) to \( z_{ss} \) is linear.

The following theorem is a summary of the above analysis.

**Theorem 6.1** In Figure 6.2, suppose that \( \Delta \in \mathcal{L}_A(\mathbb{D}, L_2(\mathcal{K}_2)) \) and that \( K \) stabilizes the perturbed system. Then for \( p_1 = 0, p_2 = 0 \) and \( w \in \mathcal{W} \), there exist unique signals \( z_{ss} \) in \( \mathcal{W}_T \) and \( z_{tr} \) in \( \mathcal{L}_2 \) such that \( z = z_{ss} + z_{tr} \). Moreover, the mapping from \( w \) to \( z_{ss} \) is linear on \( \mathcal{W} \).

Linearity of the map from \( w \) to \( z_{ss} \) allows the following definition for the stable perturbed system.

**Definition 6.1** The induced power-norm of the perturbed system in Figure 6.2 is

\[
J_{z,w}(\Delta) = \sup_{w \in \mathcal{W}_T, \|w\| \leq 1} \|z_{ss}\|. \tag{6.11}
\]

For each perturbation \( \Delta \), we have a measure of tracking given by the induced power-norm. To state our robust tracking analysis problem, we have to consider a class of perturbations. Let a prefix \( \mathcal{U} \) denote the open unit ball. We have the following definitions.

**Definition 6.2** The system in Figure 6.2 is robustly stable with respect to \( \mathcal{U} \mathcal{X}_{PTV} \) if the map

\[
\begin{bmatrix}
w \\
p_1 \\
p_2
\end{bmatrix} \mapsto \begin{bmatrix}
z \\
q_1 \\
q_2
\end{bmatrix}
\]
is bounded on $L_2$ for each $\Delta \in \mathcal{U} \mathcal{X}_{PTV}$.

**Definition 6.3** The system in Figure 6.2 has robust tracking of periodic signals with respect to $\mathcal{U} \mathcal{X}_{PTV}$ if it is robustly stable with respect to $\mathcal{U} \mathcal{X}_{PTV}$ and $J_{zw}(\Delta) \leq 1$ for each $\Delta \in \mathcal{U} \mathcal{X}_{PTV}$.

The problem of robustness analysis can then be stated as:

**Problem 6.1** Find necessary and sufficient conditions that guarantee robust tracking of the setup in Figure 6.2 with respect to $\mathcal{U} \mathcal{X}_{PTV}$.

For the solution, we need the notion of structured singular value of an operator.

### 6.4 Generalized Structured Singular Values

The notion of structured singular value was first defined for matrices in [Doy82] for the analysis of LTI feedback systems with structured uncertainties. For sampled-data systems, we need a generalization of this notion for operators on Hilbert spaces [DG93]. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces. Then the space of all bounded linear transformations from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $L(\mathcal{E}, \mathcal{F})$. If $\mathcal{E}$ and $\mathcal{F}$ are identical, we write simply $L(\mathcal{E})$, as before.

**Definition 6.4** Suppose $P \in L(\mathcal{E}, \mathcal{F})$ and $\Delta$ is a subspace of $L(\mathcal{F}, \mathcal{E})$. The structured singular value $\mu_\Delta(P)$ of $P$ with respect to the perturbation set $\Delta$ is the inverse of the largest $\delta$ such that

$$I - P\Delta \text{ is invertible in } L(\mathcal{F}) \ \forall \Delta \in \Delta, \ |\Delta| \leq \delta.$$  \hspace{1cm} (6.12)

Thus $\mu_\Delta(P)^{-1}$ is a stability margin for structured perturbations.

The Main Loop Theorem from [ZGD95] may be generalized as well. First, some definitions. Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, and $\mathcal{F}_2$ represent four arbitrary Hilbert spaces and let $\oplus$ denote the direct sum. Suppose that

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in L(\mathcal{E}_1 \oplus \mathcal{E}_2, \mathcal{F}_1 \oplus \mathcal{F}_2)$$
and two subspaces $\Delta_1 \subseteq L(F_1, E_1), \Delta_2 \subseteq L(F_2, E_2)$ which define

$$
\Delta := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \Delta_1, \Delta_2 \in \Delta_2 \right\}
$$

are given. Assuming that $I - P_{11}\Delta_1$ is invertible, define the upper linear-fractional transformation (upper LFT) by

$$
F_u(P, \Delta_1) := P_{22} + P_{21}\Delta_1(I - P_{11}\Delta_1)^{-1}P_{12}. \quad (6.13)
$$

The rationale for this terminology comes from Figure 6.5, in which $\Delta_1$ appears in the upper loop, and the corresponding input-output map is given by $F_u(P, \Delta_1)$.

![Figure 6.5: Upper LFT of $\Delta_1$ equals the input-output map.](image)

The following result is obtained.

**Theorem 6.2 (Main Loop Theorem)**

$$
\mu_{\Delta}(P) \leq 1 \iff \begin{cases} 
\mu_{\Delta_1}(P_{11}) \leq 1 \text{ and } \\
\sup_{\Delta_{11} \in \Delta_1} \mu_{\Delta_2}(F_u(P, \Delta_1)) \leq 1.
\end{cases} \quad (6.14)
$$

**Proof** ($\iff$) Let $\Delta \in \Delta$, $\|\Delta\| < 1$. We must show $I - P\Delta$ is invertible. The inequality $\|\Delta\| < 1$ implies $\|\Delta_1\| < 1$ and $\|\Delta_2\| < 1$, since $\|\Delta\| = \max(\|\Delta_1\|, \|\Delta_2\|)$. 


Since $I - P_{11} \Delta_1$ is invertible, one can write
\[
I - P \Delta = \begin{bmatrix}
I - P_{11} \Delta_1 & -P_{12} \Delta_2 \\
-P_{21} \Delta_1 & I - P_{22} \Delta_2
\end{bmatrix}
= \begin{bmatrix}
I & -P_{12} \Delta_2 \\
-P_{21} \Delta_1 (I - P_{11} \Delta_2)^{-1} & I - P_{22} \Delta_2
\end{bmatrix}
\begin{bmatrix}
I - P_{11} \Delta_1 & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
I & -P_{12} \Delta_2 \\
I - P_{22} \Delta_2 - P_{21} \Delta_1 (I - P_{11} \Delta_1)^{-1} P_{12} \Delta_2 & I
\end{bmatrix}
= \begin{bmatrix}
I & -P_{12} \Delta_2 \\
I & \begin{array}{c}
\text{invertible} \\
\text{invertible}
\end{array}
\end{bmatrix}
\begin{bmatrix}
I - P_{11} \Delta_1 & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
I - P_{11} \Delta_1 & 0 \\
0 & I
\end{bmatrix}.
\]

(6.15)

So $I - P \Delta$ is invertible.

$(\implies) \mu_{\Delta_1}(P_{11}) \leq 1$ since otherwise for some $\Delta_1 \in \Delta_1$ with $\|\Delta_1\| < 1$, $I - P_{11} \Delta_1$ is not invertible which means that for
\[
\Delta = \begin{bmatrix}
\Delta_1 & 0 \\
0 & 0
\end{bmatrix},
\]

that satisfies $\|\Delta\| = \|\Delta_1\| < 1$, $I - P \Delta$ is non-invertible. This contradicts $\mu_{\Delta}(P) \leq 1$. Since $I - P_{11} \Delta_1$ is invertible for $\|\Delta_1\| < 1$, from (6.15) one can see that for each $\Delta_1 \in U_\Delta_1$, $I - F_u(P, \Delta_1) \Delta_2$ is invertible for all $\Delta_2 \in \Delta_2$ with $\|\Delta_2\| < 1$. Therefore

$$
\sup_{\Delta_1 \in U_\Delta_1} \mu_{\Delta_2}(F_u(P, \Delta_1)) \leq 1.
$$
This section closes with a lemma which is the Small Gain Theorem in a general setting. This lemma is used in the proof of the main result of the chapter.

**Lemma 6.1** Suppose that $P \in L(E)$ and that $\Delta = L(E)$. Then $\mu_\Delta(P) = \|P\|$. 

**Proof** If $\|\Delta\| < \frac{1}{\|P\|}$, then $\|P\Delta\| < 1$ which implies the invertibility of $I - P\Delta$. This implies that $\mu_\Delta(P) \leq \|P\|$.

On the other hand, for a given $\epsilon > 0$, there is unit vector $v \in E$ so that $\|Pv\| > \|P\| - \epsilon$. Set $u := Pv/\|Pv\|$ and define the perturbation $\Delta \in L(E)$ by

$$\Delta w := \frac{1}{\|Pv\|} v < u, w >, \quad w \in E.$$ 

One observes that

$$(I - P\Delta)u = u - \frac{Pv}{\|Pv\|} < u, u > = u - u = 0,$$

which shows that $I - P\Delta$ is not invertible. Also

$$\|\Delta\| = \sup_{\|w\|=1} \|\Delta w\| = \sup_{\|w\|=1} \frac{\|v\|}{\|Pv\|} | < u, w > | < \sup_{\|w\|=1} \frac{1}{\|P\| - \epsilon} | < u, w > | = \frac{1}{\|P\| - \epsilon},$$

which gives $\mu_\Delta(P) < \|P\|$. 

---

### 6.5 The Main Result

For the solution of Problem 6.1, absorb the controller $K$ into $G$ to get the system in Figure 6.6, where $P$ is LTI and has state space representation given in (2.32-2.36).
With respect to its inputs and outputs, $P$ can be partitioned as in

$$
P = \begin{bmatrix} P_{11} & P_{12} \\
                        & P_{22} \end{bmatrix}.
$$

The following lemma then gives a frequency-domain expression for $J_{zw}(\Delta)$. This lemma can be proved exactly as in Lemma 5.3.

\begin{lemma}
\[ J_{zw}(\Delta) = \max_{0 \leq k \leq M-1} \left\| F_u \left[ \hat{p}(W^k), \hat{\Delta}(W^k) \right] \right\|. \] (6.16)
\end{lemma}

We also need to bring in the following two results from [Dul96]. The first result states that one can check for the robust stability of the setup in Figure 6.2 by performing a test on the LTI lifted system in Figure 6.6.

\begin{lemma}
The system in Figure 6.2 is robustly stable with respect to $\mathcal{U}\mathcal{X}_{PTV}$ iff for $\Delta \in \mathcal{U}\mathcal{X}_{PTV}$, $(I - P_{11}\Delta)$ is invertible in $\mathbb{L}(\ell_2(K_2))$.
\end{lemma}

The second result allows this test to be carried out in the frequency domain.

\begin{theorem}
The system in Figure 6.2 is robustly stable with respect to $\mathcal{U}\mathcal{X}_{PTV}$ iff

\[ \sup_{\lambda \in \partial D} \mu_{\Delta_{PTV}} \left[ \hat{p}_{11}(\lambda) \right] \leq 1. \] (6.17)
\end{theorem}
So the maximum value that $\mu_{\Delta_{PTV}} \left[ \hat{p}_{l1}(\lambda) \right]$ takes on the unit circle is the determining factor for robust stability.

Now, define the robust tracking uncertainty set $\Delta_{rt}$ by

$$\Delta_{rt} := \left\{ \tilde{\Delta}_{rt} \in \mathfrak{L}(\mathcal{K}_2) : \tilde{\Delta}_{rt} = \text{diag}(\tilde{\Delta}, \tilde{\Delta}_t), \text{ where } \tilde{\Delta} \in \Delta_{PTV}, \tilde{\Delta}_t \in \mathfrak{L}(\mathcal{K}_2) \right\}. \quad (6.18)$$

We note from (6.6) that in fact

$$\Delta_{rt} = \left\{ \tilde{\Delta} \in \mathfrak{L}(\mathcal{K}_2) : \tilde{\Delta} = \text{diag}(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_{l+1}), \tilde{\Delta}_i \in \mathfrak{L}(\mathcal{K}_2), i = 1, \ldots, l + 1 \right\}. \quad (6.19)$$

The solution to Problem 6.1 can then be stated as follows.

**Theorem 6.4** The system in Figure 6.2 has robust tracking with respect to $\mathcal{U}X_{PTV}$ iff it has robust stablization with respect to $\mathcal{U}X_{PTV}$ and

$$\max_{0 \leq k \leq M-1} \mu_{\Delta_{rt}} \left[ \hat{p} \left( W^k \right) \right] \leq 1. \quad (6.20)$$

**Proof** ($\Longleftarrow$) By Definition 6.3, one only has to show that

$$\forall \Delta \in \mathcal{U}X_{PTV}, \quad J_{zu}(\Delta) \leq 1.$$

Condition (6.20) implies that for $0 \leq k \leq M - 1,$

$$\mu_{\Delta_{rt}} \left[ \hat{p} \left( W^k \right) \right] \leq 1.$$

By Theorem 6.2 then

$$\sup_{\Delta \in \mathcal{U}X_{PTV}} \mu_{\mathfrak{L}(\mathcal{K})} \left\{ F_u \left[ \hat{p} \left( W^k \right), \tilde{\Delta} \left( W^k \right) \right] \right\} \leq 1, \quad 0 \leq k \leq M - 1$$

or by Lemma 6.1

$$\sup_{\Delta \in \mathcal{U}X_{PTV}} \left\| F_u \left[ \hat{p} \left( W^k \right), \tilde{\Delta} \left( W^k \right) \right] \right\| \leq 1, \quad 0 \leq k \leq M - 1.$$
So by Lemma 6.2,

\[ \sup_{\Delta \in \mathcal{U} \Delta_{PTV}} J_{zw}(\Delta) \leq 1. \]

\((\Longrightarrow)\): Robust tracking with respect to \(\mathcal{U} \mathcal{X}_{PTV}\) implies robust stabilization with respect to \(\mathcal{U} \mathcal{X}_{PTV}\). So one only has to show that condition (6.20) holds. Since in Figure 6.2

\[ \forall \Delta \in \mathcal{U} \mathcal{X}_{PTV}, \quad J_{zw}(\Delta) \leq 1, \]

by Lemma 6.2

\[ \forall \Delta \in \mathcal{U} \mathcal{X}_{PTV}, \quad \max_{0 \leq k \leq M-1} \| F_u \left[ \hat{p}(W^k), \hat{\Delta}(W^k) \right] \| \leq 1. \]

So for each \(k, 0 \leq k \leq M - 1\), and for each \(\Delta \in \mathcal{U} \mathcal{X}_{PTV}\),

\[ \| F_u \left[ \hat{p}(W^k), \hat{\Delta}(W^k) \right] \| \leq 1. \]

This means that for each \(k, 0 \leq k \leq M - 1\) and for all \(\tilde{\Delta} \in \mathcal{U} \Delta_{PTV}\)

\[ \| F_u \left[ \hat{p}(W^k), \tilde{\Delta} \right] \| \leq 1 \]

since if for \(\tilde{\Delta} \in \mathcal{U} \Delta_{PTV}\) this is not true, then for \(\hat{\Delta}(\lambda) = \lambda W^{-k} \tilde{\Delta}\), one obtains a contradiction. Therefore by Lemma 6.1

\[ \sup_{\Delta \in \mathcal{U} \Delta_{PTV}} \mu_{\mathcal{U}(\mathcal{K})} \left\{ F_u \left[ \hat{p}(W^k), \hat{\Delta}(W^k) \right] \right\}, \]

for \(0 \leq k \leq M - 1\). On the other hand, from Theorem 6.3, robust stabilization with respect to \(\mathcal{U} \mathcal{X}_{PTV}\) implies that

\[ \max_{0 \leq k \leq M-1} \mu_{\Delta_{PTV}} \left[ \hat{p}_{11}(W^k) \right] \leq 1. \]

Apply Theorem 6.2 to get (6.20).
6.6 Computational Aspects

Theorems 6.3 and 6.4, state the necessary and sufficient conditions for robust stability and robust tracking of the system in Figure 6.2 in terms of the generalized structured singular values of some operators. Computing these values is not an easy problem. In fact, there is no general solution to the matrix case yet. However, there are upper bounds that can be used to estimate an acceptable size for the perturbations to the plant. For instance, from the Small Gain Theorem, an upper bound for the singular value of an operator is its norm. In the case of structured perturbations, though, this upper bound leads to conservative results, due to the fact that, information on the perturbation structure is ignored. Similar to the matrix case, this information could be utilized to derive tighter and, hence, less conservative upper bounds. Our intention in this section is to introduce one such upper bound. This upper bound is in fact a well-established bound from the matrix case that is generalized to operators in [Dul96]. As we will see in this section, this upper bound together with our algorithm for computing the induced-norm of operators in Section 5.5 make a nice complement to our robustness analysis machinery.

Let $\mathcal{E}$ be a Hilbert space and let $\Delta$ be a subspace of $L(\mathcal{E})$. Denote by $\tilde{D}_\Delta$ the set of all invertible operators in $L(\mathcal{E})$ that commute with all members of $\Delta$, that is,

$$\tilde{D}_\Delta := \{ D \in L(\mathcal{E}) : D \text{ is invertible and } D\Delta = \Delta D, \forall \Delta \in \Delta \}. \quad (6.21)$$

The upper bound of our interest is then provided by the following result from [Dul96]. Proof of this result is similar to the matrix case, see e.g. [ZGD95].

**Proposition 6.1** Suppose that $P$ is in $L(\mathcal{E})$. Then

$$
\mu_\Delta(P) \leq \inf_{D \in \tilde{D}_\Delta} \| DPD^{-1} \|. \quad (6.22)
$$

This result states that, the infimum over a set of weighted norms of $P$ is a bound for its structured singular value $\mu_\Delta(P)$. Noting that the identity operator $I$ is in $\tilde{D}_\Delta$, one can see why the bound given by this result may in general be smaller than $\|P\|$,
which is the bound given by the Small Gain Theorem.

For $\Delta_{PTV}$ and $\Delta_{rt}$, we have

$$\tilde{D}_{\Delta_{PTV}} = \{\text{diag}(d_1 I, \ldots, d_l I), \; d_i \in \mathbb{C}, \; d_i \neq 0, \; i = 1, \ldots, l\} \quad (6.23)$$

and

$$\tilde{D}_{\Delta_{rt}} = \{\text{diag}(d_1 I, \ldots, d_l I, d_{l+1} I), \; d_i \in \mathbb{C}, \; d_i \neq 0, \; i = 1, \ldots, l, l + 1\}. \quad (6.24)$$

Thus finding the upper bounds for $\mu_{\Delta_{PTV}} \left[ \hat{P}_{11}(\lambda) \right]$ in (6.17) and $\mu_{\Delta_{rt}} [\hat{P}(W^k)]$ in (6.20) reduces to two minimization problems over $l$ and $l + 1$ complex variables, respectively. It is even possible to work with real optimization variables as opposed to complex. Define

$$\mathcal{D}_{\Delta_{PTV}} := \{\text{diag}(d_1 I, \ldots, d_l I), \; d_i > 0, \; i = 1, \ldots, l\}. \quad (6.25)$$

Dullerud shows in [Dul96] that

$$\inf_{d \in \mathcal{D}_{\Delta_{PTV}}} \left\| D^{1/2} \hat{P}_{11} \left( e^{j\omega} \right) D^{-1/2} \right\| = \inf_{d \in \mathcal{D}_{\Delta_{PTV}}} \left\| D^{1/2} \hat{P}_{11} \left( e^{j\omega} \right) D^{-1/2} \right\|. \quad (6.26)$$

With this identity, we get

$$\mu_{\Delta_{PTV}} \left[ \hat{P}_{11}(\lambda) \right] \leq \inf_{d \in \mathcal{D}_{\Delta_{PTV}}} \left\| D^{1/2} \hat{P}_{11} \left( e^{j\omega} \right) D^{-1/2} \right\|. \quad (6.26)$$

Similarly, define

$$\mathcal{D}_{\Delta_{rt}} := \{\text{diag}(d_1 I, \ldots, d_l I, d_{l+1} I), \; d_i > 0, \; i = 1, \ldots, l, l + 1\}, \quad (6.27)$$

to get

$$\mu_{\Delta_{rt}} [\hat{P}(\lambda)] \leq \inf_{d \in \mathcal{D}_{\Delta_{rt}}} \left\| D^{1/2} \hat{P}(\lambda) D^{-1/2} \right\|. \quad (6.28)$$
To use optimization algorithms to find the upper bounds in (6.26) and (6.28), we need to compute the functionals that appear in the infimums in those upper bounds. Let's concentrate on

\[ \left\| D^{1/2} \hat{P}(\lambda) D^{-1/2} \right\| \]  \hspace{1cm} (6.29)

for now, where \( \lambda \) and \( D = \text{diag}(d_1I, \ldots, d_lI, d_{l+1}I) \), \( d_i > 0 \), \( i = 1, 2, \ldots, l, l + 1 \) are given. First recall from Chapter 2 that, given the state-space representations for \( \hat{y}(s) \) and \( \hat{k}(\lambda) \), (6.2) and (6.3), respectively, that of the lifted closed-loop system \( \hat{p}(\lambda) \) is

\[ \hat{p}(\lambda) = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}, \]  \hspace{1cm} (6.30)

where \( A_{cl}, B_{cl}, C_{cl} \) and \( D_{cl} \) are obtained through (2.26-2.30) and (2.33-2.36). From here

\[ \hat{p}(\lambda) = D_{cl} + \lambda C_{cl} (I - \lambda A_{cl})^{-1} B_{cl}. \]  \hspace{1cm} (6.31)

We saw in Section 5.5 that Lemma 5.4 gives us a method for computing \( \| \hat{p}(\lambda) \| \) to any degree of accuracy, by means of fast discretization. As we will see below, this lemma can be used to compute the functional in (6.29) as well.

Define the scaling matrix

\[ D_0 := \text{diag}(d_1I_{n_1}, \ldots, d_lI_{n_l}, d_{l+1}I_{n_{l+1}}), \]  \hspace{1cm} (6.32)

where \( I_{n_i}, i = 1, \ldots, l + 1, \) are identity matrices with the same Euclidean dimension as that of the uncertainty block number \( i \) in (6.19). Then multiply (6.31) from left by \( D^{1/2} \) and from right by \( D^{-1/2} \). This gives

\[ D^{1/2} \hat{p}(\lambda) D^{-1/2} = D^{1/2} \left[ D_{cl} + \lambda C_{cl} (I - \lambda A_{cl})^{-1} B_{cl} \right] D^{-1/2} \]

\[ = D^{1/2} D_{cl} D^{-1/2} + \lambda D^{1/2} C_{cl} (I - \lambda A_{cl})^{-1} B_{cl} D^{-1/2}. \]
By examining the equations for $B_{cl}$, $C_{cl}$ and $D_{cl}$ in (2.34-2.36), we note that

$$D^{1/2} \hat{p}(\lambda) D^{-1/2}$$

is the transfer function of the lifted closed-loop system, with $B_1$, $C_1$ and $D_{12}$ in the state-space representation for $G$ replaced with $B_1 D_0^{-1/2}$, $D_1^{1/2} C_1$ and $D_0^{1/2} D_{12}$, respectively. Therefore, fast discretization can be used to compute the weighted norm (6.29) as well. The same argument is valid for the functional in (6.26); we just need to start with block $(1,1)$ of $\hat{p}(\lambda)$, $\hat{p}_{11}(\lambda)$.

**Example:** Consider the uncertain sampled-data setup of Figure 6.2. The plant $G$ has state-space realization

$$\hat{g}(s) = \begin{bmatrix}
-2 & 0 & 0.25 & 0 & 0.25 & 1 & 0 \\
0 & -3 & 0.25 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0.52 & 0 & 0 & 0 & 0 & 0.18 \\
2.2 & 1.7 & 0 & 0 & 0 & 0 \\
0.5 & 0.8 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and the periodic input $w$ is unknown of period $T = 10\text{sec}$. In the absence of perturbation $\Delta$, the controller

$$\hat{k}(\lambda) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

that receives its measurements from the plant every $h = 0.5\text{sec}$, stabilizes the closed-loop system and achieves an induced power-norm of $0.1818$ for the system from the input $w$ to the output $z$. The perturbation $\Delta$ is diagonal, i.e., $\Delta = \text{diag}(\Delta_1, \Delta_2)$, with $\Delta_1$ and $\Delta_2$ bounded on $L_2$ and $h$-periodic. We would like to find a bound for the induced power-norm of the system with plant being subject to such perturbations.

We start by examining the stability of the system. From Theorem 6.3, we need to
bound $\mu_{\Delta_P^{TV}} \left[ \hat{P}_{11} (e^{j\omega}) \right]$ for $\omega \in [0, 2\pi)$. In fact, we need to do this only for $\omega \in [0, \pi]$, since all the transfer functions are real rational. First, we concentrate on the upper bound in (6.26), with

$$D_{\Delta_P^{TV}} = \left\{ \begin{bmatrix} d_1 I & 0 \\ 0 & d_2 I \end{bmatrix}, \ d_1, d_2 > 0 \right\}.$$ 

By using the technique that was just explained and the \texttt{constr} function in MATLAB optimization toolbox, we minimize the functional in (6.26) for a vector of frequencies from 0 to $\pi$. The result is the graph shown in solid in Figure 6.6. From this graph, we obtain

$$\sup_{\omega \in [0,2\pi)} \mu_{\Delta_P^{TV}} \left[ \hat{P}_{11} (e^{j\omega}) \right] \leq 0.4901. \quad (6.33)$$

Thus perturbations with size

$$\|\Delta\| \leq \frac{1}{0.4901} = 2.0404, \quad (6.34)$$

will not destabilize the system. Also, on the same plot, but shown in dashed, is the graph of $\|\hat{P}_{11} (e^{j\omega})\|$ versus frequency $\omega$. This graph gives

$$\sup_{\omega \in [0,2\pi)} \mu_{\Delta_P^{TV}} \left[ \hat{P}_{11} (e^{j\omega}) \right] \leq 0.5304, \quad (6.35)$$

which bounds us to perturbations with

$$\|\Delta\| \leq \frac{1}{0.5304} = 1.8854. \quad (6.36)$$

In this case, we see that the bound provided by the Small Gain Theorem is only slightly more conservative than the bound obtained by minimizing the weighted norms in (6.26).

Now we can use Theorem 6.4 to find a bound for the induced power-norm of the system. First we note that $W = e^{-j \frac{2\pi}{25}}$, since $M = T/h = 20$. So we only need to bound $\mu_{\Delta_P} \left[ \hat{P} (\lambda) \right]$ for $\lambda = W^k$, $k = 0, 1, \ldots, 19$. Again, due to symmetry, we would
need to do this only for $k = 0, 1, \ldots, 10$. The bound provided in (6.28) is obtained by minimizing the weighted norms of $\hat{p}(W^k)$ over

$$
\mathcal{D}_{\Delta_{rt}} = \left\{ \begin{bmatrix} d_1 I & 0 & 0 \\ 0 & d_2 I & 0 \\ 0 & 0 & d_3 I \end{bmatrix}, \ d_1, d_2, d_3 > 0 \right\}.
$$

These bounds, are listed in Table 6.1. From this table, we gather that

$$
\sup_{k=0,1,\ldots,19} \mu_{\Delta_{rt}} [\hat{p}(W^k)] \leq 0.6942. \tag{6.37}
$$

Thus for $\Delta$ with

$$
\|\Delta\| \leq \frac{1}{0.6942} = 1.4405, \tag{6.38}
$$

the induced power-norm from $w$ to $z$ will be less than 0.6942. Note that for such perturbations, the closed-loop system will be stable as predicted by (6.34).

Table 6.1 also has the bounds predicted by the Small Gain Theorem, which give

$$
\sup_{\omega \in [0, 2\pi]} \mu_{\Delta_{PTV}} [\hat{p}_{11}(e^{i\omega})] \leq 1.8624. \tag{6.39}
$$

Figure 6.7: Plot of upper bound for $\mu [\hat{p}_{11}(e^{i\omega})]$ versus $\omega$, predicted by the Small Gain Theorem (dashed) and weighted norm minimization (solid).
If we were to use the Small gain theorem, we would have

\[ \| \Delta \| \leq \frac{1}{1.8624} = 0.5369. \]

That is, the Small Gain Theorem restricts the perturbation to a much smaller size and yields a much larger bound for the induced power-norm.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( | \hat{\mathbf{p}}(W^k) | )</th>
<th>( \inf_{D \in \Delta} | D^{1/2} \hat{\mathbf{p}}(W^k) D^{-1/2} | )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4647</td>
<td>0.1663</td>
</tr>
<tr>
<td>1</td>
<td>0.4737</td>
<td>0.1744</td>
</tr>
<tr>
<td>2</td>
<td>0.5011</td>
<td>0.1881</td>
</tr>
<tr>
<td>3</td>
<td>0.5482</td>
<td>0.2078</td>
</tr>
<tr>
<td>4</td>
<td>0.6181</td>
<td>0.2345</td>
</tr>
<tr>
<td>5</td>
<td>0.7174</td>
<td>0.2713</td>
</tr>
<tr>
<td>6</td>
<td>0.8578</td>
<td>0.3228</td>
</tr>
<tr>
<td>7</td>
<td>1.0581</td>
<td>0.3966</td>
</tr>
<tr>
<td>8</td>
<td>1.3394</td>
<td>0.5004</td>
</tr>
<tr>
<td>9</td>
<td>1.6770</td>
<td>0.6255</td>
</tr>
<tr>
<td>10</td>
<td>1.8624</td>
<td>0.6942</td>
</tr>
</tbody>
</table>

Table 6.1: Different upper bounds for \( \mu [\hat{\mathbf{p}}(W^k)] \), \( k = 0, 1, \ldots, 19 \).
Chapter 7

Conclusions

In this final chapter, we give an overview of the work presented in this thesis along with some directions for future research.

7.1 Summary

In this thesis, we introduce a performance measure for repetitive control systems, referred to as the induced power-norm. This measure, which represents the maximum power-norm of the steady-state error vector in the system for the worst-case periodic input of unit power-norm, accounts for three major contributing factors to the steady-state error: the low-pass nature of most physical systems (as was reviewed in Chapter 3), the intersample behavior associated with digital implementations and last but not least the plant uncertainties. Thus, the induced power-norm is a useful tool for comparing different repetitive controllers.

Moreover, we establish a framework for design of sampled-data repetitive controllers by proposing two methodologies. The first is based on minimizing the power-norm of the steady-state error vector for a given periodic input. We show that, under certain mild conditions in the SISO case, the resulting controller is optimal for a wide class of periodic inputs. The second methodology considers an optimal design for unknown periodic inputs through the minimization of the induced power-norm. We verify that fast discretization is a useful computational tool for obtaining suboptimal
controllers and evaluating their performance under both techniques. We also demonstrate the effectiveness of these techniques by studying active suppression of the fan noise present in an acoustic duct.

Also, in order to study the steady-state error arising from plant uncertainties, we formulate a robust tracking problem for sampled-data repetitive control systems. Specifically, we investigate whether the induced power-norm of the closed-loop system remains below a given bound in the presence of structured linear periodically time-varying perturbations. We show that the result can be stated in terms of a necessary and sufficient condition which involves Dullerud's generalized notion of structured singular values for operators. We illustrate some of the computational aspects by a numerical example.

### 7.2 Future Research

The methods developed in this thesis may be extended in order to address some other issues in repetitive control. Below we list a few possible extensions.

- **Design for minimum transient response.**

  The methods developed in Chapters 4 and 5 can be modified to incorporate the transient response of the closed-loop system into the design specifications. For example, we see in Chapter 4 that controller parametrization in terms of a stable FDLTI system $Q$ can be used to convert Problem 4.1 to the equivalent but simpler problem stated by (4.10). A solution to this problem is given by Lemma 4.1. However, as mentioned in the discussion surrounding the lemma, the solution is not unique. If we define

  \[ Q := \{ Q : Q \text{ solves (4.10)} \}, \]

  then a problem for optimal sampled-data repetitive control systems with minimum transient response can be stated as follows:
Recalling from Theorem 4.1 that \( z_{tr} \) is an \( L_2 \) signal, we note that (7.2) together with (4.10) pose a mixed \( L_2 - \mathcal{W}_T \) problem for sampled-data repetitive control systems with known periodic inputs. Similarly, a mixed problem can be posed for such systems with unknown periodic inputs, by minimizing the \( \mathcal{W}_T \) to \( L_2 \) induced-norm of the mapping from \( w \) to \( z_{tr} \) over the set of all controllers that solve (5.1).

- Design for rejection of non-periodic disturbances.

Repetitive control, which aims at rejecting periodic disturbances, does not guarantee any performance in rejecting non-periodic disturbances. In fact, Tenney and Tomizuka [TT96] show that repetitive control systems can perform poorly in response to disturbances of short duration. In order to consider non-periodic disturbances in our formulation, we propose extending the definition of \( z_{tr} \) in (7.2) such that it includes the transient responses arising from both periodic and non-periodic disturbances.

- Robustness analysis with respect to LTI perturbations.

In our robustness analysis of sampled-data repetitive control systems, we consider periodically time-varying plant perturbations. Such perturbations form a much larger class than LTI perturbations, which are a natural candidate for modeling the uncertainties in LTI plants. Thus the conditions in our analysis of tracking robustness would be too conservative. A good research problem is to see if less conservative conditions may be found.

- Design for robust performance.

Our final suggestion on future research is developing techniques which incorporate the tracking robustness criterion directly into the design.
Bibliography


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