Commutants of Composition Operators

by

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Abstract

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Let $C_\phi$ be a composition operator on $H^2(D)$. We consider which operators $A$ commute with $C_\phi$. In particular, we prove that, unless $\phi$ is an elliptic disc automorphism of finite periodicity, the only Toeplitz operators that commute must be analytic. Also, we provide a classification of the multiplication operators that commute for several particular cases of $\phi$. We prove that the outer functions are an invariant class for the composition operators. In the last two chapters we provide some results about the commutant in general when $\phi$ has an interior fixed point. In particular, we find some hyperinvariant subspaces of such operators. We also obtain a partial classification of the quasi-normal composition operators.
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Chapter 1

Introduction

Let $D$ be the unit disc in the complex plane, $\mathbb{C}$. We define $H^2(D)$, the Hardy space on the unit disc, to be the set of analytic functions on $D$ which have square summable power series coefficients. $H^2$ is the natural realization of $l^2$, the square summable one sided sequence space, in terms of analytic functions on $D$. Thus $H^2$ is a Hilbert space (see [16], [17], or [20]). If $f$ and $g$ are in $H^2$, the inner product of $f$ and $g$ may be expressed in terms of their power series coefficients, or as

$$< f, g > = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} d\theta.$$ 

If we have an analytic self map of the disc, $\phi$, we can define a composition operator on $H^2$ by $C_\phi(f) = f \circ \phi$, for all $f \in H^2$. The study of this class of bounded operators began in the 1960's with the work of Nordgren and has expanded into an area of much interest. (For basic information on these operators, see [14] or [30].)

Given a fixed operator $A$, we say an operator $B$ commutes with $A$ (written $B \leftrightarrow A$) if $AB = BA$. The set of all operators which commute with a fixed operator $A$ forms a weakly closed algebra which is called the commutant of $A$. The commutant of a particular operator is known in only a few cases (see [27] and [32]). For composition operators, Cowen discussed in [13] when two analytic self maps of the disc must commute and in [12] showed that in the commutant of certain Toeplitz operators there are non-trivial composition operators.
This appears to be all that is previously known regarding the commutant of a composition operator.
Chapter 2

Toeplitz and Hankel Operators

In [12] Carl Cowen showed that if $f$ is a covering map of $D$ onto a bounded domain in $\mathbb{C}$, then the commutant of the Toeplitz operator $T_f$ is generated by Toeplitz operators and by composition operators induced by linear fractional transformations $\phi$ which satisfy $f \circ \phi = f$. Our first result is concerned with which Toeplitz operators commute with a fixed composition operator.

2.1 Toeplitz Operators

A Toeplitz operator is a special type of generalized multiplication operator. If $f \in L^\infty(\partial D)$ and $P$ is the orthogonal projection of $L^2(\partial D)$ onto $H^2$ (under the standard identification of an $H^2$ function with its boundary function in $L^2(\partial D)$), then we may define an operator on $H^2$ by $T_f(g) = P(fg)$ for $g \in H^2$. If the inducing function, $f$, for the Toeplitz operator is in $H^\infty$, then such a Toeplitz operator is called analytic, and corresponds to the multiplication operator induced by $f$ on $H^2$.

**Lemma 1** A sequence of functions $f_n$ in $H^2$ converges weakly if and only if it is bounded (in the $H^2$ norm) and converges pointwise.

**Proof:** see [14] page 3. ■
If $\phi : D \to D$, the $n$th term in the sequence of iterates of $\phi$ under composition will be denoted $\phi^{[n]}$.

**Theorem 1 (Denjoy-Wolff)** If $\phi$ is an analytic self map of the disc which is not an elliptic disc automorphism, then there is a unique point $a \in \overline{D}$ such that the iterates $\phi^{[n]}$ converge uniformly on compacta to $a$.

**Proof:**
See [14], [7], or [30].

**Lemma 2** Let $\phi : D \to D$ be an analytic mapping which is not an elliptic disc automorphism. Let $a$ be the Denjoy-Wolff point of $\phi$. Then, for each fixed positive integer $l$, $(\phi^{[n]})' l$ converges to $a'$ weakly.

**Proof:**
By the Denjoy-Wolff Theorem, $\phi^{[n]}$ converges to $a$ uniformly on compacta and thus $(\phi^{[n]})'$ converges to $a'$ pointwise. Since $\phi$ is a self map of the disc, $(\phi^{[n]})'$ is bounded in the $H^\infty$ norm by 1 and hence also in the $H^2$ norm. Applying Lemma 1 gives the result.

Disc automorphisms may be classified into three types by the locations of their fixed points. The types are elliptic, parabolic and hyperbolic which are distinguished respectively by a fixed point in $D$, a fixed point on $\partial D$ and two distinct fixed points on $\partial D$. Elliptic disc automorphisms are just conjugate to rotations about zero (by Schwarz’s Lemma). They may be sub-classified into elliptic disc automorphisms of finite periodicity ($\phi^{[n]} \equiv z$ for some $n \in \mathbb{N}$) and of infinite periodicity (see [30]).

**Theorem 2** Let $\phi : D \to D$ be an analytic mapping which is not an elliptic disc automorphism of finite periodicity. If $f \in L^\infty(\partial D)$ and $T_f \leftrightarrow C_\phi$, then $f$ must be analytic.

**Proof:**
The proof breaks into three parts depending on the nature of $\phi$. 

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(I) First suppose that $\phi$ is neither an elliptic disc automorphism nor a constant. Let the Denjoy-Wolff point of $\phi$ be $a$. Let $f = \sum_{n=0}^{\infty} c_n z^n$ and decompose $f$ as $f_1 + f_2$ where $f_1 \in (H^2)_\perp$ and $f_2 \in (H^2)$. Then $T_f \leftrightarrow C_\phi$ implies that

$$T_f C_\phi(z) = C_\phi T_f(z)$$

which gives us

$$T_f(\phi^{[n]}) = (c_{-1} + f_2 z) \circ \phi^{[n]}.$$

Now $C_\phi T_f(1) = T_f C_\phi(1)$ implies that $f_2 \circ \phi = f_2$ and thus

$$T_f(\phi^{[n]}) = c_{-1} + (f_2)\phi^{[n]}.$$

Now taking the inner product of this equation with 1 we have

$$< T_f(\phi^{[n]}), 1 > = c_{-1} + f_2(0) \phi^{[n]}(0) \Rightarrow < \phi^{[n]}, T_f(1) > = c_{-1} + f_2(0) \phi^{[n]}(0).$$

By Lemma 2, $< \phi^{[n]}, T_f(1) >$ converges to $< a, T_f(1) > = a f_2(0)$ while by the Denjoy-Wolff Theorem $c_{-1} + f_2(0) \phi^{[n]}(0)$ converges to $c_{-1} + f_2(0)$. Thus $c_{-1} = 0$. Assume by strong induction that $c_{-1} = \ldots = c_{-l+1} = 0$ and consider $T_f(z^l)$ in the above argument. This gives us $T_f((\phi^{[n]})^l) = c_{-1} + (f_2(\phi^{[n]}))^l$. If we inner product with 1 and pass to the weak limit as above, then we have $a' f_2(0) = c_{-l} + a' f_2(0)$ which implies that $c_{-l} = 0$. Hence by induction, we have that $c_{-n} = 0$ for all $n \geq 1$, which implies that $f$ is analytic.

(II) Let $\phi(z) = b$ for all $z \in D$ where $|b| < 1$. Let $f \in L^\infty(\partial D)$ with $f = f_1 + f_2$ where $f_1 \in (H^2)_\perp$ and $f_2 \in H^2$. First, consider $T_f C_\phi(1) = C_\phi T_f(1)$ which implies $f_2 = f_2(b)$. Thus $f_2$ is a constant; let $f_2 = c$. For any $g \in H^2$ we have $T_f C_\phi(g) = C_\phi T_f(g)$, which implies $cg(b) = (P(f_1 g))(b) + cg(b)$, so $(P(f_1 g))(b) = 0$. In particular, take $g$ to be successively $z^k$ where $k \in \mathbb{N}$ to conclude that the negative Fourier coefficients of $f$ are zero. Thus $f = f_2$.

(III): Let $\phi$ be an elliptic automorphism of infinite periodicity. First, assume that the fixed
point is 0; then by Schwarz's Lemma, \( \phi(z) = e^{i\theta}z \) where \( e^{i\theta} \neq 1 \) for all integers \( n \neq 0 \).

A direct computation yields that if \( AC_\phi = C_\phi A \), then \( A \) is an operator represented by a diagonal matrix with respect to the standard basis and every such operator must commute with \( C_\phi \). If \( T_f \) is a Toeplitz operator which commutes with \( C_\phi \), then since \( T_f \) must have constant diagonals (in the standard basis), \( f \) must be a constant. Now let the fixed point of \( \phi \) be \( b \) where \( b \) is non-zero. Let \( \alpha(z) = \frac{z - b}{1 - \overline{b}z} \); note that \( \alpha \) is a self inverse. Suppose that \( T_f \) commutes with \( C_\phi \) where \( f = f_1 + f_2 \) with \( f_1 \in (H^2)^\perp \) and \( f_2 \in H^2 \). \( T_f C_\phi(1) = C_\phi T_f(1) \) implies that \( f_2 = f_2 \circ \phi \) and since \( \phi \) has infinite periodicity, this implies that \( f_2 \) is a constant. Thus \( f_1 \) induces a Toeplitz operator which commutes with \( C_\phi \). We claim \( f_1 = 0 \). Now \( T_{f_1} \) commutes with \( C_\phi \) implies \( C_\alpha T_{f_1} C_\alpha \) commutes with \( C_\alpha C_\phi C_\alpha \) but \( C_\alpha C_\phi C_\alpha = C_\alpha \circ \alpha \circ \alpha \) and \( \alpha \circ \alpha \circ \alpha \) is an elliptic disc automorphism of infinite periodicity with Denjoy-Wolff point 0.

Thus \( T_{f_1} = C_\alpha D C_\phi \) where \( D \) is a diagonal matrix in the standard basis. Let the \( k^{th} \) diagonal entry of \( D \) be \( \lambda_k \). Let \( D(\alpha) = g \). Then since \( T_{f_1}(z) \) is a constant, we have \( g \circ \alpha \) is a constant and hence \( g \) is a constant. Now

\[
g(z) = \lambda_0 b + \sum_{k=1}^{\infty} \lambda_k (\overline{b})^{k-1}(|b|^2 - 1)z^k
\]

is constant as a function of \( z \) if and only if \( \lambda_k = 0 \) for \( k \geq 1 \) and \( T_{f_1}(1) = 0 \) implies \( \lambda_0 = 0 \) so \( D = 0 \) and thus \( f_1 = 0 \) and the result follows.

As the next result shows, the previous theorem cannot be extended to all elliptic disc automorphisms.

**Theorem 3** Let \( \phi \) be an elliptic disc automorphism of period \( q \) (\( q \geq 2 \)) with \( \phi(0) = 0 \). Then \( T_f \leftrightarrow C_\phi \) if and only if the Fourier expansion of \( f \) is of the form \( \sum_{n=-\infty}^{\infty} a_n z^{-nq} \).

**Proof:**
Since \( \phi \) is an elliptic disc automorphism of period \( q \), we have \( \phi(z) = e^{i\theta}z \) with \( \theta = 2\pi \frac{p}{q} \) where \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \) and \( g.c.d(p, q) = 1 \). Note \( e^{2\pi i \frac{p}{q}} = 1 \) if and only if \( q | n \). Let \( f = \sum_{n=-\infty}^{\infty} a_n z^n = f_1 + f_2 \) with \( f_1 \in (H^2)^\perp \) and \( f_2 \in H^2 \). Now, \( T_f C_{e^{2\pi i \frac{p}{q}}} (1) = C_{e^{2\pi i \frac{p}{q}}} T_f(1) \)
implies that

\[ f_2(z) = f_2(e^{2\pi i z/z}). \]

Thus \( f_2 \) is \( q \)-fold symmetric and hence \( f_2 = \sum_{n=0}^{\infty} a_{nq}z^{nq} \). Now \( T_f C_{e^{2\pi i z/z}}(z) = C_{e^{2\pi i z/z}} T_f(z) \) implies that \( c_{-1}e^{2\pi i z/z} + ze^{2\pi i z/z} f_2(z) = c_{-1} + ze^{2\pi i z/z} f_2(z) \) which implies that \( c_{-1} = 0 \). For \( n \) such that \( q \nmid n \) assume by strong induction that if \( m < n \) and \( q \nmid m \) that \( c_{-m} = 0 \). Then

\[ T_f C_{e^{2\pi i z/z}}(z^n) = C_{e^{2\pi i z/z}} T_f(z^n) = a_{-n}e^{2\pi i z^n} + \sum_{m<n,q|m} a_{-m}z^{-m}e^{2\pi i z^n} + e^{2\pi i z^n} z^n f_2(z) \]

Comparing terms we have \( a_{-n} = 0 \). Thus \( f = \sum_{n=-\infty}^{\infty} a_{nq}z^{nq} \). Moreover, if \( f \) is \( q \)-fold symmetric in its Fourier coefficients, and \( f \in L^\infty(\partial D) \), then certainly \( T_f \leftrightarrow C_{e^{2\pi i z/z}} \). 

### 2.2 Hankel Operators

The next theorem classifies which Hankel operators commute with a composition operator (for the basics of Hankel operators see [26]).

**Theorem 4** Let \( S_f \) be a Hankel operator. Suppose that \( \phi \) is an analytic function that maps the unit disc into itself which is not an elliptic disc automorphism of finite periodicity or the identity operator and suppose \( S_f \) commutes with \( C_\phi \). If \( \phi(0) \neq 0 \), then \( S_f = 0 \), and if \( \phi(0) = 0 \), then \( S_f \) is a constant multiple of a rank one projection. Moreover, the Hankel operators which commute with a given composition operator are in the weakly closed algebra generated by that composition operator.

**Proof:**

Let \( S_f = PJM_f \) where \( J(z^n) = z^{-n} \), \( P \) is the orthogonal projection of \( L^2 \) onto \( H^2 \), and \( M_f \) is multiplication by the \( L^\infty \) function \( f \). Let \( f = \sum_{n=-\infty}^{\infty} c_n z^n \). Let \( h_k = \sum_{n=k}^{\infty} c_{-n} z^{-n-k} \). We note that \( h_k \) belongs to \( H^2 \). Now

\[ C_\phi S_f(1) = S_f C_\phi(1) = S_f(1) \]
implies that

\[ C_\phi P_J M_f(1) = P J M_f(1) \]

which implies that

\[ h_0 \circ \phi = h_0. \]

Now if \( \phi \) has an interior fixed point, then it follows that \( h_0 \) is a constant. In this case, \( S_f C_\phi(g) = C_\phi S_f(g) \) for every \( g \in H^2 \) implies that \( c_0 g(\phi(0)) = c_0 g(0) \) and thus either \( c_0 = 0 \) or \( \phi(0) = 0 \).

If the Denjoy-Wolff point of \( \phi, \alpha \), is on the boundary of the disc, then \( C_\phi S_f = S_f C_\phi \) implies that \( C_{\phi^{[n]}}(h_1) = S_f(\phi^{[n]}(z)) \). If we inner product this last equation with 1, on the left hand side we get 0 since \( h_1 = \frac{h_0 - \alpha}{z} \), \( h_0 \circ \phi = h_0 \), and \( c_0 = h_0(0) \). The right hand side is equal to \( < \phi^{[n]}(z), S_f^*(1) > \) which by Lemma 2 converges to \( < S_f(\alpha), 1 > = a h_0(0) \). Thus \( c_0 = 0 \). Assume by strong induction that \( c_0 = c_{-1} = ... c_{-(t-1)} = 0 \). Consider \( < C_{\phi^{[n]}}(S_f(\alpha^{[t]})), 1 > = < S_f((\phi^{[n]}(z))^{t+1}, 1 > \). The left hand side equals

\[ < \frac{h_0 - c_{t}(\phi^{[n]})^t}{(\phi^{[n]})^{t+1}}, 1 > = \frac{-c_{t}(\phi^{[n]}(0))}{(\phi^{[n]}(0))^{t+1}} \]

which converges to \( \frac{-c_{t}}{\alpha} \). The right hand side converges to \( a^{t+1} h_0(0) = 0 \). Thus \( c_{-t} = 0 \).

Putting the two cases together, we have if \( f \) belongs to \( z H^2 \), then \( S_f = 0 \) and if \( f \) belongs to \( H^2 \) then \( S_f \) is \( c_0 \) times the rank 1 projection \( Q \), where \( Q(1) = 1 \) and \( Q(z^n) = 0 \) when \( n \neq 0 \).

To prove the last assertion note that in the case \( \phi(0) = 0 \), the rank 1 projection \( Q \) always belongs to the weakly closed algebra generated by the powers of \( C_\alpha \). Since \( C_\phi^n(f) \) is bounded in \( H^2 \) and converges pointwise to \( f(0) \) by the Denjoy-Wolff theorem, we have \( C_\phi^n \) converges weakly to \( Q \).
We note that the theorem does not hold for elliptic disc automorphisms of finite periodicity. In particular, $S_\tau$ commutes with $C_{-\tau}$. 

Chapter 3

Multiplication Operators

3.1 Conjecture

A reasonable conjecture for the commutant of a composition operator, $C_\phi$, is the weakly closed algebra generated by the Toeplitz operators which commute with $C_\phi$ and the composition operators which commute with $C_\phi$. In the following material, we provide two examples, one where the conjecture is true and another where the conjecture is false.

Example 1 Let $\phi : D \mapsto D$ be an elliptic disc automorphism with $\phi(0) = 0$. The commutant of $C_\phi$ is the weakly closed algebra generated by $C_\phi$ and the Toeplitz operators which commute with $C_\phi$.

Proof:

The proof divides into two cases. First assume that $\phi$ is an elliptic disc automorphism of infinite periodicity; i.e., $\phi^{[n]} \neq \phi$ for all natural numbers $n$ (where $\phi^{[n]} = \underbrace{\phi \circ \phi \circ ... \circ \phi}_{\text{n times}}$).

Hence by Schwarz's lemma $\phi(z) = e^{i\theta}z$ where $e^{i\theta} \neq 1$ for all natural numbers $n$. A direct computation yields that if $AC_\phi = C_\phi A$, then $A$ is an operator represented in the standard basis by a diagonal matrix and every such operator must commute with $C_\phi$. By Theorem 2, the only Toeplitz operators which commute with such an elliptic disc automorphism are constant multiples of the identity. Thus let $A$ be the weakly closed algebra generated by $C_\phi$ and the identity operator. It is easy to verify that $A$ is just the diagonal matrices.
Second, we assume that $\phi$ is an elliptic disc automorphism of finite periodicity; i.e., $\phi(z) = e^{i\theta}z$ and $e^{ik\theta} = 1$ for some natural number $k > 1$ (let $k$ be the smallest such one). By Theorem 3, we showed the only Toeplitz operators which commute with $C_\phi$ are linear combinations of $M_{z^n}$ and $T_{z^n}$ where $n$ is any natural number. Let $H^2 = M_0 \oplus M_1 \oplus \ldots \oplus M_{k-1}$ where $M_l$ is the subspace spanned by $z^{kn+l}$ for all natural numbers $n$. The operators $C_\phi$, $M_{z^n}$, and $T_{z^n}$ decompose with respect to this direct sum. $T_{z^n}$ and $M_{z^n}$ are just unilateral backwards and forward shifts on $M_n$, respectively. Since the shift has no reducing subspaces in $H^2$, the weakly closed algebra generated by $T_{z^n}$ and $M_{z^n}$ restricted to $M_n$ is just all of the bounded linear operators on $M_n$. Let $A$ be the weakly closed algebra generated by $C_\phi$, $T_{z^n}$, and $M_{z^n}$.

$C_\phi$ is a normal operator with finite spectrum and hence it generates its Von Neumann algebra. Thus $C_\phi^*$ and the spectral projections of $C_\phi$ belong to $A$. The orthogonal projection onto $M_n$ is such a spectral projection. If $A$ commutes with $C_\phi$, its restriction to $M_n$ is in $A$, and $A$ is a sum of the restrictions to $M_l$, $0 \leq l \leq k-1$.

As an addendum to Example 1, we note that if $C_\phi$ is an elliptic composition operator, then its commutant is the conjugation by an invertible composition operator of either the algebra of all diagonal operators (with respect to the standard basis) or a finite direct sum of full operator algebras. Also, we emphasize that if $\phi$ is an elliptic disc automorphism of finite periodicity $k$, and $\phi(0) = 0$, then the commutant of $C_\phi$ does not coincide with the weakly closed algebra generated by $C_\phi$. This follows from any operator in this algebra must fix the constants, but there are operators which commute with $C_\phi$ which do not fix the constants. In particular $T_{z^n}$ commutes with $C_\phi$ and $T_{z^n}(1) = z^k$.

**Example 2** There are operators which commute with $C_{z^n}$ which are not in the algebra generated by commuting composition operators and commuting Toeplitz operators.

**Proof:**

Suppose $C_f$ commutes with $C_{z^n}$ where $f : D \mapsto D$ is an analytic function. Since $f(z^{2l}) = \ldots$
\((f(z))^2\), letting \(l \to \infty\), we have \(f(0) = 0\). Thus either \(f\) is identically 0 or we may write \(f(z) = z^n g(z)\) where \(g(0) \neq 0\) and \(n\) is some natural number greater than or equal to 1. Now \(f(z^2) = (f(z))^2\) implies that \(z^{2n} g(z^2) = z^{2n} (g(z))^2\). Hence \(g(z^2) = (g(z))^2\). If \(g\) is a constant then \(g\) is identically 1. If \(g\) is non-constant, then, by the Maximum Modulus Theorem, \(g : D \mapsto D\). Reapplying the above argument to \(g\), we have \(g(0) = 0\), which is a contradiction. Hence \(f(z) = z^n\) for some \(n\). Thus the only composition operators that commute with \(C_{z^2}\) are of the form \(C_{z^n}\).

By Theorem 2 and Theorem 7, the only Toeplitz operators that commute with \(C_{z^2}\) must be constant multiples of the identity. Let \(A\) be the weakly closed algebra generated by \(C_{z^n}\) for all natural numbers \(n\) and the constant multiples of the identity. Since \(C_{z^n}\) restricted to \(zH^2\) is strictly lower triangular in the standard basis, for \(n \geq 2\), any operator \(A\) in \(A\) when restricted to \(zH^2\) consists of a strictly lower triangular matrix in the standard basis plus a constant multiple of the identity.

Suppose \(A\) is an operator that commutes with \(C_{z^2}\) and that the matrix of \(A = (a_{l,k})\) with respect to the standard basis. Now

\[< AC_{z^2}(z^k), z^l> = < C_{z^2}A(z^k), z^l>\]

implies that

\[a_{l,2k} = < \sum_{m=0}^{\infty} a_{m,k} z^{2m}, z^l>.\]

It follows that \(a_{m,k} = a_{2m,2k}\) for all \(m\) and \(k\), and \(a_{l,2k} = 0\) whenever \(l\) is odd. If \(k = 0\), then \(a_{2m,0} = a_{m,0}\) for all \(m = 1, 2, ..., \infty\) and if \(m = 0\), then \(a_{0,2k} = a_{0,k}\) for all \(k = 1, 2, ..., \infty\). Hence by the boundedness of \(A\), it follows that \(a_{m,0} = 0\) and \(a_{0,k} = 0\) for \(m = 1, 2, ..., \infty\) and \(k = 1, 2, ..., \infty\). Conversely, any operator \(A\) that satisfies the above conditions on its matrix (in the standard basis) commutes with \(C_{z^2}\).

In particular, if we define an operator \(A\) by \(A(1) = 0\), \(A(z^{2k}) = z^{2k}\) for \(k = 0, 1, 2, 3, ..., \infty\),
and $A(z^n) = 0$ for $n \neq 2^k$, then $A$ is a bounded operator which commutes with $C_z$. However, the restriction of $A$ to $zH^2$ has a non constant diagonal in the standard basis and hence does not belong to $A$. ■

Similar results hold for $C_z^m$ (for $m$ in $\mathbb{N}$).

### 3.2 Multiplication Operators

Carl Cowen in [13] characterizes for a given function $f$ (not a conformal automorphism of $D$) which functions $g$ commute with it under composition. In Chapter 1, we showed, except for elliptic disc automorphisms of finite periodicity, that the Toeplitz operators which commute with $C_\phi$, must be analytic (and hence multiplication operators). We will now determine which multiplication operators commute in some special cases of $C_\phi$. For $f \in H^\infty$, $M_f \leftrightarrow C_\phi$ is equivalent to $f \circ \phi = f$.

**Theorem 5** Let $\alpha$ be a disc automorphism and $f$ a function in $H^\infty$. Then $C_\alpha M_f C_\alpha^{-1} = M_{f \circ \alpha}$.

**Proof:**
For every $g \in H^2$, $C_\alpha M_f C_\alpha^{-1} (g) = (f \circ \alpha) g = M_{f \circ \alpha}(g)$. ■

**Theorem 6** Let $\phi$ be an elliptic disc automorphism with fixed point $b$. If $\phi$ is of infinite periodicity, then a multiplication operator, $M_f$, commutes with $C_\phi$ if and only if $f$ is a constant. If $\phi$ is of period $q$, then a multiplication operator, $M_f$, commutes with $C_\phi$ if and only if $f \in H^\infty$ and $f$ is of the form $\sum_{n=0}^\infty a_n (\frac{b-z}{1-b\bar{z}})^n q$.

**Proof:**
Apply Theorem 2 (part III) and Theorem 3 together with Theorem 5. ■

**Theorem 7** If $\phi : D \leftrightarrow D$ has an interior fixed point and is not an elliptic disc automorphism, then $M_f \leftrightarrow C_\phi$ implies that $f$ is constant.
Proof:
Suppose $\phi(a) = a$ and $f \circ \phi = f$; then for any $z_0 \in D$, we have $f(\phi^n(z_0)) = f(z_0)$ for all $n \in \mathbb{N}$. Now since, $\phi^n(z_0) \to a$ as $n \to \infty$, by the Denjoy-Wolff Theorem, we have $f(z_0) = f(a)$ for every $z_0 \in D$ and thus $f$ is constant. 

3.3 Parabolic & Hyperbolic Disc Automorphisms

Now suppose that $\phi$ is a parabolic disc automorphism with Denjoy-Wolff point 1 or a hyperbolic disc automorphism with fixed points 1 and $-1$ where 1 is the Denjoy-Wolff point.

Define $\phi^0 = z$ and $\phi^{-n} = \underbrace{\phi^{-1} \circ \phi^{-1} \circ \ldots \circ \phi^{-1}}_{n \text{ times}}$. Let $z_n = \phi^n(0)$ and define

$$B(z) = z \prod_{n=-\infty}^{\infty} \frac{z_n - z}{|z_n|} \left(1 - \frac{z}{z_n z}ight) \quad (n \neq 0).$$

It was shown by Nordgren, Rosenthal and Wintrobe in [23] that $B$ is a Blaschke product, and that $B \circ \phi = B$ and $B \circ \phi = -B$ when $\phi$ is respectively such a parabolic disc automorphism and such a hyperbolic disc automorphism.

**Theorem 8** Let $\phi$ be a parabolic disc automorphism with Denjoy-Wolff point 1 and let $B$ be the Blaschke product defined above. Let $f \in H^2$. Then $f \circ \phi = f$ if and only if $f = \sum_{k=0}^{\infty} a_k B^k$ where $\sum_{k=0}^{\infty} |a_k|^2 < \infty$.

**Proof:**
Let $M$ be the subspace of $H^2$ spanned by $\{B^k : k = 0, 1, \ldots, \infty\}$. Now

$$< B^k, B^j > = \frac{1}{2\pi} \int_0^{2\pi} B^k(e^{i\theta})\overline{B^j(e^{i\theta})}d\theta$$

which equals

$$\frac{1}{2\pi} \int_0^{2\pi} B^{k-j}(e^{i\theta})d\theta = < B^{k-j}, 1 >$$

when $k \geq j$ or

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{B^{j-k}(e^{i\theta})}d\theta = < 1, B^{j-k} >$$

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when \( j \geq k \). Now \( B(0) = 0 \) so \( B^k \) and \( B^j \) are orthogonal when \( k \neq j \). Thus

\[
M = \{ \sum_{n=0}^{\infty} a_n B^n : a_n \in \mathbb{C}, \sum_{n=0}^{\infty} |a_n|^2 < \infty \}.
\]

We want to show that if \( f \circ \phi = f \), then \( f \in M \). Let \( P : H^2 \mapsto M \) be the orthogonal projection onto \( M \) and let \( Q : H^2 \mapsto M^\perp \) be the orthogonal projection onto \( M^\perp \). Decompose \( f \) with respect to these projections to get \( f = P(f) + Q(f) \). Now if \( g \in M \) then \( g \circ \phi = g \) (since \( B \circ \phi = B \)); thus \( P(f) \circ \phi = P(f) \). Also, since \( f \circ \phi = f \), we must have \( Q(f) \circ \phi = Q(f) \) which also implies \( Q(f) \circ \phi^{-1} = Q(f) \). Now the constants are in \( M \), so \( (Q(f))(0) = 0 \) and thus \( (Q(f))(\phi[n](0)) = (Q(f))(0) = 0 \) for all \( n \in \mathbb{Z} \). Hence we have that \( B \) divides \( Q(f) \) in the \( H^2 \) sense that \( B \) must be a factor of the inner part of \( Q(f) \). Thus \( Q(f) = Bh \) for some \( h \in H^2 \). Now for \( i \geq 1 \) we have

\[
< B^i, Q(f) > = 0 \Rightarrow < B^i, Bh > = 0 \Rightarrow < B^{i-1}, h > = 0
\]

so \( h \perp M \). Moreover, since \( Q(f) \circ \phi = Q(f) \) and \( B \circ \phi = B \), we must have \( h \circ \phi = h \). Thus we may repeat the process on \( h \) to conclude that \( B|h \). Continuing by induction we have that \( B^k|Q(f) \) for arbitrary \( k \in \mathbb{N} \) and thus \( z^k|Q(f) \) for all \( k \in \mathbb{N} \) which implies \( Q(f) = 0 \); thus \( f \in M \).

**Corollary 1** If \( \phi \) is a parabolic disc automorphism with Denjoy-Wolff point 1, and \( f \in H^\infty \), then \( M_f \leftrightarrow C_\phi \) if and only if \( f = \sum_{n=0}^{\infty} a_n B^n \) where \( \sum_{n=0}^{\infty} |a_n|^2 < \infty \).

**Proof:**
Apply Theorem 8.

**Corollary 2** If \( \phi \) is a parabolic disc automorphism with Denjoy-Wolff point 1, then \( f \) is an eigenfunction associated with eigenvalue 1 for \( C_\phi \), if and only if \( f = \sum_{n=0}^{\infty} a_n B^n \) where \( \sum_{n=0}^{\infty} |a_n|^2 < \infty \).

**Proof:**
If \( f \) is an eigenfunction associated with eigenvalue 1 for \( C_\phi \), then \( f \circ \phi = f \). Apply Theorem 8.
Corollary 3 If $\phi$ is a parabolic disc automorphism with Denjoy-Wolff point 1, then \( \{ f = \sum_{n=0}^{\infty} a_n B^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \} \) is a hyperinvariant subspace for $C_\phi$.

Proof:
Any operator which commutes with $C_\phi$ must preserve the eigenspaces of $C_\phi$. Apply Corollary 2.

Corollary 4 If $\phi$ is a parabolic disc automorphism with Denjoy-Wolff point 1, and if $h$ is an eigenfunction associated with eigenvalue $\lambda$ for $C_\phi$, then

\[
h = \frac{\left( \sum_{n=0}^{\infty} a_n B^n \right)}{f_\lambda}
\]

for some $a_n$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and $f_\lambda = e^{(idb^{-1} \gamma(z))}$ where $\gamma(z) = \frac{i(1+z)}{1-z}$, $b$ is a real number determined by $\phi$, and $d$ is a real number of the same sign as $b$ such that $e^{id} = \overline{\lambda}$. Conversely, if $h \circ \phi = h$, then $f_\lambda h$ is an eigenfunction for eigenvalue $\lambda$ where $f_\lambda$ is defined in a similar manner as $f_\lambda$.

Proof: The following construction is from [21]. Define $\gamma(z) = \frac{i(1+z)}{1-z}$. This is a conformal mapping of the unit disc onto the upper half plane. Let $\delta = \gamma \circ \phi \circ \gamma^{-1}$. Since $\delta$ has no fixed points in the upper half plane, and is a conformal map, it must be of the form $\delta(w) = w + b$ for $b \neq 0$ and $b \in \mathbb{R}$. Now the set of eigenvalues for $C_\phi$ is $\partial D$ (see [21]), so let $\mu \in \partial D$ and let $d \in [-2\pi, 2\pi]$ with $e^{id} = \mu$ and $\text{sign}(d) = \text{sign}(b)$. Define

\[
f_\mu = e^{(idb^{-1} \gamma(z))}.
\]

Now

\[
|f_\mu(z)| = e^{(-d \mu(\text{Im}(\gamma(z)))} \leq 1
\]

and thus $f_\mu \in H^\infty$. Now $\gamma \circ \phi = \delta \circ \gamma = \gamma + b$ so

\[
C_\phi(f_\mu) = e^{(idb^{-1}\gamma \circ \phi)} = e^{(id + idb^{-1} \gamma)} = \mu f_\mu.
\]
We can do this procedure for any $\mu$ in $\partial D$, in particular, for $\bar{\lambda}$ and $\lambda$. If $h \circ \phi = \lambda h$ where $h \in H^2$, $\lambda \in \partial D$, $\lambda \neq 1$, then $f_{\bar{\lambda}}h \in H^2$ (because $f_{\bar{\lambda}} \in H^\infty$) and $(f_{\bar{\lambda}}h) \circ \phi = |\lambda|^2 f_{\bar{\lambda}}h = f_{\bar{\lambda}}h$. Thus $f_{\bar{\lambda}}h = \sum_{n=0}^{\infty} a_n B^n$ for some square summable $a_n \in \mathbb{C}$ by Corollary 2. Now $f_{\bar{\lambda}}$ is never zero on $D$ but as $z \to 1$, we have $|f_{\bar{\lambda}}(z)| \to 0$. Thus $\frac{1}{f_{\bar{\lambda}}}$ is not bounded on $D$ but it is an analytic function on $D$. Thus

$$h = \frac{\sum_{n=0}^{\infty} a_n B^n}{f_{\bar{\lambda}}}$$

as analytic functions on $D$ but $h \in H^2$ so

$$\frac{\sum_{n=0}^{\infty} a_n B^n}{f_{\bar{\lambda}}}$$

is in $H^2$. Thus if $h$ is an eigenvector for $C_\phi$ associated with eigenvalue $\lambda$, then

$$h = \frac{\sum_{n=0}^{\infty} a_n B^n}{f_{\bar{\lambda}}}$$

for some $a_n \in \mathbb{C}$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Moreover, if $h \circ \phi = h$, then $(f_{\bar{\lambda}}h) \circ \phi = \lambda f_{\bar{\lambda}}h$ and hence $f_{\bar{\lambda}}h$ is eigenvector for eigenvalue $\lambda$. \[\square\]

For the parabolic disc automorphism, we note that for $f_{\mu}$ defined in Corollary 4,

$$\frac{1}{f_{\mu}} \notin H^2.$$

As is well known (see [28]), if $g \in H^2$ and $\frac{1}{g} \in H^2$, then $g$ is an outer function. Also, $f$ is outer if and only if

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

From the construction in Corollary 4, $\log|f_{\mu}(0)| = -db^{-1}$ and since $\text{Im}(\gamma(e^{i\theta})) = 0$, we have $\int_0^{2\pi} \log|f(e^{i\theta})| d\theta = \int_0^{2\pi} -db^{-1}(\text{Im}(\gamma(e^{i\theta}))) d\theta = 0$ and hence $f_{\mu}$ cannot be outer.

If $\phi$ is a parabolic disc automorphism which has Denjoy-Wolff point $e^{i\theta} \neq 1$, then we may conjugate $C_\phi$ by $\alpha(z) = e^{-i\theta}z$, so that $\alpha^{-1} \circ \phi \circ \alpha$ has Denjoy-Wolff point 1. Thus if $f \in H^2$
is such that \( f \circ \phi = f \), then \( f \) must be of the form \( g \circ \alpha^{-1} \) where \( g \) belongs to the subspace \( M \) for \( C_\alpha C_\alpha^{-1} \) from Theorem 8.

The hyperbolic case is not so easily dealt with but there are some partial results.

**Theorem 9** Let \( \phi \) be a hyperbolic disc automorphism with Denjoy-Wolff point 1. For each integer \( i \) bigger than or equal to 0, let \( B_i \) be the Blaschke product formed from the iterates of 0 under the hyperbolic disc automorphism \( \phi^{[2]} \). If \( f \circ \phi = f \), then \( f = \sum_{k=0}^{\infty} a_k B_0 \cdots B_k \), where \( \sum_{k=0}^{\infty} |a_k|^2 < \infty \).

**Proof:**
First of all, since \( B_0 \cdots B_k \) divides \( B_0 \cdots B_l \) as inner functions when \( k \leq l \) and each \( B_k \) for every \( k \) has a factor of \( z \), if we follow the integral norm procedure in Theorem 8, we get that \( B_0 \cdots B_k \) is perpendicular to \( B_0 \cdots B_l \) when \( k \neq l \).

We have to use a slightly different technique than in the parabolic case since \( B_k \) for any \( k \) is not fixed by \( \phi \) under composition. Let \( a_0 = f(0) \) and consider \( g_1 = f - a_0 \). Now \( g_1 \circ \phi = g_1 \) since \( f \) and \( a_0 \) are fixed by \( \phi \) under composition. Thus \( g_1 \) is zero at all points \( \phi^{[k]}(0) \) and hence \( B_0 | g \). Hence we may write \( f \) as \( a_0 + B_0 f_1 \). Now since \( B_0 \circ \phi = -B_0 \) and \( f \circ \phi = f \), we must have \( f_1 \circ \phi = -f_1 \). We cannot work with this directly but \( f_1 \circ \phi \circ \phi = -f_1 \circ \phi = f_1 \); thus \( f_1 \) is fixed by \( \phi^{[2]} \). Replacing \( \phi \) by \( \phi^{[2]} \) we may work with \( f_1 \) and write \( f_1 \) as \( a_1 + B_1 f_2 \) using the above procedure. Thus \( f = a_0 + a_1 B_0 + B_0 B_1 f_2 \); here \( f_2 \) is fixed by \( \phi^{[4]} \) and we may continue this procedure to write, at the \( N \)th step, \( f = \sum_{k=0}^{N} a_k B_0 \cdots B_k + B_0 \cdots B_{N+1} R_N \) where \( R_N \) is a remainder term. Now \( \|f\|^2 = \langle f, f \rangle = \sum_{k=0}^{N} |a_k|^2 + \|B_0 \cdots B_{N+1} R_N\|^2 \), since \( \{B_0 \cdots B_j : j = 0, 1, \ldots\} \) are an orthogonal set of inner functions. Hence we have \( \sum_{k=0}^{\infty} |a_k|^2 < \infty \). Now let \( R = f - \sum_{k=0}^{\infty} a_k B_0 \cdots B_k \); thus \( R - B_0 \cdots B_{N+1} R_N = \sum_{k=N+1}^{\infty} a_k B_0 \cdots B_k \). As \( N \to \infty \), we have \( \|R - B_0 \cdots B_{N+1} R_N\|^2 = \sum_{k=N+1}^{\infty} |a_k|^2 \to 0 \). Hence \( B_0 \cdots B_{N+1} R_N \to R \) in \( H^2 \) and, since \( B_0 \cdots B_{N+1} R_N \) has an \((N+1)\)-fold zero at 0, we conclude that \( R = 0 \) and \( f \) has the desired form. ■

The first question of interest is: how big is the span of \( \{B_0 \cdots B_k : k \in \mathbb{N}\} \)? Since none
of the $B_0 \cdots B_k$ for any $k$, is fixed by $\phi$ under composition, the span contains more than those functions that satisfy $f \circ \phi = f$. However, every function of the form $zk_{\phi^n(0)}$ where $n \neq 0$ is perpendicular to that span since such functions are perpendicular to constants and $B_0 \circ (\phi^n(0)) = 0$. Hence the span is not all of $H^2$.

**Corollary 5** Let $\phi$ be a hyperbolic disc automorphism with fixed points 1 and $-1$ where 1 is the Denjoy-Wolff point. If $f \in H^\infty$, and $M_f \leftrightarrow C_\phi$, then $f = \sum_{n=0}^{\infty} a_n B_0 \cdots B_n$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

**Proof:**
Apply Theorem 9. ■

**Corollary 6** If $\phi$ is a hyperbolic disc automorphism with fixed points 1 and $-1$ where 1 is the Denjoy-Wolff point and if $f$ is an eigenfunction associated with eigenvalue 1 for $C_\phi$, then $f = \sum_{n=0}^{\infty} a_n B_0 \cdots B_n$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

**Proof:**
If $f$ is an eigenfunction associated with the eigenvalue 1 for $\phi$, then $f \circ \phi = f$. Apply Theorem 9. ■

Secondly, is there, perhaps, another function which would serve instead of $B_0$ but give a complete result? Since $\phi'(1) = \frac{1-\phi(0)}{1+\phi(0)}$, by Theorem 7.21 in [14], for each $\theta \in [0, 2\pi]$ there is a function $h_\theta$ such that $h_\theta \circ \phi = e^{i\theta} h_\theta$ and, moreover, both $h_\theta$ and $\frac{1}{h_\theta}$ are in $H^2$. In place of $B_0$, one could use the function $h_\pi B_0$, since this is fixed by $\phi$. The problem with using this function is that it is not an inner function and the estimates for the powers of the norms become unbounded. There is something that $h_\theta$ does contribute to the problem, as follows.

**Theorem 10** If $C_\phi$ is a hyperbolic composition operator induced by a disc automorphism with fixed points 1 and $-1$ with the Denjoy-Wolff point at 1, then $f \in H^2$ is an eigenvector associated with eigenvalue $e^{i\theta}$ if and only if $f = h_\theta g$ where $g$ is an eigenvector associated with eigenvalue 1 and $h_\theta \circ \phi = e^{i\theta} h_\theta$ with $h_\theta$ and $\frac{1}{h_\theta}$ in $H^2$. 

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Proof:
Let $g = f/h_\phi$; then $g \in H^2$ and $g \circ \phi = g$. ■

If $\phi$ is a hyperbolic disc automorphism with fixed points other than $-1$ and $1$, we may conjugate it by a disc automorphism so that it has fixed points $1$ and $-1$, with $1$ the Denjoy-Wolff point (see [23]). Thus the above results can be translated to apply to any composition operator induced by a hyperbolic disc automorphism.

3.4 Conclusion

To our knowledge nothing else is known about the question of when the commutant of a composition operator is generated by composition and Toeplitz operators. It is open even in the case of hyperbolic and parabolic disc automorphisms. Let $C_\phi$ be a composition operator induced by either a hyperbolic or parabolic disc automorphism. In Corollary 1 and Corollary 5, we classify all Toeplitz operators which commute with the latter and give a partial classification for those which commute with the former. In [19], Heins gives a beautiful proof of the fact that the only composition operators which commute with a hyperbolic composition operator are ones which are also induced by a hyperbolic disc automorphism with the same fixed points. Unfortunately, there is yet no classification of which composition operators commute with a parabolic composition operator. One thing which is clear is that the commutant in both the hyperbolic and parabolic cases is not the weakly closed algebra generated by the composition operators which commute. For any composition operator fixes the constants, and in each case there are non trivial commuting analytic Toeplitz operators.
Chapter 4

Outer Functions

Beurling’s Theorem states that every function $f \in H^2$ may be factored into an inner part and an outer part with the factorization unique up to a unimodular constant (see [14] page 31). An inner function $I$ is one such that $|I(z)| = 1$ almost everywhere on $\partial D$. An outer function $O$ is one such that $\{p(z)O : p \text{ polynomial} \}$ is strongly dense in $H^2$. A question of interest is: what happens to the Beurling factorization of an $H^2$ function under composition? Certainly if $\phi$ is an inner function and $I$ is an inner function, then $I \circ \phi$ is an inner function: for outer functions the following is true.

**Theorem 11** If $\phi : D \to D$ is analytic and if $F \in H^2$ is an outer function, then $F \circ \phi$ is an outer function.

**Proof:**

$F$ is outer if and only if

$$\{p(z)F : p \text{ polynomial} \} = H^2.$$  

which is the same as

$$\{hF : h \in H^\infty \} = H^2.$$  

Suppose there is $g \in H^2$ such that $< h(F \circ \phi), g > = 0$ for all $h \in H^\infty$. We claim $g$ is identically zero. If we substitute $h_n^k = z^n(\phi)^k$ for $h$ in the previous equation we get

$$< z^n(\phi)^k F \circ \phi, g > = 0 \Rightarrow < (\phi)^k F \circ \phi, M^*_z g > = 0$$
$$\Rightarrow < z^k F, C^*_\phi (M^n z g) >= 0.$$ 

Now for fixed $n$, the fact that this holds for all $k = 0, 1, 2...$ implies that $C^*_\phi (M^n z g) = 0.$ since $F$ is outer. Therefore

$$C^*_\phi (M^n z g) = 0 \Rightarrow C^*_\phi (M^n z g), 1 >= 0 \Rightarrow C^*_\phi (M^n z g), 1 >= 0$$

which yields $< g, z^n >= 0$ for $n = 0, 1, 2...$ and thus $g = 0.$

Thus the outer functions are an invariant set for any composition operator.

**Corollary 7** Let $\phi : D \mapsto D$ be analytic and inner. If $f \in H^2$ has the factorization $f = IO$ where $I$ is inner and $O$ is outer, then $f \circ \phi$ has the factorization $f \circ \phi = I \circ \phi O \circ \phi$ with $I \circ \phi$ the inner part and $O \circ \phi$ the outer part.

**Proof:**
Since both outer and inner functions are invariant classes for $C_\phi$, and the factorization is unique up to a constant, the result holds. ■

**Corollary 8** Let $\phi : D \mapsto D$ be analytic and inner. If $f \in H^2$, $f \circ \phi = f$, and $f = IO$ where $I$ is inner and $O$ is outer, then $I \circ \phi = e^{i\theta} I$ and $O \circ \phi = e^{-i\theta} O$ for some $\theta \in \mathbb{R}$.

**Proof:**
Apply Corollary 7 to conclude that $IO = I \circ \phi O \circ \phi$: this implies that $I \circ \phi = \lambda I$ and $O \circ \phi = \bar{\lambda}O$ where $\lambda$ is a unimodular constant. ■

We have also considered the question: if $\phi$ is an analytic self map of $D$ and $F$ is in $H^2$ such that $F \circ \phi$ is an outer function, does $F$ itself have to be outer? It is easy to see that this is answered in the affirmative when $\phi$ is an inner function and the negative when $\phi$ is an outer function. The question is also answered in the negative if $(\phi(D))^c \cap D$ is non-empty. For if $b$ is a point in this set and $F(z) = z - b$, then $F \circ \phi$ is bounded away from zero on $D$. It follows that $\frac{1}{F \circ \phi}$ is in $H^2$, which implies that $F \circ \phi$ is outer and clearly, $F$ is not outer since it has a zero in $D$. Other than this, the author is not aware of any other information about this problem.
Chapter 5

The General Commutant Problem

We would like to consider the general commutant problem in some specific cases. Firstly, we'll assume that the inducing function for the composition operator has an interior fixed point.

**Lemma 3** If $\phi$ is an analytic self map of $D$ which is not an elliptic disc automorphism, and $\phi(b) = b$ for some $b \in D$, then for each $g \in H^2$ we have $g \circ \phi^n \rightarrow g(b)$ weakly.

**Proof:**
By Ryff's estimate (see [14]) and the fact that $\phi$ has an interior Denjoy-Wolff point, $\{g \circ \phi^n\}$ forms a bounded sequence in the $H^2$ norm and thus, by Lemma 1 and the Denjoy-Wolff Theorem, $g \circ \phi^n \rightarrow g(b)$ weakly.

**Theorem 12** Let $\phi : D \rightarrow D$ be an analytic function which is not an elliptic disc automorphism and let $\phi$ Denjoy-Wolff point $b \in D$ (so $\phi(b) = b$). Suppose $A : H^2 \rightarrow H^2$ and $A \leftrightarrow C_\phi$. Then $(A(f))(b) = f(b)A(1)$ for all $f \in H^2$.

**Proof:**
First of all,

$$C_\phi(A(1)) = A(C_\phi(1)) \Rightarrow (A(1)) \circ \phi = A(1)$$

which implies $A(1)$ is constant (by Theorem 7). Let $f \in H^2$ be fixed. By Lemma 3, $f \circ \phi^n \rightarrow f(b)$ weakly. Now examine $A(f \circ \phi^n) = C_\phi(A(f))$. If we inner product this with $k_w$ (the reproducing kernel for $w \in D$), we have
\[ < A(f \circ \phi[n]), k_w > =< C_{\phi[n]}(A(f)), k_w > \]
\[ \Rightarrow < f \circ \phi[n], A^* k_w > = (A(f))(\phi[n](w)). \]

As \( n \) tends to \( \infty \), we have \(< f \circ \phi[n], A^* k_w > \) converges. by Lemma 3. to

\[ < f(b), A^* k_w >= < A(f(b)), k_w >= A(1)f(b) \]

while \((A(f))(\phi[n](w))\) converges by the Denjoy-Wolff Theorem to \((A(f))(b)\). Thus \((A(f))(b) = f(b)A(1). \]

**Corollary 9** Let \( \phi : D \rightarrow D \) be an analytic map which is not an elliptic disc automorphism, and which has an interior fixed point at \( b \). Then \( \{ f \in H^2 : f(b) = 0 \} \) is a hyperinvariant subspace for \( C_{\phi} \).

**Proof:**
A subspace is a hyperinvariant subspace for \( C_{\phi} \) if it is invariant for every operator which commutes with \( C_{\phi} \). If \( A \leftrightarrow C_{\phi} \) and if \( f(b) = 0 \), then \((A(f))(b) = A(1)f(b) = 0 \) by Theorem 12.

**Corollary 10** Let \( \phi : D \rightarrow D \) be an analytic map which is not an elliptic disc automorphism, and which has an interior fixed point at \( b \). Then for any operator \( A \) in the commutant, the point spectrum of \( A \) is the point spectrum of \( A \) restricted to the subspace \( \{ f \in H^2 : f(b) = 0 \} \) together with \( A(1) \).

**Proof:**
\( A(f) = \lambda f \). Evaluating this at \( b \) and applying Theorem 12, we have \( A(1)f(b) = \lambda f(b) \) so either \( \lambda = A(1) \) or \( f(b) = 0 \). Moreover, since \( A(1) \circ \phi = A(1) \), it follows that \( A(1) \) is a constant and thus an eigenvalue for \( A \).

**Corollary 11** Let \( \phi : D \rightarrow D \) be an analytic mapping, which is not an elliptic disc automorphism, and which has an interior fixed point at \( b \). Then \( A \leftrightarrow C_{\phi} \) implies that \( \hat{A}(b) = A(1) \) where \( \hat{A} \) is the Berezin Symbol.
Proof:
Recall the Berezin symbol (see [2] or [22]) for an operator $A$ on $H^2$ is given by

$$\tilde{A}(b) = \langle A \frac{k_b}{\|k_b\|}, \frac{k_b}{\|k_b\|} \rangle.$$ 

By Theorem 12, this equals

$$A(1)k_b(b) \frac{1}{\|k_b\|^2} = A(1) \frac{1 - |b|^2}{1 - |b|^2} = A(1).$$

**Corollary 12** Let $\phi : D \rightarrow D$ be an analytic mapping which is not an elliptic disc automorphism, and which has an interior fixed point at $b$. If $A \in C_\phi$, then $k_b$ is a eigenvector of $A^*$. 

Proof:
Let $M$ be the strong closure of the space spanned by the functions $(z - b)^n$ where $n = 1, 2, \ldots$. Then every function in $M$ is zero at $b$ and $M^\perp$ is the space spanned by $k_b$. Now, by Theorem 12.

$$\langle A((z - b)^n)k_b \rangle = 0 \Rightarrow \langle (z - b)^nA^*(k_b) \rangle = 0$$

$$\Rightarrow A^*(k_b) \in M^\perp \Rightarrow A^*(k_b) = \lambda k_b$$

for some $\lambda \in \mathbb{C}$. 

**Corollary 13** Let $\phi : D \rightarrow D$ with $\phi(0) = 0$ be a non constant analytic map which is not an elliptic disc automorphism. Then $C_\phi$ has a reducing subspace which is hyperinvariant.

Proof:
By Corollary 12, $A^*$ takes constants to constants. Since $\phi$ has an interior fixed point and $A(1) \circ \phi = A(1)$, it follows that $A(1)$ is a constant. Thus the constants are a reducing subspace for any $A$ which commutes with $C_\phi$. 

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Theorem 13 Let \( A : H^2 \to H^2 \) be an operator such that \( A \leftrightarrow C_{\phi_n} \). Suppose that \( \phi_n : D \to D \) and \( \phi_n(z_n) = z_n \) where each \( \phi_n \) is an analytic map, none of which are elliptic disc automorphisms. Moreover suppose that \( z_n \to z_0 \) (where \( z_0 \in D \)) as \( n \to \infty \) and all of the \( z_n \) are distinct. Then \( A = cI \) for some \( c \in \mathbb{C} \) where \( I \) is the identity operator.

Proof: 
By Theorem 12, if \( f \in H^2 \), we have 
\[
C_{\phi_n} \leftrightarrow A \Rightarrow (A(f))(z_n) = f(z_n)A(1).
\]
Thus by the Identity Theorem in complex analysis, we have for all \( z \in D \)
\[
(A(f))(z) = f(z)A(1) \Rightarrow (A(f)) = (f)(A(1)) \Rightarrow A = A(1)I.
\]

Corollary 14 Let \( A \) be the algebra generated by a collection of composition operators induced by functions, \( \phi_n \), satisfying the hypothesis of Theorem 13. Then \( A' = \{cI : c \in \mathbb{C}\} \) where \( I \) is the identity operator.

Proof: 
Certainly, every operator of the form \( cI \) is in \( A' \) and Theorem 13 implies that every operator in \( A' \) must be of that form.

Corollary 15 Let \( A \) be a weakly closed self-adjoint algebra induced by a collection of composition operators satisfying the hypothesis of Theorem 13. Then \( A = B(H^2) \).

Proof: 
\( A' = \{cI : c \in \mathbb{C}\} \) by Corollary 11. By the Double Commutant Theorem, \( A'' = A \) but \( A'' = \{cC_z : c \in \mathbb{C}\}' = B(H^2) \).

We now consider the question of generating \( B(H^2) \) with a smaller number of composition operators.

Guyker proved the following (see [18])
Theorem 14 (Guyker) If \( \phi \) is univalent and \( b \in D, b \neq 0 \) and \( \phi(b) = b \), then \( C_\phi \) is irreducible.

One can weaken the hypothesis of Theorem 14 slightly without changing the proof by replacing univalent with univalent at the point \( b \). If \( f : D \to D \) is univalent at the point \( b \) if \( f(z) = b \) implies that \( z = b \). Thus under this hypothesis such a \( C_\phi \) is irreducible and the weakly closed self-adjoint algebra generated by it must be \( B(H^2) \). The question remains of what happens when \( \phi \) has a fixed point at zero. We have the following theorem.

**Theorem 15** Suppose \( \phi, \psi : D \to D, \psi(a) = a, \phi(0) = 0 \), and \( a \neq 0 \). Also suppose that \( \phi \) and \( \psi \) are not constants or elliptic disc automorphisms. \( \phi \) is univalent at 0, and one of them maps the disc to a compact subset. Then it follows that the weakly closed self-adjoint algebra generated by \( C_\psi \) and \( C_\phi \) is \( B(H^2) \).

**Proof:**

We have two cases

(1) \( \psi : D \to rD \) with \( r < 1 \) or

(2) \( \phi : D \to rD \) with \( r < 1 \).

(1) Let \( f_n = \psi(\phi^{[n]}) \). Since \( f_n : D \to rD \) (\( n \) fixed) \( f_n \) must have an interior fixed point. Let \( f_n(z_n) = z_n \) and note that \( z_n \in rD \) for all \( n \in \mathbb{N} \). We claim that there exists \( n_k \) in \( \mathbb{N} \) such that \( z_{n_k} \) is an infinite distinct sequence. If not \( \{ z : \text{there exists } n, f_n(z) = z \} \) is finite and there is \( n_t, \) an infinite sequence and some \( z_0 \in D \), such that

\[
\psi(\phi^{[n_t]}(z_0)) = z_0.
\]

Now as \( n_t \to \infty \), we have \( \phi^{[n_t]}(z_0) \to 0 \) by the Denjoy-Wolff Theorem and since \( \phi \) is univalent at 0, we have that \( \phi^{[n_t]}(z_0) \) is an infinite distinct sequence (else if \( \phi^{[n_r]}(z_0) = 0 \) for some \( n_r \in \mathbb{N} \), we would have \( z_0 = 0 \) and thus \( \psi(0) = 0 \) but \( \psi(a) = a \) which is a contradiction). Now since \( \phi^{[n_t]}(z_0) \) is an infinite sequence which converges to 0, by the Identity Theorem
we have $\psi = z_0$ which is also a contradiction. Thus $z_{n_0}$ is infinite distinct sequence and.
applying Corollary 15. we have the result.

(2) $\psi$ is continuous on $rD$ which is compact. so $\psi(rD)$ is compact. Thus $\psi(rD) \subset sD$ for some $s < 1$. This implies that $f_n = \psi(\phi^{[n]}): D \mapsto sD$. If we define $z_n$ as above. we have $z_n \in sD$ for all $n$ and hence we may apply the rest of the argument of case (1).

The previous results were for $\phi$ with an interior fixed point. It is possible to prove an
analog of Theorem 12 for $\phi$ with Denjoy-Wolff point on the boundary of $D$.

**Theorem 16** Let $\phi: D \mapsto D$ be an analytic function with Denjoy-Wolff point $a \in \partial D$. Let
$A \leftrightarrow C_\phi$. Then $A(1)$ is a constant if and only if there exists a polynomial $p$ such that $A(p)$
is continuous onto the boundary at $a$ and $p(a) \neq 0$.

**Proof:**
If $A(1)$ is a constant. then let $p(z) = 1$ for all $z \in D$.

Conversely, suppose there exists $p$, a polynomial. such that $A(p)$ is continuous onto the
boundary at $a$. Then $p \circ \phi^{[n]}$ is a bounded sequence in $H^2$ for $\| p \circ \phi^{[n]} \|_2 \leq \| p \|_\infty$. Thus by
Lemma 1 and the Denjoy-Wolff Theorem, $p \circ \phi^{[n]} \rightarrow p(a)$ weakly. Thus we have, for $\lambda \in D$.
and $k_\lambda$ the reproducing kernel function for $\lambda$.

$$< A(p \circ \phi^{[n]}), k_\lambda > = C_{\phi^{[n]}(A(p))}, k_\lambda > = A(p) (\phi^{[n]}(\lambda)).$$

This converges to $(A(p))(a)$ as $n_k \rightarrow \infty$. But

$$< A(p \circ \phi^{[n]}), k_\lambda > = p \circ \phi^{[n]} . A^*(k_\lambda) >$$

converges to

$$< p(a) . A^*(k_\lambda) > = < A(p(a)), k_\lambda > = < p(a).A(1), k_\lambda > .$$

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Since \((p(a)A(1))(\lambda) = (A(p))(a)\) for all \(\lambda\) in \(D\), it follows that \(p(a)A(1)\) is a constant, and \(p(a) \neq 0\) implies \(A(1)\) is a constant.
Chapter 6

Hyperinvariant Subspaces, Quasi-Normals and Isometries.

Let $\phi$ be an analytic self map of $D$. Suppose $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. In 1884, Koenigs showed that the sequence $\{\sigma_k\}$ with

$$\sigma_k(z) = \frac{\phi[k](z)}{(\phi'(0))^k}$$

converges uniformly on compact subsets of $D$ to a non-constant function $\sigma$, which is known as the Koenigs function for $\phi$ (see [30] or [14]). Paul Bourdon proved the following Theorem when $\phi$ was univalent. and shortly afterwards Pietro Poggi-Corradini was able to remove that hypothesis (see [4] or [24]).

**Theorem 17** Let $\phi$ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Let $\sigma$ be the Koenigs function of $\phi$ and $q$ a natural number. If $(\sigma)^q$ is in $H^2$, then the sequence $\{\sigma_k]^q\}$ converges to $(\sigma)^q$ in the $H^2$ norm.

In [6], Bourdon and Shapiro proved a sufficient condition for the Koenigs function to belong to $H^p$ as well as showing the condition to be necessary in the case that the function $\phi$ is analytic on the closed unit disc. In [25], Pietro Poggi-Corradini was able to prove the necessity of the condition without any additional conditions on $\phi$. These results together lead to the following theorem.
Theorem 18 Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Let \( \sigma \) be the Koenigs function. Then \( (\sigma)^q \) is in \( H^2 \) if and only if \( |\phi'(0)|^q \) exceeds the essential spectral radius of \( C_\phi \).

6.1 Hyperinvariant Subspaces

Theorem 19 Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Suppose \( (\sigma)^q \) is in \( H^2 \) where \( \sigma \) is the Koenigs function of \( \phi \). Then for each natural number \( k \), \( 1 \leq k \leq q \), the subspaces \( z^k H^2 \) are hyperinvariant for \( C_\phi \).

Proof:
By Corollary 9, \( zH^2 \) is a hyperinvariant subspace for such a \( C_\phi \). This covers the case \( k = 1 \). Proceeding by induction on \( n < q \), we will assume, for \( k < n \) that \( z^k H^2 \) is hyperinvariant for \( C_\phi \). Let \( A \) be an operator which commutes with \( C_\phi \) and \( z^{n+1}p \) be a function in \( z^{n+1} H^2 \) where \( p \) is a polynomial. We wish to show

\[
< A(z^{n+1}p), z^l > = 0
\]

for \( l \leq n \). Since \( z^{n+1}p \) is in \( z^n H^2 \), we have by the induction hypothesis that

\[
< A(z^{n+1}p), z^l > = 0
\]

for \( l < n \). Now

\[
< AC_\phi^m(z^{n+1}p), z^n > = < A(z^{n+1}p), (C_\phi^*)^m(z^n) >
\]

\[
= < A(z^{n+1}p), (\phi'(0))^{mn} z^n >.
\]

Dividing both sides of the equation by \( (\phi'(0))^{mn} \), it follows that

\[
< A((\sigma_m^m) n \phi[m](p \circ \phi[m])), z^n > = < A(z^{n+1}p), z^n >
\]
The left hand side of this equation is equal to
\[ <(\sigma_m)^np \circ \phi^{[m]}, A^*(z^n)>.\]

Since \( H^{2q} \subset H^{2n} \) when \( q \geq n \), \( (\sigma)^n \) is in \( H^2 \). By Theorem 17, \( (\sigma_m)^n \) is a bounded sequence in the \( H^2 \) norm. Since \( p \) is in \( H^\infty \),

\[ \{(\sigma_m)^np \circ \phi^{[m]}\}\]

is also a bounded sequence in the \( H^2 \) norm. This sequence converges pointwise to 0 as \( m \) goes to \( \infty \), and thus it converges weakly to 0. It follows that \( <A(z^{n+1}p), z^n> = 0 \).

If \( z^{n+1}f \) is in \( z^{n+1}H^2 \), we may find a sequence of polynomials \( \{p_l\} \) which converges strongly to \( f \) as \( l \) goes to \( \infty \). In particular, \( <A(z^{n+1}p_l), z^n> \) will converge to \( <A(z^{n+1}f), z^n> \) as \( l \) goes to infinity. Hence the result holds. \( \blacksquare \)

Let \( \|A\|_e \) denote the essential norm of an operator \( A \). An operator \( A \) is a Riesz operator if \( \|A^n\|_e^{\frac{1}{n}} \) tends to 0 as \( n \) tends to \( \infty \). This implies that \( A \) is Riesz if and only if the essential spectrum of \( A \) is \( \{0\} \). In [5]. Bourdon and Shapiro show that a Riesz composition operator must be induced by a function which has an interior fixed point in \( D \). For basic details of Riesz operators see [15].

**Corollary 16** Let \( \phi \) be an analytic self map of \( D \) with \( \phi(b) = b \) where \( b \) is in \( D \) and \( 0 < |\phi'(b)| < 1 \). If \( C_\phi \) is a Riesz operator, then \( C_\phi \) has a triangularizing chain of hyperinvariant subspaces.

**Proof:**
Let \( \alpha(z) = \frac{b-z}{1-bz} \). Define \( \psi = \alpha \circ \phi \circ \alpha \). Note \( \psi(0) = 0 \) and \( 0 < |\psi'(0)| < 1 \). Since \( C_\psi^n = C_\alpha C_\phi^n C_\alpha \), it follows that \( C_\psi \) is a Riesz operator. Now by Theorem 18 and Theorem 19, \( C_\psi \) has the \( z^qH^2 \) as hyperinvariant subspaces for all \( q = 0, 1, 2, \ldots \). Thus \( C_\phi \) has \( (\alpha)^qH^2 \), \( q = 0, 1, 2, \ldots \) as a chain of hyperinvariant subspaces. Moreover any operator which holds these subspaces invariant must be lower triangular with respect to the orthonormal basis.
Let $f^{(k)}$ be the $k$th derivative of $f$.

**Corollary 17** Let $\phi$ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Let $k$ be a natural number greater than or equal to $1$. Suppose $(\sigma)^{k+1}$ is in $H^2$ and let $A$ commute with $C_\phi$. Then, for all $f$ in $H^2$, \( \frac{d^k}{dz^k}(A(f))(0) = k! \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} < A(z^n), z^k >. \)

**Proof:** Let $f = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of $f$. By Theorem 12, if $A$ commutes with $C_\phi$, then $(A(f))(0) = A(1)f(0)$ where $A(1)$ is a constant. This implies that $< A(a_0), z^k > = 0$. Now
\[
< A(f), z^k > = < A(a_0), z^k > + \sum_{n=1}^{k} a_n < A(z^n), z^k > + < A(\sum_{n=k+1}^{\infty} a_n z^n), z^k >
\]
The last term is 0 since $z^{k+1} H^2$ is hyperinvariant for $C_\phi$ by Theorem 19. Since $a_n = \frac{f^{(n)}(0)}{n!}$, the result follows.

The next example shows that there are analytic self maps of $D$ with $\phi(0) = 0$ and $\phi'(0) = 0$ such that the $z^q H^2$ are not hyperinvariant for $C_\phi$ for $q \geq 2$.

**Example 3** Let $n$ be a natural number greater than $1$. For $C_{z^n}$, the subspaces $z^q H^2$ for $q \geq 2$ are not hyperinvariant.

**Proof:** In Example 2, we discussed the commutant of $C_{z^n}$ in more detail. Let $q$ be a natural number greater than or equal to 2. If we define an operator $A$ on $H^2$ by $A(z^{q \cdot n^i}) = z^{n^i}$ and $A(z^k) = 0$ when $k \neq q \cdot n^i$, then $A$ commutes with $C_\phi$. Now $A(z^q) = z$ and hence $z^q H^2$ is not a hyperinvariant subspace for $C_\phi$. ■
\section{6.2 Quasi-normals}

We now turn to quasi-normal operators. An operator $A$ is quasi-normal if $A$ commutes with $A^*A$. For more information, see [11].

\textbf{Lemma 4} If $C_\phi$ is quasi-normal, then $\phi(0) = 0$.

\textbf{Proof:}

$C_\phi C_\phi^* C_\phi(1) = C_\phi^* C_\phi C_\phi(1)$ implies that

$$\frac{1}{1 - \phi(0) \phi(z)} = \frac{1}{1 - \phi(z)}.$$

It follows that $\phi(0) = 0$. $\blacksquare$

\textbf{Lemma 5} Let $\phi$ belong to $L^2(\partial D)$ with $\phi$ non-zero and not a characteristic function of a proper subset of the unit circle. Also suppose that $\|\phi\|_{L^2} = \mu$ and $\|(\phi)^{k_l}\|_{L^2} = \mu$ where $k_l$ is a sequence of natural numbers which diverges to $\infty$ and $\mu$ is a non-zero real number. Then $|\phi(e^{i\theta})| = 1$ almost everywhere on $\partial D$.

\textbf{Proof:}

Let $\lambda = \mu^2$. Let $m(A)$ be the normalized standard Lebesgue measure of a set $A \subset \partial D$. Suppose that $|\phi(e^{i\theta})| > 1$ on a set of positive measure of $\partial D$. Then in particular there exists $\epsilon > 0$ such that the set $A_{\epsilon} = \{e^{i\theta} : |\phi(e^{i\theta})| > 1 + \epsilon\}$ has positive measure. It follows that

$$\lambda = \|\phi^{k_l}\|_{L^2}^2 \geq \frac{1}{2\pi} \int_{A_{\epsilon}} |\phi(z)|^{2k_l} dz \geq (1 + \epsilon)^{2k_l} m(A_{\epsilon}).$$

As $k_l$ tends to infinity, the right hand side also tends to infinity. This is a contradiction; thus $|\phi(e^{i\theta})| \leq 1$ almost everywhere.

Similarly, suppose that $|\phi(e^{i\theta})| < 1$ on a set of positive measure of $\partial D$. Then in particular there exists $\epsilon > 0$ such that the set $B_{\epsilon} = \{e^{i\theta} : |\phi(e^{i\theta})| < 1 - \epsilon\}$ has positive measure. Then
\[
\lambda = \frac{1}{2\pi} \int_{B_\varepsilon} |\phi(z)|^2 dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\varepsilon} |\phi(z)|^2 dz
\]

Let \( \int_{\partial D \setminus B_\varepsilon} |\phi(z)|^2 dz = \kappa \) We note \( \kappa \) is strictly less than \( \lambda \). Now

\[
\frac{1}{2\pi} \int_{B_\varepsilon} |\phi(z)|^{2k_l} dz \leq (1 - \varepsilon)^{2k_l} m(B_\varepsilon).
\]

Choose a natural number \( N \) such that \( k_l \geq N \) implies that \( (1 - \varepsilon)^{2k_l} m(B_\varepsilon) \) is strictly less than \( \lambda - \kappa \). Since \( |\phi(z)| \leq 1 \) almost everywhere, we note \( |\phi(z)|^{k_l} \leq |\phi(z)| \). Then with \( k_l \geq N \)

\[
\lambda = \| (\phi)^{k_l} \|_{L^2}^2 = \frac{1}{2\pi} \int_{B_\varepsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\varepsilon} |\phi(z)|^{2k_l} dz
\]

\[
\leq \frac{1}{2\pi} \int_{B_\varepsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\varepsilon} |\phi(z)|^{2k_l} dz
\]

\[
< \lambda - \kappa + \frac{1}{2\pi} \int_{\partial D \setminus B_\varepsilon} |\phi(z)|^2 dz = \lambda
\]

which is a contradiction; thus \( |\phi(z)| = 1 \) almost everywhere. \( \blacksquare \)

**Theorem 20** Suppose \( C_\phi \) is quasi-normal. If \( C_\phi^* C_\phi \) is a diagonal matrix in the standard basis, then either \( \phi(z) = cz \) for some constant \( c \) of modulus less than 1 or \( \phi \) is an inner function.

**Proof:**

Let the power series of \( \phi \) be \( \sum_{n=1}^{\infty} a_n z^n \) and let the \( k \)th diagonal entry of \( C_\phi^* C_\phi \) be \( \lambda_k \). First of all, if \( \phi \) is identically zero or if only one of the \( a_n \) is non-zero, then we are done.

Next suppose that only a finite number of the coefficients \( a_n \) are non-zero but more than one of them is non-zero. Thus let \( \phi(z) = z^k a_k + \cdots + a_l z^l \) where \( a_l \) and \( a_k \) are non-zero and \( k \) is the smallest power of \( z \) with a non-zero coefficient and \( l \) is the greatest. Now,

\[
< (\phi(z))^l, (\phi(z))^k > = (a_k)^l (a_l)^k
\]
which by hypothesis must be zero. Thus if only a finite number of coefficients are non-zero, we have \( \phi(z) = cz^k \) where \( c \) is a constant of modulus less than or equal to 1. If \( k \geq 2 \) and \( c \) is less than 1 in modulus, then \( C_\phi \) is not quasi-normal so in this case \( k = 1 \).

Suppose an infinite number of the coefficients \( a_n \) are non-zero. Then \( \phi(z) = \sum_{n=k}^{\infty} a_n z^n \) where \( a_k \) is the first non-zero coefficient. Now \( C_\phi^* C_\phi C_\phi(z) = C_\phi C_\phi^* C_\phi(z) \) implies that

\[
\sum_{n=k}^{\infty} \lambda_n a_n z^n = \sum_{n=k}^{\infty} \lambda_1 a_n z^n.
\]

It follows that \( \lambda_n = \lambda_1 \) for infinitely many \( n \). Since \( \lambda_1 = \langle \phi(z), \phi(z) \rangle \) and \( \lambda_n = \langle (\phi(z))^n, (\phi(z))^n \rangle \), we may apply Lemma 5 to conclude that \( \phi \) must be an inner function.

**Corollary 18** If \( C_\phi \) has \( z^q H^2 \), \( 1 \leq q < \infty \) as hyperinvariant subspaces, and \( C_\phi \) is quasi-normal, then either \( \phi(z) = cz \) for some \( c \) of modulus less than 1 or \( \phi \) is inner.

**Proof:**
If \( C_\phi \) is quasi-normal, then \( C_\phi \) commutes with \( C_\phi^* C_\phi \) as does \( C_\phi^* \). Thus \( C_\phi^* C_\phi \) is both upper and lower triangular with respect to the standard basis and thus diagonal. Apply Theorem 20.

We could apply Corollary 16 to conclude that if \( C_\phi \) is quasi-normal and Riesz that \( \phi(z) = cz \) for \( \|c\| < 1 \). This would say if \( C_\phi \) is Riesz and quasi-normal, then it is normal. As the next theorem shows this is true in greater generality.

**Theorem 21** If \( A \) is an operator on a Hilbert Space, \( H \), and \( A \) is Riesz and quasi-normal, then \( A \) is normal.

**Proof:**
If \( A \) is identically 0, then we are done. Assume \( A \) is not equal to 0. By [3], \( \ker(A) \) is reducing so we may decompose \( A \) as \( A_1 \oplus 0 \) with \( A_1 \) quasi-normal and injective. Moreover, we may decompose \( A_1 \) as \( BV \) with \( B \) self-adjoint and injective, \( V \) isometric, and \( B \) commuting with
V. Since $A_1^n = V^n B^n$, we have $\|(V^*)^n A_1^n\| = \|B^n\|$. Thus

$$\|B^n\|^\frac{1}{2} \leq \|(V^*)^n\|^\frac{1}{2} \|A_1^n\|^\frac{1}{2} \leq \|A_1^n\|^\frac{1}{2}.$$ 

The last term goes to 0 as $n$ goes to $\infty$ and thus $B$ is also Riesz. By Theorem 3.7 in [33], since $B$ is self adjoint and Riesz, it must be compact. Hence by the spectral theorem, $B$ is unitarily equivalent to $\sum \oplus \lambda_k I_k$ where $\lambda_k$ are the non-zero eigenvalues and the $I_k$ are identity operators on finite dimensional spaces. Since eigenspaces are hyperinvariant, and $V$ commutes with $B$, $V$ is unitarily equivalent to $\sum \oplus V_k$ where the $V_k$ are isometries on finite dimensional spaces and hence unitary. Thus $V$ is a unitary operator and $A_1$ is a product of commuting normals and thus normal. 

6.3 Isometries

An operator $A$ on $H^2$ is an isometry if $A^* A = I$. In [29], Schwarz proves the following theorem.

**Theorem 22** $C_o$ is an isometry on $H^2$ if and only if $o(0) = 0$ and $o$ is an inner function.

If $o$ is an elliptic disc automorphism, then the commutant of $C_o$ is well understood (see Example 1 and the remark following it). If $o$ is not an elliptic disc automorphism and $C_o$ is an isometry, Nordgren ([21]) shows that $C_o$ restricted to the constants and $C_o$ restricted to $zH^2$ are the unitary and purely isometric parts, respectively. If $A$ commutes with $C_o$, then the constants are a reducing subspace for $A$ (Corollary 13). Hence we may consider the commutant of $C_o$ as the direct sum of the commutant of the unitary part and the commutant of the purely isometric part. The purely isometric part is similar to a unilateral shift of infinite multiplicity on the wandering subspace, $M$, of $C_o$ (see [27]). The commutant of such a unilateral shift is given in terms of multiplication operators on $H^2(M)$ (see [27]).

Let $\phi = zf$ where $f$ is an inner function. Let $\{g_n : n = 0, 1, \ldots\}$ be a basis for $(fH^2)^\perp$. Then the following set
\[ \{z(\phi)^kg_n : n = 0, 1, 2..., k = 0, 1, 2...\} \]

is a basis for the wandering subspace of \( C_\phi \) on \( H^2 \).

In particular, let \( \phi = zB \) where \( B \) is a Blaschke product with zero set \( \{a_n : n = 0, 1, 2...\} \). Let \( k_\lambda \) be the reproducing kernel function for \( \lambda \) in \( D \) and let \( B_l \) be the Blaschke product with zero set \( \{a_n : n = 0, 1, ..., l\} \). Then the following is an orthonormal basis for \((BH^2)^+\) :

\[ \{g_0 = k_{a_0}, g_1 = B_0k_{a_1}, \ldots, g_n = B_{n-1}k_{a_n}\} \].

This gives us a basis for the wandering subspace of \( C_\phi \) and the commutant of \( C_\phi \) can be explicitly interpreted in terms of this basis.

**Questions**

(1): If \( \phi \) is an analytic self map of \( D \). \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). are the subspaces \( \{z^qH^2\} \), hyperinvariant subspaces for all natural numbers \( q \)? If this is true, then the only \( C_\phi \) which are quasi-normal are either isometries or are given by \( \phi(z) = cz \) for some constant \( c \) of absolute value less than 1.

(2): If \( \phi \) is an analytic self map of \( D \). \( \phi(0) = 0 \) and \( \phi'(0) = 0 \). what can be said about the hyperinvariant subspaces of \( C_\phi \)? If \( C_\phi \) is quasi-normal, what form does \( \phi \) have to take?

(3): Given a basis for the wandering subspace of an isometric composition operator can the hyperinvariant subspaces or the composition operators which commute be explicitly determined?
Bibliography


[25] P. Poggi-Corradini, The Hardy Class of Koenigs maps,


