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Influence of statistical sample size on evaluation of overall strength of cement-treated soil column

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4 cement-treated soil column
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Abstract

The quality of the cement-treated soil columns is normally assured based on the unconfined compressive strength q_u of core samples. q_u of core samples varies spatially and the statistical parameters of q_u (i.e., mean μ_{q_u} , variance $\sigma_{q_u}^2$ and autocorrelation distance θ_{q_u}) are adopted in quality assurance procedures. The statistical parameters of q_u evaluated from the core sample strengths have a statistical uncertainty depending on the statistical sample size. The present study investigates the influence of the statistical sample size on the evaluation of overall strength of a cement-treated soil column. A probabilistic framework in which a Bayesian inference analysis and a finite element method analysis are incorporated is used to calculate the overall strength while simultaneously considering the statistical uncertainty and spatial variability of core strength. The probabilistic framework is briefly described and a parametric analysis is performed to investigate the influence of the statistical sample size on the evaluation of the overall strength of a full-scale column. The numerical results show that the sample size and spatial correlation influence the variability of the overall strength, and the influence can be reasonably described using an equivalent number of independent data.

KEY WORDS: cement-treated soil; strength; statistical sample size; statistical uncertainty; spatial variability

Introduction

The deep cement mixing method is a ground improvement technology in which cement is mixed with in situ soil. Ground improvements via deep mixing have been used for several purposes, such as structure foundation, retaining walls, and liquefaction mitigation methods. The quality of cement-treated soil columns constructed via the deep mixing varies widely owing to the variability of in-situ soil properties, mixing effects, and other factors. Therefore, a major concern with the ground improvement via deep mixing is the proper assessment of the quality of cement-treated soil columns.

In current quality control programs of deep mixing, the quality of the cement-treated columns is ensured based on unconfined compressive strength q_u of core samples retrieved from the columns (e.g., CDIT 2002; Filz 2009). In the quality assurance procedure, the representative strength R_{qu} evaluated from the observed q_u values is compared with the design strength F_c using the following equation:

$$F_c \leq R_{qu} = \mu_{qu} - K\sigma_{qu} \cong \mu_{squ} - K\sigma_{squ} \quad (1)$$

where μ_{qu} is the mean of q_u , σ_{qu} is the standard deviation of q_u , μ_{squ} is the sample mean of q_u , σ_{squ} is the sample standard deviation of q_u , and K is the reduction coefficient. The value of K determines the confidence level of q_u . When the R_{qu} calculated from the mean and standard deviation of q_u satisfies the criterion defined in Eq. (1), the ground improvement is considered successful. As the q_u of core samples varies spatially in the columns, the sampling plan, including the locations of the sampling and a statistical sample size n , that is, the number of the core samples, is important to assess the success of the ground improvement properly. In the quality assurance procedure used in Japan (Kitazume and Terashi 2013), three core borings are required when the number of columns is less than 500. When the number of columns is more than 500, one core boring is added for every additional 250 columns. In each core boring, three core barrels are selected at three different levels, and three specimens are taken from each core barrel. The unconfined compressive tests are conducted for the nine specimens at each core boring. When the number of columns is less than 500, 27 q_u values ($n = 27$) are required in the quality assurance procedure. Thus, the Japanese regulation recommends n and the approximate location of samples. However, this regulation provides no evidence for the recommended sampling location and number of specimens.

The statistical parameters of q_u (e.g., μ_{qu} , σ_{qu}) evaluated from the core sample strengths have the statistical uncertainty. The statistical uncertainty may depend on n and the spatial correlation. Therefore, statistical uncertainty quantified suitably based on the sampling data should be considered when adopting the sample statistical parameters in

the quality assurance procedure. When μ_{squ} and σ_{squ} calculated from the core strength are directly adopted for μ_{qu} and σ_{qu} in Eq. (1), respectively, the statistical uncertainty should be properly accounted for in the assessment of the quality of the cement-treated columns. However, the current quality assurance procedure does not account for the statistical uncertainty in the evaluation of R_{qu} .

Several past studies have addressed the sample size determination in geotechnical site investigation. Wang et al. (2019) performed a parametric study and developed a statistical chart for sample size determination with consideration of the spatial variation using Bayesian compressive sensing or sampling. The parametric study results showed that the normalized mean relative error increases as the normalized sampling interval increases. Guan and Wang (2020) performed a parametric study using Bayesian compressive sampling and developed statistical charts for determination of sample size at different levels of both accuracy and confidence in geotechnical site investigation. The analysis results indicated that the correlation function type has a significant effect on the relationship between sample size and levels of accuracy and confidence. Those studies provided the proper understanding of the accuracy of the soil profile interpreted from sparse measurements. Some studies addressed the influence of the statistical sample size on the quality assurance of cement-treated ground. Fenton et al. (2015) examined the sampling requirement for estimating the hydraulic conductivity of cement-based solidification/stabilization construction cells. They performed a random field analysis and determined the number of samples required to maintain the probabilities of making errors acceptably small. Liu et al. (2017) investigated the representative strength of deep cement-mixed clay and proposed a method for assessing the sufficiency of the sample size in core sampling.

The Bayesian approach is an effective framework for treating the uncertainties in statistical analyses. Several studies have adopted the Bayesian approach to infer the statistical parameters of soil properties (e.g., Wang et al. 2010; Ching et al. 2016). Namikawa (2019) adopted the Bayesian approach to assess the statistical uncertainty in the estimation of the statistical parameters of cement-treated soil strength. The mean μ_{qu} , variance σ_{qu}^2 , and autocorrelation distance θ_{qu} of q_u of core samples retrieved from the deep cement mixing columns were inferred using the Bayesian approach. A Markov chain Monte Carlo (MCMC) method was adopted to draw the parameter values from their target posterior distributions. The analytical result indicated that not only n but also the spatial correlation affects the statistical uncertainty in the estimation of the statistical parameters and the variability of R_{qu} calculated from the inferred statistical parameters should be considered in the quality assurance procedure.

In Eq. (1), μ_{qu} and σ_{qu} are used to ensure the quality of the ground improvement.

However, μ_{qu} and σ_{qu} are not sufficient to characterize the variability of the strength of the cement-treated soil columns because a spatial correlation is found in the core strength data. The spatial correlation of q_u should be considered in the quality assurance procedures to assess the overall internal stability of ground improvement. The finite element method (FEM) analysis incorporated with random field theory is a powerful tool for evaluating the overall internal stability of ground improvement. Several past studies have investigated the behavior of cement-treated grounds comprised of elements having different strengths using the FEM analysis incorporated with the random field theory (e.g., Namikawa and Koseki 2013; Liu et al. 2015; Namikawa 2016; Toraldo et al. 2018; Pan et al. 2019). These studies have indicated that the spatial variability of the material properties of cement-treated soils significantly affects overall behaviors of the cement-treated ground. The present author has developed a probabilistic analysis framework to consider the statistical uncertainty and the spatial variability simultaneously in evaluating the overall behavior of ground improvement (Namikawa 2021a). In this framework, the Bayesian approach is adopted to estimate the statistical uncertainty, and the FEM analysis incorporated with the random field theory is adopted to evaluate the overall behavior of ground improvement with spatial variability.

This paper aims to examine the influence of the statistical sample size n on the evaluation of the overall strength Q_u of a cement-treated soil column. The probabilistic analysis framework proposed by Namikawa (2021a) is used to assess the influence of n on the inferred overall strength. The statistical uncertainty is estimated using the Bayesian inference framework, in which the statistical parameters μ_{qu} , σ_{qu}^2 and θ_{qu} are drawn via the MCMC method. The drawn values are used to consider the statistical uncertainty when generating the random field realizations of the cement-treated column, and the FEM analysis is conducted for the generated realizations. A parametric simulation with the various statistical sample sizes is conducted to investigate the relationship between n and the variability of the calculated Q_u values. The simulation results reveal the influence of the statistical sample size on the evaluation of Q_u . A framework of the quality assurance criterion in which the statistical sample size is properly considered is discussed on the basis of the simulation results.

APPROACH

Namikawa (2021a) proposed the Monte Carlo simulation framework to evaluate the overall behavior of cement-treated soil ground. The statistical uncertainty depending on the sample size n and the spatial variability are properly considered in this framework. In this study, that framework is adopted to assess the influence of n on the overall strength Q_u of a cement-treated soil column.

The analytical process of that framework is shown in Fig. 1. First, the statistical uncertainty included in estimating the statistical parameters is assessed using the Bayesian inference method. In this analysis, the statistical parameters μ_{qu} , σ_{qu}^2 and θ_{qu} are drawn via the MCMC method. Second, random fields representing strength distributions in cement-treated soil columns are then generated, based on the inferred values of μ_{qu} , σ_{qu}^2 and θ_{qu} . Finally, the FEM analysis is conducted for the cement-treated column models in which the strength distributions correspond to the generated random fields. In this framework, the statistical uncertainty and the spatial variability can be simultaneously considered when evaluating Q_u of a cement-treated soil column. The individual components of this framework are briefly described in the following sections.

Probability distribution for cement-treated soil strength

A probability distribution of q_u of cement-treated soils is required to estimate the statistical parameters and generate the random fields of q_u . Several past studies have discussed proper probability distributions of cement-treated soil strengths (e.g. Navin and Filz 2005; Namikawa and Koseki 2013; Liu et al. 2017). Namikawa and Koseki (2013) reported that the normal and lognormal distributions can be employed for the probability distribution of q_u . In the present study, the normal distribution is adopted for the probability distribution of q_u because of its simplicity and the familiarity of its description.

q_u is assumed to follow the multivariate normal distribution as follows:

$$p(\mathbf{q}_u | \mu_{qu}, \sigma_{qu}^2, \theta_{qu}) = \frac{1}{\sqrt{(2\pi)^m (\sigma_{qu}^2)^m |\mathbf{C}|}} \exp \left\{ -\frac{1}{2\sigma_{qu}^2} (\mathbf{q}_u - \boldsymbol{\mu}_{qu})^T \mathbf{C}^{-1} (\mathbf{q}_u - \boldsymbol{\mu}_{qu}) \right\} \quad (2)$$

$$\mathbf{q}_u = \begin{bmatrix} q_u(\mathbf{r}_1) \\ \vdots \\ q_u(\mathbf{r}_n) \end{bmatrix}, \quad \boldsymbol{\mu}_{qu} = \begin{bmatrix} \mu_{qu} \\ \vdots \\ \mu_{qu} \end{bmatrix}, \quad \mathbf{C} = \rho_{qu}(\mathbf{d}) = \exp \left(-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\theta_{qu}} \right)$$

where m denotes the number of q_u values, $\mathbf{C}$ denotes the correlation matrix, and $\mathbf{r}_i$ denotes the space vector at point i . A stationary random field with an exponential type autocorrelation function is assumed here for the spatial variability of q_u in the cement-treated soil columns.

While a stationary condition is assumed for the random field of q_u here, random field models for soil properties are generally at most non-stationary in the mean (Fenton and Griffiths 2008). That is because the strength of in-situ soils normally increases with the

depth; the soil strength increases as the effective confining pressure increases. Thus, the trend along the depth should be considered in estimating the spatial variability of the strength of in-situ soils. On the other hand, the strength of the cement-treated soil does not increase with the depth (Namikawa and Koseki 2013) because the strength of cement-treated soil depends on variable binder distribution, as well as in-situ soil properties. Therefore, the trend along the depth is not considered when estimating the statistical uncertainty of the strength parameters of the cement-treated soil columns. Several past studies addressed the consideration of non-Gaussian and non-stationary random fields in Geotechnical engineering (e.g, Montoya-Noguera et al. 2019). Griffiths et al. (2015) conducted the random finite element analysis for non-stationary random fields. Although the methods generating the non-Gaussian and non-stationary random fields could be incorporated in the analysis framework shown in Fig. 1, that topic is beyond the scope of this study.

Honjo (1982) suggested that the exponential-type autocorrelation function describes the correlation characteristics of cement-treated soil strength reasonably well. Several past studies adopted the exponential-type autocorrelation function to describe the spatial correlation of the cement-treated soil strength (e.g., Kasama et al. 2012, Toraldo et al. 2018). Thus, the exponential-type autocorrelation function is adopted as the spatial correlation function of q_u here. However, since the autocorrelation function form affects possibly sample size determination (Guan and Wang 2020), the selection of a proper autocorrelation function is a task for estimating the statistical parameters of q_u . Cao and Wang (2014) has proposed the Bayesian model comparison approach for selecting the most probable correlation function to represent the spatial correlation character of soil properties. That approach could be used to evaluate the proper autocorrelation function for the spatial correlation of q_u . A further challenge remains with respect to selecting properly the autocorrelation function describing the spatial correlation of q_u of the cement-treated ground.

Bayesian inference on statistical parameters

The statistical parameters μ_{qu} , σ_{qu}^2 and θ_{qu} in Eq. (2) are inferred using the Bayesian approach (Namikawa 2019). In this approach, the joint probability distribution of μ_{qu} , σ_{qu}^2 , and θ_{qu} after observing the q_u values is expressed as follows,

$$p(\mu_{qu}, \sigma_{qu}^2, \theta_{qu} | \mathbf{q}_u) \propto p(\mathbf{q}_u | \mu_{qu}, \sigma_{qu}^2, \theta_{qu}) p(\mu_{qu}) p(\sigma_{qu}^2) p(\theta_{qu}) \quad (3)$$

where $\mathbf{q}_u$ denotes the vector of the core strengths, and $p(\mu_{qu})$, $p(\sigma_{qu}^2)$, and $p(\theta_{qu})$ denote the prior distributions of μ_{qu} , σ_{qu}^2 , and θ_{qu} , respectively. In Eq. (3), $p(\mu_{qu}, \sigma_{qu}^2, \theta_{qu} | \mathbf{q}_u)$ is the posterior distribution that represents the probability properties of the parameters under the condition where the data values are known.

As it is difficult to calculate the probability properties directly from the joint probability density function defined by Eq. (3), the MCMC method (Gamerman and Lopes 2006; Gelman et al. 2014) is adopted to draw the values of the parameters from the posterior distributions. In the MCMC simulation, the values of each parameter are sequentially sampled from the conditional posterior distributions. When the prior and posterior distributions belong to the same class of distributions, the realization of the parameter can be directly sampled from the conditional distribution given with the other parameters. Such prior distribution is termed a natural conjugate distribution. Namikawa (2019) selected the normal distribution for the prior distribution of μ_{qu} and the inverse gamma distribution for the prior distribution of σ_{qu}^2 as natural conjugate distributions.

The conditional posterior distribution of μ_{qu} is then expressed as follows:

$$\begin{aligned}
 p(\mu_{qu} | \sigma_{qu}^2, \theta_{qu}, \mathbf{q}_u) &\propto p(\mathbf{q}_u | \mu_{qu}, \sigma_{qu}^2, \theta_{qu}) p(\mu_{qu}) \\
 &\propto \frac{1}{\sqrt{(2\pi)^n (\sigma_{qu}^2)^n |\mathbf{C}|}} \exp \left\{ -\frac{1}{2\sigma_{qu}^2} (\mathbf{q}_u - \mu_{qu})^T \mathbf{C}^{-1} (\mathbf{q}_u - \mu_{qu}) \right\} \frac{1}{\sqrt{2\pi\sigma_{\mu 0}^2}} \exp \left\{ -\frac{(\mu_{qu} - \mu_{\mu 0})^2}{2\sigma_{\mu 0}^2} \right\} \\
 &\propto \exp \left\{ -\frac{(\mu_{qu} - \mu_{\mu})^2}{2\sigma_{\mu}^2} \right\}
 \end{aligned} \tag{4}$$

$$\mu_{\mu} = \frac{2\mu_{\mu 0}\sigma_{qu}^2 + \sigma_{\mu 0}^2 \sum_{i=1}^n \sum_{j=1}^n \zeta_{ij} (q_{ui} + q_{uj})}{2(\sigma_{qu}^2 + \sigma_{\mu 0}^2 \sum_{i=1}^n \sum_{j=1}^n \zeta_{ij})}, \quad \sigma_{\mu}^2 = \left(\frac{1}{\sigma_{\mu 0}^2} + \frac{\sum_{i=1}^n \sum_{j=1}^n \zeta_{ij}}{\sigma_{qu}^2} \right)^{-1}$$

where n is denotes the number of samples, i.e., the statistical sample size, $\mu_{\mu 0}$ and $\sigma_{\mu 0}^2$ denote the mean and variance of $p(\mu_{qu})$, respectively; ζ_{ij} denotes the elements of $\mathbf{C}^{-1}$; and q_{ui} corresponds to q_u at the point i . The conditional posterior distribution of σ_{qu}^2 is expressed as follows:

$$\begin{aligned}
 p(\sigma_{qu}^2 | \theta_{qu}, \mu_{qu}, \mathbf{q}_u) &\propto p(\mathbf{q}_u | \mu_{qu}, \sigma_{qu}^2, \theta_{qu}) p(\sigma_{qu}^2) \\
 &\propto \frac{1}{\sqrt{(2\pi)^n (\sigma_{qu}^2)^n |\mathbf{C}|}} \exp \left\{ -\frac{1}{2\sigma_{qu}^2} (\mathbf{q}_u - \mu_{qu})^T \mathbf{C}^{-1} (\mathbf{q}_u - \mu_{qu}) \right\} (\sigma_{qu}^2)^{-(\alpha_{\sigma 0} + 1)} \exp \left\{ -\frac{\beta_{\sigma 0}}{\sigma_{qu}^2} \right\}
 \end{aligned}$$

$$\propto (\sigma_{qu}^2)^{-(\alpha_\sigma + 1)} \exp\left\{-\frac{\beta_\sigma}{\sigma_{qu}^2}\right\} \quad (5)$$

$$\alpha_\sigma = \frac{n}{2} + \alpha_{\sigma 0}, \quad \beta_\sigma = \beta_{\sigma 0} + \frac{1}{2}(\mathbf{q}_u - \boldsymbol{\mu}_{qu})^T \mathbf{C}^{-1}(\mathbf{q}_u - \boldsymbol{\mu}_{qu})$$

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243 where $\alpha_{\sigma 0}$ and $\beta_{\sigma 0}$ denote the parameters of the inverse gamma distribution for $p(\sigma_{qu}^2)$.

244 The selection of the natural conjugate distribution for the prior distribution of θ_{qu} is
 245 difficult in the framework of this study. The truncated normal distribution is adopted for
 246 the prior distribution of θ_{qu} . The conditional posterior distribution of θ_{qu} is expressed as
 247 follows:

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$$p(\theta_{qu} | \mu_{qu}, \sigma_{qu}^2, \mathbf{q}_u) \propto p(\mathbf{q}_u | \mu_{qu}, \sigma_{qu}^2, \theta_{qu}) p(\theta_{qu})$$

$$\propto \frac{1}{\sqrt{(2\pi)^n (\sigma_{qu}^2)^n |\mathbf{C}|}} \exp\left\{-\frac{1}{2\sigma_{qu}^2}(\mathbf{q}_u - \boldsymbol{\mu}_{qu})^T \mathbf{C}^{-1}(\mathbf{q}_u - \boldsymbol{\mu}_{qu})\right\} \frac{1}{\sqrt{2\pi\sigma_{\theta 0}^2}} \exp\left\{-\frac{(\theta_{qu} - \mu_{\theta 0})^2}{2\sigma_{\theta 0}^2}\right\} \quad (6)$$

251 where $\mu_{\theta 0}$ and $\sigma_{\theta 0}^2$ denote the mean and variance of $p(\theta_{qu})$, respectively.

252 Through the Markov chain algorithm, the values of μ_{qu} , σ_{qu}^2 , and θ_{qu} are sequentially
 253 drawn from the posterior distributions based on the last values drawn. A Gibbs sampler
 254 in which the values are directly sampled from the conditional posterior distributions is
 255 utilized to draw the values of μ_{qu} and σ_{qu}^2 . A Metropolis–Hastings algorithm is utilized
 256 to draw the values of θ_{qu} from the posterior distribution. In the Metropolis–Hastings
 257 method, a random walk algorithm is adopted to select the value of the proposal. The
 258 MCMC framework used here has been described in detail in other publications
 259 (Namikawa 2019; Namikawa 2021a). The algorithm of the MCMC method is
 260 implemented with statistical software R.

261 As mentioned above, the normal distribution is assumed for the probability
 262 distribution of q_u in the present study. In future studies, other distributions will be possibly
 263 selected for the probability distribution of q_u . Since various probability distributions can
 264 be adopted in the Baye's theorem (e.g., Wang et al. 2015), the proposed method allows
 265 other probability distributions for q_u . When adopting the other probability distributions
 266 for q_u , it is difficult to select the natural conjugate distributions for the prior distributions
 267 of μ_{qu} and σ_{qu}^2 in the Bayesian inference method. Then, the Gibbs sampler cannot be used
 268 to draw realizations of μ_{qu} and σ_{qu}^2 and the Metropolis–Hastings algorithm must be used
 269 to draw those realizations in the MCMC method.

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Finite element analysis with random field theory

A random field of q_u following the multivariate normal distribution defined by Eq. (2) is generated using the covariance matrix decomposition method (Fenton and Griffiths 2008). In this method, the random field in the presence of spatial autocorrelation is generated by multiplying a white noise vector with a standard normal distribution $\mathbf{u}$ by a lower triangle of the correlation matrix $\mathbf{C}$. The random field vector $\mathbf{q}_u$ having mean value μ_{qu} , standard deviation σ_{qu} , and autocorrelation distance θ_{qu} is derived as

$$\mathbf{q}_u = \boldsymbol{\mu} + \sigma_{qu} \mathbf{L} \mathbf{u}$$

(7)

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{qu} \\ \vdots \\ \mu_{qu} \end{bmatrix}$$

where $\mathbf{L}$ is lower triangle of $\mathbf{C}$ defined in Eq. (2). When generating $\mathbf{q}_u$, the values of parameters μ_{qu} , σ_{qu} , and θ_{qu} are normally maintained constant in the FEM analysis with random field theory. In this study, $\mathbf{q}_u$ is generated with the parameter values evaluated using the Bayesian inference approach, meaning that the values of μ_{qu} , σ_{qu} , and θ_{qu} vary for each realization.

The random field is generated only using the statistical parameters of q_u here. However, when core samples are retrieved from cement-treated soil columns, their locations and strength are obtained as known data. Such measurements can be used to predict other strengths at unobserved locations. Namikawa (2016) proposed the method of the generating spatial dependence random fields with the data known at some locations and used the generated strength field for the FEM-analysis of the cement-treated column from which core samples are retrieved. Finite-element analysis with conditional simulation predicted the overall strength of a targeted cement-treated soil column appropriately, and the overall strength variability obtained with the conditional simulation was lower than that obtained with the unconditional simulation. The results suggested that the conditional simulation offers advantages over the unconditional simulation when assessing the probability of failure of a cement-treated soil column having the core strength data. The conditional simulation would be possible to incorporate in the analysis framework shown in Fig. 1. However, a significant challenge remains with respect to the generation of random strength fields of columns from which core samples are not retrieved. In practical projects, the core samples are retrieved from a few columns, that is, almost all of columns have no core boring data. When generating the random strength fields of columns having no core boring from the core strength data conditionally, the spatial correlation in strength

between the columns is required. Since the strength of a cement-treated soil column depends on the variable binder distribution, as well as the in-situ soil variability (Larsson et al. 2005), the correlation characteristics in strength between the columns could not be simply determined from the distance. Further research is required to generate the random strength fields of columns having no core boring from the core strength data conditionally.

The random variables represented by $\mathbf{q}_u$ are assigned to elements in the FEM analysis. Three-dimensional finite element analyses are performed using the FEM software DIANA. An elasto-plastic model developed by Namikawa and Mihira (2007) is used to simulate the failure behavior of cement-treated ground. This model can appropriately describe tensile and shear failure behaviors of cement-treated soils. Several material parameters are required for this elasto-plastic model. The material parameter values for each element are determined based on the assigned q_u values. Then, the three-dimensional finite element analyses are performed for each realizations of the random field of q_u .

PROBABILISTIC SIMULATIONS

Core strength data

The probabilistic simulations are conducted for various core-sampling conditions to investigate the influence of n on the evaluation of unconfined compressive strengths Q_u of a full-scale cement-treated soils columns. The core strength data used in the simulations are generated using the covariance matrix decomposition method expressed in Eq. (7); that is, the core strength data here used are not real data obtained in practical projects. The core boring lengths L_b are set to 3, 6 and 9 m. The values of the sample mean μ_{squ} and the sample standard deviation σ_{squ} are set to 4.0 and 1.2 MPa. The values of the sample autocorrelation distance θ_{squ} are set to 0.2, 0.5, 1.0, and 2.0 m. These values are selected based on the real core strength data reported in past studies (Saitoh et al. 1996; Namikawa and Koseki 2013). A generated example distribution of q_u for $L_b = 9$ m with $\theta_{squ} = 1$ m is shown in Fig. 2. Assuming a height of 100 mm for the specimens, the q_u values are generated at intervals of 100 mm.

It should be noted that the generated core strength data is used to evaluate the statistical uncertainty in estimating the statistical parameters. Thus, the analysis results in this studied demonstrate the general feature of its influence of the statistical uncertainty on the evaluation of the overall strength of a cement-treated soil column. The real core strength data should be adopted to study its influence in practical projects. Namikawa (2021b) adopted real core strength data in the analysis framework shown in Fig.1. Further detail can be found in the reference (Namikawa 2021b).

The current quality assurance program of deep mixing method in Japan requires three core borings when the total number of columns is less than 500 (Kitazume and Terashi

2013). In each core boring, three core barrels are selected from three different levels, and three specimens are taken from each core barrel to measure q_u values. 9 core strength values are obtained for each core boring and 27 core strength values ($n = 27$) are used to verify the quality of cement-treated columns.

In this study, the number of core borings nb is set to be 1, 3, and 6 and nine core specimens are assumed to be taken from each core boring. Thus, the influence of the statistical sample size n on evaluating the overall strength of a cement-treated soil column is examined for $n = 9, 27$, and 54. The locations of the core specimens are randomly selected from the generated core strength data, as shown in Fig. 2. The locations are determined so that the values of the statistical parameters of the strengths at the selected location correspond to the target values.

The simulation cases are summarized in Table 1. The analysis is mainly conducted for the case of $L_b = 9$ m. In this case, the analysis is performed for $\theta_{squ} = 0.2, 0.5, 1.0$, and 2.0 m to investigate the influence of the spatial correlation. For the case of $L_b = 3$ m and 6 m, the analysis is performed for $\theta_{squ} = 0.5$ m to investigate the influence of the sampling interval on the analytical results. In Table 1, the case of $n = \infty$ indicates the situation where the statistical uncertainty is neglected. For the case of $n = \infty$, μ_{qu} , σ_{qu}^2 , and θ_{qu} are maintained constant when generating the random field realizations for the FEM analysis.

Statistical uncertainty of parameters

The statistical uncertainty of μ_{qu} , σ_{qu}^2 , and θ_{qu} are evaluated using the Bayesian inference method described in the previous section. In the Bayesian inference, the parameters of the prior distributions are required to draw the realization values from the posterior distribution. Namikawa (2019) determined the parameters of the prior distributions for μ_{qu} , σ_{qu} , and θ_{qu} from the core strength data obtained in the past deep mixing projects. In the present study, the mean values of the prior distributions for μ_{qu} , σ_{qu}^2 , and θ_{qu} are set to correspond to those of the generated core strength data to avoid the influence of the parameters of the prior distributions; that is, $\mu_{\mu 0}$ is 4 MPa and the mean $\mu_{var 0}$ of the prior distributions for σ_{qu}^2 is 1.44. $\mu_{\theta 0}$ is selected to correspond to the θ_{squ} values of core strength. The variance values of the prior distribution parameters are determined from the core strength data reported in a past study. Saitoh et al. (1996) reported the sample mean and variance of the core strength obtained at 27 practical projects. This data is shown in Fig. 3. The variance of the sample mean μ_{squ} is 4.3 and that of the sample variance σ_{squ}^2 is 1.2. In this study, the variance $\sigma_{\mu 0}^2$ of the prior distribution of μ_{qu} is determined as $\sigma_{\mu 0}^2 = 4$ and that of σ_{qu}^2 is determined as $\sigma_{var 0}^2 = 1$. Although the variance $\sigma_{\theta 0}^2$ of θ_{qu} calculated from the data reported in past studies is around 0.6 (Namikawa and Koseki 2013, Namikawa 2012), the coefficient of variation

of the parameters of the prior distributions of θ_{qu} is assumed to be 1 to avoid the influence of the prior distribution on the analytical results. The parameter values of the prior distributions could affect the estimated values of the statistical parameters. That influence on the estimated statistical parameters will be examined below.

In all the cases, the 11000 values of each variable, μ_{qu} , σ_{qu}^2 , and θ_{qu} were sampled using the MCMC method. The example realization values of μ_{qu} , σ_{qu}^2 , and θ_{qu} estimated using the MCMC method for Bayesian inference are shown in Fig. 4. This figure shows the realization values for the case of $L_b = 9$ m with $n = 27$ when $\theta_{squ} = 1$ m. The estimated values of μ_{qu} , σ_{qu}^2 , and θ_{qu} vary significantly, indicating that the statistical parameters estimated from the data with $n = 27$ have a large amount of the statistical uncertainty.

In the MCMC simulation, the starting values influence the initial sequence of the sampled values. In the following analyses, the values sampled during the initial 1000 iterations are discarded when estimating the statistical properties of the parameters, to eliminate the influence of the starting values. The number of samples used in the analysis is 10000. The standard deviation $SD(\mu_{qu})$ of the realization values of μ_{qu} for $L_b = 9$ m are summarized in Fig. 5. $SD(\mu_{qu})$ representing the variability of the estimated μ_{qu} value decreases as n increases. Fig. 5 also shows that $SD(\mu_{qu})$ increases as θ_{squ} increases. The large value of the autocorrelation distance induces a strong correlation between the data values (Namikawa 2019). When θ_{squ} increases, the equivalent number of independent data decreases, and the variability of the estimated parameter increases. The standard deviation $SD(\sigma_{qu}^2)$ of the realization values of σ_{qu}^2 for $L_b = 9$ m is shown in Fig. 6. Similar to the results for $SD(\mu_{qu})$, $SD(\sigma_{qu}^2)$ decreases as n increases, and $SD(\sigma_{qu}^2)$ increases as θ_{squ} increases.

The Bayesian inference analysis with the different parameter values of the prior distribution is conducted to demonstrate the influence of those on the estimation of the statistical parameters. In the analysis, the influence of $\mu_{\mu 0}$ and $\mu_{var 0}$ on the inferred statistical parameters is examined by changing these values. The analysis is conducted with $\mu_{\mu 0} = 1$ MPa for the case of $L_b = 9$ m when $\theta_{squ} = 0.5$ m to investigate the influence of $\mu_{\mu 0}$ on the estimated statistical parameter values. In the analysis, the other parameter values of the prior distribution are not changed. The mean $E(\mu_{qu})$ of the realization values of μ_{qu} is shown in Fig. 7. The $E(\mu_{qu})$ values for $\mu_{\mu 0} = 1$ MPa is smaller than those for $\mu_{\mu 0} = 4$ MPa, indicating that the estimated μ_{qu} values decrease as the $\mu_{\mu 0}$ value decreases. Fig. 7 also shows that the discrepancy between the $E(\mu_{qu})$ values for $\mu_{\mu 0} = 1$ MPa and $\mu_{\mu 0} = 4$ MPa increases as the sample size n decreases. This indicates that the influence of the parameters of the prior distribution on the estimation of the statistical parameters becomes larger when n is small.

The Bayesian inference analysis is conducted with $\mu_{var 0} = 4$ and $\sigma_{var 0}^2 = 4$ for the case

of $L_b = 9$ m when $\theta_{squ} = 0.5$ m to investigate the influence of the parameters of the prior distribution for the variance on the estimated statistical parameter values. In the analysis, the other parameter values of the prior distribution are not changed. The mean $E(\sigma_{qu}^2)$ of the realization values of σ_{qu}^2 is shown in Fig. 8. The $E(\sigma_{qu}^2)$ values for $\mu_{var0} = 4$ is larger than those for $\mu_{var0} = 1.44$, indicating that the estimated σ_{qu}^2 values increases as the μ_{var0} value increases. Similar to the result of the influence of $\mu_{\mu0}$, the discrepancy between the $E(\sigma_{qu}^2)$ values for $\mu_{var0} = 4$ and $\mu_{var0} = 1.44$ increases as the sample size n decreases.

Figs. 7 and 8 indicate that the parameters of the prior distribution affect the estimation of the statistical parameters. Therefore, the evaluation of the parameters of the prior distribution is important for estimating the statistical uncertainty properly. The core strength data obtained in some practical projects are used to determine the prior distribution parameters. More core strength data are required to evaluate the parameters of the prior distribution more appropriately. The cement-treated soil columns have been constructed using various execution techniques. A further study is required to build the databases of core strength for each deep mixing technique. Moreover, the following sections will show that the FEM analysis result for the overall strength Q_u of a cement-treated soil column depends on the estimated statistical uncertainty estimated by the Bayesian inference. Thus, the parameters of the prior distribution could affect the evaluation of Q_u . Namikawa (2021b) studied the effect of the statistical uncertainty evaluated from the real core sample strength on Q_u . That study revealed that the parameters of the prior distribution affect the calculated Q_u values. The influence of the prior distribution on the evaluation of the overall behavior of a column is a topic for future research.

Finite element analysis condition

A three-dimensional FEM analysis is performed to investigate the influence of the statistical sample size on the evaluation of the overall strength Q_u of a full-scale cement-treated soil column. An unconfined compression behavior of a cement-treated soil column is simulated using the FEM software DIANA. Namikawa and Koseki (2013) conducted the FEM analysis for a cylindrical model of diameter 1 m and height 2 m as a full-scale cement-treated column. The same model is used in this study. The FEM mesh is shown in Fig. 9. A mesh consists of eight-node isoparametric elements, and the majority of the elements are cubic with a length of 100 mm. The element size in the FEM analysis corresponds to the specimen size assumed in the generated core boring data. The boundary conditions at the top and bottom surfaces are smooth. Uniform vertical displacements are applied at the upper surface on the loading process. The imposed displacement value of the incremental step interval is 0.005 mm/step.

An elastoplastic model describing the tensile and shear behaviors of cement-treated soils is used in the FEM analysis (Namikawa and Mihira 2007). The program of the elastoplastic model was installed using DIANA software as the user subroutine. The formulation and performance of the model have been described in detail in other publications (Namikawa and Mihira 2007; Namikawa and Koseki 2013). The parameter values of the elastoplastic model for $q_u = 4.0$ MPa are listed in Table 2. These parameters were determined from the results of various laboratory tests conducted in previous studies (Namikawa and Koseki 2006; Namikawa and Mihira 2007). It is assumed that the friction angle ϕ is held constant and cohesion c varies with q_u . When ϕ is held constant, $c = q_u(1 - \sin\phi)/(2\cos\phi)$. The elastic modulus E , tensile strength T_f , and fracture energy G_f are regarded as the stochastic parameters. Several literatures (e.g., Kitazume and Terashi 2013, Filz et al. 2015) reported that E and T_f vary proportionally with q_u . It is assumed that E and T_f are proportional to q_u . Based on the experimental results (Namikawa and Mihira 2007), E / q_u is set to be 1765 and T_f / q_u is set to be 0.2235. G_f is defined as the energy consumed during the tensile failure process. Zaman et al. (2009) performed a three-point bending test to investigate G_f of a stabilized soil. Their experimental results show that G_f is approximately proportional to q_u . It is assumed that G_f is proportional to the value of q_u . Based on the experimental results (Namikawa and Mihira 2007), G_f / q_u is set to be 0.00529 mm. Other parameters, such as Poisson's ratio ν , hardening parameters α and e_y , softening parameter e_r , dilatancy coefficient D_c , localization size t_{s0} , and characteristic length l_c , are assumed to be maintained constant at the values listed in Table 2. When the regression relationships based on the experimental results are used for the evaluation of the material parameters, those relationships affect possibly the FEM analysis results. However, since the overall strength Q_u mainly depends on the q_u values assigned to elements, it is conceivable that the other material parameters do not significantly affect Q_u calculated in the FEM analysis. This study will not examine the influence of the regression relationships on the FEM analysis results.

One hundred realizations are used in the FEM analysis for all cases listed in Table 1. Although one hundred realizations are not sufficient to ignore the error involved in the Monte Carlo simulation results, the number of realizations was determined considering the balance between the reliability of the simulation result and the calculation cost. When considering the statistical sample size, the statistical parameters in the FEM analysis are determined from the realization values generated by the MCMC method. Thus, the values of μ_{qu} , σ_{qu}^2 , and θ_{qu} vary at each realization. The values of μ_{qu} , σ_{qu}^2 , and θ_{qu} adopted for the case of $L_b = 9$ m with $n = 27$ when $\theta_{squ} = 1$ m are shown in Fig. 10. The values during the initial 1000 iterations are discarded when selecting the statistical parameters, and 100 values are randomly selected from the drawn values. The FEM analysis in which the

statistical uncertainty is not considered is also conducted for the case of $n = \infty$. In this analysis, the statistical parameters of q_u are maintained constant as $\mu_{qu} = 4$ MPa, $\sigma_{qu} = 1.2$ MPa, and $\theta_{qu} = 0.2$ m, 0.5 m, 1.0 m and 2.0 m. In the FEM analysis, q_u is classified into discrete classes whose intervals are set to 0.12 MPa. The same white noise vector $\mathbf{u}$ is used when generating a random field in the analyses in all cases. The typical realizations of the full-scale column for $\theta_{squ} = 1.0$ m are shown in Fig. 11. Fig. 11(a) shows the realization for $n = \infty$, and Fig. 11(b) shows that for $L_b = 9$ m with $n = 27$. Since the same white noise vector $\mathbf{u}$ is used, Fig. 11 shows a minor discrepancy between the strength distributions for $n = \infty$ and $n = 27$.

Analysis results

The typical calculated stress-strain relationships for $\theta_{squ} = 1.0$ are shown in Fig. 12. These results are for the realizations shown in Fig. 11. The analysis result with consideration of the sample size differs slightly from that for an infinite sample size.

The overall strength Q_u of the column for the case of $L_b = 9$ m with $\theta_{squ} = 1.0$ m is summarized in Fig. 13. Q_u is defined as the peak axial stress in the stress-strain relationship. Q_u varies widely even for $n = \infty$ owing to the spatial variability of q_u . The variability of Q_u increases as n decreases. The example histogram of Q_u for the case of $L_b = 9$ m with $n = 27$ when $\theta_{squ} = 0.2$ m and 1.0 m is shown in Fig. 14. It can be observed that the calculated Q_u values follow a normal distribution. The Kolmogorov-Smirnov (K-S) test is performed to evaluate the goodness of fit for the normal distribution against Q_u (Kanji 2006). The D statistic is defined as the maximum difference between the data and the target distribution. The calculated values of the D statistic are around 0.06 in all cases. The 10% critical value of the D statistic is 0.122 for the sample size of 100 in the K-S test. The calculated values of the D statistic are lower than the critical value, indicating that the null hypothesis that Q_u follows the normal distribution can be accepted at the 10% significant level.

The sample mean $E(Q_u)$ of Q_u for $L_b = 9$ m is shown in Fig. 15. In this figure, $E(Q_u)$ is normalized by $\mu_{squ} = 4$ MPa, and the lateral axis $E(\theta_{qu})$ denotes the average of the realizations of θ_{qu} used in the FEM analysis. Fig. 15 shows that $E(Q_u)$ varies with the autocorrelation distance. The minimum value of $E(Q_u)$ is observed at around $E(\theta_{qu}) = 0.5$ m. It can be observed that the variability of $E(Q_u)$ for $n = 9$ is larger than that for the other cases. However, the influence of n on $E(Q_u)$ cannot be observed in Fig. 15, indicating that the sample size does not considerably affect the evaluation of $E(Q_u)$.

The sample standard deviation $SD(Q_u)$ of Q_u for $L_b = 9$ m is shown in Fig. 16. In this figure, $SD(Q_u)$ is normalized by $\sigma_{squ} = 1.2$ MPa, and the lateral axis represents $E(\theta_{qu})$. $SD(Q_u)$ increases as the autocorrelation distance increases. The same result was observed

in Namikawa and Koseki (2013). Fig. 16 shows that $SD(Q_u)$ decreases with n for the same autocorrelation distance. This result indicates that the variability of the evaluated Q_u values increases as n decreases.

$SD(Q_u)$ as a function of the core boring length L_b for $\theta_{squ} = 0.5$ m is shown in Fig. 17. This figure shows the effect of the sampling distance on the evaluation of $SD(Q_u)$. It can be observed that $SD(Q_u)$ decreases as L_b increases, indicating that the variability of the evaluated Q_u decreases as the sampling distance increases. This is because the equivalent number of the independent data n' increases as the sampling distance increases under the same spatial correlation condition.

It should be noted that $E(Q_u)$ and $SD(Q_u)$ are the sample statistical parameters obtained from 100 realizations; $E(Q_u)$ and $SD(Q_u)$ do not exactly correspond to the population statistical parameters. Although these values include the statistical uncertainty, it is assumed in the following sections that $E(Q_u)$ and $SD(Q_u)$ reasonably represent the population mean and standard deviation of Q_u . $E(Q_u)$ and $SD(Q_u)$ are then used as the population parameters to examine the influence of the sample size on Q_u . Analyses with a large number of realizations are required to obtain more precise solutions in future studies.

Application of analysis results

The analysis results indicated that the statistical sample size affects the variability of the evaluated Q_u values. Such influence of the sample size should be considered in the quality assurance procedures of the cement-treated soil columns. The application of analysis results to the quality assurance procedures is discussed in this section.

Figure 16 showed that n affects $SD(Q_u)$, whereas Fig. 15 showed that n does not considerably affects $E(Q_u)$. These results suggest that the influence of n on $SD(Q_u)$ should be considered in the quality assurance procedures. The influence of n could be associated with the amount of the statistical uncertainty involved in μ_{qu} , σ_{qu}^2 and θ_{qu} estimated in the Bayesian inference. Here, the standard deviation of μ_{qu} , σ_{qu}^2 and θ_{qu} that expresses the amount of the statistical uncertainty is examined before evaluating the effect of n on $SD(Q_u)$. Fig. 18 shows the comparison between $SD(\mu_{qu})$ and $SD(\sigma_{qu}^2)$ for all the cases. A unique relationship between $SD(\mu_{qu})$ and $SD(\sigma_{qu}^2)$ is observed, indicating that the effect of the variability of σ_{qu}^2 estimated in the Bayesian inference can be represented by $SD(\mu_{qu})$. $SD(\mu_{qu})$ and the standard deviation $SD(\theta_{qu})$ of θ_{qu} are shown in Fig. 19. Almost linear relationship is recognized in the relationship between $SD(\mu_{qu})$ and $SD(\theta_{qu})$ for each θ_{squ} , indicating that the effect of the variability of θ_{qu} can be represented by $SD(\mu_{qu})$ for each autocorrelation distance.

Figures 18 and 19 show that the standard deviation of σ_{qu}^2 and θ_{qu} is related to

SD(μ_{qu}). This suggest that the effect of the statistical uncertainty of the three parameters is possibly represented by SD(μ_{qu}). The relationship between SD(μ_{qu}) and SD(Q_u) for $L_b = 9$ m is shown in Fig. 20. In this figure, the values of SD(μ_{qu}) and SD(Q_u) are normalized by $\sigma_{squ} = 1.2$ MPa. For each θ_{squ} , SD(Q_u) almost linearly increases as SD(μ_{qu}) increases. The incremental standard deviation Δ SD(Q_u) of Q_u regarding the statistical uncertainty is defined as

$$\Delta$$
SD(Q_u) = SD(Q_u) – SD(Q_u) _{$n = \infty$} (8)

where SD(Q_u) _{$n = \infty$} is the SD(Q_u) evaluated without the statistical uncertainty. The relationship between SD(μ_{qu}) and Δ SD(Q_u) for all cases is shown in Fig. 21. In this figure, the values of SD(μ_{qu}) and Δ SD(Q_u) are normalized by $\sigma_{squ} = 1.2$ MPa. The Δ SD(Q_u) almost linearly increases as SD(μ_{qu}) increases. Although the inclination of the relationship between SD(μ_{qu}) and Δ SD(Q_u) for $\theta_{squ} = 2.0$ m is slightly smaller than that for the other cases, a unique linear line can be observed for the relationship between SD(μ_{qu}) and Δ SD(Q_u) when θ_{squ} is less than 2.0 m. Fig. 21 suggests that Δ SD(Q_u) could be reasonably estimated from SD(μ_{qu}) calculated using the Bayesian inference method.

Namikawa (2019) has studied the relationship between SD(μ_{qu}) and the statistical sample size. In that study, the equivalent number of the independent data n' was adopted as the effective sample size depending on the spatial correlation of the data. n' is defined as the sample size with regard to the standard deviation of the estimator of the mean (Cressie 1993). n' is expressed as

$$n' = \frac{n^2}{\sum_{i=1}^n \sum_{j=1}^n C_{ij}} \quad (9)$$

where C_{ij} represents the components of $\mathbf{C}$ defined in Eq. (2). C_{ij} is calculated from the distances between the sampling locations and θ_{qu} . When θ_{qu} approaches 0, n' approaches n , whereas when θ_{qu} approaches infinity, n' approaches 1. The relationship between $1/\sqrt{n'}$ and SD(μ_{qu}) estimated using the Bayesian inference is shown in Fig. 22. In this figure, the θ_{squ} values are used for calculating the C_{ij} values in Eq. (9). Fig. 22 shows the linear relationship between SD(μ_{qu}) and $1/\sqrt{n'}$. This suggests that n' can be used instead of SD(μ_{qu}) to evaluate Δ SD(Q_u).

When the strength distribution has the spatial correlation, not only the sample size but also sampling locations affects the evaluation of the statistical uncertainty. Several studies proposed the method for selecting efficient sampling locations for geotechnical

site characterization (Zhao and Wang 2019, Guan et al. 2020). Those studies showed that the efficient sampling locations are objectively determined by integrating the concept of information entropy and Bayesian compressive sampling method. In this study, the locations of the core specimens are randomly selected from the generated core strength data and n' is adopted to consider the effect of the core sampling location intervals. Although this study won't focus on the individual locations of the core sample, the investigation of the influence of the core sample locations on the evaluation of Q_u is a topic for future research.

The relationship between $1/\sqrt{n'}$ and $\Delta SD(Q_u)$ is shown in Fig. 23. $\Delta SD(Q_u)$ almost linearly increases with $1/\sqrt{n'}$. Although the inclination of the relationship for $\theta_{squ} = 2.0$ m is slightly smaller than that for the other cases, a unique linear line could be observed for the relationship between $1/\sqrt{n'}$ and $\Delta SD(Q_u)$ when θ_{squ} is less than 2.0 m. The linear regression relationship between $1/\sqrt{n'}$ and $\Delta SD(Q_u)$ is expressed as

$$\frac{\Delta SD(Q_u)}{\sigma_{squ}} = \frac{A}{\sqrt{n'}} \quad (10)$$

where A is the coefficient of the regression relationship. Based on the analytical results, the A value is calculated to be approximately 0.65 for $\theta_{squ} \leq 1.0$ m. The regression relationship is plotted in Fig. 23. Substituting Eq. (10) into Eq. (8), $SD(Q_u)$ can be expressed as

$$SD(Q_u) = SD(Q_u)_{n=\infty} + \frac{A}{\sqrt{n'}} \sigma_{squ} \quad (11)$$

In current quality assurance procedures for the deep mixing method (e.g. CDIT 2002; Kitazume and Terashi 2013), a representative strength R_{qu} is compared with the design strength F_c to assure the quality of cement-treated soil columns using Eq. (1). Assuming q_u follows a normal distribution, K is normally set to be 1.3 for 10 % defects. When the sample mean μ_{squ} and standard deviation σ_{squ} are used in Eq. (1), the statistical uncertainty depending on the sample size and the spatial correlation must be considered in determination of the K value.

Considering the defect rate of Q_u , the variation of K with the statistical sample size is discussed. The probability of failure is defined here as the probability that Q_u is less than R_{qu} . When assuming that Q_u follows a normal distribution, the probability that Q_u is less than R_{qu} can be easily evaluated using the normalizing transformation in which R_{qu} is transformed to the standard normal value Z_R . Z_R is defined as

$$(12)$$

$$Z_R = \frac{R_{qu} - E(Q_u)}{SD(Q_u)} = \frac{\mu_{squ} - K\sigma_{squ} - E(Q_u)}{SD(Q_u)}$$

636

637 The same value of Z_R provides the same probability of failure.

638 Figs. 15 and 16 suggest that the mean and standard deviation of Q_u depend on the
 639 autocorrelation distance. Here, it is assumed that $E(Q_u) / \mu_{squ}$ and $SD(Q_u) / \sigma_{squ}$ at $n = \infty$
 640 are expressed as a function of θ_{qu} that corresponds to $E(\theta_{qu})$ at $n = \infty$.

641

$$\frac{E(Q_u)_{n=\infty}}{\mu_{squ}} = f(\theta_{qu}), \quad \frac{SD(Q_u)_{n=\infty}}{\sigma_{squ}} = g(\theta_{qu}) \quad (13)$$

643

644 where $f(\theta_{qu})$ and $g(\theta_{qu})$ are the approximate functions. Figs. 15 and 16 show examples of
 645 $f(\theta_{qu})$ and $g(\theta_{qu})$ for $n = \infty$, where $\theta_{qu} = E(\theta_{qu})$. These functions are expressed as follows:

646

$$f(\theta_{qu}) = 1 - \frac{(\theta_{qu})^a \exp\left(-\frac{\theta_{qu}}{b}\right)}{c} \quad (14)$$

$$g(\theta_{qu}) = \frac{\theta_{qu}}{\theta_{qu} + d} \quad (15)$$

649

650 where a , b , c , and d are the coefficient of the approximate functions; $a = 0.37$, $b = 2.0$, c
 651 $= 3.6$, and $d = 0.43$ for $n = \infty$ in Figs. 15 and 16. Toraldo et al. (2018) examined the
 652 relationship between the autocorrelation distance and the standard deviation of the overall
 653 strength of the cement-treated soil. They proposed the exponential-type function for
 654 expressing the standard deviation in terms of the autocorrelation distance. Although other
 655 functions may be suitable for predicting the statistical parameters of the overall strength
 656 from the autocorrelation distance, the approximate functions will not be examined in more
 657 detail in this study.

658 As mentioned above, Fig. 15 indicates that $E(Q_u) / \mu_{squ}$ does not evidently depend
 659 on n . Thus, it is assumed that $E(Q_u) / \mu_{squ}$ is expressed by $f(\theta_{qu})$.

660

$$\frac{E(Q_u)}{\mu_{squ}} = \frac{E(Q_u)_{n=\infty}}{\mu_{squ}} = f(\theta_{qu}) \quad (16)$$

662

663 Substituting Eqs. (11), (13), and (16) into Eq. (12), Z_R is expressed as

664

$$Z_R = \frac{V_{squ}\{1 - KV_{squ} - f(\theta_{qu})\}}{\{g(\theta_{qu}) + A/\sqrt{n'}\}}$$

666

667 where V_{squ} is the sample coefficient of variation ($V_{squ} = \sigma_{squ}/\mu_{squ}$). Eq. (17) indicates
 668 that Z_R varies with n' , i.e., the probability of failure varies with n' . Eq. (17) suggests that
 669 K should be changed with n' to maintain the failure probability constant.

670 The K value is examined for various sample sizes to obtain the same probability of
 671 failure. When a regulation provides $K = K_R$ for the regulated sample size n_R , the K value
 672 should be changed for a different sample size n_I to obtain the same value of Z_R . With Eq.
 673 (17), the K value for n_I can be evaluated as

674

$$K_1 = \frac{g(\theta_{qu}) + \frac{A}{\sqrt{n_1'}}}{g(\theta_{qu}) + \frac{A}{\sqrt{n_R'}}} K_R + \frac{1 - f(\theta_{qu})}{V_{squ}} \left\{ 1 - \frac{g(\theta_{qu}) + \frac{A}{\sqrt{n_1'}}}{g(\theta_{qu}) + \frac{A}{\sqrt{n_R'}}} \right\} \quad (18)$$

676

677 where K_1 is the K value for n_1 , and n_R' and n_1' are the equivalent number of independent
 678 data for n_R and n_1 , respectively. When statistical uncertainty is not considered in the
 679 regulation, n_R is assumed to be infinity. Then, K_1 is represented as

680

$$K_1 = \left\{ 1 + \frac{A}{g(\theta_{qu})\sqrt{n_1'}} \right\} K_R - \frac{1 - f(\theta_{qu})}{V_{squ}} \frac{A}{g(\theta_{qu})\sqrt{n_1'}} \quad (19)$$

682

683 When $f(\theta_{qu})$ and $g(\theta_{qu})$ are given in Eqs. (18) and (19), the K value can be suitably
 684 determined for various statistical sample sizes.

685 The method to consider the sample size influence on the probability of overall failure
 686 has been discussed based on the current quality assurance procedures. In the current
 687 regulations (e.g., CDIT 2002), Eq. (1) defining F_c is proposed by considering the defect
 688 ratio of the core strength. However, the relationship between the defect ratio of core
 689 samples and the overall failure of the cement-treated column has not been investigated in
 690 previous studies on the deep mixing method. The relation between the variability of the
 691 core strength and the overall strength is required to determine the F_c value rationally. The
 692 analytical method presented in this study enables us to consider the spatial variability and
 693 the statistical sample size simultaneously for evaluating the overall behavior of the
 694 cement-treated column under various boundary conditions. The framework using the

presented analytical method is a topic for future research on the various boundary value problems of cement-treated soil ground.

Conclusions

This paper has investigated the influence of the statistical sample size n on the evaluation of the overall strength Q_u of a cement-treated soil column with the spatial variability of unconfined compressive strength q_u . A framework in which the statistical uncertainty and the spatial variability are considered simultaneously was used to evaluate the overall behavior of the column based on the core strength data. The statistical uncertainty representing the influence of the sample size on the statistical parameters, i.e., mean μ_{q_u} , variance $\sigma_{q_u}^2$, and autocorrelation distance θ_{q_u} , was evaluated using Bayesian inference. In the Bayesian inference procedure, the realizations of the statistical parameters were drawn by the Markov chain Monte Carlo method. The drawn realizations of the statistical parameters were adopted for generating the random fields representing the spatial variation of q_u in cement-treated soil column. The FEM analysis was conducted the cement-treated soil column model associated with the generated random fields to evaluate the overall behavior of the column with the spatial variability of q_u .

The analysis conducted for various sampling conditions revealed that the sample size and the spatial correlation affect the variability of Q_u as the statistical uncertainty depends on them. The equivalent number of the independent data n' was introduced to describe the influence of the sample size and spatial correlation on the variability of Q_u adequately. The incremental standard deviation $\Delta SD(Q_u)$ of Q_u induced by the statistical uncertainty increases proportionally with $1/\sqrt{n'}$. This result suggests that the effect of the sample size on the variability of Q_u could be reasonably estimated from n' . The influence of the statistical sample size on the overall strength was quantitatively evaluated using the regression relationship between $1/\sqrt{n'}$ and $\Delta SD(Q_u)$.

The analytical framework presented in this study enables us to consider the spatial variability and statistical sample size simultaneously for evaluating the overall behavior of a cement-treated column. This approach provides a rational evaluation of the performance of cement-treated ground whose strength varies spatially. In the future, applications of the presented analytical framework to the practical design of cement-treated ground will be investigated.

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Table 1 Simulation cases

L_b	nb	n	μ_{squ}	σ_{squ}	θ_{squ}
3 m	1, 3, 6	9, 27, 54	4 MPa	1.2 MPa	0.5 m
6 m	1, 3, 6	9, 27, 54	4 MPa	1.2 MPa	0.5 m
9 m	1, 3, 6	9, 27, 54	4 MPa	1.2 MPa	0.2 m, 0.5 m, 1.0 m, 2.0 m
-	-	∞	4 MPa	1.2 MPa	0.2 m, 0.5 m, 1.0 m, 2.0 m

Note: L_b = boring length; nb = number of borings; n = sample size; μ_{squ} = sample mean; σ_{squ} = sample standard deviation; θ_{squ} = sample autocorrelation distance.

Table 2 Material parameters for cement-treated soil with the core strength $q_u = 4.0$ MPa

Parameter	Stochastic or Deterministic	Value
E (MPa)	Stoch.	7060
ν	Deter.	0.167
ϕ (degree)	Deter.	30
c (MPa)	Stoch.	1.15
T_f (MPa)	Stoch.	0.894
α	Deter.	1.05
e_y	Deter.	0.0002
G_f (N/m)	Stoch.	21.2
e_r	Deter.	0.40
D_c	Deter.	-0.4
t_{s0} (mm)	Deter.	0.6
l_c (mm)	Deter.	100

Note: E = elastic modulus; ν = Poisson's ratio; ϕ = friction angle; c = cohesion; T_f = tensile strength; α = hardening parameter; e_y = hardening parameter; G_f = fracture energy; e_r = softening parameter; D_c = dilatancy coefficient; t_{s0} = localization size; l_c = characteristic length; Stoch. = stochastic parameter; Deter. = deterministic parameter.

Figure caption

Fig. 1 Framework for considering statistical uncertainty and spatial variability.

Fig. 2 Example of the generated strength distributions and location of core sampling for $\theta_{squ} = 1.0$ m.

Fig. 3 Sample mean μ_{squ} and sample variance σ_{squ}^2 of the core strength in practical projects (modified from Saitoh et al. 1996).

Fig. 4 Example realization values of the statistical parameters estimated using the MCMC method ($L_b = 9$ m, $n = 27$, $\theta_{squ} = 1$ m): (a) mean μ_{qu} ; (b) standard deviation σ_{qu}^2 ; (c) autocorrelation distance θ_{qu} .

Fig. 5 Standard deviation $SD(\mu_{qu})$ of mean μ_{qu} for $L_b = 9$ m.

Fig. 6 Standard deviation $SD(\sigma_{qu}^2)$ of variance σ_{qu}^2 for $L_b = 9$ m.

Fig. 7 Influence of the mean $\mu_{\mu 0}$ of prior distribution of μ_{qu} on the mean $E(\mu_{qu})$ of realization values of μ_{qu} ($L_b = 9$ m, $\theta_{squ} = 0.5$ m).

Fig. 8 Influence of the mean $\mu_{\sigma^2 0}$ of prior distribution of σ_{qu}^2 on the mean $E(\sigma_{qu}^2)$ of realization values of σ_{qu}^2 ($L_b = 9$ m, $\theta_{squ} = 0.5$ m).

Fig. 9 Finite element mesh of a full-scale column.

Fig. 10 Example of the statistical parameters used in generating random field realizations ($L_b = 9$ m, $n = 27$, $\theta_{squ} = 1$ m): (a) mean μ_{qu} ; (b) standard deviation σ_{qu}^2 ; (c) autocorrelation distance θ_{qu} .

Fig. 11 Typical realizations of strength for the finite element model ($L_b = 9$ m, $\theta_{squ} = 1.0$ m).

Fig. 12 Stress-strain relationships for typical realizations ($L_b = 9$ m, $\theta_{squ} = 1.0$ m).

Fig. 13 Summary of the overall strength Q_u for the case of $L_b = 9$ m with $\theta_{squ} = 1.0$ m.

Fig. 14 Histogram of the overall strength Q_u for the case of $L_b = 9$ m with $n = 27$: (a) $\theta_{squ} = 0.2$ m; (b) $\theta_{squ} = 1.0$ m.

Fig. 15 Relationship between the mean $E(\theta_{qu})$ of θ_{qu} and mean $E(Q_u)$ of Q_u for the case of $L_b = 9$ m.

Fig. 16 Relationship between the mean $E(\theta_{qu})$ of θ_{qu} and standard deviation $SD(Q_u)$ of Q_u for the case of $L_b = 9$ m.

Fig. 17 Influence of the core boring length L_b on the evaluation of standard deviation $SD(Q_u)$ of Q_u ($\theta_{squ} = 0.5$ m).

Fig. 18 Relationship in standard deviation between the mean $SD(\mu_{qu})$ and variance $SD(\sigma_{qu}^2)$ of q_u drawn using the MCMC method.

Fig. 19 Relationship in standard deviation between the mean $SD(\mu_{qu})$ and autocorrelation distance $SD(\theta_{qu})$ of q_u drawn using the MCMC method.

Fig. 20 Influence of the standard deviation $SD(\mu_{qu})$ of μ_{qu} on the standard deviation $SD(Q_u)$ of Q_u .

Fig. 21 Influence of the standard deviation $SD(\mu_{qu})$ of μ_{qu} on the incremental standard deviation $\Delta SD(Q_u)$ of Q_u .

Fig. 22 Relationship between the standard deviation $SD(\mu_{qu})$ of μ_{qu} and the equivalent number of independent data n' .

Fig. 23 Influence of the equivalent number of independent data n' on the incremental standard deviation $\Delta SD(Q_u)$ of Q_u .

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Draft

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