DYNAMICS OF MACRO/MICRO MANIPULATOR SYSTEMS

by

Degao Li

A thesis submitted in conformity with the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Mechanical and Industrial Engineering
University of Toronto

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Abstract

This thesis seeks to provide an effective way for developing the dynamics of a modular manipulator system consisting of a micro manipulator mounted on the tip of a macro manipulator. A modular dynamic formulation method is developed to establish the equations of motion of the macro/micro (M/m) manipulator system. The modular equations of the overall system can be constructed by directly using the equations of individual subsystems and calculating the couplings between the subsystems. The coupling torques are directly calculated using closed-form coupling equations. The modular formulation method is first applied on a rigid M/m manipulator system, and then extended to flexible M/m manipulators mounted on a flexible base (M/m+B).
The kinematic redundancy of the M/m system is solved based on the criterion of peak torque reduction. Instead of minimizing the joint torques or kinetic energy at the current instant of time, the proposed Peak Torque Reduction method uses the current torque to approach an optimum velocity at the next instant of time, which, along with joint acceleration, minimizes the torque at that time, while at the same time keeps the current torque within the limits.

To model the flexible M/m+B system, the modes of a flexible beam with a flexible joint are first derived. The obtained modes, called flexible-free modes, incorporate the link flexibility with the joint flexibility. By using the flexible-free modes, the flexible-link, flexible-joint macro manipulator can be treated as a flexible-link, rigid-joint manipulator. As a result, the flexible joint coordinates do not appear in the equations of motion and the order of the equations is reduced, while the accuracy remains unaffected. By applying the modular formulation method and the flexible-free modes, the equations of motion of the system consisting of flexible M/m manipulators mounted on a flexible base are finally established. The resultant equations are modular and order-reduced. Closed-form coupling equations are also obtained, which allow direct calculations of the couplings between the subsystems. Simulation is performed on a 3-DOF model to study the dynamic behavior of the system.
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Chapter 1

Introduction

1.1 Motivation

Hazardous materials in nuclear and industrial waste management hinder direct human operations, such as waste task remediation and waste material handling. Remote manipulation may be a more suitable and safer approach. A system consisting of macro/micro (M/m) manipulators has been proposed for such applications (Lew and Book 1990). The micro manipulator can perform dexterous operations. The macro manipulator can cover a large workspace and provide a powerful support to the micro. Sometimes, the system is mounted on a vehicle, which allows for mobility of the M/m manipulators. M/m manipulator systems have many other applications, such as in space and construction industries (Yoshikawa et al. 1993, Crane et al. 1991).

A M/m manipulator system is different from a conventional manipulator in its modular structure, kinematic redundancy, and most of times, mechanical flexibility. The modular M/m structure provides many advantages, such as increased workspace, fast speed, and en-
hanced positioning accuracy (Sharon and Hardt 1984. Lew and Book 1990. Chiang et al. 1991. Chalhoub and Zhang 1993). The kinematic redundancy allows dexterous manipulation, and makes it possible for the micro manipulator to compensate for the positioning error generated by the macro (Yoshikawa et al. 1993). Although some M/m manipulator systems can be considered rigid (Egeland and Sagli 1990). most long-reach M/m systems have significant flexibility and must be considered flexible (Jansen et al. 1991). The flexibility comes from the links and joints of the macro manipulator, and possibly from the base (a carrying truck or supporting structure). Base flexibility and joint flexibility are taken into account because a small base or macro joint flexibility can cause a significantly large end-point displacement for long-reach manipulator systems.

A few researchers have tackled the vibration problem of M/m manipulator systems. Sherf (1994) studied the active damping of a large flexible manipulator with a short-reach robot. The reaction force from the small robot to the large one is used as a control variable which allows to effectively decouple the controller design problems for the two manipulators. Yoshikawa et al. (1993) used a PD controller to control the end-point motion of a M/m manipulator system. The rigid micro manipulator is used to compensate for the flexibility of the macro. Later, they modified their PD controller to account for the system dynamics (Yoshikawa et al. 1994). The controller was validated with a system comprising a two-link flexible manipulator and a two-DOF rigid robot. Ballhaus and Rock (1992) experimented with the control of a flexible manipulator with a mini-manipulator mounted at its tip. The two manipulators were controlled independently with a PD law to realize the desired end-point motion. The results show that such a partitioned approach is limited and may lead to instability because of the dynamic coupling between the two manipulators.
To achieve satisfactory control performance, dealing with modeling is crucial. The disparate features of such M/m manipulator systems require an efficient modeling method, which deals with modular structure, kinematic redundancy and mechanical flexibility. So far, no prior work on this modeling method has been done. As the first step, this thesis proposes a systematic study on the modeling of such a M/m system. To take advantage of the modular structure, a modular formulation method is developed. Thus, the equations of motion of the overall system can be established based on the equations of the original macro and micro manipulators, plus the couplings between them. The equations show the structure of the system dynamics and the couplings. Kinematic redundancy is used to optimize the dynamic behavior of the system. Also, the effects of the flexible links, joints and the base are included in the dynamic modeling.

1.2 Literature Review

M/m manipulator systems have three major features: modular structure, kinematic redundancy and mechanical flexibility. Dynamic modeling of such systems needs to deal with these issues. In this section, a brief review of the study related to these three topics is presented.

1.2.1 Dynamics of M/m Manipulators

In the formulation of manipulator dynamics, conventional methods are based on Lagrange's equations, Newton-Euler equations, or Kane's method (Hollerbach 1980, Luh 1980, Paul 1981, Kane and Levinson 1985). The Newton-Euler method, based on the Newton's second law, is greatly complicated by link flexibility. By contrast, the Lagrange approach is based on
work, energy and generalized coordinates. The resultant equations are generally compact and provide a closed-form expression for joint generalized forces. In Kane’s method, the equations are obtained from constructing the generalized active and inertia forces with appropriate selection of the generalized speeds. The above formulation methods have been widely used for single rigid or flexible manipulator systems. When they are applied to a modular M/m manipulator system, the M/m system is considered as a new system consisting of a single manipulator. Obviously, these methods are cumbersome, since they do not make use of the known equations of motion of the subsystems. Also, the resultant equations cannot show the modular structure, which is essential information in control design. A modular formulation method based on the modular structure is desirable in order to establish the overall equations of motion and study the couplings between the subsystems.

1.2.2 Dynamics of Flexible-Link, Flexible-Joint Manipulators

To improve control performance of a flexible system, an accurate dynamic model is crucial. The dynamic model that is widely used for rigid manipulators, is not adequate for high performance demands in the presence of flexibility. This effect must be included in the dynamic model. Basically, there are two types of flexibility existing in the mechanical manipulators: link flexibility and joint flexibility. Both have individually been addressed in the literature.

Research on link flexibility has been extensive. Using the rigid-body coordinate system attached to the link and the $4 \times 4$ homogeneous coordinate transformation matrix. Book (1984) applied Lagrange equations to derive the recursive Lagrange dynamics equations of the spatial flexible manipulators with revolute joints. Oakley and Cannon (1989) implemented Kane’s Method with assumed-modes method in the derivation of equations of
motion for a two-link flexible manipulator. Fukuda (1985) investigated a flexible arm with two degrees of freedom, which was dynamically controlled to suppress vibrations in precision positioning applications. Rakhsha and Goldenberg (1985) addressed the mathematical formulation of the dynamic model of a single-link flexible robot, based on a Newton-Euler approach combined with the constrained modes representation of structural flexibility. Later, they studied the feedforward control issue of such a manipulator (Goldenberg and Rakhsha 1986). Based on the equivalent Rigid Link System dynamic model, Chang (1992) investigated the dynamics and control of an electro-hydraulically actuated flexible link, whose motion is restricted to a vertical plane.

Joint flexibility has also been a topic of many papers. Spong (1987) and Good et al. (1985) investigated the modeling and control of flexible joint robots. A simplified model was developed, in which the motion of the actuators' rotors were considered as pure rotation with respect to an inertial frame. Based on the simplified models, static feedback linearization and adaptive controller design were addressed by Khorasani and Kokotovic (1985) and Ghorbel and Spong (1990). In recent years, a research team in the Robotics and Automation Laboratory at the University of Toronto achieved significant progress, both theoretically and experimentally, on control of manipulators with harmonic drive transmission, which causes joint flexibility. Several control schemes were proposed (Lin and Goldenberg 1995, Kircanski and Goldenberg 1996, Lin and Goldenberg 1996).

Some attention has been paid to the manipulators with both link flexibility and joint flexibility. Kärkkäinen (1985) studied a 6-DOF log-loading manipulator in forestry application. The manipulator can handle very large and heavy objects with maximum lifting capability of up to 55 K`Nm. The experiments showed that the flexibility of the
system results from the hydraulic actuators at joints in addition to the flexure of the links and the base. Yang and Donath (1988) derived a simplified model based on decoupling and a detailed model incorporating cross-coupling, for a flexible link with both link flexibility and joint flexibility. It was shown that the simplified model is sufficiently accurate for small joint and structural deflections. An effective approach for developing the dynamics of a multi-link flexible manipulator was proposed by Yuan et al. (1993), based on Book's 4 × 4 transformation matrix method (Book 1984). In addition to the link flexibility, joint flexibility was taken into account in their model. Experimental and finite element methods were employed to determine the component mode shapes of the 2-DOF flexible robot. RALF. Huang and Wang (1993) studied an ITRI-U manipulator with both link flexibility and joint flexibility by using the finite element method. The model they developed based on the concept of substructure combination was then used to analyze the dynamic response of the manipulator and to determine the nonlinear effects of link flexibility, joint flexibility and gravity.

All the methods mentioned above do not integrate the joint flexibility with the link modes. Because of the unavailability of suitable modes, clamped-free or pinned-free modes for the flexible links are assumed, or finite element or experimental methods are used to determine the link modes. The joint flexibility is represented by independent coordinates in the system model. All those methods always generate high-order equations of motion. Some of them even result in poor model accuracy. To address the problems, the modes integrating the joint flexibility need to be derived. In this respect, Xi and Fenton (1994) conducted preliminary research on a simple beam connected to a rotor through a flexible joint. They studied the effect of the joint flexibility and rotor inertia on the system frequencies. However.
they did not address the effect of joint flexibility on the tip vibration and torque oscillation, and the effect of joint-hub dynamics on the system frequencies. Also, their result is limited to a one-link manipulator, and cannot be applied to any multi-link case, because the influence of other links on the link modes was not considered.

1.2.3 Redundancy Resolution

One of the basic problems that has to be solved in manipulator trajectory planning is the inverse kinematic problem, namely, to find the joint motion for the given end-effector motion. For non-redundant robots, inverse kinematics has a unique solution (at non-singular configurations). For redundant robots, there are infinitely many solutions to the inverse kinematics problem. To solve the redundancy, joint motions may be selected among infinitely many solutions to achieve an objective, while realizing the primary goal of moving the end-effector to obtain desired translational and rotational motions. This advantage of redundant manipulators has attracted extensive research and led to a great number of publications. Methods reported so far can be classified as: kinematic resolution and dynamic resolution, according to whether or not they take the manipulator dynamics into consideration.

Kinematic resolution of redundancy is the subject of the majority of papers published in this area. Whitney (1969) proposed using a pseudoinverse or a weighted pseudoinverse of the manipulator Jacobian to determine the joint rates in a redundant serial chain. Such a solution has the smallest Euclidean norm of joint rates or the smallest weighted norm. The singularity issue raised by algorithms based on such local performance functions was discussed by Baker and Wampler (1988) and Baillieu (1986). Kircanski and Petrivic (1993) proposed a combined analytical-pseudoinverse inverse kinematic solution to reduce the com-
putational complexity associated with the pseudoinverse solution. Liegeois (1977) proposed the use of the null space of the Jacobian, which corresponds to the self-motion of the manipulator. to improve the behavior of the pseudoinverse control. Baillieul (1987) proposed an extended Jacobian method which complemented the velocity equations by an additional set of equations to get the total number of motion and constraint equations equal to the number of unknown joint rates. Podhorodeski, Goldenberg and Fenton (1991) formulated local optimizations for redundant manipulator joint rates in terms of Jacobian null-space based on screw theory. An algorithm for solving the inverse kinematics using the method of generalized inverse based on a modified Newton-Raphson iterative technique was proposed by Benhabib. Goldenberg and Fenton (1985).

As mentioned above, much work has been done on redundancy resolution at the kinematic level. Incorporating the manipulator dynamics in the redundancy resolution is a more practical approach since robotic manipulators are actually controlled by specifying the joint torques to track a desired end-effector trajectory.

Khatib (1983) took the manipulator dynamics into consideration by using the inertia-weighted pseudoinverse of the manipulator Jacobian to minimize the instantaneous kinetic energy. Such a control scheme minimizing the instantaneous kinetic energy does not necessarily lead to instantaneous torque minimization. Hollerbach and Suh (1987) minimized the instantaneous driving actuator torques by resolving redundancy at the acceleration level. Vukobratovic and Kircanski (1984) developed an approach which uses the dynamic model of the manipulator and its actuators to generate the nominal trajectories in the joint space so as to be optimal with respect to total energy consumption of the actuators. Egeland (1987) proposed to maximize response bandwidth to improve dynamic performance equivalent to
the extended Jacobian technique (1985).

Generally, redundancy resolutions incorporated with the manipulator dynamics are based on two main optimization criteria: torque optimization (Hollerbach and Suh 1987) and energy minimization (Khatib 1983). The latter minimizes the kinetic energy of the system at every time instant, but it does not necessarily minimize the joint torques. The former minimizes joint torques locally, but it does not necessarily prevent torques to be larger than their limits. In cases such as tracking control, torque limit avoidance is more important than torque minimization, since large demand torques can cause torque saturation which leads to reduced tracking accuracy. Hence, avoiding torque saturation by using redundant degrees of freedom to reduce peak torques is a more useful issue, about which little has been done to date.

1.3 Contributions of the Thesis

The major contributions of this thesis are summarized as follows:

(1) Development of an effective dynamic formulation for both rigid and flexible modular M/m manipulator systems. The obtained equations of motion are modular. This proposed approach simplifies the dynamic formulation, shows the structure of the system dynamics, and allows direct calculations of the couplings between the subsystems.

(2) Development of a new method, Peak Torque Reduction, for redundancy resolution (Li et al. 1996, 1997b). The proposed method can be used not only for redundant M/m systems, but also for conventional redundant manipulators. Analysis and simulation have shown clear superiority of the proposed method over conventional methods.

(3) Derivation and analysis of flexible-free modes of a beam with a flexible joint...
(Li et al. 1995a, 1995b, 1997a). The modes incorporate the link flexibility with the joint flexibility and include the effects of payload and hub.

(4) Application of the flexible-free modes on flexible-link, flexible-joint manipulators. By applying flexible-free modes, the dynamics of a flexible-link, flexible-joint manipulator can be formulated the same way as for a flexible-link, rigid-joint manipulator. The flexible joint coordinates are no longer present in the resultant equations of motion. The order of the obtained model is reduced. The accuracy of the order-reduced model is the same as or higher than the conventional full-order models.

(5) Dynamic modeling and simulation analysis of a system consisting of flexible-link, flexible-joint M/m manipulators mounted on a flexible base. The obtained model has a modular structure and a reduced order. Important conclusions on the couplings between the subsystems have been drawn from the simulation and analysis.

1.4 Organization of the Thesis

This thesis is composed of seven chapters.

Chapter 1 introduces the historical perspective and state of the art of the study of dynamics of M/m manipulator systems.

Chapter 2 addresses the modular formulation of rigid M/m manipulator systems. The inertia matrix, non-linear force matrix and the gravity matrix of the overall system are obtained in modular form. An example is given to illustrate the proposed formulation.

Chapter 3 presents a new method of solving the kinematic redundancy in M/m manipulator systems. The basic idea of the method is first discussed. An optimization scheme is then developed to implement the idea. The proposed method is compared with
conventional methods by analysis and simulation.

Chapter 4 discusses the modes of a flexible beam with a flexible joint. The effects of the flexible joint, payload and hub on the modes are studied.

Chapter 5 investigates the dynamics of flexible-link, flexible-joint manipulators. A flexible-link, flexible-joint manipulator is treated as a flexible-link, rigid-joint manipulator by using the modes derived in Chapter 4. The order and accuracy of the resultant model are discussed in contrast with conventional methods.

Chapter 6 addresses the dynamics of a system consisting of flexible M/m manipulators mounted on a flexible base. The modular formulation method developed in Chapter 2 is extended to this system. Simulation of a 3-DOF model is conducted to study the couplings between the subsystems.

Chapter 7 summaries the results of this research and offers recommendations for future work.
Chapter 2

A Modular Formulation of M/m Manipulator Dynamics

2.1 Introduction

The objective of this chapter is to develop a modular formulation method for rigid M/m manipulator systems, as a basis for the flexible M/m system formulation to be addressed in Chapter 6. With the module concept, the system equations of motion also have a modular structure. The overall equations of motion are constructed based on the known equations of individual subsystems, plus the couplings between them.

The equations of motion of the macro manipulator may be written in the general form as

\[ M_M \ddot{q}_M + C_M \dot{q}_M + G_M = \tau_M \]  \hspace{1cm} (2.1)

where \( q_M, G_M \) and \( \tau_M \in \mathbb{R}^{n_M} \), and \( M_M \) and \( C_M \in \mathbb{R}^{n_M \times n_M} \). \( n_M \) is the number of joints of the macro manipulator.
Similarly, the micro arm has the following equations of motion

\[ M_m \ddot{q}_m + C_m \dot{q}_m + G_m = \tau_m \]  

(2.2)

where \( \dot{q}_m, G_m \) and \( \tau_m \in \mathbb{R}^{n_m}; M_m \) and \( C_m \in \mathbb{R}^{n_m \times n_m} \). \( n_m \) is the number of joints of the micro manipulator.

The modular M/m system has equations of motion of the general form

\[ M \ddot{q} + C \dot{q} + G = \tau \]

(2.3)

where \( q = [q_M^T \quad q_m^T]^T \in \mathbb{R}^n \). \( M \) and \( C \in \mathbb{R}^{n \times n} \). and \( G \) and \( \tau \in \mathbb{R}^n \). \( n \) is the total number of joints of the macro/micro manipulators. calculated by

\[ n = n_M + n_m \]  

(2.4)

The next step is to determine the structure of the matrices in Eq. (2.3), based on the matrices in Eqs. (2.1) and (2.2).

### 2.2 Inertia Matrix M

Define a body-fixed frame \( x_My_Mz_M \) at the tip of the macro arm, \( P_M \), where the macro arm holds the micro arm, as shown in Figure 2.1. Define an inertial frame \( x_0y_0z_0 \) at the base of the macro arm. The mass center location of link \( i \) of the micro arm may be expressed in the reference frame as

\[ r_{mi} = r_M + R_M \tilde{r}_{mi} \]  

(2.5)

where \( r_M \) is the position vector of \( P_M \) expressed in \( x_My_Mz_M \); \( R_M \) is the transformation matrix from frames \( x_My_Mz_M \) to \( x_0y_0z_0 \); and \( \tilde{r}_{mi} \) is the position vector of the mass center of link \( i \), with respect to frame \( x_My_Mz_M \).
Differentiating Eq. (2.5) results in the velocity

\[ \ddot{\mathbf{r}}_{mi} = \frac{d\mathbf{r}_{mi}}{dt} \]

\[ = \frac{\partial \mathbf{r}_M}{\partial \mathbf{q}_M} \dot{\mathbf{q}}_M + \mathbf{R}_M \frac{\partial \ddot{\mathbf{r}}_{mi}}{\partial \mathbf{q}_m} \mathbf{q}_m + \frac{\partial}{\partial \mathbf{q}_M} (\mathbf{R}_M \ddot{\mathbf{r}}_{mi}) \dot{\mathbf{q}}_M \]

Using the following equation

\[ \frac{\partial}{\partial \mathbf{q}_M} (\mathbf{R}_M \ddot{\mathbf{r}}_{mi}) = -\mathbf{R}_M \text{skew}(\dddot{\mathbf{r}}_{mi}) (\mathbf{J}_M^\nu) \]

it follows that

\[ \ddot{\mathbf{r}}_{mi} = (\mathbf{J}_M^\nu) \mathbf{q}_M + \mathbf{R}_M \frac{\partial \ddot{\mathbf{r}}_{mi}}{\partial \mathbf{q}_m} \mathbf{q}_m - \mathbf{R}_M \text{skew}(\dddot{\mathbf{r}}_{mi}) (\mathbf{J}_M^\nu) \dot{\mathbf{q}}_M \]

\[ = (\mathbf{J}_m^\nu) \mathbf{q}_M + (\mathbf{J}_m^\nu) \mathbf{q}_m \]

(2.7)
where

\[
(J_{mi}^v)_M = (J_M^v)_M - R_M \text{skew}(\dot{r}_{mi}) (J_M^\omega)_M
\]  

(2.8a)

\[
(J_{mi}^\omega)_m = R_M \frac{\partial \dot{r}_{mi}}{\partial \dot{q}_m}
\]  

(2.8b)

In the above equations, \text{skew}(\cdot) converts a vector to a skew-symmetric matrix as follows:

\[
\text{skew}\left(\begin{array}{c}
a_x \\
a_y \\
a_z
\end{array}\right) = \left[\begin{array}{ccc}
0 & -a_z & a_y \\
a_z & 0 & -a_x \\
-a_y & a_x & 0
\end{array}\right]
\]  

(2.9)

\((J_M^v)_M\) is the linear velocity Jacobian of the macro

\[
(J_M^v)_M = \frac{\partial \dot{r}_M}{\partial \dot{q}_M}
\]  

(2.10)

\((J_M^\omega)_M\) is the angular velocity Jacobian of the macro, and the expression for the angular velocity of the last link of the macro arm is

\[
\omega_M = (J_M^\omega)_M \dot{q}_M
\]  

(2.11)

The angular velocity of the link \(i\) of the micro arm is

\[
\omega_{mi} = \omega_M + R_M \dot{\omega}_{mi}
\]

\[
= (J_{mi}^\omega)_M \dot{q}_M + (J_{mi}^\omega)_m \dot{q}_m
\]  

(2.12)

where

\[
(J_{mi}^\omega)_M = (J_M^\omega)_M
\]  

(2.13a)

\[
(J_{mi}^\omega)_m = R_M (\dot{J}_{mi}^\omega)_m
\]  

(2.13b)
\( \omega_M \) is the angular velocity of the last link of the macro arm (or that of the base of the micro arm) with respect to the frame \( x_0y_0z_0 \); while \( \omega_{mi} \) is the angular velocity of the \( i \)th link of micro arm, with respect to the frame \( x_My_Mz_M \).

The kinetic energy of the micro arm is the sum of the translational and rotational kinetic energy of each link as follows:

\[
T_m = \sum_{i=n_M+1}^{n} \frac{1}{2} m_{mi} \dot{r}_{mi}^T \dot{r}_{mi} + \sum_{i=n_M+1}^{n} \frac{1}{2} \omega_{mi}^T I_{mi} \omega_{mi}
\]

\[
= \sum_{i=n_M+1}^{n} \frac{1}{2} m_{mi} [(J_{mi})_M \dot{q}_M + (J_{mi})_m \dot{q}_m]^T [(J_{mi})_M \dot{q}_M + (J_{mi})_m \dot{q}_m] \\
+ \sum_{i=n_M+1}^{n} \frac{1}{2} [(J_{mi})_M \dot{q}_M + (J_{mi})_m \dot{q}_m]^T I_{mi} [(J_{mi})_M \dot{q}_M + (J_{mi})_m \dot{q}_m]
\]

\[
= \frac{1}{2} \begin{pmatrix} \dot{q}_M^T & \dot{q}_m^T \end{pmatrix} \begin{bmatrix} D_{m11} & D_{m12} \\ D_{m12}^T & D_{m22} \end{bmatrix} \begin{pmatrix} \dot{q}_M \\ \dot{q}_m \end{pmatrix} \tag{2.14}
\]

where

\[
D_{m11} = \sum_{i=n_M+1}^{n} m_{mi} (J_{mi})_M^T (J_{mi})_M + \sum_{i=n_M+1}^{n} (J_{mi})_M^T I_{mi} (J_{mi})_M \tag{2.15a}
\]

\[
D_{m22} = \sum_{i=n_M+1}^{n} m_{mi} (J_{mi})_m^T (J_{mi})_m + \sum_{i=n_M+1}^{n} (J_{mi})_m^T I_{mi} (J_{mi})_m \tag{2.15b}
\]

\[
D_{m12} = \sum_{i=n_M+1}^{n} m_{mi} (J_{mi})_M^T (J_{mi})_m + \sum_{i=n_M+1}^{n} (J_{mi})_M^T I_{mi} (J_{mi})_m \tag{2.15c}
\]

In the above equations, \( I_{mi} \) is the inertia matrix of link \( i \) of the micro expressed in frame \( x_0y_0z_0 \), calculated by

\[
I_{mi} = R_M \dot{I}_{mi} R_M^T \tag{2.16}
\]

where \( \dot{I}_{mi} \) is the inertia matrix of link \( i \) expressed in frame \( x_My_Mz_M \).

The total kinetic energy of the whole system can be obtained as:

\[
T = T_M + T_m
\]
\[
= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_M \dot{\mathbf{q}}_M + \frac{1}{2} \left( \dot{\mathbf{q}}^T \mathbf{D}_{m11} \dot{\mathbf{q}}^T \mathbf{D}_{m12} \right) \left( \begin{array}{c} \mathbf{D}_{m11} \\ \mathbf{D}_{m12} \end{array} \right) \left( \begin{array}{c} \dot{\mathbf{q}}_M \\ \dot{\mathbf{q}}_m \end{array} \right) \\
= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}
\]

\( \mathbf{M} \) in (2.17) can be written in the following form

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{M}_M + \mathbf{M}_{Mm} & \mathbf{M}_{Mm} \\
\mathbf{M}_{Mm}^T & \mathbf{M}_m
\end{bmatrix}
\]

(2.18)

where \( \mathbf{M}_M \) and \( \mathbf{M}_m \) in (2.18) are inertia matrices of the respective original subsystems, and the other elements are couplings between subsystems as follows:

\[
\mathbf{M}_{Mm} = \sum_{i=n_1+1}^{n} m_{mi} (\mathbf{J}_{mi}^v)(\mathbf{J}_{mi}^v)_M + \sum_{i=n_1+1}^{n} (\mathbf{J}_{M}^v)(\mathbf{J}_{mi}^v)_M \\
\mathbf{M}_{Mm} = \sum_{i=n_1+1}^{n} m_{mi} (\mathbf{J}_{mi}^v)(\mathbf{J}_{mi}^v)_M + \sum_{i=n_1+1}^{n} (\mathbf{J}_{M}^v)(\mathbf{J}_{mi}^v)_m
\]

(2.19)

The coupling matrices \( \mathbf{M}_{Mm} \) and \( \mathbf{M}_{Mm} \) have their physical meanings. If we immobilize the micro and consider it as a payload of the macro, \( \mathbf{M}_{Mm} \) is the moment of inertia of the micro (as a payload) added to the moment of inertia of the macro. For any motion, the macro has to overcome the micro inertia \( \mathbf{M}_{Mm} \), in addition to its own inertia \( \mathbf{M}_M \). Geometrically, \( \mathbf{M}_{Mm} \) is the inertia of the micro about the axes of the macro joints. \( \mathbf{M}_{Mm} \) is the coupling moment of inertia between the macro and the micro. It can be considered as an additional inertia added to the macro joints, due to the inertia of the micro about its own joints. In terms of joint torques, both \( \mathbf{M}_{Mm} \) and \( \mathbf{M}_{Mm} \) can cause additional torques to the macro joints. The difference is that the former corresponds to the torque when the micro is immobilized, while the latter corresponds to the torque when the micro has motion about its own joints.
2.3 Non-Linear Matrix C

The non-linear matrix $C$ of the overall system corresponding to the centrifugal and Coriolis forces can be obtained from the inertia matrix $M$ as follows:

$$C = \dot{M} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n} \end{bmatrix}$$

or

$$C = \dot{M} - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial q_1} (M\dot{q}) \\ \vdots \\ \frac{\partial}{\partial q_n} (M\dot{q}) \end{array} \right]^T$$

Similarly, the non-linear matrices for the subsystems are

$$C_M = \dot{M}_M - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial q_1} (M_M\dot{q}_M) \\ \vdots \\ \frac{\partial}{\partial q_{n_M}} (M_M\dot{q}_M) \end{array} \right]^T$$

$$C_m = \dot{M}_m - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial q_{n_M+1}} (M_m\dot{q}_m) \\ \vdots \\ \frac{\partial}{\partial q_n} (M_m\dot{q}_m) \end{array} \right]^T$$

From Eq. (2.18), it follows that

$$\dot{q}^T \frac{\partial M}{\partial q_i} = \left( \dot{q}^T_M \frac{\partial}{\partial q_i} (M_M + \dot{M}_M) + \dot{q}^T_m \frac{\partial M_m^T}{\partial q_i} \dot{q}_M \frac{\partial M_m}{\partial q_i} + \dot{q}_m \frac{\partial M_m}{\partial q_i} \right)$$

\hspace{1cm} (i = 1, 2, \ldots, n)

For the macro manipulator, Eq. (2.25) becomes

$$\dot{q}^T \frac{\partial M}{\partial q_i} = \left( \dot{q}^T_M \frac{\partial}{\partial q_i} (M_M + \dot{M}_M) + \dot{q}^T_m \frac{\partial M_m^T}{\partial q_i} \dot{q}_M \frac{\partial M_m}{\partial q_i} \right)$$

\hspace{1cm} (i = 1, 2, \ldots, n_M)

where

$$\frac{\partial M_m}{\partial q_i} = 0 \hspace{1cm} (i = 1, 2, \ldots, n_M)$$
has been used. For the micro manipulator, Eq. (2.25) becomes

\[ \dot{q}_i^T \frac{\partial \mathbf{M}}{\partial q_i} = \left( \dot{q}_m^T \frac{\partial \mathbf{M}_{Mm}}{\partial q_i} + \mathbf{q}_m^T \frac{\partial \mathbf{M}_{m}}{\partial q_i} \right) \quad (i = n_M + 1, n_M + 2, \ldots, n) \tag{2.27} \]

where

\[ \frac{\partial \mathbf{M}_M}{\partial q_i} = 0 \quad (i = n_M + 1, n_M + 2, \ldots, n) \]

has been used. Substituting Eqs. (2.26) and (2.27) into Eq (2.21), it follows that

\[
\mathbf{C} = \begin{bmatrix}
\dot{\mathbf{M}}_M + \dot{\mathbf{M}}_{Mm} & \dot{\mathbf{M}}_{Mm} \\
\dot{\mathbf{M}}_{Mm}^T & \dot{\mathbf{M}}_m
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\dot{q}_M \frac{\partial \left( \mathbf{M}_M + \mathbf{M}_{Mm} \right)}{\partial q_1} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_1} & \dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_1} \\
\dot{q}_M \frac{\partial \left( \mathbf{M}_M + \mathbf{M}_{Mm} \right)}{\partial q_{n_M}} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M}} & \dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M}} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M}}
\end{bmatrix}
\]

\[ \begin{bmatrix}
\ddots & \ddots \\
-\frac{1}{2} \dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M+1}} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M+1}} & \dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M+1}} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_{n_M+1}} \\
\dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_n} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_n} & \dot{q}_M \frac{\partial \mathbf{M}_{Mm}}{\partial q_n} + \dot{q}_m \frac{\partial \mathbf{M}_{Mm}}{\partial q_n}
\end{bmatrix} \tag{2.28}
\]

Thus, \( \mathbf{C} \) is obtained from Eq (2.28) as

\[
\mathbf{C} = \begin{bmatrix}
\mathbf{C}_M + \tilde{\mathbf{C}}_{Mm} & \mathbf{C}_{Mm} \\
\mathbf{C}_{Mm} & \mathbf{C}_m + \tilde{\mathbf{C}}_{m,m}
\end{bmatrix} \tag{2.29}
\]

where

\[
\tilde{\mathbf{C}}_{Mm} = \dot{\mathbf{M}}_{Mm} - \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial q_1} \left( \dot{q}_M^T \mathbf{M}_{Mm} + \dot{q}_m^T \mathbf{M}_{Mm} \right) \\
\vdots \\
\frac{\partial}{\partial q_{n_M}} \left( \dot{q}_M^T \mathbf{M}_{Mm} + \dot{q}_m^T \mathbf{M}_{Mm} \right)
\end{bmatrix} \tag{2.30a}
\]


\[
\tilde{C}_{m,M} = -\frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_{n+1}} (\dot{q}_M^T M_{Mm}) \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{q}_M^T M_{Mm})
\end{array} \right]
\]

\[
C_{Mm} = M_{Mm} - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_M^T M_{Mm}) \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{q}_M^T M_{Mm})
\end{array} \right]
\]

\[
C_{m,M} = \dot{M}_{Mm}^T - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_{n+1}} (\dot{q}_M^T M_{Mm} + \dot{q}_m^T M_{Mm}^T) \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{q}_M^T M_{Mm} + \dot{q}_m^T M_{Mm}^T)
\end{array} \right]
\]

\[2.4 \text{ Gravity Vector } G\]

To derive the gravitational term \( G \) of the overall system, let us investigate the potential energy of the system. By definition, the elements of \( G \) are obtained from the potential energy:

\[
G_j = \frac{\partial (V_M + V_m)}{\partial q_j}
\]

The potential energy of the micro is

\[
V_m = \sum_{i=n_M+1}^{n} m_{mi} g^T r_{mi}
\]

The partial derivative of \( V_m \) with respect to \( q_j \) \( (j = 1, 2, \cdots, n) \) is obtained as:

\[
\frac{\partial V_m}{\partial q_j} = \sum_{i=n_M+1}^{n} m_{mi} g^T \frac{\partial r_{mi}}{\partial q_j}
\]

It follows that

\[
\frac{\partial V_m}{\partial q_j} = \sum_{i=n_M+1}^{n} m_{mi} g^T (J_{mi}^v)_{Mj} \quad (j = 1, 2, \cdots, n_M)
\]
where \((J^u_{mi})_{Mj}\) is the \(j\)th column of \((J^u_{mi})_M\) defined by (2.8a), and

\[
\frac{\partial V_m}{\partial q_j} = \sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{mj} \quad (j = n_M + 1, n_M + 2, \ldots, n) \tag{2.33}
\]

where \((J^u_{mi})_{mj}\) is the \(j\)th column of \((J^u_{mi})_m\) defined by (2.8b).

Based on the above equations, the vector \(G\) is obtained as

\[
G = \begin{pmatrix}
\frac{\partial (V_M + V_m)}{\partial q_1} \\
\vdots \\
\frac{\partial (V_M + V_m)}{\partial q_{n_M}} \\
\frac{\partial (V_M + V_m)}{\partial q_{n_M+1}} \\
\vdots \\
\frac{\partial (V_M + V_m)}{\partial q_{n}} \\
\end{pmatrix} = \begin{pmatrix}
\frac{\partial V_M}{\partial q_1} + \sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{M1} \\
\vdots \\
\frac{\partial V_M}{\partial q_{n_M}} + \sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{Mn_M} \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{m1} \\
\vdots \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{mn_m} \\
\end{pmatrix}
\]

\[
G = \begin{pmatrix}
G_M + G_{Mm} \\
G'_m \\
\end{pmatrix} \tag{2.34}
\]

where \(G_M\) is the gravity vector of the macro. \(G_{Mm}\) is the coupling gravity vector defined by

\[
G_{Mm} = \begin{pmatrix}
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{M1} \\
\vdots \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{Mn_M} \\
\end{pmatrix} \tag{2.35}
\]

and \(G'_m\) is the modified gravity vector of the micro defined by

\[
G'_m = \begin{pmatrix}
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{m1} \\
\vdots \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (J^u_{mi})_{mn_m} \\
\end{pmatrix} \tag{2.36}
\]
The coupling gravity term $G_{Mm}$ is the additional torque applied on the macro joints caused by the gravity of the micro arm. $G'_m$ is the gravity term of the micro, modified from its original gravity vector $G_m$ which is

$$
G_m = \begin{pmatrix}
\sum_{i=n_M+1}^{n} m_{mi} g^T (\tilde{J}^v_{mi})_{m1} \\
\vdots \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (\tilde{J}^v_{mi})_{mn_m}
\end{pmatrix}
$$

From the following transformation

$$(J^v_{mi})_m = R_M (\tilde{J}^v_{mi})_m$$

it is seen that the difference between $G_m$ and $G'_m$ lies in the change in the tip orientation of the macro arm (or the base of the micro arm). Any change in the tip orientation of the macro arm leads to the change in the gravity term of the micro. Defining this coupling as

$$G_{mM} = G'_m - G_m$$

it follows that

$$G_{mM} = \begin{pmatrix}
\sum_{i=n_M+1}^{n} m_{mi} g^T (I - R_M^T) (J^v_{mi})_{m1} \\
\vdots \\
\sum_{i=n_M+1}^{n} m_{mi} g^T (I - R_M^T) (J^v_{mi})_{mn_m}
\end{pmatrix}
$$

where $I$ is the identity matrix. When the frame $x_My_Mz_M$ is aligned with $x_o y_o z_o$. $G_{mM}$ is equal to zero and $G'_m$ is equal to $G_m$.

By introducing the coupling term $G_{mM}$, Eq. (2.34) can now be expressed as

$$G = \begin{pmatrix}
G_M + G_{Mm} \\
G_m + G_{mM}
\end{pmatrix}$$

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2.5 Modular-Form Equations of Motion

In above derivations, the inertia matrix, non-linear matrix and gravity vector of the M/m system are expressed in their original forms of the subsystems and the couplings terms between the macro and micro arms. The overall equations of motion of the M/m system expressed by Eq. (2.3) is now rewritten as

\[
\begin{bmatrix}
    M_M + \bar{M}_{Mm} & M_{Mm} \\
    M_{Mm}^T & M_m
\end{bmatrix}
\begin{pmatrix}
    \ddot{q}_M \\
    \ddot{q}_m
\end{pmatrix}
+ \begin{bmatrix}
    C_M + \bar{C}_{Mm} & C_{Mm} \\
    C_{mM} & C_m + \bar{C}_{mM}
\end{bmatrix}
\begin{pmatrix}
    \dot{q}_M \\
    \dot{q}_m
\end{pmatrix}
+ \begin{pmatrix}
    G_M + G_{Mm} \\
    G_m + G_{mM}
\end{pmatrix}
= \begin{pmatrix}
    \tau_M + \tau_{Mm} \\
    \tau_m + \tau_{mM}
\end{pmatrix}
\] (2.41)

where \( \tau_{Mm} \) and \( \tau_{mM} \) are the coupling torques between the macro and micro arms.

By subtracting Eqs. (2.1) and (2.2) from (2.41), the following equations are obtained:

\[
\begin{bmatrix}
    \bar{M}_{Mm} & M_{Mm} \\
    M_{Mm}^T & 0
\end{bmatrix}
\begin{pmatrix}
    \ddot{q}_M \\
    \ddot{q}_m
\end{pmatrix}
+ \begin{bmatrix}
    \bar{C}_{Mm} & C_{Mm} \\
    C_{mM} & \bar{C}_{mM}
\end{bmatrix}
\begin{pmatrix}
    \dot{q}_M \\
    \dot{q}_m
\end{pmatrix}
+ \begin{pmatrix}
    G_{Mm} \\
    G_{mM}
\end{pmatrix}
= \begin{pmatrix}
    \tau_{Mm} \\
    \tau_{mM}
\end{pmatrix}
\] (2.42)

which are the equations of the coupling dynamics.

Eq. (2.41) shows that when a micro manipulator is mounted on a macro manipulator to form a modular M/m manipulator system, the equations of motion of the overall system can also have a modular form. The modular-form equations can be established by directly using the equations of individual subsystems and calculating the couplings between the subsystems. The joint torques of the overall system are equal to the torques of the original subsystems plus the coupling torques between them. The coupling torques consist of three parts: the coupling torques produced by joint acceleration, by joint velocity (Centrifugal and Coriolis forces), and by link gravity. They can be directly calculated from Eq. (2.42).
In the dynamic formulation of M/m manipulator systems, the developed modular formulation method has two major advantages over the conventional direct formulation methods (Hollerbach 1980, Paul 1981). One is the simplification of the formulation. In the modular formulation method, the original equations of motion of the subsystems are directly used in the overall equations. Only the coupling terms have to be derived. In the conventional methods, the M/m system is treated as a single manipulator, and every term in the equations has to be derived. The other major advantage of the modular formulation method is the clear structure of the equations of motion and the closed form of the coupling equations. This is very useful for the dynamic analysis of the M/m system and for its controller design.

2.6 Example

In this section, a M/m manipulator system consisting of 1-DOF macro and 2-DOF micro is used as an example to demonstrate the proposed modular formulation method. For link \( i \) of the M/m system \( (i = 1,2,3) \), \( l_i \) is the length; \( l_{ci} \) is the mass center location; \( m_i \) is the mass; \( I_i \) is the moment of inertia about the mass center. The following notation is used in this section:

\[
\begin{align*}
s_i &= \sin \theta_i, \quad s_{ij} = \sin(\theta_i + \theta_j), \quad s_{ijk} = \sin(\theta_i + \theta_j + \theta_k); \\
c_i &= \cos \theta_i, \quad c_{ij} = \cos(\theta_i + \theta_j), \quad c_{ijk} = \cos(\theta_i + \theta_j + \theta_k).
\end{align*}
\]

where \( i, j, k = 1,2,3 \).

The 1-DOF macro manipulator has the equation of motion (2.1) with its matrices
as follows:

\[ q_M = \theta_1 \]  
\[ \tau_M = \tau_1 \]  
\[ M_M = \bar{I}_1 + m_1 l_{c1}^2 \]  
\[ C_M = 0 \]  
\[ G_M = m_1 g l_{c1} \delta_1 \]

For the 2-DOF micro manipulator, the equations of motion (2.2) have the matrices as follows:

\[ q_m = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} \]  
\[ \tau_m = \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \]  
\[ M_m = \begin{bmatrix} M_{m11} & M_{m12} \\ M_{m12}^T & M_{m22} \end{bmatrix} \]  
\[ C_m = \begin{bmatrix} C_{m11} & C_{m12} \\ C_{m21} & C_{m22} \end{bmatrix} \]

where

\[ M_{m11} = \bar{I}_2 + m_2 l_{c2}^2 + \bar{I}_3 + m_3 \left( l_2^2 + l_{c3}^2 + 2 l_2 l_{c3} c_3 \right) \]  
\[ M_{m12} = \bar{I}_3 + m_3 \left( l_{c3}^2 + l_2 l_{c3} c_3 \right) \]  
\[ M_{m22} = \bar{I}_3 + m_3 l_{c3}^2 \]  

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The transformation matrix from frame $x_M y_M z_M$ to frame $x_0 y_0 z_0$ is
\[
R_M = \begin{bmatrix}
c_1 & -s_1 & 0 \\
s_1 & c_1 & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{2.53}
\]

The linear and angular velocity Jacobians of the macro and micro are obtained as follows:

\[
(J^v_M)_M = \frac{\partial \mathbf{r}_M}{\partial q_M} = \begin{bmatrix}
-l_1 s_1 \\
l_1 c_1 \\
0
\end{bmatrix} \tag{2.54}
\]

\[
(J^\omega_M)_M = \frac{\partial \omega_M}{\partial q_M} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \tag{2.55}
\]

\[
(J^v_{m1})_m = R_M \frac{\partial \mathbf{r}_{m1}}{\partial q_m} = \begin{bmatrix}
-l_{c2}s_{12} & 0 \\
l_{c2}s_{12} & 0 \\
0 & 0
\end{bmatrix} \tag{2.56}
\]
The inertia matrices of the links of the micro manipulator are expressed in the global frame $x_0y_0z_0$ as follows:

\[(J^v_{m1})_M = \frac{\partial \mathbf{r}_M}{\partial \mathbf{q}_M} - \mathbf{R}_M \text{skew} (\mathbf{\dot{r}}_{m1}) (J^\omega_{m1})_M = \begin{pmatrix}
-l_1 s_1 - l_2 s_{12} \\
l_1 c_1 + l_2 c_{12} \\
0
\end{pmatrix}\]  \hspace{1cm} (2.57)

\[(J^\omega_{m1})_m = \mathbf{R}_M (J^\omega_{m1})_m \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix}\]  \hspace{1cm} (2.58)

\[(J^v_{m2})_m = \mathbf{R}_M \frac{\partial \mathbf{r}_m}{\partial \mathbf{q}_m} = \begin{pmatrix}
-l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\
l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\
0 & 0
\end{pmatrix}\]  \hspace{1cm} (2.59)

\[(J^v_{m2})_M = \frac{\partial \mathbf{r}_M}{\partial \mathbf{q}_M} - \mathbf{R}_M \text{skew} (\mathbf{\dot{r}}_{m2}) (J^\omega_{m2})_M = \begin{pmatrix}
-l_1 s_1 - l_2 s_{12} - l_3 s_{123} \\
l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\
0
\end{pmatrix}\]  \hspace{1cm} (2.60)

\[(J^\omega_{m2})_m = \mathbf{R}_M (J^\omega_{m2})_m \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1
\end{pmatrix}\]  \hspace{1cm} (2.61)

The inertia matrices of the links of the micro manipulator are expressed in the global frame $x_0y_0z_0$ as follows:

\[I_{m1} = \mathbf{R}_M \mathbf{\hat{I}}_{m1} \mathbf{R}^T_M = \mathbf{\hat{I}}_2 \begin{pmatrix}
s_{12}^2 & -s_{12}c_{12} & 0 \\
-s_{12}c_{12} & c_{12}^2 & 0 \\
0 & 0 & 1
\end{pmatrix}\]  \hspace{1cm} (2.62)
The coupling sub-matrices of the overall mass matrix are calculated based on the following equations:

\[
\bar{M}_{Mm} = D_{m11} = \sum_{i=1}^{n_m} m_{mi} (J_{mi}^v)^T (J_{mi}^v)_M + \sum_{i=1}^{n_m} (J_{M}^v)^T_i I_{mi} (J_{M}^v)_M
\]

\[
M_{Mm} = D_{m12} = \sum_{i=1}^{n_m} m_{mi} (J_{mi}^v)^T (J_{mi}^v)_m + \sum_{i=1}^{n_m} (J_{M}^v)^T_i I_{mi} (J_{M}^v)_m
\]

The overall mass matrix is obtained as follows:

\[
M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{12} & M_{22} & M_{23} \\
M_{13} & M_{23} & M_{33}
\end{bmatrix}
\]

where

\[
M_{11} = \bar{I}_1 + m_1 l_1^2 c_1 + \bar{I}_2 + m_2 \left[ l_1^2 + l_2^2 c_2 + 2l_1 l_3 c_2 \right] + \bar{I}_3 + m_3 \left[ l_1^2 + l_2^2 + l_3^2 c_3 + 2l_1 l_2 c_2 + 2l_2 l_3 c_3 + 2l_1 l_3 c_3 \right]
\]

\[
M_{12} = \bar{I}_2 + m_2 l_1 l_2 c_2 + m_2 l_2^2 c_2 + \bar{I}_3 + m_3 \left[ l_2^2 + l_3^2 + l_1 l_2 c_2 + 2l_2 l_3 c_3 + l_1 l_3 c_3 \right]
\]

\[
M_{13} = \bar{I}_3 + m_3 \left[ l_3^2 + l_2 l_3 c_3 + l_1 l_3 c_3 \right]
\]

\[
M_{22} = \bar{I}_2 + m_2 l_2^2 + \bar{I}_3 + m_3 \left( l_2^2 + l_3^2 + 2l_2 l_3 c_3 \right)
\]

\[
M_{23} = \bar{I}_3 + m_3 \left( l_3^2 + l_2 l_3 c_3 \right)
\]
\[ M_{33} = I_3 + m_3 l_{c3}^2 \]

The matrix \( C \) can be calculated from \( M \):

\[
\dot{C}_{Mm} = \dot{\mathbf{M}}_{Mm} - \frac{1}{2} \frac{\partial}{\partial \theta_1} \left( \dot{\mathbf{q}}^T M_{Mm} + \dot{\mathbf{q}}^T \ddot{\mathbf{M}}_{Mm} \right)^T = \dot{\mathbf{M}}_{Mm}
\]

\[
C_{Mm} = \ddot{\mathbf{M}}_{Mm} - \frac{1}{2} \frac{\partial}{\partial \theta_1} \left( \dot{\mathbf{q}}^T \ddot{\mathbf{M}}_{Mm} \right) = \ddot{\mathbf{M}}_{Mm}
\]

\[
\dot{C}_{mM} = -\frac{1}{2} \dot{\theta}_1 \begin{bmatrix}
\frac{\partial M_{12}}{\partial \theta_2} & \frac{\partial M_{13}}{\partial \theta_2} \\
\frac{\partial M_{12}}{\partial \theta_3} & \frac{\partial M_{13}}{\partial \theta_3}
\end{bmatrix}
\]

\[
C_{mM} = \begin{bmatrix}
\dot{M}_{12} \\
\dot{M}_{13}
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial \theta_2} \left( \dot{\mathbf{q}}^T \ddot{\mathbf{M}}_{Mm} + \dot{\mathbf{q}}^T \dddot{\mathbf{M}}_{Mm} \right) \\
\frac{\partial}{\partial \theta_3} \left( \dot{\mathbf{q}}^T \ddot{\mathbf{M}}_{Mm} + \dot{\mathbf{q}}^T \dddot{\mathbf{M}}_{Mm} \right)
\end{bmatrix}
\]

and obtained as follows:

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\]

where

\[
C_{11} = -2m_2 l_1 l_2 \dot{\theta}_2 s_2 - m_3 \left[ 2l_1 l_2 \dot{\theta}_2 s_2 + 2l_1 l_3 \left( \dot{\theta}_2 + \dot{\theta}_3 \right) s_{23} + 2l_2 l_3 \dot{\theta}_3 s_3 \right]
\]

\[
C_{12} = -m_2 l_1 l_2 \dot{\theta}_2 s_2 - m_3 \left[ l_1 l_2 \dot{\theta}_2 s_2 + l_1 l_3 \left( \dot{\theta}_2 + \dot{\theta}_3 \right) s_{23} + 2l_2 l_3 \dot{\theta}_3 s_3 \right]
\]

\[
C_{13} = -m_3 \left[ l_1 l_3 \left( \dot{\theta}_2 + \dot{\theta}_3 \right) s_{23} + l_2 l_3 \dot{\theta}_3 s_3 \right]
\]

\[
C_{21} = m_2 l_1 l_2 \left( \dot{\theta}_1 - \dot{\theta}_2 \right) s_2
\]

\[
+ m_3 \left[ l_1 l_2 \left( \dot{\theta}_1 - \frac{1}{2} \dot{\theta}_2 \right) s_2 + l_1 l_3 \left( \dot{\theta}_1 - \frac{1}{2} \dot{\theta}_2 - \frac{1}{2} \dot{\theta}_3 \right) s_{23} - 2l_2 l_3 \dot{\theta}_3 s_3 \right]
\]
The gravity matrix is calculated as

\[ C_{22} = \frac{1}{2} m_2 l_1 l_2 \hat{\theta}_1 s_2 + m_3 \left[ \frac{1}{2} l_1 l_2 \hat{\theta}_1 s_2 + \frac{1}{2} l_1 l_3 \hat{\theta}_1 s_{23} - 2 l_2 l_3 \hat{\theta}_3 s_3 \right] \]

\[ C_{23} = m_3 \left[ \frac{1}{2} l_1 l_3 \hat{\theta}_1 s_{23} - l_2 l_3 \hat{\theta}_3 s_3 \right] \]

\[ C_{31} = m_3 \left[ l_1 l_3 \left( \hat{\theta}_1 - \frac{1}{2} \hat{\theta}_2 - \frac{1}{2} \hat{\theta}_3 \right) s_{23} + l_2 l_3 \left( \hat{\theta}_1 + \hat{\theta}_2 - \frac{1}{2} \hat{\theta}_3 \right) s_3 \right] \]

\[ C_{32} = m_3 \left[ l_1 l_3 \left( \frac{1}{2} \hat{\theta}_1 \right) s_{23} + l_2 l_3 \left( \hat{\theta}_1 + \hat{\theta}_2 - \frac{1}{2} \hat{\theta}_3 \right) s_3 \right] \]

\[ C_{33} = m_3 \left[ l_1 l_3 \left( \frac{1}{2} \hat{\theta}_1 \right) s_{23} + l_2 l_3 \left( \frac{1}{2} \hat{\theta}_1 + \frac{1}{2} \hat{\theta}_2 \right) s_3 \right] \]

The gravity matrix is calculated as

\[
G_{M_m} = m_2 g^T (J_{m_1}^v T)_{m_1} + m_3 g^T (J_{m_2}^v T)_{m_2}
\] (2.69)

\[
G'_m = \begin{pmatrix}
    m_2 g^T (J_{m_1}^v T)_{m_1} + m_3 g^T (J_{m_2}^v T)_{m_1} \\
    m_2 g^T (J_{m_1}^v T)_{m_2} + m_3 g^T (J_{m_2}^v T)_{m_2}
\end{pmatrix}
\] (2.70)

and obtained as:

\[
G = \begin{pmatrix}
    G_1 \\
    G_2 \\
    G_3
\end{pmatrix}
\] (2.71)

where

\[
G_1 = m_1 g l_1 c_1 + m_2 g (l_1 c_1 + l_2 c_{12}) + m_3 g (l_1 c_1 + l_2 c_{12} + l_3 c_{123})
\]

\[
G_2 = m_2 g l_2 c_{12} + m_3 g (l_2 c_{12} + l_3 c_{123})
\]

\[
G_3 = m_3 g l_3 c_{123}
\]

Based on the above derivations, the overall equations of motion can be constructed as (2.41).
2.7 Conclusion

This Chapter presents a modular formulation for the M/m manipulator system. The equations of motion of the overall system is constructed by directly using the original equations of individual subsystems and calculating their couplings. As shown in Eq. (2.41), the joint torques of the M/m system are the torques of the original subsystems plus the coupling torques. Closed-form equations of coupling torques are also established as expressed by Eq. (2.42). The equations contain three terms: torques caused by joint accelerations, torques by joint velocities and torques by link gravitational forces. The proposed modular formulation method is illustrated by an example of 3-DOF M/m system. In contrast with the conventional modeling methods, the proposed approach simplifies the formulation, shows the structure of the system dynamics, and allows for direct calculation of the coupling torques. The modular-form equations of motion are very useful for the dynamic analysis of the M/m system and for its controller design.
Chapter 3

Dynamics-Based Redundancy Resolution

3.1 Introduction

A M/m manipulator system is generally kinematically redundant. Kinematic redundancy enhances the system capability to reach the task in different orientations and allows for dexterous manipulation. It also makes it possible for the micro to compensate for the error produced by the macro. The basic property of a redundant manipulator is its redundant motion in the joint space. In other words, for any given end-point motion in the task space, the joint motion has infinitely many solutions. This characteristic of redundant manipulators has attracted extensive research. Most work focuses on solving redundancy based on kinematics, such as obstacle avoidance, singularity avoidance, manipulability enhancement, etc. As discussed in Chapter 1, incorporating the manipulator dynamics is a more practical approach in manipulator control.
In this chapter, a new, dynamics-based redundancy resolution method is proposed to avoid large peak torques. The contributions of both velocity and acceleration to the joint torques are taken into account. The joint torques at the current time are used to generate a joint acceleration which will lead to an optimum velocity at the end of time interval. The optimum velocity, along with the optimum acceleration, will suppress the peak torques at that time.

3.2 Problem Formulation

For a redundant manipulator, the instantaneous first-order kinematic relationship between the n-dimensional joint space vector (velocity) \( \dot{q} \) and m-dimensional task space vector (end-effector velocity) \( \dot{x} \) is expressed by

\[
\dot{x} = J\dot{q}
\]

where \( J \in \mathbb{R}^{m \times n} \) (\( m < n \)) is the manipulator Jacobian.

The joint acceleration is related to the end-effector acceleration by differentiating Eq. (3.1) to obtain

\[
\ddot{x} = J\ddot{q} + J\dot{\dot{q}}
\]

where the general solution for \( \ddot{q} \) is expressed in the form (Hollerbach and Suh 1987)

\[
\ddot{q} = J^+ \left( \ddot{x} - J\dot{q} \right) + (I - J^+J) \ddot{\phi}
\]

where \( \ddot{\phi} \in \mathbb{R}^n \) is an arbitrary vector and \( J^+ \) is the pseudoinverse of \( J \). \( (I - J^+J) \) is a projection operator on the null space of \( J \).

The dynamic equation of motion can be written as

\[
\tau = M(q)\ddot{q} + c(q, \dot{q}) + g(q)
\]
where \( \tau \) is an \( n \)-element vector of the joint forces/torques \( \tau_i \) (hereafter joint torques mean torques and/or forces). \( M(q) \) is the \( n \times n \) inertia matrix. \( c(q, \dot{q}) \) is an \( n \)-element vector of Coriolis and centrifugal forces, and \( g(q) \) is an \( n \)-element vector of gravitational forces.

First, let's consider the torque optimization problem. Given a desired end-effector trajectory \( x(t) \), tracking speed \( \dot{x}(t) \) and acceleration \( \ddot{x}(t) \), we would like to find the set of joint torques that results in \( x(t) \), \( \dot{x}(t) \) and \( \ddot{x}(t) \), and at the same time reduces actuator demands. This problem can be solved at the acceleration level (Hollerbach and Suh 1987). A quadratic objective function with a linear constraint is used:

Minimize:

\[
Q = \frac{1}{2} \left( \tau - \frac{\tau^+ + \tau^-}{2} \right)^T W \left( \tau - \frac{\tau^+ + \tau^-}{2} \right)
\]  

Subject to:

\[
\ddot{x} = J\ddot{q} + \dot{J}q
\]

where \( W \) is a symmetric positive definite weighting matrix

\[
W = \text{diag}\left[ \frac{1}{(\tau^+_i - \tau^-_i)^2} \right]
\]

and \( \tau^+_i \) and \( \tau^-_i \) are the upper and lower torque limits for joint \( i \). The goal of Eq. (3.5) is to place \( \tau \) closest to \((\tau^+ + \tau^-)/2 \) in a least square sense, by minimizing the performance index \( Q \). \( Q \) in Eq. (3.5) is considered as a function of \( \tau \), which in turn is a function of \( \ddot{q} \) expressed by Eq. (3.4). Since \( \ddot{q} \) has infinitely many solutions for the given end-effector acceleration \( \ddot{x} \), it is therefore possible to choose an optimum \( \ddot{q}_{opt} \) such that \( Q \) is minimum.

The Lagrangian function is

\[
L = \frac{1}{2} \left( \tau - \frac{\tau^+ + \tau^-}{2} \right)^T W \left( \tau - \frac{\tau^+ + \tau^-}{2} \right) - \lambda^T \left( \ddot{x} - J\ddot{q} - \dot{J}q \right)
\]

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where $\lambda \in \mathcal{R}^m$ is a vector of the Lagrangian multipliers. Letting

$$\frac{\partial L}{\partial \dot{q}} = 0.$$ 

it is obtained that

$$\frac{\partial L}{\partial \ddot{q}} = M^T W \left( M \ddot{q}_{opt} + c + g - \frac{\tau^+ + \tau^-}{2} \right) + J^T \lambda = 0$$

It follows that

$$\ddot{q}_{opt} = - \left( M^T W M \right)^{-1} \left[ M^T W \left( c + g - \frac{\tau^+ + \tau^-}{2} \right) + J^T \lambda \right]$$  \hspace{1cm} (3.7)

Using Eqs. (3.2) and (3.7), the solution of $\ddot{q}$ is obtained as follows

$$\ddot{q}_{opt} = A \left( \ddot{x} - \dot{J} \dot{q} \right) + B$$  \hspace{1cm} (3.8)

where

$$A = \left( M^T W M \right)^{-1} J^T \left[ J \left( M^T W M \right)^{-1} J^T \right]^{-1}$$  \hspace{1cm} (3.9)

$$B = (AJ - I) M^{-1} \left( c + g - \frac{\tau^+ + \tau^-}{2} \right)$$  \hspace{1cm} (3.10)

and $I$ is the $n \times n$ identity matrix.

Eq. (3.8) establishes the analytical relationship between the optimal joint acceleration $\ddot{q}_{opt}$, joint velocity $\dot{q}$ and end-effector acceleration $\ddot{x}$. At any time instant when joint position $q$ and velocity $\dot{q}$ are available, $\ddot{q}_{opt}$ is a function of $\ddot{x}$ only. Any $\ddot{x}$ in the task space corresponds to a $\ddot{q}_{opt}$ in the joint space which locally minimizes the actuator torques. This
is the redundant resolution at the acceleration level, which takes \( \ddot{q} \) as a variable of \( \tau \) while keeping \( \dot{q} \) as the integration of \( \ddot{q} \).

Obviously the above-mentioned redundancy resolution at the acceleration level has the following disadvantages:

1. It minimizes the joint torques locally, without global consideration:

2. It minimizes the joint torques all the time, no matter whether the torques are above or below the torque limits: and

3. \( \ddot{q}_{opt} \) thus obtained may lead to \( \dot{q} \) which produces large nonlinear forces.

In practice, torque minimization is not an important issue when driving torques are well below the torque limits. Reducing peak torques to avoid torque saturation is more important. This is the motivation of the peak torque reduction method developed in this Chapter.

### 3.3 Peak Torque Reduction Method

We consider a redundant manipulator which moves along a given trajectory at desired velocity and acceleration. Suppose that by using conventional redundancy resolution, joint torque vector \( \tau \) would exceed the limit at time \( t^P \) (i.e., some or all variables exceed the torque limits).

\[
\tau_i^P > \tau_i^+ \quad \text{or} \quad \tau_i^P < \tau_i^-
\]

where \( \tau_i^P \) is the torque of joint \( i \) at time \( t^P \). The joint torques during the time interval \( 0 \leq t < t^P \) are assumed to be below or at their limits.

\[
\tau_i^- \leq \tau_i(t) \leq \tau_i^+
\]
We now attempt to plan a new trajectory that can reduce $\tau^p$ by more efficient use of the redundancy, where $\tau^p$ is an $n$-element vector consisting of $\tau^p_i$.

It can be seen from Eq. (3.4) that for the given position $q$, $\tau$ is a function of $\dot{q}$ and $\ddot{q}$. Joint acceleration $\ddot{q}$ affects the inertial force, the first term of (3.4), and $\dot{q}$ affects the centrifugal and Coriolis forces, the second term of (3.4). At time $t^p$ when the torque is expected to exceed the limit due to the large end-effector velocity and/or acceleration, the conventional method (Hollerbach and Suh 1987) minimizes the torque by considering the contribution of $\ddot{q}_{opt}^p$ only, with $\dot{q}^p$ produced by the integration of $\ddot{q}(t) \ (t < t^p)$. Whether or not this $\dot{q}^p$ causes large centrifugal and Coriolis forces is not taken into consideration. We consider the contribution of $\dot{q}^p$ to $\tau^p$ by introducing $\ddot{q}_{opt}^p$, which is determined by the prescribed end-effector motion and produced by $\ddot{q}(t) \ (0 \leq t < t^p)$. Obviously, the combined contributions of $\ddot{q}_{opt}^p$ and $\ddot{q}_{opt}^p$ could make $\tau^p$ even smaller than obtained by using the conventional torque optimization method (Hollerbach and Suh 1987), which includes the contribution of $\ddot{q}_{opt}^p$ only.

Since $\tau^p$ is a function expressed by Eq. (3.4) of joint velocity $\dot{q}^p$ and acceleration $\ddot{q}^p$ (for given position $q^p$) both of which have infinite many solutions for the desired end-effector motion, we would like to:

1. find the optimum velocity and acceleration, $\dot{q}_{opt}^p$ and $\ddot{q}_{opt}^p$, which would simultaneously minimize the performance index $Q$ in Eq. (3.5):

2. choose a suitable function $\ddot{q}(t)$ for $0 \leq t < t^p$ such that it leads to

$$\ddot{q}_{opt}^p = \int_0^{t^p} \ddot{q}(t) dt \quad (3.11)$$
subject to
\[ \tau_i^- \leq \tau_i (t) \leq \tau_i^+ \]

The objective of this optimization scheme is to reduce \( \tau^P \). The joint torque in the time interval \( 0 \leq t < t^P \) is not minimized, since it is assumed to be below or at the limit. The choice of \( \ddot{q}(t) \) is arbitrary, as long as it satisfies Eqs. (3.2) and (3.11). As a result, \( \tau(t) \) is also arbitrary for \( 0 \leq t < t^P \), as long as it does not exceed torque limits.

In order to get \( \dot{q}_{\text{opt}}^P \) and \( \ddot{q}_{\text{opt}}^P \), Eq. (3.8) is substituted into Eq. (3.4). It follows that
\[ \tau^P = M(q^P)A(q^P)\left[\ddot{x}^P - J(q^P)\dot{q}^P\right] + M(q^P)B(q^P, \dot{q}^P) + c(q^P, \dot{q}^P) + g(q^P) \] (3.12)
where \( \ddot{x}^P \) is the desired end-effector acceleration. Using Eq. (3.10) and after mathematical manipulation. Eq. (3.12) can be rearranged as following
\[ \tau^P = M(q^P)A(q^P)\left[J(q^P)M^{-1}(q^P)c(q^P, \dot{q}^P) - J(q^P)\dot{q}^P\right] + K(q^P) \] (3.13)
where
\[ K(q^P) = M(q^P)A(q^P)\ddot{x}^P + M(q^P)A(q^P)J(q^P)M^{-1}(q^P)\left[g(q^P) - \frac{\tau^+ + \tau^-}{2}\right] \]
\[ + \frac{\tau^+ + \tau^-}{2} \]

Eq. (3.13) indicates that \( \tau^P \) is a function of \( \dot{q}^P \) for the given position \( q^P \). which is of the second order. By substituting Eq. (3.13) into Eq. (3.5), a fourth order objective function of \( \dot{q}^P \) is obtained:
\[ Q = \frac{1}{2} \left[ \tau^P(q^P, \dot{q}^P, \ddot{x}^P) - \frac{\tau^+ + \tau^-}{2} \right]^T W \left[ \tau^P(q^P, \dot{q}^P, \ddot{x}^P) - \frac{\tau^+ + \tau^-}{2} \right] \] (3.14)
\( \dot{q}_{\text{opt}}^P \) is obtained by minimizing \( Q \). subject to linear constraint
\[ \ddot{x}^P = J(q^P)\dot{q}^P \] (3.15)
where \( \dddot{x}^P \) is the desired end-effector velocity. \( \dot{q}_{\text{opt}}^P \) is then available by using Eq. (3.8).
3.4 Implementation

Although the procedure described above provides promising advantages for peak torque reduction, it is a global optimization scheme and computationally very complex since it requires information about the entire trajectory. Also it is difficult to determine:

1. when the torque exceeds the limit; and

2. how to choose $q(t)$ for $0 \leq t < t_P$ such that it satisfies Eq. (3.11).

In this section, we attempt to develop a local optimization scheme that requires the local information of the end-effector motion, based on the idea developed above.

Instead of considering the time interval $0 \leq t \leq t_P$ as in the global algorithm, the local algorithm uses the interval $t^i \leq t \leq t^{i+1}$, where $t^i$ and $t^{i+1}$ are two immediate time instants. The basic idea of the local algorithm is to consider the following station $i + 1$ as an objective station and use the current torque $\tau^i$ to accelerate/decelerate the joint velocity from $\dot{q}^i$ to $\dot{q}_{opt}^{i+1}$, and at the same time keep $\tau^i$ within its limits.

The optimization problem can be rewritten from Eqs. (3.14) and (3.15) as follows:

Minimize

$$Q = \frac{1}{2} \left[ \tau^{i+1} (q^{i+1}, \dot{q}^{i+1}) - \frac{\tau^{+} + \tau^{-}}{2} \right]^T W \left[ \tau^{i+1} (q^{i+1}, \dot{q}^{i+1}) - \frac{\tau^{+} + \tau^{-}}{2} \right]$$

subject to

$$\dot{x}^{i+1} = J(q^{i+1}) \dot{q}^{i+1}$$

and

$$\tau^-_k \leq \tau^i_k \leq \tau^+_k$$

The problem can be stated as follows:
Find a joint torque $\tau^i$ at the current station $i$ such that it drives the manipulator to reach a velocity $q^{i+1}$ at the next station $i + 1$, which minimizes $Q$ expressed by Eq. (3.16). At the same time, $\tau^i$ satisfies (3.18) and $q^{i+1}$ satisfies (3.17).

In order to make the problem straightforward, we should attempt to formulate the problem in another form as follows.

At the current station $i$, $q'$ and $q''$ are already known from the previous station. The position $q^{i+1}$ of the next station can be calculated by

$$q^{i+1} = \int_{t'}^{t^{i+1}} q'' dt + q' \approx q' \Delta t + q'$$  \hspace{1cm} (3.19)

In order to reach a velocity $q^{i+1}$ at the following station, the acceleration $q'$ at the current station must satisfy:

$$q^{i+1} = \int_{t'}^{t^{i+1}} q' dt + q' \approx q' \Delta t + q'$$  \hspace{1cm} (3.20)

From Eqs. (3.19) and (3.20), the joint velocity $q'$ and acceleration $q'$ at the current station $i$ is defined in $[t', t^{i+1})$. Since the relationship between $q'$ and $\tau^i$ can be written from Eq. (3.4) as follows:

$$q' = M^{-1} (q') \left[ \tau^i - c \left( q', q'' \right) - g \left( q' \right) \right]$$  \hspace{1cm} (3.21)

Eq. (3.20) becomes:

$$q^{i+1} = M^{-1} (q') \left[ \tau^i - c \left( q', q'' \right) - g \left( q' \right) \right] \Delta t + q'$$  \hspace{1cm} (3.22)

which indicates that $q^{i+1}$ is a function of $\tau^i$ only, if $q'$ and $q_i$ are known.

Based on Eqs. (3.19) and (3.22), the optimization problem expressed by Eqs. (3.16), (3.17) and (3.18) is transformed into:

Minimize

$$Q = \frac{1}{2} \left[ q^{i+1} (q', q', \tau^i) - \bar{\tau}^+ + \bar{\tau}^- \right]^T W \left[ q^{i+1} (q', q', \tau^i) - \bar{\tau}^+ + \bar{\tau}^- \right]$$  \hspace{1cm} (3.23)
subject to

$$\dot{x}^{i+1} = J (\dot{q}^i \Delta t + q^i) M^{-1} (q^i) \left[ \tau^i - c (q^i, \dot{q}^i) - g (q^i) \right] \Delta t + J (\dot{q}^i \Delta t + q^i) \ddot{x}^i$$ \hspace{1cm} (3.24)$$

and

$$\tau^i_k^- \leq \tau^i_k \leq \tau^i_k^+$$ \hspace{1cm} (3.25)

The obtained torque $\tau^i$ is the optimum torque $\tau^i_{opt}$. The problem can be stated as follows:

Find a joint torque $\tau^i$ at the current station $i$ such that $Q$ expressed by Eq. (3.23)

is minimized, subject to constraints (3.24) and (3.25).

This optimization problem can be also interpreted as the problem of finding the optimum torque $\tau^i_{opt}$ at the current station $i$ to minimize the torque $\tau^{i+1}$ at the following station $i + 1$.

Based on the peak torque reduction method described above, a computational algorithm for joint trajectory planning of a given end-effector motion is proposed. It contains the following steps (3.1):

(1) Based on the information at current station $i$, calculate the position of the next station $i + 1$ using Eq. (3.19):

(2) Obtain the current torque $\tau^i_{opt}$ using Eqs. (3.23), (3.24) and (3.25). The obtained $\tau^i_{opt}$ is below the limits. It minimizes $\tau^{i+1}$ at the following station.

(3) Compute the current acceleration $\ddot{q}^i_{opt}$. Based on the known position $q^i$ and velocity $\dot{q}^i$ at the current station, the acceleration $\ddot{q}^i$ is generated by the torque $\tau^i$. The optimum torque $\tau^i_{opt}$ will produce the optimum acceleration $\ddot{q}^i_{opt}$. Eq. (3.21) can be rewritten as:

$$\ddot{q}^i_{opt} = M^{-1} (q^i) \left[ \tau^i_{opt} - c (q^i, \dot{q}^i) - g (q^i) \right]$$ \hspace{1cm} (3.26)
(4) Compute the velocity $\dot{q}_{i+1}^{opt}$ of the next station. To obtain $\dot{q}_{i+1}^{opt}$, Eq. (3.20) is rewritten as:

$$\dot{q}_{i+1}^{opt} = \ddot{q}_{i+1}^{opt} \Delta t + \dot{q}_i$$

(3.27)

It can be seen from Eqs. (26) and (3.27) that by applying the optimum torque $\tau_{i+1}^{opt}$ at the current station, the optimum velocity $\dot{q}^{i+1}_{opt}$ is achieved at the next station. This velocity will minimize the torque $\tau^{i+1}$, as can be seen from Eq. (3.16).
(5) Repeat above steps at station $i + 1$.

It is worth noting that the Peak Torque Reduction method does not minimize the current torque $\tau^i$. Instead, it uses $\tau^i$ to minimize the torque $\tau^{i+1}$ at the next station. In other words, $\tau^i$ is obtained based on the previewed information of the next station. When a large torque is needed at the next station due to the large end-effector velocity and acceleration, a large current torque $\tau^i_{opt}$ may be used, subject to the torque limits, to drive the manipulator to the optimum velocity $q^{i+1}_{opt}$ at the next station, which, along with $q^{i+1}_{opt}$, will reduce the joint torque $\tau^{i+1}$.

### 3.5 Comparison with Traditional Methods

As mentioned before, the existing redundancy resolutions incorporated with the manipulator dynamics are mainly based on two methods: Torque Optimization (TO) (Hollerbach and Suh 1987) and Energy Minimization (EM) (Khatib 1983). This section addresses the differences between those methods and the Peak Torque Reduction (PTR) method proposed in this research.

#### 3.5.1 Comparison with TO Method

The TO method, or Weighted Null-Space method (Hollerbach and Suh 1987), uses a quadratic objective function (3.5) with a linear constraint (3.2) to minimize joint torques. An optimum joint acceleration $\dot{q}_{opt}$ expressed in Eq. (3.8) is obtained, which leads to a locally optimum torque vector. The TO method minimizes joint torques at every time instant by an appropriate choice of the joint acceleration, without consideration of the contribution of the joint
velocity to torques. In fact, it is possible that the optimum joint acceleration thus obtained may produce a joint velocity at the next station which leads to large centrifugal and Coriolis forces, and hence large joint torques.

The objective of the PTR method proposed in this paper is to reduce peak torques. So the PTR method minimizes joint torques when they are large enough to exceed the torque limits, and do not minimize joint torques all the time. To implement this idea, a local optimization scheme is developed, in which the robot uses its torque at the current station $i$ to reach an optimum velocity $\hat{q}_{opt}^{i+1}$ of the next station $i + 1$, which leads to small nonlinear forces and small joint torques. The joint acceleration $\ddot{q}^{i+1}$ of station $i + 1$ produced by the optimized $\tau^{i+1}$ can produce the optimum velocity at station $i + 2$ and at the same time keeps the torque below the limits. This lead control feature of the PTR method makes the peak torque produced by the PTR method smaller than that by the TO method.

### 3.5.2 Comparison with EM Method

The EM method uses the inertia-weighted pseudoinverse to minimize instantaneous kinetic energy

$$T = \frac{1}{2} \dot{q}^T M \dot{q}$$

subject to the linear constraint (3.1). The joint velocity and acceleration are then obtained as follows

$$\dot{q} = J_{\dot{q}}^+ \dot{x}$$

$$\ddot{q} = J_{\ddot{q}}^+ (\ddot{x} - J \ddot{q})$$
where $J_M^+$ is the inertia-weighted pseudoinverse given by

$$J_M^+ = M^{-1} J^T \left( J M^{-1} J^T \right)^{-1}$$

Since the objective of this method is to minimize the kinetic energy, the joint torques are not necessarily minimized. In fact, if a large energy is needed at the next station due to the desired large end-effector velocity, energy minimization at the current station will lead to large actuator torques.

Different from the EM method, the PTR method uses the torque at the current station $i$ to approach an optimum velocity of the following station $i + 1$ where joint torques are minimized. The relationship between the work done by joint torques over $[t^i, t^{i+1}]$ and the system energy is given by

$$\int_{q^i}^{q^{i+1}} \tau^T dq = T^{i+1} \left( q^{i+1}, \dot{q}_{opt}^{i+1} \right) + P^{i+1} \left( q^{i+1} \right) - T^i \left( q^i, \dot{q}^i \right) - P^i \left( q^i \right)$$

where $T^k$ and $P^k$ are the kinetic energy and potential energy of the system at station $k$ ($k = i, i + 1$). If $\tau$ does positive work, the system energy increases ($T^{i+1} + P^{i+1} > T^i + P^i$); if $\tau$ does negative work, the energy decreases ($T^{i+1} + P^{i+1} < T^i + P^i$). In other words, the PTR method may increase the energy or reduce it (but does not minimize it), depending on the optimum velocity $q_{opt}^{i+1}$.

### 3.6 Simulation

The comparative evaluation of the proposed PTR method against the TO method and the EM method is conducted by computer simulations. The simulated model is a planar 3-DOF rotary manipulator (Hollerbach and Suh 1987), as shown in Figure 3.2. The links are all
identical and modeled as uniform thin rods with lengths $l_1 = l_2 = l_3 = 1m$ and masses $m_1 = m_2 = m_3 = 10kg$. The simulated end-effector movements are straight-line Cartesian trajectory starting and ending with zero velocity, with equal and constant acceleration and deceleration over the first and last halves of the movements, respectively. The total duration time and acceleration/deceleration of the movements are varied at each simulation. The end-effector has two-dimensional movements, translations along $x$ and $y$, with its orientation unconstrained, giving one DOF redundancy. The manipulator moves in a horizontal plane, so the gravity has no influence on the motion. The upper and lower torque limits for joint 1 – 3 are set at $\pm 54, \pm 24$, and $\pm 6.4 \, Nm$, respectively.

First, the case of a short movement is considered, where the end-effector makes a movement of 0.2 $m$ in both $x$ and $y$ directions, starting from the initial configuration $q_1 = -45^\circ$, $q_2 = 135^\circ$, and $q_3 = -135^\circ$, at an acceleration/deceleration of 2 $m/s^2$ in both $x$ and $y$ directions. Torque profiles for joints 1 – 3 and continual motion configurations are shown in Figure 3.3. The results obtained by using the PTR method are compared with
Figure 3.3: Short movement
Figure 3.4: Medium-length movement
Figure 3.5: Long movement
those obtained by using TO method and EM method. The TO method leads to the peak torque of $-56 \, Nm$ for joint 1 (the torque limits are $\pm 54 \, Nm$), while the EM method leads to the peak torque of $9.7 \, Nm$ for joint 3 (the torque limits are $\pm 6 \, Nm$). The PTR method reduces the peak torques and keeps all joint torques within their limits. Note that for the PTR method the torque for joint 3 is always at $\pm 6 \, Nm$, which means that the ability of the motor for joint 3 is fully used in order to reduce peak torques. Three right boxes in Figure 3.3 show the continual configurations for three different methods.

In the second case, the length of movement is increased to $0.5 \, m$ in both $x$ and $y$ directions, with all other conditions the same as in the first case. Simulation results are shown in Figure 3.4. In this case, the TO method produces large peak torques of $98 \, Nm$ for joint 1, $-64 \, Nm$ for joint 2, and $-6.8 \, Nm$ for joint 3, all exceeding the torque limits. As in the first case, the EM method leads to relatively small torques for joint 1 and 2, but a large torque for joint 3. One of the reasons for this "unbalance" is that the EM method does not weight the joint torques by the magnitudes of the torque ranges, but by the inertia matrix. Once again, the PTR method yields the best results, with all joint torques within their limits.

In the last case, a long movement of $0.83 \, m$ in both $x$ and $y$ directions is considered. The initial configuration remains the same as above, but the acceleration/deceleration is reduced to $1 \, m/s^2$. Simulation results (Figure 3.5) show that the TO method and the EM method produce extremely large torques for all three joints. Superior to the others, the PTR method once again keeps all joint torques within torque limits. It is interesting to note that in this case the joint 3 torque does not reach its limits all the time. In the first 0.26 seconds and the last 0.54 seconds, the joint torque is below the limits, which implies that not all
motor power is needed to reach the optimum velocity which minimizes the joint torques. This is because the end-effector acceleration/deceleration in this case is much smaller than in the first two cases and the end-effector velocity is small in the beginning and the end.

It is shown in the above three cases that the PTR method is a superior and reliable method that can yield satisfactory results for all short, medium-length and long movements. Although the PTR method is a local algorithm, it is able to globally reduce peak torques because of its tendency to minimize torques at the next station. The TO method, though locally minimizing joint torques, does not reduce peak torques from a global point of view. Of all three methods, the TO method most likely leads to an unacceptable performance. Compared with the TO method, the EM method produces smaller torques in the sense of absolute magnitudes, because the energy minimization leads to smaller joint velocities and smaller nonlinear forces. However, it cannot effectively reduce peak torques as the PTR method does, because it cannot “foresee” the energy requirement of the next station, and because the joint torques are not weighted by the magnitudes of the torque ranges, but by the inertia matrix.

3.7 Discussion and Conclusion

A new method has been presented in this Chapter for kinematically redundant manipulators in an attempt to reduce the peak torques. Instead of minimizing the joint torques or kinetic energy at the current station, the method always uses the current torque to approach an optimum velocity of the following station, which, along with joint acceleration, minimizes the torque of the following station, while at the same time keeps the current torque within the limits. In other words, the PTR method always intends to use the current power to minimize
torque at the next station. This "foreseeing" characteristic is the essential difference between the PTR method and the conventional local optimization schemes.

The basic idea of the PTR method described in Section 3.3 is a global approach, which requires complete description of the desired end-effector movement. To obtain the global solutions, the global optimization problems are usually converted into a two-point boundary value problem and solved by the methods of calculus of variations (Nakamura and Hanafusa 1987), or posed as a finite time nonlinear control problem and solved by a Newton-Raphson type algorithm (Seeresam and Wen 1993). No matter what approaches are used, the global optimization schemes are computationally intensive because of their inherent complexity. Thus, they are used for off-line programming, not for real-time control.

Instead of calculating the global minima based on the information of the entire trajectory, the local scheme of the PTR method described in Section 3.4 calculates the local minima based on the information of current station and next station. Therefore, it can greatly reduce the computations and becomes suitable for real-time control. On the other hand, because of its "foreseeing" characteristic, the PTR method can yield solutions globally better than those by truly local schemes, which calculate the local minima based on the information of current station only.

Simulation results show the clear superiority of the proposed method over the conventional TO method and EM method in three different cases: short movement, medium-length movement and long movement. In all three cases, the PTR method can effectively reduce peak torques from a global point of view, while the other methods leads to unacceptable results. The proposed PTR method is applicable to conventional redundant manipulators, not just limited to redundant M/m manipulator systems.

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Chapter 4

Natural Modes of Beams with a Flexible Joint

4.1 Introduction

Mechanical flexibility of robotic manipulators is a major problem in motion control. In many cases, such as manufacturing, high-speed manipulation, large-reach operation and space application, the slender, flexible mechanical structure may cause undesirable vibrations. In the case of flexible M/m manipulator systems, the flexibility in the macro may make the micro miss the target because of tracking errors. The flexibility may come from the manipulator links or joints or both. Study on the joint flexibility, in addition to the link flexibility, is especially meaningful for M/m manipulator systems. A small joint flexibility in the macro joints may produce a significant deflection at the micro tip, owing to the nature of long reach. Thus, the link flexibility must be integrated with joint flexibility in the modeling of the flexible M/m systems.
This Chapter and the following Chapter address the dynamics of manipulators with both link and joint flexibility. The issues of model order reduction by the integration of joint flexibility with link flexibility and the model accuracy with the proposed approach are investigated. The objective of this chapter is to derive the mode shapes of a flexible beam supported by a flexible joint, oriented towards the general multi-link cases. The payload and hub are also taken into account. An analytical approach is adopted to establish the equations of motion of the system. Analytical shape functions are obtained. A parametric study to investigate the influence of different parameters on the natural frequencies is performed. The obtained mode shapes will be used to model flexible-link, flexible-joint manipulators in the following chapter.

### 4.2 Equations of Motion

The objective of this section is to establish a dynamic model of the one-link flexible robot. Consider a uniform distributed beam driven by a motor through a flexible joint, as shown
in Figure 4.1. A payload is attached to the free end of the beam.

The system has the following parameters: $L$, $E$, $I$ and $\rho$ are the length, Young’s modulus, the moment of inertia of the cross section, and the density (mass per unit length) of the link; $m_p$ and $I_p$ are the mass and moment of inertia of the payload; $I_h$ is the moment of inertia of the hub; $I_r$ is the moment of the inertia of the rotor; and $k$ is the stiffness of the joint. The coordinate frame $xyz$ is the original inertial coordinate system with $z$ in vertical direction, while frame $x_1y_1z_1$ is the coordinate system attached to the hub, with $z_1$ coinciding with $z$.

The following assumptions (Rakhsha and Goldenberg 1985, Cetinkunt and Yu 1991) are introduced in modeling the system: (1) only the rigid motion and elastic deflection in the horizontal plane are considered; (2) the elastic deflection is small; (3) the radius of the joint hub and the payload holding distance are ignored (but their moments of inertia are still taken into account); (4) Bernoulli-Euler beam assumptions are used (rotary inertia and shear deflection are neglected); (5) the flexible joint is modeled as a linear torsional spring (Spong 1987).

Based on the above assumptions, the motion of the link is a superposition of the link’s gross motion and its transverse vibration about its nominal rigid position. The position of any point on the link is described uniquely by three quantities: $\theta$ represents the nominal position of the link with respect to the inertial frame $xyz$; $\varphi$ is the angle of the hub with respect to the rotor due to the joint deflection; and $w(x_1, t)$ is the deflection of the link measured from the line $\alpha x_1$.

For small $\theta$ and $\varphi$, the position of an arbitrary point $x$ on the link is calculated as:

$$y(x, t) = x[\theta(t) + \varphi(t)] + w(x, t)$$  \hspace{1cm} (4.1)
The kinetic energy of the system is expressed as

\[
T = \frac{1}{2} I_r \dot{\theta}^2 + \frac{1}{2} I_h \left( \dot{\dot{\theta}} + \dot{\dot{\beta}} \right)^2 + \frac{1}{2} \int_0^L \rho \left[ x \left( \dot{\theta} + \dot{\beta} \right) + \frac{\partial w}{\partial t} \right]^2 \, dx
\]

\[
+ \frac{1}{2} m_p \left[ x \left( \dot{\theta} + \dot{\beta} \right) + \frac{\partial w}{\partial t} \right]^2 \bigg|_{x=L} + \frac{1}{2} I_p \left[ \left( \dot{\theta} + \dot{\beta} \right) + \frac{\partial^2 w}{\partial x \partial t} \right]^2 \bigg|_{x=L}
\]  

\[(4.2)\]

The potential energy of the system is

\[
V = \frac{1}{2} k \beta^2 + \frac{1}{2} \int_0^L EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx
\]

\[(4.3)\]

The only non-conservative force exerted on the system is the joint torque \( \tau \). Its virtual work is

\[
\delta W = \tau \delta \theta
\]

\[(4.4)\]

where \( \delta \theta \) is the virtual displacements of the rotor.

By applying Hamilton's principle

\[
\delta \int_{t_1}^{t_2} (T - V) dt + \int_{t_1}^{t_2} \delta W dt = 0
\]

a fourth-order partial differential equation is obtained:

\[
EI \frac{\partial^4 w}{\partial x^4} + \rho \frac{\partial^2 w}{\partial t^2} = -\rho x \left( \ddot{\theta} + \ddot{\beta} \right)
\]

\[(4.5)\]

with four associated boundary conditions:

at \( x = 0 \).

\[
w_{x=0} = 0
\]

\[(4.6a)\]

\[
\left( \frac{\partial w}{\partial x} \right)_{x=0} = 0
\]

\[(4.6b)\]

at \( x = L \).

\[
EI \left( \frac{\partial^3 w}{\partial x^3} \right)_{x=L} - m_p \left[ x \left( \dot{\theta} + \dot{\beta} \right) + \frac{\partial^2 w}{\partial t^2} \right]_{x=L} = 0
\]

\[(4.6c)\]
In addition, two dynamic equations can be established for the rotor and the hub.

For the rotor:

\[ \tau + k3 = I_r \ddot{\theta} \]  

(4.7)

For the hub:

\[ EI \left( \frac{\partial^2 w}{\partial x^2} \right)_{x=L} + I_p \left( \ddot{\theta} + \dddot{\theta} \right)_{x=L} = 0 \]  

(4.6d)

Eqs. (4.5)-(4.8) describe the motion of the whole system. The accurate mode shapes of the link can be obtained from these equations. In Eqs. (4.5)-(4.8), the motion of the system is described in the coordinate system \(x_1y_1z_1\) in terms of \(w\). In order to describe the motion in the inertial coordinate system \(xyz\), a transformation is needed. From Eq. (4.1), it is obtained that

\[ w = y - x(\theta + 3) \]  

(4.9)

Substituting Eq. (4.9) into Eqs. (4.5) and (4.6a)-(4.6d), the equation of motion for vibration of the link is derived:

\[ EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} = 0 \]  

(4.10)

with the boundary conditions:

at \(x = 0\),

\[ y_{x=0} = 0 \]  

(4.11a)

\[ \left( \frac{\partial y}{\partial x} \right)_{x=0} - (\theta + 3) = 0 \]  

(4.11b)

at \(x = L\),

\[ EI \left( \frac{\partial^3 y}{\partial x^3} \right)_{x=L} - m_p \left( \frac{\partial^2 y}{\partial t^2} \right)_{x=L} = 0 \]  

(4.11c)
To derive the natural vibration modes, \( \tau \) is dropped from Eq. (4.7). Thus, the equation of motion of the rotor is

\[
\ddot{\theta} = \frac{k}{I_r} \dot{\theta}
\]  

(4.12)

By substituting Eqs. (4.9) into (4.8), the equation of motion of the hub becomes

\[
EI \left( \frac{\partial^2 y}{\partial x^2} \right)_{x=0} - k \dot{3} = I_h \left( \ddot{\theta} + \dot{3} \right)
\]

Differentiating the boundary condition (4.11b) twice with respect to time and then substituting the result into the above equation, the following equation is obtained:

\[
\dot{3} = \frac{EI}{k} \left( \frac{\partial^2 y}{\partial x^2} \right)_{x=0} - \frac{I_h}{k} \left( \frac{\partial^3 y}{\partial x \partial t^2} \right)_{x=0}
\]  

(4.13)

4.3 Eigenvalue Problem

Assume the harmonic motion of the link given by

\[
y(x, t) = \phi(x) \cos \omega t
\]  

(4.14)

where \( \phi(x) \) is the mode shape function of the link expressed in the coordinate system \( xyz \) and \( \omega \) is the natural frequency. The motion of the flexible joint can be obtained from Eq. (4.13) as follows

\[
\dot{3} = \frac{1}{k} \left[ EI \phi''(0) + I_h \omega^2 \phi'(0) \right] \cos \omega t
\]  

(4.15)

Eq. (4.15) is then substituted into Eq. (4.12) to get the motion of the rotor

\[
\ddot{\theta} = \frac{1}{I_r} \left[ EI \phi''(0) + I_h \omega^2 \phi'(0) \right] \cos \omega t
\]  

(4.16)
The solution to Eq. (4.16) is

\[ \theta(t) = H \cos \omega t + C_1 t + C_2 \]  

(4.17)

where \( H \) is given by

\[ H = -\frac{1}{I_r} \left[ \frac{EI}{\omega^2} \phi''(0) + I_h \phi'(0) \right] \]  

(4.18)

and \( C_1 \) and \( C_2 \) are integral constants.

For a flexible link, two types of natural modes are usually considered: the unconstrained and the constrained modes of vibration (Barbieri and Özgüner 1988). The unconstrained modes are defined as the modes of the link with all external influences removed. These modes correspond to the motion of a flexible link in the free space (Xi and Fenton 1994). The constrained modes are defined as those when the rigid body is constrained or fixed in an inertial frame. The constrained modes are quite natural for a flexible-link, rigid-joint system where the links are controlled by actuators. In this research, the constrained modes are considered for the flexible-link, flexible-joint system. To derive the constrained modes, the rotor is fixed, which means that \( \theta = 0 \).

By substituting Eqs. (4.14) and (4.15) into (4.10) and (4.11a)-(4.11d), the partial differential equation (4.10) becomes

\[ \phi^{(4)} - \lambda^4 \phi = 0 \]  

(4.19)

with four boundary conditions:

at \( x = 0 \),

\[ \phi(0) = 0 \]  

(4.20a)

\[ EI \phi''(0) - (k - I_h \omega^2) \phi'(0) = 0 \]  

(4.20b)
at \( x = L \).

\[
EI \phi''(L) + m_p \omega^2 \phi(L) = 0 \tag{4.20c}
\]

\[
EI \phi''(L) - I_p \omega^2 \phi'(L) = 0 \tag{4.20d}
\]

where

\[
\lambda^4 = \frac{\rho \omega^2}{EI} \tag{4.21}
\]

Eq. (4.19) and (4.20a)-(4.20d) are the governing equations of the mode shapes and mode frequencies for a flexible link supported by a flexible joint. From these equations, the characteristic equation can be derived.

The general solution to Eq. (4.19) is

\[
\phi(x) = A_1 \sin \lambda x + A_2 \cos \lambda x + A_3 \sinh \lambda x + A_4 \cosh \lambda x \tag{4.22}
\]

where \( A_i \) (\( i = 1, \cdots, 4 \)) are four constants determined by the four boundary conditions (4.20a)-(4.20d). After applying the four boundary conditions to Eq. (4.22), the following set of four algebraic equations are obtained:

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  A_1 \\
  \vdots \\
  A_4
\end{bmatrix} = 0 \tag{4.23}
\]

where the expressions of coefficients \( a_{ij} \) are given in Appendix A.

For the solution of Eq. (4.23) to be nontrivial, the following equation must hold true

\[
\det |a| = 0
\]
which is the characteristic equation or frequency equation, from which the frequencies can be calculated. After mathematical manipulation, the characteristic equation is derived as

\[
B_1 \cosh + B_2 \cosh + B_3 \cosh + B_4 \sinh + B_5 = 0 \tag{4.24}
\]

where a shorthand notation \( s = \sin \lambda L, c = \cos \lambda L, \text{sh} = \sinh \lambda L, \text{ch} = \sinh \lambda L \) is used. \( B_i \) in Eq. (4.24) are derived from \( a_{ij} \) as follows (the derivation is omitted):

\[
\begin{align*}
B_1 &= b_1 + b_2 - b_3 + b_1 b_2 b_3 \tag{4.25a} \\
B_2 &= b_1 - b_2 + b_3 - b_1 b_2 b_3 \tag{4.25b} \\
B_3 &= b_1 b_2 - 2b_1 b_3 - 1 \tag{4.25c} \\
B_4 &= -2b_2 b_3 \tag{4.25d} \\
B_5 &= -b_1 b_2 - 1 \tag{4.25e}
\end{align*}
\]

in which

\[
\begin{align*}
b_1 &= \frac{1}{3} K'_{lp} \gamma^3 \tag{4.25f} \\
b_2 &= K'_{mp} \gamma \tag{4.25g} \\
b_3 &= \frac{\gamma}{\frac{1}{3} K'_{lh} \gamma^4 - K_k} \tag{4.25h}
\end{align*}
\]

where \( K_k, K'_{lh}, K_{mp}, \) and \( K'_{lp} \) are the relative stiffness of the joint, relative moment of inertia of the hub, relative mass of the payload, and relative moment of inertia of the payload, respectively. They are all non-dimensionalized parameters with respect to the beam, defined by

\[
K_k = \frac{kL}{EI}, \quad K'_{lh} = \frac{I_h}{I_b}, \quad K_{mp} = \frac{m_p}{\rho L}, \quad K'_{lp} = \frac{I_p}{I_b} \tag{4.26}
\]
where \( I_b \) is the moment of inertia of the beam given by \( I_b = \frac{1}{3} \rho L^3 \). \( \gamma \) is the root of the characteristic equation (4.24) defined by

\[
\gamma = \lambda L \tag{4.27}
\]

which is also a non-dimensional parameter.

Substituting (4.25a)-(4.25h) into (4.24), the characteristic equation (4.24) can be expressed in the following form:

\[
K_k - \frac{1}{3} \gamma^4 K_{ih} = \gamma \frac{(3 - K_{mp} K_{lp} \gamma^4)(s ch - c sh) + 2 K_{lp} \gamma^3 c ch + 6 K_{mp} \gamma s sh}{3(c ch + 1) - K_{lp} \gamma^3(s ch + c sh) - 3 K_{mp} \gamma(s ch - c sh) - K_{mp} K_{lp} \gamma^4(c ch - 1)} \tag{4.28}
\]

It can be shown that if the payload and hub are ignored, i.e.,

\[
K_{mp} = 0, \quad K_{lp} = 0, \quad K_{ih} = 0.
\]

Eq. (4.28) reduces to the characteristic equation of the classic pinned-free beam

\[
\tan \gamma = \tanh \gamma
\]

when \( K_k = 0 \) (zero joint stiffness), and to the characteristic equation of the classic clamped-free beam

\[
1 + \cos \gamma \cosh \gamma = 0
\]

when \( K_k = \infty \) (infinite joint stiffness). In contrast to the clamped-free modes and pinned-free modes, the modes for the flexible link (beam) with a flexible joint are called flexible-free modes.

The flexible-free mode shape is derived as

\[
\phi(x) = C[(\cos \lambda x - \cosh \lambda x) + \epsilon(\sin \lambda x - \sinh \lambda x) + \zeta(\sin \lambda x + \sinh \lambda x)] \tag{4.29}
\]
where $C$ is a constant, and $\epsilon$ and $\zeta$ are given by

$$\zeta = \frac{3\gamma}{K_{th}\gamma^4 - 3K_k}$$  \hspace{1cm} (4.30a)$$

$$\epsilon = -\frac{3(c + ch) + \zeta[3(s - sh) + \gamma^3K_{fp}(c + ch)] - \gamma^3K_{fp}(s + sh)}{3(s + sh) + \gamma^3K_{fp}(c - ch)}$$  \hspace{1cm} (4.30b)$$

The root $\gamma$ reflects the natural frequency $\omega$. Their relationship can be determined from Eqs. (4.21) and (4.27) as follows:

$$\omega = \frac{\gamma^2}{L^2} \sqrt{\frac{E\ell}{\rho}}$$  \hspace{1cm} (4.31)$$

The first two mode shapes of flexible-free modes, clamped-free modes and pinned-free modes are shown in Figure 4.2.

### 4.4 Mode Analysis

The influences of the different parameters, including joint stiffness $K_k$, hub inertia $K_{th}$, payload mass $K_{mp}$, and payload inertia $K_{fp}$, are investigated in this section. For convenience, rewrite the characteristic equation (4.28) as

$$K_k = \frac{1}{3} \gamma^4 K_{th} + G$$  \hspace{1cm} (4.32)$$

where

$$G = \gamma \frac{(3 - K_{mp}K_{fp}\gamma^4)(s ch - c sh) + 2K_{fp}\gamma^3 c ch + 6K_{mp}\gamma s sh}{3(c ch + 1) - K_{fp}\gamma^3(s ch + c sh) - 3K_{mp}\gamma(s ch - c sh) - K_{mp}K_{fp}\gamma^4(c ch - 1)}$$  \hspace{1cm} (4.33)$$

#### 4.4.1 Effect of Joint Stiffness $K_k$

The relative joint stiffness $K_k$ defined in Eq. (4.26) is a non-dimensional parameter representing the relative stiffness of the joint with respect to that of the beam. Its physical
Figure 4.2: Different mode shapes
meaning can be explained by studying a flexible-joint, rigid-link system and a rigid-joint, flexible-link system. For a flexible-joint, rigid-link system, as shown in Figure 4.3(a), if the angular displacement of the link caused by a torque \( M \) is \( \alpha_1 \), the joint stiffness is

\[ k = \frac{M}{\alpha_1} \]

For a rigid-joint, flexible-link system, as shown in Figure 4.3(b), the angular displacement \( \alpha_2 \) of the link at the free end, under the action of the same torque \( M \), is

\[ \alpha_2 = \frac{ML}{EI} \]

By rearranging the above equation, the stiffness of the link, in the sense of the angular displacement at the end-point, is obtained as

\[ \frac{EI}{L} = \frac{M}{\alpha_2} \]

Thus, the relative joint stiffness can be expressed by

\[ K_k = \frac{kL}{EI} = \frac{k}{EI/L} = \frac{\alpha_2}{\alpha_1}. \quad (4.34) \]

Eq. (4.34) indicates that relative joint stiffness \( K_k \) is equal to the ratio of the angular displacement \( \alpha_2 \) of the rigid-joint, flexible link system, with respect to the angular displacement.
Relative Joint Stiffness

Relative Joint Stiffness

Relative Joint Stiffness

Relative Joint Stiffness

Mode 1

Mode 2

Mode 3

Mode 4

Figure 4.4: Effect of joint stiffness

(a) $K_{th} = 0. K_{mp} = 0. K_{fp} = 0$
(b) $K_{th} = 0.05. K_{mp} = 0. K_{fp} = 0$
(c) $K_{th} = 0.1. K_{mp} = 0. K_{fp} = 0$
(d) $K_{th} = 0.5. K_{mp} = 0. K_{fp} = 0$

$\alpha_1$ of the flexible-joint, rigid-link system, when both systems are subject to the same torque. When $K_k = 1$, the flexible joint and the flexible link are said to have equivalent stiffness. In this case, $\alpha_1 = \alpha_2$, meaning that a torque applied to the flexible-joint, rigid-link system will produce the same angular displacement at the end-point of the link as the torque is applied to the rigid-joint, flexible link system.

Figure 4.4 illustrates the effect of the relative joint stiffness $K_k$ on the first four roots.
Table 4.1: Characteristic root $\gamma$

($K_{lh} = 0, K_{mp} = 0, K_{lp} = 0$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$K_k = 0$</th>
<th>$K_k = 1$</th>
<th>$K_k = 20$</th>
<th>$K_k = 100$</th>
<th>$K_k = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.2479</td>
<td>1.7912</td>
<td>1.8568</td>
<td>1.8751</td>
</tr>
<tr>
<td>2</td>
<td>3.9266</td>
<td>4.0311</td>
<td>4.5127</td>
<td>4.6497</td>
<td>4.6941</td>
</tr>
<tr>
<td>3</td>
<td>7.0686</td>
<td>7.1341</td>
<td>7.5863</td>
<td>7.7827</td>
<td>7.8548</td>
</tr>
<tr>
<td>4</td>
<td>10.2102</td>
<td>10.2566</td>
<td>10.6609</td>
<td>10.8976</td>
<td>10.9955</td>
</tr>
</tbody>
</table>

Table 4.2: Characteristic root $\gamma$

($K_{lh} = 0.1, K_{mp} = 0, K_{lp} = 0$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$K_k = 0$</th>
<th>$K_k = 1$</th>
<th>$K_k = 20$</th>
<th>$K_k = 100$</th>
<th>$K_k = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.2281</td>
<td>1.7899</td>
<td>1.8567</td>
<td>1.8751</td>
</tr>
<tr>
<td>2</td>
<td>3.1677</td>
<td>3.3043</td>
<td>4.3573</td>
<td>4.6421</td>
<td>4.6941</td>
</tr>
<tr>
<td>3</td>
<td>5.0011</td>
<td>5.0165</td>
<td>5.5113</td>
<td>7.3107</td>
<td>7.8548</td>
</tr>
<tr>
<td>4</td>
<td>7.9190</td>
<td>7.9195</td>
<td>7.9309</td>
<td>8.0856</td>
<td>10.9955</td>
</tr>
</tbody>
</table>

$\gamma_i$, where $K_k$ varies from 0.1 to 1000. Four cases are studied for $K_{lh} = 0, 0.05, 0.1$ and 0.5, respectively, with zero payload ($K_{mp} = 0, K_{lp} = 0$). In the case of $K_{lh} = 0$, the system consisting of a simple beam supported by a torsional spring is the simplest flexible-link, flexible-joint system. The value of $\gamma_i$ for $K_{lh} = 0$ and $K_{lh} = 0.1$ is listed in Table 4.1 and Table 4.2, respectively. The roots for $K_k = 0$ correspond to the frequencies of the pinned-free
beam. where the zero root implies the rigid body mode. The roots for \( K_k = \infty \) correspond to the frequencies of the clamped-free beam.

It is also noticed from Figure 4.4 that the two neighbouring root curves approach each other and diverge. Near the approaching point, they undergo violent change. This phenomenon is called "curve veering" (Leissa 1974, Perkins and Mote 1986). Curve veering is characterized by rapid and opposite change in curve concavities. Veering reflects the coupling of eigenfunctions as the system parameter is varied.

In order to study the sensitivity of \( \gamma \) with respect to \( K_k \). Eq. (4.32) is differentiated with respect to \( K_k \). and the following equation is obtained:

\[
\frac{\partial \gamma}{\partial K_k} = \frac{3}{4\gamma^3 K_{lk} + 3 \frac{\partial G}{\partial \gamma}}
\]  

(4.35)

Define the sensitivity index of \( \gamma \) with respect to \( K_k \) by (Wang 1994)

\[
S_k = \frac{\% \text{ change in } \gamma}{\% \text{ change in } K_k} = \frac{\delta \gamma / \gamma}{\delta K_k / K_k}
\]

(4.36)

It follows from (4.35) that

\[
S_k = \frac{K_k}{\gamma^3 K_{lk} + \frac{3 \partial G}{\partial \gamma}}
\]

(4.37)

The sensitivity index \( S_k \) for the above-mentioned four cases is shown in Figure 4.5. Its value for the case of \( K_{lk} = 0 \) is listed in Table 4.3.

Following conclusions can be drawn from the simulation results:

(1) Natural frequencies increase with the joint stiffness \( K_k \) and approach constant values when \( K_k \) tends to the infinity. The constant values are the frequencies of the clamped-free beam. This phenomenon is observed directly from Figure 4.4 under all four cases. It is also shown in Figure 4.5, where \( S_k > 0 \) for all \( K_k \) and \( S_k = 0 \) as \( K_k \to \infty \). Rewriting
Figure 4.5: Sensitivity of roots vs joint stiffness

(a) $K_{lh} = 0, K_{mp} = 0, K_{fp} = 0$  
(b) $K_{lh} = 0.05, K_{mp} = 0, K_{fp} = 0$

(c) $K_{lh} = 0.1, K_{mp} = 0, K_{fp} = 0$  
(d) $K_{lh} = 0.5, K_{mp} = 0, K_{fp} = 0$

Eq. (4.36) as

$$\frac{\partial \gamma}{\partial K_k} = \frac{\gamma}{K_k} S_k$$  

(4.38)

it is clearly indicated from (4.38) that $S_k > 0$ for all $K_k$ implying that a monotonic increment type of $\gamma$, and $S_k = 0$ as $K_k \rightarrow \infty$ implying a constant $\gamma$.

(2) The fundamental frequency is relatively sensitive to the joint stiffness $K_k$ when $K_k$ is small, but is insensitive when $K_k$ is large. The sensitivity index $S_k$ varying with $K_k$
Table 4.3: Sensitivity index $S_k$

\[ S_k = 0. \quad K_{mp} = 0. \quad K_{lp} = 0 \]

<table>
<thead>
<tr>
<th>Mode</th>
<th>$K_k = 0$</th>
<th>$K_k = 1$</th>
<th>$K_k = 20$</th>
<th>$K_k = 100$</th>
<th>$K_k = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.2019</td>
<td>0.0420</td>
<td>0.0096</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.0225</td>
<td>0.0309</td>
<td>0.0090</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.0085</td>
<td>0.0239</td>
<td>0.0085</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.0043</td>
<td>0.0186</td>
<td>0.0080</td>
<td>0</td>
</tr>
</tbody>
</table>

is listed in Table 4.3 and illustrated in Figure 4.5. It should be noted that the sensitivity index of frequency $\omega$ is different from that of root $\gamma$. The former is defined as

\[
S_k' = \frac{\delta \omega / \omega}{\delta K_k / K_k}
\]

Substituting Eq. (4.31) into the above equation yields

\[
S_k' = 2 \frac{\delta \gamma / \gamma}{\delta K_k / K_k} = 2S_k
\]

which means that mode frequencies are two times as sensitive to the joint stiffness as the roots are.

(3) Even a small flexibility (large stiffness) of the joint can significantly affect the system frequencies. Define the frequency error caused by ignoring the joint flexibility as

\[
\text{Error} = \frac{\left(\omega\right)_{K_k=\infty} - \left(\omega\right)_{K_k}}{\left(\omega\right)_{K_k=\infty}} = 1 - \frac{\left(\omega\right)_{K_k}}{\left(\omega\right)_{K_k=\infty}}
\]

Using Eq. (4.31), the above equation can be expressed as

\[
\text{Error} = 1 - \left(\frac{\left(\gamma\right)_{K_k}}{\left(\gamma\right)_{K_k=\infty}}\right)^2
\]

(4.39)
Considering the case of $K_{th} = 0.1$. $K_{mp} = 0$ and $K_{lp} = 0$ (Table 4.2). When $K_k = 20$, implying a joint stiffness 20 times of the link stiffness, the frequency error is $8.89\%$ for mode 1, $13.83\%$ for mode 2, $50.77\%$ for mode 3 and $47.98\%$ for mode 4.

### 4.4.2 Effect of Hub Moment of Inertia $K_{th}$

Relative hub inertia $K_{th}$, defined by (4.26), is the ratio of the inertia of the hub to that of the beam. A large $K_{th}$ means a relatively large hub inertia and a relatively small beam inertia, and vise versa. Differentiation of Eq. (4.32) with respect to $K_{th}$ yields the following equation:

$$\frac{\partial \gamma}{\partial K_{th}} = -\frac{\gamma^4}{4\gamma^3 K_{th} + 3 \frac{\partial G}{\partial \gamma}}$$

Similar to (4.36), the sensitivity index of $\gamma$ with respect to $K_{th}$ is defined as

$$S_{th} = \frac{\delta \gamma / \gamma}{\delta K_{th} / K_{th}} = -\frac{\gamma^3 K_{th}}{4\gamma^3 K_{th} + 3 \frac{\partial G}{\partial \gamma}}$$  \hspace{1cm} (4.40)
Figure 4.6 illustrates the effect of the relative hub inertia on the first four roots \( \gamma_i \), where \( K_{Ih} \) varies from 0.01 to 100. Four cases are studied for \( K_k = 0.1, 1.0, 20 \) and 100. The sensitivity index \( S_{Ih} \) for these four cases is shown in Figure 4.7. The values of \( \gamma \) and \( S_{Ih} \) for the case of \( K_k = 10, K_{mp} = 0 \) and \( K_{Ip} = 0 \) are listed in Table 4.4 and Table 4.5, respectively.
Following conclusions are drawn from the simulation results:

1. Natural frequencies decrease with the hub moment of inertia $K_{lh}$ and approach constant values when $K_{lh}$ tends to the infinity (Figure 4.6). The constant values are the frequencies of the clamped-free beam. It is shown in Figure 4.7 that $S_{lh} \leq 0$ for all $K_{lh}$. Based on Eq. (4.40), this also implies a monotonic decrement type of $\gamma$. Note that as $K_{lh}$ goes to infinity, the fundamental frequency approaches zero, corresponding to the rigid body
mode.

(2) Fundamental frequency is not sensitive to hub moment of inertia when $K_{lh}$ is smaller than 1. In most engineering problems, $K_{lh}$ is less than 1. In this case, the natural frequencies are almost constant (Figure 4.6) and the sensitivity index is small (Figure 4.7).

### 4.4.3 Effect of $Kk$ and $K_{lh}$

It is seen from the characteristic equation (4.28) that the roots $\gamma_i$ are determined by four parameters: $K_k$, $K_{lh}$, $K_{mp}$, and $K_{fp}$. For a given payload (known $K_{mp}$ and $K_{fp}$), $\gamma_i$ are functions of $K_k$ and $K_{lh}$. As shown in the previous sections, a change in $K_k$ or $K_{lh}$ will result in a change in $\gamma_i$, hence a change in the mode frequencies $\omega_i$. An increasing $K_k$ causes increasing frequencies, while an increasing $K_{lh}$ causes decreasing frequencies. What will happen if both $K_k$ and $K_{lh}$ change?

\begin{table}
<table>
<thead>
<tr>
<th>Mode</th>
<th>$K_{lh} = 0$</th>
<th>$K_{lh} = 0.01$</th>
<th>$K_{lh} = 0.1$</th>
<th>$K_{lh} = 1$</th>
<th>$K_{lh} = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-0.0002</td>
<td>-0.0022</td>
<td>-0.0286</td>
<td>-0.25</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-0.0058</td>
<td>-0.1091</td>
<td>-0.2131</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-0.0551</td>
<td>-0.1250</td>
<td>-0.0071</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-0.1312</td>
<td>-0.0098</td>
<td>-0.0008</td>
<td>0</td>
</tr>
</tbody>
</table>
\end{table}
Variation of Eq. (4.32) leads to the following equation

\[ \delta K'_k = \frac{1}{3} \gamma^4 \delta K'_{lh} + \frac{4}{3} \gamma^3 K'_{lh} \delta \gamma + \frac{\partial G}{\partial \gamma} \delta \gamma \]

The above equation can be expressed as

\[ \delta \gamma = \frac{3 \delta K'_k - \gamma^4 \delta K'_{lh}}{4 \gamma^3 K'_{lh} + 3 \frac{\partial G}{\partial \gamma}} \]  \hspace{1cm} (4.41)

Eq. (4.41) indicates:

(1) if both \( K'_k \) and \( K'_{lh} \) change and their variations \( \delta K'_k \) and \( \delta K'_{lh} \) satisfy the relationship

\[ \delta K'_k - \frac{1}{3} \gamma^4 \delta K'_{lh} > 0. \]  \hspace{1cm} (4.42)

\( \gamma \) increases:

(2) if \( \delta K'_k \) and \( \delta K'_{lh} \) satisfy the relationship

\[ \delta K'_k - \frac{1}{3} \gamma^4 \delta K'_{lh} < 0. \]  \hspace{1cm} (4.43)

\( \gamma \) decreases; and

(3) if \( \delta K'_k \) and \( \delta K'_{lh} \) satisfy the relationship

\[ \delta K'_k - \frac{1}{3} \gamma^4 \delta K'_{lh} = 0. \]  \hspace{1cm} (4.44)

\( \gamma \) remains constant.

Eq. (4.44) and Eq. (4.32) suggest a linear relationship between \( K'_k \) and \( K'_{lh} \) for a constant \( \gamma \), as shown in Figure 4.8. A constant \( \gamma \) implies a constant frequency. Thus, the following is concluded:

There exists a linear relationship, expressed by Eq. (4.32), between the joint stiffness \( K'_k \) and the hub moment of inertia \( K'_{lh} \), under which the system frequencies remain constant.
4.4.4 Effects of Payload Mass $K_{mp}$ and Payload Moment of Inertia $K_{lh}$

Relative payload mass $K_{mp}$ is the ratio of the payload mass to the beam mass. Relative payload inertia $K_{lp}$ is the ratio of the payload inertia (with respect to its mass center) to the beam inertia (with respect to the joint). Large $K_{mp}$ and $K_{lp}$ indicate a relatively large payload and vice versa. When a flexible robot is holding a payload, will the payload significantly affect the system frequencies? How will the frequencies change with different payload?

In order to answer these questions, the effects of payload are simulated. Figure 4.9 illustrates the characteristic roots $\gamma$ versus the payload mass $K_{mp}$, where $K_{mp}$ varies from 0.1 to 100. Four cases are simulated for $K_k = 0.1, 1.0, 10$ and $\infty$, with the hub inertia and payload inertia constant ($K_{lh} = 0.1, K_{lp} = 0$). The first three cases correspond to flexible-joint, flexible-link systems with different joint stiffness, while the last case corresponds to a rigid-joint, flexible-link system. Figure 4.10 illustrates the characteristic roots versus the
payload moment of inertia $K_{lp}$, where $K_{lp}$ varies from 0.01 to 10. In all four simulated cases, $K_k$ takes different values: 0.1, 1.0, 10 and $\infty$, and $K_{lh}$ and $K_{mp}$ remain constant ($K_{lh} = 0.1$, $K_{mp} = 0.2$).

From the simulation results, following conclusions can be drawn:

(1) Payload mass affects the fundamental frequency, but does not significantly affect
higher frequencies (Figure 4.9).

(2) The fundamental frequency is insensitive to a small payload moment of inertia, but relatively sensitive to a large moment of inertia; higher frequencies are insensitive to a large payload moment of inertia, but sensitive to a small moment of inertia (Figure 4.10).

(3) Among the first two frequencies, the fundamental frequency is mainly affected by
the payload mass $K_{mp}$, while the second frequency is mainly affected by payload moment of inertia $K_{fp}$ (Figures 4.9 and 4.10). This conclusion is useful in many engineering applications, where the flexible dynamics can be represented by the first two modes (Tsujisawa and Book 1989).

### 4.5 Summary and Conclusions

In this chapter, a systematic approach is developed to model the dynamics of a flexible link with a flexible joint. Accurate modes, called flexible-free modes, of such a system are obtained. The characteristics of the flexible-free modes are studied and the following phenomena are discovered: (1) even a small joint flexibility can significantly affect the system frequencies; (2) the fundamental frequency is not sensitive to the hub moment of inertia or payload moment of inertia; (3) there exists a linear relationship between the joint stiffness and the hub moment of inertia, under which the system frequencies remain constant; (4) for a given flexible system, the fundamental frequency is mainly affected by the payload mass, while the second frequency is mainly affected by the payload moment of inertia. These characteristics are very useful in the design and control of flexible-link, flexible-joint manipulators. In the next chapter, the derived flexible-free modes will be applied to model flexible-link, flexible-joint manipulators and to study the system response.
Chapter 5

Dynamics of Flexible-Link, Flexible-Joint (FLFJ) Manipulators

5.1 Introduction

In modeling flexible-link, flexible-joint (FLFJ) manipulators, several approaches appear in the literature. Because of unavailability of suitable modes, they either assume clamped-free or pinned-free modes for the flexible links (Yu and Elbestawi 1995, Yuan et. al 1993), or use finite element or experimental methods to determine the link modes (Yuan et. al 1993, Huang and Wang 1993). Of the assumed modes methods, the flexible-clamped-free modes method (Yu and Elbestawi 1995) and the flexible-pinned-free modes method (Yuan et. al 1993) are often used. In the flexible-clamped-free method, the link assumes the classic clamped-free beam modes. The joint flexibility is considered separately, by using independent coordinates. Thus, the independent flexible joint coordinates are present in the final equations. This flexible-pinned-free method is the same as the flexible-clamped-free
method except that the classic pinned-free beam modes are assumed for the flexible links. In all those methods, the joint flexibility is represented by independent coordinates in the system model. The system generalized coordinates include rigid joint coordinates, flexible link modal coordinates and flexible joint coordinates. Therefore, all those methods always generate high-order equations of motion. Some of them even result in poor model accuracy.

In this chapter, the flexible-free modes, obtained in the last chapter for FLFJ manipulators, are used to model the FLFJ manipulator dynamics. The application of the flexible-free modes is expected to solve the above-mentioned problems. The advantages of the flexible-free modes, such as modeling simplification and model order reduction, are studied. The comparison between the flexible-free modes method and conventional flexible-clamped-free modes method and flexible-pinned-free modes method is also made. Simulation on a one-link FLFJ manipulator is used to demonstrate the superiority of the proposed flexible-free modes method.

5.2 Kinematics

Consider a multi-link manipulator with both link flexibility and joint flexibility. The position vector of any point \( x_i \) on link \( i \) can be expressed as (Figure 5.1)

\[
\mathbf{r}_i(x_i, t) = \mathbf{r}_{i-1}(L_{i-1}, t) + \mathbf{R}_{o,i} \mathbf{r}_i(x_i, t)
\]  

(5.1)

where \( \mathbf{r}_i \) is the position vector of any point \( x_i \) of link \( i \) with respect to the reference frame \((x_o, y_o, z_o)\); \( \mathbf{r}_{i-1} \) is the position vector of the distal point \((x_{i-1} = L_{i-1})\) of link \( i - 1 \) with respect to the frame \((x_o, y_o, z_o)\); \( \mathbf{R}_{o,i} \) is the \( 3 \times 3 \) transformation matrix from \((x_i, y_i, z_i)\) to \((x_o, y_o, z_o)\); and \( \mathbf{r}_i \) is the position vector of any point \( x_i \) of link \( i \) with respect to the local
coordinate system \((x_i, y_i, z_i)\).

Considering the in-plane and off-plane transverse deformations of the link \(i\), \(\tilde{r}_i\) can be written as

\[
\tilde{r}_i(x_i, t) = \begin{pmatrix} x_i \\ w_{xyi}(x_i, t) \\ w_{xzi}(x_i, t) \end{pmatrix}
\]

Figure 5.1: Coordinate systems

where \(w_{xyi}\) and \(w_{xzi}\) are the in-plane and off-plane deformations, respectively. These deformations are usually described by a truncated modal series, in terms of spatial mode shape functions and time-dependent mode amplitudes, as follows

\[
w_{xyi}(x_i, t) = \sum_{j=1}^{n_i} \hat{\phi}_{xyij}(x_i)\eta_{xyij}(t) + x_i\beta_i(t)
\]

\[
w_{xzi}(x_i, t) = \sum_{j=1}^{n_i} \hat{\phi}_{xzij}(x_i)\eta_{xzij}(t)
\]

where \(\beta_i(t)\) is the flexible deformation of joint \(i\); \(n_i\) is the number of modes adopted for link
\( i: \mathbf{\hat{o}}_{x_i j} \) and \( \mathbf{\hat{o}}_{z i j} \) are mode shapes which do not take joint flexibility into account.

From Eqs. (5.2)-(5.4), the position vector of any point on link \( i \) with respect to its local frame \((x_i, y_i, z_i)\) can be expressed as

\[
\mathbf{\dot{r}}_i(x_i) = \mathbf{x}_i(x_i) + \mathbf{\hat{r}}_i(x_i)\eta_i(t) + \mathbf{x}_{j i}(x_i)\mathbf{\dot{y}}_i(t)
\]

(5.5)

where

\[
\mathbf{\hat{r}}_i = \begin{bmatrix}
0 & 0 \\
\mathbf{\hat{o}}_{x_i 1} & \cdots & \mathbf{\hat{o}}_{x_i n_i} \\
0 & \mathbf{\hat{o}}_{z_i 1} & \cdots & \mathbf{\hat{o}}_{z_i n_i}
\end{bmatrix}
\]

(5.6a)

\[
\mathbf{x}_i = \begin{pmatrix} x_i & 0 & 0 \end{pmatrix}^T
\]

(5.6b)

\[
\eta_i = \begin{pmatrix} \eta_{x_i 1} & \cdots & \eta_{x_i n_i} \\
\eta_{z_i 1} & \cdots & \eta_{z_i n_i} \end{pmatrix}^T
\]

(5.6c)

\[
\mathbf{x}_{j i} = \begin{pmatrix} 0 & x_i & 0 \end{pmatrix}^T
\]

(5.6d)

Eq. (5.5) is the fundamental expression of the kinematics of FLFJ links, widely used in the literature (Yu et al. 1995, Yuan et al. 1993). The first term of Eq. (5.5) is the nominal position vector corresponding to a rigid link. The second term represents the deformation due to link flexibility. The third term is the deformation due to joint flexibility. \( \mathbf{\hat{r}}_i \) in Eq. (5.5) are the mode shapes of the flexible links. Usually, these mode shapes are selected as those of the clamped-free or pinned-free beam. They do not contain the joint flexibility effect. Joint flexibility of the system is represented by \( \mathbf{\dot{y}}_i \), another independent variable. Thus, each flexible joint adds another degree of freedom to the FLFJ system, leading to large equations of motion.

As discussed in the previous chapter, the flexible-free modes take the joint flexibility into account. The mode shapes of a link contain the deformation caused by the flexible joint.
In terms of the flexible-free modes, the link deformations expressed by Eqs. (5.3) and (5.4) can be written as

\[ w_{xyi}(x_i, t) = \sum_{j=1}^{n_i} \phi_{xyi}(x_i) \eta_{xyij}(t) \quad (5.7) \]

\[ w_{xzi}(x_i, t) = \sum_{j=1}^{n_i} \phi_{xzi}(x_i) \eta_{xzi}(t) \quad (5.8) \]

For small deformations, it is reasonable to assume that the effect of the joint flexibility on the off-plane deformation is small and can be neglected. Thus, it follows that

\[ \phi_{xzi} = \hat{\phi}_{xzi} \quad (5.9) \]

Based on Eqs. (5.7) and (5.8), Eq. (5.5) now has the form

\[ \bar{r}_i(x_i, t) = \bar{x}_i + \Phi_i(x_i) \eta_i(t) \quad (5.10) \]

where the first term is the rigid position vector and the second term is the flexible displacement caused by the link flexibility and joint flexibility. Substituting (5.10) into (5.1) results in

\[ r_i(x_i, t) = r_{i-1}(L_{i-1}, t) + [x_i + \Phi_i(x_i) \eta_i(t)] \quad (5.11) \]

It is interesting to note that the flexible joint coordinate \( \lambda_i \) is no longer present in Eq. (5.10), which has the same form as for links without joint flexibility. Thus, it can be said that

*When modeling the deformations of flexible links, links with joint flexibility can be treated as links without joint flexibility, if the flexible-free modes are employed.*
5.3 Equations of Motion of FLFJ Manipulators

The velocity of any point on link $i$ can be obtained by differentiating (5.11) with respect to time as follows:

$$
v_i = J_{\theta i} \dot{\theta} + J_{\eta i} \dot{\eta}
$$

(5.12)

where

$$
J_{\theta i} \equiv \frac{\partial r_{i-1}(L_{i-1}, \tau)}{\partial \theta} + \frac{\partial}{\partial \theta} \left[ R_{o,i} \left( x_i + \dot{\Phi}_i \eta_i \right) \right]
$$

(5.13a)

$$
J_{\eta i} \equiv \frac{\partial r_{i-1}(L_{i-1}, \tau)}{\partial \eta} + \frac{\partial}{\partial \eta} \left[ R_{o,i} \left( x_i + \dot{\Phi}_i \eta_i \right) \right]
$$

(5.13b)

The kinetic energy of the system is the sum of the kinetic energy of individual links and can be written as

$$
T = \sum_{i=1}^{n} \int_{0}^{L_i} \frac{1}{2} \rho_i \mathbf{v}_i^T \mathbf{v}_i \, dx_i = \frac{1}{2} \mathbf{q}^T \mathbf{M} \dot{\mathbf{q}}
$$

(5.14)

where

$$
\mathbf{M}_{\theta\theta} = \sum_{i=1}^{n} \int_{0}^{L_i} \rho_i \mathbf{J}_{\theta i}^T \mathbf{J}_{\theta i} \, dx_i
$$

(5.15a)

$$
\mathbf{M}_{\eta\eta} = \sum_{i=1}^{n} \int_{0}^{L_i} \rho_i \mathbf{J}_{\eta i}^T \mathbf{J}_{\eta i} \, dx_i
$$

(5.15b)

$$
\mathbf{M}_{\theta\eta} = \sum_{i=1}^{n} \int_{0}^{L_i} \rho_i \mathbf{J}_{\theta i}^T \mathbf{J}_{\eta i} \, dx_i
$$

(5.15c)

$\mathbf{M}_{\theta\theta}$ is the inertia of the system corresponding to the rigid body motion; $\mathbf{M}_{\eta\eta}$ is the inertia corresponding to the flexible motion of the system; $\mathbf{M}_{\theta\eta}$ is the coupling inertia between the rigid motion and flexible motion.
It can be seen from the above equations that the formulation of the kinetic energy of a FLFJ system is the same as for a FLRJ (flexible-link, rigid-joint) system, if the flexible-free mode shapes are used to model the link deformations.

The potential energy of the system includes the elastic potential and gravity potential. The elastic potential energy contains that of the links and that of the joints:

\[ V^e = V^e_{\text{link}} + V^e_{\text{joint}} \]  

(5.16)

where

\[ V^e_{\text{link}} = \sum_{i=1}^{n} \int_0^{L_i} \frac{1}{2} \left[ EI_{zi} \left( \frac{\partial^2 w_{yi}}{\partial x_i^2} \right)^2 + EI_{yi} \left( \frac{\partial^2 w_{zi}}{\partial x_i^2} \right)^2 \right] dx_i \]  

(5.17a)

\[ V^e_{\text{joint}} = \sum_{i=1}^{n} \frac{1}{2} k_i \left( \frac{\partial w_{yi}}{\partial x_i} \right)_{x_i=0}^2 \]  

(5.17b)

Eqs. (5.17a) and (5.17b) can be written in the matrix form as

\[ V^e_{\text{link}} = \frac{1}{2} \eta^T K_L \eta \]  

(5.18a)

\[ V^e_{\text{joint}} = \frac{1}{2} \eta^T K_J \eta \]  

(5.18b)

\( K_L \) and \( K_J \) are stiffness matrices for links and joints expressed as

\[ K_L = \text{diag}(K_{Li}) \]  

(5.19a)

\[ K_J = \text{diag}(K_{Ji}) \]  

(5.19b)

where

\[ K_{Li} = \int_0^{L_i} \left[ EI_{zi} \left( \frac{\partial^2 \phi_{yi}}{\partial x_i^2} \right)^2 + EI_{yi} \left( \frac{\partial^2 \phi_{zi}}{\partial x_i^2} \right)^2 \right] dx_i \]  

(5.20a)

\[ K_{Ji} = k_i \left( \frac{\partial \phi_{yi}}{\partial x_i} \right)_{x_i=0}^2 \]  

(5.20b)

Note that in Eqs. (5.19a) and (5.19b), the orthogonal property of modes has been used to eliminate the coupling terms.
The gravity potential is of the form

$$V^g = \sum_{i=1}^{n} \int_{0}^{L_i} \rho_i g \mathbf{r}_i \mathbf{d}x_i$$  \hspace{1cm} (5.21)$$

where $\mathbf{r}_i$ is expressed by Eq. (5.11).

For beam-type members, the internal structural damping can cause energy dissipation during the dynamic response. The dissipation function of links due to the structural damping is (Kojima 1988)

$$F_{\text{link}} = \sum_{i=1}^{n} \int_{0}^{L_i} \frac{1}{2} \left[ C_{\xi \epsilon y_i} I_{z_i} \left( \frac{\partial^2 w_{x y_i}}{\partial x_i^2 \partial t} \right)^2 + C_{\xi \epsilon z_i} I_{y_i} \left( \frac{\partial^3 w_{x z_i}}{\partial x_i^3 \partial t} \right)^2 \right] \mathbf{d}x_i$$  \hspace{1cm} (5.22)$$

with

$$C_{\xi \epsilon y_i} = \xi_i L_i^2 \sqrt{\frac{A_i E_i}{I_{z_i}}}$$ \hspace{1cm} (5.23a)$$
$$C_{\xi \epsilon z_i} = \xi_i L_i^2 \sqrt{\frac{A_i E_i}{I_{y_i}}}$$ \hspace{1cm} (5.23b)$$

where $\xi_i$ is the damping ratio of link $i$ and $A_i$ is the cross section area. The joint dissipation function is

$$F_{\text{joint}} = \sum_{i=1}^{n} \frac{1}{2} c_{\xi_i} \left( \frac{\partial^2 w_{x y_i}}{\partial x_i \partial t} \right)^2 \mathbf{d}x_i = 0$$  \hspace{1cm} (5.24)$$

where $c_{\xi_i}$ is the damping constant of joint $i$. It has been shown by Clough (1975) that the damping matrix satisfies the orthogonality conditions and they can be uncoupled the same way as the inertia and stiffness matrices.

The dissipation function can be written in the matrix form as

$$F_{\text{link}} = \frac{1}{2} \mathbf{\dot{\eta}}^T C_{\xi \epsilon} \mathbf{\dot{\eta}}$$  \hspace{1cm} (5.25a)$$
$$F_{\text{joint}} = \frac{1}{2} \mathbf{\dot{\eta}}^T C_{\xi J} \mathbf{\dot{\eta}}$$  \hspace{1cm} (5.25b)$$

$C_{\xi \epsilon}$ and $C_{\xi J}$ are damping matrices for links and joints expressed as

$$C_{\xi \epsilon} = \text{diag}(C_{\xi \epsilon_i})$$  \hspace{1cm} (5.26a)$$
where

\[ C_{\xi Li} = \int_0^L \frac{1}{2} \left[ C_{\xi z i} I_{zi} \left( \frac{\partial^3 \varphi_{zyi}}{\partial x_i^2 \partial t} \right)^2 + C_{\xi x i} I_{yi} \left( \frac{\partial^3 \varphi_{xzi}}{\partial x_i^2 \partial t} \right)^2 \right] dx_i \] (5.27a)

\[ C_{\xi i} = c_{\xi i} \left( \frac{\partial \varphi_{x yi}}{\partial x_i} \right)_{x_i=0} \] (5.27b)

It can be seen from above that in the formulation of a FLFJ system, the elastic potential, gravity potential, and dissipation function of links expressed by Eqs. (5.17a), (5.21) and (5.22) have the same forms as for FLRJ systems, except that the joint potential expressed by (5.17b) and joint dissipation energy expressed by (5.24) must be included for the FLFJ system.

The equations of motion of the FLFJ system can be derived based on Lagrange's equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} + \frac{\partial F}{\partial q} = Q \] (5.28)

where \( Q \) is the generalized force expressed by

\[ Q = \left( \begin{array}{c} \tau^T \ : \ 0^T \end{array} \right)^T = \left( \begin{array}{ccc} \tau_1 & \cdots & \tau_n \ : \ 0 & \cdots & 0 \end{array} \right)^T \] (5.29)

Eq. (5.28) can also be written as

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q} \right) - \frac{\partial T}{\partial q} + \frac{\partial F_{\text{link}}}{\partial q} + \frac{\partial V^g}{\partial q} + \frac{\partial V^e_{\text{link}}}{\partial q} = Q - \frac{\partial V^e_{\text{joint}}}{\partial q} - \frac{\partial F_{\text{joint}}}{\partial q} \] (5.30)

where the left side terms of the equation have the same forms as for FLRJ systems, and the last two terms of the right side, corresponding to the generalized elastic force and generalized dissipation force due to joint flexibility, are special to the FLFJ systems.

Defined a new generalized force \( Q' \) as

\[ Q' = Q - \frac{\partial F_{\text{joint}}}{\partial q} - \frac{\partial V^e_{\text{joint}}}{\partial q} \] (5.31)
Then, Eq. (5.30) becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial F}{\partial q} + \frac{\partial (V_{\text{link}}^c + V^g)}{\partial q} = Q'$$ (5.32)

which is of the same form as for FLRJ systems. By this arrangement, the equations of motion of a FLFJ system can be formulated the same way as for a FLRJ system providing that the modes adopted for the flexible links are the flexible-free modes and the generalized force is defined by (5.31).

Using Eq. (5.32), the equations of motion can be obtained as

$$\begin{bmatrix} M_{\theta \theta} & M_{\theta \eta} \\ M_{\eta \theta} & M_{\eta \eta} \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\eta} \end{pmatrix} + \begin{bmatrix} C_{\theta \theta} & C_{\theta \eta} \\ C_{\eta \theta} & C_{\eta \eta} \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\eta} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_L + K_J \end{bmatrix} \begin{pmatrix} \theta \\ \eta \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$ (5.33)

Note that the system equations (5.33) no longer contain the flexible joint coordinates. Thus, the number of the equations is reduced by n.

### 5.4 Comparison with Conventional Methods

In the conventional dynamic formulations of a FLFJ manipulator, such as assumed modes method (Yu and Elbestawi 1995), finite element method (Huang and Wang 1993) and experimental modes method (Yuan et. al 1993), the generalized coordinates include the rigid (nominal) joint coordinates, flexible link modal coordinates and flexible joint coordinates. The flexible joint coordinates are present in the equations of motion as independent coordinates, leading to high-order equations of motion. By using the derived flexible-free modes, the flexible joint coordinates are no longer present in the equations of motion, leading to an order-reduced model. This is a major advantage of the flexible-free modes. For a two-link FLFJ manipulator with two modes adopted for each link, for example, the conventional
methods result in a model with an order of 8; while the order is reduced to 6 with the proposed method.

Another major advantage of the flexible-free modes is the simplification of model derivation. Since the link deformation produced by flexible joints does not need to be explicitly calculated, the system kinetic energy is derived exactly the same way as for FLRJ manipulators. The link elastic potential energy and dissipation function are also of the same forms as for FLRJ manipulators. If the joint elastic and dissipative forces are included in the generalized force as defined in (5.31), the equations of motion can be derived based on (5.32). Thus, FLFJ manipulators can be treated the same way as FLRJ manipulators, by using the flexible-free modes.

As explained above, flexible-free modes can reduce the model order. Generally, an order-reduced model has a lower accuracy. Will the flexible-free modes reduce the model accuracy? In order to answer the question, we compare the flexible-free modes method with two widely used methods, flexible-clamped-free modes method (Yu and Elbestawi 1995) and flexible-pinned-free modes method (Yuan et. al 1993). As an assumed modes method, the flexible-clamped-free modes method is a typical conventional method. In this method, the joint flexibility is represented by independent joint variables. The link flexibility is described by clamped-free beam modes. The clamped-free modes use a "clamped" boundary condition between the links and the flexible joints, which is accurate in describing the mechanical system. Although the clamped-free modes do not incorporate the flexible joint effect, the flexible links and joints are coupled in the final equations of motion, through their own coordinates. Under the same assumptions and boundary conditions, the flexible-free modes are derived as seen in the last chapter. Eq. (4.5) associated with $w$ is the governing equation
5.5 Simulation of a One-Link FLFJ Manipulator

In this section, the flexible-free modes are applied to a one-link FLFJ manipulator. The dynamics of the manipulator under the flexible-free modes is simulated and the results under
different modes are compared.

The one-link FLFJ manipulator considered here is a uniform beam carrying a payload mass. Its parameters are listed in Table 5.1. Assume that the system has the rigid and flexible motion in the vertical plane and the off-plane motion is not excited. The dynamics of the link under four different mode shapes are simulated. Those modes include: flexible-free modes, flexible-clamped-free modes, flexible-pinned-free modes, and clamped-free modes.

The first three natural frequencies of the system under different mode shapes are listed in Table 5.2. It can be seen that the flexible-free modes and flexible-clamped-free modes yield the same frequencies (exact results). Flexible-pinned-free modes has a higher first frequency but lower second and third frequencies. By ignoring the joint flexibility, clamped-free modes have higher frequencies than flexible-free modes or flexible-clamped-free modes do.

The simulated motion is the natural vibration in the vertical plane. The rigid joint coordinate is fixed during the motion. The system is released from the horizontal position at zero speed with zero deflection. Figures 5.3 and 5.4 show the profiles of joint torque and end-point deflection in two seconds. The first two natural modes of vibration are considered.
Table 5.1: Physical parameters of the one-link manipulator

<table>
<thead>
<tr>
<th>Physical Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link Length</td>
<td>$L$</td>
<td>$m$</td>
</tr>
<tr>
<td>Link Mass</td>
<td>$m$</td>
<td>$kg$</td>
</tr>
<tr>
<td>Link Stiffness</td>
<td>$EI$</td>
<td>$N \cdot m^2$</td>
</tr>
<tr>
<td>Link Damping Constant</td>
<td>$C_f I$</td>
<td>$N \cdot m^2 \cdot s$</td>
</tr>
<tr>
<td>Joint Stiffness</td>
<td>$k$</td>
<td>$N \cdot m/rad$</td>
</tr>
<tr>
<td>Joint Damping Constant</td>
<td>$c_f$</td>
<td>$N \cdot m \cdot s/rad$</td>
</tr>
<tr>
<td>Payload Mass</td>
<td>$m_p$</td>
<td>$kg$</td>
</tr>
</tbody>
</table>

Table 5.2: Natural frequencies (Hz) under different mode shapes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Flexible-Free</th>
<th>Flexible-Clamped-Free</th>
<th>Flexible-Pinned-Free</th>
<th>Clamped-Free</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5451</td>
<td>1.5451</td>
<td>2.3298</td>
<td>2.0609</td>
</tr>
<tr>
<td>2</td>
<td>22.4301</td>
<td>22.4301</td>
<td>18.3767</td>
<td>28.2230</td>
</tr>
<tr>
<td>3</td>
<td>76.3357</td>
<td>76.3357</td>
<td>71.1010</td>
<td>89.7669</td>
</tr>
</tbody>
</table>
Figure 5.3: Joint torque under different mode shapes

Figure 5.4: Link deflection under different mode shapes
in the simulation. It can be seen that the flexible-free modes and flexible-clamped-free modes result in exactly the same torque profile. All types of modes approach the same static torque of 98 .Vm, which is the gravitational torque of the link and the payload. The flexible-free modes and flexible-clamped-free modes also produce the same deflection profile, which approaches to a static value of $-0.110 \ m$ (exact result). The flexible-pinned-free modes and the clamped-free modes approach static deflections of $-0.049 \ m$ and $-0.062 \ m$, respectively.

### 5.6 Conclusion

A new order-reduced formalism for modeling of FLFJ manipulator systems is presented. In terms of formulation, flexible manipulators which have both link flexibility and joint flexibility are treated as FLRJ manipulators by using the flexible-free modes and a new generalized force containing the joint elastic and damping forces. As a result, the flexible joint coordinates do not appear in the equations of motion and the order of the equations is reduced, while the accuracy remains unaffected. When modeling the same physical system, the flexible-free modes and flexible-clamped-free modes lead to two different sets of equations of motion, which can be converted to each other. Simulation on a one-link FLFJ manipulator shows that the flexible-free modes method and flexible-clamped-free method generate the same dynamic responses, while the flexible-pinned-free method yields less accurate results.
Chapter 6

Dynamics of Flexible Macro/Micro Manipulators Mounted on a Flexible Base (M/m+B)

6.1 Introduction

Modular long-reach macro/micro (M/m) manipulators have been proposed for applications such as waste management, space industry and construction industry (Jansen et al. 1991, Crane et al. 1991, Yoshikawa et al. 1993). For most long-reach manipulator systems, one of the basic problems in modeling and control is their flexibility. The flexibility mainly comes from the links and joints of the macro manipulator and sometimes from the base (a carrying truck or supporting structure). Base flexibility and joint flexibility are taken into account because a small base or macro joint flexibility can cause a significantly large end-point displacement for long-reach manipulators. Dynamic modeling of such systems has to
deal with all link/joint/base flexibility.

In the past, dynamic modeling of flexible manipulators has been the topic of extensive research. Lagrange's equations or Newton-Euler method has been used for dynamic formulation (Rakhsha and Goldenberg 1985). The flexible link dynamics has been modeled by the assumed mode method (Book 1984) or finite element method (Sunada and Dubowsky 1981). In addition to link flexibility, joint flexibility can also affect the overall structural flexibility, and thus it needs to be included in the dynamic model (Huang and Wang 1994). For flexible-link, flexible-joint (FLFJ) manipulators, link deformations have often been described by pinned-free or clamped-free mode shape functions (Book 1984), without consideration of the influence of the joint flexibility on the link modes. Li et al. (1997a) has incorporated the joint flexibility into the link flexibility and addressed the effect of the joint flexibility. When a flexible robot is mounted on a moving base, the dynamics characteristics change greatly. The effect of a moving base on a flexible beam has been addressed by Kane and Ryan (1987). The work mentioned above is based on single manipulators with link flexibility, link/joint flexibility, or link/base flexibility. For modular macro/micro manipulators mounted on a flexible base (M/m+M) with link/joint/base flexibility, conventional modeling methods are cumbersome. The derived equations of motion cannot show explicitly the couplings between the subsystems. The modular formulation method based on modular structure concept is desirable in order to establish the overall equations of motion and study the interaction between the subsystems.

The objective of this Chapter is to study the dynamics of flexible macro/micro manipulators mounted on a flexible base (M/m+M). The macro has both link and joint flexibility. The micro is rigid. The modular formulation method for rigid M/m manipulators
developed in Chapter 2 is extended to flexible $M/m+B$ system. The flexible-free modes derived in Chapter 4 and the associated modeling method proposed in Chapter 5 are also used. Simulation is conducted on a 3-DOF flexible $M/m+B$ system to investigate its dynamic behavior and the interaction between the subsystems.

6.2 Modular Formulation of Flexible $M/m+B$ System Dynamics

The objective of this section is to derive the equations of motion of flexible $M/m$ manipulators on a flexible base ($M/m+B$) using the modular formulation method developed in Chapter 2.

Consider the base as a rigid body supported by springs with six degrees of freedom. Its equations of motion has the following form

$$M_b \ddot{q}_b + C_b \dot{q}_b + K_b q_b = 0$$  \hspace{1cm} (6.1)

where $q_b \in \mathbb{R}^6$, $M_b$, $C_b$ and $K_b \in \mathbb{R}^{6x6}$. Note that $M_b$, $C_b$ and $K_b$ are constant matrices.

The equations of motion of the macro manipulator with both link flexibility and joint flexibility are expressed by Eq. (5.33), where flexible joint coordinates are no longer explicitly present. The equations may be written as

$$M_M \ddot{q}_M + C_M \dot{q}_M + K_M q_M + G_M = A_M \tau_M$$  \hspace{1cm} (6.2)

where $q_M$ and $G_M \in \mathbb{R}^{n_M}$, $M_M$, $C_M$ and $K_M \in \mathbb{R}^{n_M \times n_M}$, $\tau_M \in \mathbb{R}^{n_M r}$, and $A_M \in \mathbb{R}^{n_M \times n_M r}$. $n_M$ is the number of rigid joint coordinates and $n_M$ is the total number of the generalized coordinates including the rigid and flexible coordinates.
Similarly, the rigid micro arm has the following equations of motion

\[ M_m \ddot{q}_m + C_m \dot{q}_m + G_m = \tau_m \]  

(6.3)

where \( q_m, G_m \) and \( \tau_m \in \mathbb{R}^{m} \); \( M_m \) and \( C_m \in \mathbb{R}^{m \times m} \). Note that the effect of the payload has been included in the last link of the micro arm.

The modular \( M/m \) system has equations of motion of the general form

\[ M\ddot{q} + C\dot{q} + Kq + G = A\tau \]  

(6.4)

where \( q = (q_b^T, q_M^T, q_m^T)^T \in \mathbb{R}^n \), \( M, C \) and \( K \in \mathbb{R}^{n \times n} \); \( G \in \mathbb{R}^n \); \( A \in \mathbb{R}^{n \times (n_M + n_m)} \); and \( \tau \in \mathbb{R}^{n_M \times n_m} \). The next step is to determine the structure of Eq. (6.4).

Define a body-fixed frame \( x_b, y_b, z_b \) on the base at point \( P_b \), where the base and the macro arm are connected, as shown in Figure 6.1. The position of any point on link \( i \) of the
macro arm. \( \mathbf{r}_{Mi} \), can be expressed in the inertial reference frame \( x_0y_0z_0 \) as

\[
\mathbf{r}_{Mi} = \mathbf{r}_b + \mathbf{R}_b \mathbf{r}_{Mi}
\]  

(6.5)

where \( \mathbf{r}_b \) is the absolute position vector of \( P_b \) expressed in the frame \( x_0y_0z_0 \). \( \mathbf{R}_b \) is the transformation matrix from frame \( x_by_bz_b \) to frame \( x_0y_0z_0 \), and \( \mathbf{r}_{Mi} \) is the position of the point on link \( i \) expressed in frame \( x_by_bz_b \). Differentiating Eq. (6.5) leads to

\[
\dot{\mathbf{r}}_{Mi} = \dot{\mathbf{r}}_b + \mathbf{R}_b \dot{\mathbf{r}}_{Mi} - \mathbf{R}_b \text{skew}(\dot{\mathbf{r}}_{Mi}) \omega_b
\]

(6.6)

where \( \text{skew}(\cdot) \) was defined in Eq. (2.9). \( \omega_b \) is the angular velocity of the base

\[
\omega_b = \mathbf{J}_b^\omega \dot{\mathbf{q}}_b
\]

(6.7)

Where \( \mathbf{J}_b^\omega \in \mathbb{R}^{3 \times 6} \). Eq. (6.6) can also be expressed as

\[
\dot{\mathbf{r}}_{Mi} = (\mathbf{J}_{Mi})_b \dot{\mathbf{q}}_b + (\mathbf{J}_{Mi})_M \dot{\mathbf{q}}_M
\]

(6.8)

where

\[
(\mathbf{J}_{Mi})_b = \frac{\partial \mathbf{r}_b}{\partial \mathbf{q}_b} - \mathbf{R}_b \text{skew}(\dot{\mathbf{r}}_{Mi}) \mathbf{J}_b^\omega
\]

(6.9a)

\[
(\mathbf{J}_{Mi})_M = \mathbf{R}_b \frac{\partial \dot{\mathbf{r}}_{Mi}}{\partial \mathbf{q}_M}
\]

(6.9b)

The kinetic energy of the macro arm can be written as

\[
T_M = \sum_{i=1}^{n_M} \int_{0}^{L_{Mi}} \frac{1}{2} \rho_i \dot{\mathbf{r}}_{Mi}^T \dot{\mathbf{r}}_{Mi} dx_i
\]

\[
= \frac{1}{2} \left( \begin{array}{c} \dot{\mathbf{q}}_b^T \\ \dot{\mathbf{q}}_M^T \end{array} \right) \left[ \begin{array}{cc} \mathbf{D}_{M11} & \mathbf{D}_{M12} \\ \mathbf{D}_{M12}^T & \mathbf{D}_{M22} \end{array} \right] \left( \begin{array}{c} \dot{\mathbf{q}}_b \\ \dot{\mathbf{q}}_M \end{array} \right)
\]

(6.10)

where

\[
\mathbf{D}_{M11} = \sum_{i=1}^{n_M} \int_{0}^{L_{Mi}} \rho_i (\mathbf{J}_{Mi})_b^T (\mathbf{J}_{Mi})_b dx_i
\]

(6.11a)
\[
D_{M22} = \sum_{i=1}^{n_{M}} \int_0^{L_M} \rho_i (J_{M_i}^v)_{M}^T (J_{M_i}^v)_{M} \, dx_i \tag{6.11b}
\]
\[
D_{M12} = \sum_{i=1}^{n_{M}} \int_0^{L_M} \rho_i (J_{M_i}^v)_{b}^T (J_{M_i}^v)_{M} \, dx_i \tag{6.11c}
\]

\(D_{M11}\) is the inertia of the macro arm corresponding to the motion of the base; \(D_{M22}\) is the inertia of the macro arm itself; and \(D_{M12}\) is the coupling inertia between the motions of the flexible base and the macro arm.

Define a body-fixed frame \(x_M y_M z_M\) at \(P_M\), the tip of the macro arm. The mass center location of link \(i\) of the micro arm may be expressed in the reference frame as

\[
r_{mi} = r_M + R_b \hat{R}_M \hat{r}_{mi}\tag{6.12}
\]

where \(r_M\) is the position vector of \(P_M\), at which the macro arm and the micro arm are connected; \(\hat{R}_M\) is the transformation matrix between frames \(x_b y_b z_b\) and \(x_M y_M z_M\); and \(\hat{r}_{Mi}\) is the position vector of the mass center of link \(i\), with respect to frame \(x_M y_M z_M\): \(r_M\) can be expressed as

\[
r_M = r_b + R_b \hat{r}_{bm}\tag{6.13}
\]

where \(\hat{r}_{bm}\) is the position vector of \(P_M\), expressed in the frame \(x_b y_b z_b\).

Differentiating Eqs. (6.12) and (6.13) results in the velocity

\[
\dot{r}_{mi} = (J_{mi}^v)_b \dot{q}_b + (J_{mi}^v)_M \dot{q}_M + (J_{mi}^v)_m \dot{q}_m \tag{6.14}
\]

where

\[
(J_{mi}^v)_b = \frac{\partial r_b}{\partial q_b} - R_b \text{ skew} (\hat{r}_{bm}) \hat{J}_b^v - R_b \hat{R}_M \text{ skew} (\hat{r}_{mi}) J_M^v \tag{6.15a}
\]
\[
(J_{mi}^v)_M = R_b \frac{\partial \hat{r}_{bm}}{\partial q_m} - R_b \hat{R}_M \text{ skew} (\hat{r}_{mi}) J_M^v \tag{6.15b}
\]
\[
(J_{mi}^v)_m = R_b \hat{R}_M \frac{\partial \hat{r}_{mi}}{\partial q_m} \tag{6.15c}
\]

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The angular velocity of the link $i$ of the micro arm is

$$\omega_{mi} = \omega_b + R_b \omega_M + R_b \tilde{R}_M \omega_{mi}$$

$$= (J^\omega_{mi})_b \dot{q}_b + (J^\omega_{mi})_M \dot{q}_M + (J^\omega_{mi})_m \dot{q}_m$$

(6.16)

where

$$(J^\omega_{mi})_b = J_b$$

(6.17a)

$$(J^\omega_{mi})_M = R_b J^\omega_M$$

(6.17b)

$$(J^\omega_{mi})_m = R_b \tilde{R}_M J^\omega_{mi}$$

(6.17c)

$\omega_M$ is the angular velocity of the macro arm at the tip (or that of the micro arm at the base) with respect to the frame $x_b y_b z_b$; while $\omega_m$ is the angular velocity of the micro arm at the tip, with respect to the frame $x_M y_M z_M$.

The kinetic energy of the micro arm is the sum of the translational and rotational kinetic energy of each link as follows:

$$T_m = \sum_{i=1}^{n_m} \frac{1}{2} m_{mi} \dot{x}_{mi}^T \dot{x}_{mi} + \sum_{i=1}^{n_m} \frac{1}{2} \omega^T_{mi} I_{mi} \omega_{mi}$$

$$= \frac{1}{2} \left( \begin{array}{c} \dot{q}_b \\ \dot{q}_M \\ \dot{q}_m \end{array} \right)^T \left( \begin{array}{ccc} D_{m11} & D_{m12} & D_{m13} \\ D_{m12} & D_{m22} & D_{m23} \\ D_{m13} & D_{m23} & D_{m33} \end{array} \right) \left( \begin{array}{c} \dot{q}_b \\ \dot{q}_M \\ \dot{q}_m \end{array} \right)$$

(6.18)

where

$$D_{m11} = \sum_{i=1}^{n_m} m_{mi} (J^\omega_{mi})_b^T (J^\omega_{mi})_b + \sum_{i=1}^{n_m} (J^\omega_{mi})_b^T I_{mi} (J^\omega_{mi})_b$$

(6.19a)
The total kinetic energy of the whole system is the sum of the kinetic energy of each subsystem:

$$ T = T_b + T_M + T_m $$

$$ = \frac{1}{2} \ddot{q}_b^T M_b \ddot{q}_b + \frac{1}{2} \left( \ddot{q}_b^T \ddot{q}_M \right) \begin{bmatrix} D_{M11} & D_{M12} \\ D_{M12}^T & D_{M22} \end{bmatrix} \begin{bmatrix} \ddot{q}_b \\ \ddot{q}_M \end{bmatrix} + \frac{1}{2} \left( \ddot{q}_b^T \ddot{q}_M + \ddot{q}_m^T \ddot{q}_m \right) \begin{bmatrix} D_{m11} & D_{m12} & D_{m13} \\ D_{m12}^T & D_{m22} & D_{m23} \\ D_{m13}^T & D_{m23}^T & D_{m33} \end{bmatrix} \begin{bmatrix} \ddot{q}_b \\ \ddot{q}_M \\ \ddot{q}_m \end{bmatrix} $$

$$ = \frac{1}{2} \dot{q}^T M \dot{q} $$

(6.20)

$M$ in (6.20) is constructed as the following form

$$ M = \begin{bmatrix} M_b + M_{b,M} + \bar{M}_b & M_{b,M} & M_{b,m} \\ M_{b,M}^T & M_M + \bar{M}_{M,m} & M_{M,m} \\ M_{b,m}^T & M_{M,m}^T & M_m \end{bmatrix} $$

(6.21)
where $\mathbf{M}_b$, $\mathbf{M}_M$, and $\mathbf{M}_m$ in (6.21) are inertia matrices of the respective original subsystems.

and the other elements are couplings between subsystems as follows:

$$
\mathbf{M}_{bM} = \mathbf{D}_{M11} \quad (6.22a)
$$

$$
\mathbf{M}_{bm} = \mathbf{D}_{m11} \quad (6.22b)
$$

$$
\mathbf{M}_{Mm} = \mathbf{D}_{m22} \quad (6.22c)
$$

$$
\mathbf{M}_{bM} = \mathbf{D}_{M12} + \mathbf{D}_{m12} \quad (6.22d)
$$

$$
\mathbf{M}_{bm} = \mathbf{D}_{m13} \quad (6.22e)
$$

$$
\mathbf{M}_{Mm} = \mathbf{D}_{m23} \quad (6.22f)
$$

The non-linear terms corresponding to the centrifugal and Coriolis forces can be obtained from the inertia matrices. The $\mathbf{C}$ matrix of the overall system is calculated by

$$
\mathbf{C} = \dot{\mathbf{M}} - \frac{1}{2} \left[ \begin{array}{c} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial \mathbf{q}_1} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial \mathbf{q}_n} \end{array} \right] + \mathbf{C}' = \dot{\mathbf{M}} - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial \mathbf{q}_1} (\mathbf{M} \dot{\mathbf{q}}) \\ \vdots \\ \frac{\partial}{\partial \mathbf{q}_n} (\mathbf{M} \dot{\mathbf{q}}) \end{array} \right]^T + \mathbf{C}' \quad (6.23)
$$

where $\mathbf{C}'$ is the damping matrix. For the subsystems, it follows that

$$
\mathbf{C}_M = \dot{\mathbf{M}}_M - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial \mathbf{q}_1} (\mathbf{M}_M \dot{\mathbf{q}}_M) \\ \vdots \\ \frac{\partial}{\partial \mathbf{q}_{nM}} (\mathbf{M}_M \dot{\mathbf{q}}_M) \end{array} \right]^T + \mathbf{C}'_M \quad (6.24)
$$

$$
\mathbf{C}_m = \dot{\mathbf{M}}_m - \frac{1}{2} \left[ \begin{array}{c} \frac{\partial}{\partial \mathbf{q}_1} (\mathbf{M}_m \dot{\mathbf{q}}_m) \\ \vdots \\ \frac{\partial}{\partial \mathbf{q}_{nm}} (\mathbf{M}_m \dot{\mathbf{q}}_m) \end{array} \right]^T + \mathbf{C}'_m \quad (6.25)
$$

$$
\mathbf{C}_b = \mathbf{C}'_b \quad (6.26)
$$

where $\mathbf{C}'_b$, $\mathbf{C}'_M$ and $\mathbf{C}'_m$ are damping matrices of individual subsystems.
From Eq. (6.21), it follows that

\[
\mathbf{M}\ddot{\mathbf{q}} = \begin{bmatrix}
(M_b + \dot{M}_{b,M} + \dot{M}_{b,m}) \dot{q}_b + M_{b,M} \dot{q}_M + M_{b,m} \dot{q}_m \\
M_{b,M}^T \dot{q}_b + (M_M + \dot{M}_m) \dot{q}_m + M_{M,m} \dot{q}_m \\
M_{b,M}^T \dot{q}_b + M_{M,m}^T \dot{q}_M + M_m \dot{q}_m
\end{bmatrix}
\] (6.27)

Substituting Eqs. (6.27) into (6.23), \( \mathbf{C} \) is obtained as

\[
\mathbf{C} = \begin{bmatrix}
C_b + \dot{C}_{b,M} + \dot{C}_{b,m} & C_{b,M} & C_{b,m} \\
C_{M,b} & C_M + \dot{C}_{M,b} + \dot{C}_{M,m} & C_{M,m} \\
C_{m,b} & C_{m,M} & C_m + \dot{C}_{m,b} + \dot{C}_{m,M}
\end{bmatrix}
\] (6.28)

where

\[
\dot{C}_{b,M} = \dot{M}_{b,M} - \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial q_1} (\dot{M}_{b,M} \dot{q}_b + M_{b,M} \ddot{q}_M)^T \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{M}_{b,M} \dot{q}_b + M_{b,M} \ddot{q}_M)^T
\end{bmatrix}
\] (6.29a)

\[
\dot{C}_{b,m} = \dot{M}_{b,m} - \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial q_1} (\dot{M}_{b,m} \dot{q}_b + M_{b,m} \ddot{q}_m)^T \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{M}_{b,m} \dot{q}_b + M_{b,m} \ddot{q}_m)^T
\end{bmatrix}
\] (6.29b)

\[
\dot{C}_{M,b} = -\frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial q_1} (\dot{q}_b^T \dot{M}_{b,M}) \\
\vdots \\
\frac{\partial}{\partial q_n} (\dot{q}_b^T \dot{M}_{b,M})
\end{bmatrix}
\] (6.29c)

\[
\dot{C}_{M,m} = \dot{M}_{M,m} - \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial q_1} (\dot{M}_{M,m} \dot{q}_M + M_{M,m} \ddot{q}_m)^T \\
\ddots \\
\frac{\partial}{\partial q_n} (\dot{M}_{M,m} \dot{q}_M + M_{M,m} \ddot{q}_m)^T
\end{bmatrix}
\] (6.29d)
\[
\mathbf{C}_{mb} = -\frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_b^T M_{b_1}) \\
\vdots \\
\frac{\partial}{\partial q_{n_m}} (\dot{q}_b^T M_{b_m})
\end{array} \right] \\
\mathbf{C}_{mM} = -\frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_M^T M_{M_1}) \\
\vdots \\
\frac{\partial}{\partial q_{n_m}} (\dot{q}_M^T M_{M_m})
\end{array} \right]
\]

\[
\mathbf{C}_{bM} = \dot{M}_{bM} - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_b^T M_{bM}) \\
\vdots \\
\frac{\partial}{\partial q_{n_b}} (\dot{q}_b^T M_{bM})
\end{array} \right] \\
\mathbf{C}_{bm} = \dot{M}_{bm} - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_b^T M_{bm}) \\
\vdots \\
\frac{\partial}{\partial q_{n_b}} (\dot{q}_b^T M_{bm})
\end{array} \right]
\]

\[
\mathbf{C}_{Mm} = \dot{M}_{Mm} - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} (\dot{q}_b^T M_{bm} + \dot{q}_M^T M_{Mm}) \\
\vdots \\
\frac{\partial}{\partial q_{n_M}} (\dot{q}_b^T M_{bm} + \dot{q}_M^T M_{Mm})
\end{array} \right]
\]

\[
\mathbf{C}_{Mb} = \dot{M}_{bM}^T - \frac{1}{2} \left[ \begin{array}{c}
\frac{\partial}{\partial q_1} \left( (\dot{M}_{bM}^T \dot{q}_b + M_{bM} \dot{q}_M + M_{bm} \dot{q}_m)^T \right) \\
\vdots \\
\frac{\partial}{\partial q_{n_M}} \left( (\dot{M}_{bM}^T \dot{q}_b + M_{bM} \dot{q}_M + M_{bm} \dot{q}_m)^T \right)
\end{array} \right]
\]
The gravity matrix $G$ can be constructed using the same method developed in Chapter 2. It has the following form:

$$
G = \begin{pmatrix}
G_{bM} + G_{bm} \\
G_M + G_{Mb} + G_{Mm} \\
G_m + G_{mbM}
\end{pmatrix}
$$

(6.30)

where

$$
G_{bM} = \begin{pmatrix}
\sum_{i=1}^{nM} m_{Mi} g_T (J_{Mi})_{bl} \\
\vdots \\
\sum_{i=1}^{nM} m_{Mi} g_T (J_{Mi})_{b6}
\end{pmatrix}
$$

(6.31a)

and

$$
G_{bm} = \begin{pmatrix}
\sum_{i=1}^{nM} m_{mi} g_T (J_{mi})_{bl} \\
\vdots \\
\sum_{i=1}^{nM} m_{mi} g_T (J_{mi})_{b6}
\end{pmatrix}
$$

(6.31b)
The equations of motion of the M/m+B system expressed by Eq. (6.4) is now rewritten as

\[
\begin{bmatrix}
M_b + \bar{M}_{b,M} + \bar{M}_{b,m} & M_{b,M} & M_{b,m} \\
M_{b,M}^T & M_M + \bar{M}_{M,m} & M_{M,m} \\
M_{b,m}^T & M_{M,m}^T & M_m
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_b \\
\ddot{q}_M \\
\ddot{q}_m
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
C_b + \bar{C}_{b,M} + \bar{C}_{b,m} & C_{b,M} & C_{b,m} \\
C_{M,b} & C_M + \bar{C}_{M,b} + \bar{C}_{M,m} & C_{M,m} \\
C_{mb} & C_{m,M} & C_m + \bar{C}_{mb} + \bar{C}_{m,m}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_b \\
\dot{q}_M \\
\dot{q}_m
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
K_b & 0 & 0 \\
0 & K_M & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
q_b \\
q_M \\
q_m
\end{bmatrix}
= \begin{bmatrix}
G_{b,M} + G_{b,m} \\
G_M + G_{M,b} + G_{M,m} \\
G_m + G_{m,b,m}
\end{bmatrix}
\]
where $\tau'_M$ and $\tau'_m$ are the coupling torques on the macro and micro, respectively.

Subtracting Eqs. (6.1)-(6.3) from (6.32) leads to the coupling dynamic equations:

\[
\begin{bmatrix}
\ddot{\mathbf{M}}_{bM} + \ddot{\mathbf{M}}_{bm} & \mathbf{M}_{bM} & \mathbf{M}_{bm} \\
\mathbf{M}_{bM}^T & \ddot{\mathbf{M}}_{mM} & \mathbf{M}_{mM} \\
\mathbf{M}_{bm}^T & \mathbf{M}_{mM}^T & 0 \\
\end{bmatrix}
\begin{pmatrix}
\ddot{\mathbf{q}}_b \\
\ddot{\mathbf{q}}_M \\
\ddot{\mathbf{q}}_m \\
\end{pmatrix}
\]

\[
\begin{bmatrix}
\mathbf{C}_{bM} + \ddot{\mathbf{C}}_{bm} & \mathbf{C}_{bM} & \mathbf{C}_{bm} \\
\mathbf{C}_{Mb} & \mathbf{C}_{Mb} + \ddot{\mathbf{C}}_{Mm} & \mathbf{C}_{Mm} \\
\mathbf{C}_{mb} & \mathbf{C}_{mM} & \ddot{\mathbf{C}}_{mb} + \ddot{\mathbf{C}}_{mM} \\
\end{bmatrix}
\begin{pmatrix}
\dot{\mathbf{q}}_b \\
\dot{\mathbf{q}}_M \\
\dot{\mathbf{q}}_m \\
\end{pmatrix}
\]

\[
\begin{bmatrix}
\mathbf{G}_{bM} + \dddot{\mathbf{G}}_{bm} \\
\mathbf{G}_{Mb} + \dddot{\mathbf{G}}_{Mm} \\
\mathbf{G}_{mbM} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
\mathbf{A}_M & 0 \\
0 & I \\
\end{bmatrix}
\begin{pmatrix}
\tau'_M \\
\tau'_m \\
\end{pmatrix}
\]  

Eq. (6.32) shows that the equations of motion of a modular $M/m+B$ system can also have a modular form. They can be constructed using the original equations of motion of the base, the macro manipulator and the micro manipulator, plus their coupling terms. The torques due to the couplings between the subsystems can be directly calculated using
the closed-form of coupling equations (6.33). The modular form of equations of motion and the closed-form of coupling equations reveal the dynamic structure of the flexible M/m+B system and are very useful in system analysis and control.

6.3 Simulation

The model studied here is a 3-DOF system consisting of flexible macro/micro manipulators mounted on a flexible base (M/m+B), as shown in Figure 6.2. The macro manipulator is a flexible uniform beam with a flexible joint. The flexible-free modes derived in Chapter 4 are used to model the beam dynamics. The micro manipulator is rigid and has two degrees of freedom. Two rigid links of the micro manipulator are modeled as uniform rods. The flexible base consists of a mass supported by springs with two degrees of freedom. Vertical translation and rotation in the vertical plane. Thus, the overall system is a hybrid flexible/rigid system with link, joint and base flexibility. Viscoelastic damping is considered for the system. The system physical parameters are listed in Table 6.1.

The studied M/m+B system is an assembly of flexible and rigid subsystems. Each subsystem has its own dynamic characteristics. When assembled, their dynamics affects each other. The coupled dynamics of all subsystems forms the overall system dynamics. The purpose of this simulation is to study the dynamic behavior of the assembled 3-DOF M/m+B system. In doing this, two main issues are investigated: (1) how the flexible base affects the flexible M/m+B manipulator system; and (2) how the micro and macro manipulators affect each other with and without the presence of the flexible base.
Table 6.1: Physical parameters of the flexible M/m+B system

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<tr>
<th>Base:</th>
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<tbody>
<tr>
<td>Mass</td>
<td>$m_b$</td>
<td>20</td>
<td>kg</td>
</tr>
<tr>
<td>Moment of Inertia</td>
<td>$I_b$</td>
<td>75</td>
<td>kg·m$^2$</td>
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<tr>
<td>Vertical Stiffness</td>
<td>$k_{b1}$</td>
<td>$8.256 \times 10^3$</td>
<td>N/m</td>
</tr>
<tr>
<td>Vertical Damping</td>
<td>$c_{\xi b1}$</td>
<td>200</td>
<td>N·s/m</td>
</tr>
<tr>
<td>Angular Stiffness</td>
<td>$k_{b2}$</td>
<td>$3.096 \times 10^4$</td>
<td>N·m·/rad</td>
</tr>
<tr>
<td>Angular Damping</td>
<td>$c_{\xi b2}$</td>
<td>100</td>
<td>N·m·s/rad</td>
</tr>
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Link 1:

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<td>m</td>
</tr>
<tr>
<td>Link Mass</td>
<td>$m_1$</td>
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<td>kg</td>
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<tr>
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<td>$EI$</td>
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<td>N·m$^2$</td>
</tr>
<tr>
<td>Link Damping</td>
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<td>N·m$^2$·s</td>
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<td>N·m/rad</td>
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<td>N·m·s/rad</td>
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Link 2:

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Link 3:

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<td>Link Mass</td>
<td>$m_3$</td>
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</tbody>
</table>
6.3.1 Effect of Flexible Base

The base studied here is a rigid object mounted on springs with two degrees of freedom: vertical translation and rotation about the mass center. The two natural frequencies are assumed equal. It is obvious that a base of different sizes (mass and moment of inertia) and with different stiffness constants generates different effects on the flexible M/m system. In this simulation, two ratios are used to evaluate the base: inertia ratio and frequency ratio. The inertia ratio is the ratio of the mass of the base to the mass of the M/m system, or the ratio of the moment of inertia of the base to the moment of inertia of the M/m system. (An equal ratio will be used for mass and moment of inertia in this simulation.) The moment of inertia of the M/m system is calculated about the first joint of the macro manipulator.
when the M/m system is fully extended as shown in Figure 6.3. Three cases have been simulated: small base (inertia ratio 0.2), medium size base (inertia ratio 1) and large base (inertia ratio 5). The frequency ratio is the ratio of the natural frequency of the base, \( \omega_3 \), to the fundamental frequency of the M/m system, \( \omega_1 \). A small frequency corresponds to a flexible base, and a large frequency to a stiff base. Six cases have been simulated: frequency ratio equal to 0.5 (very flexible base), 1, 2, 5, 10 and \( \infty \) (rigid base). Table 6.2 lists the first two frequencies of the flexible M/m+B system versus different bases, at the configuration shown in Figure 6.3.

It is seen from Table 6.2 that when frequency ratio approaches \( \infty \) (rigid base), the frequencies of the M/m+B system do not change with the inertia ratio. They are the frequencies of the flexible M/m system (on a rigid base), which are 1.6168 Hz and 14.2911 Hz. The flexible M/m+B system, formed by mounting the flexible M/m system on a flexible base has lower frequencies. When the base inertia ratio is 0.2 and frequency ratio is 0.5, the system frequencies are 0.2462 Hz and 0.6504 Hz, much lower than the frequencies of the rigidly-based system. This suggests that a small base with large flexibility greatly affects the system dynamics. When the base inertia ratio is 5 and frequency ratio is 10 (the base frequency is close to the second frequency of the flexible M/m system), the flexible M/m+B system frequencies are 1.6139 Hz and 13.6931 Hz, close to the flexible M/m system frequencies. This suggests that a large and stiff base has a small effect on the system dynamics. When its inertia is 5 times of the M/m system inertia and frequencies are beyond the second frequency of the flexible M/m system, the effect of the base is small and may be neglected.

To further investigate the effect of the flexible base, a movement has been simulated
on three different systems: flexible M/m+B system (flexible macro, rigid micro and flexible base), flexible M/m system (flexible macro, rigid micro and rigid base), and rigid M/m system (rigid macro, rigid micro and rigid base). The physical parameters of the flexible M/m+B system are listed in Table 6.1. Its base has an inertia ratio of 5 and a frequency ratio of 2. The other two systems have the same physical parameters as the flexible M/m+B system except that the flexible M/m system has no base flexibility and the rigid M/m system is totally rigid. The simulated nominal (rigid) movement for three systems are the same. It is a straight line of 1.5 m starting and ending with zero speed. For flexible M/m+B and flexible M/m systems, the initial deflection is zero. The duration of movement is 2 seconds. plus 1 second of settlement at the end of the movement. The nominal end-point acceleration and velocity are shown in Figures 6.5 and 6.6. Simulation results of end-point trajectories and joint torques are shown in Figures 6.7 to 6.12.

It is seen from Figures 6.7 to 6.9 that either the flexible M/m+B system or flexible M/m system cannot follow the nominal trajectory, because of their flexibility. Both systems have end-point deflections. The deflection of the flexible M/m+B system is even larger because of its additional base flexibility. Comparing the trajectories of two flexible systems, it is observed that flexible base causes larger oscillations at the initial time. These oscillations are damped out with time, and both flexible systems approach their static positions.

The effect of the flexible base on the joint torques can be seen by comparing the torque profiles of the flexible M/m+B system and flexible M/m system as shown in Figures 6.10-6.12. In the first 1.2 second of movement, all three joint torques of the flexible M/m+B system oscillate with significantly larger amplitudes than those of the flexible M/m system do. This difference diminishes with time. This shows that the initial oscillations of
Figure 6.3: A fully extended configuration

Figure 6.4: Nominal end-point trajectory
Table 6.2: Frequencies (Hz) of $M/m+B$ system with different bases

Small Base. Inertia Ratio = 0.2

<table>
<thead>
<tr>
<th>$\frac{\omega_b}{\omega_1}$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>0.2462</td>
<td>0.4786</td>
<td>0.8626</td>
<td>1.3814</td>
<td>1.5492</td>
<td>1.6168</td>
</tr>
<tr>
<td>Mode 2</td>
<td>0.6504</td>
<td>1.2907</td>
<td>2.5048</td>
<td>5.5548</td>
<td>9.6163</td>
<td>14.2911</td>
</tr>
</tbody>
</table>

Medium Size Base. Inertia Ratio = 1

<table>
<thead>
<tr>
<th>$\frac{\omega_b}{\omega_1}$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>0.4614</td>
<td>0.8533</td>
<td>1.3030</td>
<td>1.5606</td>
<td>1.6026</td>
<td>1.6168</td>
</tr>
<tr>
<td>Mode 2</td>
<td>0.7674</td>
<td>1.5301</td>
<td>3.0208</td>
<td>7.1698</td>
<td>12.3728</td>
<td>14.2911</td>
</tr>
</tbody>
</table>

Large Base. Inertia Ratio = 5

<table>
<thead>
<tr>
<th>$\frac{\omega_b}{\omega_1}$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>0.6701</td>
<td>1.2007</td>
<td>1.5318</td>
<td>1.6051</td>
<td>1.6139</td>
<td>1.6168</td>
</tr>
<tr>
<td>Mode 2</td>
<td>0.7997</td>
<td>1.5982</td>
<td>3.1861</td>
<td>7.8572</td>
<td>13.6931</td>
<td>14.2911</td>
</tr>
</tbody>
</table>
Figure 6.5: Nominal end-point acceleration

Figure 6.6: Nominal end-point velocity
Figure 6.7: End-point position (horizontal) of three different systems

Figure 6.8: End-point position (vertical) of three different systems
Figure 6.9: End-point trajectory of three different systems

Figure 6.10: First joint torque of three different systems
Figure 6.11: Second joint torque of three different systems

Figure 6.12: Third joint torque of three different systems
the large base have significant effect on the flexible $M/m$ system. With the base oscillations dying out, the joint torques caused by the base tend to be small. It is also observed that the frequency of the flexible $M/m+B$ system is lower due to the base flexibility.

From the above observations and analyses, it can be concluded that a small and flexible base greatly affects the flexible $M/m+B$ system frequencies, while a large and stiff base has a small effect on the flexible $M/m+B$ system. When the mass and moment of inertia of the base are 5 times of those of the flexible $M/m$ system, and the base frequency is beyond the second frequency of the flexible $M/m$ system, the effect of the base is small and may be ignored.

### 6.3.2 Couplings between Macro and Micro

The dynamics of the macro manipulator is affected not only by the base, but also by the micro manipulator, due to their couplings. Motion of the rigid micro manipulator can excite the oscillations of the flexible macro manipulator. The couplings between the macro and micro may be large or small, depending on the inertia and the configuration of the micro. A micro with large link mass and moment of inertia implies large couplings, and vice versa. For a given micro, the couplings change with configuration.

To evaluate the inertia of the micro, two ratios are introduced: mass ratio and length ratio. Mass ratio is the ratio of the mass of the micro to the mass of the macro. Length ratio is the ratio of the maximum micro reach to the maximum macro reach. The reach of a manipulator is the distance from its end-point to the first joint axis. For a given mass ratio, a small length ratio corresponds to a small moment of inertia.

Table 6.3 lists the first two frequencies of the $M/m+B$ system with different micros.
Table 6.3: Frequencies (Hz) of M/m+B system with different micros

<table>
<thead>
<tr>
<th>Length Ratio = 0</th>
<th>( \frac{m}{V} )</th>
<th>0</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>2.9381</td>
<td>2.8343</td>
<td>2.6453</td>
<td>2.3318</td>
<td>2.0825</td>
<td>1.8921</td>
<td></td>
</tr>
<tr>
<td>Mode 2</td>
<td>3.2212</td>
<td>3.2180</td>
<td>3.2146</td>
<td>3.2109</td>
<td>3.2086</td>
<td>3.2070</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Length Ratio = 0.1</th>
<th>( \frac{m}{V} )</th>
<th>0</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>2.9381</td>
<td>2.8234</td>
<td>2.6103</td>
<td>2.2684</td>
<td>2.0089</td>
<td>1.8160</td>
<td></td>
</tr>
<tr>
<td>Mode 2</td>
<td>3.2212</td>
<td>3.2173</td>
<td>3.2130</td>
<td>3.2085</td>
<td>3.2057</td>
<td>3.2038</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Length Ratio = 0.2</th>
<th>( \frac{m}{V} )</th>
<th>0</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>2.9381</td>
<td>2.8113</td>
<td>2.5718</td>
<td>2.2026</td>
<td>1.9352</td>
<td>1.7413</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Length Ratio = 0.5</th>
<th>( \frac{m}{V} )</th>
<th>0</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>2.9381</td>
<td>2.7668</td>
<td>2.4350</td>
<td>1.9986</td>
<td>1.7212</td>
<td>1.5318</td>
<td></td>
</tr>
</tbody>
</table>
The frequencies are calculated based on the configuration as shown in Figure 6.3. The length ratio changes from 0 to 0.5 and the mass ratio changes from 0 to 1. In the case of length ratio equal to 0, the micro is equivalent to a payload to the macro. In this case, any change in the configuration of the micro will not affect the dynamics of the macro. In the case of mass ratio equal to 0, the frequencies are independent of the length ratio. The $M/m+B$ system is reduced to a flexible macro manipulator mounted on a flexible base ($M+B$). Its first two frequencies are 2.9381 Hz and 3.2212 Hz. It is seen from the table that when the length ratio changes from 0 to 0.5 and mass ratio changes from 0 to 1, the first frequency changes from 2.9381 Hz to 1.5318 Hz and the second frequency changes from 3.2212 Hz to 3.1861 Hz. This suggests that for $M/m+B$ systems with a configuration as shown in Figure 6.2, the micro manipulator significantly changes the fundamental frequency, but does not significantly change the second frequency, as long as the mass ratio is not larger than 1 and length ratio not larger than 0.5. It is also seen from the table that when the length ratio is not larger than 0.2 and mass ratio not larger than 0.5, the frequencies do not significantly change with
the length ratio. In this case, the maximum change happens to the first frequency when the mass ratio is 0.5. The frequency changes from 2.3318 Hz to 2.2026 Hz, a variation of 5.5%. This implies that a small micro with its length ratio not larger than 0.2 and mass ratio not larger than 0.5, can be treated as a payload to the macro with the error in the first two frequencies below 5.5%.

To study how the coupling dynamics is affected by motion of the micro, the nominal joint motion of the macro is fixed and the micro moves along a trajectory. The simulated system is the M/m+B system as shown in Figure 6.2, with its parameters listed in Table 6.1. Its length ratio is 0.5 and mass ratio is 1. The simulated trajectory is a straight line of 0.8 m, as shown in Figure 6.13. The end-point moves from the initial end to the final end in 1.5 seconds and stays at the final end for 0.5 seconds. The nominal end-point acceleration and velocity are shown in Figures 6.14 and 6.15. Figures 6.16 to 6.21 show the end-point motion and the joint torques of the flexible M/m+B system, in contrast to the results of the flexible M/m system and the rigid M/m system. Initially, both flexible M/m+B and flexible M/m systems are at static equilibrium with steady deflection.

As shown in Figure 6.16, the horizontal end-point displacement of the flexible M/m+B system is the same as that of the flexible M/m system, and the displacements of both systems are slightly different from the nominal displacement. Since the horizontal end-point displacement is produced by the angular deflections of the base and the macro, the result implies that the motion of the micro causes small angular deflections of the macro and the base. The vertical deflections of both the flexible M/m+B system and the flexible M/m system are much larger (Figure 6.17). The deflection of the flexible M/m+B system is even larger as a result of the flexible base. It is seen from Figure 6.18 that the end-point
Figure 6.14: Nominal end-point acceleration

Figure 6.15: Nominal end-point velocity
Figure 6.16: End-point position (horizontal) of three different systems

Figure 6.17: End-point position (vertical) of three different systems
Figure 6.18: End-point trajectory of three different systems

Figure 6.19: First joint torque of three different systems
Figure 6.20: Second joint torque of three different systems

Figure 6.21: Third joint torque of three different systems
trajectories of both flexible systems are almost straight lines. This means that in terms of the end-point motion, the micro has a negligibly small effect on the macro, although the static deflection cannot be ignored. It can be observed from the torque profiles as shown in Figures 6.19-6.21, the motion of the micro can cause large torque variations in all three systems. The joint torques of the micro are dominated by the nominal torques, while the torque of the macro is the nominal torque plus the oscillations caused by the flexibility.

From above observations and analyses, it is concluded that the size (mass, moment of inertia, and length) of the micro can greatly change the fundamental frequency of the flexible M/m+B system, but not significantly affect the second frequency. For M/m+B systems with a configuration as shown in Figure 6.2, when the micro has a length ratio less than 0.2 and a mass ratio less than 0.5, its effect on the M/m+B system dynamics is small and may be ignored.

6.4 Conclusion

In this chapter, the dynamics of flexible macro/micro manipulators mounted on a flexible base (M/m+B) is investigated. With the modular structure concept, the equations of motion of a modular M/m+B system are established using the modular formulation approach. In this method, the equations of motion of the overall system are constructed by using the original equations of motion of the subsystems, plus their coupling terms. The torques due to the couplings between the subsystems can be directly calculated using the closed-form of coupling equations (6.33). The obtained model is order-reduced, as a result of the applications of the flexible-free modes derived in Chapter 4 and the associated modeling method developed in Chapter 5. The modular, order-reduced form of equations of motion and
the closed-form of coupling equations reveal the dynamic structure of the flexible M/m+B system and are very useful in system analysis and control.

Simulation is conducted on a 3-DOF flexible M/m+B system. The coupling dynamics between the subsystems are studied. Simulation results show that:

(1) A small and flexible base greatly affects the flexible M/m+B system frequencies, while a large and stiff base has a small effect on the flexible M/m+B system. For M/m+B systems with a configuration as shown in Figure 6.2, when the mass and moment of inertia of the base are 5 times of those of the flexible M/m system, and the base frequency is beyond the second frequency of the flexible M/m system, the effect of the base is small and may be ignored.

(2) The size (mass, moment of inertia and length) of the micro can greatly change the fundamental frequency of the flexible M/m+B system, but not significantly affect the second frequency. When the micro has a length ratio less than 0.2 and a mass ratio less than 0.5, its effect on the M/m+B system dynamics is small and may be ignored.
Chapter 7

Conclusions

7.1 Conclusions

This thesis has addressed the dynamic modeling of rigid and flexible modular macro/micro (M/m) manipulator systems.

A modular dynamic formulation method has been developed in Chapter 2 to establish the modular equations of motion of the rigid M/m manipulator system. The modular equations of the overall system can be constructed by directly using the equations of individual subsystems and calculating the couplings between the subsystems. As shown in Eq. (2.41), the joint torques of the overall system are equal to the torques of the original subsystems plus the coupling torques between them. The coupling torques consist of three parts: those produced by joint acceleration, by joint velocity (Centrifugal and Coriolis forces), and by link gravity. Closed-form coupling equations are obtained as Eq. (2.42), allowing direct calculations of the coupling torques. The developed modular formulation method is then extended to flexible M/m manipulators mounted on a flexible base (M/m+B) in Chapter 6.
The modular formulation method has two major advantages over the conventional direct formulation methods (Hollerbach 1980, Luh 1980, Paul 1981, Kane and Levinson 1985). One is the simplification of the formulation. In the modular formulation method, the original equations of motion of the subsystems are directly used in the overall equations. They do not need to be repeatedly derived. Only the coupling terms have to be formulated. By contrast, the conventional methods treat the M/m system as a single manipulator. Every term in the equations has to be derived. The other major advantage of the modular formulation method is the clear structure of the system dynamics and the closed form of the coupling equations. The modular-form equations are very useful for the dynamic analysis and controller design of the M/m system.

Based on the derived model, a dynamics-based redundancy resolution method, Peak Torque Reduction (PTR) method, has been presented in Chapter 3 to solve the kinematic redundancy in the M/m manipulator system and other types of redundant manipulators as well. Instead of minimizing the joint torques or kinetic energy at the current time, the PTR method always uses the current torque to approach an optimum velocity of the end of time interval, which, along with joint acceleration, minimizes the torque at that time, while keeping the current torque within the limits. The basic characteristic of the PTR method is its intention to use the current power to minimize torque at the next station. This is the essential difference between the PTR method and the conventional local optimization schemes.

The proposed PTR method has been simulated in contrast with the conventional Torque Optimization method (Hollerbach and Suh 1987) and Energy Minimization method (Khatib 1983). Simulation results show clear superiority of the proposed method in three
different cases: short movement, medium-length movement and long movement. In all three cases, the PTR method can effectively reduce peak torques globally, while the other methods leads to unacceptable results. The proposed PTR method can be applied to conventional redundant manipulators, not just limited to redundant M/m manipulator systems.

For most long-reach M/m manipulator systems, flexibility is one of the basic problems in modeling the systems. New modes, called flexible-free modes, have been derived in Chapter 4 for a beam with a flexible joint, aiming at the application on the macro manipulator with both link and joint flexibility and other flexible-link, flexible-joint (FLFJ) manipulators as well. The characteristics of the flexible-free modes and the effects of the joint flexibility, payload mass and inertia, and hub inertia are studied. Some important phenomena are investigated and concluded regarding a beam with a flexible joint.

Associated with the derived flexible-free modes, a new order-reduced formalism of FLFJ manipulator dynamics has been presented in Chapter 5. The FLFJ manipulator with both link and joint flexibility, are treated as FLRJ (flexible-link, rigid-joint) manipulators by using the flexible-free modes and a new generalized force containing the joint elastic and damping forces. The resultant equations of motion do not contain the flexible joint coordinates. The order of the model is reduced, but the accuracy is unaffected. It has been shown by simulation that the flexible-free modes method and flexible-clamped-free method (Yu and Elbestawi 1995) generate the same dynamic responses, while the flexible-pinned-free modes method (Yuan et al. 1993) yields less accurate results.

The dynamics of flexible macro/micro manipulators mounted on a flexible base (M/m+B) has been investigated in Chapter 6. The modular formulation method developed in Chapter 2 is extended to the flexible M/m+B system to establish its equations of mo-
tion. The flexible-free modes derived in Chapter 4 and the correspondent modeling method developed in Chapter 5 are used in the modeling. The resultant model has a reduced order. The closed-form of coupling equations are also established, allowing for direct calculation of coupling dynamics. Simulation on a 3-DOF flexible M/m+B system has been conducted to study the coupling dynamics between the subsystems. It has been shown that for M/m+B systems with a configuration as shown in Figure 6.2, (1) when the mass and moment of inertia of the base are 5 times of those of the flexible M/m system, and the base frequency is beyond the second frequency of the flexible M/m system, the effect of the base is small and may be ignored: (2) the size (mass, moment of inertia and length) of the micro can greatly change the fundamental frequency of the flexible M/m+B system, but not significantly affect the second frequency. The developed model and the simulation results are very useful in the system design, analysis and control.

7.2 Recommendations for Further Research

This thesis systematically studies the dynamics of M/m manipulator systems. Due to the comprehensive nature of the topic, further work based on this thesis needs to be done. In this section, we list several open issues relevant to this topic:

- **Redundancy resolution based on flexible dynamics.** Although the dynamics-based PTR method presented in Chapter 3 is a valid approach to peak torque reduction by using kinematic redundancy, it is limited to rigid manipulators. For flexible redundant manipulators, redundancy resolution based on flexible dynamics is much more complicated. The internal motion associated with the redundancy may excite or damp
the flexural modes, and thus greatly affect the dynamic responses. Although some progress has been achieved in this area by using redundancy to control the motion of flexible-joint manipulators (Baillieul 1992), or using redundancy to compensate for the static deflection of flexible-link manipulators (Yoshikawa et al. 1993), dynamics-based redundancy resolution involving link flexibility is still an open issue, which needs further research.

- **Modeling spatial link deformation.** Chapter 5 deals with modeling of FLFJ manipulators with spatial link deformation. Besides joint deformation, both in-plane and off-plane link bending deformations are taken into account. These deformations constitute the main source of the system deformation for most flexible manipulators. For manipulators with special configuration, however, link torsional and longitudinal deformations may need to be considered in the dynamic modeling. Thus, further work is required to develop a modeling method including all these deformations.

- **Experimental study of the developed models.** The model developed in Chapters 2 and 3 for rigid \(M/m\) manipulator systems and the model developed in Chapters 4-6 for flexible \(M/m\) manipulator systems have been analyzed theoretically and tested by simulation. The models are established based on assumptions, such as viscous damping, linear joint stiffness, small elastic deformation, etc. Some other factors, such as above-mentioned link torsional and longitudinal deformations, are also neglected. Experimental study is needed to verify the model accuracy against these unmodeled factors.
References


pp. 637-658.


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### List of Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i, A, A_M$</td>
<td>coefficients</td>
</tr>
<tr>
<td>$b_i, B_i$</td>
<td>coefficients</td>
</tr>
<tr>
<td>$c$</td>
<td>$\cos \lambda L$</td>
</tr>
<tr>
<td>$c_i, c_{ij}, c_{ijk}$</td>
<td>$\cos(\theta_i), \cos(\theta_i + \theta_j), \cos(\theta_i + \theta_j + \theta_k)$</td>
</tr>
<tr>
<td>$c_{\xi i}$</td>
<td>joint damping constant</td>
</tr>
<tr>
<td>$C, C_1, C_2$</td>
<td>constants</td>
</tr>
<tr>
<td>$C_{\xi y i}, C_{\xi z i}$</td>
<td>link structural damping constants</td>
</tr>
<tr>
<td>$C_{\xi L}, C_{\xi J}$</td>
<td>link and joint damping matrices</td>
</tr>
<tr>
<td>$C$</td>
<td>non-linear matrix</td>
</tr>
<tr>
<td>$C'$</td>
<td>damping matrix</td>
</tr>
<tr>
<td>$C'_b, C'_m, C'_M$</td>
<td>damping matrices of base, micro and macro</td>
</tr>
<tr>
<td>$C_b, C_m, C_M$</td>
<td>non-linear matrices of base, micro and macro</td>
</tr>
<tr>
<td>$C_{ij}, \bar{C}_{ij}$</td>
<td>non-linear coupling matrices between coordinates $i$ and $j$</td>
</tr>
<tr>
<td>$\text{ch}$</td>
<td>$\cosh \lambda L$</td>
</tr>
<tr>
<td>$\det(\cdot)$</td>
<td>determinant of a matrix</td>
</tr>
<tr>
<td>$D_{mij}, D_{Mij}$</td>
<td>submatrices of mass matrix</td>
</tr>
<tr>
<td>DOF</td>
<td>degree of freedom</td>
</tr>
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</table>
\( E \)  \( \text{Young's modules} \)

\( \text{EM} \)  \( \text{energy minimization} \)

\( F_{\text{link}}, F_{\text{joint}} \)  \( \text{link and joint dissipation functions} \)

\( \text{FLFJ} \)  \( \text{flexible link and flexible joint} \)

\( \text{FLRJ} \)  \( \text{flexible link and rigid joint} \)

\( g \)  \( \text{gravitational acceleration} \)

\( G_j \)  \( j\text{th element of gravity vector} \)

\( G, G_m, G_M \)  \( \text{gravity vectors of } M/m. \text{ micro. macro} \)

\( G_{ij}, G_{ijk}, G_i' \)  \( \text{coupling gravity terms} \)

\( I \)  \( \text{link cross section moment of inertia} \)

\( I_b, I_h, I_p, I_r \)  \( \text{moments of inertia of beam. hub. payload and rotor} \)

\( \bar{I}_i \)  \( \text{moment of inertia of link } i \text{ about the mass center} \)

\( I \)  \( \text{identity matrix} \)

\( I_{mi}, \bar{I}_{mi} \)  \( \text{moment of inertia tensor of micro link } i \text{ w.r.t. global and local systems} \)

\( I_{yi}, I_{zi} \)  \( \text{section moments of inertia of link } i \)

\( \text{J, } \dot{\text{J}} \)  \( \text{manipulator Jacobians and its time derivative} \)

\( J^+ \)  \( \text{generalized inverse of } J \)

\( J_b^w \)  \( \text{angular velocity Jacobian of base} \)

\( J_M^+ \)  \( \text{inertia-weighted generalized inverse of } J \)

\( (J_{mi})_b, (J_{mi})_m \)  \( \text{linear velocity Jacobians of micro link } i \text{ (mass center) w.r.t. } q_b \text{ and } q_m \)

\( (\bar{J}_m^v)_m \)  \( \text{expression of } (J_m^v)_m \text{ in local coordinate system} \)

\( (J_{Mi})_b, (J_{Mi})_M \)  \( \text{linear Jacobians of macro link } i \text{ w.r.t. } q_b \text{ and } q_M \)
(J^v_{mi})_{mj} \quad jth \ column \ of \ (J^v_{mi})_m

(J^v_{mi})_M \quad \text{linear velocity Jacobian of macro link } i \ (\text{mass center}) \ w.r.t. \ q_M

(J^\omega_{mi})_{mj} \quad jth \ column \ of \ (J^\omega_{mi})_M

(J^\omega_{mi})_b \quad \text{angular velocity Jacobian of micro link } i \ w.r.t. \ q_b

(J^\omega_{mi})_m \quad \text{angular velocity Jacobian of micro link } i \ w.r.t. \ q_m

J^\omega_{mi} \quad \text{expression of } J^\omega_{mi} \ \text{in } x_My_Mz_M

(J^\omega_{mi})_m \quad \text{expression of } (J^\omega_{mi})_m \ \text{in } x_My_Mz_M

(J^\omega_{mi})_M \quad \text{angular velocity Jacobian of micro link } i \ w.r.t. \ q_M

(J^v_{Mi})_M \quad \text{linear velocity Jacobian of macro tip w.r.t. } q_M

J^\omega_{Mi} \quad \text{expression of } J^\omega_{Mi} \ \text{in } x_by_bz_b

(J^\omega_{Mi})_M \quad \text{angular velocity Jacobian of macro (last link) w.r.t. } q_M

J_{\theta i} \quad \text{defined in Eq. (5.13a)}

J_{\eta i} \quad \text{defined in Eq. (5.13b)}

k_i, \ k_i \quad \text{joint stiffness constants}

K_{lh} \quad \text{relative hub moment of inertia}

K_{lp} \quad \text{relative payload moment of inertia}

K_k \quad \text{relative joint stiffness}

K_{mp} \quad \text{relative payload mass}

K_{Li}, K_{Ji} \quad \text{link and joint stiffness constants}

K_L, K_J \quad \text{link and joint stiffness matrices}

K_b \quad \text{base stiffness matrix}

l_i, l_{ci} \quad \text{length and mass center location of link } i
\( L \)  
link length or the Lagrangian

\( m_i, m_{mi} \)  
mass of link \( i \) and mass of micro link \( i \)

\( m_p \)  
payload mass

\( M/m \)  
macro/micro manipulators

\( M/m+B \)  
macro/micro manipulators mounted on a flexible base

\( M, \dot{M} \)  
mass matrix and its time derivative

\( M_b, M_m, M_M \)  
mass matrices of base, micro and macro

\( M_{ij}, M_{ij} \)  
coupling mass matrices

\( n \)  
number of DOFs

\( n_m, n_M \)  
numbers of DOFs of micro and macro

\( n_{Mr} \)  
number of rigid DOFs of macro

\( \text{PTR} \)  
peak torque reduction

\( q_i, \dot{q_i}, \ddot{q_i} \)  
position, velocity and acceleration vectors of joint \( i \)

\( q, \dot{q}, \ddot{q} \)  
generalized coordinate, velocity and acceleration vectors

\( q_b, q_m, q_M \)  
generalized coordinate vectors of base, micro and macro

\( \dot{q}_b, \dot{q}_m, \dot{q}_M \)  
generalized velocity vectors of base, micro and macro

\( \ddot{q}_b, \ddot{q}_m, \ddot{q}_M \)  
generalized acceleration vectors of base, micro and macro

\( \dot{q}_{opt}, \ddot{q}_{opt} \)  
optimum joint velocity and acceleration

\( \dot{q}^i, \ddot{q}^i, \dddot{q}^i \)  
joint position, velocity and acceleration at station \( i \)

\( \dot{q}^i_{opt}, \ddot{q}^i_{opt} \)  
optimum joint velocity and acceleration at station \( i \)

\( \dot{q}^P_{opt}, \ddot{q}^P_{opt} \)  
optimum joint velocity and acceleration at time \( t^P \)

\( Q \)  
optimization index
generalized forces
position vector of a point on link $i$
position and velocity vectors of base
vector from point $P_b$ to point $P_M$ expressed in $x_by_bz_b$
mass center location and velocity of micro link $i$
position vector of a point on link $i$ w.r.t. local coordinate system
mass center location of micro link $i$ w.r.t. local coordinate system
position vector of tip of macro manipulator w.r.t. global system
position vector of a point on macro link $i$ in global and local systems
real space of dimension $i$
real space of dimension $i \times j$
transformation matrix from $x_by_bz_b$ to $x_0y_0z_0$
transformation matrix from $x_My_Mz_M$ to $x_0y_0z_0$
transformation matrix from $x_My_Mz_M$ to $x_by_bz_b$
transformation matrix from $x_iy_iz_i$ to $x_0y_0z_0$
s. sh
$\sin(\lambda L)$ and $\sinh(\lambda L)$
$\sin(\theta_i). \sin(\theta_i + \theta_j). \sin(\theta_i + \theta_j + \theta_k)$
sensitivity indexes of characteristic root w.r.t. $K_k$
sensitivity index of characteristic root w.r.t. $K_{lh}$
skew-symmetric matrix of a vector
time instant at station $i$
time instant of peak torque

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$T$      kinetic energy
$T_M, T_M$ kinetic energy of micro and macro
TO        torque optimization
$v_i$     velocity of a point on link $i$
$V$       potential energy
$V_g$     gravity potential energy
$V_m, V_M$ potential energy of micro and macro
$V_{li}, V_{ji}$ link and joint elastic potential energy
$w$        link deflection
$w_{xi, w_{zi}}$ in-plane and off-plane deflections of link $i$
$W$        weighting matrix
$x$        link position coordinate
$x_i$     position vector of a point on link $i$
$x_{ji}$   defined in Eq. (5.6d)
$x_b y_b z_b$ local coordinate system fixed on base
$x_o y_o z_o$ global coordinate system
$x_M y_M z_M$ local coordinate system fixed at the tip of macro
$x, \dot{x}, \ddot{x}$ tip position, velocity and acceleration
$x^P, \dot{x}^P, \ddot{x}^P$ tip position, velocity and acceleration at time $t^P$
$y$        link deflection
$\alpha_i$ link angular deflection
$\zeta, \dot{\zeta}, \ddot{\zeta}$ flexible joint displacement, velocity and acceleration

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\( \beta \) flexible joint coordinate vector
\( \delta K_k, \delta K_{th}, \delta \gamma \) variations of \( K_k, K_{th}, \gamma \)
\( \delta \theta, \delta W \) joint virtual displacement and virtual work
\( \Delta t \) time step
\( \epsilon, \zeta \) coefficients
\( \eta_{xij}, \eta_{xzi} \) modal coordinates
\( \eta, \dot{\eta}, \ddot{\eta} \) modal coordinate, velocity, and acceleration vectors
\( \gamma \) root of characteristic equation
\( (\gamma)K_k \) characteristic root for \( K_k \)
\( (\gamma)K_k=\infty \) characteristic root for \( K_k = \infty \)
\( \lambda \) coefficient in characteristic equation
\( \lambda \) vector of Lagrangian multipliers
\( \theta, \dot{\theta}, \ddot{\theta} \) rigid joint position, velocity, and acceleration
\( \theta_i, \dot{\theta}_i, \ddot{\theta}_i \) rigid joint position, velocity, and acceleration of joint \( i \)
\( \theta, \dot{\theta}, \ddot{\theta} \) rigid joint coordinate, velocity, and acceleration vectors
\( \rho \) mass density
\( \tau_i \) \( i \)th joint torque
\( \tau^P_i \) torque of joint \( i \) at time \( t^P \)
\( \tau^i \) torque vector at station \( i \)
\( \tau^P \) torque vector at time \( t^P \)
\( \tau, \tau_m, \tau_M \) joint torque vectors of M/m, micro, and macro manipulators
\( \tau'_m, \tau'_M \) joint coupling torque vectors of micro and macro
\[ \tau_{mM} \cdot \tau_{MM} \] joint coupling torque vectors
\[ \tau_i^+, \tau_i^- \] upper and lower torque limits of joint \( i \)
\[ \tau^+, \tau^- \] vectors of upper and lower torque limits
\( \phi \) mode shape function
\( \phi', \phi'', \phi^{(3)} \) first, second, third and fourth derivatives of \( \phi \) w.r.t. \( x \)
\( \phi_{xyij}, \phi_{xziij} \) flexible-free mode functions
\( \hat{\phi}_{xyij}, \hat{\phi}_{xziij} \) mode shape functions without joint flexibility effect
\( \Phi_i \) mode shape function matrix
\( \omega \) natural frequency
\( (\omega)_{K_k} \) natural frequency for \( K_k \)
\( (\omega)_{K_k=\infty} \) natural frequency for \( K_k = \infty \)
\( \omega_b, \omega_M \) angular velocity vectors of base and macro (last link)
\( \omega_{m_i} \) angular velocity vector of micro link \( i \) w.r.t. \( x_o y_o z_o \)
\( \dot{\omega}_{m_i} \) angular velocity vector of micro link \( i \) w.r.t. \( x_M y_M z_M \)
\( \dot{\omega}_M \) angular velocity vector of macro (last link) w.r.t. \( x_b y_b z_b \)
Appendix A

Coefficient Matrix of Flexible-Free Modes

The elements of coefficient matrix \( a \) in Eq. (4.23) are expressed as follows:

\[
\begin{align*}
a_{11} &= 0 & (A.1a) \\
a_{12} &= 1 & (A.1b) \\
a_{13} &= 0 & (A.1c) \\
a_{14} &= 1 & (A.1d) \\
a_{21} &= -(k - I_h \omega^2) \lambda & (A.1e) \\
a_{22} &= -EI \lambda^2 & (A.1f) \\
a_{23} &= -(k - I_h \omega^2) \lambda & (A.1g)
\end{align*}
\]
\begin{align*}
a_{24} &= EI\lambda^2 \quad \text{(A.1h)} \\
a_{31} &= -EI\lambda^3 \cos \lambda L + m_p\omega^2 \sin \lambda L \quad \text{(A.1i)} \\
a_{32} &= EI\lambda^3 \sin \lambda L + m_p\omega^2 \cos \lambda L \quad \text{(A.1j)} \\
a_{33} &= EI\lambda^3 \cosh \lambda L + m_p\omega^2 \sinh \lambda L \quad \text{(A.1k)} \\
a_{34} &= EI\lambda^3 \sinh \lambda L + m_p\omega^2 \cosh \lambda L \quad \text{(A.1l)} \\
a_{41} &= -EI\lambda^2 \sin \lambda L - I_p\omega^2 \lambda \cos \lambda L \quad \text{(A.1m)} \\
a_{42} &= -EI\lambda^2 \cos \lambda L + I_p\omega^2 \lambda \sin \lambda L \quad \text{(A.1n)} \\
a_{43} &= EI\lambda^2 \sinh \lambda L - I_p\omega^2 \lambda \cosh \lambda L \quad \text{(A.1o)} \\
a_{44} &= EI\lambda^2 \cosh \lambda L - I_p\omega^2 \lambda \sinh \lambda L \quad \text{(A.1p)}
\end{align*}