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Dynamic Mode Decomposition of Elliptical Liquid Jets in Crossflow

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Abstract

Dynamics of round and elliptical liquid jets in subsonic crossflow is studied using high-speed imaging technique. The experiments are performed at constant gaseous weber number and liquid-gas momentum flux ratio of 6.45 and 17.87 respectively, with orifices of different aspect ratios (AR) having an equivalent diameter of 0.43 mm. All cases are carried out inside an open loop subsonic wind tunnel with a test section of $100 \times 100 \times 750$ mm. For each case, dynamic modes are generated directly from the snapshots using a variant of Arnoldi method known as the dynamic mode decomposition (DMD). DMD results indicate that elliptical liquid jets have more small-scaled patterns with higher frequencies compared to the case of round liquid jets. As the first attempt to investigate the dynamics of elliptical liquid jets in crossflow, present work captures the dominant spatiotemporal structures. It is also found that the orifice AR can alter the jet wavelengths remarkably. The extracted data of this work can provide beneficial information on the behavior of elliptical liquid jets exposed to the gas crossflow in the enhanced capillary breakup regime.

Keywords: Dynamic Mode Decomposition, Elliptical jet, Circular jet, Crossflow

1 Introduction

Liquid jet injection into a gaseous crossflow is an applicable mechanism in various fields such as thermal spray, ink-jet printing, and gas turbine engines (Wu et al. 1997). Study of the near field behavior of liquid jets in crossflow is essential since any oscillations on the jet body affects the functional efficiency of the spray. Identification of the principal phenomena could be useful in finding the coherent dynamic features and enhance the system performance as the liquid jet in crossflow (LJIC) is considered as a multi-scale phenomenon. There are many governing parameters controlling the LJIC phenomenon. Weber (Taylor 1940) found the effect of liquid viscosity on liquid jet breakup. Taylor (Sallam et al. 2004) discovered the impact of gas density. Many other parameters were also investigated by analyzing their effects on spray characteristics such as penetration height, column breakup point (CBP), Sauter mean diameter (SMD), spray spread angle in the works of (Sallam et al. 2004; Lee et al. 2007; Herrmann 2010; Yunyi et al. 1998). Despite numerous researches in this area, only a few works have been performed on asymmetric nozzles due to their associated complexities. Among asymmetric nozzles, elliptical type orifices are investigated more than the others. (Gutmark and Grinstein 1999) reported the enhanced mixing and spreading as the principal benefit of elliptical jets over their circular counterparts. One of the important features of elliptical liquid jets known as axis-switching is discussed in the work of (Amini and Dolatabadi 2012). Study of such phenomena is often not straight-forward directly from raw experimental results such as time-resolved snapshots of high-speed camera. Therefore, flow structures may be analyzed by decomposing them into modes using reduced order modelling (ROM) methods such as proper orthogonal decomposition (POD) (Holmes et al. 2012), dynamic mode decomposition (DMD) (Rowley et al. 2009), discrete Fourier transform (Holmes et al. 2012), balancing modes for linear systems (Rowley 2005), global

eigenmodes for linearized dynamics (Bagheri et al. 2009) and many other variants of these techniques. Recently, (Jadidi et al. 2019) showed that the flow structures at the exit of circular and elliptical nozzles are different and the elliptical nozzles show more prominent signs of turbulence. They also observed that for a specified momentum flux ratio and weber number, the elliptical jets disintegrate earlier.

In this paper, ROM modelling of elliptical liquid jets exposed to low weber numbers in the enhanced capillary regime close to the bag breakup regime is presented based on DMD technique (Schmid 2010). Using time-resolved snapshots, we extract the oscillatory physics of the flow and derive the relevant frequencies and the associated spatial modes using DMD. Dynamic mode decomposition (DMD) is an algorithm for finding Koopman modes and a standard variant of Arnoldi method (Rowley et al. 2009). As a global stability analysis technique, DMD is applicable over a broad range of linear and nonlinear flows without any flow assumptions required (Mittal and Kumar 2003).

Previous insight into the mechanism of LJIC was provided by an analysis technique based on proper orthogonal decomposition (POD) and spectral analysis (Arienti and Soteriou 2009). DMD as the decomposition method used in this paper can be thought of as an ideal combination of spatial dimensionality-reduction techniques as POD with Fourier transforms in time. Each of the spatial modes in DMD are also associated with a given temporal frequency, possibly with a growth or decay rate (Kutz et al. 2016). Thus, modal analysis of POD modes may result in association of each mode with different frequencies while DMD decomposes the input data into modes with singular frequency values for each of them.

2 Methodology and Governing Equations

The experimental facility in this study consists of an open loop subsonic wind tunnel with a test section of $100 \times 100 \times 750$ mm, liquid injection system and an imaging system. A schematic of the experimental setup can be found in the work of (Jadidi et al. 2019). A 1.5-HP blower fan made by Aeroflo is used to blow the ambient air into the wind tunnel. A Honeywell torque controller is utilized to adjust the air speed. The turbulence intensity of the gas velocity vectors was reported in the work of (Farvardin 2013) where it was shown by particle image velocimetry (PIV) that the air velocity is constant and parallel in all regions except at the narrow boundary layers on the wall.

The liquid injection system also comprises of a pressure vessel, a flowmeter, and circular/elliptical injectors with different AR. Distilled water is the used liquid throughout the test. Injectors having plain orifices with length to diameter of 10 are used where the equivalent diameter of different orifices is 0.43 mm. ARs are defined based upon the same rule provided in the work of (Marzbali 2011) where the direction of the orifice major axis with respect to the crossflow direction (parallel/perpendicular) is considered as the criteria for the AR calculation. Elliptical orifices of ARs 0.22 and 4.47, and a circular orifice of AR=1 is used in the present work. A tapered transition from 2 mm to 0.43 mm is considered to prevent cavitation effects inside the orifice (Farvardin 2019). Mass flow rate of the liquid jet and the velocity of the wind tunnel of 38 g/min and 30 m/s are kept constant during all the tests to ensure constant gas weber number (We) and momentum flux ratio (q) of 6.45 and 17.87, respectively.

$$We = \frac{\rho U_g^2 d}{\sigma} \quad (1)$$

$$q = \frac{\rho_l U_l^2}{\rho_g U_g^2} \quad (2)$$

The imaging facilities in this test include a Photron SA1.1 high-speed digital camera set at 5000 frames per second (fps)¹ with a 1024×1024 resolution, a lens (AF Micro-Nikkor 105 mm f/2.8, Nikon) and two halogen lamps (type: T-3, 200T3Q/CL/78MM, Satco, USA).

The experimental tests were performed at atmospheric pressure and room temperature and were repeated four times for each case at its unique AR. Fig.1 reveals three random snapshots of circular/elliptical liquid jets with different ARs at the enhanced capillary breakup regime. For each of the cases, Dynamic Mode Decomposition (DMD) was applied on 500 consecutive snapshots for acquisition of coherent dynamical structures.

Figure 1 - Enhanced capillary breakup of circular/elliptical jets; gaseous weber number (We) and momentum flux ratio (q) are 6.45, and 17.87, respectively.

Using DMD, one can analyze the temporal coefficients of coherent spatial modes. Development of this algorithm to extract dynamic mode information from a flow field was carried out by (Schmid 2015) based upon the Koopman analysis of dynamical system (Rowley et al. 2009). Current work is perhaps the first research on capturing the dominant dynamical modes to look at the small-scaled oscillations around elliptical liquid jets. Using the raw snapshots of the

¹ This fps amount was considered since even the smallest patterns like bags could be captured, it's notable that higher fps amounts could result in a lower resolution level that was not desirable.

experimental data, each is represented in the form of a columnar vector and the matrix of all snapshots could be introduced by V_1^R .

$$V_1^R = v_1, v_2, v_3, \dots, v_R \quad (3)$$

(Schmid 2010) assumed that linear operator A connects the two consecutive flow fields of v_j and v_{j+1} together as:

$$v_{j+1} = Av_j \quad (4)$$

After reaching an adequate and reasonable number of snapshots, the problem becomes linearly dependent and going through further flow fields would not further improve to the accuracy of the method. In this condition, the last snapshot v_R can be expressed as a linear combination of the previous snapshots and could be described as:

$$v_R = a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_{R-1}v_{R-1} + r \quad (5)$$

or in the matrix form:

$$v_R = v_1^{R-1}a + r \quad (6)$$

with $a^T = \{a_1, a_2, a_3, \dots, a_{R-1}\}$ and r as the residual vector, matrix A could be written in the matrix form using the least square representation of the full system (Schmid 2010).

$$V_2^R = AV_1^{R-1} = V_1^{R-1}Hb + re_{R-1}^T \quad (7)$$

with $e_{R-1} \in \mathbb{R}^{R-1}$ as the (R-1)th unit vector. The matrix Hb is of the companion type and is expressed as:

$$\mathbf{H}_b = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & a_1 \\ 1 & 0 & \cdot & \cdot & \cdot & a_2 \\ \cdot & 1 & \cdot & \cdot & \cdot & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 0 & a_{R-2} \\ \cdot & \cdot & \cdot & \cdot & 1 & a_{R-1} \end{bmatrix} \quad (8)$$

The eigenvalues of $\mathbf{H}_b$ then approximate some of the eigenvalues of A . The calculation of matrix $\mathbf{H}_b$ continues as follows: the last element of a given snapshot sequence v_R is expressed as a linear combination of the previous elements of the sequence as mentioned in Eq.4 whose least-squares solution, for a full rank matrix V_1^{R-1} is expressed as (Schmid 2010; Duke et al. 2012):

$$\mathbf{H}_b = R^{-1} Q^H V_2^R \quad (9)$$

The matrix is not fully singular and cannot become inverted directly. Therefore, a Moore-Penrose pseudo-inversion is applied to R utilizing the singular value decomposition (SVD) (Schmid 2010; Duke et al. 2012) so that:

$$R^{-1} = W \Sigma^{-1} U^H \quad (10)$$

Since the experimental data has often come along with various uncertainties and noise, a more robust approach will be used which results in a full matrix $\tilde{\mathbf{H}}_b$ in relation to $\mathbf{H}_b$ with a transformation function. The modal structures are extracted from the matrix $\tilde{\mathbf{H}}_b$ using a method like the standard Arnoldi method (Schmid 2010; Duke et al. 2012). In this paper, the corresponding dynamic modes are found as:

$$\phi_i = U v_i \quad (11)$$

where v_i is the (i) th eigenvector of $\mathbf{H}_b$. The characteristics solution of $\mathbf{H}_b$ approximates some of the eigenvalues (λ_i) of A (Schmid 2010) which are called Ritz values (Seena and Sung 2011). We

can further investigate the growth or decay rates (η_r) and frequency (η_i) by performing the logarithmic transformation of the real and imaginary eigenvalues to the complex stability plane. Ritz values falling on the unit circle exhibit modes with zero growth/decay rates while Ritz values laying inside or out of the unit circle represent the damped (stable) and undamped (unstable) modes respectively (Seena and Sung 2011).

3 Results and Discussion

A low-speed water jet exposed to a low-speed gaseous crossflow in the enhanced capillary regime is investigated by DMD. As mentioned before, 500 snapshots of data were captured with a 5000 frames per second imaging speed for each of the circular/elliptical liquid jet cases among which 101 arbitrary snapshots were used to generate A matrix with a maximum rank of 100 leading to the generation of maximum 100 dynamic modes. Further, an r-truncation scheme (Kutz et al. 2016) is developed to optimize the typical Ritz spectrum to have a near-equilibrium state of the system where the flow dynamics is almost linear. The exponential growth/decay rate and the stability/instability condition of the system could be discussed by the conjugate pair of eigenmodes. In addition, linear condition happens when almost all Ritz values lay on the unit circle $\|\lambda_i\| = 1$ that informs us about the coincidence of the sample points with an attracting set (Sarkar et al. 2013). Similar observation of it was reported in the work of (Rowley et al. 2009) for a liquid jet in crossflow (LJIC) case. As it can be seen in Fig.2, Ritz spectrum diagrams have been depicted for different values of (r) equal to 60, 80, 100 in case of a circular liquid jet. There are different configurations of data points to track their effect on the stability of the reduced-order model.

Figure 2 - Ritz spectra for different r-truncation schemes; (a) r=60, (b) r=80, (c) r=100

Figure 3 - Global spectra for different r-truncation schemes; (a) r=60, (b) r=80, (c) r=100

Hereby, an optimization approach was performed to maximize the number of data points laying on the unit circle that guarantees the stability and to minimize the energy content of the modes inside the unit circle. This approach that could be represented as the optimization problem of the Frobenius norm of the error (Eq.12), has been applied on all circular/elliptical cases by using different (r) values and an optimum (r) value is found for each of the cases that is equal to 62, 72, and 67 for orifices with ARs of 0.22, 1 and 4.47, respectively. Ritz and global spectra diagrams with the optimum AR for each of the circular/elliptical liquid jets are found in Figs.4-5.

$$\|V_2^R - AV_1^R\|_F \quad (12)$$

where $\|\cdot\|_F$ is the Frobenius norm, given by:

$$\|X\|_F = \sqrt{\sum_{j=1}^n \sum_{k=1}^m X_{jk}^2} \quad (13)$$

Figure 4 - Ritz spectra; (A) AR=1, (B) AR=0.22, (C) AR=4.47 at optimum r values

Figure 5 - Global spectra; (A) AR=1, (B) AR=0.22, (C) AR=4.47 at optimum r values

In Figs.2-3, the filled circles in colors indicate the global energy norm of the associated modes. The red-colored circle at $\|\lambda_i\| = 0$ and $\|\lambda_r\| = 1$ denotes the mean flow. For the case where $r=100$, two paired modes denoted by pale blue colour indicate that this model is not well suited since it is not linear and the decaying modes inside the circle have higher energy magnitudes with respect to their other counterparts laying on the unit circle. In general, the stability grows with increasing the value of (r). However, higher values of (r) may result in modes with remarkable energy content laying inside the unit circle that is not desirable. The principal modes of the circular/elliptical cases on the Ritz spectrum in Fig.4 are shown in a larger scale in Fig.6 where the associated modes are annotated with their indices.

Figure 6 - Ritz spectra for the mean and dominant dynamic modes at different ARs; (A) AR=1, (B) AR=0.22, (C) AR=4.47

These modes dominate the entire energy spectrum with specified energy contents. Corresponding energy values are normalized with respect to the energy content of the mean mode (mode 1) using the following form:

$$E = \frac{\|\phi_i\|}{\|\phi_1\|} \quad (14)$$

The stability level of a single mode can be discussed based on its closeness to the unit circle as:

$$S = \sqrt{\lambda_{real}^2 + \lambda_{imag}^2} \quad (15)$$

From Ritz spectra of dominant eigenvalues, the continuous oscillatory frequency can be found by the relationship (Kutz et al. 2016; Sarkar et al. 2013):

$$Frequency = \frac{imag(\ln(\lambda))}{2\pi\Delta t} = \frac{arg(\lambda)}{2\pi\Delta t} \quad (16)$$

Where Δt (the sampling time between two snapshots) and $arg(\lambda)$ are defined by the following expressions (Eqs.15-16):

$$\Delta t = \frac{1}{f_s} \quad (17)$$

$$\arg(\lambda) = \arctan\left(\frac{\lambda_{imag}}{\lambda_{real}}\right) \quad (18)$$

The sampling frequency is equal to 5000 in this case. Therefore, Δt will be 0.2 milliseconds. It is notable that the maximum capturable oscillatory frequency by DMD is limited by half of the sampling frequency. Tables 1,2, and 3 show the real and complex eigenvalues of the mean and dominant modes, their associated absolute and normal energy content, stability S term for each of the modes, the angle of the modes with respect to the positive real axis and finally their corresponding frequencies. It is interesting that the dominant mode capturing the oscillatory behavior of elliptical liquid jets in the enhanced capillary breakup regime at the threshold of bag breakup regime possess higher frequency levels compared to the case of circular liquid jet. From Eq.12, the average oscillatory frequency was calculated as 286 for the circular liquid jet whereas it was estimated at 349 and 343 for the major and minor axis elliptical liquid jets represented in Table.4, respectively. These results confirm the existence of more small-scaled patterns with higher frequencies which is equivalent to smaller wavelengths associated with the elliptical jets with ARs over or under 1. This is also in line with the difference between the flow structures at the exit of circular and elliptical nozzles where obvious signs of turbulence could be seen for the elliptical nozzles.

Fig.5 portrays the global DMD spectrum for the circular/elliptical cases. The global spectrum for the circular liquid jet is presented in Fig.5A whereas Figs.5B-5C represent the same for the elliptical liquid jet. A strong alignment trend could be considered along the line of the growth/decay rate $\eta_r = 0$. This trend is strongest in the circular case and the weakest in the elliptical case with the minor axis perpendicular to the direction of the crossflow. Strong alignment indicates a nearly linear model which is corresponding with the least amount of frequency and the highest

amount of jet wavelength. This means that most of the singular modes of a circular jet are aligned around the growth/decay rate $\eta_r = 0$ line which is representative of a more linear type of behavior of the circular jet compared to the elliptical jet and being more linear means more stability and less sparsity in terms of pixel intensity in a jet crossflow image. This is in line with our previous intuition from different breakup regimes of jet crossflow where a smaller number of droplets/ligaments are detached from the core of the jet in the enhanced capillary breakup regime in comparison with regimes that contain larger aggregates of drops/ligaments. This issue could also be investigated through the S values where the closeness of each mode to the unitary circle is specified. Based on Tables.1-3, averaged S values of 0.94, 0.92, and 0.89 are estimated for the dominant modes in the circular/elliptical liquid jets the results of which can be found in Table.4. At constant Weber number and momentum flux ratios, increasing or decreasing AR could result in an increase of the average frequency and an inevitable reduction of wavelengths and the breakup lengths (Jadidi et al. 2019).

Table 1 - Eigenvalues, energy content, stability, and frequencies of the mean and dominant modes for the circular liquid jet with AR=1

Mode No.	λ_{real}	λ_{imag}	$\ \phi\ $	E	S	$arg(\lambda)$	Freq.
1	1.00	0.00	125180	1.0000	1.00	0.00	0
2,3	0.96	± 0.17	2221	0.0177	0.95	0.17	138
4,5	0.94	± 0.05	2016	0.0161	0.94	0.05	44
6,7	0.026	± 0.34	3285	0.0262	0.98	0.35	286
8,9	0.84	± 0.18	5689	0.0454	0.86	0.22	176
10,11	0.84	± 0.30	3207	0.0256	0.89	0.34	275
12,13	0.83	± 0.42	1810	0.0145	0.94	0.47	374
14,15	0.80	± 0.53	2313	0.0185	0.96	0.58	467
16,17	0.73	± 0.57	868	0.0069	0.93	0.66	527

Table 2 - Eigenvalues, energy content, stability, and frequencies of the mean and dominant modes for the elliptical liquid jet with AR = 0.22

<i>Mode No.</i>	λ_{real}	λ_{imag}	$\ \phi\ $	E	S	$arg(\lambda)$	<i>Freq.</i>
1	1.00	0.00	117920	1.0000	1.00	0.00	0
2	0.93	0.00	24	0.0002	0.93	0.00	0
3,4	0.91	± 0.13	2861	0.0247	0.92	0.14	114
5,6	0.89	± 0.22	3963	0.0336	0.92	0.24	196
7,8	0.89	± 0.35	1900	0.0161	0.96	0.38	299
9,10	0.83	± 0.38	4410	0.0373	0.92	0.43	346
11,12	0.81	± 0.49	2107	0.0179	0.95	0.54	432
13,14	0.80	± 0.55	156	0.0013	0.97	0.59	475
15	0.75	0.00	5221	0.0443	0.75	0.00	0
16,17	0.69	0.62	2295	0.0195	0.93	0.73	583

In case of the energy norms, the values of E are given in Table.1 for the circular jet and in Tables.2-3 for its elliptical counterparts. Modes with higher E play a more significant role in terms of contributing their share to the total energy of all modes in the reduced order model (ROM). For example, paired modes (8,9) in the elliptical case with AR=4.47 has a 5.13% energy norm higher than any other paired modes of interest which make them the principal dynamics of the system. This result is validated by the depiction of the spatial structures of the real and imaginary parts of the principal modes in Figs.7-8.

These modes were calculated and depicted for AR=4.47 as the representation of all other studied cases. It is obvious from the real and imaginary parts of the modes that a 90° phase shift exists between them. Strong similarity between the spatial structure of these modes with the original snapshot data depicted in Fig.1 explains their higher energy norm compared to the other modes. This higher norm could be corresponded with the liquid penetration height which is a function of orifice aspect ratio and the momentum flux ratio. Elliptical liquid jets penetrate less into the

crossflows compared to the circular jets, the reason behind which may be the effects of drag force and axis switching.

Table 3 - Eigenvalues, energy content, stability, and frequencies of the mean and dominant modes for the elliptical liquid jet with AR = 4.47

<i>Mode No.</i>	λ_{real}	λ_{imag}	$\ \phi\ $	E	S	$arg(\lambda)$	$Freq.$
1	1.00	0.00	100070	1.0000	1.00	0.00	0
2,3	0.92	± 0.17	2107	0.0211	0.94	0.18	148
4,5	0.86	± 0.06	3483	0.0348	0.86	0.07	59
6,7	0.87	± 0.37	1554	0.0155	0.95	0.40	323
8,9	0.82	± 0.32	5135	0.0513	0.89	0.37	302
10,11	0.81	± 0.44	2399	0.0240	0.92	0.49	397
12,13	0.75	± 0.54	1048	0.0105	0.93	0.62	497
14,15	0.75	± 0.38	5339	0.0534	0.84	0.47	378
16	0.64	0.00	2182	0.0218	0.64	0.00	0
17,18	0.61	± 0.64	1770	0.0177	0.89	0.80	639

Table 4 - Averaged stability and frequency of elliptical liquid jets based on their corresponding AR.

AR	S	$Freq.$
0.22	0.92	349
1	0.94	286
4.47	0.89	343

Figure 7 - Real and complex parts of mode 2 for the elliptical jet with $AR=4.47$

Figure 8 - Real and complex parts of mode 8 for the elliptical jet with $AR=4.47$

3. Conclusion

An investigation of the dynamics of the elliptical liquid jets in gaseous crossflows is performed at low weber numbers. Different elliptic-type orifices with similar equivalent diameter are used. Flow conditions are kept constant by enforcing the same air and liquid velocity for all tests. It is concluded that the flow structures coming out of the elliptical nozzles are totally different in comparison with their circular counterparts. For a given momentum flux ratio, the elliptical liquid jets contain more coherent small-scaled structures with higher frequencies compared to the circular jets. Higher frequencies lead to lower wavelengths that means faster disintegration and less

penetration into the crossflows for the elliptical jets which is perhaps because of drag force, axis-switching, and the turbulence.

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Appendix A. Nomenclature

A	Gaseous Weber Number
AR	Momentum Flux Ratio
E	Singular Value Decomposition
f	Proper Orthogonal Decomposition
f_s	Principal Component Analysis
r	Dynamic Mode Decomposition
r	Support Vector Machine
S	Stability Term
U	Velocity m/s
H_b	Companion-type matrix

Appendix B. Greek Letters

Σ	Matrix of Eigenvalues
ρ	Density (kg/m^3)
σ	Surface Tension (N/m)
ϕ	Dynamic Modes
λ	Eigenvalues
η	Logarithmic Eigenvalues
We	Gaseous Weber Number
q	Liquid-Air Momentum Flux Ratio