ESSAYS ON EQUILIBRIUM VALUATION OF OPTIONS:
THEOREM AND EMPIRICAL ESTIMATES

by

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the degree of Doctor of Philosophy

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Abstract

This thesis consists of three essays which study the valuation of options in an equilibrium framework.

The first essay uses a general equilibrium model to study the valuation of options on the market portfolio with predictable returns and stochastic volatility in a complete market. In a closed endowment economy where aggregate dividend is the only source of uncertainty, I investigate why the stock return exhibits certain predictable features. I also examine the equilibrium relationship between the price of the market portfolio and its volatility, as well as the relationship between the spot interest rate and the market volatility. Equilibrium conditions imply that the predictable feature of the market portfolio is induced by the mean-reverting of the rate of dividend growth. It is shown that there is strong interdependence between the stock price process and its volatility process. Using the Euler equation, I derive equilibrium pricing formulas for options on the market portfolio which incorporate both stochastic volatility and stochastic interest rates. Since there is only one source of uncertainty, this model preserves the completeness feature for hedging and risk management purposes. With realistic parameter values, numerical examples show that stochastic volatility and stochastic interest rates are both necessary for correcting the Black-Scholes pricing biases.

The second essay focuses on the currency options in an incomplete market where the economy is subject to shocks in aggregate dividend and money supply.
The key feature is that the exchange rate exhibits systematic jump risks which should be priced in the currency options. The closed-endowment equilibrium model in the first essay is extended to a small open monetary economy with stochastic jump-diffusion processes for both the money supply and aggregate dividend. It is shown that the exchange rate is affected by both government monetary policies and aggregate dividends. Since the jump in the exchange rate is correlated with aggregate consumption, the jump risk in the exchange rate derived from aggregate consumption must be priced by means of utility maximization. I further derive the foreign agents’ risk-neutral valuation of the European currency option and provide restrictions that ensure the law of one price in currency option pricing. In general, these restrictions depend on the agent’s risk preference.

The objective of the third essay is to empirically study the existence of systematic jump risks in exchange rates and analyze their importance for currency option pricing. The empirical study is based on the theoretical model studied in the second essay, which argues that exchanges rates are inherently correlated with the market and so must exhibit systematic jump risks. The third essay uses the maximum-likelihood method to estimate the joint distribution of exchange rates and the price of the market portfolio. Empirical results show that it is important to incorporate both systematic and non-systematic jump components in exchange rates in order to correctly price currency options.
To my parents
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CHAPTER 1

OBJECTIVES AND LITERATURE REVIEW
1. Objective of the Dissertation

The objective of this dissertation is to study derivative securities in an equilibrium framework both theoretically and empirically. There are three essays in the dissertation. The first essay examines the pricing of options written on the market portfolio with predictable returns and stochastic volatility. The second essay focuses on currency options with endogenized systematic jumps in the exchange rate. The third essay explores the empirical implications of the currency option model developed in the second essay.

The common focus of these essays is on the volatility of asset prices. This focus is clearly interesting because a good understanding of volatility is necessary for firms to efficiently manage their portfolio risks and for the market economy to maintain stability. Recent financial disasters associated with derivative trading experienced by some large financial institutions indicate that such understanding is lacking.

Since the volatility of asset prices is not constant empirically, existing option pricing models have attempted to incorporate this aspect. Notable examples are Merton (1976), who adds a jump risk into the diffusion process used by Black and Scholes (1973) for the asset price, Hull and White (1987) and Stein and Stein (1991), who directly assume that the volatility is driven by a state variable which is different from the one in the asset price process. These models share two main features. First, they are partial equilibrium models that assume exogenous processes for asset prices, interest rates and factor premia. As pointed out by Bailey and Stulz (1989), the partial equilibrium framework provides no guarantee for the results to be internally consistent with any general equilibrium. More importantly, the framework does not clearly illustrate how option
prices may respond to a change in any fundamental underlying economic variable or structural parameter. The second feature of these partial equilibrium models is that the market price of risk for the jump or stochastic volatility is usually assumed to be zero (or a constant). This assumption is violated when the security under consideration is either the market portfolio or an exchange rate. In these cases, it is more appealing to use equilibrium models to price options. Also, an equilibrium framework can guide empirical research on options since the equilibrium conditions impose cross-equation restrictions that are useful for estimating the security price processes.

The following sections broadly summarize the related literature in derivative pricing. The three essays contain more detailed comparisons with existing option models individually.

2. Stock Option Models

Most theoretical option models related to the first essay are stock option models. The well-known Black-Scholes' (1973) stock option model was invented in 1973 and is still rightly regarded by both practitioners and academics as the premier model of option valuation. However, it has some well-known deficiencies when matched with market prices. In particular, the model overprices out-of-the-money call options and underprices in-the-money calls.¹ These deficiencies are usually ascribed to the strong assumption in the Black-Scholes model (BS henceforth) that the price of the asset underlying the derivative security follows a geometric Brownian motion with a constant drift rate and constant volatility. The majority of empirical works indicates that volatility is not constant (e.g. Rosenberg 1972, Oldfield, Rogalski and Jarrow 1977). Consequently, many theoretical option models have attempted to allow for non-constant volatility while maintaining the assumption of constant interest rates. Notable examples are Merton (1976), who adds a jump risk into the diffusion

¹This evidence is documented by MacBeth and Merville (1979). Similar findings are provided by Lauterbach and Schultz (1990) in an empirical study on warrants.
process used by Black and Scholes for the stock price, Hull and White (1987) and Stein and Stein (1991), who directly assume that the volatility is driven by a state variable which is different from the one in the stock price process.

These modifications of the BS model are important attempts to incorporate non-constant volatility in option prices but they are not satisfactory for three reasons. First, these models have ignored the important empirical evidence of stock return predictability. For example, the predictability of financial asset returns has been documented by Bekaert and Hodrick (1992), Breen, Glosten and Jagannathan (1989), Campbell and Hamao (1992), Fama and French (1988a, 1988b) and Ferson and Harvey (1991). Despite of the lack of understanding of the sources of such predictability, there is a growing consensus that predictability is a genuine feature of many financial asset returns. As pointed out by Grundy (1991) and Lo and Wang (1995), the predictability of an asset will affect the prices of options on that asset, although predictability is typically induced by the drift of the stock price, which does not enter the option formula. It will be interesting to investigate why an asset return has certain predictable features and how this predictability affects option prices.

Second, the typical assumption on the jump risks in the stock price process or the stochastic volatility is that they are uncorrelated with aggregate consumption or aggregate dividend. Such assumption is problematic when the security under consideration is the market portfolio.² A considerable amount of evidence has shown that the non-constant volatility of stock prices is highly correlated with the volatility of the market as a whole. For example, Wiggins (1987) has found a significant negative correlation between volatility movements and stock prices for the highly aggregated stock indices. The estimated correlation for S&P 500 is -0.79 for an 8-day interval in which the volatility is assumed constant. Intuitively, the volatility and price of the market portfolio

²In a parallel fashion, Naik and Lee (1990) argue that the jump risk in the market portfolio should be correlated with aggregate consumption and dividend.
are both driven by the same fundamental forces such as aggregate consumption and dividend. Thus, it is desirable to specify only the processes for these fundamental forces and endogenize the processes for the market portfolio and its volatility, then examine the relation between these two processes.

The third unsatisfactory feature of previous models of non-constant volatility is that they have assumed a constant interest rate. However, the volatility of the market portfolio is in general negatively related to the interest rate (see Bailey and Stulz 1989). To incorporate this negative correlation, one must abandon the assumption of constant interest rates and simultaneously analyze the effects of stochastic volatility and stochastic interest rates. Finally, previous models of non-constant volatility have introduced a non-traded source of risk such as jumps or stochastic volatility and hence lost their completeness, i.e., the ability to hedge options with the underlying asset. Such ability is desirable for hedging and risk management purpose in the business world, as stated in Dupire (1994).

To eliminate these unsatisfactory features, I use a continuous-time extension of the Lucas (1978) equilibrium model. As in Lucas (1978), the economy is a pure exchange economy in which there is a single representative agent with an infinite lifetime horizon. In the financial market, this single agent can instantaneously trade a single risky stock, pure discount bonds and other contingent claims written on this risky stock and the discount bonds. The risky stock can be viewed as the market portfolio whose dividend is the only exogenous source of uncertainty. By appealing to the study of Marsh and Merton (1987) on the dynamic behavior of aggregate dividend, the rate of dividend growth is modelled as a general affine-class of mean reverting process. The general pricing equation (the Euler equation) is derived by solving the representative agent's maximization problem. Under the specification of dividend process and the agent's preference, the processes
for the market portfolio, its volatility, the spot interest rate and the bond price are derived in equilibrium. Equilibrium results suggest that the predictability of the stock return is induced by the mean-reverting feature of the rate of dividend growth. It is also shown that the process of the market volatility is not independent of the price process of the market portfolio, rather is related in a particular way. The negative correlation between these two processes is ensured by imposing conditions on the parameters underlying the dividend process. Finally, the equilibrium spot interest rate exhibits mean-reverting feature which is resulted from a similar process for the rate of dividend growth. As a result, stochastic interest rates and stochastic volatility are both incorporated into the European stock option formulas which include the BS model as a special case. The effects of predictability on stock option prices can be explicitly analyzed since the stock option pricing formula are functions of the parameters underlying aggregate dividend process. For example, the long-run mean of the rate of dividend growth affect the stock call option positively. However, the effect of the mean-reverting speed is ambiguous on stock call options.

In addition to its simplicity and analytical tractability, numerical exercises show that the current stock option model adjusts the BS pricing biases very well under reasonable parameter values. It provides higher prices for in-the-money calls and lower prices for out-the-money calls. Also, call prices given by the current model generate a realistic pattern of implied volatility which is consistent with the empirical study on S&P 500 index European option by Wiggins (1987) and Dumas, Fleming and Whaley (1995). These results indicate that eliminating the unsatisfactory features can certainly improve some aspects of the BS stock option model.

This essay shares the common focus with Grundy (1991) and Lo and Wang (1995) on the effects of predictable returns on option prices. These studies have made the first attempt to understand some basic issues in a partial equilibrium framework. For example, issues like whether the BS
risk-neutral log-normal assumption is consistent with a trending Ornstein-Uhlenbeck process for the actual stock return (see Grundy 1991) and how the parameters underlying a trending Ornstein-Uhlenbeck process for the actual stock return affect the BS formulas (see Lo and Wang 1995). These studies are performed in the BS environment where interest rates and volatility are constant. This paper provides further insight on these issues in an equilibrium context where both interest rates and volatility are stochastic. Since the predictable return of the market portfolio, its volatility and spot interest rates are endogenized in equilibrium, we can determine the conditions under which a constant risk-neutral drift is consistent with a mean-reverting actual stock return. Also, we can precisely analyze how the predictable features affect the option prices in an environment where the spot interest rate and the volatility of the market portfolio are mean-reverting.

The equilibrium approach for option valuation is shared with Bailey and Stulz (1989) who price options written on stock indices, Naik and Lee (1990) who address the systematic jump risks in the market portfolio, and Amin and Ng (1993) who focus on individual stock option prices with systematic volatility. These models do not explicitly model the predictable returns and stochastic interest rates. bonds.

In addition, the first essay contributes to the existing derivative literature by provides exploiting the implications of the stochastic interest rate in the bond option markets.3 Closed-form formulas for European bond option prices are obtained. The Vasicek (1979) model and the Cox-Ingersoll-Ross model (1985, CIR henceforth) can be viewed as special cases. In this sense, the first essay provides a consistent way to price options on the market portfolio and the bonds.

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3 Other well-known Markov models of yield curve include Ho and Lee (1986) and Hull and White (1990). As pointed out by Hull (1992), these models can fit the initial term structure by choosing a function of time for the drift of the instantaneous interest rate.
3. Currency Option Models

Currency options are typically priced using the model of Garman and Kohlhagen (1983) (GK hereafter), which extends Black's (1976) commodity option model to currency options. A deficiency of the GK model is that it underprices out-of-the-money currency options as compared with market prices (Bodurtha and Courtadon 1987). In an attempt to find models that correct this deficiency, various empirical studies on exchange rates suggest that there may be jump risks in exchange rate movements (See Akgiray and Booth (1988), Jorion (1988), Tucker (1991), and Ball and Roma (1993)). A number of scholars, such as Bodurtha and Courtadon (1987), Jorion (1988) and Dumas, Jennergren and Näslund (1995), suggest that replacing the Brownian motion in the GK model by Merton's mixed jump-diffusion process should improve the performance of the model.

However, there are three reasons why directly applying Merton's jump-diffusion stock option model to currency options is unsatisfactory. First, Merton's formula is derived under the assumption that the Brownian motion risk is arbitrated away, while the jump risk is uncorrelated with the market. The assumption of uncorrelated jump risks may be reasonable if the concern were instead firm specific stocks, but is problematic for currency markets. Since the exchange rate reflects one nation's purchasing power relative to another nation, the exchange rate is inherently correlated with aggregate fundamental forces that affect the market. For example, movements in aggregate dividends must simultaneously affect aggregate consumption and the exchange rate.

Second, the information arrival process in the foreign exchange market differs from that in the stock market, since the exchange rate is directly influenced by government monetary policies that do not have apparent counterparts in the stock market. In fact, the price bias pattern of the GK model for currency options is opposite to the bias pattern of the Black-Scholes model (1973) (BS hereafter) for stock options. The BS model generally overprices out-of-the-money call options and underprices
in-the-money calls (MacBeth and Merville 1979), but the GK model usually underprices out-of-the-money currency options. Thus, if Merton's jump-diffusion model with non-systematic jumps can eliminate the price distortion by the BS model for stock options, adding a similar non-systematic jump process into the GK model may not be sufficient to correct the price bias for currency options. Such an inappropriateness of applying the stock option pricing model to currency options has been recognized by researchers such as Jorion (1988, pp427-428):

> Many financial models rely heavily on the assumption of a particular stochastic process, while relatively little attention is paid to the empirical fit of the postulated distribution. As a result, models like option pricing models are applied indiscriminately to various markets such as the stock market and the foreign exchange market when the underlying processes may be fundamentally different.

A third argument against directly applying Merton's stock option model to currency options is that such an application generates seemingly paradoxical results such as the analog of the so-called Siegel's paradox in currency options which is illustrated by Dumas and Näslund (1995). In particular, if both domestic and foreign investors maintain Merton's risk-neutral formulation of the exchange rate process, then the jump-diffusion model delivers option values that are different for the two investors. This shows that the jump risk cannot be unpriced for both investors.

To eliminate these unsatisfactory consequences of applying Merton's model requires an equilibrium model of currency options. An equilibrium model, based on utility maximization, solves the first problem above by endogenously solving for the relationship between the exchange rate and the fundamental forces that underlie the market. In this case, jump risks in exchange rates are, in general, correlated with aggregate consumption and government monetary policies. The explicit modelling of the relationship between the exchange rate and monetary policies also helps

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4 The paradox (Siegel 1972) originally illustrates the discrepancy between the forward exchange rate and the expected future exchange rate. That is, when the exchange rate is expressed as the price of the domestic currency in terms of the foreign currency, the forward exchange rate is always less than the expected future rate.
to uncover the distinct nature of the exchange rate process that differs from the stock price process. This enables me to price options on the exchange rate and stock accordingly. When an investor's intertemporal decision is explicitly formulated, the specification of the utility function abandons the assumption of unpriced jump risks and hence solves the third problem with Merton's model.

There is a voluminous literature on currency option valuations in a partial equilibrium framework where the spot exchange rate is exogenously specified. Notable examples are Amin and Jarrow (1991), Biger and Hull (1983), Chesney and Scott (1989), Dumas, Jennergren and Näslund (1995), GK (1983), Grabbe (1983), Heston (1993), and Melino and Turnbull (1990, 1991). As pointed out by Bailey and Stulz (1989), the arbitrary choice of the exogenous process for any security price in the partial equilibrium models is unlikely to be consistent with the equilibrium condition or to provide important insights into how derivative prices may respond to changes in any fundamental economic variables. The main improvement of the second essay over these models is that assets prices and the exchange rate here are endogenously derived from agent's maximization behavior and market clearing conditions.

On the other hand, most international equilibrium models of exchange rates do not examine currency options. Examples include Bekaert (1994), Stulz (1987), Svensson (1985), and Grinols and Turnovsky (1994), who focus on issues such as equilibrium interest rate, exchange rate premia and forward exchange biases. An exception is Bakshi and Chen (1996), who extend Lucas's (1982) two-country endowment economy from a discrete-time environment to a continuous-time environment and study the exchange rate derivative valuations. The second essay and the Bakshi and Chen paper both share the equilibrium approach but differ in many important aspects. First, the focus of this paper is how systematic jump risks in the exchange rate inherited from aggregate dividend

\footnote{I became aware of this independent work by Bakshi and Chen (1996) after completing the first draft of this paper.}
and affect currency option prices. In contrast, jump risks are not modelled by Bakshi and Chen (1996), although their presence in exchange rates is apparent and has important implications on currency options, as discussed earlier. Second, this paper examines currency options in a small open economy, while Bakshi and Chen (1996) study a two-country economy. The distinction between a small open economy model and a two-country model has important implications on the equilibrium exchange rate. For example, in Lucas (1982) and Bakshi and Chen (1996), the equilibrium portfolio of each country is identical to its initial endowment and so the net trading in assets between the two countries is zero in equilibrium. In contrast, the net trading in foreign bonds here must be non-zero in equilibrium in order to clear the goods market. Specifically, the domestic agents in the small open economy can finance their consumption through both domestic aggregate dividend and the return to holding foreign bonds. Since the exchange rate clears the goods market, the net trading volume affects the exchange rate. Third, the role of money is introduced here through agents' utility function rather than through a cash-in-advance constraint as in Bakshi and Chen (1996). The cash-in-advance approach depends crucially on the timing of events and hence on the discrete-time structure, as stated in Sargent (1987, pp 157). For the cash-in-advance constraint to bind, all financial markets must be temporarily shut down when consumption goods are purchased with money. In a continuous-time setting where agents can instantaneously sell goods and assets for money, the cash-in-advance constraint can not bind, since the utility cost of a marginal delay in consumption induced by the cash-in-advance constraint goes to zero.

4. Empirical Evidence Related with Currency Option Prices and Exchange Rate Dynamics

Bodurtha and Courtadon (1987) tested the ability of the American Option valuation model and the GK model to explain the pricing of currency options quoted on the Philadelphia Stock Exchange.
Focusing on the relative pricing error, these models seem to underprice short-term out-of-the-money options by as much as 29 percent. At-the-money and in-the-money options are generally slightly overpriced, with the bias most pronounced for short-maturity options. These results are in contrast to what has been found for stock options. As suggested by Bodurtha and Courtadon (1987), the directions of these biases are generally consistent with a mixed jump-diffusion process. Consider the price of an out-of-the-money call option close to maturity. If the exchange rate follows a diffusion process, the chance of exercising the option at maturity may be quite small. With a jump process, however, one jump may be sufficient to move the option in the money, which implies that a diffusion model will underprice the option. Thus the issue is whether a jump-diffusion model for the exchange rate can account for the empirically observed large biases.

Many scholars have provided strong empirical evidence on jump risks in exchange rates, for example, Akgiray and Booth (1988), Jorion (1988), Tucker (1991), and Ball and Roma (1993). In particular, Jorion (1988) has investigated the existence of discontinuities in the sample path of exchange rates and a value-weighted U.S. stock market index. He found that exchange rates display significant jump components, which are more manifest than in the stock market. These discontinuities seem to arise even after explicit allowance is made for possible heteroskedasticity in the usual diffusion process and appear very strongly in weekly data but less so in monthly data. The statistical analysis was performed side by side for the foreign exchange market and the stock market, and it suggests important differences in the structure of these markets. Further, Jorion (1988) used Merton's mixed jump-diffusion stock model to illustrate the implication of jump risks for the currency options. Using the estimated parameters, numerical examples showed that about two-thirds of the 29 percent biases (i.e. 17 percent) reported for short-term out-of-the-money options can be explained by Merton's mixed jump-diffusion process with non-systematic jump risks.
There is still 12% pricing bias left unexplained. Such a large unexplained error might be attributed to the absence of the \textit{systematic jump risk} which is the main subject of the third essay.

The objective of the third essay is thus to empirically examine the existence of systematic jump risks in exchange rates and determine whether they can explain the observed mispricing in the currency options market that can not be explained by Merton's model. A particular strength of this empirical analysis is that the equilibrium model in the second essay provides conditions on the joint distribution of the exchange rate and the price of the market portfolio. These conditions suggest that cross-equation restrictions must be imposed on the coefficients when the processes for exchange rates and the price of the market portfolio are estimated.

This essay is closely related to Jorion (1988), but differs in the following aspects. First, this study incorporates both systematic and non-systematic jump risks on currency option prices, while Jorion only focuses on non-systematic jump risks. Second, the distributions of exchange rates and stock indices arise endogenously from the equilibrium conditions derived in an equilibrium model proposed in the second essay, while Jorion exogenously specifies such distributions. Third, this paper estimates the joint distribution between exchange rates and stock indices, while Jorion estimates two independent mixed jump-diffusion processes for exchange rates and stock indices. The empirical evidence suggests that the two processes are dependent and the dependence plays an important role for correcting the GK pricing bias on currency options. Finally, the current study also analyzes the implied volatility smile in the presence of systematic and non-systematic jump risks in the currency option model, which is not addressed in Jorion (1988).
CHAPTER 2

EQUILIBRIUM VALUATION OF OPTIONS ON THE MARKET PORTFOLIO
WITH PREDICTABLE RETURNS AND STOCHASTIC VOLATILITY
1. Introduction

The objective of this essay is to eliminate several unsatisfactory features of the existing stock option models discussed in the literature review. To do so, I use a continuous-time extension of the Lucas (1978) equilibrium model. As in Lucas (1978), the economy is a pure exchange economy in which there is a single representative agent with an infinite lifetime horizon. In the financial market, this agent can instantaneously trade a single risky stock, pure discount bonds and other contingent claims written on this risky stock and the discount bonds. The risky stock can be viewed as the market portfolio whose dividend is the only exogenous source of uncertainty. By appealing to the study of Marsh and Merton (1987) on the dynamic behavior of aggregate dividend, the rate of dividend growth is modelled as an affine-class of mean reverting process. The general pricing equation (the Euler equation) is derived by solving the representative agent's maximization problem. Under the specification of dividend process and the agent's preference, the processes for the market portfolio, its volatility, the spot interest rate and the bond price are derived in equilibrium.

Equilibrium results indicate that the mean-reverting feature of the rate of dividend growth generates the predictability of the stock return. Endogenizing the return of the market portfolio, its volatility and spot interest rates enables me to determine whether the BS model is consistent with a mean-reverting actual stock return. It is shown that the predictability of the return to the market portfolio requires either the volatility and/or the spot interest rate to be stochastic. Thus, the BS model with constant volatility and spot interest rate is not consistent with the predictability of the asset return on the market portfolio. In addition, I analyze how the predictable features affect the option prices in an environment where the spot interest rate and the volatility of the market portfolio are mean-reverting.
Although the actual drift rate does not explicitly appear in the option pricing formula when the equivalent martingale pricing principle is used, fundamental forces that affect the drift affect option prices as well through their effects on the endogenous interest rate and volatility.

It is also shown that the process of the market volatility and its price process are not independent. The two processes are negatively correlated, as indicated by empirical evidence, when reasonable conditions are imposed on the parameters underlying the dividend process. Finally, the equilibrium spot interest rate exhibits the mean-reverting feature. As a result, stochastic interest rates and stochastic volatility are both incorporated into the European stock option formulas which include the BS model as a special case. The predictability affects stock option prices through both the stochastic interest rate and stochastic volatility. Since changes in parameters underlying the predictable features generate opposite impacts on the interest rate and volatility simultaneously, an induced increase in the volatility or interest rate by these fundamental parameters does not necessarily increase the call price.\(^1\)

In addition to its simplicity and analytical tractability, the current stock option model adequately corrects the BS pricing biases very well under reasonable parameter values. Numerical exercises show that the current model provides higher prices for in-the-money calls and lower prices for out-the-money calls than the BS model. Also, call prices given by the current model generate a realistic pattern of implied volatility which is consistent with the empirical study on S&P 500 index European option by Wiggins (1987) and Dumas, Fleming and Whaley (1995). These results indicate that eliminating the unsatisfactory features can significantly improve some aspects of the BS stock option model.

The equilibrium approach for option valuation is shared with Bailey and Stulz (1989) who price

\(^1\) The common belief is that an increase in stock volatility will be accomplished by an increase in call price according to the BS model. Bailey and Stulz (1989) show that this common belief is not necessarily supported in an equilibrium framework. The results in the current paper confirms the observation made by Bailey and Stulz (1989).
options written on stock indices, Naik and Lee (1990) who address the systematic jump risks in the market portfolio, and Amin and Ng (1993) who focus on individual stock option prices with systematic volatility. The current model differs from these models by explicitly modelling the predictability of stock returns and stochastic interest rates. In addition, this paper provides closed-form formulas for European bond option prices which encompass the Vasicek (1979) model and the Cox-Ingersoll-Ross model (1985, CIR henceforth). It provides a consistent way to price options on the market portfolio and bonds in a one-factor context.

On the effects of predictability of the asset return on option prices, this paper is related to Grundy (1991) and Lo and Wang (1995). Using partial equilibrium frameworks, these studies have attempted to understand some basic issues such as whether the BS risk-neutral log-normal assumption is consistent with a trending Ornstein-Uhlenbeck process for the actual stock return (Grundy 1991) and how parameters underlying such a process affect the BS formulas (Lo and Wang 1995). These studies employ the BS environment where interest rates and volatility are constant. In contrast, the current paper uses an equilibrium framework where both interest rates and volatility are endogenous and stochastic, which shows that the BS model with constant volatility and interest rate is inconsistent with stock return predictability.

The remainder of this paper is organized as follows. Section 2 describes the economy, presents the general equilibrium results and then analyzes the dynamics of the price of the market portfolio, its volatility, the bond price and the spot interest rate. Section 3 examines the relations among the spot interest rate, stock return predictability and its volatility. Section 4 derives the equilibrium pricing formulas for options written on the market portfolio and bonds, and examines the effects of predictability on option prices. In addition, comparative statics are performed for both types of options. Section 5 numerically compares the option pricing for the market portfolio with the BS model and examines the pattern of the implied volatility for the market portfolio. Section 6
concludes the paper and the appendices provide necessary proofs.

2. The Economy

2.1. Structure of the Economy

Consider a continuous-time extension of the Lucas (1978) pure exchange economy in which there is a representative investor with an infinite lifetime horizon. In the financial market, the representative agent can trade a single risky stock, pure discount bonds and a finite number of other contingent claims at any time. The risky stock can be viewed as the market portfolio, whose total supply is normalized to one share and its dividend stream \( \{ \delta_t \} \) can be understood as the aggregate dividends in the economy. The contingent claims and the riskless bond are all in zero net supply. I assume that the aggregate dividend process is exogenously given by a Markov process on a given probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \). The fundamental uncertainty in the model is completely described by the process for the aggregate dividend. Denote the security prices at time \( t \) by a vector \( X_t \) and the corresponding vector of dividends by \( q_t \). The cumulative dividends up to \( t \) are defined as \( D_t \equiv \int_0^t q_r dr \).

The representative agent's information structure is given by the filtration \( \mathcal{F}_t \equiv \sigma( \delta_r, 0 \leq r \leq t) \). His preference is described by a smooth time-additive expected utility function:

\[
V(c) = E \int_0^\infty U(c_t, t) dt,
\]

where \( U : \mathbb{R}_+ \times (0, \infty) \rightarrow \mathbb{R} \) is smooth on \( (0, \infty) \times (0, \infty) \) and, for each \( t \in (0, \infty) \), \( U(\cdot, t) : \mathbb{R}_+ \rightarrow \mathbb{R} \) is increasing, strictly concave, and has a continuous derivative \( U_\cdot(\cdot, t) \) on \( (0, \infty) \). Initially, the agent is endowed with one share of the risky stock. Denote his portfolio holdings at time \( t \) as \( \theta_t = (N_t^s, N_t^B, N_t^\tau) \), where \( N_t^s, N_t^B \) and \( N_t^\tau \) represent the number of shares invested in the risky stock, the discount bond and other contingent claims, respectively. The agent's consumption over time is financed by a continuous trading strategy \( \{ \theta_t, t \geq 0 \} \). The agent's decision problem is to choose
such a trading strategy so as to maximize his expected lifetime utility. Precisely, he solves\(^2\)

\[
\max_{\{c_t, \theta_t\}} E \int_0^\infty U(c_t, t) dt
\]

s.t.

\[
\int_0^t c_r d\tau = \theta_0 \cdot X_0 + \int_0^t \theta_\tau \cdot dD_\tau + \int_0^t \theta_\tau \cdot dX_\tau - \theta_t \cdot X_t.
\]

The first order conditions give the usual stochastic Euler equation:

\[
X_t = \frac{1}{U_c(c_t, t)} E_t \left( \int_t^\infty U_c(c_\tau, \tau) dD_\tau \right).
\]

Thus, the price of any security equals the expected discounted sum of its dividends, with the marginal rate of substitution being the stochastic state price deflator.

In equilibrium, the financial market clears and so the demand for the stock equals the supply of shares, which is one share. Also, equilibrium prices are such that the representative agent holds nothing of the other claims because the corresponding net supply is zero. In addition, the goods market clears so that consumption equals dividends generated from the risky stock. Therefore, the equilibrium price of the risky stock, denoted \(S_t(\delta_t)\), is

\[
S_t(\delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left( \int_t^\infty U_c(\delta_\tau, \tau) \delta_\tau d\tau \right), \quad \forall \ t \in (0, \infty).
\]

For a riskless bond paying 1 unit of consumption goods at \(T\) and 0 at all other time, its equilibrium price at time \(t\), denoted \(B_t(T, \delta_t)\), is

\[
B_t(T, \delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left( U_c(\delta_T, T) \right), \quad \forall \ t \in (0, T).
\]

For any contingent claim \(i\) with a payoff \(q_T^i\) at maturity \(T\), its price at time \(t\), denoted \(F_t(T)\), is

\[
F_t(T, \delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left( U_c(\delta_T, T) q_T^i \right), \quad \forall \ t \in (0, T).
\]

\(^2\)All the expectations in this paper are taken with respect to the filtration specified earlier. The budget constraint is similar to that defined in Duffie (1992) for the security market equilibrium. This Euler equation approach is also adopted in Naik and Lee (1990).
In particular, \( q_T^r = \max\{S_T(\delta_T) - K, 0\} \) for a European call option written on the risky stock, and \( q_T^b = \max\{B_T(T, \delta_T) - K, 0\} \) for a European call option written on the riskless bond whose maturity is \( T \geq T \). \( K \) denotes the striking price for both options.

2.2. Equilibrium Prices under a Specific Dividend Process

To facilitate discussion and to obtain closed form solutions, let us restrict attention to a specific dividend process for the market portfolio. The specific process is chosen by appealing to the study of Marsh and Merton (1987) on the dynamic behavior of aggregate dividends (see also Lintner 1956 and Fama and Babiak 1968). Their estimation results suggest that changes in the rate of dividend tend to conform with the following description:\(^3\)

\[
\ln (\text{div}_t) - \ln (\text{div}_{t-1}) = \text{speed of adjustment} \times (\text{target ratio} \times \text{change in stock price}_t - \ln \text{div}_{t-1}).
\]

Their regression results also show that the random components in the change of dividend growth exhibit heteroskedasticity. In order to be consistent with these findings, the process for aggregate dividends is assumed as follows:\(^4\)

**Assumption 1.** The rate of aggregate dividend growth evolves according to the following stochastic process:

\[
d\ln \delta = (\beta_1 - \alpha_1 \ln \delta)dt + \sqrt{\beta_2 + \alpha_2 \ln \delta} \, dz,
\] (2.6)

where \( dz \) is the standard Wiener process. In addition, restrict \( 0 \geq \alpha_2 \geq -2\alpha_1 \).

The drift in (2.6) captures the mean-reverting feature of the rate of dividend growth while the volatility structure \( \beta_2 + \alpha_2 \ln \delta \) corresponds to a GARCH model. Note that (2.6) is an extension of the process assumed for the single state variable in CIR (1985). It is the continuous-time

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\(^3\)Lintner (1956) and Fama and Babiak (1968) study the dividend behavior for individual stocks. They use the accounting earnings variable instead of the changes in stock prices.

\(^4\)The square-root process corresponds to the limit case of a particular jump process (see Cox and Ross 1976).
counterpart of a first-order autoregressive process in discrete-time where the randomly changing rate of dividend is pulled toward a long-run mean, $\beta_1/\alpha_1$. Parameter $\alpha_1$ determines the speed of mean reversion. The restriction $0 \geq \alpha_2 \geq -2\alpha_1$ is imposed in order to guarantee the realistic negative relationship between the stock price and its volatility, as well as the negative relationship between the bond price and the spot interest rate (see later discussion).

In order to describe the distribution of $\delta_T$ conditional on $(\delta_t, t)$, I take a linear transformation $Y(\delta) = \beta_2 + \alpha_2 \ln \delta$. By Ito’s Lemma, we have

$$dY = (\alpha_1 \beta_2 + \alpha_2 \beta_1 - \alpha_1 Y)dt + \alpha_2 \sqrt{\gamma} dz.$$  \hfill (2.7)

The process implied by (2.7) has the following properties: (i) $Y$ is strictly positive if $2(\alpha_1 \beta_2 + \alpha_2 \beta_1) \geq \alpha_2^2$ and $\alpha_1 > 0$; (ii) the variance of $Y$ increases when $Y$ increases; and (iii) $Y_T$ conditional on $(Y_t, t)$ has a non-central $\chi^2$ distribution with the following density function:

$$f(Y_T, T; Y_t, t) = a(t, T)e^{-(x+\lambda)} \left(\frac{x}{\lambda}\right)^{\frac{1}{2}(v-1)} I_{v-1}(\sqrt{x\lambda})$$  \hfill (2.8)

where

$$a(t, T) \equiv \frac{2\alpha_1}{\alpha_2^2 (1 - e^{-\alpha_1(T-t)})}, \quad v \equiv \frac{2(\alpha_1 \beta_2 + \alpha_2 \beta_1)}{\alpha_2^2}, \quad x \equiv a(t, T)Y_T.$$  \hfill (2.9)

In this description, $2\lambda$ is the non-central parameter, $2v$ is the degree of freedom, and $I_{v-1}(\cdot)$ stands for the modified Bessel function of the first kind of order $v - 1$. \hfill (6) In the steady state as $T \to \infty$, the density function (2.8) converges to $f(Y_\infty, \infty; Y_t, t) = \bar{a}\Gamma(v)^{-1}e^{-\bar{a}Y_\infty} (\bar{a}Y_\infty)^{v-1}$, a central $\chi^2$ distribution. $\bar{a} = \lim_{T \to \infty} a(t, T) = \frac{2\alpha_1}{\alpha_2^2}$ and the degree of freedom is $v$.

---

5See Johnson and Kotz (1970) for the non-central $\chi^2$ distribution and Feller (1951) for the corresponding probability transition function.

6The modified Bessel function of the first kind of order $q$ is defined as:

$$I_q(y) = \left(\frac{y^2}{2}\right)^q \sum_{j=0}^{\infty} \frac{(\frac{y^2}{4})^j}{j!\Gamma(q + j + 1)},$$

where $\Gamma(a)$ is expressed as $\Gamma(a) \equiv \int_0^\infty e^{-y} y^{a-1} dy$. (see Johnson and Kotz 1994).
The conditional expected mean and variance of \( \ln \delta_T \) can be computed as:

\[
E(\ln \delta_T | \ln \delta_t) = \ln \delta_t e^{-\alpha_1 (T-t)} + \frac{\beta_1}{\alpha_1} (1 - e^{-\alpha_1 (T-t)}),
\]

\[
Var(\ln \delta_T | \ln \delta_t) = \frac{\alpha_2}{\alpha_1} \ln \delta_t (e^{-\alpha_1 (T-t)} - e^{-2\alpha_1 (T-t)}) + \frac{\beta_2}{2\alpha_1} (1 - e^{-2\alpha_1 (T-t)}) + \frac{\alpha_2 \beta_1}{2\alpha_1^2} (1 - e^{-\alpha_1 (T-t)})^2.
\]

When the reversion rate \( \alpha_1 \) goes to 0, the mean converges to \( \ln \delta_t + \beta_1 (T-t) \) and the variance converges to \( (\alpha_2 \ln \delta_t + \beta_2)(T - t) + \alpha_2 \beta_1 (T - t)^2 \). As \( T \to \infty \), the steady state mean and variance are \( \beta_1/\alpha_1 \) and \( (\alpha_1 \beta_2 + \alpha_2 \beta_1)/2\alpha_1^2 \), respectively.

For analytical tractability, I adopt the typical logarithmic assumption on the agent’s preference.\(^7\)

**Assumption 2.** The representative agent’s period utility is described by

\[
U(c_t, t) = e^{-\rho t} \ln c_t,
\]

where \( \rho > 0 \) is the rate of time preference.

Based on Assumptions 1 and 2, the equilibrium price for any financial asset can be solved through the Euler equation. The equilibrium stock price \( S_t \) can be easily computed from (2.3):

**Proposition 2.1.** Under Assumptions 1-2, the equilibrium price of the risky stock at time \( t \), is

\[
S_t = S(\delta_t) = \frac{\delta_t}{\rho}, \quad \forall \quad t \in (0, \infty).
\]

The stock price equals the present value of future dividends discounted at the rate of time preference. That is, the stock generates a constant dividend yield which is equal to the rate of time preference. The equilibrium stock price follows a similar process as the dividend:

\[
\frac{dS}{S} = (\mu_s - \delta) dt + \sqrt{V} dz = \left[ \beta_1 + \frac{\beta_2}{2} - (\alpha_1 - \frac{\alpha_2}{2}) \ln \delta \right] dt + \sqrt{\beta_2 + \alpha_2 \ln \delta} \, dz,
\]

where \( \mu_s \) is the expected stock return and \( V \) is the variance, explicitly given below. The expected stock return and its process are:

\[
\mu_s = \beta_1 + \beta_2/2 - (\alpha_1 - \alpha_2/2) \ln \delta + \rho.
\]

\(^7\)This assumption is also adopted by Merton (1971) and CIR (1985) for a similar reason.
\[ d\mu_s = \left( \alpha_1 \rho + \frac{\nu \alpha_2^2}{4} - \alpha_1 \mu_s \right) dt - \sqrt{\left( \alpha_1 - \frac{\alpha_2^2}{2} \right) \left( \alpha_2 \rho + \frac{\nu \alpha_2^2}{2} - \alpha_2 \mu_s \right) / 2} dz. \]

The speed of reversion \((\alpha_1)\) and the long-run mean of the stock return \((\rho + \nu \alpha_2^2/4 \alpha_1)\) determine the predictable features of the stock return. Clearly, the predictability is affected by all parameters \((\rho, \alpha_1, \beta_1, \alpha_2, \beta_2)\). This result provides a theoretical explanation for the predictability of the stock return. That is, such predictability is induced by the mean-reverting feature of the fundamental rate of dividend growth (See references at the beginning of this subsection).

The dividend process also endogenously generates stochastic volatility that is meaning-reverting and provides a rationale for a similar stochastic volatility process assumed by Stein and Stein (1991). The volatility and its process are:

\[ V = \beta_2 + \alpha_2 \ln \delta, \]

\[ dV = (\beta_1 \alpha_2 + \alpha_1 \beta_2 - \alpha_1 V) dt + \alpha_2 \sqrt{V} dz. \]

The instantaneous variance exhibits the mean-reverting feature, with the speed of adjustment being \(\alpha_1\) and the long-run mean being \((\alpha_1 \beta_2 + \alpha_2 \beta_1) / \alpha_1\). The mean-reverting feature of the volatility is induced by the heteroskedasticity in the dividend yield. Note that the restriction \(\alpha_2 < 0\) ensures a negative correlation between the price of the market portfolio and its volatility.

Since bond prices and the spot interest rate are useful for analyses on option prices, they are determined in the following proposition and corollary (see Appendix A for a proof):

**Proposition 2.2.** Under Assumptions 1-2, the equilibrium price of a pure discount bond with maturity \(T\) at time \(t \leq T\), \(B_t(T, \delta_t)\), is

\[ B_t(T, \delta_t) = A(t, T)v^{\left( \rho + A(t,T) e^{-\alpha_1 (T-t)-1} \frac{v}{\alpha_2} \right)}(T-t) . \]  

where \(A(t, T) = \frac{\alpha(t, T) \alpha_2}{\alpha(t, T) \alpha_2 + 1} \), \(a(t, T)\) and \(v\) are defined in (2.9).
Corollary 2.3. Denote the instantaneous interest rate at any time \( \tau \in (t, T) \) by \( r(\tau) \) and define it implicitly through \( B_t(T, \delta_t) = E_t(e^{-\int_t^T r(\tau) d\tau}) \). The spot instantaneous interest rate \( r(t) \) and the expected forward instantaneous rate \( E_t[r(T)] \) are

\[
    r(t) = \rho + \beta_1 - \alpha_1 \ln \delta_t - \frac{1}{2}(\beta_2 + \alpha_2 \ln \delta_t),
    
    E_t[r(T)] = \rho - \nu \alpha_2^2/4\alpha_1 - \frac{1}{2}(2\alpha_1 + \alpha_2)e^{-\alpha_1(T-t)}(\ln \delta_t - \frac{\beta_1}{\alpha_1}).
\]

The expected steady state interest rate is given as \( \bar{r} = \lim_{T \to \infty} E_t[r(T)] = \rho - \nu \alpha_2^2/4\alpha_1 \).

Since the spot interest rate is linear in \( \ln \delta \), I can rewrite the bond price in Proposition 2.2 in terms of the spot interest rate. That is,

\[
    B_t(T, \delta_t) = A(t, T)^v e^{-\rho(T-t)-(r-\rho-\alpha_2 v/2)A(t,T)(1-\exp(-\alpha_1(T-t)))/\alpha_1}.
\]

The restriction \( 0 \geq \alpha_2 \geq -2\alpha_1 \) ensures a negative relation between the bond price and the spot interest rate. Under such restriction, the bond price has appealing properties. For example, the bond price is a decreasing convex function of the interest rate and an increasing (decreasing) function of time (maturity). The bond price is negatively correlated with the aggregate dividend. That is, the common disturbance in aggregate dividends has opposite effects on the prices of the stock and the bond. The intuitive explanation is that a high aggregate dividend makes investments in the stock attractive and so the bond price must fall in order to induce investors to invest in bonds at all.

The dynamics of the bond price are described as

\[
    \frac{dB_t}{B_t} = \left[ r - (r - \rho - \alpha_2 v/2)A(t,T)(1-e^{-\alpha_1(T-t)})\frac{\alpha_2}{\alpha_1} \right] dt + \left[ 1 - A(t,T)e^{-\alpha_1(T-t)} \right] \sqrt{Y} dz.
\]

The volatility of the bond price is \( V_B(t, T) = \left[ 1 - A(t,T)e^{-\alpha_1(T-t)} \right]^2 Y \). As one should expect, the volatility of the bond price equals 0 when the bond is at the maturity. When the maturity goes to infinity, the volatility approaches \( Y \) which is the same as that of the stock price. The intuitive
reason is that when the bond has an infinite maturity, it is very similar to a stock. In this case, the volatility of bond mimics that of the stock price, which in turn reflects the volatility of the aggregate dividend.

The bond price is usually quoted in terms of the yield-to-maturity, \( R(r, t, T) \), which is defined through \( e^{-R(r,t,T)(T-t)} = B_t(T, \delta_i) \). We have

\[
R(r, t, T) = \rho + \frac{2(r - \rho - \alpha_2 \sigma/2)}{\alpha_2 (\alpha_2 a(t, T) + 1)(T - t)} - \frac{\nu \ln (A(t, T))}{T - t}.
\]

As the bond approaches the maturity, the yield-to-maturity approaches the spot interest rate. If the maturity goes to infinity, the yield approaches the rate of time preference. When the spot rate is below \( \rho \), the term structure is uniformly increasing. If the spot rate is above \( \rho - \nu \alpha_2^2 / 4 \alpha_1 \), the term structure decreases. For any value of the spot rate in between, the yield curve is hump shaped.

The spot interest rate here obeys a mean-reverting process similar to that of \( \ln \delta \) since it is linear in \( \ln \delta \). Under the restrictions \( 0 \geq \alpha_2 \geq -2 \alpha_1 \), the spot interest rate follows:

\[
dr = (\alpha_1 \rho - \frac{\nu \alpha_2^2}{4} - \alpha_1 r) dt + \sqrt{\alpha_2 \frac{\alpha_2}{2}(\alpha_2 \rho + \frac{\nu \alpha_2^2}{2} - \alpha_2 r)} dz.
\]

This mean-reverting process resembles the so-called affine class of the term-structure model.\(^8\) The mean-reverting speed for the spot rate is \( \alpha_1 \) and the long-run mean is \( \rho - \nu \alpha_2^2 / 4 \alpha_1 \). The Vasicek (1977) model corresponds to \( \alpha_2 = 0 \) and the CIR (1985) model to \( \rho = -\alpha_2 \nu / 2 \), respectively.

3. Relations among the Spot Interest Rate, the Predictability of Return of the Market Portfolio and its Volatility

Before getting into the details of pricing options, let us examine the relations among the spot interest rate, the stock return and its volatility. Recent studies in partial equilibrium settings

\(^8\) As stated in Duffie (1992), the process of the affine class of term-structure model is

\[
dr = (a_1 + a_2 r) dt + \sqrt{b_1 + b_2 r} dz.
\]
have attempted to find the relationship between the stock return predictability and the BS model. For example, Grundy (1991) states that the BS model can be consistent with a mean-reverting stock return. Lo and Wang (1995) cautioned that the predictability of stock returns could provide additional information on the forecast of the volatility in the BS model. With the current model, one can investigate whether the BS model can be consistent with the stock return predictability in equilibrium where the spot interest rate \((r)\), the drift to the stock price \((\mu_s)\) and its volatility \((V)\) are all endogenously determined.

As is typical in such an examination, I first present the equivalent martingale price process for the market portfolio. Using the formulas for \(\mu_s\), \(V\) and \(r\), we can rewrite \(\mu_s = r + \beta_2 + \alpha_2 \ln \delta = r + V = r + \lambda \sqrt{V}\). This implies that the market risk of \(dz\), defined commonly as \(\lambda = (\mu_s - r)/\sqrt{V}\), is \(\sqrt{V}\). The equivalent martingale process for the stock price is\(^9\)

\[
dS = rSdt + S\sqrt{V}dz^*,
\]

where \(dz^* = dz + \sqrt{V}dt\) is the equivalent martingale Wiener process. The equivalent martingale process for the volatility is

\[
dV = [\beta_1 \alpha_2 + \alpha_1 \beta_2 - (\alpha_1 + \alpha_2)V] dt + \alpha_2 \sqrt{V}dz^* ,
\]

which differs from the actual volatility process in the reversion speed and the long-run mean.

As the usual argument suggests, the actual drift of the stock price \((\mu_s)\) does not explicitly enter the option price formula here when the equivalent martingale pricing principle is used. However, it would be erroneous to infer that fundamental forces that affect the drift \(\mu_s\) do not affect the option prices. As is apparent from the formulas for \(r\) and \(V\), parameters \((\rho, \alpha_1, \beta_1, \alpha_2, \beta_2)\) that underlie the predictability affect both the volatility and the spot interest rate simultaneously. This result, which can be obtained only in an equilibrium model where \(\mu_s\), \(V\) and \(r\) are endogenous, is quite

\(^9\)The market risk of \(dz\) can be easily verified under the partial differential equation approach.
different from and much stronger than that in Lo and Wang (1995), who only argue that the stock return predictability can help the forecast on volatility. The effects of the parameters which affect predictability on the interest rate and volatility can be briefly discussed here. First, the rate of time preference, $\rho$, affects the spot interest rate positively, but has no effect on the market volatility in this context. Second, $\alpha_1$ not only determines the speed of adjustments for $r$ and $V$, but also affects the long-run means for $r$ and $V$. The parameter $\alpha_1$ affects the long-run mean of $r$ negatively, but affects that of $V$ positively. In addition, parameters ($\beta_1$, $\alpha_2$, $\beta_2$) influence the spot rate and the market volatility differently. For example, an increase in the long-run mean for $r$ could be resulted from increase in $\beta_1$ or decrease in either $\alpha_2$ or $\beta_2$. Also, an increase in the long-run mean for $V$ could be resulted from decrease in $\beta_1$ or increase in either $\alpha_2$ or $\beta_2$.

Table 1: Summary of Special Cases for Stock Return, Volatility and Spot Interest Rate

<table>
<thead>
<tr>
<th>Stochastic Volatility: SV</th>
<th>Stochastic Interest rate: SI</th>
<th>Constant Interest Rate: CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Volatility: SV</td>
<td>Generic Case (SVSI):</td>
<td>Special Case 1 (SVCI):</td>
</tr>
<tr>
<td></td>
<td>restrictions: $-2\alpha_1 &lt; \alpha_2 &lt; 0$</td>
<td>restriction: $\alpha_2 = -2\alpha_1$</td>
</tr>
<tr>
<td></td>
<td>$V = \beta_2 + \alpha_2 \ln \delta$</td>
<td>$V = \beta_2 + \alpha_2 \ln \delta$</td>
</tr>
<tr>
<td></td>
<td>$r = \rho + \beta_1 - \alpha_1 \ln \delta - (\beta_2 + \alpha_2 \ln \delta)/2$</td>
<td>$r = \rho + \beta_1 - \beta_2/2$</td>
</tr>
<tr>
<td></td>
<td>$\mu_s = \rho + \beta_1 - \alpha_1 \ln \delta + (\beta_2 + \alpha_2 \ln \delta)/2$</td>
<td>$\mu_s = \rho + \beta_1 + \beta_2/2 - 2\alpha_1 \ln \delta$</td>
</tr>
<tr>
<td>Constant Volatility: CV</td>
<td>Special Case 2 (CVSI):</td>
<td>Special Case 3 (CVCI):</td>
</tr>
<tr>
<td></td>
<td>restriction: $\alpha_2 = 0$</td>
<td>restrictions: $\alpha_2 = \alpha_1 = 0$</td>
</tr>
<tr>
<td></td>
<td>$V = \beta_2$</td>
<td>$V = \beta_2$</td>
</tr>
<tr>
<td></td>
<td>$r = \rho + \beta_1 - \alpha_1 \ln \delta - \beta_2/2$</td>
<td>$r = \rho + \beta_1 - \beta_2/2$</td>
</tr>
<tr>
<td></td>
<td>$\mu_s = \rho + \beta_1 - \alpha_1 \ln \delta + \beta_2/2$</td>
<td>$\mu_s = \rho + \beta_1 + \beta_2/2$</td>
</tr>
</tbody>
</table>

Table 1 presents a summary of the relations among $r$, $\mu_s$ and $V$ under the restriction $0 \geq \alpha_2 \geq -2\alpha_1$. In the generic case (SVSI), $\mu_s$, $V$ and $r$ are all stochastic and mean-reverting. There are three special cases. Case 3 (CVCI) corresponds to the BS model, where volatility $V$ and interest rate $r$ are constant. However, in this case the actual drift $\mu_s$ should also be constant, implying that
the BS model is not consistent with a mean-reverting drift in the current context. This result differs from Grundy (1991) who states that the BS formulas hold for an Ornstein-Uhlenbeck process for the stock return. To accommodate the predictability of the stock return, either the volatility or the spot interest rate or both must be made stochastic and mean-reverting, as in case 1 (SVCI), case 2 (CVSI) and (SVSI), respectively. Incorporating the predictability of the stock return, stochastic spot interest rate and stochastic volatility can also significantly improve the numerical performance of the option model, as we will show in Section 5 later.

The equilibrium approach also illustrates that the two endogenized equivalent martingale processes (dS and dV) are inherently interdependent. Such interdependence cannot be established in any partial equilibrium option model, where the two processes are exogenous. For example, Hull and White (1987) and Stein and Stein (1991) simply assume the two corresponding processes as

\[ dS = rSdt + \sqrt{V}Sdz_1^*, \quad dV = \theta(V)dt + \phi(V)dz_2^*, \]

where the drift in dV is not related to the stock price process (dS). Moreover, the correlation between dz_1 and dz_2 is usually assumed to be zero. In contrast, the endogenized processes for S and V indicate that the drifts of dS and dV are associated in a particular way. Also, the correlation between dz_1 and dz_2 depends on the sign of \( \alpha_2 \). They are perfectly negatively correlated if \( \alpha_2 < 0 \). Although the special relation between dz_1 and dz_2 here depends on the one-factor setting, the general message of our exercise should be valid when the model is extended into a multi-factor setting. These requirements suggest that cross-equation restrictions must be imposed on the coefficients when the processes for S and V are to be estimated.
4. Pricing European Options

4.1. Options Written on the Market Portfolio

Consider the European style stock option. The equilibrium price of such a stock option satisfies the Euler equation (2.5). We can explicitly compute (2.5), since the stock price is linear in the dividend which is a function of \( Y \) and since the density function of \( Y_T \) conditional on \( (Y_t, t) \) is known. For a European call written on the risky stock with a striking price \( K \) that matures at time \( T \), its price at time \( t \leq T \), denoted \( C_t(K, T) \), can be written as

\[
C_t(K, T) = \frac{1}{U_{c(t,t)}} E_t \left( U_c(c_T, T) \times \max(S_T - K, 0) \right) = e^{-\rho (T-t)} S_t E_t \left( \frac{\max(\delta_T - \rho K, 0)}{\delta_T} \right).
\]

Similarly, for a European put written on the risky stock with a striking price \( K \) that matures at time \( T \), its price at time \( t \leq T \), denoted \( P_t(K, T) \), can be expressed as

\[
P_t(K, T) = e^{-\rho (T-t)} S_t E_t \left( \frac{\max(\rho K - \delta_T, 0)}{\delta_T} \right).
\]

The following proposition summarizes the European stock option price formulas for the generic case (SVSI) with the restriction \(-2\alpha_1 < \alpha_2 < 0\) (see Appendix B for a proof).\(^{10}\)

**Proposition 4.1.** Under Assumptions 1-2, the equilibrium stock option prices are:

\[
C_t(K, T) = S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda_j (T-t)} \gamma(v+j, a(t,T)Y) Y(\rho K)}{\Gamma(v+j)},
\]

\[
-K B_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda_j Z(v+j, A(t,T) \lambda_j Y) Y(\rho K)} \gamma(v+j, \frac{a(t,T)}{A(t,T)}) Y(\rho K)}{\Gamma(v+j)},
\]

and

\[
P_t(K, T) = K B_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda_j Z(v+j, A(t,T) \lambda_j Y) Y(\rho K)} \gamma(v+j, \frac{a(t,T)}{A(t,T)}) Y(\rho K)}{\Gamma(v+j)}
\]

\[-S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda_j (T-t)} \gamma(v+j, a(t,T)Y(\rho K)) \Gamma(v+j)}{\Gamma(v+j)}.
\]

\(^{10}\)Note that the domain for \( \delta \) is \( \delta \in (0, e^{-\beta_2/\alpha_2}) \) under \( \alpha_2 < 0 \) since \( Y = \beta_2 + \alpha_2 \ln \delta > 0 \). However, the domain for \( \delta \) becomes \( \delta \in (e^{-\beta_2/\alpha_2}, \infty) \) under \( \alpha_2 > 0 \). The option price formulas under condition \( \alpha_2 > 0 \) are presented in the appendices.
where $\Gamma(a, x) \equiv \int_x^\infty e^{-y} y^{a-1} dy$, $\gamma(a, x) \equiv \int_0^x e^{-y} y^{a-1} dy \ (\forall x > 0)$, and $\gamma(a, x) + \Gamma(a, x) \equiv \Gamma(a)$.

The call and put prices satisfy the put-call parity condition for European options on assets with a constant dividend yield.

Since the stock option prices are functions of parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, I can explicitly examine how each parameter affects the option prices. To economize on space, I examine only the call option written on the risky stock. Similar analysis can be conducted on puts. Comparative statics show that a high $\rho$ results in a low call price. The intuitive explanation is that, since the stock dividend yield in equilibrium equals the rate of time preference, a high dividend yield ($\rho$) generates a low stock price which in turn induces a low call price. However, the signs of $\frac{\partial C_t}{\partial \alpha_1}$, $\frac{\partial C_t}{\partial \beta_1}$, $\frac{\partial C_t}{\partial \beta_2}$ and $\frac{\partial C_t}{\partial \alpha_2}$ are all ambiguous, which confirm the discussion on the effects of the predictability in the previous subsection. Changes in any of the four parameters have opposite effects on the long-run mean of $r$ and $V$. For example, a high $\beta_1$ implies a high long-run mean of $r$ and a low long-run mean of $V$. Since the spot rate and the volatility are pulled toward their long-run means, a higher $\beta_1$ is more likely to result in a higher $r$ and a lower $V$. A higher $r$ alone generates a higher call while a lower $V$ alone corresponds to a low call price, thus the overall effect on the call price is ambiguous. The effects of $\alpha_2$ or $\beta_2$ can be explained in a similar way. The effect of changing $\alpha_1$ is even more complicated since such changes not only affect the long-run means of $r$ and $V$ differently but also influence the reversion speeds of both.

Finally, the limit behavior of the call option when $S_t$ becomes very large is intuitive. When $S_t \to \infty$, a call option is almost certain to be exercised. The call option becomes very similar to a forward contract with a delivery price $K$. That is, $C_t(K, T) \to S_t e^{-\rho(T-t)} - KB_t(T)$, which is confirmed by the limit of (4.1).
4.2. Pricing European Bond Options

Consider the price of European style bond options. Denote by \( CB_t(k, T, \bar{T}) \) \((PB_t(k, T, \bar{T}))\) the value at time \( t \) of a call (put) option on a discount bond of a maturity date \( \bar{T} \), with a striking price \( k \) and an expiration date \( T \). It is understood that \( t \leq T \leq \bar{T} \). According to (2.5), we have

\[
CB_t(k, T, \bar{T}) = e^{-\rho(T-t)} \delta_t E_t \left( \delta_T^{-1} \max(B_T(\bar{T}, \delta_T) - k, 0) \right),
\]

\[
PB_t(k, T, \bar{T}) = e^{-\rho(T-t)} \delta_t E_t \left( \delta_T^{-1} \max(k - B_T(\bar{T}, \delta_T), 0) \right).
\]

Since the bond price \( B_T(\bar{T}, \delta_T) \) is a function of \( \delta_T \), we can compute the bond option prices in the same way as those of the stock options. The explicit formulas under the generic case (SVSI) are stated in the following proposition (see Appendix C for a proof):

**Proposition 4.2.** Under Assumptions 1-2, the equilibrium bond option prices are:

\[
CB_t(k, T, \bar{T}) = B_t(\bar{T}, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{D(t, \bar{T}) \lambda (D(t, \bar{T}) \lambda)}{j!}} \frac{\gamma(v + j, \frac{a(t, \bar{T})}{A(t, \bar{T})} Y(\bar{k}))}{\Gamma(v+j)}
\]

\[
+ k B_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{A(t, \bar{T}) \lambda (A(t, \bar{T}) \lambda)}{j!}} \frac{\gamma(v + j, \frac{a(t, \bar{T})}{A(t, \bar{T})} Y(\bar{k}))}{\Gamma(v+j)},
\]

and

\[
P_t(k, T, \bar{T}) = k B_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{A(t, \bar{T}) \lambda (A(t, \bar{T}) \lambda)}{j!}} \frac{\gamma(v + j, \frac{a(t, \bar{T})}{A(t, \bar{T})} Y(\bar{k}))}{\Gamma(v+j)}
\]

\[
-B_t(\bar{T}, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{D(t, \bar{T}) \lambda (D(t, \bar{T}) \lambda)}{j!}} \frac{y(v + j, \frac{a(t, \bar{T})}{D(t, \bar{T})} Y(\bar{k}))}{\Gamma(v+j)}
\]

where \( \bar{k} \) is so chosen that \( B_T(\bar{T}, \bar{k}) = k \) and \( D(t, T, \bar{T}) = \frac{A(t, \bar{T})}{A(T, \bar{T})} \). The call and put prices satisfy the put-call parity condition.

As one should expect, the call on bond is an increasing function of the bond price and a decreasing function of the striking price. Also, the call price increases with the maturity of the option. The remaining signs of comparative statics are ambiguous. These general features are similar to those of the call on bonds stated in CIR (1985), since the latter model is a special case of the current model.
5. Performance of the Model for Options on the Market Portfolio

In this section, I use numerical examples to examine the performances of the generic case (SVSI, Table 1) where both the spot interest rate and volatility are stochastic and mean-reverting, then investigate the implied volatility pattern in the current model.

5.1. Comparison with the Black-Scholes' Model

Table 2 compares the generic case SVSI with the BS model for European call options for three different maturities, assuming a stock price of $S_t = 100$. The rate of time preference, $\rho$, is set to be 4% to match the aggregate dividend yield on stock index. To compute the BS call prices, the risk-free rate is set to be the long run real interest rate $r_{BS} = 4\%$ and the volatility is set to match the observed average volatility $\sigma_{BS} = 20\%$. Choosing $\alpha_1$ to match the estimate by Marsh and Merton (1987) on the speed of adjustment of aggregate dividends gives $\alpha_1 = 0.25$. To determine $\beta_1$, we initialize the spot interest rate and instantaneous volatility to be $r_{BS}$ and $\sigma_{BS}$. Thus, $\beta_1 = 0.3666$. Further, $\alpha_2 = -0.1029$ and $\beta_2 = 0.1827$ are identified through the condition on the instantaneous volatility and restrictions to ensure positive $Y$.

Table 2, Figure 1 and Figure 2 here.

Table 2 shows the price differences between the model SVSI and the BS model for call options on the same stock with different striking prices. The generic case SVSI corrects the BS price biases, providing lower prices for the out-of-the-money calls and higher prices for the in-the-money calls for different maturities. Such biases of the BS model become more pronounced as the time to maturity increases or the degree to which the option is in- or out-of-the-money increases. These numerical results are consistent with the empirical study by MacBeth and Merville (1979) for stock options of large companies during 1976. They also confirm the result in Hull and White (1987) for the case
where the Wiener process for the stock price and the process for volatility are negatively correlated.

The current model corrects the BS bias because of the mean-reverting feature of $r$ and $V$ and the negative correlation between the stock price and its volatility. To be specific, consider a situation where the rate of dividend is very high, which indicates a low spot rate. The mean-reverting force will push the spot rate up, which induces a high call price. Also a high rate of dividend results in a high stock price which, in turn, implies a low volatility because of the negative correlation between $S$ and $V$. In this case, it is unlikely for the stock price to change by a large amount. The joint effect is that a high stock price will imply a higher call value in the current model than in a model with constant interest rates and constant volatility. Numerical exercises in Table 3 show that the effect of mean-reverting interest rates dominates the effect of negative correlation between the stock price and its volatility when the stock price is low.

Figure 1 shows that the price difference is positive and the largest for the deep-in-the-money call, which falls monotonically as the call goes less and less in the money. This pattern resembles the recent empirical results of Dumas, Fleming and Whaley (1995) on the S&P 500 index options. When the call is slightly out-the-money, the price difference is almost 0. When the call moves more to out-of-the-money, the price difference is negative and the absolute difference increases. The influence of the time to maturity on absolute price bias is insignificant. The percentage price correction is illustrated in Figure 2. Note that the largest percentage price correction is for the deep-out-of-the-money option. The time to maturity also affects the percentage price bias. The longer the maturity, the larger the percentage price bias for the same striking price.

Table 3, Figure 3 and Figure 4 here.

Table 3 compares the BS model (case 3 CVCI), case 1 (SVCI), and case 2 (CVSI). A SVCI model generates reasonable price corrections for the in-the-money call, but fails to correct the BS
price bias for the out-of-the-money call. On the other hand, a CVSI model generates a uniformly lower call prices than the BS model. This is because the stock price and bond price are negatively correlated. In this case, according to Merton (1973), stochastic interest rates reduce the call price in relation to the BS value with constant interest rates. Thus allowing for a stochastic interest rate alone can only produce reasonable prices for the out-of-the-money call.

Comparing the performance of the generic case in Table 2 and those of the other three cases in Table 3, one concludes that both stochastic volatility and stochastic interest rates are necessary to generate reasonable corrections for the bias in the BS model for in- and out-of-the-money calls.

5.2. The Pattern of Implied Volatility

Table 4 presents the implied volatility of the current model, which is calculated through the BS formula using the option prices of the generic case SVSI given in Table 2. Consistent with empirical results, the pattern of the implied volatility with respect to the striking price is a decreasing function of $K/S$. For the deep-in-the-money calls, the implied volatility is very large and decreases as the call moves to the deep-out-of-the-money, as illustrated in Figure 5. This phenomenon is consistent with the empirical findings of MacBeth and Merville (1979) and Wiggins (1987). Also, a recent empirical study by Dumas, Fleming and Whaley (1995) on S&P 500 index European option finds the same regularity. In addition, for a given maturity, the average of the implied volatility with different striking prices is higher than the instantaneous volatility.

Table 4 and Figure 5 here.

In summary, call prices given by the generic case SVSI match market prices better than the Black-Scholes formula. Important for correcting the BS price biases are the mean-reverting feature of the spot interest rates, the negative correlation between the stock price and its volatility and the negative correlation between the market volatility and the spot interest rate. These features are
also necessary for delivering a realistic pattern of implied volatility. Sensitivity analyses show that these results are robust with respect to changes in parameter values.\textsuperscript{11}

6. Conclusion

This paper has used an extension of the equilibrium model of Lucas (1978) to study the valuation of options on the market portfolio with stochastic volatility and predictability of stock return. I have investigated the equilibrium relationship between the price of the market portfolio and its volatility, as well as the relationship between the spot interest rate and the market volatility, in an endowment economy. The only uncertainty in this economy is the aggregate dividend whose growth rate follows an affine class of mean-reverting process. The equilibrium results indicate that the predictability of the stock return can be induced by the mean-reverting feature of the growth rate of aggregate dividends. In contrast to previous analyses that employ partial equilibrium settings, here we show that the BS model is not consistent with the predictability of return on the market portfolio when the interest rate and volatility are endogenously generated from the underlying dividend process. Although the actual drift of the stock price does not explicitly enter the option price formula when the equivalent martingale pricing principle is used, fundamental forces that affect the drift do affect option prices through their effects on the endogenous interest rate and volatility. It is also shown that there are strong interdependence between the price process and its volatility process for the market portfolio.

Using the Euler equation, I have derived the pricing formulas for the options on the market portfolio which incorporate both stochastic volatility and stochastic interest rate. Since there is only one source of uncertainty, this model preserves the completeness feature for the hedging and risk management purpose. Numerical examples show that the current model performs better

\textsuperscript{11}For lack of space, sensitivity analyses are not presented here but available upon request.
than the BS model with realistic parameter values. They suggest that the option valuation should incorporate both stochastic volatility and stochastic interest rates in order to correct the BS pricing bias. Moreover, stochastic volatility and stochastic interest rate are consistent with predictability of stock return.

In addition to providing pricing equations for options on the market portfolio, I have also derived closed-form formulas for European style bond options in a manner that is consistent with the prices of options written on the market portfolio. The Vasicek (1977) model and the CIR (1985) model can be viewed as special cases. These formulas have potential use for future examinations of the term structure and bond option pricing.
Appendices

A. Proofs of Proposition 2.2 and Corollary 2.3:

A.1. The Price of Pure Discount Bonds

Proof. By equation (2.4) and the specifications in section 2.2, we can express the pure discount price with maturity $T$ at time $t \leq T$ as

$$B_t(T, \delta_t) = e^{\rho t} \delta_t E_t \left( e^{-\rho T \delta_T^{-1} \times 1} \right) = e^{-\rho(T-t) \delta_t} E_t \left( \delta_T^{-1} \right), \quad \forall \ t \in (0, T).$$

Since $Y = \beta_2 + \alpha_2 \ln \delta$, we use the conditional density function for $Y_T$ in (2.8) and parameters defined in (2.9), thus

$$B_t(T, \delta_t) = e^{-\rho(T-t) \delta_t} e^{-\gamma_T \delta_T^{-1} \lambda / \alpha_2} \int_0^{\infty} e^{-\gamma_T \delta_T^{-1} \lambda / \alpha_2} \alpha(t, T) e^{-(x+y)T e^{-\rho T \delta_T^{-1} \times 1}} \sum_{j=0}^{\infty} \frac{(x \lambda)^j}{j! \Gamma(v + j)} dY_T$$

$$= e^{-\rho(T-t) \delta_t + \gamma_T \lambda \alpha_2 / \alpha_2} \sum_{j=0}^{\infty} \frac{e^{-\lambda \lambda j}}{j! \Gamma(v + j)} \Gamma(v + j) \left( \frac{a(t, T) \alpha_2}{a(t, T) \alpha_2 + 1} \right)^{v+j}$$

$$= A(t, T)^t e^{-\left( \rho + A(t, T) e^{-\gamma_T \delta_T^{-1} \lambda / \alpha_2} \right)^{T-t}},$$

where $A(t, T) = \frac{a(t, T) \alpha_2}{a(t, T) \alpha_2 + 1}$. 

A.2. The Instantaneous Interest Rate

Proof. The instantaneous interest rate is defined through $B_t(T, \delta_t) = E_t \left( e^{-\int_t^T r(s) ds} \right)$. Thus

$$r(t) = -\frac{dB_t(T, \delta_t)}{dT} \bigg|_{T=t}.$$ Since

$$\frac{dB_t(T, \delta_t)}{dT} = -B_t(T, \delta_t) \left( \rho + \frac{\gamma_T \delta_T^{-1} \alpha_2}{2} A(t, T) e^{-\gamma_T \delta_T^{-1} \lambda / \alpha_2} \right) + \left( \frac{1}{\alpha_1 \alpha_2} \right) Y_t A(t, T) e^{-(T-t) \lambda / \alpha_2},$$

therefore, the spot instantaneous interest rate is

$$r(t) = \rho + \beta_1 - \alpha_1 \ln \delta_t - \frac{1}{2} (\beta_2 + \alpha_2 \ln \delta_t).$$

It is easy to show that

$$E_t[r(T)] = \rho - \frac{\gamma_T \delta_T^{-1} \alpha_2}{4 \alpha_1} - \frac{1}{2} (2 \alpha_1 + \alpha_2) e^{-(T-t) \lambda / \alpha_2} (\ln \delta_t - \frac{\beta_1}{\alpha_1}).$$
Then, the expected steady state interest rate is $\bar{r} = \lim_{T \to \infty} E_t[r(T)] = \rho - \nu \alpha_2^2 / 4 \alpha_1$.

**B. Proof of Proposition 4.1:**

**Proof.** The domain for $\delta$ is $\delta \in (0, e^{-\beta_2/\alpha_2})$ for $\alpha_2 < 0$ since $Y = \beta_2 + \alpha_2 \ln \delta > 0$. As stated in section 4.1, the European call option with a striking price $K$ and maturity $T$ at time $t \leq T$ is computed as

$$C_t(K, T) = e^{-\rho(T-t)} S_t E_t \left( \delta_t^{-1} \times \max(\delta_T - \rho K, 0) \right)$$

$$= e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \geq \rho K) - e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} g(\delta_T | \delta_t) d\delta_T.$$

The call price of the stock is proven as follows:

$$e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \geq \rho K) = e^{-\rho(T-t)} S_t \text{Prob}[Y_T \leq Y(\rho K)]$$

$$= e^{-\rho(T-t)} S_t \int_0^{Y(\rho K)} a(t, T) e^{-(x+\lambda)} x^{-1} \sum_{j=0}^{\infty} \frac{(x \lambda)^j}{j! (v+j)} \gamma(v+j, a(t, T) Y(\rho K)) \frac{1}{\Gamma(v+j)} \, dY_T$$

and

$$e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} g(\delta_T | \delta_t) d\delta_T$$

$$= K e^{-\rho(T-t)+\frac{-\beta_2+Y_T}{\alpha_2}} \int_0^{Y(\rho K)} e^{\frac{-\beta_2+Y_T}{\alpha_2}} a(t, T) e^{-(x+\lambda)} x^{-1} \sum_{j=0}^{\infty} \frac{(x \lambda)^j}{j! (v+j)} \gamma(v+j, a(t, T) Y(\rho K)) \, dY_T$$

$$= K e^{-\rho(T-t)+\frac{Y_T}{\alpha_2}} \sum_{j=0}^{\infty} \frac{e^{-\lambda \gamma(v+j, a(t, T) Y(\rho K))}}{j! (v+j)} \gamma(v+j, a(t, T) Y(\rho K)) \, dY_T$$

The European put option with a striking price $K$ and maturity $T$ at time $t \leq T$ is computed as

$$P_t(K, T) = e^{-\rho(T-t)} K \rho S_t \int_0^{\rho K} \delta_T^{-1} g(\delta_T | \delta_t) d\delta_T - e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \leq \rho K), \quad \forall \ t \in (0, T).$$
The put price can be shown in a similar way as for the call:

\[ e^{-\rho(T-t)} K \rho K \int_0^{\rho K} \delta_T^{-1} g(\delta_T | \delta_t) d\delta_T \]

\[ = Ke^{-\rho(T-t)+\frac{\alpha_2+\lambda_1}{\alpha_2}} \int_0^\infty a(t,T) e^{(\lambda+1) (x+\lambda-1) \sum_{j=0}^{\infty} (x\lambda^j) \Gamma[v+j, \alpha(t,T) Y(\rho K)] A(t,T)^{v+j} \frac{\alpha(t,T) Y(\rho K)}{\Gamma(v+j)} \frac{1}{j!} \frac{\delta_T}{\Gamma(v+j)} dY_T \]

\[ = Ke^{-\rho(T-t)+\frac{Y_1}{\alpha_2}} \sum_{j=0}^\infty \frac{(x\lambda)^j}{j!} \Gamma[v+j, \alpha(t,T) Y(\rho K)] A(t,T)^{v+j} \frac{\alpha(t,T) Y(\rho K)}{\Gamma(v+j)} \frac{1}{j!} \frac{\delta_T}{\Gamma(v+j)} \frac{1}{\Gamma(v+j)} dY_T \]

and

\[ e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \leq \rho K) = e^{-\rho(T-t)} S_t \text{Prob}(Y_T \geq Y(\rho K)) \]

\[ = e^{-\rho(T-t)} S_t \int_0^\infty \alpha(t,T) e^{(\lambda+1) (x+\lambda-1) \sum_{j=0}^{\infty} x\lambda^j} \frac{\alpha(t,T) Y(\rho K)}{\Gamma(v+j)} \frac{1}{j!} \frac{\delta_T}{\Gamma(v+j)} \frac{1}{\Gamma(v+j)} dY_T \]

\[ = e^{-\rho(T-t)} S_t \sum_{j=0}^\infty \frac{(x\lambda)^j}{j!} \frac{\Gamma(v+j, \alpha(t,T) Y(\rho K))}{\Gamma(v+j)} \frac{1}{\Gamma(v+j)} \frac{\delta_T}{\Gamma(v+j)} \frac{1}{\Gamma(v+j)} dY_T \]

C. Proof of Proposition 4.2:

Proof. Since

\[ B_T(\overline{T}, \delta_T) = A(T,\overline{T}) e^{-\rho(\overline{T}-t)-\frac{Y_1}{\alpha_2}} (A(T,\overline{T}) e^{-(\overline{T}-t)} - 1) \]

we first compute the present value of \( B_T(\overline{T}, \delta_T) \). It is easy to show that

\[ E_t[PV(B_T(\overline{T}, \delta_T))] = e^{-\rho(T-t)} \delta_t E_t \left( \delta_T^{-1} B_T(\overline{T}, \delta_T) \right) \]

\[ = A(t,\overline{T}) e^{-\rho(\overline{T}-t)-\frac{Y_1}{\alpha_2}} (A(t,\overline{T}) e^{-(\overline{T}-t)} - 1) = B_t(\overline{T}, \delta_t). \]

As stated in section 4.1, the value of a European call option on bond, \( CB_t(k, T, \overline{T}) \), is

\[ CB_t(k, T, \overline{T}) = e^{-\rho(T-t)} \delta_t E_t \left( \delta_T^{-1} \times \text{max}(B_T(\overline{T}, \delta_T) - k, 0) \right) \quad \forall \ t \leq T \leq \overline{T}. \]

For \( \alpha_2 < 0 \),

\[ CB_t(k, T, \overline{T}) = e^{-\rho(T-t)} \delta_t \int_0^k e^{-\frac{\alpha_2}{\alpha_2} \delta_T^{-1} B_T(\overline{T}, \delta_T) - ke^{-\rho(T-t)} \delta_t \text{Prob}(\delta_T \geq \overline{k})}, \]

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where \( \bar{k} \) is chosen so that \( B_T(\bar{T}, \bar{k}) = k \). We have

\[
e^{-\rho(T-t)} \delta_t \int_{\bar{k}}^e \frac{-\delta_T + \alpha_2}{\alpha_2} \delta_T^{-1} B_T(\bar{T}, \delta_T) g(\delta_T | \delta_t) d\delta_T
\]

\[
e^{-\rho(T-t)} \frac{e^{-\kappa T} - \frac{-\delta_T + \alpha_2}{\alpha_2}}{\frac{-\alpha_2}{\alpha_2}} A(T, \bar{T}) e^{-\alpha(T-T')} \int_0^b Y(\bar{T}) A(T, \bar{T}) v e^{-\kappa T} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} f(Y_T | Y_t) dY_T
\]

\[
e^{-\rho(T-t)} \frac{e^{-\kappa T} \gamma(T, T')}{\frac{-\alpha_2}{\alpha_2}} A(T, \bar{T}) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} D(t, T, \bar{T}) Y(v+j)
\]

\[
e^{-\rho(T-t)} \frac{e^{-\kappa T} \gamma(T, T')}{\frac{-\alpha_2}{\alpha_2}} A(T, \bar{T}) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} D(t, T, \bar{T}) Y(v+j)
\]

where \( D(t, T, \bar{T}) = \frac{A(t, T)}{A(T, \bar{T})} \). It has been shown in Appendix B

\[
ke^{-\rho(T-t)} \delta_t \text{Prob}(\delta_T \geq \bar{k}) = kB_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]

Therefore

\[
CB_t(k, T, \bar{T}) = kB_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]

\[
-kB_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]

Similarly, we can show that, for \( \alpha_2 < 0 \), the value of a European put option on bond is

\[
P_t(k, T, \bar{T}) = kB_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]

\[
-B_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, T') \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]

D. The Option Valuation with \( \alpha_2 > 0 \)

D.1. Call and Put Options on Stock

The domain for \( \delta \) under \( \alpha_2 > 0 \) is \( \delta \in (e^{-\frac{1}{2} \alpha_2}, \infty) \). We can compute the call and put on the risky stock in the similar way as we do in Appendix B.

\[
C_t(K, T) = e^{-\rho(T-t)} S_t E_t \left( \delta_T^{-1} \times \max(\delta_T - \rho K, 0) \right)
\]

\[
= e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \geq \rho K) - e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{\infty} \delta_T^{-1} g(\delta_T | \delta_t) d\delta_T
\]

We have

\[
e^{-\rho(T-t)} S_t \text{Prob}(\delta_T \geq \rho K) = e^{-\rho(T-t)} S_t \text{Prob}(Y_T \geq Y(\rho K))
\]

\[
e^{-\rho(T-t)} S_t \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2} \lambda_T}}{j!} \gamma(T, \rho K) \frac{-\alpha_2}{\alpha_2} A(T, \bar{T}) e^{-\alpha(T-T')} Y(v+j)
\]
and
\[ e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{\infty} \delta_T^{-1} g(\delta_T \mid \delta_t) d\delta_T \]
\[ = Ke^{-\rho(T-t) - \frac{\delta_T + Y_t}{a_2}} \int_{\rho K}^{\infty} e^{-\frac{\delta_T - Y_T}{a_2}} a(t, T) e^{-(x+\lambda)} x x^{x-1} \sum_{j=0}^{\infty} \frac{(x \lambda)^j}{j!} dY_T \]
\[ = KB_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{\lambda(t, T) \lambda}{j!}} \gamma(\delta_T, a(t, T)) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K))}{\Gamma(v+j)} . \]

Therefore
\[ C_t(K, T) = S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda(t, T) \lambda}}{j!} \gamma(\delta_T, a(t, T)) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K))}{\Gamma(v+j)} . \]

Similarly, the European put option with a striking price $K$ and maturity $T$ at time $t \leq T$ is
\[ P_t(K, T) = KB_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{\lambda(t, T) \lambda}{j!}} \gamma(\delta_T, a(t, T)) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K))}{\Gamma(v+j)} . \]

### D.2. Call and Put Options on Bond

For $\alpha_2 > 0$, the value of a European call option on bond, $CB_t(k, T, T)$, is
\[ CB_t(k, T, T) = e^{-\rho(T-t)} \delta_t \int_k^{\infty} \delta_T^{-1} B_T(T, \delta_T) - ke^{-\rho(T-t)} \delta_t \text{Prob}(\delta_T \geq \bar{k}). \]

We have
\[ e^{-\rho(T-t)} \delta_t \int_k^{\infty} \delta_T^{-1} B_T(T, \delta_T) g(\delta_T \mid \delta_t) d\delta_T \]
\[ = e^{-\rho(T-t) - \frac{\delta_T + Y_t}{a_2}} \int_{\rho K}^{\infty} A(T, \bar{T}) e^{\frac{\delta_T}{a_2}} - (\rho(T-T) - \frac{Y_T}{a_2} A(T, \bar{T}) e^{-\rho(T-T)} ) f(Y_T \mid Y_t) dY_T \]
\[ = B_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{D(t, T, \bar{T})}{j!}} (D(t, T, \bar{T}) \lambda) \gamma(\delta_T, a(t, T) \bar{T}) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K)))}{\Gamma(v+j)} . \]

and
\[ ke^{-\rho(T-t)} \delta_t \text{Prob}(\delta_T \geq \bar{k}) = kB_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{A(t, T) \lambda}{j!}} (A(t, T) \lambda) \gamma(\delta_T, a(t, T) \bar{T}) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K)))}{\Gamma(v+j)} . \]

Therefore
\[ CB_t(k, T, T) = B_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{D(t, T, \bar{T})}{j!}} (D(t, T, \bar{T}) \lambda) \gamma(\delta_T, a(t, T) \bar{T}) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K)))}{\Gamma(v+j)} . \]

\[ -kB_t(T, \delta_t) \sum_{j=0}^{\infty} e^{-\frac{A(t, T) \lambda}{j!}} (A(t, T) \lambda) \gamma(\delta_T, a(t, T) \bar{T}) Y(\rho K)) \frac{\Gamma(v+j, \frac{a(t, T)}{A(t, T)} Y(\rho K)))}{\Gamma(v+j)} . \]
Similarly, we can show that, for $\alpha_2 > 0$, the value of a European put option on bond is

$$P_t(K, T, \overline{T}) = k B_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda(A(t,T)\lambda)j+1}}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T)}) Y(\bar{k})}{\Gamma(v+j)}$$

$$- B_t(\overline{T}, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-D(t,T)\lambda(D(t,T)\lambda)j+1}}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T)}) Y(\bar{k})}{\Gamma(v+j)}.$$
Comparison between the General Case SVS\(I\) and the Black-Scholes Model

Option Parameters: \(S = 100, \alpha_1 = 0.25, \beta_1 = 0.3666, \alpha_2 = -0.1029, \beta_2 = 0.1827, \rho = 0.04\).

<table>
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<th>B-S</th>
<th>% Correction</th>
<th>SVSI</th>
<th>B-S</th>
<th>% Correction</th>
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Figure 1: Price Correction, \(T = 12\) months

Figure 2: Percentage Price Correction, \(T = 12\) months
Table 3

Comparison between the Special Cases and the Black-Scholes Model

Option Parameters: $S = 100$, $\alpha_1 = 0.25$, $\beta_1 = 0.3666$, $\rho = 0.04$.

For SVCII: $\alpha_2 = -2\alpha_1$, $\beta_2 = 2\beta_1$.

For CSVII: $\alpha_2 = 0$, $\beta_2 = 0.04$.

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Figure 3: Price Correction for SVCII, $T = 12$ months

Figure 4: Price Correction for CSVII, $T = 12$ months
Table 4

Implied Volatilities Calculated by Black-Scholes Formula from the Call Prices
Given by the General Case SVSI in Table 2

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Average Implied Volatility

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<tbody>
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Figure 5: Implied Volatility for T = 12 month
CHAPTER 3

EQUILIBRIUM VALUATION OF CURRENCY OPTIONS

IN A SMALL OPEN ECONOMY
1. Introduction

The objective of this essay is to eliminate several unsatisfactory consequences of directly applying Merton's mixed jump-diffusion stock option model to currency options, as discussed in the literature review. This essay employs a continuous-time extension of the Lucas (1978) asset pricing model to a small open monetary economy, where money has a non-trivial role in the agents' utility function. The home government controls the money supply which follows an exogenously given stochastic process. There is one domestic risky stock which represents the ownership of the productive technology for the single good traded worldwide. The dividends on this stock can be understood as aggregate dividends, which follow an exogenously given stochastic process. In the financial market, there is one real foreign pure discount bond whose real return is taken as given by the domestic agent. Given the processes for the money supply, the dividends on the aggregate stock and the foreign bond price, prices of all other assets including currency options are derived by solving the domestic agent's maximization problem. In particular, the exchange rate, the domestic nominal interest rate and the prices of other assets are endogenized in equilibrium. The option pricing formula in the current model encompasses Merton's (1976) formula as a special case where there is no jump risk in aggregate consumption. In this special case, the only jump uncertainty underlying the exchange rate arises from the money supply and is not priced. Also, the stock option valuation in the current model extends the Naik and Lee (1990) option model on the market portfolio with jump risks and logarithmic utility into an international context.

The equilibrium analysis provides solutions for the above mentioned problems associated with directly applying Merton's stock option model to currency options. First, the exchange rate,
expressed as the relative price of foreign currency in terms of home currency, is a function of both the domestic money supply and aggregate consumption. Thus, in contrast to Merton’s assumption, the dynamics of exchange rates and aggregate consumption are strongly correlated and the correlation arises endogenously. Second, there are important differences between the exchange rate and the stock price: The real price of the domestic stock is only affected by aggregate consumption where the money supply plays no role. Currency option prices are affected by the dynamics of both the money supply and aggregate consumption, but only the parameters underlying aggregate consumption affect stock option values. This suggests that exchange rate movements have more discontinuities than stock price movements, a result consistent with the evidence in Jorion (1988). Moreover, the parameters underlying aggregate consumption affect currency options and stock options differently.

The analysis also provides consistent restrictions to eliminate the analog of Siegel’s paradox in currency options. The restrictions are found by answering the following hypothetical question: What adjustments must be made to the risk-neutral process of the exchange rate in order to produce prices consistent with equilibrium? Besides the directional adjustments suggested by Bardhan (1995), the current analysis suggests that an additional adjustment be made on the jump size to reflect the fact that the jump component in the exchange rate is related to aggregate consumption.

In general, all these adjustments depend on the domestic agent’s risk preference.

The remainder of this paper is organized as follows. Section 2 describes the economy and presents general equilibrium results. Section 3 examines the endogenized exchange rate and derives equilibrium pricing formulas for European currency options from the view of the domestic risk-averse agent. Section 4 identifies the adjustments on the risk-neutral process of the exchange rate that help to solve the analog of Siegel’s paradox in currency options. Section 5 extends the model
to allow for a correlation between the money supply and aggregate dividends. Section 6 concludes the paper and the appendices provide necessary proofs.

2. A Small Open Monetary Economy

Consider a small open economy with perfect capital mobility between itself (termed the domestic country) and the rest of the world (termed the foreign country). This economy consists of a single risk-averse representative agent whose lifetime horizon is infinite. I adopt the standard formulation of a small open economy, which employs the following characteristics. First, the agent in the small economy has perfect access to the international goods and assets markets. Since the small economy has little influence on the foreign country, it takes the price of any foreign asset as given. Second, the domestic currency and domestic assets held by the foreign country are assumed to be negligible, implying that the supplies of these assets or currency are cleared by domestic demands. Third, domestic aggregate consumption is financed through both domestic aggregate output (dividend) and the return to holding foreign assets (which is paid in consumption goods). When the sum of aggregate dividend and the return to foreign assets exceeds aggregate consumption, the goods market is cleared by an increased holding of foreign assets (i.e., a current account surplus); when the sum of aggregate dividend and the return to foreign assets falls short of aggregate consumption, the residual is financed by a reduction in the holding of foreign assets (i.e., a current account deficit).

This feature distinguishes a small open economy from a closed economy.

I will first describe the primitives of the economy and then solve the agent's maximization problem. Equilibrium asset prices, including the domestic nominal interest rate and the exchange rate, are determined by requiring goods, money and financial markets to clear, as in Lucas (1982).

---

1 For a reference to a deterministic model of a small open economy, see Obstfeld (1982). An example in the stochastic environment is Grinols and Turnovsky (1994).
2.1. Structure of the Economy

There is a single good traded worldwide with no barriers, which can be used for consumption and investment. The nominal price of the good at home at time $t$ is denoted $p_t$. Let $P^*$ be the foreign price level measured in the foreign currency. According to the law of one price in the good market, $p$ equals the spot exchange rate times $P^*$. Since the home country is small, it takes $P^*$ as given and so we can simplify the discussions by normalizing $P^* = 1$. Then, $p_t$ equals the spot exchange rate expressed as the relative price of the foreign currency in terms of the home currency.

The home government controls the domestic money supply, which is taken as given by each domestic agent. The real money balance held by the domestic agent at time $t$ is defined as $m_t = M_t / p_t$, where $M_t$ is the domestic money demanded by home agents. To assign a non-trivial role to money, I follow Sidrauski (1967) to assume that real money balances yield utility to agents in addition to their purchasing power. In particular, the agent's period utility function, $U(c_t, m_t, t)$, depends positively on the agent's real money balance, $m_t$, as well as consumption, $c_t$. The rationale is that a larger real money balance reduces the transaction time in the goods market and hence allows the agent to enjoy more leisure. As long as leisure yields positive marginal utility to the agents, real money balances yield utility.

The government's purchase of goods and services is assumed to be constant and so the change in the money supply is injected into the economy as lump-sum monetary transfers. As in Lucas (1982), I assume that the agent is endowed with one unit of a claim on these monetary transfers. Denote the real price of this equity claim at time $t$ as $L_t$. The money transfer measured in real terms, $l$, can be understood as the "dividend" for this claim. Therefore, $L$ is the present value of

\[^2\text{Allowing } P^* \text{ to follow a stochastic process complicates the analysis without changing the qualitative results, provided that the process for } P^* \text{ is independent of the processes for domestic dividends and domestic money supply.}\]
future real monetary transfers. Note that monetary transfers are lump-sum and hence are taken as given by individual agents. The dynamics of the domestic money supply are described in the following assumption.

**Assumption 1.** The domestic money supply, $M^s$, is assumed to evolve according to the following mixed diffusion-jump process:

$$\frac{dM^s}{M^s} = (\mu_m - \lambda_m k_m)dt + \sigma_m dz_1 + (Y_m - 1)dQ_m, \quad \forall \ t \in (0, \infty). \quad (2.1)$$

Here, $\mu_m$ is the instantaneous expected growth rate of the money supply; $\sigma_m^2$ is the instantaneous variance of the growth rate, conditional on no arrivals of new important shock and $dz_1$ is a one-dimensional Gauss-Wiener process. The element $dQ_m$ is a jump process with a jump intensity parameter $\lambda_m$ and $Y_m - 1$ is the random variable percentage change in the money supply if the Poisson event occurs. The logarithm of $Y_m$ is normally distributed with mean $\theta_m$ and variance $\phi_m^2$. The expected jump amplitude, $k_m = E(Y_m - 1)$, is equal to $\exp(\theta_m + \phi_m^2/2) - 1$. Also, $\bar{k}_m = E(\frac{1}{Y_m} - 1)$ is equal to $\exp(-\theta_m + \phi_m^2/2) - 1$. The random variables $\{z_{1t}, \ t \geq 0\}$, $\{Q_{mt}, \ t \geq 0\}$ and $\{Y_{mj}, \ j \geq 1\}$ are assumed to be mutually independent. Also, $Y_{mj}$ is independent of $Y_{mj'}$ for $j \neq j'$. The parameters $(\mu_m, \sigma_m, \lambda_m, \theta_m, \phi_m)$ are constant.

The above money supply process incorporates both frequent fluctuations in the money supply, which correspond to the diffusion part $dz_1$, and infrequent large shocks to the money supply, which correspond to the jump part $dQ_m$. Both capture changes in government monetary policies.

There is only one domestic risky stock, which represents the ownership of the home productive technology for the single good. The total supply of this risky stock is normalized to one. Denote its real price at time $t$ as $S_t$ and the dividend as $\delta_t$. The dividend stream $\{\delta_t\}$ can be understood
as aggregate dividends in this small economy, which are exogenously given as:

\[ \frac{d\delta}{\delta} = \mu(\delta)dt + \sigma(\delta)dz_2 + (Y_\delta - 1)dQ_\delta, \]  

(2.2)

where \(dz_2\) is a one-dimensional Gauss-Wiener process and \(dQ_\delta\) is an independent jump process, described more precisely later.

The specification of aggregate dividend process corresponds to an economy which is infrequently subject to real shocks of unpredictable magnitude. The shocks on dividends could result from output shocks or shocks due to technological innovations. The mixed jump-diffusion formulation of the aggregate dividend process is also adopted by Naik and Lee (1990) and Ma (1994). For most of the discussion, the dividend process and the money supply process are assumed to be independent, measured with respect to a given probability space \((\Omega, \mathcal{F}, \mathcal{P})\). Section 5 will extend the discussion to allow for a correlation between the two processes.

There is a single foreign pure discount bond available for trading to the home agent at any time. That is, the net trading in assets between this small economy and the foreign country is positive and time-varying. The agent diversifies his portfolio internationally by holding the foreign bonds and the domestic financial assets. Since the country is small, the real price of the foreign bond at time \(t\), \(F_t\), is taken as exogenous by the home agent. The dynamics of \(F_t\) are assumed below.

**Assumption 2.** \(F_t\) evolves as \(dF = rFdt\), where \(r\) is a positive constant.

The processes for the money supply, the foreign bond price and the aggregate dividend are the primitives of the economy. Together with the specification of the utility function described below, they induce equilibrium prices for other assets. Among these other assets, there is a single domestic

---

\(^3\)Although dividends are not continuously distributed in reality, one may be able to find reasonable proxies for aggregate dividends used here. Aggregate output and dividends on stock indices are the examples.
nominal pure discount bond in zero net supply, with nominal rate of return $i$. Denote $B_t$ as the
nominal price of the discount bond at time $t$. Then, $dB = iB dt$, where $i$ is endogenously determined
in equilibrium. The real price of the domestic bond at time $t$, $b_t$, is given as $b_t = B_t/p_t$. In addition,
there are many other contingent claims on the risky domestic stock and the spot exchange rate
available for trading at any time in the economy. These contingent claims are all in zero net supply.
Denote the real prices of the contingent claims at time $t$ by a vector $x_t$ and the corresponding
vector of real dividends by $\delta_t^x$.

2.2. The Agent’s Optimization Problem

The representative agent’s information structure is given by the filtration $\mathcal{F}_t \equiv \sigma(M^*_t, \delta_t; 0 \leq \tau \leq t)$. 
As described earlier, the period utility at time $t$ is $U(c_t, m_t, t)$, where $U(\cdot, \cdot, t) : \mathcal{R}^2_+ \to \mathcal{R}$ is increasing
and strictly concave and satisfies the following properties:

$$\lim_{x_j \to -\infty} U_j(x_1, x_2) = 0 \quad \text{and} \quad \lim_{x_j \to 0} U_j(x_1, x_2) = \infty, \quad j = 1, 2.$$ 

The agent’s intertemporal utility is described by

$$V(c, m) = E_0 \int_0^\infty U(c_t, m_t, t) dt.$$ 

Initially, the agent is endowed with $N^F_0$ units of the foreign bond, one share of the domestic
risky stock, money holdings $M_0$ and one share of the equity claim for domestic monetary transfer.
His consumption over time is financed by a continuous trading strategy $\{M_t, N_t, \forall t \geq 0\}$, where
$M_t$ is the money holding at time $t$ and $N_t = (N^L_t, N^F_t, N^S_t, N^b_t, N^x_t)'$ is a vector which represents
the portfolio holdings consisting of all the financial assets traded in financial markets at time $t$. For
example, $N^F_t$ is the quantity of foreign bonds held by the domestic agent at time $t$. Denote the real
prices of all financial assets at time $t$ by a vector $X_t = (L_t, F_t, S_t, b_t, x_t)'$ and the corresponding
vector of real dividends by \( q_t \). The cumulative dividends up to \( t \) are defined as \( D_t = \int_0^t q_r \, dr \). At any point \( \tau \geq 0 \), the agent's wealth is \( W_\tau = N_\tau \cdot X_\tau + M_\tau / p_\tau \) and the flow budget constraint is

\[
c_\tau \, d\tau = M_\tau \cdot \frac{1}{p_\tau} + N_\tau^X \cdot (dD_\tau + dX_\tau) - dW_\tau. \tag{2.3}
\]

This constraint intuitively states that the sum of the wealth increase \( (dW_\tau) \) and consumption flow \( (c_\tau \, d\tau) \) is bounded by the dividend and capital gain from the portfolio \( \{M_\tau, N_\tau\} \).

With this flow budget constraint, one can use the technique of optimal control to derive the partial differential equations that are satisfied by the assets prices. In the presence of the jump components in the money supply process and the dividend process, these partial differential equations turn out to be very complicated. In contrast, the Euler equation approach appears much simpler and is adopted here.\(^4\) To do so, transform the flow budget constraint into an integrated one (see Duffie for a similar constraint):

\[
\int_0^t c_\tau \, d\tau = \frac{M_0}{p_0} + \int_0^t M_\tau \cdot \frac{1}{p_\tau} + N_\tau^X \cdot X_0 + \int_0^t N_\tau^X \cdot (dD_\tau + dX_\tau) - N_\tau^X \cdot X_\tau. \tag{2.4}
\]

The agent chooses an optimal portfolio trading strategy \( \{M_t, N_t, \forall \, t \geq 0\} \) so as to maximize his expected lifetime utility. Precisely, he solves:\(^5\)

\[
\max_{\{c_t, M_t, N_t\}} E \int_0^\infty U(c_t, m_t, t) \, dt \quad \text{s.t. (2.4) holds.}
\]

The first order conditions (Euler equations) are:

\[
\frac{1}{p_t} = \frac{1}{U_c(c_t, m_t, t)} E_t \left( \int_0^\infty U_m(c_\tau, m_\tau, \tau) \frac{1}{p_\tau} \, d\tau \right). \tag{2.5}
\]

\(^4\)The Euler equation approach has been used in Naik and Lee (1990) and the two approaches are equivalent in the sense that they lead to the same assets prices.

\(^5\)All expectations in the paper are taken with respect to the filtration specified earlier. The budget constraint is the continuous version of that in Lucas (1992). Accordingly, the agent's optimal decisions are described by the Euler equations.
\[ X_t = \frac{1}{U_c(c_t, m_t, t)} E_t \left( \int_t^\infty U_c(r, m_t, \tau) dD\tau \right). \] (2.6)

That is, the reciprocal of the exchange rate equals the expected discounted sum of future real wealth of one dollar, with the state price deflator being the marginal rate of substitution between consumption and the real money balance. The price of any other asset equals the expected discounted sum of dividends, with the stochastic state price deflator being the marginal rate of substitution between consumption at different dates.

As is typical for a small open economy, the exogenous foreign interest rate, the rate of time preference and the parameters describing consumption must satisfy certain restriction in order to ensure existence of an equilibrium. Such a restriction can be obtained by examining an agent's trade-off between consuming at time \( t \) and purchasing the foreign bond. The net utility gain from purchasing bond is \( \frac{dF}{dt} E_t \left( \frac{dU_s}{dt} \right) \), where \( \frac{dF}{dt} = r \) is the rate of return to holding the bond and \( E_t \left( \frac{dU_s}{dt} \right) \) is the utility loss due to the delay in consumption. Since optimality requires the net utility gain to be zero, the equilibrium restriction is \( r = E_t \left( \frac{dU_c}{dt} \right) \).\(^6\)

2.3. Equilibrium Exchange Rate and Asset Prices under Logarithmic Utility

Market clearing conditions are described as follows. The domestic currency held by the foreign country is negligible, and so the money market clearing condition requires that money demanded by domestic agents equal the money supply. That is, \( M^d = M \). Similarly, the demand for the risky stock equals the supply of shares, which is one share, and the demand for the claim on monetary transfers equals the supply, which is also one. Also, equilibrium prices are such that the representative agent holds neither the domestic nominal bonds nor any other contingent claims, because the net supply of each such asset is zero. Note that the supply of a domestic asset (or

\(^6\)This restriction can be formally derived from the Euler equation (2.6). When there is no uncertainty, this restriction becomes the well-known equality between the real interest rate \( (r) \) and the rate of time preference \( (\rho) \).
money) equals the domestic demand for the asset (or money). This equality holds here not because the economy is closed but rather because the economy is small relative to the outside world and so the foreign demand for its asset (or money) is negligible, as discussed at the beginning of Section 2.

On the other hand, the goods market clearing condition is quite different here from that in a closed economy. Since the country can have current account surplus or deficit, as discussed in the introduction, aggregate consumption does not necessarily equal the aggregate dividend generated from the domestic stock. Since the country can export the goods to the foreign country to increase its holdings on foreign bonds, the total expenditure on goods is \( c dt + df \), where \( f_t = N_t^F F_t \) is the value of foreign bonds. The total supply of goods is the sum of domestic dividends, \( \delta dt \), and return to holding foreign bonds, \( rf dt \). Thus, the goods market clearing condition is

\[
df = (\delta + rf - c)dt.
\]

This goods market clearing conditions also differs from that in Lucas’s (1982) two-country assets pricing model and its application in currency options by Bakshi and Chen (1996). In these models, the equilibrium portfolio of each country is identical to its initial endowment and so the net trading in assets between the two countries is zero in equilibrium. In contrast, here the net trading in foreign bonds must be non-zero in equilibrium as \( \delta \) and \( c \) vary over time. This difference not only makes it more challenging to solve for the equilibrium portfolio here but also leads to important differences in the behavior of the exchange rate: since the exchange rate clears the goods market, the net trading volume affects the exchange rate.

For analytical tractability, I assume that preferences are given by:\(^7\)

\(^7\)A similar assumption on the agent's period utility is also used in Bakshi and Chen (1996).
Assumption 3. The risk-averse agent’s period utility is described by

\[ U(c_t, m_t, t) = e^{-\rho t} [\alpha \ln c_t + (1 - \alpha) \ln m_t], \quad \alpha \in (0, 1). \]  

(2.7)

Using (2.5), (2.6) and the money market clearing condition \( M^* = M \), we can derive the equilibrium exchange rate and the nominal interest rate, which are summarized in the following proposition (see Appendix A for proof).

**Proposition 2.1.** Under Assumptions 1-3, the equilibrium exchange rate is \( p_t = \frac{\alpha}{1-\alpha} \frac{M}{c_t} i \) and the nominal interest rate is \( i = \rho + \beta_m \) where \( \beta_m \equiv \mu_m - \sigma_m^2 - \lambda_m k_m - \lambda_m \bar{k}_m \) is defined through \( E_t(\frac{M}{M_T}) = e^{-\beta_m(T-t)} \).

The exchange rate is proportional to the ratio of aggregate money supply and aggregate consumption, as in a typical small open economy model. This is a consequence of the representative agent’s optimal condition, \( \frac{U_m}{U_c} = i \), which states intuitively that the marginal rate of substitution between the real money balance and consumption must equal the opportunity cost of holding money (the foregone nominal interest income). Under the logarithmic utility function form, this general relation implies that the flow of services derived from holding money is proportional to the level of consumption. That is, \( i \frac{M}{p_t} = \frac{1 - \alpha}{\alpha} c_t \), which leads to the expression for equilibrium exchange rate in Proposition 2.1. An important implication of this result is that the exchange rate is affected by the uncertainties underlying both the money supply and consumption.

Proposition 2.1 also states that the nominal interest rate is constant and equal to the sum of the rate of time preference and the expected growth rate of money supply after adjusting the uncertainties, \( \beta_m \). This relation arises from the agent’s optimal trade-off between consuming today and purchasing a nominal bond today. Holding a nominal bond for an arbitrarily short period time
and then spending the return on consumption goods has a net gain \( i + E_t \left( \frac{dp^{-1}}{dt} - \frac{1}{p^{-1}} \right) + E_t \left( \frac{dU_c}{dt} \right) \), where \( E_t \left( \frac{dp^{-1}}{dt} - \frac{1}{p^{-1}} \right) \) is the capital loss resulted from inflation and \( E_t \left( \frac{dU_c}{dt} \right) \) is the utility loss from the delay in consumption. Since optimality requires the agent to be indifferent between consuming now and holding a nominal bond at the margin, \( i = -E_t \left( \frac{dp^{-1}}{dt} - \frac{1}{p^{-1}} \right) - E_t \left( \frac{dU_c}{dt} \right) \). Under the logarithmic utility function and the exchange rate \( p \) in Proposition 2.1, this implies \( i = \rho + \beta_m \).

The equilibrium also requires the real wealth, \( f_t + S_t \), to be equal to the expected present value of future consumption stream, \( c_t / \rho \). This condition is related to the goods market clearing condition. Together with Proposition 2.1, it helps to determine the equilibrium price of the domestic risky stock, \( S_t \), the real price of the claim on monetary transfers, \( L_t \), and the equilibrium quantity of the foreign bonds held by the domestic agent, \( f_t \). The results are summarized in the following proposition (See Appendix B for a proof):

**Proposition 2.2.** Under Assumptions 1-3, the equilibrium real price of the domestic risky stock at time \( t \), \( S_t \), is \( S_t = S(\delta_t) = \delta_t / \rho \). \( \forall \ t \in (0, \infty) \). The equilibrium real price at any time \( t \) of the claim for monetary transfers is \( L_t = (i - \rho)m_t / \rho \) and the equilibrium value of foreign bonds held by the domestic agent is \( f_t = N^F_t F_t = f_0 e^{(r - \rho)t} \).

In contrast to the exchange rate, the real price of the risky stock is not affected by the government monetary policy, given the logarithmic utility function in Assumption 3. Precisely, the stock price equals the present value of future dividends discounted at the rate of time preference.\(^8\) Also, the real price of the claim on monetary transfers is proportional to the real money balance, i.e., the present value of future real monetary transfers is proportional to current real money balances in equilibrium. The quantity of foreign bonds held by the domestic agent in equilibrium evolves at

\(^8\)A similar relation is also obtained in the closed endowment economies in Naik and Lee (1990) and in the first essay of this dissertation.
a constant rate of \( r - \rho \). Equivalently, the level of investment in foreign bonds at time \( t \) in equilibrium is determined as \( N_t^F = N_0^F e^{-\rho t} \). Therefore, the market portfolio in this small open economy is international diversified and consists of the domestic risky stock and \( f_t \) amount of foreign bonds.

Proposition 2.2 illustrates a version of the neutrality of money. Since \( c_t = \rho(S_t + f_t) \) and since real prices of the stock and foreign bonds are independent of the money supply process, equilibrium consumption is independent of the money supply process. The domestic agent consumes the dividends generated from the domestic risky stock and the foreign bond. Since the foreign bond price evolves exogenously in equilibrium, equilibrium consumption is determined by the stock dividend process. Under the general process for dividends (2.2), consumption follows a complicated stochastic process. This makes it difficult to compare the results of the current model with those in previous models such as GK (1983) and Merton (1976), who assume that the exchange rate follows a diffusion or jump-diffusion process. To facilitate comparison, let us restrict the dividend process by the following assumption, which allows me to derive currency option pricing formulas that encompass GK (1983) and Merton (1976) as special cases.

**Assumption 4.** The dividend process (2.2) evolves as:

\[
\begin{align*}
\text{dd} & = (\mu_{\delta} - \lambda_{\delta} k_{\delta})(\delta + \rho f)dt - \rho(r - \rho)f dt \\
& + \sigma_{\delta}(\delta + \rho f)dz_2 + (Y_{\delta} - 1)(\delta + \rho f)dQ_{\delta}.
\end{align*}
\]

Assumption 4 implies the following mixed jump-diffusion process for consumption:

\[
\frac{dc}{c} = (\mu_{\delta} - \lambda_{\delta} k_{\delta})dt + \sigma_{\delta}dz_2 + (Y_{\delta} - 1)dQ_{\delta}.
\]  \hspace{1cm} (2.8)

Here, \( \mu_{\delta} \) is the instantaneous expected growth rate; \( \sigma_{\delta}^2 \) is the instantaneous variance of the growth rate, conditional on no arrivals of new important shock. The element \( dQ_{\delta} \) is a jump process with
a jump intensity parameter $\lambda_\delta$ and $Y_\delta - 1$ is the random variable percentage change in aggregate consumption if the Poisson event occurs. The logarithm of $Y_\delta$ is normally distributed with mean $\theta_\delta$ and variance $\phi_\delta^2$. The expected jump amplitude, $k_\delta = E(Y_\delta - 1)$, is equal to $\exp(\theta_\delta + \phi_\delta^2/2) - 1$. Also $\bar{k}_\delta = E(\frac{1}{Y_\delta} - 1)$, is equal to $\exp(-\theta_\delta + \phi_\delta^2/2) - 1$. The random variables $\{z_{2t}, t \geq 0\}$, $\{Q_{\delta t}, t \geq 0\}$ and $\{Y_{\delta j}, j \geq 1\}$ are assumed to be mutually independent. Also, $Y_{\delta j}$ is independent of $Y_{\delta j'}$ for $j \neq j'$. The parameters $(\mu_\delta, \sigma_\delta, \lambda_\delta, \theta_\delta, \phi_\delta)$ are constant.

Under the logarithmic utility function and the above assumption, the restriction on the foreign interest rate, discussed at the end of subsection 2.2, becomes $r = \rho + \beta_\delta$, where $\beta_\delta = \mu_\delta - \sigma_\delta^2 - \lambda_\delta k_\delta - \lambda_\delta \bar{k}_\delta$ is defined through $E_t(\frac{\sigma_\delta}{\gamma}) = e^{-\beta_\delta(T-t)}$.

3. Pricing Currency Options

3.1. Dynamics of the Exchange Rate

Let us examine the dynamics followed by the exchange rate from the domestic agent’s perspective.

Since $p_t$ is a function of $M$ and $c$, applying Ito’s Lemma yields

$$\frac{dp}{p} = (\mu_p - \lambda_\delta \bar{k}_\delta - \lambda_\delta \bar{k}_\delta)dt + \sigma_\delta dz_1 - \sigma_\delta dz_2 + (Y_m - 1)dQ_m + (\frac{1}{Y_\delta} - 1)dQ_\delta, \quad (3.1)$$

where $\mu_p = \mu_m - \beta_\delta$. Under the equilibrium conditions for the nominal interest rate and the rate of time preference, the exchange rate dynamics can be rewritten as:

$$\frac{dp}{p} = (i - r + \sigma_m^2 + \lambda_\delta \bar{k}_\delta - \lambda_\delta \bar{k}_\delta)dt + \sigma_\delta dz_1 - \sigma_\delta dz_2 + (Y_m - 1)dQ_m + (\frac{1}{Y_\delta} - 1)dQ_\delta.$$

The key feature of the above exchange rate is that it is derived endogenously from the underlying money supply and dividend processes. This endogeneity is in stark contrast with the arbitrariness in the existing currency option models mentioned in the introduction. Clearly, the exchange rate is affected by the domestic government monetary policy and domestic consumption.
In this sense, Merton's assumption that the jump risk is uncorrelated with aggregate consumption is inappropriate for the exchange rate.

The domestic government monetary policy and domestic consumption affect the real price of the domestic risky stock and the exchange rate differently. The difference is crystal clear under the logarithmic utility. The real price of the domestic risky stock is solely determined by the aggregate dividend, where monetary policies play no role. The exchange rate incorporates jump components from both aggregate consumption and the money supply, while the stock price is only affected by the jump risk from the aggregate consumption. Thus, the current model is able to explain why discontinuities in exchange rate movements are more valent than in stock prices, a feature empirically documented by Jorion (1988). Examining the sample paths of exchange rates and a value-weighted U. S. stock market index, Jorion finds that exchange rates display significant jump components, while discontinuities are harder to detect in the stock market.

Specifically, the expected growth rate of the exchange rate, \( \mu_p \), is associated with the drifts of the money supply and aggregate consumption. It is also affected by the instantaneous variance of the growth rate of consumption and the jump component in consumption. The exchange rate dynamics incorporate the two independent jump components from the money supply and consumption. Since the money supply is assumed to be independent of consumption, the jump underlying the money supply will not be priced by a risk-neutral agent. Obviously, the jump in consumption generated by aggregate dividends must be priced. The instantaneous variance of the growth rate of the exchange rate is the sum of the variances in the money supply and consumption, \( \sigma_m^2 + \sigma_\delta^2 \). On the other hand, the stock price is completely described by the parameters underlying aggregate consumption. These requirements suggest that cross-equation restrictions must be imposed on the coefficients when the
processes for the exchange rate and the stock price are to be estimated.

3.2. Domestic Risk-Averse Agent's Equilibrium Valuation of Currency Options

Now consider the valuation of European style currency options. According to the agent's maximization condition (2.6), for any contingent claim with maturity $T$ and dividend $q_T$, its real price at time $t \leq T$, $x_t(T)$, is

$$x_t(T) = \frac{1}{U_c(c_t, m_t, t)} E_t \left( q_T U_c(c_T, m_T, T) \right).$$

For a European call written on the spot exchange rate with a striking price $K$ that matures at time $T$, its nominal price at time $t \leq T$, $CC_t(p_t, T)$, is

$$CC_t(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left( \frac{1}{c_T} \max(p_T - K, 0) \right).$$

Similarly, for a European put written on the spot exchange rate with a striking price $K$ that matures at time $T$, its nominal price at time $t \leq T$, $CP_t(p_t, T)$, is

$$CP_t(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left( \frac{1}{c_T} \max(K - p_T, 0) \right).$$

The joint density function for $(c_T, M_T)$ conditional on $(c_t, M_t)$, $f(c_T, M_T, T | c_t, M_t, t)$, is known. We can explicitly compute the prices of the European call and put, since the exchange rate is a function of $c$ and $M$. To facilitate the presentation of equilibrium prices of call and put options, let $C_{GK}$ and $P_{GK}$ be, respectively, the currency call and put prices derived by GK (1983) with the following expressions

$$C_{GK}(p_t, \tau; K, r_f, r_D, \sigma_E) = p_t e^{-r_f \tau} N(d_1) - Ke^{-r_D \tau} N(d_2),$$

$$P_{GK}(p_t, \tau; K, r_f, r_D, \sigma_E) = Ke^{-r_D \tau} N(-d_2) - p_t e^{-r_f \tau} N(-d_1),$$

where

$$\tau = T - t, \quad d_1 = \frac{\ln p_t/K + (r_D - r_f + \frac{1}{2} \sigma_E^2)\tau}{\sigma_E \sqrt{\tau}}, \quad d_2 = d_1 - \sigma_E \sqrt{\tau}.$$
Then, the option prices in the current model are described as follows (See Appendix C for a proof):

**Proposition 3.1.** Under Assumptions 1-4, equilibrium nominal prices of currency call and put are:

\[
CC_i^D (p_t, T) = \sum_{n_s=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) C_{GK} (p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m})
\]  

(3.2)

and

\[
CP_i^D (p_t, T) = \sum_{n_s=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) P_{GK} (p_t, \tau; K, r_\delta, i_m, \sigma_{\delta,m})
\]  

(3.3)

where \( P(\cdot, \cdot) \) is defined as

\[
P(a, b) = \frac{e^{-a\tau} (a\tau)^{n_s}}{n_s!} \frac{e^{-b\tau} (b\tau)^{n_m}}{n_m!}
\]

and

\[
r_\delta = r + \lambda_\delta \bar{k}_\delta + \frac{n_s(\theta_\delta - \frac{1}{2}\phi^2_\delta)}{\tau} = \rho + \mu_\delta - \lambda_\delta \bar{k}_\delta - \sigma^2_\delta + \frac{n_s(\theta_\delta - \frac{1}{2}\phi^2_\delta)}{\tau},
\]

\[
i_m = i + \lambda_m \bar{k}_m + \frac{n_m(\theta_m - \frac{1}{2}\phi^2_m)}{\tau} = \rho + \mu_m - \lambda_m \bar{k}_m - \sigma^2_m + \frac{n_m(\theta_m - \frac{1}{2}\phi^2_m)}{\tau},
\]

\[
\sigma_{\delta,m} = \sqrt{\sigma^2_\delta + \frac{n_s\phi^2_\delta}{\tau} + \sigma^2_m + \frac{n_m\phi^2_m}{\tau}}.
\]

Consider the call price for example. \( C_{GK} \) is an increasing function of the conditional domestic interest rate, \( i_m \), and the conditional exchange rate volatility, \( \sigma_{\delta,m} \), but a decreasing function of the conditional foreign interest rate, \( r_\delta \). The currency option prices depend intuitively on the fundamental parameters. First, an increase in the conditional consumption volatility, \( \sigma_\delta \), or the volatility of jump size, \( \phi_\delta \), induces a lower \( r_\delta \) and a higher \( \sigma_{\delta,m} \): the joint consequence is a higher currency call price. Second, a higher conditional volatility of money supply, \( \sigma_m \), or higher volatility of the corresponding jump, \( \phi_m \), does not necessarily imply a higher call price. This is because an increase in \( \sigma_m \) or \( \phi_m \) reduces \( i_m \) and increases \( \sigma_{\delta,m} \) simultaneously, while the increase in \( \sigma_{\delta,m} \) tends to increase the call price and the reduction in \( i_m \) tends to reduce the call price. Further, the call
value is positively related to the instantaneous expected growth rate of the money supply, $\mu_m$, and negatively related to the instantaneous expected growth rate of aggregate consumption, $\mu_\delta$. Call prices also depend ambiguously on $(\lambda_\delta, \lambda_m, \theta_\delta, \theta_m)$.

Note that if there were no jump component in aggregate consumption, the currency call and put prices in Proposition 3.1 would reduce to Merton’s (1976) price equations. In this case, the only jump uncertainty underlying the exchange rate would be from the money supply and this jump uncertainty is not priced.

The Euler equation (2.6) can also be used to price European style options on the domestic risky stock. Denote the real price of a call (put) on the risky stock at time $t$ with a striking price $k$ and an expiration date $T$ by $C_t(k, S_t, T)$ ($P_t(k, S_t, T)$). As shown in Appendix D, the stock option prices are completely described by the parameters underlying aggregate consumption. The explicit valuations are stated in the following proposition.

**Proposition 3.2.** Under Assumptions 1-4, $C_t(k, S_t, T)$ and $P_t(k, S_t, T)$ are:

$$C_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} C_{GK}(S_t + f_t, \tau; k + f_t e^{(r - \rho)^\tau}, \rho, r_\delta, \sigma^2_\delta + \frac{n_\delta \rho^2_\delta}{\tau})$$

and

$$P_t(k, S_t, T) = \sum_{n_\delta=0}^{\infty} \frac{e^{-\lambda_\delta \tau}(\lambda_\delta \tau)^{n_\delta}}{n_\delta!} P_{GK}(S_t + f_t, \tau; k + f_t e^{(r - \rho)^\tau}, \rho, r_\delta, \sigma^2_\delta + \frac{n_\delta \rho^2_\delta}{\tau})$$

where $r_\delta$ is defined in Proposition 3.1. The nominal values of the call and put are $p_t C_t(k, S_t, T)$ and $p_t P_t(k, S_t, T)$.

In contrast to currency options, real prices of stock options are independent of the uncertainty underlying the domestic money supply and nominal prices of stock options are affected by the money supply only through the price level $p_t$. Although aggregate consumption affects both the
stock price and the exchange rate, the parameters describing the dynamics of consumption affect stock options and currency options differently. For example, the instantaneous expected growth rate of consumption, $\mu_\delta$, positively affects the price of a call on the stock but negatively affects the price of a call on the exchange rate. An increase in $\sigma_\delta$ or $\phi_\delta$ increases the currency call prices as discussed earlier, but does not necessarily increase the stock call price. For the call price on the stock, increasing $\sigma_\delta$ or $\phi_\delta$ implies a higher instantaneous stock volatility $\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}$, which in turn induces a higher call price. However, an increase in $\sigma_\delta$ or $\phi_\delta$ also reduces $r_\delta$ at the same time. Since $r_\delta$ is positively related to the call price, the joint effect of a lower $r_\delta$ and a higher $\sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}$ on the call price, is ambiguous. This further illustrates the difference between currency options and stock options.\(^9\)

Note that the market portfolio in this small open economy consists of the domestic stock and the foreign bond. If the domestic agent did not hold foreign bond in equilibrium, this small open economy would be similar to a small closed economy in which the market portfolio is the domestic stock. In this case, the stock option formulas in Proposition 3.2 would reduce to those on the market portfolio in Naik and Lee (1990) with jump risks and logarithmic utility.

4. Foreign Agent's Risk-Neutral Valuation

I now use the above framework to examine the analog of Siegel's paradox in currency option valuation. The purpose is to identify the necessary restrictions that must be imposed on the risk-neutral process of the exchange rate if foreign agents use the risk-neutral approach.

The analog of Siegel's paradox in currency option valuation refers to the violation of the parity

\(^9\)The common belief is that an increase in stock volatility will be accomplished by an increase in call price according to the risk-neutral based Black-Scholes model (1973). Bailey and Stulz (1989) show that this common belief is not necessarily supported in an equilibrium context. Our result confirms the observation made by Bailey and Stulz (1989).
conditions between domestic and foreign investors’ valuations. A call option from the domestic agent’s point view is a put option from the foreign investor’s perspective. A call gives the domestic agent the right to buy the foreign currency from the foreign agent. On the other hand, a put option from the point of view of the foreign agent is an option to sell the domestic currency to obtain the foreign currency. In fact, the expression of “the call option value from the domestic agent’s view” is the same as the expression of “the put option value from the foreign agent’s view”. The foreign agent’s risk-neutral valuation of the put option is

$$CP_t^F(1/p_t, T) = e^{-r(T-t)} E_t^F \left( \max \left(1 - \frac{K}{p_T}, 0 \right) \right),$$

where $E_t^F()$ is the risk-neutral expectation operator conditional on the information at time $t$ available to the foreign investor. According to the law of one price, $CP_t^F(1/p_t, T)$ converted into the domestic currency at the spot exchange rate should be the same as $CC_t^D(p_t, T)$. That is

$$p_t CP_t^F(1/p_t, T) = CC_t^D(p_t, T).$$

(4.1)

Similarly, the put value from the domestic agent’s point should equal the call value from the foreign agent’s point, once the price is converted into the domestic currency at the spot exchange rate. That is

$$p_t CC_t^F(1/p_t, T) = CP_t^D(p_t, T),$$

(4.2)

where $CC_t^F(1/p_t, T) = e^{-r(T-t)} E_t^F \left( \max \left(\frac{K}{p_T} - 1, 0 \right) \right)$.

As pointed out by Dumas and Näslund (1995), if both the domestic and foreign investors assume their own risk neutral processes, even in the case where the jump component in the exchange rate is uncorrelated with the consumption, applying Merton’s formula generates an analog to Siegel’s paradox that either (4.1) or (4.2) is violated. The reason is that both investors use different
probability measures for the exchange rate. To see this, let \( x \) be the risk-neutral exchange rate expressed as the relative price of the foreign currency in terms of the home currency. The risk-neutral process is usually assumed to be

\[
\frac{dx}{x} = (i - r - \lambda_x E(Y_x - 1))dt + \sigma dw_x + (Y_x - 1)dQ_x,
\]

where the difference between the domestic and the foreign interest rate, \( i - r \), is the risk-neutral drift rate. The foreign agent observes the same exchange rate dynamics but instead expresses the spot rate as \( y = 1/x \), the relative price of the home currency expressed in terms of the foreign currency. The risk-neutral process for \( y \) is usually assumed by the foreign investor to be

\[
\frac{dy}{y} = (r - i - \lambda_y E(Y_y - 1))dt + \sigma dw_y + (Y_y - 1)dQ_y,
\]

where the difference between the foreign and the domestic interest rate, \( r - i \), is the risk-neutral drift rate. Obviously,

\[
\frac{dx^{-1}}{x^{-1}} = \left( r - i + \sigma_x^2 + \lambda_x E\left(\frac{(Y_x - 1)^2}{Y_x}\right) - \lambda_x E\left(\frac{1}{Y_x} - 1\right)\right)dt - \sigma_x dw_x + \left(\frac{1}{Y_x} - 1\right)dQ_y \neq \frac{dy}{y},
\]

with \( \sigma_x = \sigma_y \) and \( Y_y = \frac{1}{Y_x} \). The extra term, \( \sigma_x^2 + \lambda_x E\left(\frac{(Y_x - 1)^2}{Y_x}\right) \), appears in the drift for \( dx^{-1}/x^{-1} \).

Bardhan (1995) calls this extra term the "directional adjustments" and suggests that the foreign investor use \( dx^{-1}/x^{-1} \) as his risk-neutral process for \( y \), or vice versa.\(^{10}\) Strictly speaking, \( dx^{-1}/x^{-1} \) is not the risk-neutral process for \( y \) since the drift for \( y \) is no longer the risk-neutral drift \( r - i \). Instead, the drift is \( r - i + \sigma_x^2 + \lambda_x E\left(\frac{(Y_x - 1)^2}{Y_x}\right) \). One may interpret \( dx^{-1}/x^{-1} \) as the domestic risk-neutral process for \( y \). Bardhan's directional adjustments would eliminate the paradox if the jump risk in the exchange rate were uncorrelated with consumption. However, they are insufficient to eliminate the paradox when the exchange rate is correlated with consumption, as in our case.

\(^{10}\) The "directional adjustments" are sometimes referred to as the quanto adjustments or the convexity effects.
To examine the necessary restrictions on the risk-neutral process of the exchange rate when the
jump component in the exchange rate is correlated with aggregate consumption, denote \( \omega_t = \frac{1}{p_t} = \frac{1 - \alpha}{\alpha t} \). The actual process for \( \omega \) viewed by both domestic and foreign investors is
\[
\frac{d\omega}{\omega} = (r - i + \sigma_\delta^2 + \lambda_\delta k_\delta^2 - \lambda_m k_m^2)dt + \sigma_\delta dz_2 - \sigma_m dz_1 + (Y_\delta - 1)dQ_\delta + (\frac{1}{\gamma_m} - 1)dQ_m.
\] (4.3)

If the foreign agent uses the risk-neutral valuation to price the currency options, we can identify
the restrictions on the risk-neutral process for \( \omega \) by comparing the risk-neutral valuation of the
options with (3.2) and (3.3). Denote the risk-neutral process for \( \omega \) as follows:
\[
\frac{d\omega^*}{\omega^*} = (r - i - \lambda_\delta k_\delta^* - \lambda_m k_m^*)dt + \sigma_\delta dz_2^* - \sigma_m dz_1^* + (Y_\delta^* - 1)dQ_\delta^* + (\frac{1}{\gamma_m} - 1)dQ_m^*.
\] (4.4)

The following proposition details the foreign agents' risk-neutral valuations of the corresponding
currency options (See Appendix E for proof):

**Proposition 4.1.** Under the risk-neutral process of the exchange rate (4.4), the foreign agents' valuations of \( CP_t^F(1/p_t, T) \) and \( CC_t^F(1/p_t, T) \) are:
\[
CP_t^F(1/p_t, T) = \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta^*(k_\delta^* + 1), \lambda_m^*) C_{GK}(1, \tau; \frac{K}{p_t}, r_\delta^*, i_m^*, \sigma_{\delta,m}^*) \] (4.5)
and
\[
CC_t^F(1/p_t, T) = \sum_{n_\delta=0}^{\infty} \sum_{n_m=0}^{\infty} (\lambda_\delta^*(k_\delta^* + 1), \lambda_m^*) P_{GK}(1, \tau; \frac{K}{p_t}, r_\delta^*, i_m^*, \sigma_{\delta,m}^*) \] (4.6)
where \( C_{GK}(\cdot), P_{GK}(\cdot) \) and \( P(\cdot, \cdot) \) are defined in previous section and
\[
r_\delta^* = r + \lambda_\delta^* k_\delta^* + \frac{\sigma_\delta^* (\sigma_\delta^2 + \frac{\sigma_m^2}{r})}{r},
\]
\[
i_m^* = i + \lambda_m^* k_m^* + \frac{\sigma_m^* (\sigma_m^2 + \frac{\sigma_\delta^2}{r})}{r},
\]
\[
\sigma_{\delta,m}^* = \sqrt{\sigma_\delta^2 + \sigma_m^2 + \frac{\sigma_\delta^2 \sigma_m^2}{r} + \frac{\sigma_\delta^2 \sigma_m^2}{r}}.
\]
In order to ensure the parity conditions (4.1) and (4.2), the following restrictions on the risk-neutral process (4.4) must be satisfied:

\[
\begin{align*}
\lambda_m^* &= \lambda_m, \\
\bar{k}_m^* &= \bar{k}_m, \\
\theta_m^* &= \theta_m, \\
\phi_m^* &= \phi_m,
\end{align*}
\]

where

\[
\begin{align*}
\lambda_s^* &= \lambda_s(1 + k_s), \\
k_s^* &= E(Y_s^* - 1) = -\frac{k_s}{k_s + 1}, \\
\theta_s^* &= \theta_s - \sigma^2, \\
\phi_s^* &= \phi_s.
\end{align*}
\]

(4.7)

Under these restrictions, the actual probability is transformed into the risk-neutral or the equivalent martingale measure. In this case, the risk-neutral process can be expressed as:

\[
\frac{d\omega^*}{\omega^*} = \frac{d\omega}{\omega} - \sigma^2 dt - Y_s^*(1 - e^{-\phi_s^2}) dQ_s.
\]

In light of (4.3) and (4.4), this implies

\[
dz_2^* = dz_2 - \sigma^2 dt, \\
dz_1^* = dz_1, \\
(Y_s^* - 1) dQ_s = (Y_s e^{\phi_s^2} - 1) dQ_s.
\]

In fact, no adjustment is needed for the money supply process since it is assumed to be independent of the consumption process. For the consumption process, one needs to adjust not only the risk from the diffusion \((dz_2)\) and jump intensity parameters \((\lambda_s, k_s)\), but also from the jump size \((\theta_s)\).

The adjustments on \((dz_2, \lambda_s, k_s)\) are the directional adjustments suggested by Bardhan (1995) for the case where the jump in the exchange rate is not correlated with aggregate consumption.

The additional adjustment on \(\theta_s\) reflects the fact that the jump risk in exchange rate is related to aggregate consumption. Note that in the special case where the jump size in consumption is certain, i.e., \(\phi_s = 0\), no adjustment is needed for the jump size and so the jump component in consumption can be hedged away (see Bardhan 1995).

The above adjustments are specific to the utility function (2.7), but the general message of the exercise should be valid for a wider class of utility functions. That is, if the jump components in the exchange rate are related to those in consumption, at least one investor (either the domestic or the foreign investor) must use the utility-based equilibrium model to price the currency options. The
appropriate risk-neutral or the equivalent martingale process for the exchange rate should be based on an equilibrium model in an international context in order to ensure the parity conditions (4.1) and (4.2). Adjustments for the risk-neutral process must be made on all uncertainties, including the Brownian motion, the jump intensity and the jump size. Making only the directional adjustments is not enough.

5. An Extension of the Model

The above discussions have employed the assumption that the government monetary policy is independent of aggregate dividend. In this section, I extend the framework in previous sections to incorporate a correlation between the money supply and aggregate dividend. This correlation arises when the government uses the monetary policy to react to shocks in aggregate output. I capture this possible active monetary policy by allowing for a correlation between the shock \( dz_1 \) in the money supply and the shock \( dz_2 \) in aggregate dividends to be correlated, with a correlation coefficient \( \rho_{12} \).\(^{11}\)

With this correlation structure, the jump component in the money supply is still independent of aggregate dividend. Because of the separability between consumption and real money balances in the utility function, the exchange rate, the nominal interest rate, the restriction on the rate of time preference, the risky stock price and the equilibrium quantity of foreign bonds held by the domestic agent are the same as in previous sections. More importantly, the stock option valuation in Proposition 3.2 is unchanged and so is still independent of the money supply. In contrast, the correlation between \( dz_1 \) and \( dz_2 \) affects currency option valuations from the domestic agent’s view.

\(^{11}\)I thank John Hull for suggesting this extension. Although in principle one can also allow the money supply and aggregate dividends to be correlated through the jumps, analyzing this type of correlation is not tractable.
To see this, one can verify that Proposition 3.1 still holds with the following modification:

\[
\sigma_{\delta,m}^2 = \sigma_\delta^2 + \frac{n_{\delta}\phi_\delta^2}{\tau} - 2\rho_{12}\sqrt{(\sigma_\delta^2 + \frac{n_{\delta}\phi_\delta^2}{\tau})(\sigma_m^2 + \frac{n_m\phi_m^2}{\tau}) + \sigma_m^2 + \frac{n_m\phi_m^2}{\tau}}.
\]

Since the parameter \(\rho_{12}\) influences the currency option price only through \(\sigma_{\delta,m}\), a call on the exchange rate with \(\rho_{12} < 0\) will have a higher value than when \(\rho_{12} = 0\), because the call price is an increasing function of \(\sigma_{\delta,m}\).

One can also examine the analog of Siegel's paradox through the hypothetical exercise in Section 4. The risk neutral valuations in Proposition 4.1 are modified through the conditional instantaneous variance below:

\[
\sigma_{\delta,m}^2 = \sigma_\delta^2 + \frac{n_{\delta}\phi_\delta^2}{\tau} - 2\rho_{12}\sqrt{(\sigma_\delta^2 + \frac{n_{\delta}\phi_\delta^2}{\tau})(\sigma_m^2 + \frac{n_m\phi_m^2}{\tau}) + \sigma_m^2 + \frac{n_m\phi_m^2}{\tau}}.
\]

The restrictions imposed on the risk-neutral process of the exchange rate are the same as in (4.7). The risk-neutral process now is expressed as:

\[
\frac{d\omega^*}{\omega^*} = \frac{d\omega}{\omega} - (\sigma_\delta^2 - \rho_{12}\sigma_\delta\sigma_m)dt - Y_\delta(1 - e^{-\sigma_t^2})dQ_\delta.
\]

In light of (4.3), this implies \(d\omega^*=dz_2 - \sigma_\delta dt\), \(dz_1^*=dz_1 - \rho_{12}\sigma_\delta dt\), \((Y_\delta^* - 1)dQ_\delta = (Y_\delta e^{-\sigma_t^2} - 1)dQ_\delta\).

Compared with the adjustments made for the risk-neutral process (4.4) where the correlation is zero, an additional adjustment on \(dz_1\) in the magnitude of \(-\rho_{12}\sigma_\delta dt\) is needed to reflect the fact that the money supply is correlated with aggregate consumption. In this case, the exchange rate is correlated with aggregate consumption, not only directly, but also indirectly through the correlation between the money supply and aggregate consumption. Both correlations must be priced in currency option valuations. This exercise reinforces the key message that currency options must be priced by means of utility maximization if the risks in exchange rates are correlated with aggregate consumption.
6. Conclusion

This paper uses an equilibrium model to investigate exchange rates and currency options in a small open monetary economy where the jump-diffusion money supply and the jump-diffusion aggregate dividend processes are the sources of uncertainties. It is known that the exchange rate is affected by both government monetary policies and aggregate dividends, while the real price of the domestic market portfolio is determined only by aggregate dividends. The model is thus able to capture the empirical feature that discontinuities in exchange rates are more manifest than in stock prices (see Jorion 1988). Since the jump in the exchange rate is correlated with aggregate consumption, ignoring the systematic jump risks in exchange rates would be inappropriate and so directly applying Merton's jump-diffusion stock option model to currency options would be deficient. If there is jump risk in aggregate consumption, the jump risk in the exchange rate from aggregate consumption must be priced through a utility maximization model. The European currency option formulas derived in this paper encompass Merton's jump-diffusion model as a special case where there is no jump risk in aggregate consumption and the jump components from the money supply are independent of the aggregate dividend. I have further examined the foreign agents' risk-neutral valuation of the European currency option and provide necessary conditions to ensure the parity conditions (4.1) and (4.2) on options. In general, these restriction depend on the agent's risk preference. The general framework in this paper also provides an extension of Naik and Lee (1990) to an international context for option valuation on the domestic risky stock with discontinuous returns.
Appendices

A. Proof of Proposition 2.1:

**Proof.** Since the money supply process (2.1) is independent of the consumption process (2.8), the joint distribution of \((M_T, c_T)\) conditional on \((M_t, c_t)\) is:

\[
f(M_T, c_T, T \mid M_t, c_t, t) = g(M_T, T \mid M_t, t) h(c_T, T \mid c_t, t),
\]

where

\[
g(M_T, T \mid M_t, t) = \sum_{n_m=0}^{\infty} \frac{e^{-\lambda_m \tau (\lambda_m \tau)^{n_m}}}{\sqrt{2\pi \Sigma_m}} e^{-\frac{(\ln M_T - \psi_m)^2}{2\Sigma_m}},
\]

\[
h(c_T, T \mid c_t, t) = \sum_{n_{\delta t}=0}^{\infty} \frac{e^{-\lambda_{\delta t} \tau (\lambda_{\delta t} \tau)^{n_{\delta t}}}}{\sqrt{2\pi \Sigma_{\delta t}}} e^{-\frac{(\ln c_T - \psi_{\delta t})^2}{2\Sigma_{\delta t}}},
\]

with

\[
\psi_m = \ln M_t + (\mu_m - \lambda_m k_m - \frac{1}{2} \sigma_m^2)(T - t) + n_m \theta_m, \quad \Sigma_m = \sigma_m^2(T - t) + n_m \phi_m^2,
\]

\[
\psi_{\delta t} = \ln c_t + (\mu_{\delta t} - \lambda_{\delta t} k_{\delta t} - \frac{1}{2} \sigma_{\delta t}^2)(T - t) + n_{\delta t} \theta_{\delta t}, \quad \Sigma_{\delta} = \sigma_{\delta t}^2(T - t) + n_{\delta t} \phi_{\delta t}^2.
\]

According to the first order condition (2.5) and utility function (2.7),

\[
\frac{1}{p_t} = \frac{1}{U_{c_t}} E_t \left( \int_t^\infty U_m \frac{1}{p_T} dT \right) = \frac{1 - \alpha}{\alpha} c_t e^{\rho t} \int_t^\infty e^{-\rho T} E_t(\frac{1}{M_T})dT.
\]

Since \(E_t(\frac{1}{M_T}) = \frac{1}{M_t} e^{-(\mu_m - \sigma_m^2 - \lambda_m k_m - \lambda_m k_m)(T - t)}\), then we have

\[
p_t = \frac{\alpha}{1 - \alpha} \frac{M_t}{c_t} (\rho + \mu_m - \lambda_m k_m - \sigma_m^2 - \lambda_m k_m), \quad \forall \ t \in (0, \infty).
\]

The first order conditions (2.5) and (2.6) imply \(i = \frac{U_m}{U_c}\). Under the logarithmic utility function (2.7), \(i = \frac{\mu_m}{1 - \alpha} c_t\). Therefore, \(i = \rho + \mu_m - \lambda_m k_m - \sigma_m^2 - \lambda_m k_m\).

B. Proof of Proposition 2.2:

**Proof.** The risky stock price follows from the utility function (2.7) and the first order condition (2.6).
For the expression on $L$, note from Proposition 2.1 that $\im t = \frac{1-\alpha}{\alpha} c_t$. Since the expected present value of services ($\im$) generated by money equals

$$m_t + L_t = \frac{1}{U_t} E_t \left( \int_t^\infty U_T \frac{iM_T}{p_T} dT \right) = \frac{\im_t}{\rho}.$$ 

That is $L_t = \frac{1-\rho}{\rho} m_t$.

For the foreign bond, denote the equilibrium quantity of foreign bond held by the domestic agent as $f = N F F$. Since the real wealth in equilibrium, $S + f$, equals to the expected present value of future consumption stream, $c/\rho$, thus $c = \rho S + \rho f$. From the flow budget constraint (2.3), we have

$$df = (\delta + rf - c)dt = (\delta + rf - \rho S - \rho f)dt.$$ 

Since $S = \delta/\rho$, therefore, $df = (r - \rho) f dt$. It follows that $f_t = f_0 e^{(r-\rho)t}$. ■

C. Proof of Proposition 3.1:

Proof. For a European call written on the spot exchange rate with a striking price $K$ that matures at time $T$, its nominal price at time $t \leq T$, $CC_t^D(p_t, T)$, is

$$CC_t^D(p_t, T) = p_t e^{-\rho(T-t)} c_t E_t \left( \frac{1}{c_T} \frac{1}{p_T} \max(p_T - K, 0) \right).$$ 

Since $p = \frac{\alpha}{1-\alpha} \frac{M}{c} = A \frac{M}{c}$, then

$$CC_t^D(p_t, T) = p_t e^{-\rho t} c_t E_t \left( \max\left( \frac{1}{c_T} - \frac{K}{A M_T}, 0 \right) \right)$$

$$= p_t e^{-\rho t} c_t \int_{-\infty}^{\infty} \left( \int_{\frac{K}{A M_T}}^{\infty} g(M_T | M_t) dM_T \right) h(c_T | c_t) dc_T.$$ 

Tedious exercises show that

$$\int_{-\infty}^{\infty} \left( \int_{\frac{K}{A M_T}}^{\infty} g(M_T | M_t) dM_T \right) h(c_T | c_t) dc_T$$

$$= \sum_{n_t=0}^{\infty} \sum_{n_m=0}^{\infty} P(\lambda_\delta, \lambda_m) e^{-\psi_0 + \frac{1}{2} \Delta \delta} \int_{-\infty}^{\infty} z(v) dv \int_{\rho_\delta}^{\infty} z(w) dw,$$ 

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where $P(\cdot, \cdot)$ is defined in proposition 3.1, $z(\cdot)$ is the standard normal density and $w_1 = \frac{d_1^m - \varphi}{\sqrt{1 - \varphi^2}}$ with
\[
d_1^m = \frac{\ln p_t/K + (r_\delta - i_m + \frac{1}{2}\sigma^2_{\delta,m})\tau}{\sigma_{\delta,m}\sqrt{\tau}}, \quad \varphi = -\sqrt{\frac{\Sigma}{\Sigma^2 + \Sigma m}} = -\frac{\Sigma}{\sigma_{\delta,m}\sqrt{\tau - \tau}}.
\]
$\sigma_{\delta,m}$ is defined in proposition 3.1. According to Abramowitz (1972), the probability function for bivariate normal with correlation $\varphi$ is defined as
\[
\int_{\varphi}^{\infty} z(u)du \int_{w_1}^{\infty} z(w)dw = L(a, b, \varphi).
\]
Thus
\[
\int_{\varphi}^{\infty} z(u)du \int_{w_1}^{\infty} z(w)dw = L(-\infty, -d_1^m, \varphi) = N(d_1^m),
\]
where $N(a) = \int_{-\infty}^{a} z(u)du$. Therefore,
\[
\int_{-\infty}^{\infty} \left(\int_{\varphi}^{\infty} \frac{1}{c_T}g(M_T \mid M_t)\,dM_T\right) h(c_T \mid c_t)\,dc_T
\]
\[
= \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P(\lambda_\delta, \lambda_m)e^{-\psi_\delta + \frac{1}{2}\psi_m} N(d_1^m).
\]
Similarly,
\[
\int_{-\infty}^{\infty} \left(\int_{\varphi}^{\infty} \frac{1}{c_T}g(M_T \mid M_t)\,dM_T\right) h(c_T \mid c_t)\,dc_T
\]
\[
= \frac{K}{A} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P(\lambda_\delta, \lambda_m)e^{-\psi_m + \frac{1}{2}\psi_m} N(d_2^m),
\]
where $d_2^m = d_1^m - \sigma_{\delta,m}\sqrt{\tau}$. Rearrange terms, we have
\[
CC_t^D(p_t, T) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P(\lambda_\delta, \lambda_m)CG_K(p_t; \tau, K, r_\delta, i_m, \sigma_\delta, m).
\]
For a European currency put, we have
\[
CP_t^D(p_t, T) = p_t e^{-\rho(T-t)}c_tE_t\left(\max\left(\frac{K}{A} \frac{1}{M_T} - \frac{1}{c_T}, 0\right)\right)
\]
\[
= p_t e^{-\rho(T-t)}c_t \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\frac{K}{A} \frac{1}{M_T} - \frac{1}{c_T}\right)g(M_T \mid M_t)\,dM_T\right) h(c_T \mid c_t)\,dc_T.
\]
The same tedious exercises will give us
\[
CP_t^D(p_t, T) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P(\lambda_\delta, \lambda_m)PG_K(p_t; \tau, K, r_\delta, i_m, \sigma_\delta, m).
\]
D. Proof of Proposition 3.2:

Proof. For a European call written on the stock with a striking price \( k \) that matures at time \( T \), its real price at time \( t \leq T \), \( C_t(k, S_t, T) \), is \( C_t(k, S_t, T) = e^{-\rho t} c_t E_t \left( \frac{1}{c_T} \max(S_T - k, 0) \right) \). Since \( S_t = \frac{S_t}{\rho} = \frac{c_t - \rho f_t}{\rho} \) and \( f_t = f_0 e^{(r - \rho)t} \), then

\[
C_t(k, S_t, T) = e^{-\rho t} c_t E_t \left( \frac{1}{c_T} \max(\frac{c_T}{\rho} - f_T - k, 0) \right) \\
= e^{-\rho t} c_t \int_{-\infty}^{\infty} \max(\frac{1}{\rho} - \frac{f_t e^{(r - \rho)t} + k}{c_T}, 0) h(c_T \mid c_t) dc_T.
\]

Tedorious exercises show that

\[
C_t(k, S_t, T) = \sum_{n_\delta = 0}^{\infty} \frac{e^{-\lambda_\delta T}(\lambda_\delta \tau)^{n_\delta}}{n_\delta !} C_{GK}(S_t + f_t, \tau; k + f_t e^{(r - \rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}).
\]

Similarly, we have

\[
P_t(k, S_t, T) = \sum_{n_\delta = 0}^{\infty} \frac{e^{-\lambda_\delta T}(\lambda_\delta \tau)^{n_\delta}}{n_\delta !} P_{GK}(S_t + f_t, \tau; k + f_t e^{(r - \rho)\tau}, \rho, r_\delta, \sigma_\delta^2 + \frac{n_\delta \phi_\delta^2}{\tau}). \quad \blacksquare
\]

E. Proof of Proposition 4.1:

Proof. Based on the risk-neutral process (4.4), the distribution of \( \omega_T \) conditional on \( \omega_t \) is:

\[
G^* (\omega_T \mid \omega_t, t) = \sum_{n_\delta = 0}^{\infty} \sum_{n_m = 0}^{\infty} P(\lambda_\delta^*, \lambda_m^*) \frac{1}{\sqrt{2\pi \Sigma^*}} e^{-\frac{(\ln \omega_T - \psi^*)^2}{2 \Sigma^*}},
\]

\[
\psi^* = \ln \omega_t + (r - i - \frac{1}{2} \sigma_m^2 - \frac{1}{2} \sigma_\delta^2 - \lambda_\delta^* k_m - \lambda_m^* k_\delta^*) \tau + n_\delta \theta_\delta^* - n_m \theta_m^*,
\]

where

\[
\Sigma^* = (\sigma_m^2 + \sigma_\delta^2) \tau + n_m \phi_m^2 + n_\delta \phi_\delta^2.
\]

For the European put currency option from the perspective of the foreign agent, \( CP_t^F(1/p_t, T) \), we can compute according to the risk-neutral probability density stated above. That is

\[
CP_t^F (\omega_t, T) = e^{-r(T-t)} E_t^F (\max(1 - \omega_T K, 0)).
\]

Also, the European call in the view of the foreign agent can be computed as

\[
CC_t^F (\omega_t, T) = e^{-r(T-t)} E_t^F (\max(\omega_T K - 1, 0)).
\]

Then it is straightforward to prove proposition 4.1. \( \blacksquare \)
CHAPTER 4

EMPIRICAL IMPLICATION OF SYSTEMATIC JUMP RISKS IN CURRENCY OPTION PRICES
1. Introduction

This essay is a sequel to the second essay. The objective is to empirically examine the existence of systematic jump risks in exchange rates and determine whether they can explain the observed mis-pricing in the currency options market that cannot be explained by Merton's mixed jump-diffusion stock option model with non-systematic jump risks. A particular strength of this empirical analysis is that the equilibrium model in the second essay provides conditions on the joint distribution of the exchange rate and the price of the market portfolio. These conditions suggest that cross-equation restrictions must be imposed on the coefficients when the processes for exchange rates and the price of the market portfolio are estimated.

In this study, stock indices are used as the proxy for the market portfolio. The maximum-likelihood method is used to estimate the parameters underlying the joint distribution. Since the equilibrium model in the second essay describes a small open economy, the empirical investigation is first carried out for a small open economy like the Canadian economy, and then extended to economies with different sizes. The purpose of such extension is to confirm that exchange rates exhibit systematic jump risks regardless of the sizes of the economies. There are three cases under investigation.

Case 1: A small open economy versus a big economy:

The Canadian economy is selected as the small open economy against the big US economy. Value-weighted $TSE$ 300 index is chosen as the proxy for the Canadian market portfolio while the exchange rate under concern is the Canadian dollar price of the US dollar ($C$/US$).

Case 2: A large economy versus a mid-sized economy:
The US economy is selected as the large economy against the mid-sized German economy. Value-weighted CRSP index is chosen as the market portfolio for the US economy while the exchange rate under study is the time series of the US dollar price of the Deutch Mark (US$/DM).

Case 3: A small open economy versus a mid-sized economy:

The Canadian economy again is chosen as the small open economy against the mid-sized German economy. Value-weighted TSE 300 index is chosen as the market portfolio for the Canadian economy while the exchange rate under study is the Canadian dollar price of the Deutch Mark (C$/DM).

The maximum-likelihood procedure is used for estimation for the weekly and four-weekly data. The null hypothesis is that there is no systematic jump in exchange rates. The likelihood test strongly rejects the null hypothesis in three different cases, suggesting that all exchange rates under consideration exhibit systematic jump risks regardless of the size of the economy. The likelihood tests based on the weekly data provides stronger evidence against the null hypothesis, since higher frequency data reflect more detailed fluctuations in the sample.

With the estimated parameters, I examine the implication of systematic jump risks for currency option prices. The unrestricted estimates correspond to the parameters underlying my model, while the restricted estimates correspond to the parameters underlying Merton's model which is used by Jorion (1988) for currency option pricing. The numerical exercises show that the unrestricted model provides higher prices for out-of-the-money short-maturity call than those by the restricted model. For short-maturity call options written on the three exchange rates (C$/US$, US$/DM and C$/DM), the unrestricted model provides a 28% upward correction on the price generated.
by the GK model, a magnitude which almost corrects the entire the price bias (29%) of the GK model. In contrast, the restricted model provides only 16% upward price correction on the GK model, which is consistent with the finding in Jorion (1988). The 12% difference in price bias correction shows that systematic jump risks are important for currency option pricing. I further illustrates such importance by examining implied volatilities for these three different exchange rates. The unrestricted model generates a much clearer volatility smile than the restricted model, because both systematic and non-systematic jump risks contribute a significant portion to the total variance of exchange rates. The numerical exercises clearly suggest that systematic risks should be incorporated in currency option pricing models.

The remainder of this essay is organized as follows. Section 2 summarizes the underlying equilibrium model in the second essay and presents the equilibrium joint distribution for the exchange rate and the market portfolio. Section 3 discusses the methodology and empirical procedures. Section 4 presents estimation results from the four-weekly and weekly data for the three different cases outlined above. Section 5 illustrates the importance of systematic jump risks for currency option pricing by comparing the unrestricted model with the restricted model and the GK model. Section 6 provides some concluding remarks. Tables and necessary information are collected in the appendices.

2. The Underlying Equilibrium Model

In this section, I briefly review the equilibrium model in the second essay, whose implications will be tested here. The model describes a continuous-time small open economy, where agents make intertemporal choices on consumption and asset holdings. The rest of the world is termed foreign country. There is a single good traded worldwide with no barriers, which can be used for
consumption and investment. The nominal price of such a good at home at time $t$ is denoted $p_t$ and the spot exchange rate is expressed as the relative price of foreign currency in terms of the home currency. The law of one price holds in the economy and so foreign and domestic prices of the good equal each other when expressed in the same currency. Without loss of generality, the foreign price level is normalized to one, making the domestic nominal price $p_t$ equivalent to the spot exchange rate. The real price of any asset is thus expressed in foreign currency.

The home government controls the domestic money supply, $M^s$, which is taken exogenously by domestic agents. In the financial market, there is a single foreign pure discount bond available for trading to the home agents at any time. The price of this foreign bond expressed in foreign currency is taken exogenously by the home agent. There is only one domestic risky stock whose supply is normalized to one. It represents the ownership of the home productive technology for the single good. The real price of such a risky stock at time $t$ is denoted $S_t$, which is expressed in foreign currency. The dividend stream of this domestic risky asset, understood as aggregate dividends, is exogenous to the home agents. In addition, there is a single domestic nominal pure discount bond whose price is expressed in domestic currency. The supply of the domestic bond is zero. There are also other zero-supply contingent claims written on the risky domestic stock and the exchange rate. The economy is composed of a single risk-averse representative agent whose lifetime horizon is infinite. The agent chooses consumption, real money balances and portfolio to maximize his expected lifetime utility with the following form

$$E \int_0^\infty e^{-\rho t} [\alpha \ln c_t + (1 - \alpha) \ln m_t] dt,$$

where $c_t$ is the consumption flow at time $t$, $m_t$ the real money balance at time $t$ and $\rho$ the rate of time preference.

In equilibrium, the money market clearing condition requires that money demanded by domestic
agent, $M$, equal the money supply, $M^s$. The financial market clears so that the demand for the risky stock equals one share and the demand for the claim on monetary transfers equals one share. The good market clears so that the real wealth equals the expected present value of future consumption stream. The equilibrium endogenously generates aggregate consumption, the quantity of foreign bonds held by the domestic agent, the domestic nominal interest rate, the exchange rate and the price of any financial asset. The equilibrium spot exchange rate, expressed as the relative price of foreign currency in terms of home currency, is equal to $p = a\frac{M}{\zeta}$, where $a$ is a constant. That is, the spot exchange rate is proportional to the ratio between the domestic money demand and aggregate consumption. The equilibrium market portfolio consists of the domestic risky stock and the foreign bond. The total value of such a portfolio at time $t$ is equal to $\xi_p$.

The central message of the equilibrium model is that the equilibrium exchange rate depends on the market portfolio through aggregate consumption, as well as on the money supply process. To directly test this equilibrium implication, however, is difficult since high-frequency data for aggregate consumption are not available. In the empirical analysis below, I will take the stock index ($X$) as the proxy for such a market portfolio. Thus $X = \xi$. The spot exchange rate then becomes a function of the money supply $M$ and the stock index level $X$. That is, $p = a\rho\frac{M}{X}$. For the empirical investigation, I propose a more general relation:

$$p = AMX^\gamma, \quad -1 < \gamma < 0.$$  

If $\gamma = 0$, the exchange rate would not be directly correlated with the market. That is, there is no

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The relation $p = AMX^\gamma$ can be derived when the utility function takes the following form:

$$E \int_0^\infty e^{-\rho t} [\alpha \frac{C_t^{\gamma+1}}{\gamma+1} + (1 - \alpha) \ln m_t] \, dt, \quad -1 < \gamma < 0.$$  

Appendix A presents equilibrium exchange rate, stock price and option pricing formula under this more general utility function. The option prices reduce to Merton's jump-diffusion model when $\gamma = 0$. In this case, the domestic agent is risk-neutral with respect to real consumption.
systematic jump risks in the exchange rate movements. I will treat $\gamma = 0$ as the null hypothesis in the estimation.

The exogenous domestic money supply is assumed to evolve according to the following process

$$\frac{dM^s}{M^s} = (\mu_m - \lambda_m k_m) dt + \sigma_m dz_1 + (Y_m - 1) dQ_m, \quad \forall \ t \in (0, \infty).$$

Here, $\mu_m$ is the instantaneous expected growth rate of the money supply; $\sigma_m^2$ is the instantaneous variance of the growth rate, conditional on no arrival of new important shock, and $dz_1$ is a one-dimensional Gauss-Wiener process. The element $dQ_m$ is a jump process with a jump intensity parameter $\lambda_m$ and $Y_m - 1$ is the random variable percentage change in the money supply when the Poisson event occurs. The logarithm of $Y_m$ is normally distributed with mean $\theta_m$ and variance $\phi_m^2$. The expected jump amplitude, $k_m = E(Y_m - 1)$, is equal to $\exp(\theta_m + \phi_m^2/2) - 1$. Also, $\bar{k}_m = E(Y_m - 1)$ is equal to $\exp(-\theta_m + \phi_m^2/2) - 1$. The random variables $\{z_{1t}, t \geq 0\}, \{Q_{mt}, t \geq 0\}$ and $\{Y_{mj}, j \geq 1\}$ are assumed to be mutually independent. Also, $Y_{mj}$ is independent of $Y_{mj'}$ for $j \neq j'$. The parameters $(\mu_m, \sigma_m, \lambda_m, \theta_m, \phi_m)$ are constant.

Equilibrium aggregate consumption $c$ is linear in the exogenous aggregate dividend. Given a suitable process for aggregate dividends, aggregate consumption $c$ can be endogenously derived as the following mixed jump-diffusion process

$$\frac{dc}{c} = (\mu_\delta - \lambda_\delta k_\delta) dt + \sigma_\delta dz_2 + (Y_\delta - 1) dQ_\delta.$$

Here, $\mu_\delta$ is the instantaneous expected growth rate of aggregate dividends and $\sigma_\delta^2$ is the instantaneous variance of the growth rate, conditional on no arrival of new important shock. The element $dQ_\delta$ is a jump process with a jump intensity parameter $\lambda_\delta$ and $Y_\delta - 1$ is the random variable percentage change in aggregate consumption when the Poisson event occurs. The logarithm of $Y_\delta$ is normally distributed with mean $\theta_\delta$ and variance $\phi_\delta^2$. Define $\bar{k}_\delta(\gamma) = E(Y_\delta^\gamma - 1)$, which is
equal to \( \exp(\gamma \theta \delta + \gamma^2 \phi_\delta^2/2) - 1 \). Thus the expected jump amplitude, \( k_\delta = E(Y_\delta - 1) \), is equal to \( \exp(\theta \delta + \phi_\delta^2/2) - 1 \). The random variable \( \{z_{2t}, t \geq 0\}, \{Q_{5t}, t \geq 0\} \) and \( \{Y_\delta, j \geq 1\} \) are assumed to be mutually independent. Also, \( Y_\delta \) is independent of \( Y_{\delta'} \) for \( j \neq j' \). The parameters \( (\mu_\delta, \sigma_\delta, 
abla \delta, \theta_\delta, \phi_\delta) \) are constant.

The money supply process is correlated with the aggregate dividend (or aggregate consumption) only through \( dz_1 \) and \( dz_2 \), with a correlation coefficient \( \rho_{12} \). The joint distribution of \( (M_T, c_T) \) conditional on \( (M_t, c_t) \) is as follows:

\[
f(\ln M_T, \ln c_T | \ln M_t, \ln c_t, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(\lambda_\delta, \lambda_m) B(\ln M_T/M_t, \ln c_T/c_t; \psi_m, \psi_\delta, \Sigma_m, \Sigma_\delta, \rho_{12}),
\]

where

\[
P(\lambda_\delta, \lambda_m) = \frac{e^{-\lambda_\delta \tau \lambda_\delta \tau} e^{-\lambda_m \tau \lambda_m \tau}}{\tau ! j!},
\]

and \( B(\ln M_T, \ln c_T; \psi_m, \psi_\delta, \Sigma_m, \Sigma_\delta, \rho_{12} | \ln M_t, \ln c_t, t) \) is the bivariate normal distribution expressed as

\[
B(\ln M_T/M_t, \ln c_T/c_t; \psi_m, \psi_\delta, \Sigma_m, \Sigma_\delta, \rho_{12}) = \frac{1}{2\pi \sqrt{\Sigma_m \Sigma_\delta (1-\rho_{12}^2)}} \exp \left( -\frac{1}{2(1-\rho_{12}^2)} \left( \frac{(\ln M_T/M_t-\psi_m)^2}{\Sigma_m} - 2\rho_{12} \frac{(\ln M_T/M_t-\psi_m)(\ln c_T/c_t-\psi_\delta)}{\sqrt{\Sigma_m \Sigma_\delta}} + \frac{(\ln c_T/c_t-\psi_\delta)^2}{\Sigma_\delta} \right) \right)
\]

with

\[
\tau = T - t,
\]

\[
\psi_m = (\mu_m - \lambda_m k_m - \frac{1}{2} \sigma_m^2) \tau + j \theta_m,
\]

\[
\Sigma_m = \sigma_m^2 \tau + j \sigma_m^2,
\]

\[
\psi_\delta = (\mu_\delta - \lambda_\delta k_\delta - \frac{1}{2} \sigma_\delta^2) \tau + i \theta_\delta,
\]

\[
\Sigma_\delta = \sigma_\delta^2 \tau + i \sigma_\delta^2.
\]

I do not directly estimate this joint distribution of \( c \) and \( M \), since high-frequency data on aggregate consumption are not available and since it is controversial what monetary aggregate one should use to identify the money supply. Instead, I estimate the joint distribution of \( (X, p) \) implied

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2The notation \( B(x, y; \mu_x, \mu_y, \sigma_x, \sigma_y, \rho) \) stands for bivariate normal distribution where \( x \) and \( y \) are the two random variables. \( \mu_x \) and \( \sigma_x \) are the mean and standard derivation for \( x \), respectively. \( \mu_y \) and \( \sigma_y \) are the counterparts for \( y \). \( \rho \) is the correlation coefficient between \( x \) and \( y \).
by the distribution of \((c, M)\). Since the stock index and the spot exchange rate are functions of aggregate consumption and the money supply, the processes for the stock index and the spot exchange rate can be derived from Ito's Lemma

\[
\frac{dX}{X} = (\mu_\delta - \lambda_\delta k_\delta)dt + \sigma_\delta dz_2 + (Y_\delta - 1)dQ_\delta;
\]

\[
\frac{dp}{p} = [\mu_m - \lambda_m k_m + \gamma(\mu_\delta - \lambda_\delta k_\delta) + \frac{1}{2} \gamma(\gamma - 1)\sigma_\delta^2]dt
\]

\[+ \gamma \sigma_\delta dz_2 + \sigma_m dz_m + (Y_m - 1)dQ_m + (Y_\delta - 1)dQ_\delta.
\]

Note that the two processes are dependent and the nature of the dependence endogenously arises from the more fundamental variables, aggregate consumption \(c\) and the money supply \(M\), rather than being exogenously assumed. This dependence provides the cross-equation restrictions for estimation. The joint distribution of \((X, p)\), computed from the joint distribution of \((M, c)\) in (2.1), is expressed below:

\[
G(\ln p_T, \ln X_T, T \mid \ln p_t, \ln X_t, t)
\]

\[
= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} P(\lambda_\delta, \lambda_m)B(\ln p_T/p_t - \gamma \ln X_T/X_t, \ln X_T/X_t; \psi_m, \psi_\delta, \Sigma_m, \Sigma_\delta, \rho_{12})
\]

where \(\psi_m, \psi_\delta, \Sigma_m\) and \(\Sigma_\delta\) are defined in (2.2). The joint distribution of \((X, p)\) in (2.5) is the equilibrium conditions imposed on the dynamics of the exchange rates and the stock index level.

Maximum-likelihood analysis in the following section is based on this joint distribution.

3. Methodology

This section presents the maximum-likelihood estimation procedure. Define the logarithmic difference of the stock index as \(x_t = \ln(X_t/X_{t-1})\). For notational purpose, we rewrite the process for the stock index in equation (2.3) as

\[
\ln X = \alpha_x dt + \sigma_x dz + (Y_x - 1)dQ_x,
\]
where $\alpha_x = \mu_x - \lambda_x \kappa_x - \sigma_x^2/2$ and $\sigma_x = \sigma_x$. The Poisson process $dQ_x$ is the same as $dQ_y$, which is characterized by a mean number of jumps occurring per unit time, $\lambda_x$, and a jump size, $Y_x$. The jump size is independently lognormally distributed with $\ln Y_x \sim N(\theta_x, \phi_x^2)$. Thus

$$x_t = \ln\left(\frac{X_t}{X_{t-1}}\right) = \alpha_x + \sigma_x z_x + \sum_{i=1}^{n_t} \ln Y_x^i,$$

(3.1)

where $n_t$ is the actual number of jumps during the interval. The diffusion risk $dz_x$ and the jump uncertainty $dQ_x$ are inherited from aggregate dividend.

Similarly, define the logarithmic difference of exchange rates as $y_t = \ln(p_t/p_{t-1})$. We rewrite the process for the spot exchange rate in equation (2.4) as

$$d \ln p = (\alpha_y + \gamma \alpha_x) dt + \gamma \sigma_x dz_x + \gamma \sigma_y dz_y + (Y_y - 1) dQ_y + (Y_x - 1) dQ_x,$$

(3.2)

where $\alpha_y = \mu_y - \lambda_y \kappa_y - \sigma_y^2/2$ and $\sigma_y = \sigma_y$. The Poisson process $dQ_y$ is the same as $dQ_m$, which is characterized by a mean number of jumps occurring per unit time, $\lambda_y$, and a jump size, $Y_y$. The jump size is assumed independently lognormally distributed with $\ln Y_y \sim N(\theta_y, \phi_y^2)$. Thus

$$y_t = \ln\left(\frac{p_t}{p_{t-1}}\right) = (\alpha_y + \gamma \alpha_x) + \gamma \sigma_x z_x + \gamma \sigma_y z_y + \gamma \sum_{i=1}^{n_t} \ln Y_x^i + \gamma \sum_{j=1}^{\omega_t} \ln Y_y^j,$$

where $\omega_t$ is the actual number of jumps in the money supply during the interval. The diffusion risk $dz_y$ and jump uncertainty $dQ_y$ are inherited from the domestic money supply. The correlation coefficient between $dz_x$ and $dz_y$ is $\rho_{xy}$.

Denote the parameters by a vector $\beta = (\alpha_x, \sigma_x, \lambda_x, \theta_x, \phi_x, \alpha_y, \sigma_y, \lambda_y, \theta_y, \phi_y, \rho_{xy}, \gamma)'$. From equation (2.5), we can show that the joint density for $x_t$ and $y_t$ is

$$f(x_t, y_t; \beta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\lambda_x \lambda_x^i} e^{-\lambda_y \lambda_y^j}}{i! j!} \frac{1}{2\pi \sqrt{\Sigma_x \Sigma_y (1 - \rho_{xy}^2)}} g(x_t, y_t - \gamma x_t; \psi_x, \psi_y, \Sigma_x, \Sigma_y, \rho_{xy}),$$

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where \( g(x_t, y_t - \gamma x_t; \psi_x^i, \psi_y^i, \Sigma_x^i, \Sigma_y^i, \rho_{xy}) \) is defined as

\[
g(x_t, y_t - \gamma x_t; \psi_x^i, \psi_y^i, \Sigma_x^i, \Sigma_y^i, \rho_{xy}) = \exp \left( \frac{-1}{2(1-\rho^2)} \left( \frac{(x_t - \psi_x^i)^2}{\Sigma_x^i} - 2\rho_{xy} \frac{(x_t - \psi_x^i)(y_t - \gamma x_t - \psi_y^i)}{\sqrt{\Sigma_x^i \Sigma_y^i}} + \frac{(y_t - \gamma x_t - \psi_y^i)^2}{\Sigma_y^i} \right) \right)
\]

with

\[
\psi_x^i = \alpha_x + i\theta_x, \quad \Sigma_x^i = \sigma_x^2 + i\phi_x^2, \\
\psi_y^i = \alpha_y + j\theta_y, \quad \Sigma_y^i = \sigma_y^2 + j\phi_y^2.
\]

The marginal densities for \( x_t \) and \( y_t \) are

\[
f(x_t; \beta) = \sum_{i=0}^{\infty} \frac{e^{-\lambda_x^i \lambda_x^i}}{i! \sqrt{2\pi \Sigma_x^i}} \exp \left( \frac{-(x_t - \psi_x^i)^2}{2\Sigma_x^i} \right),
\]

\[
f(y_t; \beta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\lambda_x^i \lambda_y^j}}{i! \sqrt{2\pi \Sigma_y^j}} \exp \left( \frac{-(y_t - \psi_y^j - \psi_x^i)^2}{2\Sigma_y^j} \right)
\]

where \( \Sigma_y^j = \gamma^2 \Sigma_x^i + 2\rho_{xy} \gamma \sqrt{\Sigma_x^i \Sigma_y^j} + \Sigma_y^i \).

If the correlation between \( dz_x \) and \( dz_y \), \( \rho_{xy} \), is zero, then the first and second moments for the logarithmic of the stock index and the exchange rate are as follows:

\[
E(x) = \alpha_x + \lambda_x \theta_x \quad E(y) = \alpha_y + \lambda_y \theta_y + \gamma E(x)
\]

\[
Var(x) = \sigma_x^2 + \lambda_x (\theta_x^2 + \phi_x^2) \quad Var(y) = \sigma_y^2 + \lambda_y (\theta_y^2 + \phi_y^2) + \gamma^2 Var(x)
\]

The parameters of interest are estimated by numerical maximization of the likelihood function of the parameter vector \( \beta = (\alpha_x, \sigma_x, \lambda_x, \theta_x, \phi_x, \alpha_y, \sigma_y, \lambda_y, \theta_y, \phi_y, \rho_{xy}, \gamma)' \) given the observation \( X = (x, y) \), \( L(\beta; X) = \prod_{t=1}^{T} f(x_t, y_t; \beta) \). The log-likelihood function is

\[
l(\beta; X) = \ln L(\beta; X) = -T \ln 2\pi - T \lambda_x - T \lambda_y - \frac{T}{2} \ln(1 - \rho_{xy}^2)
\]

\[
+ \sum_{t=1}^{T} \ln \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_x^i \lambda_y^j}{\sqrt{\Sigma_x^i \Sigma_y^j}} g(x_t, y_t - \gamma x_t; \psi_x^i, \psi_y^j, \Sigma_x^i, \Sigma_y^j, \rho) \right).
\]

The estimates are consistent, with normal asymptotic distributions of known parameters.\(^3\) In addition, maximum-likelihood estimation permits formal tests of the relative fit of various distributions.

\(^3\)See Judge et. al (page 202) for more detailed discussions regarding the properties of maximum-likelihood estimators and the corresponding regularity conditions.
Nested hypotheses can be tested using the generalized likelihood ratio of the maximized-likelihood functions under the null and under the enlarged parameter space $\Omega$, which also includes the alternative hypothesis $\Omega_1$.

$$\Lambda = \sup_{\beta \in \Omega_0} L(\beta; X) / \sup_{\beta \in \Omega_1} L(\beta; X)$$

Under the null $\Omega_0$, the statistic $-2 \ln \Lambda$ has a chi-square distribution with the degree of freedom equal to the difference between the numbers of parameters in the two models. Thus, the improvement in the maximized likelihood indicates to what extent an enlarged specification helps in fitting the data. The null hypotheses test in this study is $\gamma = 0$, which requires that the exchange rate is not correlated with the market and there be no systematic jump risks in exchange rates.\(^4\) Necessary conditions for the existence of maximum likelihood estimators $\hat{\beta}$ are provided by the solution to

$$\frac{\partial l(\beta; X)}{\partial \beta} = 0.$$ 

Corresponding sufficient conditions require the matrix $-H(\beta; X)$ to be positive definite, where the Hessian matrix $H(\beta; X)$ is defined by

$$H(\beta; X) = \frac{\partial^2 l(\beta; X)}{\partial \beta \partial \beta'}.$$

Unfortunately, the infinite summation in (3.4) cannot be perfectly computed numerically. Truncation on the sum is commonly used for approximation, where the precision of the approximation is provided by a bound on the truncated error. To illustrate this, let us consider the numerical calculation of the infinite sum in (3.4) for a single density $f(\beta; X)$. Clearly, adequate approximation of $f(\beta; X)$ will imply adequate approximation of the likelihood function given by (3.4). The

\(^4\)An alternative test of non-existence of systematic jump risks in the exchange rate is to test the existence of jump risks in the movement of the stock index. Since Jorion (1988) has provided empirical evidence that shows discontinuities in the stock index, this alternative test is not performed here. Also, existing studies have considered jump risks in the market portfolio, such as Naik and Lee (1990).
infinite sum in \( f(\beta; X) \) shall be truncated at \( I \) and \( J \), the resultant approximation error denoted by \( B(I, J) \).\(^5\) Since all terms are positive and

\[
\sup_{X \in \mathbb{R}^2, i \in I, j \in J} g(x_t, y_t - \gamma x_t; \psi^i, \psi^j, \Sigma^i, \Sigma^j, \rho) \leq 1,
\]

it follows that

\[
0 \leq B(I, J) \leq \sum_{i=I+1}^{\infty} \sum_{j=J+1}^{\infty} \frac{e^{-\lambda_x} \lambda_x^i}{i!} \frac{e^{-\lambda_y} \lambda_y^j}{j!} \frac{1}{2\pi \sqrt{\Sigma_x \Sigma_y (1 - \rho^2)}}.
\]

Noticing that \( \frac{1}{\sqrt{\Sigma_x \Sigma_y (1 - \rho^2)}} \leq \frac{1}{\sigma_x \sigma_y} \), we have

\[
0 \leq B(I, J) \leq \sum_{i=I+1}^{\infty} \sum_{j=J+1}^{\infty} \frac{e^{-\lambda_x} \lambda_x^i \cdot e^{-\lambda_y} \lambda_y^j}{2\pi \sqrt{1 - \rho^2} \sigma_x \sigma_y \cdot i! \cdot j!} = \frac{e^{-\lambda_x} e^{-\lambda_y}}{2\pi (1 - \rho^2) \sigma_x \sigma_y} \left( \sum_{i=I+1}^{\infty} \frac{\lambda_x^i}{i!} \right) \left( \sum_{j=J+1}^{\infty} \frac{\lambda_y^j}{j!} \right).
\]

Using the Taylor series expansion for the exponential function and applying integration by parts \( I \) times sequentially, we can show that

\[
\sum_{i=I+1}^{\infty} \frac{\lambda_x^i}{i!} = \int_0^{\lambda_x} \frac{(\lambda_x - u)^I e^u}{I!} du.
\]

However, for \( 0 \leq u \leq \lambda_x \), we have

\[
\sum_{i=I+1}^{\infty} \frac{\lambda_x^i}{i!} \leq e^{\lambda_x} \int_0^{\lambda_x} \frac{(\lambda_x - u)^I}{I!} du = e^{\lambda_x} \frac{\lambda_x^{I+1}}{(I + 1)!}.
\]

Similarly, we can show

\[
\sum_{j=J+1}^{\infty} \frac{\lambda_y^j}{j!} \leq e^{\lambda_y} \frac{\lambda_y^{J+1}}{(J + 1)!}.
\]

That is,

\[
0 \leq B(I, J) \leq \frac{1}{2\pi \sqrt{1 - \rho^2} \sigma_x \sigma_y} \frac{\lambda_x^{I+1}}{(I + 1)!} \frac{\lambda_y^{J+1}}{(J + 1)!}.
\]

With this error bound as a criterion, a series of experiments are performed to select the optimal truncation point. Although the truncation error is a function of parameters, for plausible values of

\(^5\)A similar approach to establish the upper bound on the truncation error for the approximation of the likelihood function has been used by Ball and Torous (1985) in a study of a single jump risk on common stock prices.
parameters, truncation at $I = 7$ and $J = 7$ provides satisfactory accuracy. For example, the upper bound of the truncation error on the log-likelihood is $2.1553 \times 10^{-9}$ in case 1, $3.1053 \times 10^{-9}$ in case 2 and $3.7771 \times 10^{-9}$ in case 3 for the four-weekly data. It is worth noting that much smaller values of $I$ and $J$ are needed in order to provide satisfactory accuracy for small values of $\lambda_x$ and $\lambda_y$.

On the basis of the above presentation and the experimental evidence, we can provide the approximate maximum likelihood estimates by maximizing the truncated log-likelihood function

$$l_{(I,J)}(\beta; X) = \ln L(\beta; X) = -T \ln 2\pi - T\lambda_x - T\lambda_y - \frac{T}{2} \ln(1 - \rho^2)$$

$$+ \sum_{i=1}^T \ln \left( \sum_{i=0}^I \sum_{j=0}^J \frac{\lambda_x^i \lambda_y^j}{\sqrt{\Sigma_x^i \Sigma_y^j}} g(x_t, y_t; \psi_x^i, \psi_y^j, \Sigma_x^i, \Sigma_y^j, \rho) \right).$$

Necessary conditions for a maximum become

$$\frac{\partial l_{(I,J)}(\beta; X)}{\partial \beta} = 0,$$

with analogous sufficient conditions for the truncated likelihood function.

The nonlinear maximization problem is solved using the quasi-Newton method with a mixed quadratic and cubic line search procedure. The programming is done with Matlab Version 4.2 on a Unix system.

4. Empirical Results

The following data sources were used in the empirical analysis. Daily observations for exchange rates $C$/US$, US$/DM$ and $C$/DM were obtained from Bank of Montreal for the period January 1980 to May 1993. Daily U.S. stock market returns were taken from the Center for Research in Security Prices (CRSP) database, which provides a value-weighted market index of all quoted NYSE and AMEX stocks. Daily Canada stock market returns were taken from the TSE-WESTERN database, which provides a value-weighted TSE 300 index. End-of-four-week and end-of-week data were sampled from the daily file. Given the complex distributional changes observed for different days
of the week, daily data are not studied in this paper. The analysis focuses instead on four-weekly and weekly data.

Tables 1, 2 and 3 here.

Tables 1, 2 and 3 summarize the statistics for four-weekly and weekly logarithmic changes in exchange rates and in value-weighted stock markets for Case 1 (C$/US$ & TSE 300 index), Case 2 (US$/DM & NYSE index) and Case 3 (C$/DM & TSE 300 index), respectively. In all three cases, the high excess kurtosis coefficients indicate the departure from the normal density. Normal densities require zero coefficients of skewness and excess kurtosis. The high asymptotic z-statistics clearly reveals "fat-tailed" distributions. The "fat" tails are consistent with a continuous-time process with time-varying parameters or with a mixed diffusion-jump process. Jorion (1988) shows that the discontinuities arise even after the explicit allowance is made for possible heteroskedasticity in the usual diffusion process. In the current study, the "fat" tail distributions are modelled by mixed-jump diffusion processes. In addition, the pattern of autocorrelation indicates little serial correlation in any of the cases.

Tables 4, 5 and 6 here.

Tables 4, 5 and 6 show the 4-weekly estimated coefficients for (3.1) and (3.2) as well as tests of non-existence of systematic jump risks as in the null hypotheses for Cases 1, 2 and 3, respectively. First, it should be noted that the C$/US$ exchange rate is less volatile than the US$/DM exchange rate or the C$/DM exchange rate. The total variance of exchange rates can be computed from the moment condition given in equation (3.3) with the estimated parameter values. Annualizing 4-weekly variance for the exchange rates by \( \sqrt{13} \), the volatility for C$/US$ is 4.55% compared

\footnote{Jorion (1988) finds a similar feature of fat-tailed distributions for $/DM and NYSE index from January 1974 to December 1985.}
with 12.62% for US$/DM and 11.94% for C$/DM. One possible explanation is related with the European Monetary System (EMS). Although the controlled fluctuation of exchange rates in EMS in terms of the US dollar gave way to the present freely-floating regime after the 1971 Smithsonian agreement. However, links between European currencies remained and formed the basis of the 1978 EMS. These links were formalized through the EMS exchange mechanism which limits and controls the behavior of the participating currencies' bilateral exchange rates. These limits and controls generate large variances for the bilateral exchange rates, and hence have direct consequences on the floating exchange rates outside the EMS, such as US$/DM and C$/DM (see Ball and Roma 1993). This big difference in volatilities will be adjusted accordingly for the measure of moneyness in the next section for comparison among different currency option models.

Second, since the estimated $\gamma$ in all cases are between $-1$ and 0, this indicates that the exchange rates are negatively correlated with the market. The $\chi^2$ tests indicate that the description with systematic jump risks for the exchange rates in (3.2) ($\gamma \neq 0$) is a significant improvement over the restricted case where there is no systematic jump risks involved in the exchange rates. The likelihood ratio tests strongly reject the null hypotheses $\gamma = 0$ at 1 percent significant level in all cases.

Third, it is interesting to note that the jump intensity for the non-systematic jump risks underlying the exchange rates, namely the money supply implied by the equilibrium model, has a much larger magnitude than that of the jump risk underlying the market portfolio in all cases. For example, the jump intensity for TSE 300 in Case 1 is 0.5168 in four weeks. Namely, we should expect one jump in two months for TSE 300. In contrast, the jump intensity for the non-systematic jump is 1.0641 in four weeks which is about two times of the jump intensity for the systematic risks underlying the market. Since exchange rates are positively correlated with the non-systematic risk while
negatively correlated with the systematic risk, sometimes jumps occurred in the non-systematic risk are offset by jumps occurred in the systematic risk. Overall, we would only observe the joint effects from these two different jumps with less frequency. If we measure the jump intensity for exchange rates independently, we would expect a lower occurrence of jumps than that of non-systematic jumps. This is confirmed by the restricted estimation for exchange rates. In the restricted case, \( \gamma = 0 \), the parameters underlying exchange rate dynamics are estimated independently from the parameters for the market portfolio. For example, the jump intensity observed in the \( C$/US$ is 0.7588 in four weeks, which represents the aggregate occurrence of jumps in the exchange rates. Certainly, the jump intensity 0.7588 for \( C$/US$ in four weeks is less than 1.0641 in four weeks for the non-systematic risks. A similar feature is reflected in Case 2 and Case 3.

The estimation results for weekly data are presented in Table 7 for case 1, Table 8 for case 2 and Table 9 for case 3. The above three observations made based on the four-weekly data estimates are also valid in the evidence for weekly data. More importantly, the estimates for \( \gamma \) using weekly data are almost identical to those using four-weekly data for all three cases. Thus the dependence on the market portfolio for exchange rates is persistent regardless of the frequency of the sample data. The likelihood ratio tests also strongly reject the null hypothesis \( \gamma = 0 \) at 1 percent significant level. The rejection is even stronger than that for four-weekly data. This is not surprising at all since higher frequency data review more detailed fluctuations in the realizations for exchange rates and stock indices.

5. Comparison among the Currency Option Models and Implications

This section explores the important implication of systematic jump risks in exchange rates for currency option pricing. Bodurtha and Courtadon (1987) empirically test the performance of the
American Option valuation model to explain the pricing of currency options quoted on the Philadelphia Stock Exchange. The model appears to underprice short-term out-of-the-money options by as much as 29% in relative terms. At-the-money and in-the-money options are slightly overpriced, with the bias most pronounced for short-maturity options. As suggested by Bodurtha and Courtadon (1987), the directions of these pricing biases are generally consistent with a mixed jump-diffusion process. Inspired by this suggestion, Jorion (1988) assumes that exchange rates follow a mixed jump-diffusion process with only non-systematic jump risks and empirically estimates the parameters of the assumed process for US$/DM. He finds that Merton’s model can generate a higher price for the out-of-the-money call options compared with the prices given by the GK model. The relative price correction for the short-maturity out-of-the-money is about 17% which is below the empirically observed price bias 29%.7

In this section, I compare my model and Merton’s model with the GK model. Denote $C_{GK}$, $C_c$ and $C_m$ as the call prices given by the GK model, my model and Merton’s model. $C_{GK}$, $C_c$ and $C_m$ are expressed in (A.1), (A.2) and (A.4) in Appendix A. The parameters used in my model is the estimation of the unrestricted cases for the three exchange rates, while the parameters used in Merton’s model is the estimation of the restricted cases for the corresponding exchange rates. As for the GK model, the volatility of the exchange rates is the only unobserved parameter. An investor ignoring the jump component would estimate the total variance of a pure diffusion process as the total variance presented in equation (3.3). Namely, the total variance can be computed with the unrestricted estimation through $\text{Var}(y) = \sigma_y^2 + \lambda_y(\theta_y^2 + \phi_y^2) + \gamma^2\text{Var}(x)$ or with the restricted estimation through $\text{Var}(y) = \sigma_y^2 + \lambda_y(\theta_y^2 + \phi_y^2)$. The total variance computed by the unrestricted

7Strictly speaking, it is not appropriate to compare the relative price correction of Merton’s model based on the GK European currency model, since the price bias pattern documented by Bodurtha and Courtadon (1987) is for the American currency options. Jorion (1988) argues that the difference in the valuation of American and European option is small for most call options, especially in cases where the foreign interest rate is lower than the domestic interest rate. For further reference on this argument, see Shastri and Tandon (1986).
estimation is almost identical to that given by the restricted estimation. Therefore the volatility used in the GK model is 4.55% for C$/US$, 12.62% for US$/DM$ and 11.94% for C$/DM$. In order to directly relate my results to those in Jorion (1988), I follow Jorion (1988) and set domestic and foreign interest rates at 5% for all calculations in the analysis of comparison.

The extent of the mispricing can be measured by the relative difference \( \frac{C_c - C_{GK}}{C_c} \) and \( \frac{C_m - C_{GK}}{C_m} \), using the parameters previously estimated from 4-weekly data over the period 1980 to 1993. The analysis is performed for the three exchange rates under concern.

Tables 10 and 11 here.

Table 10 and Table 11 show relative and absolute mispricing errors for typical option parameters. The options are classified according to time to maturity and to the moneyness of the options. The ratio \( \frac{p}{K} \) measures the moneyness of call options. Out-of-the-money, at-the-money and in-the-money call options written on exchange rate US$/DM$ correspond to \( \frac{p}{K} = 0.95, 1.00, \) and \( 1.05 \).

To reflect the lower volatilities in exchange rates C$/US$ and C$/DM$, different measures of the out-of- and in-the-money options are used. In particular, for options written on C$/US$, the amount by which the options written on the exchange rate US$/DM$ (0.05) was scaled by the ratio of the two corresponding volatilities to yield \( 0.05 \times \frac{4.55}{12.62} = 0.018 \). Similarly, the adjusted measure of moneyness is \( 0.05 \times \frac{11.94}{12.62} = 0.0473 \) for option written on the exchange rate C$/DM$.\footnote{I thank Alan White for suggesting the adjusted measure of moneyness for exchange rates C$/S$ and C$/DM$ in order to account for the lower volatilities.} Using the estimated parameters, the GK model underprices short-maturity out-of-the-money currency options by about 16% for US$/DM$ and 19% for C$/DM$ in relation to Merton's model, 28% for US$/DM$ and 26% for C$/DM$ in relation to my model. The price corrections made by these two models can be explained by the proportion of the variance caused by the jump risks with respect
to the total variance. Table 12 summarizes the relevant ratios for the three exchange rates under concern.

Table 12 is here.

Under the restricted case $\gamma = 0$, the fraction of the variance caused by all jump risks in relation to the total variances is $R(1) = \frac{\lambda_y(\sigma_y^2+\sigma_z^2)}{\sigma_y^2+\lambda_y(\sigma_y^2+\sigma_z^2)}$. This ratio is 10.1746% for C$/US$, 17.722% for US$/DM, and 17.7219% for C$/DM$. As a result, compared with Merton’s model, the GK pure diffusion model underestimates the likelihood of a jump that would bring one of the out-of-the-money short-lived call options into the money.

The price correction generated by my model can also be explained by a similar argument. Let us first examine the ratio $R(1)$, which measures the fraction of the variance caused by both the non-systematic and systematic jump risks. This ratio is 41.8508% for C$/US$, 45.2741% for US$/DM, and 33.3682 for C$/DM$. These ratios are much higher than those for Merton’s case. Therefore a higher price correction by my model should be expected. Then we decompose the price correction into two parts: the first part is contributed by the systematic jump risks, the second by the non-systematic jump risks. The fraction of jump variance contributed by the systematic jump risks is $R(2) = \frac{\gamma^2 \lambda_x(\sigma_y^2+\sigma_z^2)}{\lambda_y(\sigma_y^2+\sigma_z^2)+\gamma^2 \lambda_x(\sigma_y^2+\sigma_z^2)}$. The fraction of jump variance caused by the non-systematic jump risks is $R(3) = \frac{\lambda_y(\sigma_y^2+\sigma_z^2)}{\lambda_y(\sigma_y^2+\sigma_z^2)+\gamma^2 \lambda_x(\sigma_y^2+\sigma_z^2)}$. The magnitude of $R(2)$ for the three exchange rates is above 35%. These high ratios indicate that the price correction contributed by the systematic risk cannot be ignored.

Table 13 here.

To examine further the important of systematic jump risks for option pricing, I compare the implied volatility pattern generated by the unrestricted model and the restricted model. The
implied volatility is calculated through the GK formula using option prices given by (A.2) and (A.4). Table 13 shows the result and Figure 1 depicts the implied volatility "smile" for both models. It is obvious that the implied volatility pattern generated by the unrestricted model is much clearer than that by the restricted model.

6. Concluding Remarks

This paper estimates an equilibrium model that I have developed in the second essay to study currency option pricing with systematic jump risks. Based on the equilibrium conditions imposed on the joint distribution of exchange rates and the price of the market portfolio in this previous model, I empirically investigate the correlation and the existence of discontinuities in the exchange market and in the stock market. The maximum-likelihood method is used for the estimation of the underlying parameters. Likelihood tests strongly reject the null hypothesis that there is no systematic jump risk in exchange rates. Further, I compare my currency model with Merton's model and the GK model. Numerical exercises show that my model can perform better than the GK pure diffusion model and than Merton's non-systematic jump-diffusion model. For short-maturity call options written on the three exchange rates (C$/US$, US$/DM and C$/DM), my model provides a 28% upward correction on the price generated by the GK model, a magnitude close to the price bias (29%) suggested by evidence. In contrast, Merton's model provides only 16% upward price correction on the GK model. The 12% difference in price bias correction shows that systematic jump risks are important for currency option pricing. Thus, for a currency option pricing model to correctly reflect the observed price pattern, it is important to incorporate systematic jump risk as well as non-systematic jump risk in exchange rates.

For the estimation of parameters, I have omitted the correlation between the continuous diffu-
sions $dz_x$ and $dz_y$ in exchange rates. This reflects the emphasis of the current study on systematic jump risks. Such correlation does not alter the currency option model as shown in the second essay.

The estimation has also omitted the estimation error in parameters that are caused by truncation. Such an approximation procedure is common in the literature and it is not clear how much the estimation results can be improved by taking into account of the estimation error associated with the truncation. It is likely that incorporating these additional elements would not change the central message of the current paper that systematic jump risks must be reflected in the currency option prices.
Appendices

A. Equilibrium Results under a more General Utility Function

This appendix provides equilibrium results where the utility function takes the form

\[ U(c_t, m_t, t) = e^{-\rho t}[\alpha \frac{c_t^{\gamma+1}}{\gamma} + (1 - \alpha) \ln m_t]. \]

(1) The domestic interest rate: \( r_d = \rho + \mu_m - \sigma_m^2 - \lambda k_m - \lambda \bar{k}_m. \)

(2) Equilibrium exchange rate: \( p_t = \frac{\sigma_d}{\lambda - \rho} M_t c_t^\gamma. \) Note that if \( \gamma = 0, \) the agent is risk-neutral over the consumption. In this case, there is no systematic jump in exchange rates.

(3) The foreign interest rate: \( r_f = \rho - A(\gamma) \) where

\[ A(\gamma) = \gamma(\mu_\delta - \lambda_\delta k_\delta) + \frac{\gamma(\gamma - 1)}{2} \sigma_\delta^2 + \lambda_\delta \exp(\gamma \theta_\delta + \frac{\gamma^2 \phi_\delta^2}{2} - 1). \]

(4) The domestic risky stock price: \( S = \frac{\delta}{\rho - A(\gamma+1)}. \)

(5) Equilibrium quantity of foreign bonds: \( f_t = f_0 e^{A(\gamma+1)-A(\gamma)t}. \)

(6) The market portfolio is \( X = S + f = c/r_f. \)

(7) The European currency option price formula:

To facilitate the presentation of equilibrium prices of call and put options, let \( C_{GK} \) and \( P_{GK} \) be, respectively, the European style currency call and put prices derived by GK (1983) with the following expressions

\[ C_{GK}(p_t, \tau; K, r_f, r_d, \sigma) = p_t e^{-r_f \tau} N(d_1) - K e^{-r_d \tau} N(d_2) \tag{A.1} \]
\[ P_{GK}(p_t, \tau; K, r_f, r_d, \sigma) = K e^{-r_d \tau} N(-d_2) - p_t e^{-r_f \tau} N(-d_1) \]
where $T$ is the time to maturity, $K$ is the exercise price, $r_f$ is the foreign interest rate, $r_d$ is the domestic interest rate, $\sigma$ is the volatility of the exchange rate and

$$
\tau = T - t, \quad d_1 = \frac{\ln p_t/K + (r_d - r_f + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}.
$$

Then, the option prices in my model are described as follows

$$
C_c(p_t, T, K) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\lambda_f \tau} (\lambda_f \tau)^i}{i!} \frac{e^{-\lambda_m \tau} (\lambda_m \tau)^j}{j!} C_{GK}(p_t, T; K, r_f, r_d, \sigma_{i,j}), \quad (A.2)
$$

and

$$
P_c(p_t, T, K) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\lambda_f \tau} (\lambda_f \tau)^i}{i!} \frac{e^{-\lambda_m \tau} (\lambda_m \tau)^j}{j!} P_{GK}(p_t, T; K, r_f, r_d, \sigma_{i,j}), \quad (A.3)
$$

with

$$
r_f' = r_f + \lambda_f \delta \tau (\gamma) - \frac{i(\gamma_{\delta} + \frac{1}{2}\gamma^2 \phi_m^2)}{\tau}, \quad r_d' = r_d + \lambda_m \delta \tau - \frac{j(-\theta_m + \frac{1}{2}\phi_m^2)}{\tau}
$$

$$
\sigma_{i,j} = \left(\frac{\sigma_m^2}{\tau} + \frac{j\phi_m^2}{\tau} + 2\gamma\rho(\frac{\sigma_m^2}{\tau} + \frac{\phi_m^2}{\tau}) + \gamma^2 (\frac{\sigma_m^2}{\tau} + \frac{\phi_m^2}{\tau})\right)^{\frac{1}{2}}.
$$

8. When $\gamma = 0$, call and put prices in (A.2) and (A.3) reduce to

$$
C_m(p_t, T, K) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_m \tau} (\lambda_m \tau)^j}{j!} C_{GK}(p_t, T; K, r_f, r_d, \sigma_j), \quad (A.4)
$$

and

$$
P_m(p_t, T, K) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_m \tau} (\lambda_m \tau)^j}{j!} P_{GK}(p_t, T; K, r_f, r_d, \sigma_j), \quad (A.5)
$$

with $\sigma_j = \sqrt{\sigma_m^2 + \frac{j\phi_m^2}{\tau}}$.

Call and put prices in (A.4) and (A.5) correspond to Merton's mixed jump-diffusion model.
### Table 1


<table>
<thead>
<tr>
<th></th>
<th>$C$/US$ Rate</th>
<th>TSE 300 Index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4-weekly</td>
<td>Weekly</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000525</td>
<td>0.000125</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.012656</td>
<td>0.006455</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.471926</td>
<td>0.233761</td>
</tr>
<tr>
<td></td>
<td>(2.504616)</td>
<td>(2.483076)</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>1.099798</td>
<td>5.839505*</td>
</tr>
<tr>
<td></td>
<td>(2.918439)</td>
<td>(31.014496)</td>
</tr>
<tr>
<td>Autocorrelation k-lags</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.125</td>
<td>-0.018</td>
</tr>
<tr>
<td>2</td>
<td>0.031</td>
<td>0.016</td>
</tr>
<tr>
<td>3</td>
<td>0.003</td>
<td>-0.056</td>
</tr>
<tr>
<td>4</td>
<td>0.005</td>
<td>0.011</td>
</tr>
<tr>
<td>5</td>
<td>-0.011</td>
<td>-0.050</td>
</tr>
<tr>
<td>6</td>
<td>-0.036</td>
<td>-0.014</td>
</tr>
<tr>
<td>7</td>
<td>0.016</td>
<td>0.002</td>
</tr>
<tr>
<td>8</td>
<td>0.102</td>
<td>-0.022</td>
</tr>
<tr>
<td>9</td>
<td>0.147</td>
<td>-0.044</td>
</tr>
<tr>
<td>10</td>
<td>0.005</td>
<td>-0.001</td>
</tr>
<tr>
<td>11</td>
<td>0.180</td>
<td>0.015</td>
</tr>
<tr>
<td>12</td>
<td>0.018</td>
<td>0.038</td>
</tr>
<tr>
<td>13</td>
<td>-0.089</td>
<td>0.029</td>
</tr>
<tr>
<td>Standard Error (k)</td>
<td>0.085</td>
<td>0.030</td>
</tr>
<tr>
<td>Box-Pierce (P-value)</td>
<td>15.390</td>
<td>7.719</td>
</tr>
<tr>
<td></td>
<td>[0.284]</td>
<td>[0.861]</td>
</tr>
<tr>
<td>Number of Observations</td>
<td>169</td>
<td>677</td>
</tr>
</tbody>
</table>

Asymptotic z-statistics in parentheses and marginal significance level in brackets.
Under normality assumption, the skewness and excess kurtosis should be zero.
* significant at the 1 percent level.
Table 2


<table>
<thead>
<tr>
<th></th>
<th>US$/DM Rate</th>
<th>CRSP Index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4-weekly Weekly</td>
<td>4-weekly Weekly</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000364 0.000090</td>
<td>0.010719 0.002698</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.035108 0.017555</td>
<td>0.057283 0.026414</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.075903 0.130237</td>
<td>-1.041804 -0.668301</td>
</tr>
<tr>
<td>(k)</td>
<td>(-0.402834 1.383417)</td>
<td>(-5.529092 -7.098891)</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>-0.205460 1.421982*</td>
<td>3.995053* 3.069967*</td>
</tr>
<tr>
<td>(k)</td>
<td>(-0.545212 7.552361)</td>
<td>(10.601328 16.305062)</td>
</tr>
</tbody>
</table>

Autocorrelation k-lags

| 1 | 0.054 | 0.047 | 0.174 | 0.073 |
| 2 | 0.070 | -0.035 | 0.003 | 0.031 |
| 3 | 0.085 | 0.032 | -0.069 | 0.110 |
| 4 | 0.040 | 0.034 | 0.005 | 0.036 |
| 5 | -0.081 | 0.004 | -0.079 | -0.006 |
| 6 | 0.051 | -0.034 | -0.035 | 0.059 |
| 7 | -0.035 | 0.002 | 0.024 | 0.020 |
| 8 | 0.099 | 0.070 | -0.104 | -0.029 |
| 9 | -0.042 | -0.019 | -0.074 | 0.038 |
| 10 | 0.037 | 0.031 | 0.021 | -0.013 |
| 11 | 0.143 | 0.046 | 0.061 | -0.023 |
| 12 | -0.054 | -0.025 | -0.105 | -0.054 |
| 13 | 0.049 | 0.025 | 0.003 | -0.001 |
| Standard Error (k)  | 0.066 | 0.034 | 0.078 | 0.046 |
| Box-Pierce (P-value) | 11.112 | 11.057 | 12.580 | 20.092 |
| [0.601] | [0.606] | [0.481] | [0.093] |
| Number of Observations | 169 | 677 | 169 | 677 |

Asymptotic z-statistics in parentheses and marginal significance level in brackets. Under normality assumption, the skewness and excess kurtosis should be zero. * significant at the 1 percent level.
Table 3
Summary of Statistics for C$/DM and TSE 300 for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th></th>
<th>C$/DM Rate</th>
<th>TSE 300 Index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4-weekly</td>
<td>Weekly</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000889</td>
<td>0.000215</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.033108</td>
<td>0.016800</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.135530</td>
<td>0.070432</td>
</tr>
<tr>
<td></td>
<td>(0.719289)</td>
<td>(0.748155)</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>-0.330592</td>
<td>0.908746</td>
</tr>
<tr>
<td></td>
<td>(-0.877264)</td>
<td>(4.826490)</td>
</tr>
<tr>
<td>Autocorrelation k-lags</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.092</td>
<td>0.061</td>
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<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>0.068</td>
<td>0.019</td>
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<td>4</td>
<td>0.113</td>
<td>0.034</td>
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<tr>
<td>5</td>
<td>-0.071</td>
<td>0.015</td>
</tr>
<tr>
<td>6</td>
<td>0.020</td>
<td>-0.026</td>
</tr>
<tr>
<td>7</td>
<td>0.051</td>
<td>0.035</td>
</tr>
<tr>
<td>8</td>
<td>0.009</td>
<td>0.029</td>
</tr>
<tr>
<td>9</td>
<td>-0.052</td>
<td>-0.014</td>
</tr>
<tr>
<td>10</td>
<td>0.028</td>
<td>0.028</td>
</tr>
<tr>
<td>11</td>
<td>0.141</td>
<td>0.025</td>
</tr>
<tr>
<td>12</td>
<td>-0.056</td>
<td>-0.030</td>
</tr>
<tr>
<td>13</td>
<td>0.068</td>
<td>0.028</td>
</tr>
<tr>
<td>Standard Error (k)</td>
<td>0.066</td>
<td>0.030</td>
</tr>
<tr>
<td>Box-Pierce (P-value)</td>
<td>11.580</td>
<td>8.763</td>
</tr>
<tr>
<td></td>
<td>[0.562]</td>
<td>[0.791]</td>
</tr>
<tr>
<td>Number of Observations</td>
<td>169</td>
<td>677</td>
</tr>
</tbody>
</table>

Asymptotic z-statistics in parentheses and marginal significance level in brackets. Under normality assumption, the skewness and excess kurtosis should be zero. * significant at the 1 percent level.
Table 4
Joint Stochastic Process for C$/US$ and TSE 300 Index
Four-Weekly Data for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process Parameters</td>
<td>Process Parameters</td>
</tr>
<tr>
<td>$\alpha_x(\times 10^2)$</td>
<td>0.5168 (1.3059)</td>
</tr>
<tr>
<td>$\sigma_x^2(\times 10^4)$</td>
<td>14.6303* (19.6332)</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.5011* (3.5248)</td>
</tr>
<tr>
<td>$\theta_x(\times 10^2)$</td>
<td>-0.1756 (-0.2081)</td>
</tr>
<tr>
<td>$\phi_x^2(\times 10^4)$</td>
<td>35.9046* (25.2547)</td>
</tr>
<tr>
<td>$\sigma_y(\times 10^2)$</td>
<td>0.2708* (3.4162)</td>
</tr>
<tr>
<td>$\sigma_y^2(\times 10^4)$</td>
<td>0.6023* (296.7204)</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>1.0641* (4.0724)</td>
</tr>
<tr>
<td>$\theta_y(\times 10^2)$</td>
<td>0.0088 (1.1760)</td>
</tr>
<tr>
<td>$\phi_y^2(\times 10^4)$</td>
<td>0.4556* (218.9179)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.1249* (56.6262)</td>
</tr>
</tbody>
</table>

Log-likelihood Hypothesis Test
$-2 \ln \Lambda \sim \chi^2_1$

<table>
<thead>
<tr>
<th></th>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>776.9795</td>
<td>755.7843</td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>776.9795</td>
<td>755.7843</td>
</tr>
<tr>
<td>Hypothesis Test</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2 \ln \Lambda \sim \chi^2_1$</td>
<td>42.3904*</td>
<td>[0.0000]</td>
</tr>
</tbody>
</table>

The mixed jump-diffusion process for the stock index:
\[
d\ln X = \alpha_x dt + \sigma_x dz_x + \ln Y_x dQ_x.
\]
The jump intensity for the stock index is $\lambda_x$, jump size $\ln Y_x \sim N(\theta_x, \phi_x^2)$.
The mixed jump-diffusion process for the exchange rate consists the risks from the money supply and the market:
\[
d\ln p = (\alpha_y + \gamma \alpha_x) dt + \sigma_y dz_y + \gamma \sigma_x dz_x + \ln Y_y dQ_y + \gamma \ln Y_x dQ_x.
\]
The jump intensity inherented from the money supply is $\lambda_y$, and jump size $\ln Y_y \sim N(\theta_y, \phi_y^2)$.
Asymptotic $t$-statistics in parentheses and marginal significance levels in brackets.
The 1 percent critical levels for $\chi^2_1$ with 1 degree of freedom is 6.6. This statistics tests the hypothesis of restricted case against unrestricted case. The degree of freedom corresponds to the additional number of parameters in the alternative hypothesis. Number of observation is 169 for 4-weekly data.
* significant at the 1 percent level.
Table 5
Joint Stochastic Process for US$/DM and CRSP Index
Four-Weekly Data for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th>Process Parameters</th>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_x (x10^2)$</td>
<td>1.0062*</td>
<td>(2.3740)</td>
</tr>
<tr>
<td>$\sigma_x^2 (x10^4)$</td>
<td>19.5141*</td>
<td>(17.5245)</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.5781*</td>
<td>(2.9368)</td>
</tr>
<tr>
<td>$\theta_x (x10^2)$</td>
<td>0.5089</td>
<td>(0.7088)</td>
</tr>
<tr>
<td>$\phi_x^2 (x10^4)$</td>
<td>22.2065*</td>
<td>(11.9088)</td>
</tr>
<tr>
<td>$\alpha_y (x10^2)$</td>
<td>0.4902*</td>
<td>(2.4950)</td>
</tr>
<tr>
<td>$\sigma_y^2 (x10^4)$</td>
<td>3.4057*</td>
<td>(56.1524)</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>1.0954*</td>
<td>(4.4022)</td>
</tr>
<tr>
<td>$\theta_y (x10^2)$</td>
<td>0.0182</td>
<td>(0.1029)</td>
</tr>
<tr>
<td>$\phi_y^2 (x10^4)$</td>
<td>3.0755*</td>
<td>(69.4689)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.4181*</td>
<td>(15.7860)</td>
</tr>
</tbody>
</table>

Log-likelihood                              616.9087  575.8050
Hypothesis Test                             
$-2 \ln \Lambda \sim \chi^2_1$              82.2074* [0.0000]

The mixed jump-diffusion process for the stock index:
$$d\ln X = \alpha_x dt + \sigma_x dz_x + \ln Y_x dQ_x.$$  

The jump intensity for the stock index is $\lambda_x$, jump size $\ln Y_x \sim N(\theta_x, \phi_x^2)$.  

The mixed jump-diffusion process for the exchange rate consists the risks from the money supply and the market:
$$d\ln p = (\alpha_y + \gamma \alpha_z)dt + \sigma_y dz_y + \gamma \sigma_x dz_x + \ln Y_y dQ_y + \gamma \ln Y_x dQ_x.$$  

The jump intensity inherented from the money supply is $\lambda_y$, and jump size $\ln Y_y \sim N(\theta_y, \phi_y^2)$.  

Asymptotic $t$-statistics in parentheses and marginal significance levels in brackets.  

The 1 percent critical levels for $\chi^2_1$ with 1 degree of freedom is 6.6. This statistics tests the hypothesis of restricted case against unrestricted case. The degree of freedom corresponds to the additional number of parameters in the alternative hypothesis. Number of observation is 169 for 4-weekly data.  

* significant at the 1 percent level.  

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### Table 6

Joint Stochastic Process for C$/DM and TSE 300 Index
Four-Weekly Data for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th>Unrestricted Case Process Parameters</th>
<th>Restriction: $\gamma = 0$ Process Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_x (\times 10^2)$</td>
<td>0.5396</td>
</tr>
<tr>
<td>$\sigma_x^2 (\times 10^4)$</td>
<td>21.5838*</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.5855*</td>
</tr>
<tr>
<td>$\theta_x (\times 10^2)$</td>
<td>-0.1424</td>
</tr>
<tr>
<td>$\phi_x^2 (\times 10^4)$</td>
<td>24.3767*</td>
</tr>
<tr>
<td>$\alpha_y (\times 10^2)$</td>
<td>0.2043</td>
</tr>
<tr>
<td>$\sigma_y^2 (\times 10^4)$</td>
<td>4.421*</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>1.1337*</td>
</tr>
<tr>
<td>$\theta_y (\times 10^2)$</td>
<td>0.0074</td>
</tr>
<tr>
<td>$\phi_y^2 (\times 10^4)$</td>
<td>2.6754*</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.3131*</td>
</tr>
</tbody>
</table>

Log-likelihood: 603.9058
Hypothesis Test

$-2 \ln \Lambda \sim \chi^2_1$

$-2 \ln \Lambda \sim \chi^2_1$

\[ 60.9266^* \]

\[ 0.0000 \]

The mixed jump-diffusion process for the stock index:

$$d \ln X = \alpha_x dt + \sigma_x dz_x + \ln Y_x dQ_x.$$  

The jump intensity for the stock index is $\lambda_x$, jump size $\ln Y_x \sim N(\theta_x, \phi_x^2)$.  

The mixed jump-diffusion process for the exchange rate consists of the risks from the money supply and the market:

$$d \ln p = (\alpha_y + \gamma \alpha_x) dt + \sigma_y dz_y + \gamma \sigma_x dz_x + \ln Y_y dQ_y + \gamma \ln Y_x dQ_x.$$  

The jump intensity inherited from the money supply is $\lambda_y$, and jump size $\ln Y_y \sim N(\theta_y, \phi_y^2)$.  

Asymptotic $t$-statistics in parentheses and marginal significance levels in brackets.  

The 1 percent critical levels for $\chi^2_1$ with 1 degree of freedom is 6.6. This statistics tests the hypothesis of restricted case against unrestricted case. The degree of freedom corresponds to the additional number of parameters in the alternative hypothesis. Number of observation is 169 for 4-weekly data.  

* significant at the 1 percent level.
Table 7
Joint Stochastic Process for C$/US$ and TSE 300 Index
Weekly Data for the Period January 1980 - May 1993

<table>
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<tr>
<th>Process Parameters</th>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
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<tbody>
<tr>
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<td>(1.3059)</td>
</tr>
<tr>
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<td>(-0.2081)</td>
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<td>(25.2547)</td>
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<td>(4.0724)</td>
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<td>(1.1760)</td>
</tr>
<tr>
<td>$\phi_y^2 (\times 10^4)$</td>
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<td>(218.9179)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.1249*</td>
<td>(56.6262)</td>
</tr>
</tbody>
</table>

Log-likelihood Hypothesis Test
$-2 \ln \Lambda \sim \chi_1^2$

$776.9795 \quad 755.7843$

$42.3904^* \quad [0.0000]$
### Table 8

Joint Stochastic Process for US$/DM and CRSP Index
Weekly Data for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_x (\times 10^2)$</td>
<td>1.0062*</td>
<td>(2.3740)</td>
</tr>
<tr>
<td>$\sigma^2_x (\times 10^4)$</td>
<td>19.5141*</td>
<td>(17.5245)</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.5781*</td>
<td>(2.9368)</td>
</tr>
<tr>
<td>$\theta_x (\times 10^2)$</td>
<td>0.5089</td>
<td>(0.7088)</td>
</tr>
<tr>
<td>$\phi^2_x (\times 10^4)$</td>
<td>22.2065*</td>
<td>(11.9088)</td>
</tr>
<tr>
<td>$\alpha_y (\times 10^2)$</td>
<td>0.4902*</td>
<td>(2.4950)</td>
</tr>
<tr>
<td>$\sigma^2_y (\times 10^4)$</td>
<td>3.4057*</td>
<td>(56.1524)</td>
</tr>
<tr>
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<td>(4.4022)</td>
</tr>
<tr>
<td>$\theta_y (\times 10^2)$</td>
<td>0.0182</td>
<td>(0.1029)</td>
</tr>
<tr>
<td>$\phi^2_y (\times 10^4)$</td>
<td>3.0755*</td>
<td>(69.4689)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.4181*</td>
<td>(15.7860)</td>
</tr>
</tbody>
</table>

Log-likelihood 616.9087

Hypothesis Test

$-2 \ln \Lambda \sim \chi^2_1$

575.8050

82.2074* [0.0000]

The mixed jump-diffusion process for the stock index:

$$d\ln X = \alpha_x dt + \sigma_x d\mu_x + \ln Y_x dQ_x.$$  

The jump intensity for the stock index is $\lambda_x$, jump size $\ln Y_x \sim N(\theta_x, \phi^2_x)$.  

The mixed jump-diffusion process for the exchange rate consists the risks from the money supply and the market:

$$d\ln p = (\alpha_y + \gamma \alpha_x) dt + \sigma_y d\mu_y + \gamma \sigma_x d\mu_x + \ln Y_y dQ_y + \gamma \ln Y_x dQ_x.$$  

The jump intensity inherented from the money supply is $\lambda_y$, and jump size $\ln Y_y \sim N(\theta_y, \phi^2_y)$.  

Asymptotic t-statistics in parentheses and marginal significance levels in brackets.  

The 1 percent critical levels for $\chi^2_1$ with 1 degree of freedom is 6.6. This statistics tests the hypothesis of restricted case against unrestricted case. The degree of freedom corresponds to the additional number of parameters in the alternative hypothesis. Number of observation is 169 for 4-weekly data.

* significant at the 1 percent level.
Table 9
Joint Stochastic Process for C$/DM and TSE 300 Index
Weekly Data for the Period January 1980 - May 1993

<table>
<thead>
<tr>
<th>Process Parameters</th>
<th>Unrestricted Case</th>
<th>Restriction: $\gamma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_x (\times 10^2)$</td>
<td>0.5396</td>
<td>(1.2064)</td>
</tr>
<tr>
<td>$\sigma_x^2 (\times 10^4)$</td>
<td>21.5838*</td>
<td>(11.4849)</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.5855*</td>
<td>(3.0663)</td>
</tr>
<tr>
<td>$\theta_x (\times 10^2)$</td>
<td>-0.1424</td>
<td>(-0.1921)</td>
</tr>
<tr>
<td>$\phi_x^2 (\times 10^4)$</td>
<td>24.3767*</td>
<td>(9.0096)</td>
</tr>
<tr>
<td>$\alpha_y (\times 10^2)$</td>
<td>0.2043</td>
<td>(0.9797)</td>
</tr>
<tr>
<td>$\sigma_y^2 (\times 10^4)$</td>
<td>4.421*</td>
<td>(83.8950)</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>1.1337*</td>
<td>(3.5851)</td>
</tr>
<tr>
<td>$\theta_y (\times 10^2)$</td>
<td>0.0074</td>
<td>(0.0407)</td>
</tr>
<tr>
<td>$\phi_y^2 (\times 10^4)$</td>
<td>2.6754*</td>
<td>(20.5689)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-0.3131*</td>
<td>(19.1384)</td>
</tr>
</tbody>
</table>

Log-likelihood: 603.9058
Hypothesis Test
$-2 \ln \Lambda \sim \chi^2_1$

The mixed jump-diffusion process for the stock index:
$$d\ln X = \alpha_x dt + \sigma_x dZ_x + \ln Y_x dQ_x.$$  

The jump intensity for the stock index is $\lambda_x$, jump size $\ln Y_x \sim N(\theta_x, \phi_x^2)$.  

The mixed jump-diffusion process for the exchange rate consists the risks from the money supply and the market:
$$d\ln p = (\alpha_y + \gamma \alpha_x) dt + \sigma_y dZ_y + \gamma \sigma_x dZ_x + \ln Y_p dQ_y + \gamma \ln Y_x dQ_x.$$  

The jump intensity inherented from the money supply is $\lambda_y$, and jump size $\ln Y_y \sim N(\theta_y, \phi_y^2)$.  

Asymptotic $t$-statistics in parentheses and marginal significance levels in brackets.  

The 1 percent critical levels for $\chi^2_1$ with 1 degree of freedom is 6.6. This statistics tests the hypothesis of restricted case against unrestricted case. The degree of freedom corresponds to the additional number of parameters in the alternative hypothesis. Number of observation is 169 for 4-weekly data.  

* significant at the 1 percent level.
### Table 10

**Hypothetical Options: Relative Pricing Correction in Percent**

**Option Parameters:** $K = 100$, $R_d = 5\%$, $R_f = 5\%$

<table>
<thead>
<tr>
<th></th>
<th>$t = 0.5$ month</th>
<th>$t = 1$ month</th>
<th>$t = 3$ months</th>
<th>$t = 6$ months</th>
<th>$t = 9$ months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Percentage</td>
<td>Percentage</td>
<td>Percentage</td>
<td>Percentage</td>
<td>Percentage</td>
</tr>
<tr>
<td></td>
<td>Correction</td>
<td>Correction</td>
<td>Correction</td>
<td>Correction</td>
<td>Correction</td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.982$</td>
<td>28.0447%</td>
<td>4.7898%</td>
<td>-0.1041%</td>
<td>-0.1509%</td>
<td>-0.1074%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-2.1388%</td>
<td>-1.1406%</td>
<td>-0.3872%</td>
<td>-0.1886%</td>
<td>-0.0018%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.018$</td>
<td>0.1626%</td>
<td>0.0769%</td>
<td>-0.0734%</td>
<td>-0.0791%</td>
<td>-0.0667%</td>
</tr>
<tr>
<td><strong>Merton's Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.982$</td>
<td>16.0848%</td>
<td>2.2255%</td>
<td>0.4715%</td>
<td>0.2716%</td>
<td>0.2188%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-0.1810%</td>
<td>-0.0450%</td>
<td>0.0530%</td>
<td>0.0785%</td>
<td>0.0826%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.018$</td>
<td>0.0066%</td>
<td>-0.0113%</td>
<td>-0.0082%</td>
<td>0.0145%</td>
<td>0.0291%</td>
</tr>
<tr>
<td><strong>Exchange Rate: C$/US$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.95$</td>
<td>28.2813%</td>
<td>2.9886%</td>
<td>-0.9139%</td>
<td>-0.5115%</td>
<td>-0.3308%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-2.7819%</td>
<td>-1.4410%</td>
<td>-0.4805%</td>
<td>-0.2355%</td>
<td>-0.1537%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.05$</td>
<td>0.2672%</td>
<td>0.2003%</td>
<td>0.0089%</td>
<td>-0.0260%</td>
<td>-0.0273%</td>
</tr>
<tr>
<td><strong>Merton's Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.95$</td>
<td>15.7681%</td>
<td>2.2530%</td>
<td>-0.0325%</td>
<td>-0.0508%</td>
<td>-0.0312%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-0.9031%</td>
<td>-0.4790%</td>
<td>-0.1564%</td>
<td>-0.0703%</td>
<td>-0.0411%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.05$</td>
<td>0.0788%</td>
<td>0.0355%</td>
<td>-0.0268%</td>
<td>-0.0274%</td>
<td>-0.0212%</td>
</tr>
<tr>
<td><strong>Exchange Rate: US$/DM</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.9527$</td>
<td>29.5069%</td>
<td>5.8054%</td>
<td>0.5854%</td>
<td>0.4379%</td>
<td>0.4340%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-0.1661%</td>
<td>-0.9346%</td>
<td>-0.4056%</td>
<td>-0.2687%</td>
<td>-0.2227%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.0473$</td>
<td>0.1911%</td>
<td>0.1346%</td>
<td>0.0535%</td>
<td>0.0914%</td>
<td>0.1263%</td>
</tr>
<tr>
<td><strong>Merton's Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: $S/K = 0.9527$</td>
<td>15.6739%</td>
<td>2.3110%</td>
<td>0.0262%</td>
<td>-0.0034%</td>
<td>0.0112%</td>
</tr>
<tr>
<td>At-the-money: $S/K = 1.00$</td>
<td>-0.8698%</td>
<td>-0.4494%</td>
<td>-0.1301%</td>
<td>-0.0449%</td>
<td>-0.0160%</td>
</tr>
<tr>
<td>In-of-the-money: $S/K = 1.0473$</td>
<td>0.0768%</td>
<td>0.0382%</td>
<td>-0.0183%</td>
<td>-0.1540%</td>
<td>-0.0075%</td>
</tr>
<tr>
<td><strong>Exchange Rate: C$/DM</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Stochastic process parameters are taken from Tables 2, 5 and 6 for exchange rates C$/US$, US$/DM and C$/D$ respectively. Percentage correction is computed as $(C_c - G_K)/C_c$ and $(C_m - G_K)/C_m$.  

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### Hypothetical Options: Absolute Pricing Correction in Cents

**Option Parameters:** K = 100, Rd=5%, Rf=5%

<table>
<thead>
<tr>
<th></th>
<th>t = 0.5 month</th>
<th>t = 1 month</th>
<th>t = 3 months</th>
<th>t = 6 months</th>
<th>t = 9 months</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Absolute Correction</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.982</td>
<td>0.0034</td>
<td>0.0025</td>
<td>-0.0003</td>
<td>-0.0008</td>
<td>-0.0008</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0077</td>
<td>-0.0059</td>
<td>-0.0035</td>
<td>-0.0024</td>
<td>-0.0018</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.018</td>
<td>0.0029</td>
<td>0.0014</td>
<td>-0.0015</td>
<td>-0.0018</td>
<td>-0.0017</td>
</tr>
<tr>
<td><strong>Merton’s Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.982</td>
<td>0.0019</td>
<td>0.0011</td>
<td>0.0013</td>
<td>0.0015</td>
<td>0.0017</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0007</td>
<td>-0.0002</td>
<td>0.0005</td>
<td>0.0010</td>
<td>0.0013</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.018</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>-0.0002</td>
<td>0.0003</td>
<td>0.0007</td>
</tr>
<tr>
<td><strong>Exchange Rate: C$/US$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.95</td>
<td>0.0086</td>
<td>0.0039</td>
<td>-0.0065</td>
<td>-0.0076</td>
<td>-0.0070</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0278</td>
<td>-0.0206</td>
<td>-0.0119</td>
<td>-0.0082</td>
<td>-0.0064</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.05</td>
<td>0.0134</td>
<td>0.0103</td>
<td>0.0050</td>
<td>-0.0017</td>
<td>-0.0019</td>
</tr>
<tr>
<td><strong>Merton’s Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.95</td>
<td>0.0041</td>
<td>0.0029</td>
<td>-0.0002</td>
<td>-0.0008</td>
<td>-0.0007</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0092</td>
<td>-0.0069</td>
<td>-0.0039</td>
<td>-0.0024</td>
<td>-0.0017</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.05</td>
<td>0.0039</td>
<td>0.0018</td>
<td>-0.0015</td>
<td>-0.0018</td>
<td>-0.0015</td>
</tr>
<tr>
<td><strong>Exchange Rate: US$/DM</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>My Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.9527</td>
<td>0.0087</td>
<td>0.0075</td>
<td>0.0040</td>
<td>0.0062</td>
<td>0.0088</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0177</td>
<td>-0.0109</td>
<td>-0.0011</td>
<td>0.0049</td>
<td>0.0085</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.0473</td>
<td>0.0091</td>
<td>0.0065</td>
<td>-0.0029</td>
<td>0.0056</td>
<td>0.0085</td>
</tr>
<tr>
<td><strong>Merton’s Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out-of-the-money: S/K = 0.9527</td>
<td>0.0039</td>
<td>0.0029</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0002</td>
</tr>
<tr>
<td>At-the-money: S/K = 1.00</td>
<td>-0.0084</td>
<td>-0.0061</td>
<td>-0.0031</td>
<td>-0.0015</td>
<td>-0.0006</td>
</tr>
<tr>
<td>In-of-the-money: S/K = 1.0473</td>
<td>0.0037</td>
<td>0.0019</td>
<td>-0.0010</td>
<td>-0.0010</td>
<td>-0.0005</td>
</tr>
</tbody>
</table>

Stochastic process parameters are taken from Tables 2, 5 and 6 for exchange rates C$/US$, US$/DM and C$/D respectively. Absolute correction is computed as (Cc-GK) and (Cm-GK).
**Table 12**

<table>
<thead>
<tr>
<th></th>
<th>Relative Importance of Variance Caused by Jumps Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R(1) = \frac{\text{Var}(jump)}{\text{Var}(p)}$</td>
</tr>
<tr>
<td><strong>C$/US$</strong></td>
<td></td>
</tr>
<tr>
<td>My model</td>
<td>47.9724%</td>
</tr>
<tr>
<td>Merton's model</td>
<td>10.1746%</td>
</tr>
<tr>
<td><strong>US$/DM</strong></td>
<td></td>
</tr>
<tr>
<td>My model</td>
<td>45.2741%</td>
</tr>
<tr>
<td>Merton's model</td>
<td>17.7220%</td>
</tr>
<tr>
<td><strong>C$/DM$</strong></td>
<td></td>
</tr>
<tr>
<td>My model</td>
<td>40.4138%</td>
</tr>
<tr>
<td>Merton's model</td>
<td>17.7219%</td>
</tr>
</tbody>
</table>

Total variance of the exchange rate: $\text{Var}(p) = \sigma_y^2 + \lambda_y(\theta_y^2 + \phi_y^2) + \gamma^2(\sigma_x^2 + \lambda_x(\theta_x^2 + \phi_x^2))$

Total variance caused by all jump risks: $\text{Var}(\text{jump}) = \lambda_y(\theta_y^2 + \phi_y^2) + \gamma^2\lambda_x(\theta_x^2 + \phi_x^2)$

Variance caused by systematic jump risk: $\text{Var}(\text{sys. jump}) = \gamma^2\lambda_x(\theta_x^2 + \phi_x^2)$

Variance caused by non-systematic jump risk: $\text{Var}(\text{non-sys. jump}) = \lambda_y(\theta_y^2 + \phi_y^2)$
Table 13

Implied Volatility Calculated by GK Formula from the Call Prices Given by My Model and Merton's Model

Option Parameters: K = 100, Rd = 5%, Rf = 5%

<table>
<thead>
<tr>
<th>Exchange Rate: C$/US$</th>
<th>Exchange Rate: US$/DM</th>
<th>Exchange Rate: C$/DM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GK Call</td>
<td>My Model</td>
</tr>
<tr>
<td>S/K</td>
<td>Price</td>
<td>Call Park</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.80</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.85</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.90</td>
<td>0.0004</td>
<td>0.0008</td>
</tr>
<tr>
<td>0.95</td>
<td>0.0721</td>
<td>0.0736</td>
</tr>
<tr>
<td>1.00</td>
<td>1.2520</td>
<td>1.2497</td>
</tr>
<tr>
<td>1.05</td>
<td>4.9674</td>
<td>4.9680</td>
</tr>
<tr>
<td>1.10</td>
<td>9.7545</td>
<td>9.7548</td>
</tr>
<tr>
<td>1.25</td>
<td>24.3827</td>
<td>24.3827</td>
</tr>
</tbody>
</table>

Figure 1. Implied Volatility for Exchange Rate US$/DM
CHAPTER 5

CONCLUSIONS AND REFERENCES
Conclusions

This thesis uses an equilibrium framework to study derivative pricing both theoretically and empirically. The first essay focuses on options written on the market portfolio with predictable returns and stochastic volatility. The second essay studies currency options with endogenized systematic jump risks. The third essay investigates the empirical implications of systematic jump risks in currency option prices. As demonstrated in the thesis, the equilibrium approach is useful not only because it provides a consistent way of pricing derivative assets, but also because it offers guidance for empirical estimation.

Issues concerned in this thesis can still be studied at a more general level. For example, the model in the first essay can be extended into a multi-factor model to incorporate more realistic features of the real economy. The second essay can be extended to a two-country setting. Furthermore, the empirical implication of the theoretical model proposed in the first essay can be carried out as an interesting research in the future.
References


