CONTRIBUTIONS TO SEISMOGRAM MODELING BY
CLASSICAL MASLOV AND
NEW MASLOV-KIRCHHOFF METHODS

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
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To my mother, brother and sisters, 
and 
to the memory of my grandmother and father
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NEW MASLOV-KIRCHHOFF METHODS

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Department of Physics, University of Toronto

Abstract

Seismic modeling is important in exploring the earth’s structure. Although numerical integration of the wave equations is now beginning to be practical, approximate theories of wave propagation often are more useful. For instance, ray theory (GRT) has been widely used since seismology’s early days. Maslov (1965) developed an asymptotic theory that promises a modeling of many informative waveforms that GRT cannot model, while is nearly as efficient as GRT. In this thesis, a variety of contributions have been made to demonstrating and improving the accuracy, robustness, and automation of the Maslov technique, and to developing a more robust modeling tool.

I compared the Maslov integral (MI) and finite difference solutions. The results show that MI can accurately predict high-frequency waves, and also some diffracted/distorted low-frequency waves that GRT cannot give, provided a wavefront is sampled properly, and pseudocaustics, if any, can be eliminated by phase partitioning. Low-frequency inaccuracies can be roughly anticipated from a knowledge of the wavefront geometry.

The role of the weighting functions in blending GRT and MI solutions for a uniform seismogram was examined. A robust weighting scheme which invokes the trigonometric function was devised. It was shown stable and is therefore suitable for automation. The blending method should be used routinely, if caustics and pseudocaustics are separate.

Rotation of MI was shown useful for sampling wavefront anomalies such as bends, kinks and folds that cause waveform diffraction/distortion, in addition to its role known in re-
moving pseudocaustics. Its implementation can be easily performed by using a systematic method that I devised.

I proposed a new Maslov-Kirchhoff method. This method invokes Maslov theory to avoid caustics, and Kirchhoff theory to reduce pseudocaustic errors. It was shown more robust than Maslov theory, more accurate than ray-Kirchhoff theory or extended Kirchhoff-Helmholtz theory. It is applicable to many realistic, complicated modelings.

I also applied Maslov theory to Kirchhoff extrapolation and conducted two experiments. The results have shown that the wavefield can be reversely extrapolated accurately if the medium varies smoothly, and satisfactorily if the medium varies relatively rapidly.
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Chapter 1

General Introduction

1.1 Seismic modeling

Seismology (the study of wave motions following earthquakes or man-made disturbances) is the most important geophysical method of remotely sensing the interior structure of the earth (e.g., Howell 1990). It is applied at many scales, from planetary to locally over a few tens of meters. However, it is unlike optical vision through space or the atmosphere, because inhomogeneity of the propagating medium seriously distorts the seismic wavefields, and because relatively long wavelength waves have to be used to minimize absorption and random scattering processes. Thus, constructing a picture of earth structure from a set of observed seismograms can be a complex task. In certain cases, when a seismic data set is very complete and many aspects of the earth structure are already known approximately, it is possible to convert the data directly into a structural image, a process known as inversion or imaging. But very often, interpretation has to be based on seismic modeling (or, alternatively, computation of synthetic seismograms), a procedure whereby the physics of mechanical wave propagation is used to predict the wave motion that would be recorded at some given points (receiver point, RP) on the earth following a source disturbance at another point (source point, SP). It amounts to solving the elastic (viscoelastic, acoustic) wave equation somehow for the given source and receiver points in a given spatial velocity-density model; where the model, at least in some respects, has characteristics like those of the real earth. In all cases, the solution must provide a way of computing the time series of observable wave motion at the receiver point (seismogram), assuming a reasonable time-structure for the source motion. Since it is the aim of geophysicists to derive quantitative information about earth structure from the observation of seismic waves on the earth, it is obvious that convenient and reliable
seismic modeling methods are a necessary prerequisite. If forward modeling cannot be done adequately, believable inversion is impossible.

1.1.1 Wavefield solutions

Seismic modeling methods are necessarily based on solving the wave equation for models of the propagation medium that can satisfactorily resemble the earth. The mathematical tools may involve general, exact analytical solutions, asymptotic approximations, or numerical approximations. Because earth structure has arbitrary geometrical form (at least in detail), an exact solution to the wave equation is hard to find. Thus, numerical and asymptotic approximations are widely used in practice.

The most commonly used numerical method is the finite difference (FD) technique (e.g., Alterman & Karal 1968; Alford, Kelly & Boore 1974; Kelly et al. 1976 and Clayton & Engquist 1977; Strikwerda 1989). It offers two advantages: the underlying theory is simple and the computation of complete synthetic seismograms for a complex subsurface structure is robust. But, accurate results can only be obtained if the desired wave frequency components are lower than a limit set by the size of a practical mesh. Currently, FD methods are impractical for computing relatively high-frequency wave components in 3 dimensions because the computation would be impossible on practical computers, or at least unaffordably expensive. Besides, in many applications, we only want to focus on individual wave components, but FD predicts all parts of the seismic signals, and this may complicate seismic interpretation.

From seismology's earliest days, and long before electronic computation was available, the high-frequency approximate description of seismic body-wave propagation known as geometrical ray theory (GRT) has been widely used (also called zero-order asymptotic ray theory (ART), e.g., Červený & Ravindra 1971, Chapter 2). Despite advancing computer capacity, this theory has been and will remain very useful in the future, for the following reasons. (1) It involves simple mathematical approaches in which only the first term of the asymptotic ray series is considered, and simple physical concepts such as wavefronts that propagate along Fermat's stationary-phase paths with wave energy confined inside ray-tubes (Figure 1.1). (2) Many observational data can be reasonably and economically described using ray theory (Bullen & Bolt 1985; Dix 1952; Grant & West 1965). And (3), even if geometrical ray theory fails to describe certain non-geometrical (i.e., finite-frequency) signals such as those associated with (as partially sketched in Figure 1.2) caustics, shadows, critical points, head waves, interference head waves, edge, point and
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Figure 1.1: In geometrical ray theory, a ray tube is defined by neighboring rays (lines with arrows) that bend according to Snell's law. According to GRT, wave energy is simply transported without loss or gain along the ray tubes at the local phase velocity of the medium. Thus, the intensity (or amplitude) of the wavefield is inversely proportional to the square root of the cross-sectional area of the ray tube (shadowed region).

interface diffractions, gradient coupling, etc. (Chapman 1985), extensions can be made that remedy many of its problems while still retaining its basic simplicity.

However, non-geometrical signals are of great significance in seismology because they are a result of the earth's compositional and tectonic structures. Although specialized methods have been developed to treat certain ray problems, they often become inefficient in regions where different kinds of wavefield structure overlap. Thus, to obtain general methods that are valid for a majority of realistic situations while remaining simple to use continues to be an important aim of theoretical seismologists.

As computational methods improved in the 1970's, four important alternative methods of modeling body wave propagation in horizontally stratified models became important. Based on classical analytical solutions of linear partial differential equations in various coordinate systems, they are the "reflectivity method" (reviewed by Fuchs & Müller 1971), the "Cagniard-de Hoop-Pekeris method" (reviewed by Wiggins & Helmberger 1974), the "full-wave theory" (Richards 1973, 1976, Choy 1977, Cormier & Richards 1977, etc.), and the "WKBJ seismogram" (Chapman 1978; Dey-Sarkar & Chapman 1978; Garmany 1988). All these methods share a common approach (integral transformation of the wavefield), which reduces the partial differential equations of motion and constitution in a laterally homogeneous medium to one or more ordinary differential equations. But they differ either in the method and the order of the inverse transforms, or in the type
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Figure 1.2: Some problems with ray theory: (a) a caustic at which two nearby rays propagating through a low-velocity region cross and the ray-tube width vanishes, causing an infinite amplitude; (b) a shadow where nearby rays propagating through a high-velocity region disperse and where no wavefield intensity can be given.

of solution used in the transform domain. Also, they are most easily applied to layered structures. The Cagniard and WKBJ methods invoke the “slowness method” of evaluating the inverse transforms in which the inverse frequency transform is completed first with either a complex (Cagniard) or a real (WKBJ) wave number contour, but the former assumes homogeneous layers whereas the latter can treat vertically inhomogeneous layers in which a zeroth-order asymptotic solution, WKBJ, is constructed. On the other hand, the reflectivity and full-wave theory methods use the “spectral method” to evaluate the inverse transform in which the inverse wave number transform is performed first with a real (reflectivity) or a complex (full-wave theory) contour, but the former deals only with homogeneous layers, while the latter allows for vertical variation. A thorough review of these methods has been given in Chapman & Orcutt (1985) and textbooks (e.g. Aki & Richards 1980). They have been widely and successfully used in seismic modeling and interpretation for layered structures.

The unfortunate fact is that while real subsurface structures often may be approximately layered, they are always laterally heterogeneous, at least in detail. It is then necessary to generalize the “layered-earth” methods (or other formulations) of wave solution to accommodate lateral variation. Thus, in the 1980’s, several prominent methods have been developed, such as the “ray-Kirchhoff method” (Haddon & Buchen 1981), the “Gaussian beam summation” (Červený, Popov & Pšenčík 1982 and Popov 1982), the “coherent-state method” (Klauder 1987 and Foster & Huang 1991), the “EWKBJ seismogram” (Frazer & Phinney 1980), and the “Maslov seismogram” (Chapman & Drummond 1982
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The ray-Kirchhoff seismogram method was developed by Haddon & Buchen (1981) in order to extend the applicability of simple ray theory to caustics, diffractions and shadows. Unlike many other ray approaches in which rays are traced directly from the source to the receiver, here, Haddon & Buchen introduced one or more intermediate surfaces between the source and receiver. Rays are traced from the source to the first intermediate surface, then from the first intermediate surface to the second, and so on to the receiver. Because the rays for each step propagate a shorter distance, caustics and other problems are less likely to occur and ray theory may be applicable. The intermediate surfaces are treated as Kirchhoff integration surfaces. Because the wave motion at a farther distance at all frequencies can be thoroughly described by Kirchhoff's integration (algebraic counterpart of Huygens' principle), the ray-Kirchhoff theory is considered as a "wave-ray" theory. Haddon & Buchen (1981) applied it to calculate the PKP-wavefield in a spherically symmetric earth model. Nevertheless this method can suffer from time-consuming ray tracing if many intermediate integration surfaces are required. A similar but more efficient modeling method will be proposed in this thesis.

Working along the same line as Haddon and Buchen (1981), Frazer and Sinton (1984) proposed the extended Kirchhoff-Helmholtz (EKh) method. By invoking reciprocity the EKh method needs only a single intermediate integration surface. In it rays are traced from both the source and the receiver towards the intermediate surface. The surface must be selected such that caustics of the source receiver ray fields do not intersect on it. Also, ray amplitudes must be parameterized so that their singularities at the caustics become integrable. However, Frazer and Sinton's demonstration of the EKh method is not fully convincing; waveforms near the caustics are still in obvious error.

Červený, Popov & Pšenčík (1982) and Popov (1982) proposed a method of computing wavefields in generally inhomogeneous media based on Gaussian beams. Gaussian beams are ray tubes of finite width across which energy distributes like a bell function. They are obtained by extending geometrical rays from real space to complex space. They can penetrate stably through smooth, inhomogeneous structures, even arbitrary overlaps of geometrically singular regions, without special modification. The wavefield is then simulated by a suite of Gaussian beams instead of rays and, at any point of the medium, it is determined by a superposition of the beams passing near the point. This approach can always provide an estimate of waves such as those at caustics, shadows, and exhibits critical angle phenomena and head waves, etc. It has been frequently used both in
global seismology and in exploration seismology (e.g., Červený 1985; Weber 1988). But unfortunately, Gaussian beams are controlled by two free parameters and it is difficult to relate the accuracy of the result to these parameters. The accuracy degrades when the parameters are not chosen appropriately. For some values of the parameters, several real phenomena such as edge diffractions and head waves fail to be predicted (Nowack & Aki 1985 and Červený 1983). The accuracy also becomes poor where strong lateral inhomogeneities exist (Müller 1984 and White et al. 1987).

On the other hand, the applicability of the WKBJ seismogram has been extended from stratified media to laterally-varying media by Frazer and Phinney (1980). Called EWKB by Sinton & Frazer (1982), this method provides a finite estimation to waves at caustics and shadows, but it is troubled by its own singularities at “telescopic points” (places where neighboring rays become parallel). Synthetic waveforms at such points become inaccurate; significant errors are shown 18% high in waveform amplitude in the example of Frazer & Phinney (1982) or Brown (1994). Also, spurious linear phases are caused across the synthetic seismograms, as presented numerically in Sinton & Frazer (1982).

In fact, a wavefield representation similar to the EWKB was obtained earlier by Maslov (1965, 1972) (Maslov & Fedoriuk 1981) in searching for the semiclassical approximation to the Schrödinger wave equation. This representation was introduced into the seismic community by Chapman & Drummond (1982). By combining it with Chapman's (1978) slowness method of evaluating analytically the inverse frequency and slowness transforms, they formulated the Maslov seismogram method. Maslov asymptotic theory (MAT) also allows for lateral medium variation, but compared with the EWKB theory, it is more complete; it “systematically combines the ideas of asymptotic ray theory, Hamilton's canonical equations, phase space and canonical transformations, and Fourier transforms” (Chapman & Drummond 1982). Further, in theory it provides a way of removing its singularities (i.e., telescopic points) in order to obtain a globally-valid wavefield representation.

GRT is based on Snell's law tracing of one or a few rays in ordinary geometrical space from the source point to the receiver point. Instead, Maslov theory involves rays constructed in a different space, a spatial transform domain. Then, by switching between ray solutions obtained from different spaces, singularities of each solution can be avoided. This technique results from Liouville's theorem, and is understood by considering the ray trajectories in a phase space which is composed of up to 3 space coordinates and 3 slowness coordinates and their projections onto different subdomains (Maslov & Fedoriuk
1981, pp100).

It is known that when the propagation velocity in the model varies smoothly in space, raypaths from a point source diverge or converge smoothly and systematically through most parts of the medium, smoothly changing their directions due to velocity variations according to Snell’s law. A ray can be described more completely at any point by its position and slowness vector. By slowness we mean the inverse of local phase velocity; in isotropic media a slowness vector is locally tangent to a ray and points the direction of wave propagation. Then, in a 3D space and the Cartesian coordinates, a ray is locally described by 3 space coordinates \((x_1, x_2, x_3)\) and 3 slowness components \((p_1, p_2, p_3)\). or a point \((x_1, x_2, x_3, p_1, p_2, p_3)\) in 6D phase space consisting of all the mixed space and slowness coordinates \((x \times p)\).

When a wave equation is Fourier transformed in one space coordinate, a slowness component is introduced naturally as the transform variable. For example \(p_1\) is the transform partner of \(x_1\). In Maslov’s original work, the ray method was also used to find an approximate solution of the transformed wave equation in the transform domain (i.e., mixed space/slowness domain). “Rays” are created as a result, but are not real ones like those propagating in physical space; instead, they propagate in the transform domain. and at any point, they are designated by mixed space and slowness coordinates (Figure 1.3). For convenience we know them as transform rays. In order to differentiate from ordinary GRT solutions, solutions obtained based on transform rays are called transform solutions.

According to Liouville’s theorem, ray trajectories in phase space never cross (Maslov & Fedoriuk 1981 and Thomson & Chapman 1985). The hypersurface that is spread by them, i.e. the so-called Lagrangian manifold, never folds. But the projections of ray trajectories onto the real space (i.e. the physical rays) on which the GRT solution is based can sometimes cross, yielding a GRT solution of infinite amplitude. The places where this happens (lines in 2D or surfaces in 3D) are termed caustics, because physically the wavefield has a strong focusing of energy there (burning point in optics). In the terminology of geometry, a “catastrophe” is introduced into the projection (Poston & Stewart 1978). Likewise, ray trajectories can also be projected onto a mixed space/slowness subdomain, or simply, slowness domain or transform domain that can be obtained by spatial Fourier transformation. Their projections or transform rays on which the transform solution is based also can cross, giving the transform solution of infinite amplitude. The places where this happens are called “pseudocaustics” (Klauder 1978b). The terminology is prefixed with “pseudo” probably because of two considerations. Firstly, the focusing of
Figure 1.3: A ray in ordinary space and transform space. The relationship between depth coordinate $x_3$ and vertical slowness $p_3$ is one to one along the horizontal distance, $-p_3(x_1, x_3)$ or $x_3(x_1, p_3)$, for a given medium. This functional transformation of the ray space is equivalent to Fourier (or Radon) transformation of the wave equation.

Transform rays is artificial. It does not correspond to that of true wave energy, instead, it is a place in space where the wavefield is smooth and neighboring rays become parallel (i.e., a telescopic point). Secondly, the position of such a focusing is changeable with different transformations or projections that we may use. Because the Lagrangian manifold is always unfolded, places where caustics arise and places where pseudocaustics arise never coincide, as illustrated in Figure 1.4. As a result, invalid GRT solutions at caustics can be replaced by transform solutions. And vice versa, invalid transform solutions at pseudocaustics can be replaced by GRT solutions. This leads to Maslov's uniform wavefield representation.

The Maslov method has been applied recently to provide seismic constraints on large-scale mantle structure and anisotropy. Efforts have been made to understand the finer variations of sinking lithospheric slabs and the core-mantle boundary by analysing the waveform distortions due to the possible structure and by comparing them with observational data (e.g., Kendall 1992; Kendall, Masters & Shearer 1992; Kendall et al. 1994; Nangini & Kendall 1994; Liu & Tromp 1996). This significantly extends the ability of geometrical ray theory in seismic interpretation.
Klauder (1987a, b) proposed a so-called “coherent-state transform” solution which is independent of the complexity of caustics and pseudocaustics. The coherent-state transform can loosely be considered as a modified version of the spatial Fourier transform wherein the wavefield is weighted by a spatial Gaussian function. Similar to the Gaussian beam method, the coherent-state transform introduces a complex phase (also a complex Hamiltonian and a complex eikonal equation). In the Gaussian beam method, there are two free parameters and the source position is complex, whereas in the coherent-state transform method, there is usually only one true parameter (there can also be two) and the receiver position is complex (Foster & Huang 1991). The free parameter (inversely proportional to the width of the Gaussian function) plays a central role in the coherent-state transform method. In the limiting cases when the parameter tends to zero and the width of the Gaussian function to infinity, the coherent-state transform reduces to the spatial Fourier transform and the solution to the Maslov description - summation over nearby real rays. When the parameter tends to infinity and the width of the Gaussian function zero, no transform takes place and the solution in effect reduces to the ray theory description - depending on individual real rays. It appears that this parameter connects the ray and Maslov integral solutions, yielding a wave amplitude which is never singular (Klauder 1987b) except in the limiting cases. This presumably resembles

\[ X_1 \]
\[ X_3 \]
\[ \text{caustic} \]
\[ \text{Rays in space (a)} \]
\[ \text{p} \]
\[ \text{pseudocaustic} \]
\[ \text{Rays in mixed transform domain (b)} \]
Maslov's blending method which connects the ray solution and the transform solution for a uniform (i.e., globally-valid) wavefield solution.

Foster & Huang (1991) introduced the coherent-state transform solution to seismology with three test examples. But unfortunately, the last two interesting examples (waveguide and edge diffraction) in fact fail to demonstrate the advantages of this solution over the Gaussian beam summation. (Accuracy still remains unpredictable and whether the accuracy is necessarily better than that of the Gaussian beam or Maslov theory is still unknown.) For a waveguide model, the solution reduces to the exact form without the parameter (equation (78) of Foster & Huang (1991)), but it actually breaks down at the caustics. The edge diffraction is shown in severe error for large values of the parameter, whereas when the value is decreased the approximation tends toward the Maslov solution. Also it is not clear whether decreasing the parameter will improve accuracy for all problems. Further investigations are necessary but are beyond the scope of this thesis.

1.2 Thesis objectives

Since the introduction of Maslov asymptotic theory to the seismic community beginning 1982 by Chapman & Drummond (1982) and Thomson & Chapman (1985, 1986), further improvements have been made to generalize it to anisotropic environments by Kendall & Thomson (1993) and Guest & Kendall (1993), and to remedy the caustic/pseudocaustic problems by Kendall & Thomson (1993). However, the Maslov method has not so far been very widely adopted in either earthquake seismology or exploration seismology. There appear to be three reasons. (1) Practicing seismologists find the theory of the Maslov technique somewhat daunting so they fail to appreciate its underlying simplicity, (2) The Maslov method solves the problem of caustics that plagues the GRT method but itself suffers from problems at pseudocaustics. Problems involving joint caustics and pseudocaustics can often occur, and they have not yet been treated in a practical manner. (3) Insufficient work has been done to investigate the accuracy and improve the robustness and automation of the method. Unless a good deal of good judgement and care is applied in each case, modeled seismograms are prone to inclusion of spurious and inaccurate events. Also, because it is based on high frequency asymptotic assumptions, it is difficult to know how much faith to put into predicted diffractions and other finite frequency phenomena.
1.2.1 Modeling objectives

Interpretation of seismic data usually requires repeated modeling of the observational data, with the model being altered until a satisfactory resemblance is achieved between field and model data. Human time spent on the modeling is of the essence, thus computer modeling methods must be relatively robust. A principal objective of my work has been to find ways of improving this aspect of the Maslov method. My research has therefore proceeded in the following steps:

1. Accuracy of the Maslov integral (MI) seismogram is tested by comparing it with the accurate finite differences. Test models are idealizations of typical earth structures. They include two vertical velocity gradient models, a horizontal low-velocity channel and a high-velocity slab. The tests focus on how the Maslov method and its associated implementation techniques model the complex waveforms produced by these structures.

2. Maslov's uniform seismogram method (globally valid, in theory) blends ray and transform solutions through weighting functions. In practice, it suffers from two problems. One is when caustics and pseudocaustics cannot be far separated compared with a wavelength. The other relates to the robustness of the weighting method. In this thesis, I investigate the effects of Brown's (1994) weighting method. I then propose a more robust weighting scheme and apply it in a wavefield extrapolation example.

3. Rotation of the Maslov integration axis is known to be useful in avoiding pseudocaustics of a simple wavefront (e.g. Kendall & Thomson 1993). But its role in constructing more accurate waveform distortion is not clear. A simple way of performing the rotated Maslov integral is developed.

4. If caustics and pseudocaustics arise at nearly the same place, Maslov's uniform (blending) method breaks down. Also, techniques based on Lagrangian equivalence, such as Kendall & Thomson's (1993) phase partitioning become more difficult and need careful human supervision. For such cases, I propose a new "Maslov-Kirchhoff" (MK) modeling tool which uses the Kirchhoff summation (Huygens') principle to suppress pseudocaustic errors. I derive the MK equation for 2D acoustic propagation, and carry out a set of numerical experiments for an idealized case to test its capabilities. I then apply it to modeling wave propagation in a realistic cross-well environment where the medium is complicated.
1.2.2 Backward wavefield extrapolation

In seismic exploration, seismic signals recorded on the surface are seismic waves reflected from subsurface interfaces (velocity and density contrasts) between strata. From structural discontinuities (e.g., faults), and other spatially-isolated perturbations in geological structure. Wavefield extrapolation is a procedure of propagating computationally the recorded signals or wavefields back to the propagation medium. Geological structures function as many “secondary sources” (Huygens’ principle). In exploration seismology, they are treated as “exploding reflectors” (Claerbout 1985). By a change of time scale the wavefield recorded in zero source-receiver offset seismic sections can be synthesized as if the reflectors explode at time zero, emitting the waves that we can observe as reflections. Reversely, the reflectors can be located by backward extrapolating the reflections and by mapping the extrapolated energy at time zero. This procedure is called migration (e.g., Stolt & Benson 1986; Yilmaz 1987), as is illustrated schematically in Figure 1.5. Clearly, wavefield extrapolation is an essential step in terms of economy and quality.

Migration techniques are differentiated very much by the ways in which extrapolation is done. Before the age of computers, migration was done by geometrical techniques. In these techniques, only the phases (travel time curves) of the recorded wavefields were projected to image the geological structures with the aid of wavefront charts and diffraction curves (Hagedoorn, 1954). “Conscientiously applied, they could unravel true structure better than is often done today (Stolt & Benson 1986).”

Use of computers allows recorded trace amplitudes to be input into the geometrical migration procedure. The amplitudes can be either mapped onto the wavefronts at depth and then summed, or summed directly along the diffraction hyperbolae that are governed by the constant rms (root-mean-square) velocity. Different traces have been assumed to make equally-weighted contributions to a depth point. Efforts were made to extend the constant or 1D velocity assumption (French 1975) to 2D (e.g., Hu & McMechan 1986).

In the 1970’s, recorded wavefield information was used more faithfully by extrapolating acoustic wave equations. One such “wave equation migration” technique is the pioneering method of Claerbout (1970, 1971) and Claerbout & Doherty (1972). In it, the full wave equation is reduced to an approximate one way ±15 degree wave equation before finite difference computation. The recorded wavefields are downward continued and then mapped to geological structures according to Claerbout’s (1971) imaging principle, which states that “reflectors exist at points in the Earth where an upgoing wave is time
Figure 1.5: Schematic migration theory. A recorded seismic data is processed into a zero-offset seismic section (b), which is an equivalence recorded by a coincident source and receiver running along the survey line. The processed data also can be viewed as "observed" on an exploding reflector model (a), where the geological surface consists of many reflectors that explode in unison (at time zero) and emit waves going up at a halved velocity. This data is downward extrapolated and imaged, yielding a map (c) which can closely resemble the real geological section.
coincident with the first arrival of a downgoing wave. The migration techniques by Fourier transform (Stolt 1978) and the phase-shift method (Gazdag 1978) were further proposed to improve the computation efficiency and the limitation of medium variation.

Another important approach to wave equation migration was formulated based on Kirchhoff's theorem (Huygens' principle) of extrapolating wavefields from one surface to another (Schneider 1978). Unlike the simple diffraction summation approach, the recorded wavefields are appropriately weighted in wave-equation Kirchhoff migration. Reconstructed wavefields are much closer to images of the true reflectivity in space. This technique has been accommodated to handle nonplanar data (Wiggins 1984), weak lateral velocity variation (Carter & Frazer 1984), strong velocity and density gradients (Zhu 1988b), generally varying velocity by the use of Gaussian beams (Costa, Raz & Kosloff 1989; Wapenaar et al. 1989; Kinneging et al. 1989; Lazaratos & Harris 1990; Hill 1990; Hill et al. 1991; and Hale 1992).

1.2.3 Wavefield extrapolation

A satisfactory wavefield extrapolation operator should be able to handle complex media such as laterally heterogeneous, even anisotropic media, and be efficient in dealing with large amounts of data, such as those acquired in 3D surveys. The second objective of this thesis is to test the efficiency of the Maslov-Kirchhoff method in extrapolating waves through continuously inhomogeneous media. Two inverse extrapolation examples will be given. The first one deals with a slowly variable low-velocity channel where Maslov's blending method is applicable. The second one concerns a medium where the velocity varies rapidly, and where only the pure Maslov integral solution is used because the blending approach breaks down.

1.3 Thesis organization

This thesis consists of two parts. Chapters 2, 3 and 4 are devoted to the classical Maslov method. Chapters 5, 6 and 7 are devoted to formulation, testing, and application of my Maslov-Kirchhoff method.

Chapter 2 carefully reviews: the mathematical backbone of the thesis; the formulation of geometrical ray theory and Maslov asymptotic theory for the general case of a continuously inhomogeneous, anisotropic medium.
Chapter 3 presents an extension of Maslov asymptotic theory from the usually adopted fixed coordinates to rotated coordinates. The advantages that this extension can provide in avoiding pseudocaustics and constructing waveform distortion are also presented. The practical accuracy of the high frequency asymptotic Maslov integral is examined by comparing its predictions with those of the finite difference method.

In Chapter 4, the accuracy of Maslov’s GRT/MI switching method is studied by testing the effects of the weighting functions. An alternative, more robust weighting scheme is designed so that intensive wavefield calculations can be completed automatically.

Chapter 5 reviews the Kirchhoff extrapolation formulas, and then couples them with Maslov asymptotic theory to formulate the Maslov-Kirchhoff wavefield extrapolation operator.

Chapter 6 tests the advantages of the Maslov-Kirchhoff reverse wavefield extrapolation method over the traditional Kirchhoff method. Two simplified numerical examples are given.

Chapter 7 investigates the applicability of the Maslov-Kirchhoff method to seismic forward modeling, in particular, the efficiency of this method in suppressing pseudocaustic errors. Its ability to model a realistic cross-well wave propagation through a complex medium is demonstrated.

Finally, thesis contributions, conclusions and suggestions are summarized in Chapter 8.
Chapter 2

The Maslov Seismogram for a Continuous, Anisotropic Medium

The Maslov seismogram method has been described in Chapman & Drummond (1982) and Thomson & Chapman (1985) for isotropic media, and in Kendall (1991) and Kendall & Thomson (1993) for anisotropic media. This chapter reviews the method to provide a foundation for the thesis, and to establish notations and details of computation for what follows.

2.1 Geometrical ray theory

The geometrical ray theory (GRT) solution to the wave equation for a continuous, anisotropic elastic medium has been presented in many articles, e.g. Červený (1972), Vlaar (1968), and Kendall & Thomson (1989). The particle motion $\mathbf{u} = (u_1, u_2, u_3)$ is governed by the elastodynamic wave equation

$$\partial_{x_i}(c_{ijkl}\partial_{x_k}u_l) - \rho \partial_t^2 u_j = 0,$$

(2.1)

where $\mathbf{x} = (x_1, x_2, x_3)$ denotes the Cartesian coordinates, $t$ the time, $c_{ijkl}$ the elastic moduli and $\rho$ the density of the medium. (Note: Einstein’s notation of summation has been followed and $c_{ijkl}$ and $\rho$ may be functions of position $\mathbf{x}$.) The Fourier transform from time to frequency ($\omega$) which will be used throughout the thesis is defined by

$$\hat{u}(\omega, \mathbf{x}) = \int_{-\infty}^{+\infty} u(t, \mathbf{x}) e^{i\omega t} dt$$

(2.2)

and the inverse transform is

$$u(t, \mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\omega, \mathbf{x}) e^{-i\omega t} d\omega.$$  

(2.3)
In the frequency domain, the wave equation (2.1) becomes
\[ \partial_x (c_{ijkl} \partial_x u_k) + \omega^2 \rho u_j = 0. \] (2.4)

A solution of (2.4) may be expressed as an "asymptotic ray series"
\[ \hat{u}(\omega, x) = \sum_{n=0}^{\infty} \frac{\mathcal{U}(n)(x)}{(-i\omega)^n} e^{i\omega T(x)}, \] (2.5)

where \( T(x) \) denotes the geometrical travel time and where we assume \( T(x_0) = 0 \). The subscript 0 always indicates the values at the source. Note \( i = \sqrt{-1} \) should not be confused with the index subscript \( i \) denoting 1, 2, or 3. In many geophysical modeling problems, the wave frequency is relatively high; in other words, the hypothetical medium and the wave field amplitude and phase change smoothly and slowly within a wavelength. Then the contributions of the second and higher terms of the ray series can be neglected, and the wave propagation can be approximately described by the first (zeroth-order or geometrical) term. To find this zeroth-order approximation, we substitute the first term of the ray series (2.5), \( \mathcal{U}(0)(x) e^{i\omega T(x)} \), in the wave equation (2.4), and obtain
\[ -\omega^2 [c_{ijkl} \mathcal{U}(0)_k \mathcal{P}_l - \rho \mathcal{U}(0)_j] + i\omega [c_{ijkl} \partial_x \mathcal{U}(0)_k \mathcal{P}_l + \partial_x (c_{ijkl} \mathcal{U}(0)_k \mathcal{P}_l)] + \partial_x (c_{ijkl} \partial_x \mathcal{U}(0)_k) = 0 \] (2.6)

where \( \mathcal{P}_l \equiv \partial_x T \) is the slowness vector at \( x \). It is normal to the wavefront (contour of constant \( T \)). Equating the coefficient of \( \omega^2 \) to zero leads to a set of linear equations for \( \mathcal{U}(0) \):
\[ (a_{ijkl} \mathcal{P}_l - \delta_{jk}) \mathcal{U}(0)_k = 0, \] (2.7)

where \( a_{ijkl} \equiv c_{ijkl} / \rho \) are the density-normalized elastic parameters. Non-trivial solutions of (2.7) for \( \mathcal{U}(0)_k \) require that
\[ \det |a_{ijkl} \mathcal{P}_l - \delta_{jk}| = 0, \] (2.8)

which is an ordinary equation for the slowness vector \( \mathcal{P} \), or that, since \( \mathcal{P}_l \equiv \partial_x T \),
\[ \det |a_{ijkl} \partial_x T \partial_x T - \delta_{jk}| = 0, \] (2.9)

which is a non-linear first-order partial differential equation for the travel time function \( T(x) \). Either of the two equations governs the paths of high-frequency energy propagation and wavefront geometry, and is referred to as the characteristic equation or eikonal equation of wave propagation.

In general, there are three distinct types of waves associated with the eikonal equation. To see this more clearly, we rewrite equation (2.8) as
\[ \det |a_{ijkl} n_i n_l - V^2 \delta_{jk}| = 0, \] (2.10)
where \( \mathbf{n} \) is the unit wavefront normal and \( V \) is the normal or phase velocity related to \( p_i \) and \( n_i \) by
\[
p_i \equiv n_i / V.
\]
(2.11)
Then, it turns out that in general we can solve for three possibly distinct "eigenvalues". \( V^{(m)} (m=1, 2, 3) \), from (2.10) and accordingly three distinct slowness vectors, \( p^{(m)} \), from (2.11). We can also obtain three distinct eigenvectors, \( g^{(m)} \), from
\[
(a_{ijkl} p_i p_l - \delta_{jk}) g_k^{(m)} = 0,
\]
(2.12)
or
\[
(\Gamma_{jk}^{(m)} - \delta_{jk}) g_k^{(m)} = 0,
\]
(2.13)
where the Christoffel matrix \( \Gamma_{jk} \equiv a_{ijkl} p_i p_l \). These eigenvectors are the polarizations of the displacement vector, \( L^{(0)} \), and they are orthogonal to each other for constant \( n_i \). The three distinct sets of solutions \( (m=1, 2, 3) \) are interpreted as one quasi-compressional and two quasi-shear waves. In order to find each polarization, more restrictions are required. For simplicity of notation, I will henceforth omit the superscript \( (m) \). One quasi-compressional and two quasi-shear waves will be implied unless otherwise specified.

The eikonal equation (2.8) can be solved using the method of characteristics (Courant & Hilbert 1962). I define a Hamiltonian
\[
H(\mathbf{x}, \mathbf{p}) \equiv \frac{1}{2} \det | a_{ijkl} p_i p_l - \delta_{jk} |,
\]
(2.14)
for which the eikonal equation is \( H(\mathbf{x}, \mathbf{p}) = 0 \). From the characteristic system
\[
\frac{dx_i}{\partial p_i H} = - \frac{dp_i}{\partial x_i H} = \frac{dT}{p_s \partial p_s H},
\]
(2.15)
we see that \( T \) may be selected as a 'parameter' of the system. Thus, the system (2.15) can be rewritten as the so-called ray equations
\[
\frac{dx_i}{dT} = \frac{\partial p_i H}{p_s \partial p_s H} = a_{ijkl} p_i \frac{D_{jk}}{D},
\]
\[
\frac{dp_i}{dT} = - \frac{\partial x_i H}{p_s \partial p_s H} = - \frac{1}{2} \partial x_i a_{ijkl} p_i p_l \frac{D_{jk}}{D}
\]
(2.16)
where \( D_{jk} \) are the cofactors of the matrix \( (\Gamma_{jk} - \delta_{jk}) \), and \( D \equiv D_{ii} \). Explicit expressions can be found for them, e.g., in Červený (1972). With the definition of the Hamiltonian

\[\text{In the case of degeneracy (i.e. in isotropic media), two eigenvalues of (2.10) are equivalent, so we have two equal slowness vectors and two equal ray velocities, and can only uniquely find one polarization and a plane on which the other two polarizations lie.}\]
in equation (2.14) and the assumption that \( T(x_0) = 0 \), \( T(x) \) is the geometrical travel time for the initial wavefront to arrive at \( x \) (Chapman & Drummond, 1982, p. S287). (This explains the use of the arbitrary factor 1/2 in (2.14).) Using the relation between the cofactors and polarizations, \( D_{jk}/D = g_j g_k \), we obtain an alternative for (2.16) as

\[
\frac{dx_i}{dT} = a_{ijkl} p_i g_j g_k \\
\frac{dp_i}{dT} = -\frac{1}{2} \partial_x a_{ijkl} p_i p_s g_j g_k
\]  

(2.17)

which will become very useful in the solution of the transport equation in the transform domain. The ray or group velocity \( \nu \) is obtained simply from

\[
\nu_i \equiv \frac{dx_i}{dT} = \frac{\partial p_i H}{p_s \partial x_i H} = a_{ijkl} p_i g_j g_k.
\]  

(2.18)

Equating the coefficient of \( \omega^1 \) in (2.6) to zero yields the transport equation for the zeroth-order term of the ray series in the \( x \)-domain

\[
a_{ijkl} p_i \partial_x U_{ik}^{(0)} + \rho^{-1} \partial_x (\rho a_{ijkl} p_i U_{ik}^{(0)}) = 0.
\]  

(2.19)

As we have mentioned before, the amplitude vector \( U^{(0)} \) has the direction of the unit eigenvector \( g \) defined in (2.12) or (2.13), so we may write

\[
U^{(0)}(x) = \phi^{(0)}(x)g(x)
\]  

(2.20)

for any of the three types of waves, with \( \phi^{(0)}(x) \) known as the ray or GRT amplitude. Substituting (2.20) into (2.19) and multiplying by \( g_j \), we have

\[
a_{ijkl} p_i \partial_x (\phi^{(0)} g_k) g_j + \rho^{-1} \partial_x (\rho a_{ijkl} p_i \phi^{(0)} g_k) g_j = 0.
\]  

(2.21)

which can be rearranged as

\[
\rho^{-1} \phi^{(0)} \partial_x (\rho v_i) + 2v_i \partial_x \phi^{(0)} = 0,
\]  

(2.22)

where the ray velocity \( \nu \) defined above in (2.18) has been introduced. Note that \( v_i \equiv \partial_T x_i, \)

\[
v_i \partial_x \phi^{(0)} = \partial_T \phi^{(0)} \quad (\partial_T \) denotes henceforward the ordinary derivative w.r.t. \( T \). Thus (2.22) can be rewritten

\[
\partial_T \phi^{(0)} + \frac{\phi^{(0)}}{2\rho} \partial_x (\rho v_i) = 0.
\]  

(2.23)

Using Smirnov's lemma (Smirnov 1964, p. 422) in the ray coordinates \( (T,q_1,q_2) \), which consist of the travel time and the initial take-off angles, we have

\[
\partial_x (\rho v_i) = \rho d_T \ln(\rho V J)
\]  

(2.24)
where
\[ J \equiv \left| x_{q_1} \times x_{q_2} \right| \quad (2.25) \]
is the Jacobian describing the geometrical spreading. Substituting (2.24) into (2.23) yields a simple ordinary differential equation along the ray
\[ d_T \phi^{(0)} + \frac{\phi^{(0)}}{2} d_T \ln(\rho V J) = 0. \quad (2.26) \]
Integration of (2.26) gives the solution
\[ \phi^{(0)}(T) = \phi^{(0)}(T_0) \left( \frac{(\rho V J)T_0}{(\rho V J)T} \right)^{1/2}. \quad (2.27) \]
This result applies to all the three types of waves by substituting different solutions of the eikonal equation.

Although rays from a source in a uniform medium diverge from one another, in a non-uniform medium they may locally converge and cross each other. Geometrically, the line (2D) or surface (3D) at which a ray bundle folds back on itself and crosses is known as a caustic line or surface, or for short, a caustic. The outer side beyond the caustic which the ray bundle cannot penetrate is termed a shadow (Figure 2.1). In the vicinity of a caustic, waveforms become complicated. Those inside the caustic (or the illuminated side) undergo a distortion, and those in the shadow are diffracted signals. Ray theory fails to describe these waveforms. This is because at a caustic, the ray tube vanishes (mathematically, the Jacobian \( J \) equals zero and the solution (2.27) becomes infinite), and in a shadow, no ray information is available.

But nevertheless, the waveform of a ray passing a caustic and returning to the lit region only undergoes a complete Hilbert transform when moving far enough (roughly, a wavelength) from a caustic. Each of the waveform frequency components suffers a phase delay of \( \pi/2 \). Then use of what is called the KMAH index \( \sigma (T, T_0) \) allows ray theory to connect the waveforms on either side of a ray across a caustic (e.g. Babić & Kirpičnikova 1979, Chapter 3), i.e.
\[ \phi^{(0)}(T) = \phi^{(0)}(T_0) \left( \frac{(\rho V J)T_0}{(\rho V J)T} \right)^{1/2} e^{-i \pi \text{sgn}(\omega) \sigma (T, T_0) \pi/2}. \quad (2.28) \]
The KMAH index usually is initially zero at the source point, constant between caustics on a ray, and increases by an integer to account for a Hilbert transform of the waveform whenever a caustic (indicated by a zero of the Jacobian) is passed.

Noting that
\[ (V J)T_0 = \frac{\partial(x_{q_1}, x_{q_2}, x_{q_3})}{\partial(T, q_1, q_2)}, \]
Figure 2.1: Wave behavior near the fold caustic, where an impulsive waveform approaching the caustic (dashed line) will be distorted in the geometrically-illuminated side, diffracted into the shadowed side, and Hilbert transformed when leaving the caustic.
\[ (VJ)_T = \frac{\partial(x_1, x_2, x_3)}{\partial(T, q_1, q_2)}. \]  

(2.29)

and

\[ \left( \frac{(\rho VJ)_T}{(\rho VJ)_0} \right)^{1/2} = \left( \frac{\rho_0}{\rho} \frac{\partial z_0}{\partial x} \right)^{1/2}. \]  

(2.30)

(2.28) can be rewritten

\[ \phi^{(0)}(z) = \phi^{(0)}(z_0) \left( \frac{\rho_0}{\rho} \frac{\partial z_0}{\partial x} \right)^{1/2} e^{-i\omega \text{sgn}(\omega)} \sigma(z-z_0)^{1/2}. \]  

(2.31)

Expressions (2.28) and (2.31) have a simple physical interpretation. The quantity in the former, \((\rho VJ)_T/(\rho VJ)_0\), is the ratio of energy flux being carried by the relevant wave type at \(z_0\) and \(z_0\), and the quantity in the latter, \((\partial x/\partial z_0)_T\), is the fractional change in cross-sectional area of the ray tube from \(z_0\) to \(z\) (at which the traveltime is \(T\)). Note that they both are frequency independent.

Evaluation of the amplitude-related Jacobian in (2.25) or (2.29) involves the derivatives of the position and slowness of a ray path with respect to the initial shooting angles, i.e., \(\partial q_r x_i\) and \(\partial q_r q_i\) (where \(i = 1, 2, 3\) and \(r = 1, 2\)). In the same manner as we solve the ray paths by the ray equations (2.16), these derivatives can be obtained using the so-called dynamic ray equations (also geometrical spreading equations). The dynamic ray equations are derived simply by differentiating the ray equations (2.16) w.r.t. the initial shooting angles \(q_r\) (e.g., Červený (1972) and Kendall & Thomson (1989)), giving

\[ \frac{d}{dT}(\partial q_r x_i) = (F_{ij}^{(1)} \partial q_r x_j + F_{ij}^{(2)} \partial q_r p_j)/2D \]

\[ \frac{d}{dT}(\partial q_r p_i) = -(F_{ij}^{(3)} \partial q_r x_j + F_{ij}^{(4)} \partial q_r p_j)/2D, \]  

(2.32)

where

\[ F_{ij}^{(1)} = D_{kl} \partial^2_{p,x_j} \Gamma_{kl} + D \partial_{p_1} \Gamma_{kl} \partial x_j (D_{kl}/D) \]

\[ F_{ij}^{(2)} = D_{kl} \partial^2_{p,p_j} \Gamma_{kl} + D \partial_{p_1} \Gamma_{kl} \partial p_j (D_{kl}/D) \]

\[ F_{ij}^{(3)} = D_{kl} \partial^2_{x,x_j} \Gamma_{kl} + D \partial x_j \Gamma_{kl} \partial x_j (D_{kl}/D) \]

\[ F_{ij}^{(4)} = D_{kl} \partial^2_{x,p_j} \Gamma_{kl} + D \partial x_j \Gamma_{kl} \partial p_j (D_{kl}/D). \]  

(2.33)

Numerical implementation of this system, (2.32) and (2.32), is described in more detail in Appendix E of the thesis.

The time-domain GRT wavefield is given by inversely Fourier transforming

\[ \hat{u}(\omega, z) \approx \hat{U}^{(0)}(\vec{z}) e^{i\omega T(\vec{z})}. \]
Chapter 2: The Maslov Seismogram

w.r.t. frequency, as

$$u(t, \mathbf{x}) \approx \sum \text{Re} \left\{ \mathcal{I}^{(0)}(\mathbf{x}) \Delta(t - T(\mathbf{x})) \right\}$$

$$= \sum \left( \frac{\rho_0}{\rho} \frac{\partial \xi_0}{\partial \mathbf{x}} \right)^{1/2} q(\mathbf{x}) \text{Re} \left\{ \phi^{(0)}(\mathbf{x}_0) \Delta(t - T(\mathbf{x})) e^{-i\pi \sigma(\mathbf{x}_0 \cdot \mathbf{v})^2} \right\}, \quad (2.34)$$

where the $\Delta(t)$ analytic time series is defined as

$$\Delta(t) \equiv \delta(t) + i \tilde{\delta}(t) = \delta(t) - i \frac{1}{\pi t} \quad (2.35)$$

corresponding to the Dirac delta function, $\delta(t)$.

In many situations, the solution of the eikonal equation for the traveltime, (2.16), and the solution of the transport equation for the amplitude, (2.32), constitute a very satisfactory high frequency asymptotic solution (2.28) for the wave motion. However, there are some circumstances in which geometrical ray theory is clearly inadequate, e.g. when the receiver is at a caustic where (2.27), (2.28), and (2.31) are singular, or in a shadow region where no solution is provided. It is to treat these situations that Maslov theory is applied.

### 2.2 Maslov asymptotic theory

Instead of solving the wave equation in the spatial $\mathbf{x}$-domain, Maslov (1965, 1972) explored the wave solution in a spatial Fourier transform domain. For a general non-separable wave equation, he defined a Fourier transform of the frequency domain wavefield $\hat{u}(\omega, \mathbf{x})$ from the real space $\mathbf{x} = (x_1, x_2, x_3)$ to the mixed transform space $\mathbf{y} = (p_1, x_2, x_3)$ as

$$\hat{u}(\omega, \mathbf{y}) \equiv \left( -\frac{i\omega}{2\pi} \right)^{1/2} \int_{-\infty}^{+\infty} \hat{u}(\omega, \mathbf{x}) e^{-i\omega p_1 x_1} \, dx_1 \quad (\omega > 0) \quad (2.36)$$

with the inverse

$$\hat{u}(\omega, \mathbf{x}) \equiv \left( \frac{i\omega}{2\pi} \right)^{1/2} \int_{-\infty}^{+\infty} \hat{u}(\omega, \mathbf{y}) e^{i\omega p_1 x_1} \, dp_1; \quad (2.37)$$

and the transforms of the derivatives of $\hat{u}(\omega, \mathbf{x})$ and $\hat{u}(\omega, \mathbf{y})$ as

$$i\omega p_1 \hat{u}(\omega, \mathbf{y}) \equiv \left( -\frac{i\omega}{2\pi} \right)^{1/2} \int_{-\infty}^{+\infty} \hat{u}(\omega, \mathbf{x}) e^{-i\omega p_1 x_1} \, dx_1 \quad (2.38)$$

and

$$\hat{f}(\mathbf{p}_1) \equiv \left( \frac{-i\omega}{2\pi} \right)^{1/2} \int_{-\infty}^{+\infty} f(\mathbf{x}) \hat{u}(\omega, \mathbf{x}) e^{-i\omega p_1 x_1} \, dx_1. \quad (2.39)$$
Here, \( \hat{f}((-i\omega)^{-1}\partial_{p_1}) \) is now a function of the \( \partial_{p_1} \) differential operator (a pseudo differential operator, in fact). Applying these transforms to the wave equation (2.4), we obtain the wave equation in the Maslov transform \( y \)-domain

\[
\partial_t[\hat{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3)\partial_t\hat{u}_k] + \omega^2 \hat{\rho}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3)\hat{u}_j = 0 \tag{2.40}
\]

where the operator, \( \partial_i \), is defined as \( \partial_i \equiv (i\omega p_1, \partial x_2, \partial x_3) \) for \( i=1, 2, \) and \( 3 \). Note that \( \hat{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3) \) and \( \hat{\rho}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3) \) are the functions of the \( \partial_{p_1} \) differential operator.

By analogy with the geometrical ray theory in the \( x \)-domain, the Maslov wavefield solution is expressed in an asymptotic ray series consisting of vector amplitudes \( \hat{\hat{u}}^{(n)}_j(y) \) and phase \( \hat{T}(y) \) as

\[
\hat{u}(\omega, y) = \sum_{n=0}^{\infty} \frac{\hat{U}^{(n)}(y)}{(-i\omega)^n} e^{i\omega \hat{T}(y)}. \tag{2.41}
\]

The phase \( \hat{T}(y) \) will be called the Maslov traveltime (of a ray in the transform domain). It is, as will be shown in (2.49), related to but not the same as the ordinary traveltime \( T(x) \). To make the zeroth-order approximation, we substitute the first term of the last expression into (2.40) and obtain

\[
\partial_t[\hat{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3)\partial_t(\hat{U}_k^{(0)}e^{i\omega \hat{T}})] + \omega^2 \hat{\rho}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3)(\hat{U}_j^{(0)}e^{i\omega \hat{T}}) = 0. \tag{2.42}
\]

Afterward, the coefficients of \( \omega \) and \( \omega^2 \) in equation (2.42) need be sorted out. To outline this, I follow Thomson and Chapman’s (1985) work for isotropic media. Because the algebra for anisotropy is extremely complicated, it cannot be shown in detail in a reasonable space. More can be found in Kendall (1991).

We begin with making explicit the pseudo differential operators of the transformed asymptotic wave equation (2.42). This is implemented following Hörmander (1979, p. 20), or Thomson & Chapman (1985, eq. (B3)). For instance, the first pseudo differential operator involving \( \hat{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3) \) is defined explicitly as

\[
\hat{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3)\{\partial_t[\hat{U}_k^{(0)}(p_1, x_2, x_3)e^{i\omega \hat{T}(p_1, x_2, x_3))}\}_{p_1=p_1'} \\
\sim \sum_{m=0}^{\infty} \frac{1}{m!} [\partial_{x_1}^m \hat{c}_{ijkl}(x_1, x_2, x_3)][(-i\omega)^{-1}\partial_{p_1}]^m \\
\cdot \{\partial_t[\hat{U}_k^{(0)}(p_1, x_2, x_3)e^{i\omega N(p_1, p_1', x_2, x_3))}\}_{x_1=-\partial_{p_1} \hat{T}(p_1, x_2, x_3)} \tag{2.43}
\]

where the operators act on the terms in brackets \( \{ \) and \( N(p_1, p_1', x_2, x_3) \) is the non-linear part of \( \hat{T} \) in the vicinity of \( p_1' \), i.e.

\[
\hat{T}(p_1, x_2, x_3) = (p_1 - p_1')\partial_{p_1} \hat{T}(p_1, x_2, x_3)|_{p_1=p_1'} + N(p_1, p_1', x_2, x_3).
\]
Also note that for \( p_1 = p'_1 \), we have \( N = \dot{T}, \partial_{p_1} N = 0 \), and \( \partial_{p_1}^m N = \partial_{p_1}^m \dot{T} \) as \( m \geq 2 \). From (2.43), we may derive

\[
\dot{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1} x_2, x_2, x_3)\{i\omega p_1 \dot{U}^{(0)}_k(p_1, x_2, x_3) e^{i\omega \dot{T}(p_1, x_2, x_3)}\} \\
= i\omega \dot{c}_{ijkl}((-i\omega)^{-1}\partial_{p_1} x_2, x_2, x_3)\{p_1 \dot{U}^{(0)}_k(p_1, x_2, x_3) e^{i\omega \dot{T}(p_1, x_2, x_3)}\}.
\]

Similar expressions for the pseudo differential operator of \( \dot{\rho}((-i\omega)^{-1}\partial_{p_1}, x_2, x_3) \) can be obtained. In fact, we only need the first three terms in (2.43), i.e., \( m=0, 1, \) and \( 2 \), in order to get the terms of \( \omega \) and \( \omega^2 \) in (2.42), to which the others make no contribution.

Substituting (2.43) into (2.42) and equating the coefficients of the powers \( \omega^2 \) and \( \omega \) to zero, we obtain the linear equations for \( \dot{U}^{(0)}_k \)

\[
(\dot{c}_{ijkl}\dot{p}_i \dot{p}_l - \rho \delta_{jkl})\dot{U}^{(0)}_k = 0.
\]

as well as the transport equation

\[
\dot{c}_{ijkl}\dot{p}_i \dot{p}_l \partial_{\dot{T}} \dot{U}^{(0)}_k + \partial_{\dot{T}}(\dot{c}_{ijkl}\dot{p}_i \dot{U}^{(0)}_k) \\
- \partial_{x_1}\dot{c}_{ijkl}\dot{p}_i \partial_{p_1}(\dot{p}_l \dot{U}^{(0)}_k) - \frac{1}{2}\dot{p}_i \dot{p}_l \dot{U}^{(0)}_k \partial_{p_1}[\partial_{x_1}\dot{c}_{ijkl}(x_1, x_2, x_3)|_{x_1=\delta_{p_1} \dot{T}}] \\
+ \partial_{x_1}\dot{\rho} \partial_{p_1} \dot{U}^{(0)}_j + \frac{1}{2}\dot{U}^{(0)}_j \partial_{p_1}[\partial_{x_1}\dot{\rho}(x_1, x_2, x_3)|_{x_1=\delta_{p_1} \dot{T}}] = 0.
\]

in which

\[
\partial_{\dot{T}} \equiv (0, \partial_{x_2}, \partial_{x_3}) , \\
\dot{p}_i \equiv (p_1, \partial_{x_2} \dot{T}, \partial_{x_3} \dot{T}) , \\
\dot{c}_{ijkl} \equiv \dot{c}_{ijkl}(-\partial_{p_1} \dot{T}, x_2, x_3) . \\
\dot{\rho} \equiv \dot{\rho}(-\partial_{p_1} \dot{T}, x_2, x_3).
\]

Letting \( \tilde{a}_{ijkl}(-\partial_{p_1} \dot{T}, x_2, x_3) \equiv \dot{c}_{ijkl}(-\partial_{p_1} \dot{T}, x_2, x_3)/\dot{\rho}(-\partial_{p_1} \dot{T}, x_2, x_3) \) and then rewriting (2.44) in a form similar to the linear equation in the \( x \)-domain, (2.7), we obtain

\[
(\tilde{a}_{ijkl}\dot{p}_i \dot{p}_l - \delta_{jkl})\dot{U}^{(0)}_k = 0.
\]

Non-trivial solutions of (2.47) require that

\[
det|\tilde{a}_{ijkl}\dot{p}_i \dot{p}_l - \delta_{jkl}| = 0,
\]

which is the \( y \)-domain eikonal equation.

Before introducing the characteristic system of (2.48), it is necessary to first discuss the relations between the eikonal equations in the spatial and transform domains. Comparing the two non-linear, first-order partial differential equations, (2.8) and (2.48), with
Courant & Hilbert’s (1962. p.35) equations (7) and (8), we see that \( \dot{T}(p_1, x_2, x_3) \) and \( T(x_1, x_2, x_3) \) must be related by the partial Legendre transformation

\[
\dot{T}(p_1, x_2, x_3) = T(x_1(p_1, x_2, x_3), x_2, x_3) - p_1 x_1(p_1, x_2, x_3) \tag{2.49}
\]

with

\[
p_1 \equiv \partial_{x_1} T, \tag{2.50}
\]

\[
x_1 \equiv -\partial_{p_1} \dot{T}, \tag{2.51}
\]

showing that \( p_1 \), as given in (2.50), is actually the slope of the \( x_1 \) travel time curve or \( x_1 \) slowness, and that the phase function of (2.41) or the Maslov traveltime. \( \dot{T} \) is the familiar “intercept” time denoted usually as \( \tau \). The Legendre transformation maps the phase in the real space to the phase in the transform space, with slope constituting its independent coordinate (equation (2.50)); whereas reversely, it maps the phase in the transform space back to the phase in the real space, with negative slope returning to the spatial coordinate (equation (2.51)).

Differentiating (2.49) w.r.t. \( x_2 \) and \( x_3 \), respectively, and noting \( x_1 = x_1(p_1, x_2, x_3) \), and then substituting (2.50), we obtain

\[
\partial_{x_2} \dot{T} = \partial_{x_2} T, \quad \partial_{x_3} \dot{T} = \partial_{x_3} T. \tag{2.52}
\]

From (2.46), (2.50), and (2.52), we have the equality

\[
\tilde{p}(y) = p(x) \tag{2.53}
\]

showing that \( \tilde{p}(p_1, x_2, x_3) \) is in fact the slowness vector of a ray arriving at the point \( (x_1, x_2, x_3) \big|_{x_1 = -\partial_{p_1} \dot{T}(p_1, x_2, x_3)} \). Therefore, we conclude that all the solutions of the \( y \)-domain eikonal equation (2.48) at the point \( (p_1, x_2, x_3) \) are equivalent to those of the \( x \)-domain eikonal equation (2.8) at the point \( (x_1, x_2, x_3) \big|_{x_1 = -\partial_{p_1} \dot{T}(p_1, x_2, x_3)} \).

Returning to the solution of the \( y \)-domain eikonal equation (2.48), I first define the Hamiltonian

\[
\hat{H}(\tilde{p}_1, x_2, x_3, q, \tilde{p}_2, \tilde{p}_3) \equiv \hat{H}(\tilde{p}_1, x_2, x_3, \partial_{\tilde{p}_1} \dot{T}, \partial_{x_2} \dot{T}, \partial_{x_3} \dot{T}) \\
\equiv \frac{1}{2} \det |\tilde{\alpha}_{ijkl}(q, x_2, x_3)\tilde{p}_i \tilde{p}_j - \delta_{ik}| \tag{2.54}
\]

with six independent variables \( (\tilde{p}_1, x_2, x_3, q, \tilde{p}_2, \tilde{p}_3) \). (Note that we have used \( \partial_{p_1} \dot{T} = \partial_{\tilde{p}_1} \dot{T} \equiv q \) in denoting the variables of \( \tilde{\alpha}_{ijkl} \) because \( \tilde{p}_1 \equiv p_1 \).) Then, \( \hat{H} = 0 \) corresponds
to the \( y \)-domain eikonal equation. The characteristic system of (2.54) can be written

\[
\frac{dp_1}{dq} = \frac{dx_2}{dp_2} = \frac{dx_3}{dp_3} = -\frac{dq}{\partial T} = -\frac{dp_2}{\partial T} = -\frac{dp_3}{\partial T} = \frac{d\hat{T}}{\hat{D}}
\]  

(2.55)

where \( \hat{D} = q\partial T + p_2\partial T + p_3\partial T \). Selecting \( \hat{T} \) as a parameter, we can rewrite the system as

\[
\frac{dp_1}{d\hat{T}} = \frac{\partial \hat{T}}{\hat{D}} = \frac{1}{2} \partial_q \hat{a}_{ijkl} \hat{p}_j \hat{p}_k \hat{g}_j \hat{g}_k
\]

\[
\frac{dx_2}{d\hat{T}} = \frac{\partial \hat{T}}{\hat{D}} = \hat{a}_{2ijkl} \hat{p}_j \hat{g}_j \hat{g}_k
\]

\[
\frac{dx_3}{d\hat{T}} = \frac{\partial \hat{T}}{\hat{D}} = \hat{a}_{3ijkl} \hat{p}_j \hat{g}_j \hat{g}_k
\]

\[
\frac{dq}{d\hat{T}} = -\frac{\partial \hat{T}}{\hat{D}} = -\hat{a}_{ijkl} \hat{p}_j \hat{g}_j \hat{g}_k
\]

\[
\frac{dp_2}{d\hat{T}} = -\frac{\partial \hat{T}}{\hat{D}} = -\frac{1}{2} \partial_{x_2} \hat{a}_{ijkl} \hat{p}_j \hat{p}_k \hat{g}_j \hat{g}_k
\]

\[
\frac{dp_3}{d\hat{T}} = -\frac{\partial \hat{T}}{\hat{D}} = -\frac{1}{2} \partial_{x_3} \hat{a}_{ijkl} \hat{p}_j \hat{p}_k \hat{g}_j \hat{g}_k
\]

(2.56)

where \( \hat{a}_{ijkl} = \hat{a}_{ijkl}(-q, x_2, x_3) \). The polarization, \( \hat{g} \), is given by

\[
(\hat{a}_{ijkl}(-\partial_{x_1} \hat{T}(y), x_2, x_3)\hat{p}_i \hat{p}_l - \delta_{jk}) \hat{g}_k(y) = 0,
\]

(2.57)

and therefore equal to \( g \) at the point \( (x_1, x_2, x_3) |_{x_1=-\partial_{x_1} \hat{T}(p_1, x_2, x_3)} \), i.e.

\[
\hat{g}(y) = g(x).
\]

(2.58)

To find the amplitude solution, similar to (2.20), we express the zeroth-order approximation of the displacement vector in the \( y \)-domain, which will be called the Maslov displacement vector, as

\[
\hat{\hat{u}}^{(0)}(y) = \hat{\phi}^{(0)}(y) \hat{g}(y),
\]

(2.59)

where \( \hat{\phi}^{(0)}(y) \) is the Maslov amplitude. Following Thomson & Chapman (1985), we can write a modified Smirnov's lemma for the present problem. Note that the first three equations of the system (2.56) may be rewritten

\[
\frac{dp_1}{d\hat{T}} = -\frac{1}{2} \partial_{x_1} \hat{a}_{ijkl} \hat{p}_j \hat{p}_k \hat{g}_j \hat{g}_k
\]

\[
\frac{dx_2}{d\hat{T}} = \hat{a}_{2ijkl} \hat{p}_j \hat{g}_j \hat{g}_k
\]

\[
\frac{dx_3}{d\hat{T}} = \hat{a}_{3ijkl} \hat{p}_j \hat{g}_j \hat{g}_k
\]
from which we can derive the modified lemma as

$$d_T \ln \left( \frac{\partial (p_1, x_2, x_3)}{\partial (x_1, x_0, x_0)} \right) = \frac{1}{2} \frac{\partial}{\partial (x_1, x_0, x_0)} \left( \partial_{\|} ( \partial_{\|} p_1 \partial_{\|} g_1 \partial_{\|} g_1 ) + \partial_{\perp} ( \partial_{\perp} p_1 \partial_{\perp} g_1 \partial_{\perp} g_1 ) \right).$$

(2.60)

Substituting (2.59) into (2.45), using Smirnov's lemma (2.60), after some tedious steps, we finally obtain a simple equation similar to (2.26)

$$d_T \tilde{\phi}^{(0)} + \frac{1}{2} \partial \tilde{\phi}^{(0)} d_T \ln \left( \frac{\partial (p_1, x_2, x_3)}{\partial (x_0, x_0, x_0)} \right) = 0.$$  

(2.61)

The integral of (2.61) is simply

$$\tilde{\phi}^{(0)}(y) = \tilde{\phi}^{(0)}(y_0) \left( \frac{\partial (p_1, x_2, x_3)}{\partial (x_0, x_0, x_0)} \right)^{1/2} \left( \frac{\partial (p_1, x_2, x_3)}{\partial (x_0, x_0, x_0)} \right)^{-1/2}$$

$$= \tilde{\phi}^{(0)}(x_0) \left( \frac{\partial y_0}{\partial y} \right)^{1/2}$$

$$= \tilde{\phi}^{(0)}(y_0) \left( \frac{\partial y_0}{\partial y} \right)^{1/2}.$$  

(2.62)

where $\tilde{\rho} = \tilde{\rho}(-\partial_{\|} \tilde{T}, x_2, x_3)$, and the Maslov Jacobian $\tilde{J}_0$ and $\tilde{J}$

$$\tilde{J}_0 \equiv \frac{\partial (p_0, x_0, x_0)}{\partial (T, q_1, q_2)},$$

$$\tilde{J} \equiv \frac{\partial (p_1, x_2, x_3)}{\partial (T, q_1, q_2)}$$

(2.63)

have been used.

### 2.3 Maslov integral seismogram

In the previous section, we have described a solution for harmonic wave motion in a form valid in a continuous, inhomogeneous, anisotropic medium in the $y$-domain. The harmonic wave field in the $x$-domain is obtained using the inverse slowness transform (2.37). Then the time domain solution can be obtained using the inverse frequency transform (2.3). In practice, the two transforms (2.37) and (2.3) can be combined using Chapman's slowness method of transforming the frequency before the slowness to yield an analytic expression (Chapman 1978, Chapman & Drummond 1982)

$$y(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{i \omega}{2\pi} \right)^{1/2} \int_{-\infty}^{+\infty} \tilde{u}(\omega, y) e^{i\omega p_1 x_1} e^{-i\omega t} dp_1 d\omega$$

$$\approx \int_{-\infty}^{+\infty} \tilde{u}^{(0)}(y) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{i \omega}{2\pi} \right)^{1/2} e^{-i\omega (t-\tilde{\rho}(p_1, x_1))} dp_1$$
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\[ \Lambda(t) \equiv \lambda(t) + i\tilde{\lambda}(t) = \frac{H(t)}{t^{1/2}} + i\frac{H(-t)}{(-t)^{1/2}}, \quad (2.66) \]

where \( H(t) \) denotes the Heaviside step function, and the overbar on \( \tilde{\lambda}(t) \) indicates the Hilbert transform. Note that the term \((i\omega/2\pi)^{1/2}\) in Maslov’s inverse Fourier transform (2.3) has been factored as

\[ \left( \frac{i\omega}{2\pi} \right)^{1/2} = \frac{(-i\omega)(-i\text{sgn}(\omega))}{2\sqrt{2\pi}} \left( \frac{\pi}{\omega} \right)^{1/2} e^{i\pi/4} \]

and used in deriving (2.64); for example, the term \((\pi/\omega)^{1/2}e^{i\pi/4}\) has produced the real part of the analytic time series, \( \lambda(t) = H(t)t^{-1/2} \).

The intermediate integral over \( p_1 \) after inverse frequency transform is in fact the familiar inverse Radon transform when the zeroth-order approximation is made to the wavefield in the time-slowness domain (Chapman & Drummond 1982; Brown 1994). (The corresponding forward Radon transform, or slant stack, is an integral obtained by integrating expression (2.36) over \( \omega \). It transforms the wavefield from the space-time domain to time-slowness domain.) Representing the wavefield solution by the slowness integral allows the neighboring rays arriving at the vicinity of an observational point (with each denoted by \( p_1 \)) to make their contribution to the wanted wavefield. Specifically, it can be seen from (2.64) that at time instant \( t \), the rays with \( p_1 \) values satisfying \( t - \tilde{\theta}(p_1, \xi) = 0 \) are responsible for the wavefield construction. Then contributions from a bundle of neighboring rays will build up a time history of particle motion, forming a waveform. In fact, only singularities of \( 1/|\partial_{p_1}\tilde{\theta}(p_1, \xi)| \), as shown on the last line of (2.64) (where the summation is over \( p_1 \) values which solve \( t = \tilde{\theta}(p_1, \xi) \)) make an effective contribution provided the \( p_1 \) integral is unlimited, and the amplitude \( \tilde{U}^{(0)}(y) \) is smooth. The important consequence of integrating the neighboring rays is that the wavefield obtained is stable at caustics and may contain some low frequency wave components. This can be seen clearly later in other sections.
Also, the phase function $\tilde{\mathbf{\varphi}}(p_1, x)$ defined in (2.65) can be considered roughly as the arrival time at the receiver coordinate $x_1$ of a plane that is transverse to the neighboring $p_1$ ray, when the ray makes its contribution in an isotropic medium. From this point of view, a $p_1$ ray is sometimes called a plane $p_1$ wave (Chapman & Drummond 1982) or Snell wave (Kendall & Thomson 1993). Such a wave has a front with constant component of slowness (i.e., constant $p_1$ in this context) that propagates at a uniform apparent phase velocity. In general, it is curved in inhomogeneous media; but it can be considered plane in the proximity of the $p_1$ ray. The phase function $\tilde{\mathbf{\varphi}}(p_1, x)$ becomes the geometrical traveltime of the $p_1$ ray if measured at the position where the ray arrives, i.e. at $x_1 = x_1(y)$. Note that $x_1$ should not be confused with $x_1(y)$; the former is the receiver coordinate and is independent of $p_1$, whereas the latter is the $x_1$ coordinate of the rays exhibiting the slowness $p_1$.

From (2.64), we see that geometrical arrivals occur at the stationary points of the phase function where

$$\partial_{p_1} \tilde{\mathbf{\varphi}}(p_1, x) = 0. \quad (2.67)$$

From (2.67) and (2.65), we obtain

$$x_1 = -\partial_{p_1} \hat{T}(p_1, x_2, x_3), \quad (2.68)$$

which is the same as (2.51), mapping the ray paths from the $y$-domain to the $x$-domain. (Similarly, (2.50) can also be obtained from (2.67) and (2.65).) An alternative form of equation (2.67) is

$$x_1 - x_1(y) = 0. \quad (2.69)$$

At the stationary point, the $p_1$ values solve (2.69). Then, according to (2.65), the plane wave travel time reduces to the geometrical travel time

$$\tilde{\mathbf{\varphi}}(p_1, x) = T(x) \quad \text{where} \quad x_1 = x_1(y). \quad (2.70)$$

Expanding about these geometrical values yields the geometrical approximation for (2.64) (otherwise called the first motion approximation)

$$u(t, x) \simeq \sum_{x_1 = x_1(y)} \text{Re} \left\{ \frac{\hat{U}^{(0)}(y)}{\sqrt{2}} \Delta(t - T(x)) e^{i \frac{\pi}{2} - \text{sgn}(\partial_{p_1} x_1(y))} \right\}, \quad (2.71)$$

where the $\Delta(t)$ analytic time series is defined in (2.35).

In regions where both the zeroth-order transform (GRT) result (2.34) and the first motion approximation (2.71) of the first-order transform result (2.64) are valid, equations
(2.71) and (2.34) must agree. This leads to the canonical transformation between the displacement vectors in the $x$- and $y$-domains

$$
\dot{U}^{(0)}(y) = U^{(0)}(x) \left| \partial_{\rho} x_{1}(y) \right|^{1/2} e^{-i\pi[1 - sgn(\partial_{\rho} x_{1}(y))]y^2 - i\tilde{\sigma}(y)/2}
$$

$$
= \phi^{(0)}(x_{0}) \left( \rho_0 \frac{\partial x_0}{\partial x} \right)^{1/2} \left| \partial_{\rho} x_{1}(y) \right|^{1/2} \hat{g}(\tilde{x}) e^{-i\tilde{\sigma}(y)/2}
$$

$$
= \phi^{(0)}(x_{0}) \left( \rho_0 \frac{\partial x_0}{\partial y} \right)^{1/2} \hat{g}(y) e^{-i\tilde{\sigma}(y)/2}
$$

(2.72)

with $x_1 = x_1(y)$, $x_1 = -\partial_{\rho} \tilde{T}(y)$, or $p_1 = \partial_{x_1} T$. For simplicity of notation, we will omit $\rho$ whether $\rho$ is a function of argument $x_1$ or $p_1$ can be figured out easily from the context. The Maslov index $\tilde{\sigma}(y)$ is defined as

$$
\tilde{\sigma}(y) \equiv \sigma(x, \tilde{x}_0) + [1 - sgn(\partial_{\rho} x_{1}(y))]y^2/2.
$$

(2.73)

Note that we have used $\hat{g}(y) = \hat{g}(\tilde{x})$ at the geometrical arrivals. Comparing this result with (2.59), we have

$$
\hat{\phi}^{(0)}(y) = \phi^{(0)}(x_{0}) \left( \frac{\partial x_0}{\partial y} \right)^{1/2} e^{-i\sigma(y)/2}.
$$

(2.74)

At the initial point,

$$
\hat{\phi}^{(0)}(y_0) = \phi^{(0)}(x_{0}) \left( \frac{\partial x_0}{\partial y} \right)^{1/2}.
$$

(2.75)

Then (2.74) can be rewritten

$$
\hat{\phi}^{(0)}(y) = \phi^{(0)}(y_{0}) \left( \frac{\partial y}{\partial x_{0}} \right)^{1/2} \left( \frac{\partial y}{\partial x_{0}} \right)^{-1/2} e^{-i\tilde{\sigma}(y)/2}
$$

$$
= \phi^{(0)}(y_{0}) \left( \frac{\partial y}{\partial y} \right)^{1/2} e^{-i\tilde{\sigma}(y)/2}
$$

$$
= \phi^{(0)}(y_{0}) \left( \frac{\partial j}{\partial y} \right)^{1/2} e^{-i\tilde{\sigma}(y)/2}.
$$

(2.76)

Comparing (2.76) and (2.62), it can be seen that we have reached the same result using the two different approaches, except that (2.76) involves the Maslov KMAH index in the transform domain. It should be pointed out that although the canonical transformation (equations (2.72) to (2.76)) was derived for geometrical rays only, it is certainly suitable for neighboring rays because (2.76) appears the same as (2.62) (except for the KMAH index) which is valid for both types of rays.

The canonical transformation can also be expressed in terms of the ray and Maslov Jacobians, which is often used in computations. It is found by noting the canonical
transformation expressed in terms of the two Jacobians, which is given by substituting (2.30) and the last line of (2.76) into (2.72), and by using the initial condition following (2.75)

\[ \tilde{\phi}^{(0)}(y_0) = \phi^{(0)}(x_0) \left( \frac{V}{J} \right)_{T_0}^{1/2}. \]  

(2.77)

The form of the transformation obtained reads

\[ \tilde{J}(y) = J(x_1(y), x_2, x_3) \partial_x p_1. \]  

(2.78)

Note that for convenience of calculation, in (2.78), we have used \( J \) to denote the ray Jacobian into which the velocity of a ray, \( V \), has been absorbed

\[ J = V \frac{\partial(x_1, x_2, x_3)}{\partial(T, q_1, q_2)}, \]  

(2.79)

and which should not be confused with the conventional definition as in (2.25).

Substituting (2.72) in (2.64) we obtain the expression

\[ u(t, \omega) \simeq -\frac{1}{2V^{2/3}} \frac{d}{dt} \text{Im} \left\{ \Lambda(t) * \sum_{t=\tilde{\theta}(p_1, \omega)} \phi^{(0)}(x_0) \left( \frac{\rho_0 \partial_x x_0}{\rho \partial_x T} \right)_{T}^{1/2} \frac{\partial x_1(y)}{\partial \tilde{\theta}(p_1, \omega)} \right\} \cdot \hat{g}(y)e^{-i\pi \partial(y)/2} \]  

(2.80)

as the first-order Maslov integral (MI) seismogram, where \( \tilde{\theta}(p_1, \omega) \), \( x_1(y) \), and \( \hat{g}(y) \) are defined in (2.65), (2.68), and (2.58), respectively. This result applies in some circumstances where ray theory breaks down. In addition, it allows us to avoid the two-point ray tracing problem when the wave field is wanted at certain specified points.

Very often, it is in fact more convenient to employ the Maslov integral presented in the frequency domain and in terms of the Maslov Jacobian. We finally provide it here:

\[ \hat{u}(\omega, \omega) = \left( \frac{i\omega}{2\pi} \right)^{1/2} \int \frac{\tilde{\phi}_0}{\rho(p_1)\tilde{J}(p_1)} \frac{\hat{g}(p_1)e^{-i\pi \tilde{\theta}(p_1)/2}e^{i\omega \tilde{\theta}(p_1, \omega)}}{1/2} dp_1 \]  

(2.81)

where \( \tilde{\phi}_0 \) is a constant denoting \( \tilde{\phi}^{(0)}(y_0)(\rho_0\tilde{J}_0)^{1/2} \)

### 2.4 Maslov uniform seismogram

Maslov's Fourier transform (2.36) can be considered as a decomposition of the wave field into local plane waves, with each of them denoted by the slowness component \( p_1 \). This can
be further understood by examining the Legendre transformation (2.49). It decomposes the travel time curve into a series of tangent planes described by the slopes \( \partial_{x_1} T \) or the slownesses \( p_1 \) (recall equation (2.50)). The validity of this decomposition requires that the travel time curve be inflexion-free (its curvature must never degenerate, so that the transformation from the travel time to intercept time is one-on-one). Otherwise, at the place where the curvature degenerates, i.e. where

\[
\partial_{x_1} p_1 = \partial^2_{x_1} T = 0, \tag{2.82}
\]

da curve of intercept time \( \dot{T}(y) \) in the \( y \)-domain will bifurcate and become multiple-valued (Sewell 1977). Also, the canonical transformation (2.72) will map the finite GRT amplitude into an infinite Maslov amplitude, because the multiplier, \( \partial p_1 x_1 \), in (2.72) becomes infinite (see (2.82)). As a result, the Maslov integral solution (2.80) will break down.

Consider rays in the \( x \)- and \( y \)-domains. In the \( x \)-domain, neighboring rays become parallel at a wavefront inflection point, whereas in the \( y \)-domain, their trajectories appear focused at the corresponding place, and the Maslov amplitude becomes infinite there. This exactly resembles the situation in the \( x \)-domain where if a ray is at a caustic, the GRT amplitude becomes infinite. In order to distinguish the two types of singularities, a caustic is sometimes called an \( x \)-caustic. Analogously, the places where the singularities in the \( y \)-domain occur are referred to as the \( y \)-caustics. Because energy focusing at a \( y \)-caustic is not real – in fact, the wavefield at the corresponding physical position is smooth (Brown 1994) – and the \( y \)-caustic itself moves with different coordinate systems, a \( y \)-caustic is often termed a pseudocaustic (Klauder 1987b).

A cosmetic removal of the \( y \)-caustic adopted by Frazer & Phinney (1980) is to change the integration variable from the receiver slowness \( p_1 \) to the source slowness \( p_{01} \). Following this approach, we can re-express the Maslov transform solution (2.64) as

\[
\mathcal{U}(t, x) \sim \int_{-\infty}^{+\infty} \hat{U}^{(0)}(y) \partial_{p_{01}} p_1 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{i\omega}{2\pi} \right)^{1/2} e^{-i\omega(t-\dot{t})} d\omega dp_{01}
\]

\[
= \int_{-\infty}^{+\infty} \hat{U}^{(0)}(\hat{y}) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{i\omega}{2\pi} \right)^{1/2} e^{-i\omega(t-\dot{t})} d\omega dp_{01}
\]

\[
= -\frac{1}{2\sqrt{2\pi}} \frac{d}{dt} \text{Im} \left\{ \Lambda(t) * \sum_{t \neq \dot{t}(p_{01}, x)} \frac{\hat{U}^{(0)}(\hat{y})}{\partial_{p_{01}} \dot{y}(p_{01}, x)} \right\} \tag{2.83}
\]

where \( \hat{y} \equiv (p_{01}, x_2, x_3) \), \( \dot{y}(p_{01}, x) \) is still evaluated by (2.65) but tabulated as a function of
\[ p_{01} \text{ as} \]
\[ \dot{\theta}(p_{01}, \bar{z}) \equiv \ddot{T}(\bar{y}) + p_1(p_{01})x_1 = T(x_1(\bar{y}), x_2, x_3) + p_1(p_{01})(x_1 - x_1(\bar{y})). \]  
(2.84)

and the new Maslov displacement vector \( \dot{\vec{e}}^{(0)}(\bar{y}) \) which I call the normalized Maslov displacement vector is also expressed as a function of \( p_{01} \):
\[
\dot{\vec{e}}^{(0)}(\bar{y}) = \frac{\mu^{(0)}(\bar{x})}{\rho} \frac{\partial p_1(\bar{y})}{\partial \bar{x}} \left( \frac{\rho_0}{\rho} \frac{\partial x_0}{\partial \bar{x}} \right)^{1/2} \left[ \frac{\partial p_0}{\partial \bar{y}} \right]^{1/2} \left[ \frac{\partial p_0}{\partial \bar{y}} \right]^{1/2} \dot{\bar{y}}(\bar{y}) e^{-i\hat{\theta}(\bar{y})/2}. \]  
(2.85)

where the Maslov index \( \hat{\theta}(\bar{y}) \) is given by
\[ \hat{\theta}(\bar{y}) = \sigma(\bar{x}, \bar{x}_0) + [1 - \text{sgn}(\partial p_1 x_1(\bar{y}))]/2. \]  
(2.86)

Substituting (2.85) in (2.83) yields the Maslov integral (MI) seismogram in the source slowness domain
\[
\dot{u}(t, \bar{z}) \simeq -\frac{1}{2\sqrt{2\pi} dt} \text{Im} \left\{ \Lambda(t) \ast \sum_{t=\hat{\theta}(p_{01}, \bar{z})} \phi^{(0)}(\bar{x}_0) \left( \frac{\rho_0}{\rho} \frac{\partial x_0}{\partial \bar{x}} \right)^{1/2} \left[ \frac{\partial p_0}{\partial \bar{y}} \right]^{1/2} \left[ \frac{\partial p_0}{\partial \bar{y}} \right]^{1/2} \dot{\bar{y}}(\bar{y}) e^{-i\hat{\theta}(\bar{y})/2} \right\}. \]  
(2.87)

We note that in this approach, the normalized Maslov amplitude (the magnitude of the vector (2.85)) absorbs the term \( \partial p_0, p_1 \), and is regular at \( y \)-caustics. There, it has a zero magnitude of order 1/2, because the term \( \partial p_0, p_1 \),
\[ \partial p_0, p_1 = \partial x_1, p_1 \partial p_0, x_1 = 0. \]  
(2.88)

overbalances the singularity of \( \partial p_1 x_1 \) (to a lower-order of zero, \( O(0^{1/2}) \), in fact). Also, in the source slowness domain, the normalized Maslov amplitude and the Maslov phase often appear single-valued, provided the source is not located at a turning point. All these bring about an integrable Maslov seismogram representation (2.87) which rarely breaks down at \( y \)-caustics. Nevertheless, once a \( y \)-caustic occurs, the result will contain serious linear phase error that is caused by the stationary point of the phase function \( \hat{\theta}(p_{01}, \bar{z}) \) at the \( y \)-caustic (Frazer & Phinney 1980, Chapman & Drummond 1982, and Brown 1994). More will be seen in the subsequent chapters.

In Maslov asymptotic theory, the \( y \)-caustic problem is, instead, handled by switching the wave solution to the GRT results at \( y \)-caustics. This is possible because Liouville's
Theorem states that $x$-caustics and $y$-caustics never coincide (Maslov & Fedoriuk 1981, Thomson & Chapman 1985). Then by blending the GRT (zeroth-order) solution (2.34) and the MI (first-order) solution (2.87) through weighting factors $e_0$ and $e_1$, we can obtain the Maslov uniform wave solution

$$u(t, x) = \sum e_0(p_{01})(\frac{\rho_0}{\rho} \frac{\partial x_0}{\partial x} \bigg|_T)^{1/2} q(x) \Re\{\phi^{(0)}(x_0) \Delta(t - T(x))e^{-i\sigma(x_0)\sqrt{2}}\}
- \frac{1}{2\sqrt{2}\pi} \frac{d}{dt} \Im\{A(t) \ast \sum_{t=\hat{t}(p_{01}, x)} e_1(p_{01}) \phi^{(0)}(x_0) \left(\frac{\rho_0}{\rho} \frac{\partial x_0}{\partial x} \bigg|_T\right)^{1/2} \frac{\partial p_{01} x_1(y)}{\partial x} \frac{\partial p_{01} p_1(y)}{\partial y} \frac{\partial \hat{y}}{\partial x} e^{-i\hat{y}y^{1/2}}\},$$

which is valid everywhere and eliminates, theoretically speaking, the problems due to caustics in any domain (Chapman & Drummond 1982). Clearly for every ray, the weighting functions $e_0$ and $e_1$ must satisfy

$$e_0(p_{01}) + e_1(p_{01}) = 1.$$  

(2.90)

The functional form and argument of the weighting functions are chosen so $e_0$ or $e_1$ will be zero in regions where the corresponding asymptotic expression is invalid. Between these regions, the functions vary smoothly (infinitely differentiable) between zero and unity. As the first-order solution (2.87) is in general more accurate than the zeroth-order solution (2.34), it is preferable to choose $e_1 = 1$ and $e_0 = 0$ except near the $y$-caustics (Chapman & Drummond 1982).

### 2.5 Lagrangian equivalence: phase partitioning

Occurrence of caustics of the GRT solution and pseudocaustics of the Maslov integral solution can be understood by considering the Lagrangian manifold of rays in phase space and its projections in various subdomains. A Lagrangian manifold is a hypersurface that is spread by ray trajectories from a source or initial wavefront in the six-dimensional $(x \times p)$ phase space described by the ray coordinates $x$ and the slownesses $p$ (Maslov & Fedoriuk 1981, pp100). (The phase space reduces to four dimensions for 2D propagation.) According to Liouville's theorem, ray trajectories in phase space never cross and the Lagrangian manifold is always regular (Maslov & Fedoriuk 1981 and Thomson & Chapman 1985). But if the manifold in phase space bends appropriately relative to the direction of projection, then its projection from phase space into a subdomain may
fold. Rays will cross along the fold line. In the real space, crossing rays form \( x \)-caustics (or simply caustics) at which the GRT amplitude becomes infinite. In the transform domain, they form \( y \)-caustics at which the MAT amplitude becomes infinite. In the terminology of geometry, a “catastrophe” is introduced into the projection by the folding (Poston & Stewart 1978). Because \( x \)- and \( y \)-caustics always have different positions, invalid GRT solutions at \( x \)-caustics can be replaced by Maslov transform solutions. Vice versa, invalid Maslov transform solutions at \( y \)-caustics can be replaced by GRT solutions. This leads to Maslov’s uniform wavefield representation (equation (2.89)).

An alternative approach of removing pseudocaustics is implied by origination of pseudocaustics. It is by use of the *Lagrangian equivalence* transformation. A Lagrangian equivalence means two Lagrangian manifolds can be deformed into one another by either changing the spatial coordinates or adding a function on the spatial coordinates. The transformation involved is said to be “canonical” because it preserves intact the form of the Hamiltonian system. That is, the deformed ray trajectories in the new phase space satisfy Hamilton’s equations and naturally form a new Lagrangian manifold (Arnold, Gusein–Zade & Varchenko 1985). For example, rays described in the Cartesian-based phase space \( (x_1, x_2, x_3, \partial_{x_1} T, \partial_{x_2} T, \partial_{x_3} T) \) can be mapped onto a new phase space \( (r, \theta, \phi, \partial_{r} T, \partial_{\theta} T, \partial_{\phi} T) \) by transforming coordinates from Cartesian to spherical. In the new phase space they are also controlled by a Hamiltonian. The effect of applying the Lagrangian equivalence transformation, or for short, Lagrangian equivalence, is simple. It is to put a Lagrangian manifold into a regular form so that no catastrophe occurs during the expected projection.

Kendall & Thomson (1993) have devised a technique for “adding a function on the spatial coordinates”, which they called the “phase partitioning” approach. This approach involves adding a reference phase \( T^r(x_1) \) on the phase-distance space \( T(x_1) \sim x_1 \), and obtaining a residual phase \( \tilde{T}(x_1) \) by subtracting the reference phase from the real phase \( T(x_1) \). Because the new residual phase may appear regular, in other words, inflection-free, the Legendre transformation can be applied over the whole phase range, and the wavefield solution in the transform domain no longer breaks down. Note that the system of the rays that are associated with the new residual phase still remains Hamiltonian. Their solution in the new transform domain can be derived in the same way as the rays associated with the real phase. Following is a brief outline for the formulation of the Maslov integral with phase partitioning.

From the real phase \( T(x_1) \), subtract a reference phase \( T^r(x_1) \) such that the residual phase
\( \tilde{T}(x_1) \) given by

\[
\tilde{T}(x_1) = T(x_1) - T'(x_1)
\]

will be inflection-free, i.e., \( \partial^2_{x_1} \tilde{T}(x_1) \neq 0 \). From this we immediately obtain a similar relationship between the reference slowness \( p'_1(x_1) = \partial_{x_1} T'(x_1) \) and the residual slowness \( \tilde{p}_1(x_1) = \partial_{x_1} \tilde{T}(x_1) \), as

\[
\tilde{p}_1(x_1) = p_1(x_1) - p'_1(x_1).
\]

and a relationship between the reference curvature \( \partial_{x_1} p'_1 \) and the residual curvature \( \partial_{x_1} \tilde{p}_1 \) as

\[
\partial_{x_1} \tilde{p}_1 = \partial_{x_1} p_1 - \partial_{x_1} p'_1.
\]

Note that the new residual slowness contains no stationarity, i.e., \( \partial_{x_1} \tilde{p}_1(x_1) \neq 0 \).

Writing the Maslov integral in terms of the residual slowness constitutes defining a new transform domain \( \bar{y} = (\tilde{p}_1, x_2, x_3) \). In such a domain, rays still satisfy the Hamiltonian constraint. And most importantly, the Lagrangian manifold has been shifted by an appropriate amount \( \tilde{p}_1(x_1) \) along the so-called \( \text{fibres} \) (lines that pass through the point \( (x_1, x_2, x_3) \) and lie parallel to the \( (p_1, p_2, p_3) \) axes), and has become regular. (That is, it looks unfolded in the direction transverse to \( \tilde{p}_1 \).) Then, the same procedure as used to formulate the Maslov integral (2.81) in the original \( y \)-domain can be followed in finding the one in the \( \bar{y} \)-domain. This gives (Kendall & Thomson 1993)

\[
\hat{u}(\omega, \varepsilon) = \left( \frac{i\omega}{2\pi} \right)^{1/2} e^{i\omega \tilde{T}(\varepsilon)} \int \frac{\tilde{\phi}_0}{\rho(\tilde{p}_1) J(\tilde{p}_1)} e^{-i\pi \varepsilon/2} e^{i\omega \tilde{\phi}(\tilde{p}_1, \varepsilon)} d\tilde{p}_1
\]

where \( \tilde{\phi}_0 \) is a constant, and the Maslov index

\[
\tilde{\sigma} \equiv \sigma(x, x_0) + [1 - \text{sgn}(\partial_{x_1} x_1(\bar{y}))] / 2.
\]

The new Maslov phase function is simply

\[
\tilde{\phi}(\tilde{p}_1, x) \equiv \tilde{T}(\bar{y}) + \tilde{p}_1 x_1 = \tilde{T}(x_1(\bar{y}), x_2, x_3) + \tilde{p}_1(x_1 - x_1(\bar{y})).
\]

Similar to (2.78), the modified Maslov Jacobian is related to the (velocity absorbed) ray Jacobian through the canonical transformation, as

\[
J = \frac{\partial \tilde{p}_1 / \partial x_1}{\partial (T, q_1, q_2)}.
\]

Phase partitioning becomes crucial in removing pseudocausistics near which there are causistics, because in this case, Maslov's approach of switching the solution from MI to GRT
is invalid. This is because wave behavior at finite frequency cannot be treated as a point effect. The wavefield at any point involves the contributions of rays traveling through a zone of finite width which connects the source and the observational point. Accordingly, a caustic point expands to a caustic region, over which the wavefields predicted by GRT are all inaccurate. And once a pseudocaustic point occurs inside a caustic region, Maslov’s blending breaks down.

There exist certain facts that may limit the applicability of phase partitioning. For instance, during the implementation of phase partitioning, attention must be paid to understanding the geometry of the wavefront. A manual intervention is also required. For some cases of wave propagation, phase partitioning may not always be able to deform a Lagrangian manifold into a regular one. However, this technique has been shown successful in several applications made by Kendall & Thomson (1993). It will be applied later in the thesis.

## 2.6 Important quantities and expressions used in computation

Restating briefly the important quantities and expressions that are used in computation from those presented before can actually provide us with appreciable convenience. They are as follows:

1. **Eikonal ray tracing** to find ray paths in phase space, \((x_1, x_2, x_3, p_1, p_2, p_3)\), and their traveltimes \(T(x)\) from the ray equations (2.16)

   \[
   \frac{dx_i}{dT} = a_{ijkl}p_l \frac{D_{jk}}{D}
   \]

   \[
   \frac{dp_i}{dT} = -\frac{1}{2} \frac{\partial_x a_{ijkl} p_l p_s D_{jk}}{D}.
   \]

2. **Dynamic ray tracing** to calculate the geometrical spreading of rays in phase space, \((\partial_q, x_1, \partial_q, x_2, \partial_q, x_3, \partial_q, p_1, \partial_q, p_2, \partial_q, p_3)\) (where \(r = 1, 2\)), from the dynamic ray equations (2.32)

   \[
   \frac{d}{dT}(\partial_q, x_i) = (F^{(1)}_{ij} \partial_q, x_j + F^{(2)}_{ij} \partial_q, p_j)/2D
   \]

   \[
   \frac{d}{dT}(\partial_q, p_i) = -(F^{(3)}_{ij} \partial_q, x_j + F^{(4)}_{ij} \partial_q, p_j)/2D.
   \]
3. **Jacobian J** (with ray velocity absorbed) for the amplitude of the GRT wavefield solution from (2.79)

\[ J = \frac{\partial (x_1, x_2, x_3)}{\partial (T, q_1, q_2)}. \]

4. **Legendre transformation** mapping the phase from the real space to transform domain by (2.49)

\[ \hat{T}(p_1, x_2, x_3) = T(x_1(p_1, x_2, x_3), x_2, x_3) - p_1 x_1(p_1, x_2, x_3), \]

which is used in computing the Maslov phase function (or plane wave traveltime) \( \hat{\theta} \), as given by (2.65)

\[ \hat{\theta}(p_1, \xi) \equiv \hat{T}(p_1, x_2, x_3) + p_1 x_1. \]

5. **Maslov Jacobian \( \hat{J} \)** for the amplitude of the MI wavefield solution given by (2.78) (a result of the canonical transformation)

\[ \hat{J} = \frac{\partial (p_1, x_2, x_3)}{\partial (T, q_1, q_2)}. \]

6. **Partial derivative** \( \partial_{x_1} p_1 \) or \( \partial_{p_1} x_1 \) for the determination of the Maslov index \( \delta(y) \) in (2.73) can be obtained from (2.78) by the ratio of the Maslov and ray Jacobians. i.e.

\[ \partial_{x_1} p_1 = \hat{J} / J. \]

7. **Maslov integral seismogram** computed using the inverse Radon transform in (2.64), i.e.

\[ u(t, \xi) = -\frac{1}{2\sqrt{\pi}} \frac{d}{dt} \text{Im} \left\{ \Lambda(t) * \int_{-\infty}^{+\infty} \hat{u}^{(0)}(y) \delta(t - \hat{\theta}(p_1, \xi)) dp_1 \right\}. \]

where the integral over \( p_1 \) can easily be evaluated using a numerical technique.

8. **Analytic time series** \( \Delta(t) \) and \( \Lambda(t) \) that may be used in the GRT and MI seismogram construction. They are given in (2.35) and (2.66), respectively, i.e.

\[ \Delta(t) \equiv \delta(t) + i \tilde{\delta}(t) = \delta(t) - i \frac{1}{\pi t}, \]

\[ \Lambda(t) \equiv \lambda(t) + i \tilde{\lambda}(t) = \frac{H(t)}{t^{3/2}} + i \frac{H(-t)}{(-t)^{3/2}}, \]

where the Hilbert transform can be performed in the frequency domain via the fast Fourier transform.
9. *Blended GRT/MI Maslov seismogram* requires weighting functions $\epsilon_0$ and $\epsilon_1$ that are subject to $\epsilon_0 + \epsilon_1 = 1$ for every ray. The GRT and MI components are combined as in (2.89). Note that the Maslov integral is actually evaluated as an inverse Radon transform with respect to the source slowness $\rho_{01}$ (so that the integral is always integrable).

10. *Maslov integral seismogram with phase partitioning* is performed in a similar way as the one without phase partitioning. Of course, one needs to choose a reference phase function $T^r(x)$. The main difference is that the integral with phase partitioning is evaluated in the residual slowness domain $\tilde{\rho}_1$. The fashion used is like the inverse Radon transform in (2.64), but with reference phase $T^r(x)$ (see (2.94)) included in order to shift the phase-partitioned waveform from the residual phase $\tilde{T}(x_1)$ back to the true one $T(x_1)$.

More details on the computation of the GRT and Maslov seismograms can be found in some later chapters and Appendix E.
Chapter 3

Testing the Accuracy of the Maslov Integral Seismogram

3.1 Introduction

Since the introduction of Maslov asymptotic theory, examples of modeling waveforms have been provided in papers by Chapman & Drummond (1982); Thomson & Chapman (1985); Thomson & Chapman (1986); Kendall & Thomson (1993); Brown (1994); Guest & Kendall (1993); Hanyga, Lambaré & Lucio (1995); Keers & Chapman (1995); Liu & Tromp (1996). Nevertheless, a question associated with this method still remains unanswered. How accurate in practice are the wave components predicted by the Maslov integral?

Observed seismograms are always band limited. Few contain frequency components with wavelengths less than $10^{-3}$ of the path length. On the other hand, Maslov theory is asymptotic, only provably accurate in a high frequency sense. and it is very difficult to estimate what practical frequency this corresponds to. Furthermore, asymptotic seismograms contain low frequency components, and it is especially difficult to assess their validity. I therefore devote this chapter to testing the accuracy of the Maslov integral.

As it is hard to find either an analytic method that can assess the accuracy of the integral or an exact solution for a general medium to compare with, I examine the accuracy of Maslov integral seismograms by comparing them with seismograms computed by the finite difference (FD) method. Four earth models that are idealized but typical of subsurface structures have been used. The first is a medium with a positive vertical gradient in velocity that generates a triplicated traveltime curve. The second is a low-
velocity waveguide in which wavefront bending, kinking, and folding will be investigated. The third is also a gradient model but is used to test the role of a rotated Maslov integration axis. The last is a high-velocity (anti-waveguide) slab that diffracts waves passing through it.

Prior to presenting the test examples, a useful step that can improve the accuracy of the Maslov modeling is described. It is to rotate the axis of Maslov integration. This step should be considered when modeling waveforms near wavefront anomalies. By anomalies I mean wavefront bends, kinks, and folds. Waves can be diffracted from these anomalies, and in turn can distort waveforms nearby. Therefore, modeling of waveform distortion requires rays arriving at the vicinity of the observational point and the wavefront anomaly point be non-linearly superposed through Maslov integration. This can be achieved by rotating the Maslov integral. A method of computing a rotated Maslov integral will be described.

Pseudocaustics will be removed only using Kendall & Thomson's (1993) phase-partitioning technique, because we are currently concerned with the pure Maslov integral. Errors made by switching GRT and MI solutions in dealing with pseudocaustics will be addressed in the next chapter.

3.2 Rotation of the Maslov integral

3.2.1 Lagrangian manifolds in rotated coordinates: a geometrical example

As stated in section 2.5, rays in the physical space can be developed into a Lagrangian manifold in \((x, p)\) phase space. There, the ray trajectories never cross one another (a consequence of Liouville's theorem). However when the manifold is projected onto a subdomain, ray trajectories may cross each other and cause a failure of geometrical ray theory. A simple approach to avoiding the failure is to choose a subdomain in which rays do not cross.

To illustrate this, a simple example of 2D wave propagation in a medium with a uniform vertical velocity gradient is shown in Figure 3.1. Part (a) shows the physical ray fan (thin lines between AB and DC) and its wavefronts. In this case, a fold with a pair of caustics has developed. We cannot draw the Lagrangian manifold of these rays in 4D phase space but we can plot their submanifolds in subdomains. Figures 3.1b and c show respectively
Figure 3.1: (a) Rays (thin lines) and wavefronts (thick lines) in a 2D physical \((x_1, x_3)\) domain. The medium velocity varies only vertically with an increasing gradient. These rays exhibit a pair of caustics developing from the same point, on which GRT breaks down. Parts (b) and (c) show their projections in the 3D mixed subdomains \((x_1, x_3, p_1)\) and \((x_1, x_3, p_3)\), respectively. Both have been smoothly unfolded, but they have different geometries.
Figure 3.2: Further projection of the Lagrangian submanifolds from 3D to 2D mixed domains. (a) Projection from Figure 3.1b to \((p_1, x_3)\), where rays each fold back on themselves exactly. The places where this happens are often known as the turning points. (b) Projection from Figure 3.1c to \((x_1, p_3)\), which folds on two lines.
the Lagrangian submanifolds in the 3D \((x_1, x_3, p_1)\) and \((x_1, x_3, p_3)\) subdomains, where the ray fan is unfolded onto smooth surfaces ABCD and rays do not cross themselves. More interestingly, geometry of the two submanifolds appears different, because each is a different projection. However, both necessarily project the same way into the \((x_1, x_3)\) domain. Figure 3.2 shows the projections into (a) \((x_3, p_1)\) and (b) \((x_1, p_3)\). This time both are folded. The projection in (a) contains only one pseudocaustic fold. Beyond the fold, rays do not cross and pseudocaustics do not occur. In contrast, the projection in (b) has a pair of folds, putting pseudocaustics in several parts of the domain.

The Maslov integral over the ray data collected along the \(x_1\)-axis in the \((x_1, x_3)\) space uses a projection onto the \((x_3, p_1)\) domain. Specifically, the integral sums the ray information sampled along the \(p_1\)-axis in \((x_3, p_1)\). No pseudocaustic is encountered in the \(p_1\) range shown. The Maslov integral solution for a receiver in the middle of the \(x_1\) range in which the ray summation is carried out over a vertical line encounters the pair of pseudocaustic folds shown in Figure 3.2b. This is very unsatisfactory. It is therefore advisable that the rays in this model be summed in the \(x_1\) direction, provided the receiver is well inside or well beyond the turning points.

Integrations in other directions can be viewed as a result of coordinate or integral rotation. As this happens, rays in the real space and in their Lagrangian manifold in phase space do not alter their geometry. Nevertheless, the mixed subdomain and the submanifold will both change as the slowness component of the transformation/integration is rotated. Thus, one should try to rotate the direction of transformation/integration so that the obtained submanifold appears simple and the Maslov integral is integrable. In fact the direction in which the medium properties are relatively homogeneous is generally favorable. One example of rotation is the familiar “WKBJ condition” (Guest & Kendall 1993).

### 3.2.2 Rotated Maslov integral

Technically, a rotated Maslov integral can be evaluated by rotating the coordinate system. But this is inconvenient in many ways. For instance, the coordinates of the model grid points have to be changed. Here I describe a more convenient method of computing a rotated integral. In this method, model parameters are a function of fixed coordinates, and rays are traced in the same coordinates. All vectors used, e.g. slowness, are resolved into the three components in the fixed coordinate directions. Nothing differs from the usual ray tracing in a horizontal/vertical Cartesian system except that the ray
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Geometry of a Slanted MIA Relative to a Ray

Figure 3.3: Geometry of a slanted Maslov integration axis (MIA) relative to a physical ray in the \((x_1, x_3)\) plane, where bold \(U\) denotes the displacement vector, bold \(P\) the slowness vector, and bold \(p_l\) the component of the slowness vector along the MIA.

information required is sampled along a rotated integration line.

In a fixed Cartesian system \(x = (x_1, x_2, x_3)\), consider a slanted line in the \((x_1, x_3)\) plane along which neighboring rays will be summed. I call this line the Maslov integration axis (MIA). Let \(x_{0l}\) be its origin, \(l\) its directional vector, \(x_l\) its running coordinate (Figure 3.3) and \(x_{\perp l}\) the coordinate in the \((x_1, x_3)\) plane perpendicular to the axis. Thus \(x_l = (x_l, x_{\perp l})\) is a rotated axis. Transforming the slanted \(x_l\) axis onto the slowness axis \(p_l\) leads to a rotated transform domain \(y_l = (p_l, x_{\perp l})\).

The Maslov integral along the rotated \(x_l\) axis can be written directly as we write the one along the coordinate \(x_1\) axis; just treat the \(x_l\) axis as an axis of the virtually rotated coordinates. (In fact, rotation of the Maslov integral or, equivalently, rotation of coordinates, is canonical (Arnold et al. 1985, pp 294-5); the functional form of the Maslov integral along the slanted MIA remains unchanged.) Then routine steps used for performing a
Maslov integral can be followed in the case of MIA rotation. The component of the slowness vector along the MIA, $p_l$, constitutes the transform domain, and is used as a variable of the Maslov integration. Neighboring rays are traced to the rotated MIA. There, their traveltimes $T(p_l)$ and offsets $x_l(p_l)$ are sampled and are functions of $p_l$. Then similarly, for a receiver point of coordinate $x_l$, the Maslov phase and the Maslov Jacobian can be written respectively

$$\tilde{\theta}(p_l, x) = \tilde{T}(y_l) + p_l x_l = T(x_l(p_l), x_{\perp l}) + p_l(x_l - x_l(p_l))$$

and

$$\tilde{J} = J \frac{\partial p_l}{\partial x_l}.$$  

with the ray Jacobian $J$ defined in (2.79). With these two terms we can write the rotated Maslov integral as

$$u(\omega, x) = \left(\frac{i\omega}{2\pi}\right)^{1/2} \int \frac{\tilde{\phi}_0}{(p(p_l)\tilde{J}(p_l))^{1/2}} \tilde{q}(p_l) e^{-i\pi \tilde{\sigma}(p_l)/2} e^{i\omega \tilde{\phi}(p_l, x)} dp_l,$$

where the Maslov index $\tilde{\sigma}$ is given as

$$\tilde{\sigma} = \sigma + (1 - \text{sgn}(\partial x_l / \partial p_l))/2.$$  

The only difficulty that may arise in evaluating the rotated integral is associated with computing the rotated Maslov Jacobian, $\tilde{J}$. One possibility is to calculate the unrotated ray Jacobian $J$ and $\partial p_l / \partial x_l$ first, and then apply (3.2). But here I outline an alternative that I have used. Note that

$$\frac{\partial p_l}{\partial x_l} = \nabla (p \cdot l) \cdot l = l^T \left\{ \frac{\partial p_l}{\partial x_j} \right\} l = l^T (J_l^{-1} \tilde{J}_l) l \quad i, j = 1, 2, 3$$

where $l$ is the direction cosine vector of the integration $l$-axis, and matrix $\tilde{J}_l$ is given by

$$\tilde{J}_l = \{ \tilde{J}_{ij} \}, \quad \tilde{J}_{ij} = \frac{\partial y_{ij}}{\partial U} \quad i, j = 1, 2, 3.$$  

Note in (3.6) we have introduced the symbol $\nu$ to denote the ray coordinate system $(T, q_1, q_2)$, and $y_{ij}$ $(i, j = 1, 2, 3)$ to denote any of nine mixed domains made up of two spatial coordinates and one slowness coordinate. For instance, $y_{32} = (x_1, p_3, x_3)$, with $x_2$ replaced by $p_3$. By substituting (3.5) into (3.2) we can obtain the Maslov Jacobian in the coordinate-rotated transform domain $y_l$

$$\tilde{J} = l^T \tilde{J}_l l.$$  

The rotated Maslov Jacobian is then computed from the Jacobian components $\tilde{J}_{ij}$ evaluated in the fixed coordinate directions. The ratio of the Maslov and ray Jacobians in turn gives the derivative $\partial p_l / \partial x_l$, which is needed in (3.4).
3.3 Testing the accuracy of the Maslov integral

Since the FD seismograms could only be computed for 2D cases (2D model geometry and 2D source configuration), the MI seismograms must model this case, not the more usual point source case. There are two cases in my calculations where I used J-M. Kendall’s original 3D ray tracing package (Kendall & Thomson 1989). There is no difficulty in handling 1D, 2D or 3D velocity-density variations in this software, but it always assumes a point source and a 3D model geometry. However, it is not difficult in principle to convert seismograms back or forth between the case of a 3D cylindrical model with a point source on the axis and a 2D cartesian model with a line source (Dampney 1971), and I have derived a counterpart for the Maslov solution (Appendix A). Later in my research, I produced a modified version of Kendall’s software (Appendix E) to compute results directly for the 2D acoustic case.

In handling general velocity-density variations (1, 2 or 3D), even an asymptotic modeling system requires a discretization of the model, because the velocity and density are required to be a computable, smooth function of position. Interpolation functions provide this analytical form within each discretization cell, with continuity between functions and their first and second derivatives being required at the cell boundaries. In this chapter, all the test models (2D or 3D) are interpolated by the quintic spline method that I have developed (Appendix D).

Four models will be tested in this chapter, two elastic and two acoustic. I have used two FD modeling packages: one is for elastic models originally written by R. W. Clayton (Clayton & Engquist 1977) based on the explicit fourth-order scheme and the other is for acoustic models coded by X. Zeng (1996) based on the explicit second-order scheme. In all cases, I have used a source wavelet to which sufficient high frequency band limiting has been applied that grid dispersion and other similar errors will be negligible. Pseudo-caustics have been encountered in all the Maslov modelings, but have been removed by using the phase partitioning technique. Phases of diffractions mis-positioned after phase partitioning have been corrected and addressed where necessary.

3.3.1 Test 1: Triplications and shadows

A wavefront folding can be caused by any sharp change in velocity structure. An example of this, generated in a one dimensional velocity structure, is shown in Figure 3.1. Where
Figure 3.4: An elastic, isotropic, vertical gradient model. Rapid changes in velocity gradient generate a variety of waveform complexities. (a) Velocity, density model. (b) Ray propagation (solid and dashed lines) and wavefront geometry (dots or thick lines). The source is at (0.21,0.75). Note the pair of caustics due to the increasing gradient, and the divergence shadow due to the decreasing gradient. A narrow, denser ray bundle has been used to ensure the fine sampling of rays required by MI. The wavefronts fold on the pair of caustic lines. (c) The emerging travel-time/distance curve, reduced by $T = T_{real} - x_1/1.17$. The EF branch in the divergence shadow is shown dashed.

the wavefront folds, three arrivals may arise instead of a single one. The traveltime curve recorded through such a region is still continuous, but it consists of two faster branches and one slower branch, and these branches form a triplication. The faster branches merge with the slower branch at the cusps of the triplication. The $x_1$ position of the cusps indicates the position of the fold caustics at the observation level. The regions beyond them are geometrical shadows (Figure 2.1). GRT cannot be applied to model waveforms in regions of cusps and shadows.

The model used is elastic, with the P-wave velocity $\alpha$, the S-wave velocity $\beta$ and the density $\rho$ varying only vertically (3.4a). Effectively, the source is a line parallel to the
Figure 3.5: The horizontal components of the GRT amplitudes (top row) and MAT amplitudes (bottom row). Note that both GRT and MAT amplitudes exhibit singularities (point C and D, and B respectively) at distinct locations in the space domain (left column (a) and (b)), or slowness domain (right column (c) and (d)).
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horizontal $x_2$-axis. However, the MI modeling was done initially for a point source and later converted to the line case. The parameters are presented in dimensionless units, scaled to the depth and properties at the first gradient change, where the depth is chosen as the zero level. They increase linearly down to depth 0, more rapidly from depth 0 down to depth $-0.5$ and then remain constant down to depth $-1$.

For a source in the shallow gradient layer and receivers on a horizontal line near the source depth, the ray fan\(^1\) directed towards the receivers (shown in Figure 3.4b) exhibits many of the previously mentioned problems. Firstly, the fan of rays entering the steep gradient region (between depth 0 and $-0.5$) folds on a pair of caustic lines. The pair merge at the kink point at $(2.25, 0)$ and extend to intercept the receiver line at C and D. By "kink" we mean a position at which a waveform begins to fold abruptly. It is also termed the cusped caustic, a higher-order cusp catastrophe (Brown 1986) in the vicinity of which the wavefield is asymptotically described by the Pearcy function (Pearcy 1946). Along the pair of caustic lines the wavefronts (large dots or thick lines) fold on themselves, forming a series of cascading triplications.

Also interesting are the geometrical shadows in the ray fan. There are two types: one is the caustic shadow and the other the divergence shadow. The caustic shadow (as described earlier in Chapter 2 Figure 2.1) exists to the left of caustic line D and to the right of caustic line C. The divergence shadow exists beyond the last ray to be bent back towards the surface. E marks the edge of the shadow. Rays leaving the source at steeper angles than the ray to E do not return to the surface. However, we do expect some diffracted energy in this region, and its arrival time can be calculated from additional rays associated with the edge ray to E that do satisfy Snell's law but include a horizontal segment of indefinite length along the transition from steep to flat velocity gradient. These are shown dashed in Figure 3.4b.

Lastly, a pseudocaustic occurs at point B, a transition point from upwards-emitted to downwards-emitted rays. The information of rays on both sides of B is a double-valued function of the horizontal slowness. This leads to the failure of the original Maslov integral. Remedying this problem is definitely important.

The travel-time vs distance curve for receivers from A to F along $x_3=1$ is displayed in Figure 3.4c. It consists of a triplication BCDE, a divergence shadow EF, and a simple

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\(^1\)The model (3D for ray tracing) is sectionalized into cubic block of size $0.178 \times 0.178 \times 0.089$ units. The ray tracing time step is about at $0.071$ units. Note the P-wave heterogeneity, the relative variation of the P-wave velocity per wavelength $\Delta \alpha / \alpha$, is smaller than 0.2, so the use of GRT in the propagation of waves is appropriate.
hyperbolic type branch AB. Clearly, no travel time is continued beyond the cusps C and
D of the triplication. The curve changes from concave to convex upwards running from
A through the point B. This indicates a pseudocaustic at B.

Figure 3.5 presents the 3D GRT and MAT amplitudes of the horizontal displacement
component in the space and slowness domains. GRT gives smooth amplitudes in the
vicinity of the pseudocaustic B. But it becomes singular at the caustics C and D, and
underestimates the amplitudes in the divergence shadow beyond E (Figures 3.5a and c).
In contrast, the Maslov amplitude is finite everywhere except at the pseudocaustic B
(Figures 3.5b and d). From E to F it is roughly constant and small. Note that the curves
shown are somewhat distorted from their true theoretical form by numerical sampling
and interpolation. The GRT amplitudes at C and D and Maslov amplitude at B are true
singularities.

Kendall & Thomson’s (1993) phase-partitioning approach will be used to remove the
pseudocaustic B. Note at this moment we are not considering Maslov’s blending approach
yet, although it may apply because the pseudocaustic B is fairly separate from the caustics
C and D, as shown by the left or right column of Figure 3.5.

In phase partitioning, we need to obtain a new phase which is free of inflection points.
or pseudocaustic-free. But it is difficult to define a reference phase just by viewing the
geometry of the real phase. Instead, as suggested by Kendall & Thomson (1993), it is
much easier to begin by defining the curvature of an expected reference phase. A curve
without an inflection point has a curvature which is nowhere zero. Vice versa, a function
of curvature which is nowhere zero defines a function of phase that is free of inflection
points. Therefore, choosing a line of reference curvature that does not intersect the real
curvature promises the finding of a usable reference phase. Kendall & Thomson’s (1993)
procedure of defining a reference curvature and then integrating it successively will be
followed in the following way:

1. Display the line of real curvature \( \frac{\partial^2}{\partial x_1^2} T(x_1) \) vs distance on the computer screen, the
   value of which is computed, e.g., by \( \frac{\partial^2}{\partial x_1^2} T = \frac{\dot{J}}{J} \) (see the last section of Chapter
   2);

2. Enter a set of master points such that their linear interpolation forms a piecewise
   line that does not intersect the real curvature. The line itself defines a function of
   the wanted reference curvature;

3. Integrate piecewisely the function of the reference curvature with respect to \( x_1 \), once
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giving the reference slowness and twice the reference phase. Both are analytically continuous.

Note that the obtained reference phase is second-order differentiable. The function of the reference curvature is not everywhere differentiable, but this higher-order discontinuity should not have any appreciable effect on waveform modeling (Kendall 1991).

Figures 3.6 and 3.7 provide the detailed results of the phase partitioning I have used. The former shows the real phase curvature (solid line) as displayed on a monitor. The curvature changes sign continuously at B and discontinuously (through infinity) at C and D. Only the change through zero at B is important (a pseudocaustic). Note that the small positive spike near E is an artifact caused by the spline interpolation of the model. The dotted line in the drawing represents the chosen reference curvature. The dots are the values at the ray arrivals and are obtained by linear interpolation between the graphically-entered master-point values. Note that the reference curvature we use does cross the real EF branch, but this will not cause any singularity in the Maslov integral because this branch shrinks to a point in the slowness domain (Figure 3.5d). Elsewhere, the important portion ABCDE of the reference curvature does not cross the real curve, yielding in 3.7(e) a relative curvature which no longer passes zero.

Figure 3.7b shows the real (or physical) slowness (solid line), the reference slowness obtained by integrating the reference curvature once (dotted line), and their difference – the relative slowness (dashed line). The relative slowness has no stationary point at B. Part (a) exhibits the real phase (solid line), the reference phase given by integrating the reference curvature twice (dotted line), and the relative phase (dashed line). The relative phase contains no inflection point at B. Part (d) shows the Maslov phase function in the original slowness domain that bifurcates at B (solid line). (The dotted line denotes the relative phase. The large values and slopes of the reference phase on the right side of (a) make large the range of the reference Maslov phase function.) After phase partitioning, the new phase function in the residual slowness domain, shown in (f), becomes single-valued. So does the new Maslov amplitude displayed in (g), with value at B regularized.

Figure 3.8 shows the horizontal components of the MI and FD² waveforms. The source signal for both sets is a band-limited, symmetric Gaussian wavelet. Comparison in (a)

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2The model (2D for FD) is discretized by square grid with cells of size 0.025 x 0.025 units, and the time is sampled at an interval of 0.0075 units. For the considered source wavelet, most wavelengths span more than 6 grid points, and the dominant period (0.28 units) contains about 37 sampling points. The waves near the triplication have propagated along their raypaths between 10 to 20 wavelengths.
Figure 3.6: The interactively displayed real phase curvature (solid line) and the interactively entered reference phase curvature (dots). The object is to find a reference phase curvature which will lie evenly between the curvatures for each of the triplication branches. Note here the attempt has been made to place the reference curvature evenly between the important branches ABC and CD. Its intercept with EF will not cause appreciable effect because the EF branch shrinks in the Maslov integral.
Figure 3.7: Phase partitioning used to remove a pseudocauastic. (a) The real (solid), reference (dotted) and relative (dashed) phases. (b) The real (solid), reference (dotted) and relative (dashed) slownesses. (c) The real (solid) and reference (dotted) curvatures. (d) The real (solid, bifurcated) and reference (dotted) phase functions at location $x_1 = 4$. (e) The relative curvature. Note that the important branches ABC and CD do not cross zero, particularly at B. (f) The single-valued relative phase function (for a receiver at $x_1 = 4$) used in the new Maslov integral (over the residual slowness $p'_1$). Note the two stationary points near $p'_1 = -2.25$. (g) The MAT amplitude used in the new integral, now becoming regular between B and F.
shows that the MI and FD waveforms agree well not only for the simple geometrical signals near B, but also for the strongly interfering waves within the triplication. The accuracy of the MI diffractions is better revealed in the enlarged portion shown in (b). The two diffractions just beyond the triplication cusp C and at $x_1=5.46$ and 5.71 are satisfactorily given by MI. The one at $x_1=5.71$ is roughly two wavelengths away from the cusp (one dominant wavelength is about 0.2 units long). Beyond $x_1=5.71$, the caustic diffractions estimated by MI have too large amplitudes. Of the MI diffractions into the divergence shadow EF, the one close to the divergence edge E and at $x_1=3.96$ (also see part (a)) is accurate, to a distance of less than two wavelengths from the divergence edge. (Note it merges there with another waveform.) Events beyond that distance have insignificant amplitudes. Nevertheless, the diffractions two wavelengths away from either the triplication cusp or the divergence edge look reasonable in (a), in the sense that they decay and deform with distance, and arrive at the correct times.

Figure 3.8c shows the amplitude spectra of the MI and FD waveforms. Both are in a good agreement, with higher-frequency parts decaying beyond the range $x_1=5$. The degradation of MI at very low frequency is illustrated in Figure 3.9, where the dominant frequency has been reduced by 3.6 times.

It is necessary to make some remarks on the moveout effect of phase partitioning on diffraction waveforms, although examples are omitted for brevity. Essentially, the diffractions in MI are end-point signals, albeit for a narrow spread of rays rather than a true single ray for the divergence shadow edge E. As only real rays (i.e., rays in real number space, see, e.g., Thomson 1994; Zhu & Chun 1994) are used to synthesize the Maslov seismogram, events in ray shadows are predicted by analytic continuation of the real ray information from the lit region (i.e. the coordinate of the receiver in the Maslov integral is simply moved into the shadow and the integral is found to decay like the Airy function). As a result, the obtained diffractions distribute in the direction of the triplication cusp or the divergence edge.

In a phase partitioning, however, an intermediate Maslov seismogram is synthesized first in the residual-phase/distance domain. Such a seismogram contains all types of waveforms that can be given by the Maslov integral, including diffractions. Then, the final seismogram is obtained simply by moving the intermediate waveforms to the true phase position. This is done automatically by adding the reference phase to the residual phase. But the procedure is valid only for geometrical arrivals not for diffractions, because we only define the reference phase for the former. This consequently alters the phase of
Figure 3.8: (a) The 2D horizontal MI (dashed) and PD (solid) waveforms around a

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diffractions. A naive way of correction is to manually place the diffractions along the direction of the cusp or end-point of the real phase curve. Those shown in Figure 3.8 were corrected like this.

Sometimes correction of diffractions is not worthwhile because firstly, diffractions in deep shadows are neither important, nor predictable by MI (Figure 3.8b); Secondly, phase partitioning may not appreciably misposition important shallow-shadow diffractions. Results given later are examples.

### 3.3.2 Test 2: wavefront bending, kinking and folding

In the previous example, we saw that it is very easy for an initially smooth wavefront in a continuously inhomogeneous medium to become seriously but still smoothly (or continuously) bent, then to become kinked and finally folded. Rays converge at the smooth bend, then strongly focus at the abrupt kink and eventually moderately focus on the cusps of the folded wavefront.

Figure 3.10 shows a source in a 2D acoustic waveguide in which the velocity is laterally constant but varies hyperbolically with depth. Interestingly, a series of wavefront foldings and ray caustics develop because of the guiding-effect of the model.
Figure 3.10: A 2D acoustic waveguide in which the velocity (shaded in grey) increases hyperbolically away from the zero depth. The rays (thin lines) emitted at the lowest velocity depth ($z_3=0$) are internally refracted, forming a series of caustics. The wavefront (thick lines) propagates initially as a circle, then bends, kinks and folds. Subsequent kinking and folding occur on successive reversed branches of the folded wavefronts. The waveform variation and modeling by MI through the first kink ($z_1=1$) are investigated. Open circles mark the observational points and five vertical lines indicate the Maslov integration axes used.
Figure 3.11: Wavefield snapshots for the low-velocity waveguide model given by FD. A circular wavefront at $t=0.25$ (a), becomes elliptic at $t=0.50$ (b), bilobal at $t=0.75$ (c), kinked at $t=1.00$ (d), initially triplicated at $t=1.25$ (e), subsequently triplicated at $t=1.50$ (f). Afterwards the reversed branch of the folded wavefront develops inflection points (corresponding to pseudocaustics) at $t=1.75$ (g), another kink at $t=2.00$ (h), and another tiny triplication at $t=2.25$ (i). Note energy concentration at wavefront bends, kinks, folds and cusps.
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Figure 3.12: The MI (dashed) and FD (solid) waveforms around the first kink in the low-velocity waveguide model. The top row shows the waveforms from an impulsive source. The bottom row shows the frequency-limited version. The waveforms are regular in (a), but become distorted in (b) and (c), and finally become separated through the kink point in (d) and (e). Panels (a) – (e) correspond to the snapshots shown in Figures 3.11b, c, d, e, and f, respectively. The dotted lines are the ray traveltimes.

Figure 3.11 shows the snapshots of the wavefield obtained using Zeng’s (1996) 2D acoustic FD code\(^3\). An initially circular wavefront (a), bends at time (c), kinks at time (d), and then folds at times (e) and thereafter. The wavefront kinks and folds again on the reversed branch at times (h) and (i). Note energy concentration at bends, kinks, folds and cusps.

\(^3\)In this FD calculation, the model is discretized uniformly by 701 x 701 grid points. The computation has been run for 3500 steps at a time interval \(\Delta t=0.000588\) satisfying the condition for stability of calculation, i.e. \(\Delta t \leq 0.707 \Delta x/v_{\text{max}}\) (with \(\Delta x=0.00588\) units and \(v_{\text{max}}=7\) units). The wave propagates a maximum distance of roughly ten wavelengths. Note the spatial sampling rate is very high, because the wavefield near the kink changes rapidly. (Consequently, the temporal sampling rate becomes high.) For instance, for a typical frequency of 4 units and velocity of 1 unit, the wavelength is 0.25 units and spans 43 grid points. This is four times higher than the commonly-used sampling rate (for second-order FD). We have actually used FD to compute high-frequency seismograms.
Maslov waveforms are calculated in the region around the first kink \((x_1=1.0, \text{ Appendix B})\). The sampled observational points are openly circled in Figure 3.10. Neighboring rays are summed along the vertical lines of observational points. Phase partitioning has been performed for all the integration lines removing the effects of pseudocaustics. The detail will be omitted this time and thereafter.

Figure 3.12 shows the waveforms of the MI and FD seismograms (dashed and solid lines, respectively). From the top row, we see that both the MI and FD impulsive waveforms show distortion before the kink, an effect geometric ray theory would not predict. (See the trace at depth 0 in (b).) Beyond the kink the waveform tails evolve into distinct geometrical and diffracted arrivals (top (c) – (e)). Most of these features have been satisfactorily predicted by MI. Prediction of diffractions can be better understood from the frequency-band-limited waveforms shown at the bottom. Clearly, the diffractions given by MI at depths 0.12 and 0.24 in bottom (d) and 0.24 in bottom (e) are fairly accurate. They are within a range of one wavelength beyond the cusps. The diffraction at depth 0.35 in bottom (e) is about a wavelength away from the triplication cusp; it has been over-estimated by MI. The one at depth 0.35 in bottom (d) is delayed by phase partitioning.

Note that no phase partitioning could be found to remove the pseudocaustics beyond the second kink \((x_1=2)\). There, the reversed branch of the wavefront fold develops two more pseudocaustics. Although the second folding of the wavefront is almost a repeat of the first, but phase partitioning fails. Finding an alternate Lagrangian equivalence technique perhaps based on a change of the coordinate system is the topic of interesting future work.

To understand better the physics of waveform distortion, I have compared Maslov waveforms with different integration axes (MIA) near the first kink. Figure 10 shows the waveforms at the position \((1.0, 0.24)\). MI does a better job of predicting the distortion if the integration axis crosses the kink. This indicates that diffraction causes distortion effects. Note that this is an acoustic medium, and the diffracted tail has the same sign as the primary impulse. But this is not always the case. For vector waves, diffracted amplitude may sometimes be negative and make a waveform tail smaller.

To reinforce the view that the way distortion is predicted by MI is important, I consider another test. The model is also a one dimensional velocity structure, i.e., laterally homogeneous, elastic, isotropic. Velocity and density decrease with depth down to -84 m in the upper part and then remain constant in the rest (Figure 3.14a). Note that this model
Figure 3.13: Maslov waveforms (dashed lines along the bottom) generated using three rotated integration axes (MIAs) (as shown along the top). The solid-line waveform is the accurate FD solution. All these waveforms are calculated just above the first kink (Figure 3.10), i.e., at the circled point (1.0, 0.24). The best one is obtained from the vertical axis (b).

has been discretized and then made continuous using the quintic spline interpolation of Appendix D. Part (b) displays a P-wave propagation emitted from a point source at (14, -105). Because the wavefield becomes bent after the gradient change (dotted line), one may naturally wonder how the waveforms there can be affected by the surrounding environment?

The waveform at a wavefront bending point, circled in (b) at (77, -84), is then tested with 4 choices of the MIA. Part (c) gives the traveltimes registered along the vertical MIA (solid line in (b)), showing more obviously the development of the pseudocaustic A at depth -74.9 m, C at -85.5 m, and bend B at depth -78.8 m.

Figure 3.15 gives the ray fields used in MI, showing the variation of ray spreading along
various rotated MIA’s from none (a) to weak in (b), moderate in (c) and strongest in (d). All Maslov waveforms are obtained using phase partitioning, and are shown dashed in Figure 3.16. Comparison with the FD waveform\(^4\) (solid line) shows that for these four cases, MI predicted respectively such waveforms that are (a) standard (i.e. non-distorted) but poorest; (b) slightly distorted but poor; (c) distorted and good; (d) further distorted and best.

**3.3.3 Test 3: high-velocity slab**

We have investigated several examples of convergent focusing. But, it is also necessary to understand divergent propagation as might be generated by a high-velocity slab of sinking lithosphere. The test model used is 2D acoustic (Figure 3.17). (Thus Zeng’s (1996) 2D FD code and the modified 2D ray tracing code (Appendix E) will be used, and no conversion of seismograms will be invoked.) The model has a constant background velocity of 1 unit and is perturbed by a high-velocity slab that extends vertically from the original point to the depth of -1 unit. The slab has a Gaussian velocity cross-section with peak velocity 50% above the background.

As rays propagate into the high-velocity region, the slab acts as an anti-waveguide and refracts rays away from itself (Figure 3.17). A smooth bend in the wavefront develops just above the bottom of the slab (depth 1 unit). It eventually becomes a kink, and finally the wavefront is separated into faster- and more slowly-propagating portions, folding on either side of the slab. The fold on the left half is marked by ABCD, of which AB and CD are faster branches and BC is the slower branch. The travel-time curve develops an early-arriving phase over the slab, forming a symmetrical pair of triplications on either side. Only the triplication associated with the left wavefront fold ABCD will be considered next.

Figures 3.18 and 3.19 show the seismograms at a depth of 0.375 units above the top of the slab. For compactness, the traveltime scale has been reduced. More rays were used in construction than are shown in Figure 3.17. Note two pseudocaustics arise on the forward branches AB and CD, somewhere near the cusps B and C. Phase partitioning has been used to remove them.

Waveforms modeled by GRT, MI and FD for a relatively low frequency source are dis-

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\(^4\)It is obtained by transforming with (A.12) in Appendix A the 2D response which is computed based on the 2D model discretized by the same square mesh of size 0.7x0.7 m and time interval of 0.07 ms.
Figure 3.14: The second test on the effect of the rotated Maslov integration axis. (a) A laterally homogeneous model. It is 3-dimensional for MI but 2-dimensional for FD; and the FD result will be converted to 3D. The solid line denotes the P-wave velocity, $\alpha$, the dashed line the S-wave velocity, $\beta$, and the dotted line the density, $\rho$. (b) P-rays (solid lines) and wavefronts (thick lines and dots). They are emitted from a point source at $(14, -105)$. Above the gradient (dotted line) the rays focus and the wavefronts (plotted every 5 ms or five time steps) bend and then kink. (c) The traveltime curve along the vertical line through the interesting point.
Figure 3.15: P-rays used in the four rotated Maslov integrals. They exhibit different degrees of focusing: (a) none; (b) weak; (c) moderate; (d) strongest.
Figure 3.16: The Maslov waveforms (3D vertical components, dashed line) obtained from the rays with different degrees of focusing given in Figure 3.15. The solid line denotes the accurate result converted from the 2D FD calculation. We see a variation in the waveform modeling with the concentration of rays used in the integration; the best result (d) is generated by the greatest ray concentration (see Figure 3.15d).
Chapter 3: Accuracy of the Maslov Integral

Figure 3.17: A 2D acoustic high-velocity "slab" model. The background is homogeneous with a velocity of 1 unit. The slab is vertical, and its velocity is described horizontally by a Gaussian function of $0.5e^{-\left(x_{1}/0.1\right)^{2}}$, with a maximum perturbation of 50%. It is 1 unit deep and 0.2 units wide (measured at $1/e$ of the maximum). The rays (line) and wavefronts (dot) emanate right beneath the slab at point (0, -2.5). The Maslov modeling is performed along a horizontal line at depth of 0.375 units.

played in Figure 3.18. Although affected by the triplication, the waveforms are too slow to resolve it. The accurate FD waveforms are slightly stronger and distorted relative to the simple geometrical arrivals through the homogeneous background (e.g. the one at $x_{1}=-2.5$). The form of the waves is more sensitive to the slab than the traveltme. GRT gives a good approximation only to those around the center of the triplication but not near the cusps B and C; but MI does a good job for all of them.

Diffractions occur just beyond the two cusps B and C: one group to the right of B and the other to the left of C. Interestingly, the one beyond B is stronger, because it results from the strong forward branch AB; the other is relatively weak because of the weak forward branch CD involved. The group beyond B has isolated waveforms. Whereas

\[^{5}\text{Between } 1 > x_{1} > -1, \text{ the direct event through the slab becomes invisible because the rays are so divergent. The only significant events are the diffractions associated with the cusps of the triplications (e.g. B).}\]
Figure 3.18: Seismic modeling made by GRT (dotted), MI (dashed), and FD (solid) for the left triplication ABCD (Figure 3.17). (a) Waveforms for an impulsive source. (b) Waveforms for a frequency-limited source. The traveltime (solid) has been reduced by $T = T_{\text{real}} + x_t/1.92$. Note that the frequency limit is low, and the triplication is not resolvable.
Figure 3.19: Seismic modeling made by GRT (dashed) and MI (solid). (a) Waveforms for an impulsive source. (b) Waveforms for a frequency-limited source. Note that the frequency limit is relatively high, and the triplcation is resolvable.
the group beyond C mostly merges with the waveforms of the earlier branch near A. thus it only distorts the waveforms slightly. GRT cannot predict diffractions, but both groups of diffractions have been reasonably well estimated by MI. Diffraction events running across the upper-right corners of the drawings result from the complementary triplication at positive $x_1$.

Clearly, the diffractions predicted by MI within about a wavelength (0.38 units) from the triplication cusps (such as those at $x_1 = -0.25, -1.5, \text{and} -1.75$) are in good agreement with those predicted by FD. However, the diffractions given by MI in the deep shadows beyond a wavelength are erroneous. The worst case occurs at $x_1 = 0$, right above the slab, where the event is mispositioned by the phase partitioning.

GRT shows itself to be unsuitable in predicting all the above-mentioned diffractions and it over-estimates the waveform amplitudes at the cusps. The failure of GRT is very severe in this case, because diffractions dominant the wavefield above the slab ($x_1 = 0$).

It is also interesting to know the propagation of short period waves such that the same triplication can be resolved by the waveforms; in a tele-seismic observation the case is quite like so. The frequency range now is more than ten times higher than the previous one. Thus the FD modeling has not been made – but in any case, the fact that MI gave satisfactory results at lower frequency gives us confidence at higher frequency.

From the wavefield modeled by GRT and MI (Figure 3.19), we see three branches of waveforms inside the triplication (between B and C). Those on the reversed branch BC have a Hilbert transform (a mirror image in the 2D case). Those of the forward branch AB through the homogeneous background have strong amplitudes. Whereas those of the forward branch CD through the slab are almost invisible. Unlike the previous test, the diffractions beyond the left cusp C are separate from the earlier arrivals. Also, all the diffractions on both shadows attenuate faster. They even disappear near the top of the slab (D). One should be reminded that the sampled diffractions may be over-estimated, although they look reasonable. The displayed diffractions (on the right and left four traces) are all located beyond a wavelength ($< 0.05$ units) from the cusps. Nevertheless they are indeed a good reference in learning the wavefield of interest. As usual, GRT failed at places near and beyond the cusps B and C.
3.4 Chapter conclusions

Wavefields propagating in a structured medium often contain interesting diffractions into geometrically-shadowed regions, or waveform distortions when waves of different branches overlap. Cases studied in this chapter include (1) traveltime triplications and divergence shadows due to vertical velocity gradients; (2) wavefront bending, kinking and folding in a low-velocity waveguide, and wavefront bending in a vertical gradient model; and (3) wave propagation through a high-velocity slab. Waveforms have been predicted by MI with phase partitioning and have been mostly compared with the finite difference solutions.

It has been shown that in the high-frequency range (for which medium variation is smooth, or relative medium variation over a wavelength is much smaller than 1), waveforms near wavefront anomalies such as bends, kinks, folds, or ray field edges can be fairly accurately given by MI, provided pseudocaustics, if any, can be completely removed. This is very satisfactory because these waveforms are often prominent signals on a seismogram. Waveform diffractions and distortions far from the wavefront anomalies given by MI are not reliable. It appears appropriate to define the shallow shadow a region that is within a wavelength from the wavefront anomaly, in which we consider MI accurate. Outside this region the shadow is said to be deep and MI becomes unreliable. (For instance, it over-estimates the caustic diffractions in Figures 3.8b, 3.12e (bottom), and 3.18, or it under-estimates the divergence edge diffractions in Figure 3.8b.) Fortunately, diffractions (or distortions due to diffractions) in a deep shadow are usually weak features and are of little importance in seismic interpretation. Essentially, high frequency waveforms are controlled by wavefield geometry.

Although my examination has been restricted to continuous, isotropic media, the results definitely apply to discontinuous, anisotropic cases (Guest & Kendall 1993). Further, when a wavefront contains rapid changes in three dimensions, then the 2D Maslov integral wavefield representation is needed (Chapman & Drummond 1982; Kendall & Thomson 1993; Keers & Chapman 1995; Liu & Tromp 1996).

I have shown by example that accuracy of the Maslov integral often can be improved by rotating the Maslov integration axis (MIA). This is because correct prediction of waveform diffractions or distortions requires that the MIA cross-cut the wavefront anomalies. In practice the MIA should cross-cut the nearest wavefront anomaly. I have described a method for the calculation of a rotated Maslov Jacobian which makes it easier to calcu-
late a rotated Maslov integral. The examples also clarify a second way in which rotation affects Maslov integrals. Changing the direction of the Maslov integration can help move a pseudocaustic out of the integration range if the medium variation is one-dimensional.

A pseudocaustic can be easily introduced into the Maslov integral if a medium contains a structure. The phase-partitioning approach is useful in removing pseudocaustics occurring in noncomplex cases; this chapter shows a few of them. However, it is in general not easily incorporated into an automated waveform modeling package, although an interactive graphics implementation can reduce the necessary effort. More severely, there are many cases where phase partitioning breaks down altogether. In Section 3.3.2 it was shown that a low-velocity waveguide will quickly produce a situation where multiple caustics and pseudocaustics arise and phase partitioning does not apply.
Chapter 4

Finding a Robust Weighting Method for Blending the GRT and Maslov Integral Solutions

4.1 Introduction

Although the Maslov integral solution has been shown in Chapter 3 to be a sufficiently accurate method for modeling most features of high-frequency wavefields, phase partitioning requiring a manual intervention is often required to avoid problems at pseudocaustics.

In Maslov’s (1965) original work, he suggested that the wavefield solution be calculated by GRT at places where the transform (integral) solution breaks down. Thus (as described in Section 2.4) a uniform wavefield solution (i.e., a solution that is valid everywhere) combines the GRT and MI solutions. For convenience, it will be known as the MAT solution.

In MAT, the GRT and MI components are combined through use of weight functions $e_0$ and $e_1$, respectively. So far, rather little research has been published on how weight functions should be constructed. One early work can be found in the paper by Chapman & Drummond (1982, Figs. 13 through 16). The exact functions used for weighting were not given, probably because, as they reported, “the exact definition of the (weight) functions is not crucial”, as both solutions (GRT and MI) are valid asymptotically in the intermediate regions between caustics and pseudocaustics and only asymptotic accuracy is wanted. But they noted that the result is not perfect because an erroneous smooth pulse is produced by the smooth tail of the Maslov integral. Obviously, a smooth transition
between the solution methods is desirable. But for real seismological problems in which the recorded wavefields are frequency-band-limited, separations between caustics and pseudocaustics may not always be much longer than the dominant wavelength. Then the choice of the weight functions can become important.

Brown (1994) suggested weight functions which are hyperbolic functions of the local slope of Lagrangian manifold cross-sections. His example showed a satisfactory switching performed. However, real applications of his weighting method need further support. For instance, a judgement on the stability of the method must be given.

In this chapter, I explore the effects of the weighting functions on the accuracy of Maslov seismograms, and search for a robust weighting method. In particular, I first investigate the applicability of the Brown's weighting technique. Having realized its limitations, I then propose an alternative. Suggestions on the use of the two techniques are finally provided. All tests are performed on a guided wavefield that travels in a low velocity channel and exhibits interesting distorted waveforms.

4.2 Model
Chapter 4: A Robust Weighting Method

Fig. 4.1 shows the testing model, a 2D acoustic waveguide that is laterally homogeneous but vertically varying in velocity, as described by a hyperbolic function in (a). This is the same case used in Section 3.3.2 for the comparison with FD except for the units of coordinates. Note the density is assumed constant. The source is situated at the symmetrical depth $x_3=110$ m at lateral distance $x_1=20$ m. Rays (thin lines) and wavefronts (thick lines) emitted from this source are mainly trapped inside the waveguide, and a series of ray caustics and folded wavefronts are developed, as shown in (b). The wavefront along the symmetrical depth appears convex near the source point, becomes sharply kinked at $x_1=104.8$ m, singly folded beyond 104.8 m, and doubly folded beyond 189.6 m. The caustic lines (in 2D) along the wavefront’s first and second folds also emanate from depth $x_3=110$ and horizontal ranges $x_1=104.8$ and 189.6 m, respectively (Fig. 4.1b). This model is simple, but the wavefield in it exhibits many interesting waveform features that are commonly present in many observational data. In particular, I consider the three section lines shown dashed in Fig. 4.1b. These observe (1) the kinked wavefront at $x_1=104.8$ m; (2) the singly folded wavefront at $x_1=130$ m; and (3) the doubly folded wavefront at $x_1=220$ m.

4.3 Caustics and pseudocaustics in the problem

Fig. 4.2 shows in detail the GRT singularities involved on the three observational sections, and more interestingly, the pseudocaustics happening for all the cases. At the kinked wavefront point (at $x_1=104.8$ m and at the symmetrical depth) the traveltime curve (a) is accordingly kinked. But besides it inflects. A singularity of cuspoid type appears in the GRT amplitude at the kink point marked by $C_x$ (b). And two singularities, instead, occur in the MAT amplitude at the inflection points marked by $C_p$ (c).

By the distance ($x_1=130$ m) the wavefront develops a simple fold, the traveltime curve (d) forms a familiar triplication, but it inflects somewhere on each of the forward branches (circled points). The GRT amplitude (e) jumps at the cusps of the triplication (caustics of fold type). The MAT amplitude (f) also jumps at the inflection points.

The more complicated asymptotic information of the doubly-folded wavefront at $x_1=220$ m can be seen in the bottom row. There the solid line from A to E denotes the rays shown in Fig. 4.1, and the dashed line from F to J is for the other symmetric half. We note more caustics and pseudocaustics are generated. The traveltime curve in (g) contains two cascading triplications. Starting from A, the solid portion goes inflected
Figure 4.2: Asymptotic information across the wavefields that are kinked (top row), singly-folded (middle row), and doubly-folded (bottom row). Note that caustics occur at the cusps of traveltime curves (left column), and pseudocaustics at the inflections (circled points). The GRT amplitudes become singular at caustics (central column), and the MAT amplitudes at pseudocaustics (right column). See text for detail.
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Figure 4.3: Errors of caustic type (a) and pseudocaustic type (b). The former is a very large waveform; whereas the later a moderately-erroneous linear phase. Note that the GRT arrival “at” the wavefront kink was computed just after the kink so its waveform there suffers a Hilbert transform.

at B, folded at C, inflected at D, and folded at E. The dashed part undergoes the same changes symmetrically, folding at F, inflecting at G, folding and inflecting again at H and I (which overlaps with B). Four caustic (C, E, F, and H) and four pseudocaustic (B, D, G, and I; two of them overlapping) singularities can be seen in (h) and (i), respectively.

Fig. 4.3 demonstrates errors of caustic and pseudocaustic types. The example is computed for the wavefront kink and nearby pseudocaustics at the range \( x_1 = 104.8 \) m, and is compared with the result obtained by the phase-partitioning Maslov integral method (PPMI, dashed lines). (See Kendall & Thomson (1993) for the development of PPMI, and Chapter 3 for its accuracy.) We see that the caustic error in the GRT result (solid lines in (a)) is a very large waveform at the kink point. Whereas the pseudocaustic error in the MI seismogram (solid lines in (b)) is a moderately-erroneous linear phase. Note that in (a), the GRT arrival “at” the wavefront kink was computed just after the kink so its waveform there suffers a Hilbert transform. Detail on the MI calculation follows shortly.

As these caustics and pseudocaustics are fairly separate, Maslov’s blending seismogram method may apply. But in order to reasonably combine the GRT and MI components it becomes a question how to define \( e_0 \) and \( e_1 \) in eqs. (2.89) and (2.90) (or Appendix C eq. (C.12) for the current 2D acoustic case). I therefore begin with testing Brown’s weighting functions.
4.4 Brown's hyperbolic weighting functions

Brown (1994) recommended the weighting functions be made of "simple functions of the slope $\partial x_3/\partial p_3$ (in our notation) of the Lagrangian manifold (cross-section) in the vicinity of $p$ caustics"; and he has used hyperbolic functions $\tanh$ and $\text{sech}$ (by virtue of $\tanh^2 + \text{sech}^2 = 1$). In our notation, they are written as

$$e_0 = \tanh^2\left(\gamma \frac{\Delta p_3}{\Delta x_3} \frac{\partial x_3}{\partial p_3}\right), \quad e_1 = \text{sech}^2\left(\gamma \frac{\Delta p_3}{\Delta x_3} \frac{\partial x_3}{\partial p_3}\right)$$

between adjacent caustic ($\partial x_3/\partial p_3 = 0$) and pseudocaustic ($\partial x_3/\partial p_3 = \infty$) points; otherwise they are

$$e_0 = 0, \quad e_1 = 1.$$

Here the local slope ($\partial x_3/\partial p_3$) is scaled by the ratio of the total measures in slowness and depth of the Lagrangian manifold cross-section ($\Delta p_3/\Delta x_3$), and a coefficient $\gamma$. Especially, $\gamma=0.5$ in Brown (1994).

4.5 Effects of Brown's rule

The ray bundle I use spans $\pm 75.5$ degrees about the horizontal near the source. In order to evaluate Brown's weighting functions, we need the geometry of the Lagrangian manifold cross-section. Presented in Fig. 4.4 are the cross-sections used in the calculations. They are sections of the projected Lagrangian manifold (i.e. projected from phase space $(x_1, x_3, p_1, p_3)$ onto the mixed subdomain $(x_1, x_3, p_3)$) which is crossed at (a) $x_1=104.8$ m, (b) $x_1=130$ m, and (c) $x_1=220$ m. The cross-sections become more complicated as the rays propagate farther. But note that caustics (dotted stationary points) and pseudocaustics (openly-circled turning points) all occur at well separated locations.

We begin with computation of seismograms along a vertical line of receiver points at $x_1=104.8$ m where a caustic (cuspoid type) first appears on the symmetry line of the waveguide. The top row of Fig. 4.5 gives (a) the GRT amplitude, (b) the normalized MAT amplitude, and (c) the MAT phase function in the source slowness domain. (Recall that contributions of MI are estimated in the $p_{03}$ domain, and that multiple arrivals in the GRT solution (triplication in travel time as a function of $x_3$) can be easily sorted out as a function of $p_{03}$). One caustic ($C_x$) and two pseudocaustics ($C_y$'s) arise at $p_{03}$ values marked by the dotted lines. Note that the normalized MAT amplitude is zero at the pseudocaustics, but because the MAT phase function introduces pseudo stationarities
Brown's weighting functions have been used to blend the GRT and MI components. The top row of Fig. 4.5 also shows the weights for MI (solid line) and GRT (dashed line) calculated from the scaling ratio $\Delta p_3/\Delta x_3=5.33 \times 10^{-3}$ ms/m$^2$ (Fig. 4.4a) and three choices of the coefficient value: $\gamma=0.5$ in (d), 5 in (e), and 15 in (f). Particularly in Fig. 4.5d where $\gamma=0.5$ (the value used by Brown (1994)), we see that $\epsilon_1$ is tapered only over a very narrow range around the pseudocaustics. Consequently the MI component will dominate the synthetic everywhere and give unattenuated pseudocaustic errors, as shown by comparing it with the accurate PPMI. In Fig. 4.5e where $\gamma=5$, $\epsilon_1$ is reduced over a wide range near the pseudocaustics, and there, the MI contributions have been replaced by GRT. This leads to a good MAT estimation (Fig. 4.6b). As we further increase the coefficient to $\gamma=15$ in Fig. 4.5f, $\epsilon_1$ is more widely removed at the pseudocaustic and the pseudocaustic errors have been fully attenuated. But because less of a contribution from neighboring rays is included in the MI component, the distorted waveform at the kink point (depth 110 m) is under-estimated. In summary, for this section, blending with $\gamma=3.0$ provides the most satisfactory seismogram.

The middle row of Fig. 4.5 concerns the wavefield at $x_1=130$ m. By this distance, the wavefield has been simply folded. Figs. 4.5g through i show the information employed
Figure 4.5: Asymptotic information in the source slowness ($p_{03}$) domain (left side) and Brown's hyperbolic functions (right side). The three rows of the figure correspond to modeling sections at $x_1=104.8$, 130, and 220 m. The asymptotic information is: 

- GRT amplitude, showing singular behaviour at caustic and kink points (col. 1);
- normalized MAT amplitude, showing zeros and inflections at pseudocaustics (col. 2);
- MAT phase showing stationarity at pseudocaustics. Note that the stationary point at $p_{03}=0$ is not a pseudocaustic. Weighting functions are shown for 3 values of $\gamma$.

The central display with optimum value. The GRT weighting function $e_0$ is shown dashed and the MI weighting function $e_1$ is shown solid. Note when $\gamma$ is too small, the solution only switches momentarily to the GRT solution about pseudocaustics. Conversely, when $\gamma$ is too large it only uses the MI solution in very localized regions about caustics.
Figure 4.6: (An enlargement on the next page.) The MAT synthetics (solid lines) compared with PPMI synthetics (dashed lines) where possible. The columns are sections at different distances; the rows are for different values of $\gamma$. Where $\gamma$ is too small and the GRT solution is used only very close to pseudocaustics, constant velocity artifacts appear, connected to the pseudocaustic on the traveltime inflections. Where $\gamma$ is too large, the MI solution is used in a very narrow range about caustics. Then the kink event is under-predicted (in (c)) and the fold-caustic event is over-estimated (in (f) and (i)). Also, a diffraction-like artifact is generated. Note that PPMI seismograms could not be produced for the $x_1=220$ section because the case where two pseudocaustics surround a caustic arises.
for the calculation of the GRT and MI solutions, with two caustics $C_x$'s and two pseudocaustics $C_p$'s marked by the dotted lines. The weights associated with the scaling ratio $\Delta p_3/\Delta x_3 = 7.45 \times 10^{-3}$ ms/m$^2$ (Fig. 4.4b) and the coefficient values $\gamma = 0.5, 3, \text{ and } 30$ are presented in (j), (k), and (l), respectively. Clearly, the pseudocaustics are removed slightly in (j), more in (k) and (l). The corresponding MAT seismograms are shown in the middle column of Fig. 4.6 (solid lines). Their pseudocaustic errors are visible in (d), disappear in (e) and (c). But on the other hand, more GRT components are used around the caustics in Fig. 4.5, and caustic errors begin to appear in Fig. 4.6f. The amplitude near the cusp at depth 100 m is much larger than it should be. Besides, the sharp change (like a truncation) in the MI weight $e_1$ in Fig. 4.5l causes "truncation" error beyond the triplication cusp shown in Fig. 4.6f. At this range, values of $\gamma$ near 3 serve the weighting best.

Finally, we consider the double-folded wavefield at $x_1 = 220$ m. Ray traveltime and amplitude, which are in terms of vertical offset $x_3$, are unfolded in the source slowness domain, as shown in the bottom row of Fig. 4.5. The four caustics C, E, F, and H are marked by the dotted lines, with corresponding singularities seen in the GRT amplitude in (m). The four pseudocaustics B, D, G, and I are also marked by the dotted lines, and indicated by the open circles in (n) where the normalized MAT amplitude becomes zero, and in (o) where the MAT phase becomes stationary.

Figs. 4.5p, q, and r give the weighting functions corresponding to $\Delta p_3/\Delta x_3 = 1.45 \times 10^{-2}$ ms/m$^2$ (Fig. 4.4c) and $\gamma = 0.1, 1.5, \text{ and } 5$, respectively. As the value of $\gamma$ becomes large, the supporting range of MI for pseudocaustic B, D, G, or I reduces, whereas the supporting range of GRT at caustic C, E, F, or H increases. The corresponding MAT synthetics are shown in the right column of Fig. 4.6, with part (g), (h), and (i) obtained from the weights in Figs. 4.5p, q, and r, respectively. Note that the PPMI solution used as a reference case in the previous examples itself breaks down at this distance because pseudocaustics are located on either side of the caustic (e.g., pseudocaustics B and D surrounding caustic C). However, a qualitative judgement can be based on the types of errors seen in the previous synthetics. Clearly, the seismogram in Fig. 4.6g contains pseudocaustic errors across D and G. (Actually, if $\gamma < 0.1$, pseudocaustic errors across B and I will also become visible.) However, removal of pseudocaustic errors has been satisfactorily achieved by the coefficient values 1.5 and 5 (Figs. 4.6h and i). But for a value as large as 5, a strong caustic error appears at the cusp H in Fig. 4.6i. Erroneous diffraction across the cusp H is due to the rapid variation of the MI weight, which in effect behaves as a cutoff. The best result comes from $\gamma = 1.5$. 
Brown's use of $\partial x_3 / \partial p_3$ as the control factor seems very reasonable: it reflects how close a neighboring ray is to a caustic or pseudocaustic point. And this certainly relates to how much credit should be given to GRT and MI components. But there is a tolerance range for the value of $\gamma$ associated with this weighting method. Outside this range errors may occur in a synthetic seismogram: those of pseudocaustic type for smaller $\gamma$ values (e.g. the top row of Fig. 4.6) or those of caustic type for larger $\gamma$ values (e.g. the bottom row of Fig. 4.6). Unfortunately, as has been shown here by the simple model, the tolerance range changes with the position of the seismic section. It also depends on the size of the ray bundle used in the calculation and, this varies from model to model. The value preferred in Brown (1994), $\gamma = 0.5$, results in pseudocaustic errors at the first two sections of our modeling (Figs. 4.6a and d). As a result, Brown's weighting method cannot automatically guarantee good accuracy. Trial and error testing is often needed even for a single seismic modeling. This problem becomes severe when an intensive modeling job is to be carried out. In order to remedy this problem, I therefore propose an alternative weighting method.

### 4.6 Trigonometric weighting functions

I design the new weighting method according to the following rules:

1. Caustics and pseudocaustics are detected by numerical root finding methods according to the change of the sign of $\partial x / \partial p_0$ and $\partial p / \partial p_0$, respectively;
2. MI is used if no caustics or pseudocaustics exist (i.e. $e_1=1, e_0=0$);
3. Otherwise, MI is used entirely at caustics (i.e. $e_1=1, e_0=0$), and GRT entirely at pseudocaustics (i.e. $e_1=0, e_0=1$). Between the caustic and pseudocaustic points weighting functions provide a mixed solution;
4. One of the weighting functions varies as a cosine (or sine) between two adjacent caustic and pseudocaustic, and the other is obtained from $e_1 + e_0 = 1$;
5. Weights are held constant at 1 or 0 between adjacent caustics or adjacent pseudocaustics.

For rule 4, we make $e_0 = (1 - \cos(\pi \delta p_0 / \Delta p_0)) / 2$ where $\Delta p_0$ is the separation between adjacent caustic and pseudocaustic points, and $\delta p_0$ is the separation between the ray and the caustic.
4.7 Effects of the trigonometric weighting function

Fig. 4.7 shows the results of using the trigonometric weighting function for all three of the previously analysed sections. The left column shows the weighting functions, where all switch completely between adjacent caustics and pseudocaustics. For section $x_1=104.8$ m, the weights in (a) yield the MAT synthetic (solid line) in (b). Comparison with the PPMI (dashed line) reveals that MAT still underestimates the kinked wavefield at depth 110 m. Compared with the optimum solution in Figs. 4.5e and 4b, less use has been made of the MI solution. It seems that a broader range of the MI solution near the kink would be required in order to get more low-frequency components into the event at this place. For section $x_1=130$ m, the wavefield is singly folded. Parts (c) and (d) respectively display the new weighting functions and the obtained MAT synthetic which is well consistent with the accurate PPMI.

Modeling of the double-folded wavefield at distance $x_1=220$ m is much more challenging. (See Figs. 4.5m, n, and o for the detail of the phase and amplitude.) Fig. 4.7e depicts the new weights. It should be noted that the solution at the pseudocaustics B, D, G, and I is fully provided by GRT; while the solution at the caustics C, E, F, and H by MI. In the range between E and F, only MI is in use because no pseudocaustic exists. These result in a MAT prediction (shown solid in (f)) that contains no linear pseudocaustic errors. The geometrical waves beyond the caustics E and H are smoothly distributed. The wavefields near caustics E and H are relatively strong. The diffractions near E (at depth 100 m) and H (at depth 55 m) are reasonably strong, then away from E (e.g., at depth 95 m) and H (e.g., at depth 50 m), they rapidly decay. They look very reasonable, although no direct comparison could be made because PPMI is not applicable here.

4.8 Chapter conclusions

Interesting waveform distortion occurs where the Fermat wavefront becomes singular (i.e., rapidly changing due to bending, kinking, folding, or ending), and where a wide range of neighboring rays within the Fresnel zone of a geometrical ray may contribute significant low-frequency components to the wavefield. In this study, I have illustrated how the number of the neighboring rays included in the Maslov integral can affect the accuracy of the Maslov seismogram. Roughly, as fewer neighboring rays are used – in other words, MI is less or more locally weighted – waveform amplitudes can be underestimated. (Recall
Figure 4.7: The trigonometric weighting functions ($e_0$ for GRT, dashed lines; and $e_1$ for MI, solid lines) for the same cases as in Figs. 4.5 and 4.6, with the corresponding MAT synthetics (solid lines) for the three sections. As before, the PPMI solutions (dashed lines) are provided for the first and second sections. On the first section, MAT and PPMI agree very well except at the kink point (depth 110) where MAT under-estimates the wavefield because the caustic region was under-estimated. No pseudocaustic error is visible. In (f), diffractions beyond the cusps E and H appear reasonable, but cannot be checked by PPMI.
the waveform at the kink in Figs. 4.6c and 5b.) To ensure accuracy of MI, a sufficiently large bundle of integrable neighboring rays – rays involving no pseudocaustic – is required. Similarly, to remove the effects of pseudocaustics, the contribution of a certain number of neighboring rays near the pseudocaustic needs to be smoothly truncated from the Maslov integral; otherwise the pseudocaustic error will remain, just as shown in the top row of Fig. 4.6. Use or removal of the MI component needs a sufficiently large range of neighboring rays for accuracy at finite frequency. A study of detailed qualitative constraints on the range of integration should be made in the future; a rough guide can be obtained by estimating the size of the Fresnel zone, as implied by Guest & Kendall (1993).

Brown (1994) used the argument \( \partial x/\partial p \) to map the slope of the Lagrangian manifold cross-section onto the weighting functions; the slope quantifies the applicability of GRT or MI, while the weighting functions the opportunity for GRT or MI to contribute to the wavefield construction. Such a mapping is capable of reflecting the subtle geometry of the Lagrangian manifold. It allows for an appropriate sampling of the GRT or MI component when the switching between GRT and MI cannot simply be described by functions of the positions of caustics and pseudocaustics. One observed example is the accurate prediction of the strong waveform at the kink point, as shown in Fig. 4.6a or b. However, this method does not automatically provide a reasonable switching between GRT and MI, because the desirable value of the arbitrary coefficient \( \gamma \) changes for different modelings and its tolerance range is limited. To optimize the procedure, \( \gamma \) needs to be set. This is an obstacle if seismogram calculations are to be automated.

In this chapter I suggest the weighting functions be a simple function of the \( p_0 \) positions of the caustics and pseudocaustics, and propose use of a trigonometric function (sine or cosine) connecting adjacent caustic and pseudocaustic pairs. In comparison to Brown’s weighting functions, this gives more weight to MI at the caustic and correspondingly less weight to MI at the pseudocaustic in a symmetric way, and it switches between unity and zero smoothly in between. When a wavefront is smoothly bent (i.e., just before a kink), no real caustic occurs (meaning that \( \partial x/\partial p_0 \) does not change sign). However, the wavefield will undergo a certain convergence and waveform distortion, and GRT fails to give accurate waveform and amplitude. For MI to be accurate, enough neighboring rays are needed. Unfortunately, the weighting scheme I suggest does not recognize this opportunity because it is based on finding caustics and pseudocaustics. However, this scheme is usually accurate and stable, as demonstrated above, and should be considered first, particularly in intensive modelings.
Chapter 5

A Hybrid Maslov-Kirchhoff Method

5.1 Introduction

5.1.1 Problem formulation: joint caustics

Ray manifolds in a generally inhomogeneous medium are more likely to contain multiple caustics and pseudocaustics if the velocity inhomogeneity is proportionally stronger, if the distance from source to the receiver line is longer or if the ray paths tend to run along rather than across the structure. And the likelihood of encountering caustics and pseudocaustics that are inseparable in terms of the shortest wavelength in the practical seismic signal grows similarly. (For convenience I refer to such a pair as a joint caustic.)

Once joint caustics occur, the simple-to-use Maslov techniques for waveform modeling suffer from two problems. Firstly, Maslov’s original treatment for caustics and pseudocaustics by blending the GRT and MI components breaks down, because nowhere between a caustic and a nonseparable pseudocaustic can either of the two components be valid. Also, problems arise in recognizing and adequately sampling the real and pseudo caustics. Then, techniques based on Lagrangian equivalence, such as the phase partitioning are of limited value.

5.1.2 A new remedy: coupling Kirchhoff summation with the Maslov method

In this section of the thesis I propose a new technique for dealing with joint caustics. It is to couple the Maslov technique with the Kirchhoff method (or Huygens’ principle). More
specifically, the wavefield is first propagated onto an intermediate Kirchhoff integration surface located somewhere between the source and the receiver. There, the primary wave is considered to stimulate a number of secondary sources. Then the true wavefield at the receiver is given by the summation of all the waves emitted from the secondary sources (Huygens’ principle). In each step, pseudocaustic errors may be generated, but they may also arise randomly. Then, the Kirchhoff summation over the secondary sources will boost the real signals but suppress the numerical “noise” that is associated with the pseudocaustics.

Kirchhoff’s theorem has frequently been used in modeling wavefields before (e.g., Haddon & Buchen 1981; Frazer & Sinton 1984). But the motivation of these authors was to extend the ability of simple geometrical ray theory to include non-geometrical signals by separating the source/receiver distance into a few such short propagation ranges that real caustics were avoided. Whereas in my Maslov-Kirchhoff hybrid method, the Maslov technique itself already ameliorates the catastrophic caustic errors of ray theory and includes low-frequency wave components. Kirchhoff’s theorem is only used to suppress moderate pseudocaustic errors. Hence the former may require many more intermediate integration surfaces than I need.

Three chapters of the thesis are dedicated to the Maslov-Kirchhoff hybrid method. The method is formulated in the current chapter (5). Its use in backward wavefield extrapolation is demonstrated in Chapter 6. Its use in forward modeling is studied in theoretical detail in Chapter 7, together with a short example of modeling a realistic cross-well wave propagation in a complicated medium.

### 5.2 Kirchhoff extrapolation formulas

Although the wavefield extrapolation formulas based on various forms of the Kirchhoff integral theorem have been widely used, their formulation involves many aspects that might be easily missed, mis-understood, or mis-interpreted. Thus, I carefully review the theorem’s formulation for the case of periodic acoustic waves in a continuous but spatially variable medium.
5.2.1 Huygens-Fresnel principle

Huygens (1690) (Baker & Copson 1950; and Blok, Ferweda & Kuiken 1992) proposed a principle for the propagation of light; that each element of the wavefront at an early instant can be regarded as the centre of a new disturbance, and the wavefront at a later instant is the envelope of the secondary waves from the new disturbances. Huygens' principle is, however, a purely geometrical theory of wave propagation and takes no account of any of the dynamical properties of light. Therefore it can only account for the rectilinear propagation of light in space or a uniform medium by assuming that a secondary wave has no effect except at the point where it touches the envelope. Furthermore, the envelope of the secondary spherical waves (in a homogeneous medium) consists of two sheets (one on each side of the surface on which the secondary sources of disturbance lie) and one of them according to this principle is a nonphysical backward effect.

In 1818 (more than a century later), Fresnel made an important extension to Huygens' principle in his memoir on Diffraction which won the Paris Academy's Prize of that year (Fresnel 1826). He replaced Huygens' isolated spherical waves by purely periodic spherical waves and made use of the theory of interference. With this theory, Fresnel was able to account, not only for the rectilinear propagation of light of very short wavelength and the laws of reflection and refraction, but also for certain diffraction phenomena, depending on the reinforcement or destruction of the secondary waves at each point, not necessarily on the envelope of the secondary waves. Fresnel further proposed that each element of the primary surface acts as a secondary source, which in the case of mechanical waves is sustained by both the displacement and velocity given to the particles of the surface element by the primary wave. The resultant effect at a distant point is produced by the interference of these secondary waves. In this way, Fresnel's assumption produces not only the correct forward propagation effect but also a null backward propagation effect. Unfortunately, Fresnel was unable to carry out this analysis completely. But, in 1883, the complete Huygens-Fresnel principle (often known as Huygens' principle) on the dynamical process of the propagation of waves was formulated exactly as an integral by Kirchhoff (1882, 1883) using Green's identity theorem. Although the original theorems considered the propagation medium to be homogeneous, the theorems are applicable (in principle, at least) to propagation in a more general medium. I review this case below.
5.3 Kirchhoff’s integral formulas

Let $\Phi(x, \omega)$ be an acoustic wave (pressure) function in the domain of the (angular) frequency $\omega$ and the 3D Cartesian coordinates $x = (x_1, x_2, x_3)$. In a continuous inhomogeneous acoustic medium, consider a volume of source-free space $V$ bounded by a closed surface $S$, as illustrated in Figure 5.1. Then within $V$, the wave function $\Phi$ satisfies the homogeneous Helmholtz type wave equation (Berkhout 1982, pp 334, C.19, and Wapenaar et al. 1989)\(^1\)

$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla \Phi \right) + \left( \frac{\omega}{v} \right)^2 \Phi = 0; \quad x \in V, \tag{5.1}$$

where $v(x)$ is the wave velocity, and $\rho(x)$ the density. The notation here differs somewhat from the corresponding equation for an elastodynamic medium presented in (2.4).

In order to obtain Kirchhoff’s integral representation of the wavefield at an observational point $x_0$ within the volume $V$, we need to invoke Green’s identity. This identity requires an introduction of the Green’s function $G(x, x_0, \omega)$ that satisfies the inhomogeneous Helmholtz equation

$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla G \right) + \left( \frac{\omega}{v} \right)^2 G = \rho \delta(x - x_0). \tag{5.2}$$

Mathematically, this function represents the wavefield from a point source $\delta(x - x_0)$ situated at place $x_0$. But it should be recalled that in this context, the wavefield and the source of the Green’s function are both virtual, because there is no real source at place $x_0$. The real sources are all distributed beyond the volume $V$, and their physical wavefield within this volume, $\Phi$, is described by equation (5.1). Note that the unity delta source strength in equation (5.2) is scaled by $\rho$ to assign $G$ the dimensions pressure per unit volume\(^2\). Note that $G(x, x_0)$ is only a function of $(x - x_0)$ in a homogeneous medium, otherwise it depends on $x$ and $x_0$ separately. It is, however, always reciprocal, i.e., $G(x, x_0) = G(x_0, x)$ and $\nabla G(x, x_0) = -\nabla G(x_0, x)$. This is often symbolized by the notation $G(x|x_0)$.

\(^1\)It reduces to the standard Helmholtz equation when the density $\rho$ is constant.

\(^2\)The physical wavefield $\Phi$ has the unit of pressure, as shown by the equation it satisfies in the source region, i.e., $\rho \nabla \cdot \left( \nabla \Phi / \rho \right) + (\omega/v)^2 \Phi = \rho \nabla \cdot E$, where $E$ is the body force (force per unit mass) and $\nabla \cdot E$ the pressure dipole. After scaling this equation with a volume element $\Delta V$, we notice that the r.h.s. of the scaled equation is of the same unit as the r.h.s. of equation (5.2), therefore, the l.h.s.’ of the two equations must also have the same unit. This indicates that the unit of the Green’s function is pressure per unit volume.
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Figure 5.1: Geometry for Kirchhoff’s integral expression of the Huygens-Fresnel principle. An integration surface $S$ encloses an observational point $O$ (dot), to which the source (star) is exterior. Note $n$ denotes the outward normal.

Following Wapenaar et al. (1989), we can construct a vector $Q(x)$ where

$$Q = \frac{\nabla G}{\rho} \Phi - G \frac{\nabla \Phi}{\rho}. \quad (5.3)$$

By invoking the theorem of Gauss on the volume $V$ in Figure 5.1, where the integrals are with respect to the $x$ space variable

$$\int_V \nabla \cdot Q \, dV = \oint_S Q \cdot n \, dS, \quad (5.4)$$

we obtain a modified form of Green’s identity

$$\int_V \left( \nabla \cdot \left( \frac{1}{\rho} \nabla G \right) \Phi - G \nabla \cdot \left( \frac{1}{\rho} \nabla \Phi \right) \right) \, dV = \oint_S \left( \frac{\nabla G}{\rho} \Phi - G \frac{\nabla \Phi}{\rho} \right) \cdot n \, dS \quad (5.5)$$

where $n$ denotes the outward normal to the closed surface $S$ (Figure 5.1). Note that if $\rho$ is constant, equation (5.5) reduces to the standard Green’s identity

$$\int_V (\nabla^2 G \Phi - G \nabla^2 \Phi) \, dV = \oint_S (\nabla G \Phi - G \nabla \Phi) \cdot n \, dS. \quad (5.6)$$

Using equations (5.1) and (5.2) for $\Phi$ and $G$ and the integral property of the $\delta$-function to evaluate the volume integral in (5.5), we obtain the value of the closed surface integral accordingly as $x^O$ is inside or outside $V$

$$\oint_S \frac{1}{\rho} (\nabla G \Phi - G \nabla \Phi) \cdot n \, dS = \begin{cases} \Phi(x^O, \omega), & x^O \in V; \\ 0, & x^O \notin V. \end{cases} \quad (5.7)$$
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The l.h.s. is usually called a Kirchhoff integral of the wavefield $\Phi$. Note that it applies only when the source of $\Phi$ is exterior to $V$. The result shows that such a Green’s function weighted integral of a primary wave $\Phi$ (together with its normal derivative) over any closed surface will give the value of the primary wave at the interior point $\vec{x}^0$ of $S$ or a null effect outside. It is a complete, analytical representation of the Huygens-Fresnel principle formulated first by Kirchhoff (1883).

The Kirchhoff integral (5.7) can also be formulated with an acausal Green’s function. Such a function is the time-reverse of the time domain causal Green’s function (i.e., it is zero at positive rather than negative time) and it Fourier transforms to the complex conjugate of $G$. It will be denoted as $G^\ast$. Because it obeys the same inhomogeneous Helmholtz equation (5.2) as $G$, we can also use it to formulate Kirchhoff’s integral expression. The integral will read, if the real sources are outside the closed surface $S$.

$$\int_S \frac{1}{\rho} (\nabla G^\ast \Phi - G^\ast \nabla \Phi) \cdot \mathbf{n} \, dS = \begin{cases} \Phi(\vec{x}^0, \omega), & \vec{x}^0 \in V; \\ 0, & \vec{x}^0 \notin V. \end{cases} \tag{5.8}$$

with the normal vector $\mathbf{n}$ directed out of $S$ (Figure 5.1).

Equations 5.7 and 5.8 are Kirchhoff’s general analytical formulations of the Huygens-Fresnel principle. Kirchhoff’s integral implies that the wavefield can be considered as due to two types of secondary sources distributed on the closed surface $S$, both of which are excited by the primary wave from the source. They are a simple source of strength $\partial \Phi / \partial n$ per unit area and a normally directed doublet source of strength $\Phi$ per unit area. Kirchhoff’s integral formula asserts that the real wave effect of the primary source can be produced by the superposition of waves from these secondary sources.

The Huygens principle was initially proposed to construct waves at a later instant in terms of the waves on a surface at an earlier instant. But Kirchhoff’s formulation implies that a wavefield can in fact be reconstructed from waves at later times. The former, forward process is conducted using an outward propagating Green’s function; while the latter, backward process requires an inward propagating Green’s function. Both are very useful in earth sciences: one in forward modeling and the other in backward extrapolation or migration. Descriptions of both processes are made more explicit below.

5.3.1 Kirchhoff’s integral for forward propagation

The wavefield from a point source at a given observation point can also be constructed from the (primary) wavefield observed on a surface that encloses the source by using
Figure 5.2: Geometry used to obtain Kirchhoff’s integral for forward wave propagation. The integral over the outer sphere $\Sigma$ of radius $R$ becomes zero if the Green’s function is outward propagating. Thus, the wavefield at $O$ can be given by the Kirchhoff integral over the inner surface $S$ which encloses the source.

Kirchhoff’s formula and the causal Green’s function. To do so, we construct in Figure 5.2 a hollow volume $V$ bounded internally by $S$ and externally by a sphere $\Sigma$ of radius $R$. Denoting by $n$ the normal vector out of $V$, the total surface integral gives, following (5.7), the real wavefield solution $\Phi$ if $x^0$ is within $V$; or otherwise zero:

$$( \oint_S + \oint_{\Sigma} ) \frac{1}{\rho} ( \nabla G \Phi - G \nabla \Phi ) \cdot n \, dS = \begin{cases} \Phi(x^0, \omega), & x^0 \in V ; \\ 0, & x^0 \notin V. \end{cases} $$

(5.9)

In fact because $G$ is outward propagating (or causal) from $x^0$, the external surface integral $\oint_{\Sigma}$ is always zero so long as $x^0$ is inside $\Sigma$. This becomes clear as we make $R \to \infty$. In this case, equation (5.9) still holds while the integral $\oint_S$ tends to zero according to the Sommerfeld radiation condition (Bleistein 1984, pp. 182):

$$G = O\left( \frac{1}{R} \right), \quad \partial_R G = o\left( \frac{1}{R} \right) + \left( \frac{i\omega}{v} \right) G; \quad R \to \infty ,$$

(5.10)

(meaning that $G$ must decay to zero at the rate of $1/R$ as $R \to \infty$ and must propagate outwards). Likewise, $\Phi$ must have the same outward-propagating behavior. Thus in the integrand of (5.9), the terms relating the gradient of the amplitude are of $O(1/R^3)$ but the terms for the gradient of the phase (of order $O(1/R^2)$) cancel. This assigns an order of $O(1/R^3)$ to the integrand; while the integration surface is of $O(R^2)$. Then, no matter whether $\Sigma$ is at finite distance or infinity, we are left in equation (5.9) with only the integral over the internal surface $S$ which encloses all the real sources, and Kirchhoff’s
integral formula becomes
\[
\int_S \frac{1}{\rho} (\nabla G\Phi - G \nabla \Phi) \cdot \mathbf{n} \, dS = \begin{cases} \Phi(x^0, \omega), & x^0 \text{ outside } S; \\ 0, & x^0 \text{ inside } S, \end{cases}
\] (5.11)
with \(\mathbf{n}\) the inward normal to \(S\), which is the same form as equation (5.7) except that \(S\) surrounds the primary source rather than the observational point. This shows that Kirchhoff’s distribution of secondary sources gives the same effect at a point outside \(S\) as does the primary source, and a null effect inside \(S\), ensuring an expanding wavefront.

Note that because \(G^*\) does not satisfy the Sommerfeld radiation condition (5.10), the wave effect at a point outside \(S\) cannot be computed only from the integral over \(S\) with \(G^*\). This special consequence is attributed to the acausal property of \(G^*\) which allows the contribution of the secondary sources on the outer surface \(\Sigma\) to be reversely propagated to the observational point.

### 5.3.2 Rayleigh-type integral formula: forward propagation

In practical seismic forward modeling, the integration surface is often an open plane. Then Kirchhoff’s theorem can be modified accordingly. In this case, the source and receiver are respectively positioned on the different sides of the integration plane. Figure 5.3 shows an example where a real source is situated in the negative half-space \((x_1 < 0)\) and an observational point \(x^0\) in the positive half-space \((x_1 > 0)\). This time the closed surface surrounding the observational point is made of a plane \(S_0\) at \(x_1 = 0\) and a hemisphere \(S_1\) of radius \(R\) in the positive half-space. Then the real wave solution can be obtained using Kirchhoff’s integral formula (5.7) with the causal Green’s function, as

\[
\Phi(x^0, \omega) = \left( \int_{S_0} + \int_{S_1} \right) \frac{1}{\rho} (\nabla G\Phi - G \nabla \Phi) \cdot \mathbf{n} \, dS, \quad x^0 \in V
\] (5.12)

with the normal \(\mathbf{n}\) directed out of \(V\). When \(R \to \infty\), this formula remains true simply because \(V\) remains source-free. On the other hand, the integral \(\int_{S_1}\) tends to zero according to the Sommerfeld radiation condition (5.10). So, eventually, the surface of integration reduces from a closed one to an open plane \((x_1 = 0)\), and (5.12) becomes

\[
\Phi(x^0, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\rho} (\nabla G\Phi - G \nabla \Phi) \cdot \mathbf{n} \, dx_2 dx_3, \quad x^0_1 > 0.
\] (5.13)

This version of Kirchhoff’s integral provides an operator for wavefield forward extrapolation from the waves on an intermediate plane (Figure 5.3b). Note the integral vanishes if \(x^0_1 < 0\).
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Figure 5.3: Geometry for constructing Kirchhoff's integral over an open plane (Rayleigh-type) for forward wave propagation. (a) This time the closed surface surrounding the observational point O is made of a plane at \( x_1 = 0 \) and a hemisphere \( S_1 \) of radius \( R \) in the positive half-space \( x_1 > 0 \). (b) When \( R \rightarrow \infty \), the integral over \( S_1 \) vanishes, and only the integral over the open plane remains, giving the wavefield at \( O \).

Evaluation of equation (5.13) seems to require recording of both the wavefield and its normal variation \( (\nabla \Phi \cdot n) \). But this is not a problem if the propagation direction of all of the primary wavefield \( \Phi \) is known, as the wavefield and its normal gradient are then directly related. Then, equation (5.13) can be transformed to the so-called Rayleigh-type integral which involves only the value or the normal derivative of the wavefield on the plane of integration. This is very easy to prove if the propagation medium is homogeneous everywhere, but much more subtle if the medium is complicated. It can be achieved by making the following assumptions (refer to Figure 5.3b for the geometry):

1. The Green's wave \( G \) only propagates in a reference medium, which is the same as the real medium carrying the primary wave \( (\Phi) \) in the source-free half-space \( (x_1 > 0) \) but can be different from the real medium in the half-space with real sources \( (x_1 < 0) \) (Berkhout & Wapenaar 1989);

2. The value and derivative of the primary wave \( (\Phi) \) on the integration plane will not be changed by the choice of reference medium.

The following is an example. Assume the reference medium in the source half-space \( (x_1 < 0) \) to be the mirror image of the real medium in the source-free half-space \( (x_1 > 0) \),
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with the Kirchhoff integration plane $x_1 = 0$ as the symmetrical interface. In addition to
the Green's source situated at the observational point O, introduce its mirror image in
the opposite half-space ($x_1 < 0$). Then on the interface $x = 0$, and depending on the sign
of the mirrored source, the total Green's function $G_2$ will exhibit either the Neumann
condition

$$\partial x_1 G_2 = 0;$$

(5.14)

or the Dirichlet condition

$$G_2 = 0.$$  

(5.15)

With the Neumann condition, equation (5.13) can be reduced to the Rayleigh I integral

$$\Phi(x^O, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\rho} G_2 \partial x_1 \Phi dx_2 dx_3$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2}{\rho} G \partial x_1 \Phi dx_2 dx_3, \quad x^O_1 > 0. \quad (5.16)$$

While with the Dirichlet condition, equation (5.13) becomes the Rayleigh II integral

$$\Phi(x^O, \omega) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\rho} \partial x_1 G_2 \Phi dx_2 dx_3$$

$$= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2}{\rho} \partial x_1 G \Phi dx_2 dx_3, \quad x^O_1 > 0. \quad (5.17)$$

(Note that the coefficient 2 in equation (5.16) or (5.17) results from the symmetrical
property of $\partial x_1 G$ or $G$ on the integration plane, respectively.)

The Kirchhoff integration plane with Neumann's condition (5.14) also behaves as a
perfectly-reflecting rigid-interface with reflection coefficient +1. Therefore, introduction
of a rigid reference medium in the half-space with sources can also result in the
Rayleigh I integral (5.16). While with Dirichlet's condition (5.15), the plane responds as
a perfectly-reflecting free-interface with a reflection coefficient of -1, and the substitu-
tion of a free reference medium in the half-space with sources can lead to the Rayleigh
II integral (5.17) as well.

However, I should mention that the above approach probably conflicts with Kirchhoff's
theorem, or the Huygens-Fresnel principle; for this theorem is based on the dynamical
properties of the wave and correctly describes the secondary sources and their effects at
any observational point in the whole space. Kirchhoff's theorem requires that both the
real wave and the Green's wave propagate in the same real medium, and these waves
and the medium be continuous in value, first- and second-derivatives on the surface
of integration. The substitution of the reference medium for $G$ can deteriorate these
conditions and alter Kirchhoff's unique and real distribution of the secondary sources. By virtue of this point, the Rayleigh-type integrals (5.16) and (5.17), derived according to the above two assumptions, should not be considered as a complete formulation of the Huygens-Fresnel principle, but rather, a convenient mathematical approach to Kirchhoff's theorem when only the wave effect in part of the region is of concern.

5.3.3 Rayleigh-type integral formula: backward propagation

A key step in seismic interpretation is to extrapolate the wavefield from the data acquisition plane back to the medium region through which the wave has propagated. One should note that usually the observed wavefield is a reflected wavefield emerging at the surface of the earth, and the objective is to extrapolate it back in steps to the reflection surfaces. However, there is no need to deal with this complexity in describing the basic process. Furthermore, one is only interested in back propagating the primary wavefield, not including any local side or back scattered components. Consequently, the medium for which the Green's function is defined is always one in which velocity and density vary smoothly with respect to all useful seismic wavelengths. The Rayleigh-type integral, along with the Green's function, provides a useful tool. Consider the geometry in Figure 5.4, where the observational point O and the real source are both within the same half-space ($x_1 < 0$) to the left of the Kirchhoff plane at $x_1 = 0$. The point O can be bounded separately from the source by a closed surface which consists of the Kirchhoff plane $S_0$, an intermediate plane $S_1$ between the observational point and the source, and a cylindrical surface $S_2$ of radius $R$. Then the exact wave effect at $x^O$ is given from (5.8) as.

$$
\Phi(x^O, \omega) = \left( \int_{S_0} + \int_{S_1} + \int_{S_2} \right) \frac{1}{\rho} \left( \nabla G^* \Phi - G^* \nabla \Phi \right) \cdot n \, dS ,
$$

(5.18)

with normal $n$ directed out of volume $V$. Let $R \to \infty$, then $G^*$ decays at a rate of $1/R$ while the area of surface $S_2$ increases at a rate of $R$ ($R^2$ for a sphere), and then the integral $\int_{S_2} \to 0$. This property can also be understood intuitively by imagining that the radiation angle of the energy confined between the two open planes $S_0$ and $S_1$ vanishes as $R \to \infty$.

In many cases, the integral $\int_{S_1}$ over $S_1$ may not be zero because of the evanescent waves (Esmersoy & Oristaglio 1988 and Wapenaar et al. 1989) or propagating waves scattered or turned backwards earlier. In seismic exploration, the wavefield on the plane $S_1$ cannot be recorded, therefore the integral $\int_{S_1}$ has to be ignored. As a result, the wavefield (Figure 5.4) can only be reconstructed approximately by the integral over the acquisition plane.
Figure 5.4: Geometry for constructing Kirchhoff's integral over an open plane (Rayleigh-type) for backward extrapolation. (a) The enclosed surface surrounding the observational point O is made of an acquisition plane $S_0$ (at $x_1 = 0$), an intermediate plane $S_1$ (separating the source and the observational point), and a cylindrical surface $S_2$ of radius $R$. (b) By expanding the cylindrical surface $S_2$ to infinity and neglecting the integral over $S_1$, the wavefield at O can be approximately reconstructed by the integral over the acquisition plane $S_0$.

\[ \Phi(x^O, \omega) = \int_{S_0} \frac{1}{\rho} \left( \nabla G^* \Phi \right) \cdot n \, dS. \]  

(5.19)

In general, this integral will only reconstruct the part of the wavefield that propagated out to the integration/acquisition plane $S_0$, but it has nevertheless been proved very useful in solving a large number of problems in seismic exploration. It can also be reduced to the practical Rayleigh-type integrals by considering the Green's source to be imaged in the negative half-space. The Rayleigh I and II integrals so obtained are respectively

\[ \Phi(x^O, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2}{\rho} G^* \partial z_1 \Phi \, dx_2 \, dx_3, \quad x_1^O < 0; \]  

(5.20)

or

\[ \Phi(x^O, \omega) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2}{\rho} \partial z_1 G^* \Phi \, dx_2 \, dx_3, \quad x_1^O < 0, \]  

(5.21)

which differ only in the substitution of $G^*$ for $G$ from (5.16) and (5.17) for forward extrapolation.

Finally, it is necessary to mention some consideration on one-way wave propagation. In an inhomogeneous medium, a forward propagating wave can be scattered or turned back. Appreciable multiple reflections easily occur in media with several large contrasts. Kirchhoff integrals do actually yield the full (two-way) wave effect at the point of interest.
from the full wave field on the integration plane and with the two-way Green's function. (Note that the backward extrapolation operators (5.20 and 5.21) are in general an approximation). The two-way formulation requires an accurate description of both the wave field on the integration plane and knowledge of the medium; otherwise it becomes unstable (Berkhout & Wapenaar 1989). The one-way versions of the Kirchhoff integral have been formulated by Berkhout & Wapenaar (1989) and Wapenaar et al. (1989), in which only the one-way components of the wavefields are extrapolated, and constructed or reconstructed. They are in general not exact but are more robust than the two-way versions. But because this one-way formulation is restricted to a certain class of media, a more generally valid theory of one-way wave extrapolation still remains to be explored. In this thesis, I only concern myself with smooth media in which high frequency waves propagate mainly in one direction (though not always, such as turning waves), and hence the two-way versions of the Kirchhoff integral automatically reduces to one-way.

5.4 Formulation of the Maslov-Kirchhoff hybrid formulas

The Kirchhoff and Maslov integrals are combined by using the Maslov integral to calculate approximately the Green's function in the Kirchhoff integral. Consider in Figure 5.5 an observational point O and a Kirchhoff integration line (KIL) in a 2D acoustic medium \((x_1, x_3)\). If the real source is located to the right, then the wavefield at point O can be constructed (or forward extrapolated) from the waves arriving at the KIL using the Kirchhoff integral (5.17). Otherwise, if the real source is located to the left of O, we reconstruct the wavefield at O (or backward extrapolate) from the waves arriving at the KIL by the Kirchhoff integral (5.21).

The Green's function inside the integrals will be calculated by the much improved ray method, i.e., the Maslov solution. To calculate, for instance, the Green's wave response at a particular point K \((x^K_1, x^K_3)\) on the KIL from the observational point O, a fan of rays need be traced from point O (now a testing source point of the Green's function) to the KIL (now a Maslov integration axis, MIA). Figure 5.5 shows one of the rays that starts with the initial vertical slowness \(p_3^O\) and arrives at point M \((x^M_1, x^M_3)\) with final slowness \(p^M = (p^K_1, p^K_3)\). Following equation C.11 in Appendix C, this Green's function can be
Figure 5.5: Geometry of Maslov-Kirchhoff wavefield extrapolation in which the Green's wave is shot from the observational point O (dot) to the Kirchhoff integration line (KIL)(now consistent with the Maslov integration axis (MIA)). The dashed lines illustrate the wavefronts. The thin line denotes a ray that starts with the initial slowness $P^O$ and ends at point M with final slowness $P^M$. Particularly, the initial vertical slowness component $P^O_3$ serves as the variable of the Maslov integration, and the final horizontal slowness component $P^M_1$ accounts for the normal variation of the Green's wavefield (or the contribution in the normal direction). The open circle K denotes a secondary-source point.

written as

$$G^{MAT}(\omega, \vec{x}_K, \vec{x}_O) = \sum_{x^M_3=x^O_3(p^O_3)} e_0 A e^{-i\text{sgn}(\omega)T/2} e^{i\omega T} + \left(\frac{i\omega}{2\pi}\right)^{1/2} \int e_1 B_0 e^{-i\pi\delta/2} e^{i\omega\tilde{\theta}} dp^O_3,$$

(5.22)

where, the first term provides the weighted contributions of the geometrical rays of amplitude $A$ which solve $x^K_3 = x^M_3(p^O_3)$, or arrive at point K, and the second term is the weighted MI solution which includes the low-frequency effects of the non-geometrical rays of the normalized Maslov amplitude $B_0$. (See Appendix C for more.)

To evaluate the normal derivative of $G$ in the Kirchhoff integral, differentiate (5.22) w.r.t. $x^K_1$, or equivalently $x^M_1$. By noting that the amplitude variations $\partial_{x^M_1} A$ and $\partial_{x^M_1} B_0$ in a continuous medium are negligible compared to the phase variations $\partial_{x^M_1} T$ and $\partial_{x^M_1} \tilde{\theta}$ (Zhu 1988a), and that

$$\partial_{x^M_1} T = \partial_{x^M_1} \tilde{\theta} = p^M_1,$$

(5.23)
we obtain
\[
\frac{\partial}{\partial x_1^K} G^{\text{MAT}}(\omega, \bar{x}^K, \bar{x}^O) = i\omega \left[ \sum_{x_3^K = x_3^O(p_3^O)} e_0 p_1^M A e^{-isgn(\omega)\pi\sigma/2} e^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p_1^M B_0 e^{-i\pi\delta/2} e^{i\omega \hat{\theta}} dp_3 \right].
\] (5.24)

where, \(p_1^M\) is the vertical slowness at the KIL and is also referred to as the directivity factor or oblique factor. It should be pointed out again that the first term is pure geometrical, and the second includes non-geometric wave effects.

By substituting equation (5.24) into the Rayleigh II integral (5.17) or (5.21), we can obtain respectively the Maslov-Kirchhoff wavefield extrapolation operator for forward extrapolation, i.e.
\[
\Phi(\omega, \bar{x}^O) = \int \frac{2}{\rho} \left[ \sum_{x_3^K = x_3^O(p_3^O)} e_0 p_1^M A e^{-isgn(\omega)\pi\sigma/2} e^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p_1^M B_0 e^{-i\pi\delta/2} e^{i\omega \hat{\theta}} dp_3 \right] \cdot (-i\omega) \Phi(x_3^K, \omega) dx_3^K.
\] (5.25)

or for backward extrapolation, i.e.
\[
\Phi(\omega, \bar{x}^O) = \int \frac{2}{\rho} \left[ \sum_{x_3^K = x_3^O(p_3^O)} e_0 p_1^M A e^{-isgn(\omega)\pi\sigma/2} e^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p_1^M B_0 e^{-i\pi\delta/2} e^{i\omega \hat{\theta}} dp_3 \right]^{*} \cdot (-i\omega)^* \Phi(x_3^K, \omega) dx_3^K.
\] (5.26)

In the above formulas, rays are required to be traced from the observational point to the Kirchhoff integration line. It has been realized that when a large number of observational points are concerned with (e.g. in migration), it is more efficient to trace rays from the KIL to the observational points (Hu & McMechan 1986). This common practice is achieved by virtue of the reciprocity theorem, that the response of the Green’s wave from \(\bar{x}^O\) to \(\bar{x}^K\) is the same as from \(\bar{x}^K\) to \(\bar{x}^O\). The reciprocity theorem is often believed to be followed asymptotically by the Maslov solution (Chapman & Drummond 1982). That is,
\[
G^{\text{MAT}}(\omega, \bar{x}^O, \bar{x}^K) \sim G^{\text{MAT}}(\omega, \bar{x}^K, \bar{x}^O).
\] (5.27)

\(^3\)To support such a view, I have conducted, though not presented, a numerical experiment to test the reciprocity of waves at caustics. It appears that ordinary arrivals and strong diffractions very close to the caustic obey the theorem exactly, but weak diffractions predicted far beyond the caustic do so only poorly. These are the same results that MI predicts inaccurately.
Chapter 5: A Hybrid Maslov-Kirchhoff Method

Figure 5.6: Geometry of the Maslov-Kirchhoff wavefield extrapolation in which the Green’s wave is shot from the secondary-source position $K$ (open circle) to the MIA through the observational point $O$ (dot). The thin line denotes a ray that starts with the initial slowness $P^K = (P_1^K, P_3^K)$ and ends at point $M$. The vertical slowness component $P_3^M$ serves as the variable of the Maslov integration, and the horizontal slowness component $P_1^K$ accounts for the normal variation of the Green’s wavefield (or the contribution in the normal direction).

Then the source of the Green’s function can be changed from the observational point $x^O$ to the point $x^K$ on the Kirchhoff integration line. The secondary source and the Green’s source now become consistent in position. The new Green’s wave (of a $\delta$-source wavelet) will radiate from the position of the secondary source. To accommodate this change, the way of evaluating the Maslov solution needs to be clarified in the following.

Figure 5.6 shows the geometry on the calculation of the Green’s response from a secondary-source point $K$ to the observational point $O$ $(x_1^O, x_3^O)$. Note that rays are traced from point $K$ to the MIA which crosses point $O$ but differs from the KIL. Depicted is a ray that starts with initial slowness $(p^K_1, p^K_3)$ and ends at point $M$ $(x_1^M, x_3^M)$. The Green’s response (5.22) is then rewritten

$$G^{\text{MAT}}(\omega, x^O, x^K) = \sum_{x_3^O=x_3^M(p^K_3)} e_0 A e^{-i \text{sign}(\omega) x_3^O/2} e^{i \omega T} + \left(\frac{i \omega}{2 \pi}\right)^{1/2} \int e_1 B_0 e^{-i \pi /2} e^{i \omega \tilde{\sigma}} dp^K_3.$$  

(5.28)

Here, equation $x_3^O = x_3^M(p^K_3)$ simply means the geometrical rays that arrive at the observational point.
In this case, the normal derivative of the Green's function is asymptotically
\[
\frac{\partial}{\partial x_1^K} G^{\text{MAT}}(\omega, x^O, x^K) \\
\sim i\omega \left[ \sum_{x_3^O=x_3^M(p^K_c)} e_0 p^K_0 \exp(-\text{sgn}(\omega)\pi\alpha/2) \epsilon^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p^K_1 B_0 e^{-i\beta/2} \epsilon^{i\omega \hat{\theta}} dp_3^K \right]. \tag{5.29}
\]

Substituting (5.29) into the Rayleigh II integrals (5.15) and (5.21) yields the Maslov-Kirchhoff formulas for 2D acoustic media
\[
\Phi(\omega, x^O) \\
= \int \frac{2}{\rho} \left[ \sum_{x_3^F=x_3^M(p^K_k)} e_0 p^K_0 A e^{-\text{sgn}(\omega)\pi\alpha/2} \epsilon^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p^K_1 B_0 e^{-i\beta/2} \epsilon^{i\omega \hat{\theta}} dp_3^K \right] \\
\cdot (-i\omega) \Phi(x_3^K, \omega) dx_3^K, \tag{5.30}
\]
for forward extrapolation, and
\[
\Phi(\omega, x^O) \\
= \int \frac{2}{\rho} \left[ \sum_{x_3^F=x_3^M(p^K_k)} e_0 p^K_0 A e^{-\text{sgn}(\omega)\pi\alpha/2} \epsilon^{i\omega T} + \left( \frac{i\omega}{2\pi} \right)^{1/2} \int e_1 p^K_1 B_0 e^{-i\beta/2} \epsilon^{i\omega \hat{\theta}} dp_3^K \right]^* \\
\cdot (-i\omega)^* \Phi(x_3^K, \omega) dx_3^K. \tag{5.31}
\]
for backward extrapolation, in which rays are emitted from the KIL.

The implementation of the Maslov-Kirchhoff formulas is outlined as follows:

1. Trace a fan of rays from one secondary-source point on the KIL to the MIA that crosses the observational point(s);
2. Calculate the weighted GRT and normalized MAT amplitudes \( e_0 p^K_0 A \) and \( e_1 p^K_1 B_0 \);
3. Evaluate the impulse time series of the GRT and MI responses (of which, MI by Chapman's (1978) slowness method) and then blend them together to obtain the Green's derivative response (\( \partial x^K G \));
4. Convolve the Green's derivative response with the time variation of the secondary source by FFT to yield the contribution of this secondary source, in which the Fourier transform of the response is complex conjugated, if extrapolated reversely by equation (5.31);
5. Apply the above steps to the next secondary source;
6. Sum the contributions of all the secondary sources to produce the complete wave effect(s) at the observational point(s);

7. Differentiate the time series of the wavefield to account for the effect of $-i\omega$ in the Kirchhoff integrals.
Chapter 6

Backward Wavefield Extrapolation by the Maslov-Kirchhoff Method: Two Examples

Because backward wavefield extrapolation is important in processing seismic exploration data, it is interesting to test the efficiency of the newly-formulated Maslov-Kirchhoff extrapolation operator in this role. But because wavefield extrapolation or migration is not the main theme of the thesis, I only focus on the important difference between the traditional (GRT) Kirchhoff and Maslov-Kirchhoff methods, which arises when the ray method breaks down in a continuously inhomogeneous medium. In particular, two simple examples are investigated.

6.1 Example 1: moderate lateral variation

"Moderate lateral variation" is used here to identify a class of media that vary sufficiently slowly in the direction of the Maslov integration that caustics and pseudocaustics separate and the MAT (blended) solution is applicable.

A convenient example is the low-velocity channel given in Figure 4.1. Here I consider the section at $x_1 = 130$ m because it contains a typical bowtie wavefield triplication for a point source at distance $x_1 = 20$ m.

Figure 6.1 shows the point source wavefields (a) and (c) and their backward extrapolation results (b) and (d), respectively. The top row is for a single point source at the symmetrical depth of the waveguide ($x_3 = 110$ m). The bottom row is for triple point
sources; the central source is the same as the former but the other two have the same strength but less than that of the central source. Note that the wavefields to be extrapolated were accurately calculated by MAT with the trigonometric weighting functions, as demonstrated in Figure 4.7; but conspicuous errors would have occurred around the cusps if only GRT had been used. They consist of 101 traces spaced 10 m from depth 60 m to 160 m. In (c), the lower triplication corresponds to the upper source (at 90 m) and the upper triplication the lower source (at 130 m). The results (b) and (d) are obtained by backward extrapolation with MAT (equation (5.31)). Clearly, they are well focused point events at the source positions and at zero time. The small erroneous cross pattern is caused by the loss of evanescent waves and the truncation of the Kirchhoff integral.

It is common practice to reverse extrapolate the wavefield of plane (line) sources. Here I consider a finite length plane source simulated by a number of point sources deployed on the vertical line at \( x_1 = 20 \) from depth 90 to 130 m, with source strength tapered at both ends. Its wavefield at \( x_1 = 110 \) m as modeled by MAT is depicted in Figure 6.2a. (Note that, although this plane-source response appears smooth, GRT does not predict it accurately because the elementary point-source responses contain caustics). Reverse extrapolation by GRT, as shown in Figure 6.2b, contains strong caustic errors. A V-shape phase (corresponding to the track of the caustics) precedes the wanted linear event at the source line. In contrast, backward extrapolation with MAT gives a much improved result, as presented in Figure 6.2c, although there is some weak background noise. It is due to a relatively wide sampling of the recorded wavefield, and can be reduced by inputting more traces into the Kirchhoff integration. This is shown in Figure 6.2d where the number of input traces is doubled.

### 6.2 Example 2: strong lateral variation

In many cases, lateral variation can be so strong that caustics and pseudocaustics are not separable compared with wavelength; and then pseudocaustics cannot be removed by Maslov's blending approach. Moreover, because pseudocaustics only generate moderate errors in the Maslov integral over the source slowness domain, it is interesting to see how they affect the Kirchhoff extrapolation.

The test model, shown in Figures 6.3a and b, is a 2D constant-density, acoustic medium. It is negatively perturbed up to 30% by a circular gaussian function centered at \((0.4, 1)\). The source is a limited length line consisting of 21 point sources evenly distributed from
Figure 6.1: MAT backward extrapolation for point sources through the low-velocity acoustic channel given in Figure 4.1. The left column shows the wavefields emitted from the point sources at distance $x_1=20$ m and recorded at $x_1=130$ m, and the right column shows the extrapolated results, of which (b) and (d) were from (a) and (c), respectively.
Figure 6.2: MAT backward extrapolation for a plane source through the low-velocity acoustic channel (Figure 4.1). The wavefield (a) modeled by MAT at $x_1 = 130$ m has been extrapolated to the source line at $x_1 = 20$ m. The result (b) was by GRT; and the results in (c) and (d) were by MAT. The amplitude distribution of the event (d) is shown in (e). Note GRT gave an erroneous V-shape structure, but MAT does a better job.
depth 0.9 to 1.1 km at the horizontal distance 0.1 km (marked by crosses). The circles represent 61 discrete receivers with a separation of 0.01 km at 0.53 km. The wavefield (modeled by PPMI) displays a "bowtie" (Figure 6.3c).

The Kirchhoff backward extrapolation requires summation of the inverse responses of all the individual traces. In this model, source responses are folded wavefields for which GRT breaks down. Figures 6.3a and b show an example of the inverse operator for a trace at depth 0.95 km. It can be seen in (6.3d) that the sampled rays (thin lines) and wavefronts (thick lines) "emitted" from this trace position contain a fold. The inverse traveltime curve (6.3e) is accordingly triplicated. Then the inverse operator for this trace calculated by GRT contains, as shown in Figure 6.4a, very large amplitudes at the cusps but no diffractions beyond the cusps. Note that the two phases in the inverse operator correspond directly to the two arrivals in the considered trace at depth 0.95 km (see Figure 6.3c). In contrast, as displayed in Figure 6.4b, MI gives a much more reasonable response. It shows converging but finite wavefields at the cusps and diffractions. It, however, is moderately contaminated by two spurious linear phases due to the pseudocaustics (inflections near the cusps in Figure 6.3e). The purpose of this experiment is to check the effects of both the caustic and pseudocaustic errors on the inverse response of all the recorded traces.

The top row of Figure 6.5 presents the GRT inverse extrapolation at the source line (by equation (5.31) with \( e_0 = 1 \) and \( e_1 = 0 \)). We can see the erroneous individual spikes to the left of the required phase at zero time in (a). Even the required phase does not appear smooth, as shown by the noisy amplitude distribution in (b).

The bottom row of Figure 6.5 illustrates the MI inverse extrapolation at the source line (by equation (5.31) with \( e_0 = 0 \) and \( e_1 = 1 \)). Two points should be noted in (c). Firstly, the set of linear pseudocaustic phases across the whole wavefield have been partially removed; they mainly remain in the negative time. Secondly, the required phase at zero time is very smooth. Part (d) proves that the amplitude distribution is quite satisfactory and the inverted result is better than that by GRT.

6.3 Chapter conclusions

The two simple experiments described in this chapter have demonstrated the advantages of the Maslov-Kirchhoff backward wavefield extrapolation over the traditional Kirchhoff (GRT) extrapolation. In a medium that generates separate caustics and pseudocaustics,
Figure 6.3: A 2D acoustic velocity model negatively perturbed up to 30% by a circular gaussian function centered at (0.4, 1.0) in an otherwise homogeneous medium. The crosses at $x_1 = 0.1$ km denote the 21 point sources. The circles at $x_1 = 0.53$ km denote the 61 receivers. (b) The velocity scale. (c) The plane-source response at $x_1=0.53$ km (modeled by PPMI). (d) Rays (thin lines) and wavefronts (thick lines) reversely propagating from a sample trace at depth 0.95 km. (e) The folded traveltime curve, with two caustics at the cusps, and two pseudocaustics on the forward branches (somewhere near the cusps).
backward extrapolation based on the MAT solution (mixed GRT/MI solution) is very satisfactory. The only error is that due to the truncation of the Kirchhoff integral; but this is the same for any extrapolation method. Secondly, in a relatively strongly varying medium where MAT can suffer from caustic and pseudocaustic errors, the pure MI solution is satisfactory for extrapolation. Part of the pseudocaustic errors may still remain in the extrapolated result, but the accuracy can be appreciably improved in comparison with the result obtained using GRT.

Although the testing was on continuously varying environments, the conclusions should also hold for discrete layered media where the interfaces are smooth. The Maslov method is expected to handle such wavefields more satisfactorily than GRT, and then their extrapolation better. Moreover, a detailed investigation, along with a real application, is needed. It is certainly an interesting topic for future research.
Figure 6.5: Backward extrapolation for the plane source through the circular low-velocity perturbation. The result (a and b) by GRT contains discrete errors and the wanted event is also irregularly disturbed. Instead, MI gives a better result (c) with an improved amplitude distribution (d). Note that pseudocaustic errors are still generated by MI but have been partially reduced; also, that the plot gain in (c) is higher than in (a) so the artifacts are amplified.
Chapter 7

Modeling Seismograms by the Maslov-Kirchhoff Method

The last chapter has shown that pseudocaustic errors can cancel in many places during the Kirchhoff summation in backward wavefield extrapolation. In this chapter I demonstrate that the same happens in forward modeling, and thus a robust Maslov-Kirchhoff seismogram method can be devised. Modeling of a realistic cross-well survey shows that that in a complicated medium, the method not only suppresses the pseudocaustic errors efficiently, but also prevents ray field sampling problems by sampling the wavefield before the rays become randomly dispersed by multiple caustic folds and shadow zones.

7.1 The modeling device

In order to model the wavefield propagated from a source to a line of receivers, the usual approach is to trace rays from the source to the receiver line in one step and then to calculate the GRT/MI seismogram. Instead, in the Maslov-Kirchhoff method, a Kirchhoff integration line (KIL) is introduced between the source point and the receiver line, and the calculation is completed in two steps. The procedure is illustrated in Figure 7.1. The first step is to propagate the wavefield from the (primary) point source (•) to the intermediate KIL. The second is to extrapolate the intermediate wavefield from the KIL to the line of receivers (○). In both steps only the MI solution for computing point-source wave propagation is used, and any pseudocaustic errors arising in either step are expected to cancel in the second step. Figure 7.2 illustrates the flow chart for the computation of the Maslov-Kirchhoff seismogram.
Figure 7.1: Geometry of the Maslov-Kirchhoff seismogram for a point source, where, an intermediate KIL/MIA (Kirchhoff integration line/Maslov integration axis) is introduced between the primary source (●) and the receivers (●) on MIA, and primary rays and secondary rays are respectively traced from the primary source to KIL/MIA and from the secondary sources to MIA. The point-source responses on the KIL/MIA and MIA are all obtained by MI (the pure Maslov integral solution).
Figure 7.2: Flow chart for the computation of the Maslov-Kirchhoff seismogram method
7.2 Theoretical experiments

To demonstrate the effectiveness of the Maslov-Kirchhoff seismogram method, I examine 2D acoustic wave propagation through a circular low-velocity gaussian perturbation in an otherwise uniform medium (Figures 7.3a and b). The perturbation acts as a converging lens and produces the familiar butterfly-wavefront (triplication) with cusps and ray caustics where the waveforms become distorted and stronger. Wavefront inflexions also arise, where rays become parallel and pseudocaustics occur.

7.2.1 Geometrical problems of the wavefield

The wave propagation in Figure 7.3 provides a good example of joint caustics. Figure 7.4 shows the ray amplitudes in the space and slowness domains. Parts (a) and (b) show respectively the GRT and MAT amplitudes. There, caustics $C_x$ and $C_x''$ occur at the bifurcations, while pseudocaustics $C'_p$ and $C''_p$ just appear inside the bifurcations. Parts (c) and (d) show that in the slowness domain, caustics $C_x'$ and $C''_x$ occur just inside the bifurcations but pseudocaustics $C'_p$ and $C''_p$ at the bifurcations. Because of the bifurcation
Figure 7.4: Ray amplitudes in space and slowness domains for Figure 7.3, an example of joint caustics. (a) and (c) The GRT amplitude in the space and slowness domains, respectively. (b) and (d) The MAT amplitude in the space and slowness domains, respectively. Caustics are marked by $C'_z$ and $C''_z$ and pseudocaustics by $C'_p$ and $C''_p$. 
Figure 7.5: Comparison of receiver and source slowness asymptotic data for the model in Figure 7.3 (left and right column, respectively). Parts (a) and (f), (b) and (g), (c) and (h), (d) and (i) are, respectively, graphs of traveltime $T$, intercept time $\tau$, Maslov phase and Maslov amplitude. Part (e) shows the relationship between receiver and source slownesses.

and singularity in the MAT amplitude (d) a Maslov integral over the receiver slowness $p_3$ fails.

Changing the Maslov integration variable to the source slowness can make the integral integrable. The detail is illustrated in Figure 7.5. The geometrical problems arising in the receiver slowness domain are depicted in the left column. They are cusps $C'_x$ and $C''_x$ and inflections $C'_p$ and $C''_p$ (on the forward branches) of the travelt ime (a); inflections $C'_x$ (on the reversed branch) and $C''_x$ and cusps $C'_p$ and $C''_p$ of the intercept-time $\tau$ (b); multi-evaluation of the phase function $\theta$ (e.g., for receiver at $x_3=1$ km) (c); and multi-evaluation and singularity of the amplitude (d).

In the new slowness domain, all the above functions become single-valued. This is shown by the traveltime in (f), intercept-time in (g), the phase function in (h), and the normalized MAT amplitude in (i). (In particular, the caustics (cusps) of the travel-time in (a) change to the stationarities $C'_x$ and $C''_x$ in (f). The pseudocaustics (cusps) of the intercept-time in (b) change to the stationarities in (g).) The new Maslov amplitude is everywhere finite (i). However, the artificial stationarities of the phase function (h) resulting from the pseudocaustics (marked by the dashed lines) may yet cause appreciable error in the MI seismogram (Frazer & Phinney 1980; Brown 1994).

### 7.2.2 Kirchhoff summation, error reduction, and optimum sampling

Figure 7.6 shows the effect of Kirchhoff summation in reducing the pseudocaustic error arising in the one-step MI calculation. Zeng's (1996) explicit 2D 4th-order FD software has been employed to compute the true wavefield (solid lines)\(^1\). Note the wavefield is band limited at a relatively low frequency (about 30 Hz), thus it is smoothly extended from the cusp into the shadow zone.

Part (a) shows the pseudocaustic error as two linear phases nearly tangent to the two cusps. The two-step MK results are displayed in (b), (c), (d) and (f). They are all modeled by extrapolating an unfolded intermediate wavefield at $x_1 = 0.3$ km. The first three of them have a sufficiently small time sampling step (0.2 ms), but the last one has a longer step (exactly, five times longer or 1 ms). In particular, different numbers of intermediate wavefield traces have been input into the Kirchhoff summation: 26 for

\(^1\)The FD modeling size is $1801 \times 1201 \times 3500$ (space interval 0.5 m; time step 0.2 ms). The job took about 11 hours on a SUN SPARCstation 10.
Comparison between the results by MK and MI demonstrates that the pseudocaustic errors preceding the traveltime (a) have been satisfactorily reduced in all the Kirchhoff summation. The late spurious oscillations in (b) are due to coarse discretization of the Kirchhoff integral, but can be quickly suppressed by adding more traces into the summation, as shown in (c), (d) and (f). The results in (c) and (d) are satisfactory.

Part (f) shows that a systematic error arises if the time sampling is made too coarse. A spurious jump appears along the earlier branch. Subtle waveform features fail to be accurately resolved; the waveform at depth 1 km is one such an example. We see that MK requires a finer time sampling rate than one-step MI (or GRT).

It is interesting to compare MK and RK (the ray-Kirchhoff seismogram method. Haddon & Buchen (1981)). The wavefield modeled by RK through the same intermediate wavefield as in Figure 7.6c is given in (e). We see that RK does produce cusp diffractions; but the waveforms of the late branch contain errors, and a more serious error appears at the depth 0.88 km (behind the later branch). The errors are caused by the caustics of the secondary-source rayfields. Their elimination requires more modeling steps and substantially more computation time.

The whole purpose of asymptotic methods is to model seismograms at the higher frequencies often employed in seismology. The above test on MK has also been carried out for 5-times higher frequencies. Results are given in Figure 7.7. Note that because the traveltime contains pseudocaustics only on the forward branches, PPMI (Phase-partitioning Maslov integral) is applicable and has been employed to compute the true wavefield (solid lines). In (a) we see that a pair of linear pseudocaustic phases deteriorate the accuracy of the MI seismogram. Interestingly, the error level in amplitude is similar to that in the previous low frequency case. The MK seismogram was computed based on 231 secondary sources spaced 2 m at the same KIL position as before ($x_1=0.3$ km). As shown in (b), it is much more acceptable than that by MI. The MK result contains errors only in a part area (between depth 0.90 and 0.96 km). The diffraction event modeled by MK at depth 0.8 km is inconsistent with that by PPMI, but the event is of little importance.

In both the high- and low-frequency wavefield modelings, errors appear at a similar position in the MK seismograms (between depth 0.90 and 0.96 km). In order to track their origin the wavefields of individual secondary sources have been examined. Examples are displayed in Figure 7.8. The secondary sources are located at depths 0.932, 0.960 and 0.988 km for (a) and (b), and at depths 0.922, 1.000 and 1.078 km for (c) and (d). We
Figure 7.6: MI, MK and RK wavefields at the receiver line (RL) $x_1 = 0.75$ km, compared with the accurate FD result (all solid lines). (a) Linear pseudocaustic errors produced by one-step MI (phases marked by the arrows). (b), (c) and (d) MK wavefields with fine time sampling but different number of secondary sources (SS). (e) RK wavefield for the same case as in (c). (f) MK wavefield with a longer time sampling interval than all the other cases (a) – (e).
Figure 7.7: High-frequency wavefields at $x_1 = 0.75$ km modeled by MI (dashed lines in (a)), MK (dashed lines in (b)) and accurate PPMI (all solid lines). Linear pseudocaustic errors in (a) (marked by the arrows) have been mostly suppressed in (b) except for the small region between depth 0.90 and 0.96 km. The diffraction at 0.82 km has been improved.

see that pseudocaustic errors are generated for each of the sources; some of them actually focus. The focal point in (a) is different from the true events, whereas in (c) it merges with the true event. The bottom results show that summation of individual wavefields can easily reduce the pseudocaustic errors provided they do not focus. If focusing occurs the errors remain, no matter how many secondary sources are used. The remaining errors can be isolated events (b), or merge with the true arrivals (d). Both have been observed in the previous high- and low-frequency MK seismograms.

### 7.2.3 Tolerance of the intermediate KIL position

To test the tolerance of the intermediate KIL position, two different KIL positions are investigated, one close to the source at a distance $x_1 = 0.2$ km and the other in the central region but across the folded wavefield at $x_1 = 0.5$ km. The resulting seismograms are depicted in Figure 7.9. (Note the frequency is as low as the previous one.) The intermediate wavefield close to the source is simple. The MK seismogram using 81 traces 5 m apart of this intermediate wavefield is shown in (a). There, two pseudocaustic phases still exist, one crossing the window from the bottom-left corner to the upper-right corner, and the other is simply its mirror image. MK has provided little improvement in the amplitude of artifacts but, interestingly, the spurious phases have been shifted away from the physical event.
Figure 7.8: Focusing of pseudo (caustic) phases. Individual wavefields (a) and (c) emitted from several secondary sources and their summation results (b) and (d), respectively. The sources in (a) lie at depths 0.932, 0.960, and 0.988 km; and the sources in (c) lie at depths 0.922, 1.000, and 1.078 km. In (a), pseudo phases focus separately from real signals (near the point (0.58, 0.94)); in (c), they occur with real signals (at points (0.586, 0.9) and (0.586, 1.1)). Therefore, errors in (b) are isolated and prominent; but in (d) they are less obvious and merge with real signals.
Figure 7.9: Wavefields modeled by MK (dashed lines) via different KIL positions and compared with the accurate FD result (solid lines in (a) and (c)). For (a) the KIL is close to the source at $x_1 = 0.2$ km; for (c) it is in the central region at $x_1 = 0.5$ km. We see that in (a), pseudocaustic errors have only been shifted away from the real waveform. But in (c), errors have been very satisfactorily suppressed, although the intermediate wavefield (81 traces) contains pseudocaustic errors (i.e., those marked by the arrows in (b)).
Unlike the previous cases, the intermediate wavefield (computed by MI) at the second KIL position \(x_1 = 0.5 \text{ km}\) contains pseudocaustic errors (Figure 7.9b); testing the MK effect seems more attractive. The result from 81 traces is given in (c). Comparison with FD shows that the MK result (c) only contains minor errors and is free from the effect of the intermediate pseudocaustic errors. Note that the weak spurious arrivals between the direct and the diffracted waves and marked by an arrow are due to the cutoff of the Kirchhoff integral and could be remedied.

### 7.3 Modeling a realistic cross-well wave propagation in a complicated medium

#### 7.3.1 Background: data and model

A pair of crosswell seismic surveys were conducted by Chevron Petroleum Technology at its oil sand site near Fort McMurry, Alberta, Canada in order to monitor the spreading of steam injected into the formations; one prior to injection and the other after. Interpretations of the data are given in Paulsson et al. (1994). This study has many facets, for instance learning how seismic velocities change with increased temperature. However, the objective in this chapter is to test the applicability of the Maslov-Kirchhoff seismogram method on a realistic case.

W. Liu (Huang, Liu & West 1996) provided a tomographic model of the pre-steam formation he obtained from 6000 P-wave rays using Jackson and Tweeton's (1994) imaging codes (Figure 7.10a). The model shows two low velocity layers at the top, and a high velocity region near the bottom. It is not known how correct the tomographic model is. But much credence can be obtained from the similarity of the field data and a synthetic data. Since the tomographic model is derived mainly from first arrival data, a relatively close match between the field data and modeled first arrival times is expected if the tomographic model is correct. However, similarity of first arrival amplitudes and any later events provides an independent check. FD can fairly accurately model all the arrivals but is very laborious. It is interesting to examine the ability of MK to model the wavefield in such a complicated medium, in which rays exhibit complex geometry. In particular, I model a common source fan of seismograms where the source is at depth 180 m. The observed seismogram is provided in Figure 7.11a.
Chapter 7: Modeling Seismograms by the Maslov-Kirchhoff Method

7.3.2 Modeling results

The ray field for this shot is displayed in Figure 7.10b. Many rays are strongly deflected by the high velocity gradients, some two or more times. At the receiver well, rays appear focused or shadowed at many depths. So GRT is totally inapplicable. Because pseudocaustics also occur frequently, the classical Maslov technique also breaks down. The seismogram predicted by MI is shown in Figure 7.11b and is full of spurious linear phases. The result obtained by the Maslov-Kirchhoff method is shown in Figure 7.11c, and is compared with the FD result (d) computed by W. Liu (Huang, Liu & West 1996) using Zeng's (1996) 2D acoustic 4th-order FD software. Note both MK and FD results are computed from the same 2D acoustic model. The FD and MK results should differ only in the FD prediction of reflections from sharp contrasts (which are much suppressed in the tomographic model). Thus, strong similarity is expected if MK is accurate. Comparison between MK, FD and the field observation yields the following points:

1. The phase of the direct waves is accurately modeled by MK and FD. Their amplitude, strong at depth but decreasing near depth 200 m, is essentially predicted by MK and FD; FD gives more continuous amplitude variation;

2. The reflection from the high velocity gradient at depth 275 m is predicted by MK. It shows slightly stronger than the observational and FD seismograms;

3. Three phases above depth 180 m are produced by MK, corresponding to the fast direct waves; the waves reflected between the velocity contrast at depth 193 m near the source well and at depth 177 m near the receiver well; and the latest waves reflected at the interface at depth 208 m (Figure 7.10);

4. The secondary phase between depths 200 and 210 m is quite similarly predicted by MK and FD at a nearly correct time (peak at 63 ms), although the amplitude is overestimated by both. This phase corresponds to the waves bounced from the top-left region (Figure 7.10).

Essentially, the first few events of the field data have been very satisfactorily modeled by both MK and FD. MK shows a close similarity with FD. Note that although the FD seismogram shows a closer agreement with the field data in some places, but it was obtained at a much higher cost than that by MK. The former required 24 min on an IBM RS/6000-370 and the latter only 10 min on a SUN Sparc station 10, about a 10:1
Figure 7.10: P-wave tomographic model (a) and P-wave ray field shot from depth 180 m (b). Note the two low-velocity layers at the top and a high-velocity region at the bottom. The media below depth 200 m are mostly limestone with P-wave velocity about 3000 m/sec. Rays are randomly dispersed by the distance of the receiver well.
Figure 7.11: The observed, MI, MK and FD seismograms of the cross-well wave propagation shot from depth 180 m. The MI result is full of spurious linear phases generated by pseudocaustics. See text for other details.
Chapter 7: Modeling Seismograms by the Maslov-Kirchhoff Method

7.4 Chapter conclusions

The Maslov-Kirchhoff seismogram method has been tested by modeling a wavefield propagating through a low-velocity velocity anomaly, which causes the wavefield to fold and develop diffractions and joint caustics. Modeling by the traditional Maslov integral (MI) technique generates serious spurious pseudocaustic errors at many places. MK modeling with appropriate parameters models the wavefield well. Use of the Kirchhoff method with GRT gives much poorer results.

Tests were performed to see how pseudocaustics are removed by using the MK method. They are strongly affected by the position of the intermediate Kirchhoff integration line (KIL). In particular, they can only be shifted within the seismogram window if the KIL is placed near the source; but they can be suppressed in many places if the KIL is placed in the middle of the source-receiver distance, no matter whether pseudocaustics exist or not in the wavefield on the KIL.

The accuracy of the MK seismogram is affected by the sampling interval of the secondary sources. Because incoherent pseudocaustic errors can be easily suppressed by the use of a few secondary sources, one must pay attention to the accuracy of the Kirchhoff summation. A sampling rate of at least 4 traces per dominant wavelength was required in the example.

Use of an intermediate KIL requires no viewing of the wavefield geometry. This makes the MK method more robust than the phase-partitioning Maslov integral (PPMI) method in dealing with the troublesome pseudocaustic problem. Because a single KIL has been shown sufficient in removing most of pseudocaustic errors, the MK method should be more efficient than a ray-based RK method which likely requires many intermediate lines for a similar result.

Modeling of a realistic cross-well wave propagation has clearly demonstrated the practicality of the new Maslov-Kirchhoff method in modeling wavefields in a complicated medium (i.e., its efficiency in suppressing the pseudocaustic errors and improved sampling of the shorter range ray fields). At a practical level, it provides better results than classical ray theory or Maslov theory, and incurs a much lower cost than a finite difference
scheme. It is therefore applicable to many realistic wave propagation calculations.
Chapter 8

Summary of the Thesis
Contributions and Suggestions

Calculation of seismic wavefields is important in imaging complex subsurface structures from the very complete wavefield data provided by modern data acquisition techniques. Numerical methods such as finite differences are able to calculate full spectrum wavefields, but are very cumbersome, slow and expensive. Asymptotic methods are increasingly needed because they are efficient and economical, and because the dominant wavelengths in seismic wavefields are often two or more orders of magnitude smaller than the propagation distances and the region of interest. Therefore, much of this thesis has been devoted to trying to make forward modeling based on advanced asymptotic methods more robust and practical. Also, backward wavefield extrapolation (otherwise known as migration) is increasingly required in order to image the structure of complicated media from large data sets at a low cost. Thus, asymptotic methods for backward wavefield extrapolation have also been a focus of this thesis.

8.1 The classical Maslov seismogram method

Among the various economical asymptotic modeling methods, Maslov asymptotic theory (MAT) promises a globally-valid high-frequency wave solution based on clear physical principles. It extends the ability of simple geometrical ray theory (GRT) to model wavefields containing caustics, shadows, critical points, head waves, etc. Since the introduction of this theory to the seismic community by Chapman & Drummond (1982), it has been enhanced by several contributions such as end-point correction (Thomson & Chapman 1986), generalization to anisotropic environments (Kendall & Thomson 1993, Guest

Nevertheless, the Maslov technique is not yet in widespread use in seismology, apparently because insufficient work has been done to demonstrate its accuracy, and to improve its robustness and automation. These are important aspects that affect practical application and, the first part of this thesis has been devoted to them by the specific contributions summarized below. But, firstly, it is convenient to remind readers of the wave propagation problems we have been concerned with in this thesis: practical seismic wavefields always have finite frequency range and propagate in structured media. Geometrical optics ray-path and traveltime analysis do not predict many seismic events correctly. Seismograms often contain interesting diffractions into geometrically-shadowed regions, or waveform distortions where waves of different traveltime branches overlap. We wonder how accurately these waveform distortions or diffractions can be predicted by the Maslov method, and what steps can be employed to make the method robust and automated.

Accuracy of the Maslov integral (MI) seismogram

Wavefields propagating through several idealized earth models were modeled by MI using phase partitioning where necessary. The results were compared with accurate finite difference solutions. This study showed that in the high-frequency range (for which medium variation is smooth, or relative medium variation over a wavelength is much smaller than 1), waveforms near wavefront anomalies such as bends, kinks, folds, or ray field edges can be fairly accurately given by MI, provided pseudocaustics, if any, can be completely removed. This is very satisfactory because these waveforms are often prominent signals on a seismogram. Waveform diffractions or distortions far from the wavefront anomalies given by MI are not reliable, but fortunately, they are usually weak features and therefore of little importance in seismic interpretation. Essentially, high-frequency wavefields are controlled by the geometry of the ray field.

Although all testing was done using 2D propagation in smoothly continuous media, the conclusions should be more general. They should hold equally for discontinuous and anisotropic structures. But if a medium contains 3D structures then the 2D Maslov integral wavefield representation is needed in order to cure the higher-order caustic problem and to include diffractions in more than one directions into the seismogram (Chapman & Drummond 1982; Kendall & Thomson 1993; Keers & Chapman 1995; Liu & Tromp 1996).
Rotation of the Maslov integration axis (MIA)

I have shown by example that accuracy of the Maslov integral often can be improved by rotating the Maslov integration axis (MIA). This is because correct prediction of waveform diffractions or distortions requires that the MIA cross-cut the wavefront anomalies. I have described a method for the calculation of a rotated Maslov Jacobian which makes it easier to calculate a rotated Maslov integral. The examples also clarify a second way in which rotation affects Maslov integrals. Changing the direction of the Maslov integration can help move a pseudocaustic out of the integration range if the medium variation is one-dimensional.

Accuracy of mixed GRT/MI seismograms and a robust weighting method

I have illustrated how the number of the neighboring rays (range of \( p \) values) included in the Maslov integral affects the accuracy of the Maslov seismogram. Roughly, the more MI is locally weighted (i.e., as fewer neighboring rays are used), the more waveform amplitudes can be underestimated. To ensure accuracy of MI, a sufficiently large bundle of integrable neighboring rays – rays involving no pseudocaustic – is required. Similarly, to remove the effects of pseudocaustics, the contribution of a certain number of neighboring rays near the pseudocaustic needs to be smoothly removed from the Maslov integral; otherwise a pseudocaustic artifact will remain. The lower the frequency of interest, the more serious is the problem. A more detailed study of constraints on the range of integration should be made in the future; a rough guide can be obtained by estimating the size of the Fresnel zone, as implied by Guest & Kendall (1993).

Brown (1994) used the argument \( \partial x/\partial p \) to map the slope of the Lagrangian manifold cross-section onto the weighting functions; the slope quantifies the applicability of GRT or MI, while the weighting functions the opportunity for GRT or MI to contribute to the wavefield construction. Such a mapping is capable of reflecting the subtle geometry of the Lagrangian manifold. It allows for an appropriate sampling of the GRT or MI component when the switching between GRT and MI cannot simply be described by functions of the positions of caustics and pseudocaustics. One observed example is the accurate prediction of the strong waveform at the kink point. However, this method does not automatically provide a reasonable switching between GRT and MI, because the desirable value of the arbitrary coefficient \( \gamma \) changes for different modelings and its tolerance range is limited. To optimize the procedure, \( \gamma \) needs to be set. This is an obstacle if seismogram calculations are to be automated.
In this thesis I have suggested the weighting functions be a simple function of the $p_0$ positions of the caustics and pseudocaustics, and proposed use of a trigonometric function (sine or cosine) connecting adjacent caustic and pseudocaustic pairs. In comparison to Brown’s weighting functions, this gives more weight to MI at the caustic and correspondingly less weight to MI at the pseudocaustic in a symmetric way, and it switches between unity and zero smoothly in between. When a wavefront is smoothly bent (i.e., just before a kink), no real caustic occurs (meaning that $\partial x / \partial p_0$ does not change sign). However, the wavefield will undergo a certain convergence and waveform distortion, and GRT fails to give accurate waveform and amplitude. For MI to be accurate, enough neighboring rays are needed. Unfortunately, the weighting scheme I suggested does not recognize this opportunity because it is based on finding caustics and pseudocaustics. However, it usually is accurate and stable and should be considered first, particularly in intensive modelings.

8.2 The Maslov-Kirchhoff seismogram method

None of the techniques discussed above make routine modeling possible because many earth structures generate inseparable joint caustics and pseudocaustics in the ray field. Thus the second part of this thesis presents a new Maslov-Kirchhoff modeling tool in which a Kirchhoff summation (Huygens’ principle) is used to suppress the pseudocaustic errors. A Kirchhoff integration line (KIL) is introduced between the source point and the receiver line, and the calculation proceeds in two steps. The first is to propagate the wavefield from the real point source to the intermediate KIL. The second is to extrapolate the intermediate wavefield from the KIL to the line of receivers. In both steps only the MI solution for computing point-source wave propagation is used.

Chapter 5 presents the theoretical basis of the MK method. Numerical experiments presented in Chapter 7 show that the use of intermediate source points tends to randomize pseudocaustic errors so they really are reduced in the summation. However they may not always be made random. They will only be shifted within the seismogram window if the KIL is laid too near the source region; but they can be randomized and suppressed in many places if the KIL is placed centrally, no matter whether pseudocaustics develop on the intermediate seismograms or not. Incoherent pseudocaustic errors can be easily suppressed by the use of a few secondary sources but more are required to maintain accuracy of the Kirchhoff integral. A sampling rate of about 4 traces per wavelength is desirable.
Chapter 7: Summary of the Thesis

The ability of the MK method to model realistic cross-well wave propagation was tested. This example demonstrated the two advantages of the new method in modeling wavefields in a complicated medium. One is, as expected, that pseudocaustic errors can be satisfactorily suppressed; and the other is that the ray fields are better sampled (i.e. sampled at shorter range before they become dispersed in a complicated manner) than with the classical one-step Maslov method. It is therefore applicable to many realistic, complicated modelings.

The Maslov-Kirchhoff method has been shown better than several other methods. It can provide better results than classical ray theory or Maslov uniform theory. It requires a much lower cost than a finite difference scheme. Because introduction of an intermediate KIL requires no viewing of wavefield geometry, the new method is more robust than the phase-partitioning Maslov integral method in ameliorating the pseudocaustic problem. Compared with other Kirchhoff modeling methods, it is more efficient than Haddon & Buchen's (1981) ray-Kirchhoff method, because a single KIL has been shown sufficient to remove most pseudocaustic errors. It is also more accurate than Frazer & Sinton's (1984) extended Kirchhoff-Helmholtz (EKH) method, although the latter requires only a single integration line and is similarly efficient computationally for a single receiver.

8.3 Backward wavefield extrapolation

Backward wavefield extrapolation through a specified inhomogeneous medium is an important component in migration. The commonly-used Kirchhoff method invokes simple ray theory and assumes it to be accurate even if the medium is inhomogeneous and causes important focusing. It has been widely used because of its economy and ability to deal with well layered structures. But it often breaks down if the medium varies laterally either in the form of gradients or interface structure.

In view of the ability of Maslov theory to deal with lateral variation, I have applied it to Kirchhoff wavefield extrapolation and formulated a true-amplitude wavefield extrapolation operator for a general smoothly varying medium. Practical effectiveness has been demonstrated in two examples of backward extrapolation along a variable structure. The first example treats propagation along a slowly varying velocity channel, and shows that the wavefield can be reversely extrapolated fairly accurately, because Maslov's uniform (blending GRT/MI) seismogram method is applicable and accurate in this case. But when the medium varies moderately rapidly, joint caustics can occur and Maslov's uni-
form seismogram method should not be used. A second example dealing with a local
gaussian velocity perturbation reveals that the result obtained using only the pure Maslov
integral (over the source slowness) is still satisfactory, because any possible pseudocaustic
ergains can partially cancel during the Kirchhoff summation.

8.4 Suggestions for future work

1. The test models discussed in this thesis are mostly 2D acoustic cases. However,
   all the principles involved should apply similarly to elastic waves in isotropic and
   anisotropic media. Extensions to 3-dimensional geometry are straightforward but
   in the case of Kirchhoff integration requires surface rather than line integrals. It
   would be interesting and worthwhile to apply the improved implementations of the
   Maslov method and the Maslov-Kirchhoff method to 3D anisotropic problems.

2. The Maslov-Kirchhoff wavefield extrapolation operator developed in this thesis
   shows good potential for dealing with any models with continuous velocity variation
   or sharp velocity changes where reflections are to be neglected. It is of significance
   to apply this operator in practical migration procedures.
References


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Appendix A

Transformation Between 2D and 3D Seismograms

Transforming seismograms between 2D and 3D is common practice through which the effects of 3D wave propagation can be found with smaller effort. Models of special structure are required, but very often, the case is when the 3D model considered is symmetrical about a vertical axis. In this case, transformations derived from simple GRT have been used, as in Červený, Popov & Pšenčík (1982) and Frazer & Sinton (1984), but they are applicable only at places where GRT is valid. At many interesting places such as caustics and shadows, they break down and a new transformation, which is of the same order of accuracy as Maslov theory, must be worked out.

We formulate such a transformation by comparing the 2D and 3D WKBJ solutions. We do so because the WKBJ seismogram representation is itself a Maslov integral for a horizontally layered medium, and its complete form (including the source functions) can be found in literature (e.g., Chapman et al. 1988) and is ready to use. In this context, 2D means a source and medium structure that is constant in the \( x_2 \) direction, and 3D means a source and medium structure that exhibits cylindrical symmetry about the \( x_3 \) axis. In either case, consider for simplicity the source and receiver components in the propagation plane \( \mathbf{z} = (x_1, 0, x_3) \) in a 3D elastic medium. The 3D particle motion (for a point P-source at the origin) and the 2D particle motion (for a line P-source which is transverse to the propagation plane through the origin) can be written respectively (see, e.g., Singh 1987 and Chapman et al. 1988)

\[
y^{(3)}(t, \mathbf{z}) \simeq \frac{d}{dt} \mathbf{m}(t) : * \frac{d}{dt} \text{Im} \left[ \Lambda(t) * \sum_{t=\tilde{t}(p_1, \mathbf{z})} \frac{A^{(3)}(\mathbf{y})}{\partial \mathbf{y}, \tilde{t}(p_1, \mathbf{z})} \right]
\]
where $t$ is the time, $y$ is the mixed domain $(p_1, 0, x_3)$, $\hat{\theta}$ is the plane P-wave traveltime. $m(t)$ is the source moment tensor, or the volume integral of the source glut (Gilbert 1971 and Backus & Mulcahy 1976)

$$m(t) = \begin{pmatrix} m_{11}(t) & m_{13}(t) \\ m_{13}(t) & m_{33}(t) \end{pmatrix}, \quad \text{and} \quad f(t) = \begin{pmatrix} f_1(t) \\ f_3(t) \end{pmatrix}. \quad \text{(A.3)}$$

$A^{(3)}(y)$ and $A^{(2)}(y)$ can be respectively viewed as the 3D and 2D WKBJ analytic signal solutions in the transform domain for a step-function seismic moment source, and $B^{(3)}(y)$ and $B^{(2)}(y)$ for a delta-function force source. They are given by

$$A^{(3)}(y) = -\frac{p_1^{1/2}}{2^{3/2} |x_1|^{1/2} \pi^2} \mathcal{M}(p_1) \hat{N}(y) R(p_1),$$  

$$B^{(3)}(y) = -\frac{p_1^{1/2}}{2^{3/2} |x_1|^{1/2} \pi^2} \mathcal{E}(p_1) \hat{N}(y) R(p_1),$$  

$$A^{(2)}(y) = -\frac{1}{2\pi} \mathcal{M}(p_1) \hat{N}(y) R(p_1),$$  

$$B^{(2)}(y) = -\frac{1}{2\pi} \mathcal{E}(p_1) \hat{N}(y) R(p_1), \quad \text{(A.5)}$$

where the source components are combined using the tensor $\mathcal{M}(p_1)$ and vector $\mathcal{E}(p_1)$,

$$\mathcal{M}(p_1) = \frac{1}{(2\rho_0 p_0)^{1/2}} \begin{pmatrix} p_1^2 & \pm p_1 p_0 \\ \pm p_1 p_0 & p_0^2 \end{pmatrix}, \quad \text{and} \quad \mathcal{E}(p_1) = \frac{1}{(2\rho_0 p_0)^{1/2}} \begin{pmatrix} p_1 \\ \pm p_0 \end{pmatrix}, \quad \text{(A.6, A.7)}$$

and the receiver components are given by the vector $\hat{N}(y)$,

$$\hat{N}(y) = \frac{1}{(2\rho p_3)^{1/2}} \begin{pmatrix} p_1 \\ \pm p_3 \end{pmatrix}. \quad \text{(A.8)}$$
Appendix A: Seismogram Transformation

Subscript "zero" denotes the values at the source depth, \( x_3 = x_{03} \), and the upper sign is for a ray leaving the source or arriving at the receiver upwards and the lower sign downwards. The function \( R(p_1) \) is the product of the reflection and transmission coefficients encountered by a ray. The operators : * and : · * denote convolution in the time domain together with summation over moment and body source components, respectively. The summation notation in (A.1) or (A.2) is over all possible plane P-waves arriving at \( \vec{x} \), at time \( t \). \( \Lambda(t) \) and \( \Delta(t) \) are the two analytic time series defined previously in (2.66) and (2.35), respectively. For theoretical and numerical details, the reader is referred to Chapman et al. (1988).

A quick glance at equations (A.5) shows that these variables can be related by

\[
A^{(2)}(y) \simeq \pi \left| \frac{2x_1}{p_1} \right|^{1/2} A^{(3)}(y),
\]

\[
B^{(2)}(y) \simeq \pi \left| \frac{2x_1}{p_1} \right|^{1/2} B^{(3)}(y).
\]

(A.9)

Noticing that the Fourier transform of \( H(t)/t^{1/2} \) is \( (\pi/\omega)^{1/2} e^{i\pi/4} \) (Chapman 1978, Table 2, or Aki & Richards 1980, pp428), the relationship between the 2D and 3D analytic time series can be easily shown as (also Aki & Richards 1980, pp428)

\[
\frac{1}{\pi} \frac{d}{dt} \Lambda(t) * \frac{H(t)}{t^{1/2}} = \Delta(t).
\]

(A.10)

At the geometrical arrival, when the first motion approximation for equations (A.1) and (A.2) are valid, we have the transformations of geometrical signals between 2D and 3D

\[
\underline{u}^{(2)}(t, \vec{x}) \simeq \left| \frac{2x_1}{p_1} \right|^{1/2} \frac{H(t)}{t^{1/2}} * \underline{u}^{(3)}(t, \vec{x}),
\]

(A.11)

and

\[
\underline{u}^{(3)}(t, \vec{x}) \simeq \frac{1}{\pi} \left| \frac{p_1}{2x_1} \right|^{1/2} \frac{d}{dt} \frac{H(t)}{t^{1/2}} * \underline{u}^{(2)}(t, \vec{x})
\]

(A.12)

where \( x_1 \) and \( p_1 \) are evaluated at the geometrical arrival. The factor \( |x_1/p_1|^{1/2} \) compensates for the azimuthal geometrical spreading. \( H(t)/t^{1/2} \) and \( d(H(t)/t^{1/2})/dt \) account for the difference between the waveforms of line and point sources.

The transformations (A.11) and (A.12) can also be obtained by comparing the ray ansatzes for a point and a line source in laterally homogeneous media. These relations are not new. See, e.g., Frazer & Sen (1985, eq. (B3)). Note, however, that only for straight rays in homogeneous media can the factor \( |x_1/p_1|^{1/2} \) reduce to the product of the velocity \( V \) and the source-to-receiver distance \( r \), i.e. \( Vr \). Therefore, for bent rays...
in inhomogeneous media, the standard procedure involving the factor $\nu r$ as was used in Červený, Popov & Pšenčík (1982) and Frazer & Sinton (1984) is inappropriate.

It is interesting to mention that although they share the same notation $m$ for brevity, the seismic moments for 2D and 3D have different dimensions: the former is moment per length and the latter moment. In a similar vein, the seismic forces for 2D and 3D have the same notation $f$ but different dimensions: Newton per length and Newton. The term $|2x_1/p_1|^{1/2}H(t)/t^{1/2}$ in (A.11) has dimensions of length, and the term $1/\pi p_1/2x_1^{1/2}d(H(t)/t^{1/2})/dt$ in (A.12) dimensions of inverse length, where dimensions of time are cancelled by convolution. Equations (A.11) and (A.12) connect the 2D and 3D solutions only by magnitudes.

A novel transformation that is valid even in places where GRT breaks down can be obtained by combining the 2D and 3D WKBJ/MAT solutions, i.e. (A.9) and (A.1) or (A.9) and (A.2). For instance, the 2D solution is related to 3D by

$$u^{(2)}(t, x) \simeq 2^{1/2} \pi \left\{ \frac{d}{dt} m(t) : * \text{Im} \left[ \Delta(t) \ast \sum_{t=\delta(p_1, x)} \frac{|x_1|^{1/2} A^{(3)}(y)}{|p_1|^{1/2} |\partial_{p_1} \theta(p_1, x)|} \right] ight. $$

$$+ \left. f(t) \cdot * \text{Im} \left[ \Delta(t) \ast \sum_{t=\delta(p_1, x)} \frac{|x_1|^{1/2} B^{(3)}(y)}{|p_1|^{1/2} |\partial_{p_1} \theta(p_1, x)|} \right] \right\} . \quad (A.13)$$

For a point explosion with a Heaviside-step seismic moment, it reduces to

$$u^{(2)}(t, x) \simeq - \text{Im} \left[ \Delta(t) \ast \sum_{t=\delta(p_1, x)} \frac{|x_1|^{1/2} \hat{L}^{(3)}(y)}{|p_1|^{1/2} |\partial_{p_1} \theta(p_1, x)|} \right] . \quad (A.14)$$

where the 3D Maslov amplitude $\hat{L}^{(3)}(y) = -2^{1/2} \pi A^{(3)}(y)$ (by comparing (2.64) and the first term of (A.1)) has been used. Note in numerical implementation, the convolution $\Delta(t) \ast = (\delta(t) + i \delta(t))^*$ is performed in the frequency domain in order to avoid the singularity of $\delta(t) = -1/\pi t$ in the time domain.
Appendix B

Range of the Wavefront Kink

Rays in a low velocity channel are trapped and cross to form caustics, and wavefronts progressively develop kinks and fold upon themselves (e.g., Figure 3.11). To understand better the phenomenon of a wavefront kink, we analyse the place where it happens.

In a 2D laterally homogeneous medium, the range at which a ray initially directed downwards or upwards towards a higher velocity will return to the source level \( x_3 = 0 \) is given by, in general,

\[
X_1 = 2 \int_0^{X_3} \frac{p_1}{\sqrt{v^{-2}(x_3) - p_1^2}} dx_3 \tag{B.1}
\]

where \( v(x_3) \) is the velocity, \( p_1 \) the constant horizontal slowness, and \( X_3 \) the maximum vertical distance of penetration. For a hyperbolic velocity model described by

\[
v(x_3) = a\sqrt{x_3^2 + b^2} \tag{B.2}
\]

where \( a \) and \( b \) are constant, the range of the kink point can be solved as a complete elliptic integral of second kind by immediate substitution of the velocity in equation (B.1) (Kaufman 1953). However this can be achieved by a simpler integral. Because a kink point is the starting point of a wavefront fold or ray caustic, rays that cross it are those propagating near the source level \( x_3 = 0 \). For them, we can approximate the hyperbolic velocity equation (B.1) with a parabolic function, i.e.,

\[
v^{-2}(x_3) \simeq (ab)^{-2}(1 - (x_3/b)^2), \quad x_3/b \ll 1, \tag{B.3}
\]

and, by noting \( X_3 = \sqrt{(ap_1)^{-2} - b^2} \), we have

\[
X_1 \simeq 2ab^2p_1\arcsin\left(\frac{1}{abp_1}\right), \quad x_3/b \ll 1. \tag{B.4}
\]
Then the range of the wavefront kink is obtained by taking the limit of equation (B.4) as rays are shot towards horizontal with $p_1 \to 1/ab$, as

$$X_1 = \pi b.$$  \hfill (B.5)

Figure B.1 shows a close-up of the rays traveling near the source level and passing through the wavefront kink in a model with $a = 37$ and $b = 27$. The rays are traced with a take-off angle interval of 0.1 degrees: they all clearly focus at the kink point $(\pi, 0)$ when the model is scaled by $b$.

**Figure B.1:** A close-up of the rays passing through the wavefront kink at $(\pi, 0)$ in a hyperbolic velocity model (equation (B.2)) scaled by $b$. Note the large vertical exaggeration.
Appendix C

Maslov Uniform Solution for a 2D Constant-density, Acoustic Medium

The Maslov uniform solution for elastic wave propagation has been discussed in Chapter 2 section. But in several testing cases, the Maslov method has been studied for acoustic waves. Here, I give the Maslov uniform solution for the scalar Green’s function for a 2D \((x_1, x_3)\) constant-density, acoustic medium (with most of the notations self-contained). This solution consists of the GRT and MI solutions. For the inhomogeneous Helmholtz wave equation in the considered medium

\[
\left( \partial^2_{x_1} + \partial^2_{x_3} + (\omega/c)^2 \right) \Phi = -4\pi \delta(x_{01}) \delta(x_{03}).
\]

the GRT harmonic response at a receiver point \(x_r = (x_{r1}, x_{r3})\) is simply

\[
\Phi^{GRT}(\omega, x_r) = \sum_{\text{rays}} \frac{2^{1/2}c(x_r)}{|J(x_r)|^{1/2}} \left( \frac{\pi}{-i\omega} \right)^{1/2} e^{-i\phi(x_0)\sigma/2} e^{i\omega T(x_r)},
\]

where \(\omega\) denotes the frequency, \(c\) the velocity, the subscript “zero” the source point, \(\sigma\) the KMAH index, and \(\sum\) the sum over all rays that can arrive at \(x_r\). \(J\) is the ray Jacobian, given by

\[
J(x_r) = \frac{\partial x}{\partial \nu} = \frac{\partial(x_{1}, x_{3})}{\partial(T, q)}
\]

in the ray coordinate system \(\nu = (T, q)\) that consists of the travel-time \(T\) and the initial take-off angle \(q\) of a ray. Note that in the isotropic case, it actually reduces to

\[
J(x_r) = c|\partial_q x| = c\sqrt{(\partial_q x_1)^2 + (\partial_q x_3)^2}.
\]

And, for a homogeneous environment, \((C.2)\) becomes \(\Phi^{GRT} = (2\pi c/\omega|x_r-x_0|)^{1/2}\exp(i\omega T + \pi i/4)\). See, e.g., Hudson (1980) and Frazer & Sinton (1984).
The MI solution, on the other hand, is obtained by making a zeroth-order approximation to the spatially \((x_3, \text{say})\) Fourier transformed Helmholtz equation (C.1) (in which general velocity variation is allowed), and by inversely transforming this approximation with respect to the vertical slowness \(p_3\), as

\[
\Phi^{\text{MI}}(\omega, x_r) = \frac{i}{2^{1/2}} \int \frac{2^{1/2}c(p_3)}{|J(p_3)|^{1/2}} e^{-i\sigma/2} e^{i\omega \hat{d}(x_3, p_3)} \, dp_3. \tag{C.5}
\]

where, \(\sigma(p_3)\) is the Maslov index given by \(\sigma = \sigma + (1 - \text{sgn}(\partial x_3/\partial p_3))/2\). The purpose of the Maslov index is to ensure continuity of the physical waveform across a pseudocaustic. For this to happen, \(\sigma\) must jump discontinuously by one when any such a point where \(\partial p_3/\partial x_3 = 0\) is passed; otherwise it remains constant (e.g. Brown 1994). Note especially that the plane wave (or Snell wave) travel-time \(\hat{d}(x_r, p_3)\) (MAT phase function) in (C.5) is obtained by

\[
\hat{d}(x_r, p_3) = \tau(p_3) + p_3 x_3 \tag{C.6}
\]

which is related to the ray travel-time \(T(p_3)\) (GRT phase function) in (C.2) via the Legendre transformation

\[
\tau(p_3) = T(p_3) - p_3 x_3(p_3), \tag{C.7}
\]

the familiar \(\tau - p\) transformation which processes the physical wavefront with a series of tangent planes (see Kendall & Thomson 1993, Fig. 1). The Maslov Jacobian \(\dot{J}\) is related to the (GRT) Jacobian \(J\) by the canonical transformation

\[
\dot{J}(p_3) = J(x_r) \frac{\partial p_3}{\partial x_3} = \frac{\partial(x_1, p_3)}{\partial(T, q)}. \tag{C.8}
\]

Mathematically, the MI representation can also be rewritten by expressing the integral over the source slowness \(p_{03}\), i.e. replacing \(dp_3\) with \(dp_{03}\), as

\[
\Phi^{\text{MI}}(\omega, x_r) = \frac{i}{2^{1/2}} \int \frac{2^{1/2}c(p_{03})}{|\dot{J}(p_{03})|^{1/2}} e^{-i\sigma/2} e^{i\omega \hat{d}(x_r, p_{03})} \, dp_{03}. \tag{C.9}
\]

This representation indeed yields certain advantages. It always remains integrable because the ray information can usually be sampled as a one-on-one function of \(p_{03}\) (Frazer & Phinney 1980, Chapman & Drummond 1982, and Brown 1994). More importantly, the new Maslov Jacobian is not zero at a pseudocaustic but infinite, i.e.

\[
\dot{J}(p_{03}) = J(p_3) \frac{\partial p_{03}}{\partial p_3} \frac{\partial p_{03}}{\partial x_3} = \dot{J}(p_3) \left( \frac{\partial p_{03}}{\partial p_3} \right)^2. \tag{C.10}
\]

(Note that we have retained the same notations for the Maslov Jacobians and phases in the receiver and source slowness domains for simplicity. The phase \(\hat{d}(x_r, p_{03})\) in equation
(C.9) is still given by equations (C.6) and (C.7) but be aware that $p_3$ is now a function of $p_{03}$.

Finally, the Maslov uniform solution is completed by blending the GRT and MI components, equations (C.2) and (C.9), as

$$
\phi^{\text{MAT}}(\omega, x_r) = \sum_{\text{rays}} e_0(p_{03}) \frac{2^{1/2} c(x_r)}{|J(x_r)|^{1/2}} \left( \frac{\tau}{-i\omega} \right)^{1/2} e^{-i\text{sgn}(\omega)\tau \sigma/2} e^{i\omega T(x_r)}
+ \frac{i}{2^{1/2}} \int e_1(p_{03}) \frac{2^{1/2} c(p_{03})}{|J(p_{03})|^{1/2}} e^{-i\tau \sigma/2} e^{i\omega \hat{\theta}(x_{r3}, p_{03})} dp_{03},
$$

with the weighting functions $e_0(p_{03})$ and $e_1(p_{03})$ subject to $e_0 + e_1 = 1$ for every ray.

The wavefield solution in the space-time domain is obtained by inverse Fourier transform of equation (C.11) with respect to the frequency. In particular, the inverse MI (a double integral over slowness and frequency) is accomplished by means of Chapman's (1978) slowness method. The impulsive wave response is exactly

$$
\phi^{\text{MAT}}(t, x_r) = \sum_{\text{rays}} \text{Re} \left\{ e_0(p_{03}) \frac{2^{1/2} c(x_r)}{|J(x_r)|^{1/2}} \Lambda(t - T(x_r)) e^{-i\sigma \tau/2} \right\}
- \text{Im} \left\{ \sum_{t=\hat{\theta}(x_{r3}, p_{03})} e_1(p_{03}) \frac{c(p_{03}) e^{-i\tau \sigma/2}}{|J(p_{03})|^{1/2} |\partial_{p_{03}} \hat{\theta}(x_{r3}, p_{03})|} \right\}
$$

with the analytic time series $\Lambda(t)$ and $\Delta(t)$ given in Chapter 2 (equations (2.66) and (2.35), respectively). Note that in equation (C.12), the waveforms are displayed respectively by the series $\Lambda(t)$ in the first term (for GRT), and by the summation $\sum$ along the phase function $\hat{\theta}$ in the second term (for MI). The time series $\Delta(t)$ in the second term only responds to those of waveform components given by $\sum$ that require a Hilbert transform (a mirror image in 2D). The reader is referred to Kendall & Thomson (1989) for dynamic ray tracing, and Chapman et al. (1988) for the numerical techniques of evaluating equation (C.12).
Appendix D

Quintic Spline Interpolation

D.1 Introduction

Numerical representation of the model parameters $\rho(x)$, $c_{ijkl}(x)$, or $V(x)$ is fundamental to a wide variety of computational problems. When propagation of waves in a smooth medium is treated analytically, continuity of the medium parameter values and their first and second derivatives is required. In general, it is impractical to describe real earth models in terms of simple analytic functions, such as in Kaufman (1953). Thus models are discretized and represented by values at grid nodes, with interpolation function providing intermediate values. If models are discretized on a dense grid, variations of parameters over a grid cell may be so small that the values and derivatives can be reasonably approximated by linear interpolation. However, it is often desirable to keep the number of model parameters low, especially in seismic inversion. Then it becomes necessary for the values and derivatives of parameters at any point to be interpolated smoothly.

Polynomials are widely used for smooth interpolation, among which are the Lagrange polynomials and cubic splines. A Lagrange polynomial is a single function based on the total number of the nodes. It is flexible (can be shaped arbitrarily), but can easily become unstable (oscillates strongly) and complicated in expression and computation when the number of nodes is large. The cubic spline is a cubic polynomial designed to piecewisely approximate the true function within each nodal interval, at the same time ensuring continuous first and second derivatives at the node points (De Boor 1962; Bhattacharyya 1969; Pretlová 1976; Schultz 1973). It is stable, with the integral square measure of the approximation to the second derivative reduced to a minimum (Holladay 1957; Walsh et al. 1962). But it usually behaves “globally” because the interpolated value at any
Appendix D: Quintic Spline Interpolation

point depends on the values at all the nodes, and also on boundary constraints, such as assumed zero second derivative at the outer boundary node. This spline was used, e.g. by Bhattacharyya (1969) and Pretlová (1976).

To improve the computational efficiency and eliminate long range effects, Schultz (1973) proposed a local interpolation method of constructing the cubic Lagrange polynomials from only five node values. Lamontagne (Boerner & West 1984) devised a four-point quintic spline, but it only ensured continuity of values and first derivatives in order to preserve other desirable features. By making a local assumption that the first derivatives at any four successive nodes satisfy a linear relation, Smith (Thomson & Gubbins 1982) developed a four-point Hermite (cubic) interpolation. Unfortunately, the Smith scheme can also lead to discontinuous second derivatives, unless the model being interpolated can actually be described by second-order polynomials. (Also see Hale 1992b.) Therefore, I have formulated a new four-point quintic spline which can guarantee the continuities of values, first and second derivatives.

D.2 Formulation

I first define a four-point quintic spline interpolation formula. Given a set of at least four nodes in one dimension, say \( x \), that are equispaced with an interval \( \Delta_x \), and the values of a physical parameter \( f \) at these nodes. Denote, for a particular point \( x_a \), the four closest nodes as \( x_1, x_2, x_3, x_4 \), and their values as \( f_1, f_2, f_3, f_4 \). Then, for \( x_a \) between \( x_1 \) and \( x_4 \), the continuous function \( f(x) \) is given by

\[
f(x_a) = f_1C_1(\delta_x) + f_2C_2(\delta_x) + f_3C_3(\delta_x) + f_4C_4(\delta_x).
\]  

(D.1)

Here, \( \delta_x = (x_a - x_1)/\Delta_x \in [0, 3] \) is the normalized distance of point \( x_a \) measured from the first node, and \( C_1, \cdots, C_4 \) are the so-called cardinal splines.

To guarantee the value, the first and second derivatives are continuous, I let the function \( f(x) \) given by (D.1) satisfy the following:

1. If \( x_a \) is situated in the central interval, \( x_a \in [x_2, x_3] \), then at the two inner nodes \( x_2 \) and \( x_3 \), its values are assigned the given values \( f_2 \) and \( f_3 \), and its first and second derivatives \( f' \) and \( f'' \) are approximated by the second-order central-point finite differences, i.e.

\[
f'_2 = \frac{f_3 - f_1}{2\Delta_x}
\]
If \( x_a \) is situated in the boundary interval between a boundary node and an inner node, say, \( x_a \in [x_1, x_2] \), then at the boundary node \( x_1 \), only its value is evaluated and assigned as the given value \( f_1 \). At the inner node \( x_2 \), all its value, first and second derivatives are evaluated. The value is assigned the given value \( f_2 \). The first and second derivatives are approximated by the second-order central-point finite differences from \( f_1, f_2, \) and \( f_3 \), as given in assumption (1).

The new cardinal splines can be formulated based on these two assumptions. For \( x_a \) located in the inner interval, i.e., \( x_a \in [x_2, x_3] \) or equivalently \( \delta_x \in [1, 2] \), assumption (1) provides six constraints for each of the cardinal splines, as shown by the following column pairs on the l.h.s. and r.h.s.

\[
\begin{pmatrix}
  C_1(1) & C_2(1) & C_3(1) & C_4(1) \\
  C_1(2) & C_2(2) & C_3(2) & C_4(2) \\
  C_1'(1) & C_2'(1) & C_3'(1) & C_4'(1) \\
  C_1'(2) & C_2'(2) & C_3'(2) & C_4'(2) \\
  C_1''(1) & C_2''(1) & C_3''(1) & C_4''(1) \\
  C_1''(2) & C_2''(2) & C_3''(2) & C_4''(2)
\end{pmatrix}
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
= 
\begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -2(2\Delta_x)^{-1} & 0 & (2\Delta_x)^{-1} & 0 \\
  0 & -(2\Delta_x)^{-1} & 0 & (2\Delta_x)^{-1} \\
  \Delta_x^{-2} & -2\Delta_x^{-2} & \Delta_x^{-2} & 0 \\
  0 & \Delta_x^{-2} & -2\Delta_x^{-2} & \Delta_x^{-2}
\end{pmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  \delta_x \\
  0 \\
  0 \\
  0
\end{pmatrix}
\]  
(D.3)

where \( ' \) and \( '' \) denote the first and second derivatives, respectively. These constraints require the cardinal splines be quintic. Then after some tedious derivation, these splines are obtained respectively as

\[
C_1(\delta_x) = (\delta_x - 1)(2 - \delta_x)^3\left(\frac{1}{2} - \delta_x\right)
\]

\[
C_2(\delta_x) = (2 - \delta_x)[1 + (\delta_x - 1) + \frac{1}{2}(\delta_x - 1)^2 - 2(\delta_x - 1)^3 - \frac{15}{4}(\delta_x - 1)^3(2 - \delta_x)]
\]

\[
C_3(\delta_x) = (\delta_x - 1)[1 + (2 - \delta_x) + \frac{1}{2}(2 - \delta_x)^2 - 2(2 - \delta_x)^3 - \frac{15}{4}(2 - \delta_x)^3(\delta_x - 1)]
\]

\[
C_4(\delta_x) = -(2 - \delta_x)(\delta_x - 1)^3\left[\frac{1}{2} + \frac{5}{4}(2 - \delta_x)\right]
\]  
(D.4)

where \( \delta_x \in [1, 2] \).

For \( x_a \) situated in the boundary interval, i.e., \( x_a \in [x_1, x_2] \) or equivalently \( \delta_x \in [0, 1] \), assumption (2) provides four constraints for each of the four cardinal splines, shown as
Figure D.1: The quintic cardinal splines, $C_1$ (solid line), $C_2$ (dashed line), $C_3$ (dotted line), $C_4$ (dash-dot-dot-dot line), their values (top), first derivatives (middle) and second derivatives (bottom).
a column equality in the following system

\[
\begin{pmatrix}
C_1(0) & C_2(0) & C_3(0) & C_4(0) \\
C_1(1) & C_2(1) & C_3(1) & C_4(1) \\
C_1'(1) & C_2'(1) & C_3'(1) & C_4'(1) \\
C_1''(1) & C_2''(1) & C_3''(1) & C_4''(1)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-(2\Delta_x)^{-1} & 0 & (2\Delta_x)^{-1} & 0 \\
\Delta_x^{-2} & -2\Delta_x^{-2} & \Delta_x^{-2} & 0
\end{pmatrix}
\]  
(D.5)

These constraints only require a cubic polynomial for each cardinal spline. From them each of the splines is obtained as

\[
\begin{align*}
C_1(\delta_x) &= \frac{1}{2} (1 - \delta_x)(2 - \delta_x) \\
C_2(\delta_x) &= \delta_x(2 - \delta_x) \\
C_3(\delta_x) &= -\frac{1}{2} \delta_x(\delta_x - 1) \\
C_4(\delta_x) &= 0.
\end{align*}
\]  
(D.6)

Note that they have actually reduced to quadratics. It comes out that at the boundary node, the first derivative appears as if calculated by the second-order one-sided finite difference, \( f'_i = (-3f_i + 4f_{i+1} - f_{i+2})/2\Delta_x; \) while the second derivative \( f''_i \) appears the same as \( f''_2 \) given in (D.2). (In fact according to assumption (2), the second derivative is held constant within the boundary interval.)

In like manner, the cardinal splines for the other boundary interval \( \delta_x \in [2, 3] \) (corresponding to \( x_a \in [x_3, x_4] \)) can be obtained from assumption (2). They are

\[
\begin{align*}
C_1(\delta_x) &= 0 \\
C_2(\delta_x) &= -\frac{1}{2} (\delta_x - 2)(3 - \delta_x) \\
C_3(\delta_x) &= (3 - \delta_x)(\delta_x - 1) \\
C_4(\delta_x) &= \frac{1}{2} (\delta_x - 2)(\delta_x - 1).
\end{align*}
\]  
(D.7)

Systems (D.4), (D.6), and (D.7) constitute the complete cardinal splines \( \{C_i\}_{i=1}^4 \) which maintain continuity up to second derivative in the full range \([0, 3]\). Then, the value, first and second derivatives of the parameter at \( x_a \) can be computed by substituting in the interpolation formula (D.1) the values, first and second derivatives (w.r.t. \( \delta_x \)) of these systems, respectively. If the parameter is described by a quadratic function, its exact value, first and second derivatives at any point can all be interpolated with this scheme. Note that in evaluating the first and second derivatives, the interpolation formula (D.1) needs to be scaled by \( 1/\Delta_x \) and \( 1/\Delta_x^2 \), respectively, because the dimensionless variable \( \delta_x \) includes the interval constant \( \Delta_x \). Figure D.1 shows the quintic cardinal splines and
their first and second derivatives, where the magnitudes in the central region \([x_2, x_3]\) serve the interpolating point lying in any of the inner intervals, while the magnitudes in the boundary region \([x_1, x_2]\) or \([x_3, x_4]\) serve the interpolating point in the boundary interval.

### D.3 3D interpolation

The above one-dimensional \((x)\) four-point quintic spline interpolation can be easily generalized to two or three dimensions. In three dimensions, for instance, the interpolation formula reads

\[
f(x_a, y_a, z_a) = \sum_{i,j,k} f_{ijk} c_i(\delta_x)c_j(\delta_y)c_k(\delta_z)
\]

where, \(f_{ijk}\) are the given values of a parameter at 64 nodes adjacent to the considered point \((x_a, y_a, z_a)\), \(\delta_y = (y_a - y_1)/\Delta_y\), and \(\delta_z = (z_a - z_1)/\Delta_z\). In real computation, the cardinal splines and their derivatives at an arbitrary point \((\delta_x, \delta_y, \delta_z)\) \(\{C_i\}_{i=1}^4\), \(\{C'_i\}_{i=1}^4\) and \(\{C''_i\}_{i=1}^4\) are not evaluated analytically. Instead, they are linearly interpolated from the tabulated values and derivatives of the splines, which are sampled at 301 equally-spaced points in the range \([0, 3]\) (a sampling rate of 100 sub-intervals for each model grid size).

### D.4 Numerical illustration

Figures D.2 and D.3 present the interpolated results of two functions, a step function and a sine function, obtained by the quintic spline and Smith’s cubic spline. The curves (top), slopes (middle), and curvatures (bottom) interpolated by the quintic and Smith splines are denoted by dashed and dotted lines, respectively. In Figure D.2, the step function is defined by eight equispaced points (closed dots). Clearly the quintic and Smith splines give the similar, continuous results for the curve (top) and slope (middle). The quintic spline ensures continuity of the curvature (bottom) at the nodes 2, 3, 4, and 5, whereas the Smith spline does not. In Figure D.3, the period of the sine function is divided into ten intervals and the true values and derivatives are denoted by closed dots. It is shown that for the curve and slope, the quintic and Smith spline interpolations predict the similar, continuous results. But for the curvature, the former gives a continuous result, whereas the latter a discontinuous result.
Figure D.2: The value (top), slope (middle) and curvature (bottom) of an interpolated discretely sampled step function using quintic (dashed line) and Smith (dotted line) splines. Shown for reference (solid line) are linear interpolations of the function and its finite difference first and second derivatives.
Figure D.3: Similar to Figure D.2 except that the function interpolated is a sine function.
Appendix E

Computational Aspects of the Maslov Seismogram

I have coded a Fortran-77 package MATSEIS for the computation of the GRT seismogram, Maslov integral (MI) seismogram and Maslov uniform (MAT) seismogram for a continuous, 2D acoustic medium. This package consists of three major parts: ray tracing, GRT seismogram synthesis, and MI seismogram synthesis. A computational flow chart is provided in Figure E.1 to outline their relationships. MATSEIS has two novel features: one is allowance for an arbitrary rotation of the Maslov integral line (MIL), and the other is in the scheme for the calculation of the weighting function. It can be applied to many research and application cases. Because it shares many of the implementation skills used to compute the synthetic seismograms for 3D anisotropic media, it can be conveniently extended to model seismograms for this kind of environment. The following sections briefly describe the important computational concepts of each part.

E.1 Ray tracing

The core course of ray tracing (Figure E.2) is to compute the ray path (eikonal) system (e.g. (2.16) in Chapter 2, or (13) in Kendall & Thomson (1989)) that determines how rays propagate, and the geometrical spreading (transport) system (e.g. (18) in Kendall & Thomson (1989)) that describes how wavefield intensity changes. The two systems are first-order partial differential equations. In the computation, we need to evaluate the coefficients of these equations and to perform their integration.
Figure E.1: Flow chart of the package MATSEIS. MATSEIS consists of three major parts: ray tracing, GRT seismogram and MI seismogram. To obtain the normal derivative of the wavefields, the GRT and/or MAT amplitudes need to be multiplied by the normal source slowness \( p_{on} \) (perpendicular to the direction of the Maslov integration). If the Maslov uniform seismogram (i.e. the blended GRT/MI seismogram denoted as the MAT seismogram) is wanted, then both the GRT and MAT amplitudes should be weighted. This package provides the option of only computing one of the three kinds of seismograms.
E.1.1 Evaluation of the ray and geometrical spreading equations

The ray path system concerns the position and direction (exactly, slowness) of a ray \( dx/dT \) and \( dp/dT \). Both are positions of phase space. This system involves medium parameters (e.g. velocities) scaled by density, and implicitly their first derivatives for the change of position with time \( (dx/dT) \), and explicitly their first derivatives for the change of slowness with time \( (dp/dT) \). These derivatives are calculated using the analytical formulation or quintic spline interpolation (Appendix D). And the evaluation of the two derivatives is carried out by subroutines RAYEQN or ISOFNC coded by J-M. Kendall (Kendall & Thomson 1989).

The geometrical spreading system deals with the width of a ray tube in both the spatial and transform domains (i.e. ray tube in phase space) as a function of propagation time: \( d(dx/dq)/dT \) and \( d(dp/dq)/dT \) (equation (18) in Kendall & Thomson (1989)). Note \( d(dx/dq)/dT \) explicitly depends on the first derivatives of the medium parameters but implicitly on the second derivatives. While \( d(dp/dq)/dT \) explicitly depends on both the first and second derivatives. These derivatives are evaluated either analytically or discretely using the quintic spline interpolation. In the package MATSEIS, the coefficients of the system are computed by J-M. Kendall's subroutine ISOFNC only.

E.1.2 Integration of the ray and geometrical spreading equations

The common fourth-order Runge-Kutta (RK) method of accuracy of order \( O(\Delta t^4) \) where \( \Delta t \) denotes the time step size has been used to integrate the ray and geometrical spreading equations. This method is quite accurate, stable, and easy to program (Mathews 1987). Note a higher-order Runge-Kutta method is not useful because its better accuracy is offset by additional computational effort, and the accuracy of the fourth-order method can be maintained high by adopting a small step size. In the computation, four function evaluations per step are required to generate the discrete approximations to the ray paths and geometrical spreadings \( x, p, dx/dq, \) and \( dp/dq \). Kendall's subroutine RKAMP implements this task.
Figure E.2: Flow chart of ray tracing. RK denotes the fourth-order Runge-Kutta method, MIL the Maslov integration line, and $J_x$ and $J_y$ the ray and transform Jacobians.
E.1.3 Ray and rotated Maslov Jacobians

Geometrical changes of ray tube cross sections must be evaluated to calculate the wavefield intensity. In particular if a GRT seismogram is wanted, then only the geometrical spreadings of rays in real space are used, and computed from $dx/dq$. But if a MI seismogram is needed, we must calculate the geometrical spreadings in the mixed transform domain based on both $dx/dq$ and $dp/dq$. Usually the transform Jacobian is calculated for the horizontal or vertical integration line. However this has been generalized in MATSEIS. By using a general expression for the calculation of the transform Jacobian for an arbitrarily-rotated integration line (equation (3.7) in Chapter 3), we can calculate the MI seismogram along a slant integration line without any extra work. I have coded the subroutine SPREADING to perform the calculations of the ray and Maslov transform Jacobians.

E.2 GRT seismogram

The seismogram approximated by GRT is given by the traveltime, ray amplitude, and the time history of the source function. The traveltimes and amplitudes can be calculated at places where rays are traced or interpolated. In MATSEIS they are interpolated from the neighboring rays that may also be used for the computation of the MI seismogram. In fact rays are interpolated as a function of the source slowness (or shooting angle) so that multiple arrives (triplication) can be resolved.

The remaining work in synthesizing the GRT seismogram is mainly to obtain the discrete temporal response for the delta source function. To achieve this, the analytical response, the inverse square root of $(t - T)$ for two dimensions (equation (C.12) in Appendix C), needs to be smoothed first because of its singularity at the geometrical arrival time $T$. Here I follow Chapman (1978) using a symmetrical boxcar window to smooth the singular response. This window is defined as

$$\frac{B(t/\Delta t)}{\Delta t} = \frac{H(t + \Delta t) - H(t - \Delta t)}{2\Delta t} . \tag{E.1}$$

Its property can be understood in the frequency domain, that its Fourier transform (a sinc function) equals one at zero frequency and decays at large frequencies. Convolution of the boxcar window with a delta response removes the very high frequency components of the response and smooths its singularity – making the response wider. This operation
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is performed analytically in order to avoid unnecessary numerical calculations and errors, yielding a continuous, singularity-free function

\[
\frac{B(t/\Delta t)}{\Delta t} \ast \frac{H(t)}{t^{1/2}} = \begin{cases} 
0 & \text{if } t \leq -\Delta t \\
(t + \Delta t)^{1/2}/\Delta t & \text{if } -\Delta t < t \leq \Delta t \\
(t + \Delta t)^{1/2} - (t - \Delta t)^{1/2}/\Delta t & \text{if } \Delta t < t.
\end{cases}
\]  

(E.2)

A discrete response can then be obtained by sampling the above smooth function at step size \( \Delta t \). Further smoothing can be easily performed by convolving the discrete response numerically with a roof function (convolution of two boxcars) one or more times. This will further attenuate the high frequency components but keep the very low frequency components unchanged.

Another task in synthesizing the GRT seismogram is to Hilbert transform the discrete seismogram. In the frequency domain, the Hilbert transformation simply corresponds to multiplication by a factor \(-\text{sgn}(\omega)\). Thus, it may be implemented numerically using the fast Fourier transform acting on the discrete seismogram (which is generally a complex function). In MATSEIS the subroutine, HILBT, performs this task.

E.3 MI seismogram

The MI seismogram in the space-time domain is represented by an analytical fashion (2.64), (2.80), (2.87), or the second term of (C.12) obtained using Chapman's (1987) inverse transform method. It contains singularities at places where signals occur. Similar to the calculation of the GRT seismogram, the calculation of the MI seismogram involves two important steps: discretization and Hilbert transformation.

Chapman (1987) has provided a method of smoothing the analytical response by the use of a boxcar window \( B(t/\Delta t)/\Delta t \). Convolution of the boxcar with the analytical response (or the inverse radon transform in the \( \tau-p_0 \) domain) leads to an integral over those of \( p_0 \) regions that correspond to the temporal band \( (t - \Delta t, t + \Delta t) \) as the contribution to the wavefield at instant \( t \). (Note because MATSEIS computes the MI over the source slowness \( p_0 \), the discussion here is restricted to the \( p_0 \)-domain.) Let \( A(p_0) \) be the MAT amplitudes as a function of the source slowness for 2D or 3D. Then the operation is
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The Maslov integral (E.3) displays the waveform propagated from a delta time-space source in a 2D medium – an inverse square root of time. To obtain the 3D response, this waveform needs to be convolved with the derivative of the inverse square root of time, \( \frac{d}{dt}(\sqrt{t}/t^{1/2}) \), as

\[
\Phi^{\text{ML}}(t, x_r) \ast \frac{B(t/\Delta t)}{\Delta t} \simeq -\text{Im}\left\{ \Delta(t) \ast \frac{1}{2\Delta t} \sum_{t=\dot{\theta}(p_0) \pm \Delta t} A(p_0) dp_0 \right\}.
\]  

(E.5)

I have coded a subroutine \text{IMDELT}A in \text{MATSEIS} to perform \(-\text{Im}\{\Delta(t)\ast\), which calls the subroutine \text{HILBT} to Hilbert transform the complex seismogram, and combines the real part of the post-transformed seismogram with the imaginary part of the pre-transformed seismogram to produce the realistic waveform.

Described algebraically

\[
\frac{B(t/\Delta t)}{\Delta t} \ast \sum_{t=\dot{\theta}(p_0)} A(p_0) \left| \frac{\partial p_0}{\partial \dot{\theta}(p_0)} \right| = \frac{B(t/\Delta t)}{\Delta t} \ast \int A(p_0) \delta(t-\dot{\theta}(p_0)) \, dp_0 = \frac{1}{2\Delta t} \sum_{t=\dot{\theta}(p_0) \pm \Delta t} \int A(p_0) \, dp_0.
\]

(E.3)

where the integral is evaluated over the \( p_0 \) intervals defined by \( t \pm \Delta t = \dot{\theta}(p_0) \) (see Chapman 1978, Figure 14). Chapman (1987) has coded an efficient subroutine \text{THETAC} to evaluate this integral to produce a discrete complex seismogram. The reader is referred to Chapman et al. (1987) for the details of this subroutine. Dey–Sarkar & Chapman (1978) demonstrated how expression (E.3) stabilizes the numerical computations and allows small features in \( \dot{\theta} \) to be ignored. Thomson & Chapman (1986) has described a method for correcting the end-point errors of the Maslov integral and implemented it using their code \text{ENDPOINT}. Both \text{THETAC} and \text{ENDPOINT} are called in package \text{MATSEIS}.
E.4 MAT seismogram

Once both the GRT and MI seismograms are synthesized, the Maslov uniform (MAT) seismogram can be readily obtained by blending them together (Figure E.1). Note that the GRT and MAT amplitudes must be weighted prior to the calculations of the GRT and MI components. A robust trigonometric weighting scheme has been designed to accomplish this task (Chapter 4). The corresponding subroutine is coded as WEIGHTF.
## Appendix F

### Descriptions of Symbols

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<thead>
<tr>
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<th>Description</th>
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<tr>
<td>$\partial$</td>
<td>Partial derivative operation</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular frequency</td>
</tr>
<tr>
<td>$\nu(x)$</td>
<td>Wavefront normal</td>
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<tr>
<td>$q(x)$</td>
<td>Particle motion direction</td>
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<td>$x = (x_1, x_2, x_3)$</td>
<td>Cartesian coordinates</td>
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<tr>
<td>$y = (p_1, x_2, x_3)$</td>
<td>Mixed receiver slowness/space</td>
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<tr>
<td>$\dot{y} = (p_{01}, x_2, x_3)$</td>
<td>Mixed source slowness/space</td>
</tr>
<tr>
<td>$\nu = (T, q_1, q_2)$</td>
<td>Ray</td>
</tr>
<tr>
<td>$c_{ijkl}(x)$</td>
<td>Elasticity medium parameter</td>
</tr>
<tr>
<td>$a_{ijkl}(x)$</td>
<td>Density normalized elasticity</td>
</tr>
<tr>
<td>$\rho(x)$</td>
<td>Density</td>
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<table>
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<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$u(x, t)$</td>
<td>Particle motion wavefield</td>
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<tr>
<td>$\Phi(x, t)$</td>
<td></td>
</tr>
<tr>
<td>$H(x, \bar{p})$</td>
<td>Space domain Hamiltonian</td>
</tr>
<tr>
<td>$H(y, \bar{p})$</td>
<td>Transform domain</td>
</tr>
<tr>
<td>$J(x)$</td>
<td>Ray/GRT Jacobian</td>
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<tr>
<td>$\dot{J}(p)$</td>
<td>Maslov</td>
</tr>
<tr>
<td>$\dot{J}(p_0)$</td>
<td>Normalized Maslov</td>
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<tr>
<td>$\ddot{J}(\bar{p})$</td>
<td>Residual slowness domain Maslov</td>
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<tr>
<td>$v(x)$</td>
<td>Ray/group velocity</td>
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<td>$c(x)$</td>
<td>Acoustic</td>
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<tr>
<td>$V(x)$</td>
<td>Normal</td>
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<td>Reference ray</td>
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<td>Intercept</td>
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<tr>
<td>$\tau(p)$</td>
<td>Intercept</td>
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<td>Reference receiver</td>
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<td>Mixed-space ray series</td>
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<td>$\dddot{U}(\bar{y})$</td>
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<td>$\phi(x, t)$</td>
<td>Maslov</td>
</tr>
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<td>$\dot{\phi}(x, t)$</td>
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<td>$\ddot{\phi}(x, t)$</td>
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<tbody>
<tr>
<td>$\sigma(x)$</td>
<td>KMAH Maslov index</td>
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<td>$\hat{\sigma}(p)$</td>
<td>Residual slowness domain Maslov</td>
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<tr>
<td>$\tilde{\sigma}(p)$</td>
<td>GRT Maslov weight function</td>
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<tr>
<td>$\delta(t)$</td>
<td>Delta Heaviside step time series</td>
</tr>
<tr>
<td>$H(t)$</td>
<td>Heaviside step time series</td>
</tr>
<tr>
<td>$\lambda(t) = H(t)/t^{1/2}$</td>
<td>Inverse square root time series</td>
</tr>
<tr>
<td>$\Lambda(t) = \lambda(t) + i\dot{\lambda}(t)$</td>
<td>Analytic signal inverse square root time series</td>
</tr>
<tr>
<td>$\Delta(t) = \delta(t) + i\dot{\delta}(t)$</td>
<td>Analytic signal delta time series</td>
</tr>
<tr>
<td>$B(t) = (H(t + 1) - H(t - 1))/2$</td>
<td>Normalized box-car window</td>
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<tr>
<td>${C_i(\delta_x)}_{i=1}^{4}$, $\delta_x \in [0,3]$</td>
<td>Quintic cardinal spline</td>
</tr>
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