ESSAYS ON MONEY, BANKING AND PAYMENTS

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Abstract

The history of money has always been intertwined with the history of banking. Nevertheless, very few papers have studied banking in a rigorous monetary environment. This thesis demonstrates that it is crucial to integrate these two literatures. I present three theories of money and banking, each generating results that are drastically different from those of the traditional banking models without microfoundations for money.

Chapter 1 addresses the problem of monitoring the monitor in a model with private information and aggregate uncertainty. There is no need to monitor a bank if it requires loans to be repaid partly with money. A market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. The mechanism also applies when multiple banks exist. With multiple banks, a prohibition on private money issuing not only eliminates welfare-improving money competition but also triggers free-rider problems among banks.

In Chapter 2, I develop a dynamic model to address the following question: when both individuals and banks have private information, what is the optimal way to settle debts? I establish two main results: first, markets can improve upon the optimal dynamic contract in the
presence of private information. Prices fully reveal the aggregate states and help solve the incentive problem of the bank. Secondly, it is optimal for the bank to require loans to be settled with short-term inside money, i.e., bank money that expires immediately after debt settlement. Short-term inside money makes it less costly to induce truthful revelation and achieves more efficient risk sharing.

Chapter 3 studies bank runs in a model with coexistence of fiat money and private money. When fiat money is the only medium of exchange, a bank run equilibrium coexists with an equilibrium that achieves optimal risk sharing. In contrast, when private money is also a medium of exchange, there exists a unique equilibrium where no one demands early withdrawals of fiat money and agents in need of liquidity only use private money to finance consumption. This unique equilibrium achieves the first-best outcome and eliminates bank runs without having to resort to any government intervention.
Dedication

I would like to dedicate this work to my parents and my husband, who supported me in the completion of this task. Thank you for your endless love.
Acknowledgments

I am deeply indebted to my supervisor, Professor Shouyong Shi, for an enormous amount of guidance and inspiration. I acknowledge the valuable comments by the members of my thesis committee, Professor Andres Erosa, Professor Miquel Faig and Professor Angelo Melino. I also thank the external appraiser, Professor Stephen Williamson, for carefully reading this dissertation and making insightful suggestions. Last but not the least, I appreciate the financial support by Professor Shouyong Shi's Bank of Canada Fellowship.
# Contents

## 1 Aggregate Uncertainty, Money and Banking

1.1 Introduction .......................................................................................................................... 1

1.2 The Model .......................................................................................................................... 6

1.3 The Banking Contract ........................................................................................................ 10

1.3.1 Terms of the contract ...................................................................................................... 10

1.3.2 Timing of events .............................................................................................................. 11

1.4 The Banking Equilibrium ................................................................................................... 11

1.5 Multiple Banks .................................................................................................................. 20

1.5.1 Banking with inside money ............................................................................................ 20

1.5.2 Banking with outside money .......................................................................................... 26

1.5.3 Welfare .......................................................................................................................... 29

1.6 Conclusion .......................................................................................................................... 31

## 2 Banking, Inside Money and Outside Money

2.1 Introduction .......................................................................................................................... 33

2.2 The Environment ............................................................................................................... 38

2.3 The Banking Arrangement ................................................................................................ 41

2.4 Banking with Outside Money ............................................................................................. 43
2.4.1 Timing of events..........................................................................................44
2.4.2 The banking equilibrium.............................................................................45
2.4.3 The contract design problem......................................................................48
2.5 Banking with Inside Money...........................................................................53
  2.5.1 One-period inside money.........................................................................53
  2.5.2 Inside money with longer durations.........................................................56
2.6 Co-circulation of Inside and Outside Money..................................................59
  2.6.1 Inflation and incentives.............................................................................63
2.7 Existence and Uniqueness of Equilibrium......................................................65
2.8 Conclusion......................................................................................................66

3 Private Money and Bank Runs ........................................................................69
  3.1 Introduction....................................................................................................69
  3.2 The Environment............................................................................................73
    3.2.1 The planner’s problem............................................................................74
  3.3 Fiat Money and Bank Runs............................................................................75
    3.3.1 The demand deposit contract.................................................................75
    3.3.2 Timing .....................................................................................................77
    3.3.3 Equilibrium .............................................................................................78
  3.4 Private Money and Bank Runs.......................................................................79
    3.4.1 Timing .....................................................................................................80
3.4.2 Equilibrium ...........................................................................................................81
3.5 Private Money and Fiat Money ..................................................................................87
3.6 Conclusion ..................................................................................................................89

References .........................................................................................................................91

A. Appendix to Chapter 1 ...............................................................................................94
B. Appendix to Chapter 2 ...............................................................................................100
C. Appendix to Chapter 3 ...............................................................................................111
Chapter 1
Aggregate Uncertainty, Money and Banking

1.1 Introduction

The main goal of this paper is to address the problem of monitoring the monitor for banks with undiversifiable risks. The problem of monitoring the monitor refers to the incentive problem of a bank. As recognized by the literature on banking, one of the significant roles of a bank is being the delegated monitor. If the bank does all the work of monitoring borrowers, it avoids the duplication of efforts of lenders monitoring borrowers. Nevertheless, lenders may still find it necessary to monitor the bank with undiversifiable risks. Otherwise, the bank can always claim receiving an adverse aggregate state and default on payments to the lenders. Hence, there remains duplication of monitoring costs unless some mechanism is available to efficiently discipline the bank. Finding such a mechanism becomes the key to solving the problem of monitoring the monitor.

To tackle the incentive problem of banks, I study a model of money and banking with private information and aggregate uncertainty. Banks arise endogenously to offer loans of money and function as the delegated monitor. Banking is competitive. There are strategic interactions between banks and borrowers and among banks themselves. Due to aggregate uncertainty, banks are facing undiversifiable portfolio risks. There are incentive problems caused by private information of both individual borrowers and banks.

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I establish the following results: first, there is no need to monitor the bank if it requires loans to be repaid partly with money. Borrowers must trade goods for money to clear debts. As a result, a market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. Thus the mechanism costlessly overcomes the incentive problem of a bank. This mechanism can be readily applied to scenarios of multiple banks. If private issue of money is permitted, at the repayment stage there will arise as many markets as the number of existing banks. In each market, banknotes of a specific bank are traded. The market prices reveal the information of the corresponding banks and discipline them respectively. If private money issue is prohibited, only one market will arise at the repayment stage, where outside money is traded. The market price disciplines banks collectively.

Second, a prohibition on private money issue causes inefficient outcomes in the presence of multiple banks. Without the prohibition, banks use loans to support their own banknotes. The competition of private monies drives up the equilibrium returns of banknotes and improves welfare. With the prohibition, however, bank loans are used to support the same kind of money, outside money. The value of outside money is determined by the aggregate of the returns offered by banks. Any value added to outside money by one bank benefits all. Banks get free rides from one another on making outside money more valuable. In this case, the equilibrium return of outside money is inefficiently low as banks make private decisions. Therefore, prohibiting private money issue reduces welfare. If it has to be done, then it is a good idea to make outside money differentiable so that each
bank can work with a unique kind of outside money. Then the outcome would be the same as with inside money.

The model of this paper is based on Williamson (1986) and Andolfatto and Nosal (2003). Agents are spatially separated. Money serves as a medium of exchange due to lack of double coincidence of wants and limited communication. Banks lend to borrowers and monitor those who default. In both Williamson (1986) and Andolfatto and Nosal (2003), the cost of monitoring a bank is saved by perfect diversification. In contrast, here an agent’s endowment is the result of an idiosyncratic shock and an aggregate shock. Due to the aggregate risk, one must look for ways other than perfect diversification to tackle the problem of monitoring the monitor. As in Williamson (1986), the optimal contract is proven to be a debt contract. A borrower either makes a fixed level of repayment or defaults and gets monitored by the bank. Nonetheless, there is a twist to this debt contract: some of the repayment must be made in money and the rest in real goods. As is explained before, this twist is the key to solving the incentive problem of a bank. In addition to facilitating trades, here money also has a role in helping generate information for banking. As a result, even outside money can circulate in this model of finite horizons, which is rarely seen in monetary models.

My work is complementary to the literature that addresses the problem of monitoring the monitor. Diamond (1984) and Williamson (1986) first recognized this problem and show that it can be overcome by perfect diversification. However, one is naturally concerned about the fact that the real-world financial intermediaries cannot perfectly diversify portfolio risks. Krasa and Villamil (1992a,b) and Winton (1995) study finite-sized banks.
According to these papers, portfolio diversification and bank capitalization can help reduce the cost of monitoring a bank. My work contributes to this literature by showing that the incentive problem of a bank with undiversifiable risk, can be costlessly overcome if the bank requires loans to be repaid partly with money.

Gorton and Haubrich (1987) analyze issues of banking deregulation in a model focusing on the interaction of banks and markets. As they pointed out, if a secondary market of bank loans can be created, it will reveal bank-specific risks and eliminate the information asymmetry that causes banking panics. The opening of such a secondary market requires improvement in the monitoring technology of the banking system. Their idea is similar to the mechanism studied in this paper. Here as borrowers are required to sell some goods for money, they are essentially trading off the corresponding part of the repayment. It is as if money were traded for claims on bank loans. The market induced at the repayment stage resembles a secondary market of bank loans and it reveals bank-specific information. However, in contrast to Gorton and Haubrich (1987), inside money arises endogenously in my model and the creation of the market at the repayment stage comes naturally with banking and does not hinge on any change of monitoring technology.

The economic audience has seen quite a few examples of the phenomenon that additional trading opportunities can be detrimental to incentive compatibility and reduce welfare. For instance, Hammond (1979, 1987) and Jacklin (1987) all seem to suggest that individual incentive compatibility of a mechanism is not sufficient for truthful revelation of types, provided that there exist frictionless markets. In contrast, my paper presents a model in which markets are a favorable instrument for incentives of truthful revelation. The cre-
ation of a market at the loan repayment stage makes it incentive compatible for the bank not to misrepresent its solvency. Furthermore, the revelation of information through prices is unambiguously welfare-improving.

Shi (1996) has already pointed out that insisting on debt repayments in money instead of goods is beneficial to the economy. In particular, by studying coexistence of money and credit in a search model with divisible commodities, Shi shows that when debt is repaid with only money, the competition of money and credit eliminates the inefficient monetary equilibrium where money has a weak purchasing power. My paper identifies a new dimension of the advantage of nominal repayments over real repayments. That is, as long as debt is repaid partly with money, the incentive problem of a bank will be solved at no cost. This, however, cannot be achieved if repayments are made in goods.

This paper is also related to the literature that examines the functioning of inside money and outside money, e.g. Cavalcanti and Wallace (1999), Williamson (2004) and He, Huang and Wright (2005, 2006). According to these models, inside money has the following advantages: the private issue of money is flexible so that agents are not constrained by trading histories and they can respond to unanticipated shocks better than with outside money. Moreover, bank liabilities are a safer instrument than cash. This paper of mine identifies another advantage of inside money over outside money: it naturally promotes the efficiency of banking. The competition of banks’ private monies improves welfare. The imposition of outside money on banking not only removes the benefit of money competition but also introduces free-rider problems among banks, resulting in a lower level of welfare.
The remainder of the paper is organized as follows. Section 1.2 describes the environment of the model. Section 1.3 introduces the bank loan contract. Section 1.4 characterizes the banking equilibrium and the optimal contract. Section 1.5 studies alternative monetary regimes in the existence of multiple banks. Section 1.6 concludes the paper.

1.2 The Model

There are three islands in the economy, namely $A$, $B$ and $C$. Each island is populated by a $[0, 1]$ continuum of agents. The economy lasts for four time periods, $t = 0, 1, 2, 3$. Communication across islands is limited throughout time: at $t = 0$ no communication is available across islands; agents of islands $A$ can visit island $B$ at $t = 1$; agents of islands $B$ can visit island $C$ at $t = 2$; and lastly agents of island $C$ and island $A$ can get in touch at $t = 3$. Traveling agents return to their native island at the end of the period.

Each agent owns a project, which receives endowment of goods. The timing of endowment differs across islands: island $B$ receives endowment at $t = 1$, island $C$ at $t = 2$, and island $A$ at $t = 3$. Endowment is realized at the beginning of a period and prior to the arrival of any traveling agent. Goods are divisible but perishable across time. On both island $B$ and island $C$, the endowment of a project is deterministic at $\bar{y}$, where $0 < \bar{y} \leq 1$. The endowment of a type $A$ project is stochastic: $Y = SW$, where $S$ and $W$ are both random variables. Here, $S$ is an aggregate shock, which is common to all projects on island $A$. It is distributed according to the probability density function $f(s)$ and the cumulative distribution function $F(s)$. The variable $W$ is an idiosyncratic shock and is i.i.d. across type $A$ agents, according to PDF $g(w)$ and CDF $G(w)$. Both $f(\cdot)$ and $g(\cdot)$ are differen-
tiable on support $[0, 1]$. Let $h(y)$ and $H(y)$ denote the PDF and CDF of $Y$, respectively. By Rohatgi’s well-known result,$^2$ 

$$h(y) = \int_y^1 f(s) g\left(\frac{y}{s}\right) \frac{1}{s} ds.$$ 

The realization of $y$, not $s$ or $w$ specifically, is costlessly observable only to the agent himself. However, all agents know about $f(s), g(w)$ and hence $h(y)$.

Agents’ preferences are as follows:

$$U_A = c^1_A + \varepsilon c^3_A - e_A$$

$$U_B = c^2_B + c^1_B$$

$$U_C = c^3_C + \ln c^2_C - e_C$$

where $0 < \varepsilon < 1$, $c^t_i$ denotes a type $i$ agent’s date-$t$ consumption and $e_i$ is the amount of effort expended by the agent. Type $A$ and type $C$ agents are endowed with an unbounded amount of effort. Any type $A$ agent can exert effort to learn other type $A$ agents’ project outcomes, each project costing the agent $\mu > 0$ units of effort. The application of type $C$ agents’ effort will be specified later. Consumption always takes place at the end of a period, allowing for state verification.

Since $0 < \varepsilon < 1$, type $A$ agents prefer to consume at $t = 1$ rather than at their own "harvesting" date, $t = 3$. However, there is lack of intertemporal double coincidence of wants in this economy. To obtain $t = 1$ goods, type $A$ agents must get type $C$ agents to give up some $t = 2$ endowment for $t = 3$ consumption so that type $B$ agents are willing to trade $t = 1$ goods for $t = 2$ goods. Due to limited communication, there is no way for all three types of agents to arrange multilateral exchanges. Therefore, money is potentially

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$^2$ For the distribution of the product of two continuous random variables, see Rohatgi (1976).
useful to facilitate trades as illustrated by Figure 1.

![Figure 1 Potential monetary trades](image)

Further scrutiny, however, makes one realize that personal IOUs cannot circulate in this economy. Here type A agents are the ones who have the incentive to issue personal IOUs. However, because type C agents cannot verify type A project outcomes, type A agents will always default on payment. Knowing that, none of type B or type C agents will value those IOUs. If type A agents are endowed with outside money, they might want to trade outside money for type B goods. Unfortunately, it will not work, either, because of the finite horizons. That being said, money, be it inside or outside, cannot be valued on its own in this environment. However, banking can make money valuable by supporting it with bank loans.

Banking is competitive. At $t = 0$, type A agents each post a contract. By competition, the one who offers the most desirable terms becomes the bank and contracts with all type
A agents.³ To qualify as the bank, one must yield his/her project to public surveillance while maintaining the ownership to one’s endowment.⁴ The banking profit refers to the utility of the banker netting the utility of him being a project owner. In equilibrium the banking profit is driven down to zero, which implies that the banker has the same expected utility as any other type A agent. The bank issues and lends banknotes to type A agents. Banknotes circulate across islands. In the end, type A agents make payments back to the bank, who uses these payments to redeem banknotes and cover the cost of monitoring. The bank monitors the defaulting type A agents.

There are incentive problems due to private information. In addition to the private information of borrowers, portfolio returns are also the private information of the bank. With aggregate uncertainty, the returns depend on the aggregate state and are stochastic in nature. The bank has the incentive to default on redemption unless it is somehow disciplined. The optimal contract must be incentive compatible both for individual borrowers and for the bank.⁵ I assume that any type C agent can monitor the bank at the cost of \( \theta > 0 \) units of effort. Monitoring the bank typically involves duplication of efforts and can be very costly. One of the main tasks of this paper is to find a way to solve the two-sided incentive problem while saving the cost of monitoring the bank.

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³ I assume that if more than one agent post the same optimal contract, only one of them is chosen randomly as the bank. It will become clear later that borrowers always prefer a unique bank as long as it is feasible.

⁴ The story can be that the banker must give up the key to his storage so that his endowment becomes publicly observable. This requirement is meant to resolve the conflict of interests of the banker for being a monitor and a project owner at the same time.

⁵ Collusion, the collective deviation of any parties related by the contract, is not considered in this paper.
1.3 The Banking Contract

The bank loan contract takes the following form:

1.3.1 Terms of the contract

1. Each borrower is entitled to \( M \) units of banknotes at \( t = 0 \).\(^6\) After receiving endowment at \( t = 3 \), the borrower reports \( \tilde{y} \) to the bank. If \( \tilde{y} \in \Omega \subseteq [0, 1] \), he/she will be monitored by the bank; otherwise, monitoring does not occur.\(^7\) Repayment costs \( z(\tilde{y}) \) units of goods. The borrower is required to sell \( t(\tilde{y}) \) units of goods in the market and then repay his/her loan with \( z(\tilde{y}) - t(\tilde{y}) \) units of goods and \( p_A t(\tilde{y}) \) units of banknotes, where \( p_A \) is the market price of goods for banknotes on island \( A \).

2. Each unit of banknote promises an expected value of \( R \). After collecting repayments and monitoring, the bank announces \( \tilde{s} \) as the aggregate state and devotes \( D(\tilde{s}) \) units of goods to redeeming banknotes. Each redeemed banknote is entitled to \( D(\tilde{s})/m^R \) units of goods, where \( m^R \) is the redemption demand. Redemption is served simultaneously.\(^8\) Since the bank monitors the defaulting borrowers with probability one, any hidden goods are bound to be discovered. Part of the hidden goods should have been supplied to the market. The bank must transfer this corresponding amount to those who have

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\(^6\) To avoid the issue of private money over-issue, I assume that the bank can issue notes only at \( t = 0 \).

\(^7\) Following Williamson (1986), here I restrict my attention to non-stochastic monitoring.

\(^8\) This assumption is intended for the simplicity of analysis. An alternative assumption is that redemption is served sequentially. This alternative assumption complicates the analysis considerably in two dimensions. First, now the bank must describe the payments for redemption as a function of an agent’s place in line. Second, the price of goods may vary over time as redemption is served. Although these issues are interesting, they are secondary to the main issue in this paper that prices can reveal the information about the aggregate state of the economy. I thank a referee for pointing out this difference between sequential and simultaneous redemption.
traded their banknotes for goods in the market. Redeemers and buyers retain the option of verifying $\tilde{s}$.

### 1.3.2 Timing of events

All terms of the contract are public information. Without loss of generality, normalize $M = 1$. All market trades are competitive. According to the contract, the timing of events is summarized as follows:

1. $t = 0$, (i) a bank arises on island $A$; (ii) the bank contracts with $As$;
2. $t = 1$, competitive trades between $As$ and $Bs$;
3. $t = 2$, competitive trades between $Bs$ and $Cs$;
4. $t = 3$, (i) $Cs$ each decide whether to trade banknotes in the market or redeem them at the bank; (ii) competitive trades between $Cs$ and $As$; (iii) market closes and the bank collects repayments and monitors according to $\Omega$; (iv) the bank announces $\tilde{s}$ and redeems banknotes; (v) $Cs$ monitor the bank if necessary.

### 1.4 The Banking Equilibrium

By competitive banking, the bank designs the loan contract, $C = (z, t, \Omega, D)$, to maximize the expected utility of a borrower. Since only incentive compatible contracts are considered, in the rest of the paper the reported values and the true values are not distinguished unless otherwise stated.
First consider the type $C$ agents. Let $p_i$ denote the market price of type $i = A, B, C$ goods for banknotes. Recall that $R$ is the expected net returns of a banknote. It follows that $R = \int_0^1 [D (s) + T (s)] f (s) ds - \Theta$, where $T (s) = \int_0^1 t (sw) g (w) dw$ is the aggregate supply of goods in the market and $\Theta$ is the expected cost of monitoring the bank. Banknotes are valued as long as $R > 0$. Type $C$ agents can either redeem banknotes at the bank or use them to purchase goods in the market. The expected net returns of a banknote are equal to the expected benefit minus the expected cost of monitoring the bank. Let $m_C$ be a type $C$ agent’s holdings of banknotes. Taking $p_C$ and $R$ as given, a representative type $C$ agent chooses $(c^2_C, m_C)$ to maximize his/her expected utility, $\ln c^2_C + m_C R$, subject to the budget constraint, $p_C c^2_C + m_C = p_C \bar{y}$. Let $(c^2_C, m^*_C)$ denote the optimal choices of a type $C$ agent.

Similarly, let $m_B$ be a type $B$ agent’s holdings of banknotes. Taking $p_B$ and $p_C$ as given, a representative type $B$ agent chooses $(c^1_B, m_B)$ to maximize his/her expected utility, $c^1_B + m_B p_C$, subject to the budget constraint, $p_B c^1_B + m_B = p_B \bar{y}$. Again, let $(c^1_B, m^*_B)$ denote the optimal choices of a type $B$ agent.

Now given $c^1_B$, the bank’s problem is

\[
V_1 = \max_{c} \left\{ \bar{y} - c^1_B + \varepsilon \int_0^1 [y - z (y)] h (y) dy \right\}
\]

s.t. \[
\int_0^1 z (y) h (y) dy - \int_0^1 T (s) f (s) ds - \int_0^1 D (s) f (s) ds - \mu \int_{\Omega} h (y) dy = 0
\]

The constraint holds with equality because competition drives the banking profit down.
to zero. The expected banking profit is equal to the expected aggregate loan repayment in kind, \( \int_0^1 z(y) h(y) \, dy - \int_0^1 T(s) f(s) \, ds \), minus the expected cost of redemption \( \int_0^1 D(s) f(s) \, ds \) and the expected cost of monitoring borrowers \( \mu \int_0^1 h(y) \, dy \). Note that the timing is such that the bank starts collecting repayment after the market closes. Therefore, the bank cannot use repayments of banknotes to buy goods in the market. It can only benefit from the repayments in kind.

It is natural to focus on symmetric equilibria because agents are ex ante identical:

**Definition 1** A banking equilibrium consists of market prices \( \{p_A, p_B, p_C\} \), a contract \( C \) and the associated expected net returns of a banknote \( R \), the redemption demand \( m^R \), and individual agents’ choices \( \{c^1_B, m_B, c^2_C, m_C\} \) such that: (i) given prices and \( R \), the choices of a representative agent of type \( i = B, C \) are optimal; (ii) given \( c^1_B \), the bank designs \( C \) to maximize the expected utility of a type \( A \) agent; (iii) given \( m^R \), type \( C \) agents are indifferent between redeeming banknotes and selling them in the market; (iv) all markets clear.

According to condition (iii) in the above, the equilibrium redemption demand \( m^R \) must satisfy:

\[
\frac{\int_0^1 D(s) f(s) \, ds}{m^R} = \frac{\int_0^1 T(s) f(s) \, ds}{1 - m^R},
\]

which implies that

\[
m^R = \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 [D(s) + T(s)] f(s) \, ds}.
\]
Then the equilibrium market price on island $A$ is given by:

$$p_A = \frac{1 - m^R}{T(s)} = \frac{\int_0^1 T(s) f(s) \, ds}{T(s) \int_0^1 [D(s) + T(s)] f(s) \, ds}.$$ 

Thus,

$$T(s) = \frac{\int_0^1 T(s) f(s) \, ds}{p_A \int_0^1 [D(s) + T(s)] f(s) \, ds}.$$ 

Obviously, as long as the function $T(s)$ is fully revealing, agents are able to pin down the exact aggregate state $s$ simply by observing $p_A$. Hence they know the exact amount of goods the bank should spend on redemption, provided that $D(s)$ is also a fully revealing function. The bank will be monitored if it posts any $\bar{s} \neq s$. Note that this aggregate information is not available to individual agents unless a market exists on island $A$. Therefore, the cost of monitoring the bank is completely saved as long as borrowers are required to sell in the market. The market price $p_A$ fully reveals the aggregate state and keeps the bank perfectly disciplined, which proved the following proposition:

**Proposition 1**  The bank is perfectly disciplined and thus $\Theta = 0$, provided that

(i) $T(s), D(s) \geq 0$ for all $s$;

(ii) $T(s) \neq T(s')$ and $D(s) \neq D(s')$ for any $s \neq s' \in [0, 1]$.

Recall that the expected gross returns of banknotes are $\int_0^1 [D(s) + T(s)] f(s) \, ds = R + \Theta$. Given $R$, the expected profit of the bank, the right-hand side of the constraint in
(1.1), is the highest with $\Theta = 0$ because there is no need to compensate banknote-holders for the expected cost of monitoring the bank. Therefore, it is in the interest of the bank to implement the mechanism in Proposition 1 to discipline itself.

In equilibrium, $p_B = \frac{1}{y - c_B^x}$ and $p_C = \frac{1}{y - c_C^x}$. It is straightforward to derive that $c_B^{x^*} = c_C^{x^*} = \frac{y}{1 + R}$. Given $c_B^{x^*}$ and $\Theta = 0$, (1.1) becomes

$$V_1 = \max_{y \in [0,1] \setminus \{0\}} \left\{ \frac{yR}{1 + R} + \varepsilon \int_0^1 \left[ y - z(y) \right] h(y) \, dy \right\}$$

$$s.t. \int_0^1 z(y) h(y) \, dy - \int_1^y h(y) \, dy - R = 0.$$  \hfill (1.2)

**Proposition 2** The optimal repayment schedule is given by

$$z^*(y) = \begin{cases} x, & \text{if } y \geq x \\ y, & \text{if } y < x \end{cases},$$

where $x \in [0,1]$ is a constant. Correspondingly, the optimal monitoring state is $\Omega^* = \{y : 0 \leq y < x\}$.

Proof of Proposition 2 is provided in the Appendix.\(^9\)

By Propositions 1-2, the bank’s problem becomes

$$V_1 = \max_{x \in [0,1]} \left\{ \frac{yR(x)}{1 + R(x)} + \varepsilon \int_x^1 (y - x) h(y) \, dy \right\}$$

\hfill (1.3)

---

\(^9\) This proof follows the proof of Proposition 1 in Williamson (1986), pp166.
where

\[ R(x) = \int_0^x yh(y) \, dy + x \left[1 - H(x)\right] - \mu H(x). \tag{1.4} \]

The solution to the above pins down \( x \) and hence \( R(x) \). After that, it remains to determine \( t(y) \) and \( D(s) \) so as to completely characterize the banking contract. Notice that the contract must be incentive compatible for type A agents to sell exactly \( t(y) \) units of goods in the market. That is, the bank must make sure that a borrower does not have the incentive to sell more than \( t(y) \) units and then demand redemption for the extra banknotes. Given \( D(s) \) and any equilibrium \( \{p_A, m^R\} \), the expected payoff for a borrower to sell one unit goods and then demand redemption on the banknotes is \( \frac{p_A D(s)}{m^R} \). In the equilibrium,

\[ p_A = \frac{1 - m^R}{T(s)} \]

and

\[ m^R = \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 [D(s) + T(s)] f(s) \, ds}. \]

A borrower will sell exactly \( t(y) \) units of goods as required provided that \( \frac{p_A D(s)}{m^R} \leq 1 \) for any \( s \). That is,

\[ \frac{D(s)}{T(s)} \leq \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 T(s) f(s) \, ds}, \quad \forall \ s. \tag{1.5} \]
1.4 The Banking Equilibrium

**Proposition 3** Given \( x, t (y) \in [0, z (y)] \) must satisfy (i) \( T (s) \neq T (s') \) for any \( s \neq s' \in [0, 1] \); (ii) \( \int_0^1 T (s) f (s) ds = \frac{R(x)}{1 + \phi} \), where \( \phi \geq 0 \) is a constant.

**Proposition 4** \( D (s) = \phi T (s) \), for all \( s \).

By Proposition 3-4, the choices of \( t (y) \) and \( D (s) \) are indeterminate and can take on a continuum of values. However, the amount of goods devoted to redemption must be proportional to the aggregate supply of goods to market \( A \). Let \( x_1^* \) be the optimal solution and \( V_1^* \) be the optimum of (1.3). The optimal contract is given by \( C^* (x^*) = \{ z^* (y; x^*), t^* (y; x^*), \Omega^* (x^*), D^* (s; x^*) \} \). Define

\[
\Psi_1 (x) = \frac{\gamma R' (x)}{[1 + R (x)]^2} - \varepsilon [1 - H (x)]
\]

and

\[
\Psi_2 (x) = \frac{R (x) R' (x)}{1 + R (x)} + \Psi_1 (x).
\]

**Assumption 1** \( \varepsilon < \gamma \left[ 1 - \mu g (0) \int_0^1 f (s) \frac{1}{s} ds \right] \).

**Assumption 2** \( \Psi''_1 (x) > 0 \) and \( \Psi''_2 (x) > 0 \) for all \( x \in [0, 1] \).

Assumptions 1-2 ensure the following proposition:

**Proposition 5** A unique banking equilibrium exists and the optimal contract \( C^* \) is characterized by \( x_1^* \in (0, 1) \).
The optimal contract is essentially a debt contract as in Williamson (1986). A borrower either makes a fixed level of repayment or defaults and gets monitored by the bank. Here what is not standard about the debt contract is that a specified amount of the repayment must be made in banknotes and the rest in real goods. As a result, borrowers must trade for banknotes in order to repay their loans. The induced market at the repayment stage generates information revealing prices that perfectly discipline the bank. The cost of monitoring the bank is completely saved even in an environment with aggregate uncertainty.

**Inside money and outside money.** Trivially, there will be no incentive problem of a bank if it does not promise redemption. That is, \( D(s) = 0 \) for any \( s \in [0, 1] \). Note that redeemability is not a necessary condition for banknotes to be valued. Indeed, the bank has two ways of making its notes valuable: the redemption channel and the repayment channel. With \( D(s) = 0 \) for any \( s \in [0, 1] \), banknotes can only reflux through the repayment channel. Nonetheless, banknotes are still valued because borrowers are obliged to trade goods for banknotes at the repayment stage.

This further indicates that the optimal contract can also be implemented with outside money. Suppose the bank is prohibited from issuing banknotes (i.e. inside money) while each type \( A \) agent is endowed with \( M \) units of outside money. The bank can write a contract with type \( A \) agents, which is essentially \( C^* \) with \( D(s) = 0 \) for any \( s \in [0, 1] \) and with outside money replacing every role of banknotes in the contract. The outcome will be exactly the same. That is, the cost of monitoring the bank can also be completely saved with the help of outside money.
Monitoring technology. In this model, it is assumed that type $A$ projects can only be verified by type $A$ agents although type $C$ agents are able to monitor the bank on island $A$. This assumption allows me to focus on the main issue being addressed: solving the incentive problem of a bank. Note, however, the conclusions of this paper do not hinge on this assumption.

Without this assumption, type $C$ agents can monitor as efficiently as type $A$ agents. Type $A$ agents’ personal IOUs can be accepted by type $C$ agents. By assuming a larger population on islands $B$ and $C$ than on island $A$, one can show that this arrangement involves duplication of monitoring efforts by type $C$ agents. This result corresponds to a well established point in the literature (Diamond [1984] and Williamson [1986]) that direct lending is typically associated with duplication of monitoring costs.

A seemingly promising solution is to have a type $C$ bank, which collects type $A$ agents’ IOUs from type $C$ agents and monitors type $A$ agents on behalf of all type $C$ agents. Duplication of monitoring is eliminated in this dimension. Nevertheless, now the $C$ bank single-handedly handles all the type $A$ goods. With aggregate uncertainty, it has every incentive to misreport to type $C$ agents. Therefore, type $C$ agents must monitor their bank. Duplication of monitoring efforts still exists and the problem of monitoring the monitor prevails. As has been established, banking by type $A$ agents gains access to an efficient device to cope with the problem of monitoring the monitor, which is to let market prices discipline the bank. This cannot be done by type $C$ banking. Thus type $A$ banking strictly dominates type $C$ banking because it completely saves the cost of monitoring the bank and hence improves efficiency.
1.5 Multiple Banks

Previously it has been established that the cost of monitoring the bank can be completely saved if money is required as part of the loan repayment. In this section, I study scenarios of multiple banks. In particular, I address whether the previous result is robust to the existence of multiple banks and whether the outcome will differ if inside money and outside money are implemented, respectively, in the banking contract.

1.5.1 Banking with inside money

Now suppose there are \( n \) communities on island \( A \), where \( n \geq 2 \) is an integer. Each community has a population of measure \( \frac{1}{n} \). The endowment of an agent in community \( j = 1, \ldots, n \) is given by \( Y_j = S_jW \). The idiosyncratic shock \( W \) is defined as before, while the local shock \( S_j \) for each community \( j \) is i.i.d. from \( f (\cdot) \). The endowment of a project in community \( j \) can only be verified by someone from community \( j \). That is, type \( A \) agents cannot monitor projects across communities. The rest of the model is characterized the same as previously.

Naturally, \( n \) banks arise on island \( A \), one bank for each community. Banks write contracts with local community members. Contracts take the same form as specified in the previous section. Banks require loans to be repaid with goods and banknotes of their own. Normalize the amount of banknotes issued by a bank to \( M_j = \frac{1}{n} \) for all \( j \). Banknotes of various banks are identifiable and they cocirculate across islands. There will be \( n \) markets on each island. In each market, goods are traded for notes of a particular bank. As before,
all terms of the contract are public information. Agents take prices as given and trade competitively in all markets.

Now consider type $C$ agents’ responses. The expected net returns of one unit of $j$-banknote are

\[ R_j = \int_0^1 \left[ D_j(s_j) + T_j(s_j) \right] f(s_j) \, ds_j - \Theta_j. \]

Taking $\{p^j_C, R_j\}_{j=1}^n$ as given, a representative type $C$ agent maximizes his/her expected utility subject to the budget constraint:

\[
\max_{(c^2_C, m^j_C)} \ln c^2_C + \sum_{j=1}^n m^j_C R_j \\
\text{s.t.} \quad c^2_C + \sum_{j=1}^n \frac{m^j_C}{p^j_C} = \bar{y}
\]

where $m^j_C$ is the agent’s holdings of $j$-banknotes.

Taking $\{p^j_B, p^j_C\}_{j=1}^n$ as given, a representative type $B$ agent maximizes his/her expected utility subject to the budget constraint:

\[
\max_{(c^1_B, m^j_B)} c^1_B + \sum_{j=1}^n \frac{m^j_B}{p^j_C} \\
\text{s.t.} \quad c^1_B + \sum_{j=1}^n \frac{m^j_B}{p^j_B} = \bar{y}
\]
where \( m^j_B \) is the agent’s holdings of \( j \)-banknotes. Once again I will focus on symmetric equilibria.

**Definition 2** A symmetric equilibrium with \( n \) banks and inside money consists of prices \( \{p^j_A, p^j_B, p^j_C\}_{j=1}^n \), contracts \( \{C_j\}_{j=1}^n \) and the associated expected net returns on banknotes \( \{R_j\}_{j=1}^n \), the equilibrium redemption demand \( \{m^R_j\}_{j=1}^n \) and individual agents’ choices \( \{c^1_B, \{m^j_B\}_{j=1}^n, c^2_C, \{m^j_C\}_{j=1}^n\} \) such that: (i) given prices and returns, the choices of a representative agent of type \( i = B, C \) are optimal; (ii) given \( \{R_k \neq j\}_{k=1}^n \) and the best responses \( \{c^1_B, \{m^j_B\}_{j=1}^n\} \), bank \( j \) designs \( C_j \) to maximize the expected utility of a local borrower; (iii) given \( m^R_j \), type \( C \) agents are indifferent between redeeming \( j \)-banknotes and selling them in market \( j \); (iv) all markets clear; (v) symmetry of banks: \( C_j = C \) and \( R_j = R \) for all \( j \).

Similar to the case with one bank, in the equilibrium

\[
m^R_j = \frac{\int_0^1 D_j(s_j) f(s_j) ds_j}{\int_0^1 [D_j(s_j) + T_j(s_j)] f(s_j) ds_j}
\]

and

\[
p^j_A = \frac{1 - m^R_j}{T_j(s_j)}.
\]

Hence,

\[
T_j(s_j) = \frac{\int_0^1 T_j(s_j) f(s_j) ds_j}{p^j_A \int_0^1 [D_j(s_j) + T_j(s_j)] f(s_j) ds_j}.
\]
Analogous to Proposition 1, one can prove the following proposition:\(^{10}\)

**Proposition 6** With \( n \) banks and inside money, bank \( j \) is perfectly disciplined by the corresponding market price \( p_j^A \) and thus \( \Theta_j = 0 \) for all \( j \), provided that (i) \( T_j(s_j), D_j(s_j) \geq 0 \) for all \( s_j = 0 \); (ii) \( T_j(s_j) \neq T_j(s'_j) \) and \( D_j(s_j) \neq D_j(s'_j) \) for any \( s_j \neq s'_j \in [0, 1] \).

By Proposition 6, the expected net returns of one unit of \( j \)-banknote are given by

\[
R_j = \int_0^1 [D_j(s_j) + T_j(s_j)] f(s_j) ds_j \text{ for all } j.
\]

Market clearing requires \( m^*_B = m^*_C = \frac{1}{n} \).

Let \( q_{ji} \) be the amount of goods sold by a type \( i = B, C \) agent in market \( j \). It is straightforward to derive that for all \( j, q_{jB}^{i*} = q_{jC}^{i*} = \frac{\overline{R}_j}{n + \sum_{k=1}^{n} R_k} \). Notice that \( \frac{q_{jB}^{i*}}{q_{jB}^{i*}} = \frac{q_{jC}^{i*}}{q_{jC}^{i*}} = \frac{R_j}{R_k} \) for any \( j, k \). Intuitively, banknotes with higher expected returns have higher purchasing power.

Define \( \overline{R}_{-j} = \sum_{j \neq k=1}^{n} R_k \). Given \( q_{jB}^{i*} \) and \( \overline{R}_{-j} \), bank \( j \) seeks to maximize the expected utility of a local borrower:

\[
V_n = \max_{\mathcal{C}_j} \left\{ \frac{n\overline{R}_j}{n + R_j + \overline{R}_{-j}} + \varepsilon \int_0^1 [y_j - z_j(y_j)] h(y_j) dy_j \right\} \tag{1.6}
\]

s.t. \( \frac{1}{n} \int_0^1 z_j(y_j) h(y_j) dy_j - \frac{\mu}{n} \int_{\Omega_j} h(y_j) dy_j - \frac{R_j}{n} = 0 \).

where the first term in the objective is simply \( q_{jB}^{i*} / \left( \frac{1}{n} \right) \), which is the amount of type \( B \) goods a borrower of bank \( j \) can buy. The constraint is the same as in the one-bank case,

\(^{10}\) Implicitly, here I only consider inside money that can be redeemed. As mentioned before, when \( D_j(s_j) = 0 \) for all \( s_j \), there is no incentive problem of a bank.
except that here each bank has borrowers of measure $\frac{1}{n}$ and issues banknotes of measure $\frac{1}{n}$.

Similarly to the one bank case, we have the following proposition:

**Proposition 7**  With $n$ banks, the optimal repayment schedule of any bank $j$ is given by

$$z^*_j(y) = \begin{cases} x_j, & \text{if } y_j \geq x_j \\ y_j, & \text{if } y_j < x_j \end{cases},$$

where $x_j \in [0, 1]$ is a constant. Correspondingly, the optimal monitoring state is $\Omega^*_j = \{y_j : 0 \leq y_j < x_j\}$.

Now (1.6) is simplified to:

$$V_n = \max_{x_j \in [0,1]} \left\{ \frac{n\bar{y} R_j(x_j)}{n + R_j(x_j) + \bar{R}_{-j}} + \varepsilon \int_{x_j}^{1} (y_j - x_j) h(y_j) \, dy_j \right\}$$

where

$$R_j(x_j) = \int_{0}^{x_j} y_j h(y_j) \, dy_j + x_j [1 - H(x_j)] - \mu H(x_j).$$

Analogous to the one-bank case, the main task of characterizing the rest of the contract is to pin down the optimal $x_j$. Given $x_j$, $t_j$ and $D_j$ must satisfy the conditions in Propositions 3-4.

Assuming interior solutions, the first-order condition of (1.7) is:

$$\frac{n\bar{y} (n + \bar{R}_{-j}) R'_j(x_j)}{[n + R_j(x_j) + \bar{R}_{-j}]^2} - \varepsilon [1 - H(x_j)] = 0$$

(1.8)
In a symmetric equilibrium, \( x_k = x \) and \( R_k(x_k) = R(x) \) for all \( k \), which implies \( R_j = (n - 1) R(x) \). Substituting these into (1.8), it becomes

\[
\frac{1 + \frac{n-1}{n} R(x)}{[1 + R(x)]^2} y R'(x) - \varepsilon [1 - H(x)] = 0. \tag{1.9}
\]

Let \( x_n^* \) be the optimal choice and \( V_n^* \) be the optimum of (1.7).

**Proposition 8** If \( \mu < E(y) + \frac{n}{n-1} \), a unique symmetric equilibrium with \( n \) banks and inside money exists and the optimal contract \( C_n^* \) is characterized by \( x_n^* \in (0, 1) \).

**Proposition 9** \( V_n^* < V_1^* \); \( x_n^* > x_1^* \); \( R(x_n^*) > R(x_1^*) \).

Propositions 8-9 demonstrate that banks compete for higher purchasing power of their own banknotes, which results in higher equilibrium returns of banknotes than with a single bank \( (R[x_n^*] > R[x_1^*]) \). This is because type B and type C agents value banknotes according to their returns. A bank tends to offer higher returns on its banknotes so that its borrowers can purchase more goods. However, the higher the banknote returns, the higher the repayment level required for borrowers. As a result, banks strive for an optimal market share of banknotes at the expense of the welfare of their borrowers \( (V_n^* < V_1^*) \). This also shows that borrowers prefer to have a unique bank as long as it is feasible.
1.5.2 Banking with outside money

Suppose banks are prohibited from issuing banknotes. Each type $A$ agent is endowed with $M$ units of outside money at the beginning of time.$^{11}$ Outside money is identical. Once again, normalize $M = 1$. Outside money is not redeemable, i.e. $D_j(s_j) = 0$ for all $s_j$.\textsuperscript{12}

The rest of the terms of a typical contract $C_j$ are the same as stated before. Banks announce local shocks $\{\tilde{s}_j\}_{j=1}^n$ simultaneously.

Accordingly, outside money is valued because agents know that type $A$ agents will need it to fulfill their contractual obligations. Since outside money is identical, there will be only one market on every island, where goods are traded for outside money.

**Definition 3**  A symmetric equilibrium with $n$ banks and outside money consists of market prices $\{p_A, p_B, p_C\}$, contracts $\{C_j\}_{j=1}^n$ and the associated expected net returns on outside money $\{R_j\}_{j=1}^n$, and individual agents’ choices $\{c^B_1, m_B, c^C_2, m_C\}$ such that: (i) given prices and returns, the choices of a representative agent of type $i = B, C$ are optimal; (ii) given $\{R_{k\neq j}\}_{k=1}^n$ and the best responses $\{c^B_1, m_B\}$, bank $j$ designs $C_j$ to maximize the expected utility of a local contracted agent; (iii) all markets clear; (iv) symmetry of banks: $C_j = C$ and $R_j = R$ for all $j = 1, \cdots, n$.

The aggregate supply of goods in market $A$ is given by

---

\textsuperscript{11} Here money is not actually lent to type $A$ agents by banks. But they still write contracts stipulating that type $A$ agents should make payments to banks at $t = 3$, as if they had borrowed from banks. As is explained previously, in this model money needs the support of banking in order to be valued.

\textsuperscript{12} This comes naturally with the model because type $A$ and type $C$ agents do not meet until $t = 3$. If they could meet upfront, banks might contract with type $C$ agents as well, promising to redeem outside money for goods. In that case, $D_j(s_j)$ could be chosen in a variety of ways as with inside money. However, the result would not be any different as has been shown that the equilibrium outcome only depends on the repayment level $x$. 

\[
\sum_j T_j(s_j) = \sum_j \frac{1}{n} \int_0^1 t_j(s_j w) g(w) \, dw.
\]

As long as \( T_j \) is fully revealing of \( s_j \) for all \( j \), \( p_A \) can be used to discipline banks. To see this, suppose \( \tilde{s}_k = s_k \) for all \( k \neq j \), then

\[
T_j(\tilde{s}_j) + \sum_{k \neq j} T_k(\tilde{s}_k) = T_j(s_j) + \sum_{k \neq j} T_k(s_k) = \frac{1}{p_A}, \quad \text{if and only if } \tilde{s}_j \neq s_j.
\]

Therefore, the price \( p_A \) nests the information of all banks. Given that all the other banks choose to announce their local shocks truthfully, it is also optimal for a bank to announce the true local shock it has received. Hence the following proposition:

**Proposition 10** With \( n \) banks and outside money, the market price \( p_A \) is information revealing if (i) \( T_j(s_j) \geq 0 \) for all \( s_j \); (ii) \( T_j(s_j) \neq T_j(s'_j) \) for any \( s_j \neq s'_j \in [0, 1] \). Banks are disciplined collectively by \( p_A \) and the expected cost of monitoring a bank \( \Theta_j = 0 \) for all \( j \).

Similarly to the case of a single bank, in equilibrium \( p_B = \frac{1}{\bar{y} - c_B^1} \) and \( p_C = \frac{1}{\bar{y} - c_C^2} \). It is straightforward to derive

\[
c_B^{1*} = c_C^{2*} = \frac{\bar{y}}{1 + \frac{1}{n} \sum_{k=1}^n R_k}.
\]
Note that the value of outside money is determined by the aggregate returns \( \frac{1}{n} \sum_{k=1}^{n} R_k \). Again define \( \bar{R}_{-j} = \sum_{j \neq k=1}^{n} R_k \). Given \( c_j^* \) and \( \bar{R}_{-j} \), the problem of bank \( j \) is:

\[
V_n' = \max_{c_j} \left\{ \bar{y} - \frac{\bar{y}}{1 + \frac{1}{n} (R_j + \bar{R}_{-j})} + \varepsilon \int_0^1 [y_j - z_j (y_j)] h (y_j) dy_j \right\}
\]

s.t. \( \frac{1}{n} \int_0^1 z_j (y_j) h (y_j) dy_j - \frac{\mu}{n} \int_{\Omega_j} h (y_j) dy_j - \frac{R_j}{n} = 0. \)

It is straightforward to prove that Proposition 7 also applies here. (The proof is omitted for brevity.) Accordingly, the above problem can be simplified to:

\[
V_n' = \max_{x_j \in [0, 1]} \left\{ \bar{y} \left[ R_j (x_j) + \bar{R}_{-j} \right] \frac{1}{n + R_j (x_j) + \bar{R}_{-j}} + \varepsilon \int_0^1 [y_j - z_j (y_j)] h (y_j) dy_j \right\}, \tag{1.10}
\]

where \( R_j (x_j) = \int_0^{x_j} y_j h (y_j) dy_j + x_j [1 - H (x_j)] - \mu H (x_j) \). Assuming interior solutions, the first-order condition is:

\[
\frac{n \bar{y} R_j' (x_j)}{[n + R_j (x_j) + \bar{R}_{-j}]^2} - \varepsilon [1 - H (x_j)] = 0.
\]

In a symmetric equilibrium, the above becomes

\[
\frac{\bar{y} R' (x)}{n \left[ 1 + R (x) \right]^2} - \varepsilon [1 - H (x)] = 0. \tag{1.11}
\]
Let $x_n^{*}$ be the optimal choice and $V_n^{*}$ be the optimum.

**Proposition 11** If $\varepsilon < \frac{\mu}{n} \left[ 1 - \mu g(0) \int_0^1 f(s) \frac{1}{s} ds \right]$, a unique symmetric equilibrium with $n$ banks and outside money exists and the optimal contract $C_n^{*}$ is characterized by $x_n^{*} \in (0, 1)$.

**Proposition 12** $V_n^{*} < V_1^{*}$; $x_n^{*} < x_1^{*} < x_n^{*}$; $R(x_n^{*}) < R(x_1^{*}) < R(x_n^{*}).$

When private provision of money is permitted, a bank issues its own banknotes to the benefit of its own borrowers. In contrast, here the use of outside money is not exclusive of contracted agents of any bank. The returns of outside money are given by the aggregate of the returns offered by all banks, $\frac{1}{n} \sum_{k=1}^{n} R_k$. The support for outside money by any one bank $R_k$ benefits all type $A$ agents. The equilibrium returns on outside money are inefficiently low as banks make private decisions on the amount of goods to offer for outside money. Intuitively, banks get free rides from one another on supporting outside money, causing a less than optimal level. Moreover, outside money does not change the result that borrowers are worse off with the presence of multiple banks ($V_n^{*} < V_1^{*}$).

### 1.5.3 Welfare

Assign equal weights to all agents in the economy. Given the equilibrium level of repayment $x \in (0, 1)$, the welfare level is given by $W(x) = U_A(x) + U_B(x) + U_C(x)$.

**Proposition 13** $W(x_n^{*}) < W(x_1^{*}) < W(x_n^{*})$. 
A numerical example. Set $f(s) = 2s$, $g(w) = 1$, $\mu = 0.2$, $\varepsilon = 0.2$, $\overline{y} = 0.8$.

Figure 2 illustrates the equilibrium outcomes of this example. Welfare $W(x)$ is maximized at $x = 0.45$ and the expected utility of a type A agent $V(x)$ is maximized at $x = 0.41$.

With multiple banks, a prohibition on private money issue will shift the economy from the region of $x \in (0.41, 0.45)$ to the region of $x \in (0, 0.41)$, resulting in a strictly lower level of welfare.

![Welfare analysis graph]

In this model, type $B$ and type $C$ agents are essentially lenders. Although they do not directly lend to banks, they are willing to trade goods for money, which is (direct or indirect) liabilities of banks. It is as if type $B$ agents provided credit to banks and then sold the credit to type $C$ agents. Eventually, type $C$ agents benefit from money like a deserving lender.
Proposition 13 establishes that with multiple banks, it is inefficient to prohibit the issue of private money. While the inside money regime fosters money competition, the outside money regime eliminates competition and, to make things worse, triggers incentive problems of free-riding among banks. The equilibrium returns of money $R$ decline with the imposition of outside money. Lenders (type $B$ and type $C$ agents together) are worse off because their welfare is strictly increasing with $R$. Although the welfare of borrowers (type $A$ agents) may not be decreased, the overall effect on social welfare is negative ($W [x^*_n] < W [x^*_n]$). To summarize, it is inefficient to prohibit banks from issuing banknotes. If it has to be done, then it is a good idea to make outside money differentiable. That is, the policy of prohibition should be accompanied with the imposition of bank-specific outside money. As a result, each bank can work with a unique kind of outside money. Then the outcome would be the same as with inside money.

1.6 Conclusion

This paper studied money and banking in a model with private information and aggregate uncertainty. The cost of monitoring a bank can be completely saved if the bank requires money as a means of loan repayment. As a result, a market arises at the repayment stage and generates an information-revealing price that perfectly disciplines the bank. Thus the incentive problem of a bank is overcome costlessly. This mechanism can be readily applied to scenarios of multiple banks. If private issue of money is permitted, there will arise as many markets at the repayment stage as the number of banks. In each market, a unique kind of inside money is traded for goods. The market price fully reveals the bank-specific infor-
mation. If private money issue is prohibited, only one market will arise at the repayment stage, where outside money is traded. The market price disciplines banks collectively.

The model suggests that in the presence of multiple banks, a prohibition on private money issue gives rise to inefficient outcomes. While the inside money regime fosters money competition, the outside money regime eliminates competition and triggers incentive problems of free-riding among banks. As a result, the equilibrium returns of outside money are inefficiently low as banks make private decisions. The overall effect on welfare is negative. The policy of prohibition should be accompanied with the imposition of bank-specific outside money to avoid the negative effect on welfare.)
Chapter 2  
Banking, Inside Money and Outside Money

2.1 Introduction

The main goal of this paper is to integrate banking theory with monetary theory. I address the following question: given that both individuals and banks have private information, what is the optimal way to settle debts? This is a fundamental question concerning any modern economy, where both outside money (fiat money) and inside money (created by banks and payment systems) are used to facilitate trades. How to settle debts efficiently is critical for the performance of the banking system as a major source of lending. There are several aspects to this issue. For example, why should debts be settled with money? Which is a better instrument for settlements, inside money or outside money?

To answer these questions, I develop a dynamic model with micro-founded roles for banks and a medium of exchange. There are two types of frictions in the economy. The first one is lack of intertemporal double coincidence of wants. This, along with spatial separation and limited communication, gives rise to the role of money as the medium of exchange. The second friction is two-layered private information. On one hand, agents have private information about their random endowments. Hence banking has a role in providing risk-sharing. In particular, bankers can offer dynamic contracts to help agents smooth consumption over time. However, the contracts must be incentive compatible for individuals to truthfully make payments. On the other hand, bankers have private information about
the uncertain aggregate endowments because they can filter out the idiosyncratic shocks by aggregating the reports of individual agents. This creates a role for markets to help solve the incentive problem on the bank’s side. Indeed, markets at the settlement stages generate information-revealing prices such that bankers cannot lie about the aggregate states.

In the model, a banking sector arises endogenously at the beginning of time and provides dynamic contracts to agents. According to the contract, bankers lend money to agents at the beginning of a period and agents settle the current debt with bankers as they receive endowments at the end of the period. In each period, the amount of the loan entitlement of an agent depends on the individual’s history of past settlements (i.e., his/her history of reported endowments) and the sequence of prices at settlement stages.

I establish two main results in this paper. First, markets can improve upon the optimal dynamic contract in the presence of private information on the bank’s side. Markets of goods for money at the settlement stages generate prices that fully reveal the aggregate states. This costlessly solves the incentive problem of bankers. However, if debts are required to be settled with real goods, no market will arise at the settlement stages. Therefore, debt settlements must involve money in order to efficiently discipline bankers.

Second, the optimal instrument for settlements is the kind of inside money that expires immediately after each settlement. I call it one-period inside money. Induction of truthful revelation is less costly with one-period inside money than with outside money or inside money of any longer durations, which leaves agents better insured against idiosyncratic risks. Agents cannot benefit from holding one-period inside money across periods because it expires right after a settlement (which happens at the end of a period). In this
case, the only profitable way for one to default is to save and consume one’s own endowments, which is not very desirable. In contrast, when outside money is valued, an agent finds it more profitable to default by carrying outside money across periods rather than saving endowments. The reason is that the agent can use the hidden outside money to buy his/her preferred consumption goods. Thus the gain of default is higher with outside money than with one-period inside money. The same argument applies to inside money of longer durations. Longer-termed inside money functions similarly to outside money and involves higher incentives to misrepresent in periods when the current issue of money does not expire. Therefore, one-period inside money helps the optimal dynamic contracts implement better allocations. In equilibrium, more efficient risk-sharing is achieved and welfare is improved.

The key to the above result is the timing of the expiration of inside money, which is exactly when each settlement of debts is done. Once an agent obtains such inside money for the settlement, making the payments to the bank is nothing but giving up some worthless objects. However, this is not true if outside money is required for settlements. Outside money will still be valuable to the agent after the settlement stage. Hence the incentives to default are much stronger with outside money. Not surprisingly, inflation of outside money can be used to correct incentives. As outside money gets less valuable with time, inducing truthful revelation tends to get less costly.

The model of this paper is built upon Andolfatto and Nosal (2003) and Sun (2007). Andolfatto and Nosal (2003) construct a model with spatial separation, limited communication friction and limited information friction. They explain why money creation is
typically associated with banking. Sun (2007) addresses the problem of monitoring banks with undiversifiable risks and shows that there is no need to monitor a bank if it requires loans to be repaid partly with money. A market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. This result is strengthened in the current paper, which features an enduring relationship between bankers and the contracted agents. In contrast to the static contract studied in Sun (2007), here I show that even dynamic contracts can use the help of markets to deal with the incentive problem of bankers.

My work is complementary to the literature that examines the functioning of inside money and outside money, e.g., Cavalcanti and Wallace (1999), Williamson (2004), He, Huang and Wright (2005, 2006) and Sun (2007). Cavalcanti and Wallace (1999) study a random matching model of money and prove that inside money has the advantage of facilitating trades between bankers and non-bankers because with inside money bankers are not constrained by trading histories. One of the issues addressed by Williamson (2004) is the implication of private money issue for the role of outside money. Inside money has the advantage of being flexible and it responds to unanticipated shocks better than outside money. He, Huang and Wright (2005, 2006) study money and banking in a money search model. Bank liabilities are identified as a safer instrument than cash while cash is less expensive to hold. In equilibrium, agents may find it optimal to hold a mix of both. Sun (2007) establishes that with multiple banks, inside money helps achieve better outcomes than outside money does. The reason is that the competition of private monies drives up the equilibrium returns of money and improves welfare. A prohibition on inside
money issue not only eliminates money competition but also triggers free-rider problems among bankers, which decreases welfare. All the above papers focus on the roles of inside money and outside money as alternative instruments to facilitate trades. In contrast, this paper takes a new, yet no less important, perspective, which is the efficiency of alternative monetary instruments for settling debts.

This paper develops an integrated theory of money, banking and dynamic contracts, which is far a rare effort. A related previous work is by Aiyagari and Williamson (2000). They study money, credit and dynamic contracts. In their model, financial intermediaries write long-term contracts with consumers. Money is essential because of limited participation in the financial market. There are incentive problems due to private information and limited commitment. With limited commitment, inflation has a large impact on the distribution of welfare and consumption. In contrast, here incentive problems are caused by private information and aggregate uncertainty. It is essential to have contracts that require settlements to be made with money, in order to cope with the incentive problems of bankers. Both inside money and outside money are examined to derive the most efficient payment system for induction of truthful revelation.

The remainder of the paper is organized as follows. Section 2.2 describes the environment of the model. Section 2.3 studies banking with outside money. Section 2.4 examines banking with inside money. Section 2.5 explores banking with co-circulation of inside money and outside money. Section 2.6 studies the existence and uniqueness of the banking equilibrium and Section 2.7 concludes the paper.
2.2 The Environment

Time is discrete and infinite, \( t = 0, 1, \ldots, \infty \). Each period \( t \) consists of three sub-periods, indexed by \( \tau = 1, 2, 3 \). There are three islands indexed by \( i = a, b, c \). Each island is populated by a continuum of agents who have unit mass, live forever and discount across time \( t \) with factor \( \beta \in (0, 1) \). At any point in time, there are only two islands in communication, from which agents can freely visit each other. The sequence of communication at any date \( t \) is the following: islands \( a \) and \( b \) at \( \tau = 1 \), islands \( b \) and \( c \) at \( \tau = 2 \), and islands \( c \) and \( a \) at \( \tau = 3 \). Travelling agents return to their native islands at the end of the sub-period.

Agents on island \( i \) receive endowments of type \( i \) goods. Type \( b \) goods are endowed at \( \tau = 1 \) of all \( t \), type \( c \) goods at \( \tau = 2 \) of all \( t \) and type \( a \) goods at \( \tau = 3 \) of all \( t \). For individual type \( b \) and type \( c \) agents, the endowment is deterministic at \( \bar{y} \) for all \( t \), where \( 0 < \bar{y} < 1 \). However, the endowment of a type \( a \) agent is stochastic: \( y_t = s_t \theta_t \), where \( s_t \) and \( \theta_t \) are both random variables and \( E (y_t) = \bar{y} \). Here \( s_t \) is an aggregate shock, which is common to all type \( a \) agents. It is i.i.d. across time according to the probability density function \( f(s) \) and the cumulative distribution function \( F(s) \). The variable \( \theta_t \) is an idiosyncratic shock. It is i.i.d. over time and drawn in such a way that the law of large numbers applies across type \( a \) agents, according to PDF \( g(\theta) \) and CDF \( G(\theta) \). Both \( f(\cdot) \) and \( g(\cdot) \) have support \([0, 1] \). Let \( h(y) \) and \( H(y) \) denote the PDF and CDF of \( y_t \), respectively. By a well-known result,\(^\text{13}\) \( h(y) = \int_y^1 f(s) g \left( \frac{y}{s} \right) \frac{1}{s} ds \). The realization of \( y_t \), not \( s_t \) or \( \theta_t \) specifically, is private information of the agent. All agents know about \( f(s) \) and \( g(\theta) \). The aggregate endowment of type \( a \) goods is not publicly observable.

\(^{13}\) For the distribution of the product of two continuous random variables, see Rohatgi (1976).
2.2 The Environment

Endowments are received prior to the arrival of any travelling agent at the start of each \( \tau \). All goods are perishable. In particular, type \( b \) and type \( c \) goods can last for only one sub-period and cannot be stored across sub-periods. Type \( a \) goods, however, can last for two sub-periods. That is, the endowment of type \( a \) goods at \( \tau = 3 \) of \( t \) becomes inconsumable starting \( \tau = 2 \) of \( t + 1 \).

Agents’ preferences are as follows:

\[
U_a = E \sum_{t=0}^{\infty} \beta^t u \left( C_{t,b}^a + \varepsilon C_{t-1,a}^a \right)
\]

\[
U_b = E \sum_{t=0}^{\infty} \beta^t \left( C_{t,c}^b + C_{t,b}^b \right)
\]

\[
U_c = E \sum_{t=0}^{\infty} \beta^t \left( C_{t,a}^c + C_{t,c}^c \right)
\]

where the function \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \) is twice continuously differentiable with \( u' > 0 \) and \( u'' < 0 \), and \( C_{t,j}^i \) denotes a type \( i = a, b, c \) agent’s consumption of date-\( t \) type \( j = a, b, c \) goods. That is, the superscript characterizes the agent and the subscripts describe the consumption goods. It is given that \( C_{-1,a}^a = 0 \). Note that agents can either consume their own endowments or another particular type of goods. In contrast to type \( i = b, c \) agents, type \( a \) agents only consume their own endowments at one sub-period over.\(^{14}\) The preference pa-

\(^{14}\) This assumption, along with the assumption that type \( a \) goods can last for two sub-periods, is intended to simplify analysis but is not critical for the main results. As a result of these assumptions, a type \( a \) agent’s current-period decision of truthfully settling debts is independent of his consumption of type \( b \) goods earlier in this period.
rameter $\varepsilon$ is a very small positive number, i.e., $0 < \varepsilon \ll 1$. That is, type $a$ agents strongly prefer type $b$ goods to their own endowments.

Figure 3 Monetary trades

There is a lack of intertemporal double coincidence of wants among various types of agents. In particular, type $a$ agents would like to trade endowments for type $b$ goods. However, type $b$ agents do not value type $a$ goods. Type $b$ agents can consume type $c$ goods, but type $c$ agents do not value type $b$ goods. Similarly for type $c$ and type $a$ agents. This lack of double coincidence of wants, together with the limited communication friction, generates a role for money. At the beginning of time, each type $a$ agent is endowed with $M$ units of storable fiat objects called outside money. Agents can trade money for goods other than their own endowments (see Figure 3). With random endowments, type $a$ agents’
money incomes will also be random. Banking has a role in providing risk-sharing so as to efficiently insure type $a$ agents against idiosyncratic risks.

### 2.3 The Banking Arrangement

A banking sector arises endogenously at the beginning of $t = 0$. Each type $a$ agent chooses to be a banker or a non-banker. Bankers offer long-term contracts to non-bankers, to help them smooth consumption over time. Banking is competitive and the bankers end up offering the same equilibrium contract. Because of the free entry to banking, the equilibrium contract is such that individual bankers and non-bankers earn the same expected lifetime utility. Without loss of generality, it is convenient to think of bankers working together as one intermediary, i.e., the bank. Both the bank and non-bankers commit to the contract. All terms of the contract are public information. Market trades are competitive.

The bank aims to insure type $a$ agents against idiosyncratic endowment shocks. Perhaps the most straightforward banking arrangement is as follows. At each $\tau = 1$, the bank offers money in exchange for the endowments of type $b$ agents and then allocates type $b$ goods efficiently among type $a$ agents. Then at each $\tau = 3$, the bank collects type $a$ endowments, gives the endowments to type $c$ agents in exchange for money, and then allocates the rest of the type $a$ goods (if any) efficiently among type $a$ agents.

There are two-sided incentive problems associated with a banking arrangement as described above. On one hand, incentive problems arise because of private information at the individual level. For type $a$ agents, none of the individual endowments, consumption and money holdings are observable. I focus on incentive compatible allocations. That is,
any banking arrangement must be such that individual type $a$ agents (both bankers and non-bankers) will truthfully reveal their endowments at any time. On the other hand, the bank has the incentive to lie about the aggregate state. Note that the bank collects type $a$ endowments and hence gets to know exactly what the aggregate endowment is based on the reports of individual endowments. In other words, the aggregate endowment level becomes the private information of the bank. Therefore, the bank always has the incentive to misrepresent the aggregate information unless otherwise disciplined. For example, the bank can claim an adverse aggregate state and keep the hidden goods to benefit its bankers, instead of transferring the goods to type $c$ and type $a$ agents as it should. The incentive problem on the bank’s side is known as the problem of monitoring the monitor.

Note that the bank cannot be actually monitored here because there is no state verification technology in this model. (Even if there was, state verification would be costly.) One way to induce truthful revelation of the bank is to design a contract that makes the banking profits depend on the aggregate state announced by the bank. That is, to reward the bank (with higher profits) as it announces a high aggregate state and to punish it (with lower profits) for claiming a low state. However, this mechanism will also be costly because it distorts the optimal allocations.

In a finite horizon model of banking, Sun (2007) shows that the bank is perfectly disciplined if loans are required to be repaid with money. This result can be readily applied here in the current model. Instead of the bank managing all the allocations of goods, the optimal contract requires that at least part of the allocations are done through monetary payments (from the bank to non-bankers and vice versa). As agents are obliged to make
2.4 Banking with Outside Money

For now, assume that private money issue is prohibited. The banking contract requires that monetary payments be made with only outside money. The contract specifies that (i) at the beginning of each $t \geq 1$, the bank pays the non-banker $m_t \in \mathbb{R}^+$ units of outside money to finance his date-$t$ consumption of type $b$ goods; (ii) at $\tau = 3$ of each $t \geq 0$, the non-banker must sell a fraction $z$ of his/her endowments $y_t$ for outside money and then contributes to the bank his/her money income $p_t^a z y_t$ and the rest of the endowments $(1 - z) y_t$, where $p_t^a$ is the market price of type $a$ goods for outside money. Then the bank reallocates the collected type $a$ goods among type $a$ agents. Trivially, a non-banker’s date-0 consumption of type $b$ goods is financed by his/her endowment of $M$ units of outside money.

After money payments to non-bankers, bankers use the residual money to finance their own consumption of type $b$ goods. Each banker is allocated $m_t^B \in \mathbb{R}^+$ units of outside money at the onset of each period. At each $\tau = 3$, each banker must also sell $z y_t$
units of endowments and contribute the income \( p_t^a z y_t \) and the rest of his/her endowments \((1 − z) y_t\). Then bankers divide the type \( a \) goods among themselves after the allocations to non-bankers.

### 2.4.1 Timing of events

Timing of events is illustrated by Figure 4. In any \( t \), at the beginning of \( \tau = 1 \), the bank allocates money among non-bankers and its bankers. Then type \( a \) agents visit island \( b \) and trade money for type \( b \) goods. At \( \tau = 2 \), type \( b \) agents trade money for type \( c \) goods. At \( \tau = 3 \), first type \( c \) agents trade money for type \( a \) goods. Then type \( a \) agents make payments to the bank, which is called the settlement. The bank reallocates the collected type \( a \) goods (if any) among type \( a \) agents. The above procedure is repeated for all \( t \).
2.4.2 The banking equilibrium

Let $v_0$ be a non-banker’s expected lifetime utility prescribed by the contract. Correspondingly, $W_0$ is a banker’s expected lifetime utility. Let $\alpha \in [0, 1]$ be the equilibrium measure of bankers (i.e., the size of the bank); hence $1 - \alpha$ is the equilibrium measure of non-bankers.

**Definition 4** A banking equilibrium consists of a contract with the initial promised value $v_0$ to a representative non-banker and the associated initial value $W_0$ to a representative banker, an aggregate measure $\alpha$, allocations $\{C_{t,a}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c\}_{t=0}^\infty$ market prices $\{p_t^a, p_t^b, p_t^c\}_{t=0}^\infty$ such that: (i) given $v_0$ and $\alpha$, the contract maximizes $W_0$ while delivering the promised $v_0$; (ii) $\alpha$ clears the market of contracts, that is, $W_0 = v_0$; (iii) given prices and the contract, allocations $\{C_{t,a}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c\}_{t=0}^\infty$ maximize type $b$ and type $c$ agents’ utilities; (iv) prices $\{p_t^a, p_t^b, p_t^c\}$ clear goods markets for all $t \geq 0$.

Before examining the banking contract, it is helpful to first study the equilibrium decisions of type $b$ and type $c$ agents. Consider type $c$ agents’ best responses. Taking $(p_t^a, p_t^c)$ as given, a representative type $c$ agent maximizes his/her expected lifetime utility:

$$\max_{(C_{t,a}^c, C_{t,c}^c, d_{t+1}^c)} E \sum_{t=0}^{\infty} \beta^t \left( C_{t,a}^c + C_{t,c}^c \right)$$

s.t. \hspace{1cm} $p_t^a C_{t,a}^c + d_{t+1}^c = d_t^c + p_t^c (y - C_{t,c}^c)$

where $d_t^c$ is the type $c$ agent’s beginning-of-$t$ money holdings. Let $(C_{t,a}^{ca}, C_{t,c}^{ca}, d_{t+1}^{ca})$ de-
2.4 Banking with Outside Money

note the optimal choices. Similarly, taking \((p^b_t, p^c_t)\) as given, a representative type \(b\) agent maximizes his/her expected lifetime utility:

\[
\max_{(C^b_{t,c}, C^b_{t,b}, d^b_{t+1})} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left( C^b_{t,c} + C^b_{t,b} \right) \right]
\]

\[s.t. \quad p^c_t C^b_{t,c} + d^b_{t+1} = d^b_t + p^b_t (\bar{y} - C^b_{t,b})\]

where \(d^b_t\) is the type \(b\) agent’s beginning-of-\(t\) money holdings. Let \((O^b_{t,c}, C^b_{t,b}, d^b_{t+1})\) denote the optimal choices.

The equilibrium prices are \(p^b_t = D^b_t / (\bar{y} - c^b_{t,b})\), \(p^c_t = D^c_t / (\bar{y} - c^c_{t,c})\) and \(p^a_t = D^c_t / Z_t\) for all \(t\), where \(D^i_t\) is the aggregate money supply to the market by type \(i\) agents and \(Z_t = z Y_t\) is the aggregate supply of type \(a\) goods to the market when the aggregate endowment is \(Y_t\). It is straightforward to derive that \(d^b_{t+1} = d^c_{t+1} = 0\) and \(C^b_{t,b} = C^c_{t,c} = \bar{y} - E[Z_t] = (1 - z) \bar{y}\) for all \(t\). Neither type \(b\) nor type \(c\) agents hold money across periods because they receive a constant stream of endowments.

Now I proceed to study the optimal banking contract. First the bank must decide the optimal fraction of the aggregate type \(a\) endowments to be traded in the market, \(z\). \textit{Ex ante} the expected amount of type \(a\) goods to be saved and consumed by type \(a\) agents every period is \((1 - z) \bar{y}\), which is equivalent to consuming \(\varepsilon (1 - z) \bar{y}\) units of type \(b\) goods. Suppose instead of saving it up, the bank also requires the fraction \(1 - z\) of the aggregate endowment to be sold to type \(c\) agents. According to \(C^b_{t,b}\), this will get type \(b\) agents to sell \((1 - z) \bar{y}\) more units of goods to type \(a\) agents. Since \(\varepsilon < 1\), it is efficient for the bank to
require \( z = 1 \). As a result, type \( a \) agents must sell all their endowments to type \( c \) agents. In return, the aggregate consumption of type \( a \) agents is maximized and equal to \( \bar{y} \) units of type \( b \) goods every period.

Now let \( c_t \) denote a non-banker’s date-\( t \) consumption financed by the contract. Thus,
\[
c_t = \frac{m_t^b}{p_t^a}
\]
where \( p_t^b \) is the date-\( t \) price of type \( b \) goods for outside money and \( m_0 = M \).

Without loss of generality, normalize \( M = 1 \). The contract prescribes \( v_0 = E \sum_{t=0}^{\infty} \beta^t u(c_t) \).

Correspondingly, \( c_t^B = \frac{m_t^b}{p_t^c} \) denote a banker’s date-\( t \) consumption of type \( b \) goods and hence \( W_0 = E \sum_{t=0}^{\infty} \beta^t u(c_t^B) \). Again \( m_0^B = 1 \).

Because of the private information of individuals, payments from the bank to a non-banker must be based on the latter’s reported history of endowments. Recall that I focus on incentive compatible contracts. Unless otherwise stated, reported values also represent true values. Denote a non-banker’s history of reported endowments up to period \( t \) as \( h_t = (y_0, y_1, \cdots, y_t) \in [0, 1]^{t+1} \). Since \( z_t = z_t^B = y_t \) for all \( t \), the equilibrium price is \( p_t^a = \frac{1}{Y_t} = \frac{1}{s_t \int_{\theta \in [0, 1]} g(\theta) d\theta} \) for all \( t \). Hence the market price at the settlement stage \( (r = 3) \) fully reveals the aggregate state, i.e., \( s_t = \frac{1}{p_t^a E[\theta]} \). In other words, agents can infer the true aggregate state simply by observing the market price. As a result, the bank cannot misrepresent the aggregate information to benefit its bankers. Denote the price sequence of settlement stages up to period \( t \) as \( P_t = (p_0^a, p_1^a, \cdots, p_t^a) \in \mathbb{R}_+^{t+1} \). The banking contract can be formally defined as follows.

**Definition 5** A contract \( \sigma \) is a constant \( \sigma_0 \) and a sequence of functions \( \{\sigma_t\}_{t=1}^{\infty} \) where \( \sigma_t : [0, 1]^t \times \mathbb{R}_+^t \to \mathbb{R}_+ \). The consumption stream to a non-banker depends on his reported
history of endowments and the price sequence of settlement stages. That is, \( c_0 = \sigma_0 \) and \( c_t = \sigma_t(h_{t-1}, P_{t-1}) \) for all \( t \geq 1 \).

### 2.4.3 The contract design problem

The contract design problem of the bank can be formulated recursively. At the end of \( \tau = 3 \) of any \( t \geq 0 \), non-bankers report current endowments and make the corresponding payments to the bank. Then the bank makes decisions on future payments and promised values according to what non-bankers have reported. For any \( t \geq 1 \), each non-banker is identified with a number \( v_{t+1} \), which is his/her discounted future value starting at \( t + 1 \) and it was promised to him/her by the bank at \( t - 1 \). The bank delivers \( v_{t+1} \) by financing a state-dependent next-period consumption \( c_{t+1} \) and a promised value \( v_{t+2} \) starting at period \( t + 2 \). Let the density function \( \mu_{t+1}(v_{t+1}) \) characterize the distribution of the promised values made by the bank to be delivered starting at \( t + 1 \). Then \( \mu_{t+1} \) is the state variable for the bank’s recursive problem at the end of each period \( t \). Note that the \( t = 0 \) consumption of type \( b \) goods is financed by the agent’s endowment of outside money. Thus \( c_0 = \bar{y} \). Since \( v_0 \) is the lifetime expected value promised by the contract, it follows that \( v_1 = v_0 - u(\bar{y}) \) and

\[
\mu_1(v_1) = \begin{cases} 
1, & \text{if } v_1 = v_0 - u(\bar{y}) \\
0, & \text{otherwise}
\end{cases}.
\] (2.12)
The objective of the recursive contract design problem is to maximize a representative banker’s expected discounted value $W_{t+1}$ starting at $t+1$, while delivering the distribution of promised values $\mu_{t+1}$. Dropping time subscripts and letting $+1$ denote $t+1$ and $+2$ denote $t+2$, the bank’s end-of-period-$t$ objective can be formulated by the following functional equation:

\[
(TW_{+1})(\mu_{+1}) = \max_{(c_{+1}^{B}, c_{+1}, v_{+2})} \int_{0}^{1} \int_{0}^{1} \{ u[c_{+1}^{B}(y, s)] + \beta W_{+2}(\mu_{+2}) \} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s) ds.
\] (2.13)

The maximization problem is subject to the following conditions:

\[
u_{c_{+1}(y, s, v_{+1})} + \beta v_{+2}(y, s, v_{+1}) \\
\geq u\left[c_{+1}(\bar{y}, s, v_{+1}) + \max_{\gamma \in [0,1]} \left\{ \varepsilon \gamma (y - \bar{y}) + (1 - \gamma) (y - \bar{y}) \frac{p^a(s)}{p_{+1}} \right\} \right] + \beta v_{+2}(\bar{y}, s, v_{+1}), \quad \forall \ s, v_{+1}, \ \forall \ \bar{y} < y
\] (2.14)

\[
u_{c_{+1}(y, s)} \\
\geq u\left[c_{+1}(\bar{y}, s) + \max_{\gamma \in [0,1]} \left\{ \varepsilon \gamma^{B} (y - \bar{y}) + (1 - \gamma^{B}) (y - \bar{y}) \frac{p^a(s)}{p_{+1}} \right\} \right] \\
\forall \ s, \ \forall \ \bar{y} < y
\] (2.15)
2.4 Banking with Outside Money

\[
v_{+1} = \int_0^1 \int_0^1 \{ u[c_{+1}(y, s, v_{+1})] + \beta v_{+2}(y, s, v_{+1}) \} g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) f(s) ds, \quad \forall v_{+1} (2.16)
\]

\[
\mu_{+2}(w_{+2}; s) = \int_{\{(y, v_{+1}) : w_{+2} = v_{+2}(y, s, v_{+1})\}} g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \mu_{+1}(v_{+1}) dv_{+1}, \quad \forall s (2.17)
\]

\[
\overline{y} = \alpha \int_0^1 c_{+1}^B(y, s) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right)
\]

\[
+ (1 - \alpha) \int_0^1 \int_{-\infty}^{\overline{V}} c_{+1}(y, s, v_{+1}) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \mu_{+1}(v_{+1}) dv_{+1}, \quad \forall s (2.18)
\]

\[
c_{+1}(y, s, v_{+1}) \geq 0, \quad \forall y, s, v_{+1} (2.19)
\]

\[
c_{+1}^B(y, s) \geq 0, \quad \forall y, s (2.20)
\]

\[
v_{+2}(y, s, v_{+1}) \in (-\infty, \overline{V}], \quad \forall y, s, v_{+1} (2.21)
\]

where \(\overline{V} = u(\overline{y}) / (1 - \beta)\) is the value of the unconstrained first-best contract that finances consumption of \(\overline{y}\) units of type \(b\) goods in every period.

Constraints (2.14) and (2.15) are the incentive compatibility constraints for a non-banker and a banker respectively. Incentive compatibility requires that both bankers and non-bankers are induced to sell the entire endowments and turn over the entire incomes every period. Here \(c_{+1}(y, s, v_{+1})\) and \(v_{+2}(y, s, v_{+1})\) are a non-banker’s next period consumption and promised value starting the period after the next, given that he/she is currently promised \(v_{+1}\), his/her current endowment is \(y\) and the current aggregate state is \(s\). For a banker, \(c_{+1}^B(y, s)\) is his/her next-period consumption given his/her current endowment \(y\) and the current aggregate state \(s\).
For both parties, the payoff of truthful revelation must be no lower than the payoff of any possible deviation. The right-hand side of (2.14) is the payoff if the non-banker reports \( \tilde{y} < y \) instead of the truth \( y \). (Note that it is not feasible for an agent to claim \( \tilde{y} > y \) because he/she would not have \( p^a \tilde{y} > p^a y \) units of money to submit to the bank when the true endowment is \( y \).) The misreported endowment can either be stored for next-period consumption or be traded for money to buy type \( b \) goods. The non-banker chooses \( \gamma \), the fraction of endowment to be stored, to maximize his/her gain of default. The first term in the maximization problem (on the right-hand side of [2.14]) is the extra consumption of stored endowment \( \varepsilon \gamma (y - \tilde{y}) \). The second term is the extra consumption of type \( b \) goods purchased with the misreported money, which is \( (1 - \gamma) (y - \tilde{y}) \frac{p^a(s)}{p_{b+1}^a} \). Similar logic for the right-hand side of (2.15). Given prices, an agent optimally chooses \( \gamma^* = 0 \) (or \( \gamma^{B*} = 0 \)) if \( \frac{p^a}{p_{b+1}} > \varepsilon \). In equilibrium, \( p^a = \frac{1}{sE(\theta)} \) and \( p_{b+1} = \frac{1}{\tilde{y}} = \frac{1}{E(s)E(\theta)} \). Therefore, we have \( \gamma^* = \gamma^{B*} = 0 \) provided that \( \varepsilon < E(s) \). Now constraints (2.14) and (2.15) can both be simplified:

\[
\begin{align*}
u \left[ c_{b+1} (y, s, v_{b+1}) \right] + \beta v_{b+2} (y, s, v_{b+1}) \\
\geq u \left[ c_{b+1} (\tilde{y}, s, v_{b+1}) + (y - \tilde{y}) \frac{p^a(s)}{p_{b+1}^a} \right] + \beta v_{b+2} (\tilde{y}, s, v_{b+1}) \\
\forall s, v_{b+1}, \forall \tilde{y} < y
\end{align*}
\]
Constraint (2.16) is the promise-keeping constraint. All the values promised to non-
bankers must be delivered. Constraint (2.17) characterizes the law of motion of the state
variable $\mu$, i.e., the distribution of the promised values. Constraint (2.18) is the resource
constraint. Consumptions of bankers and non-bankers exhaust $\bar{y}$ units of type $b$ goods every
period. Constraints (2.19)–(2.21) define the choice sets for the choice variables.

Let $W^* (\mu)$ denote the fixed point of $T$ in (2.13). One can show that $W^* (\mu)$ is a
strictly increasing, concave function from the fact that $T$ is a contraction mapping that maps
the space of increasing, concave functions to itself. The policy functions $\{c_{t+1} (y, s, v_{t+1}),
\}$
together with the initial consumption $c_0$ and the associated initial promised value $v_0$, com-
pletely characterize the lifetime contract to a non-banker. Hence, $v_0 = u (\bar{y}) + E \sum_{t=1}^{\infty} \beta^t u (c_t)$. 
Similarly, the policy function $c_{t+1}^B (y, s)$ pins down the initial value of a representative
banker, $W_0 = u (\bar{y}) + E \sum_{t=1}^{\infty} \beta^t u (c_t^B)$. The equilibrium condition $v_0 = W_0$ implies that
given $\alpha$,

$$v_0 = u (\bar{y}) + W_1 (\mu_1; \alpha), \quad (2.24)$$

where $\mu_1$ is given by (2.12). The above condition defines $v_0$ as a function of $\alpha$, i.e., the
relationship between the initial value and the aggregate measure that clears the market of
contracts. The equilibrium contract must be the one that offers the highest achievable $v_0^*$. 
Therefore, $v_0^* = \max_{\alpha \in [0,1]} v_0 (\alpha)$. 

So far I have set up the contract design problem and described the banking equi-
librium for the banking contract that requires payments of outside money. The following
section examines the contract that requires payments to be made exclusively with inside money. Then I compare the implications of the two contracts and show that it matters whether inside money or outside money is used as the settlement instrument.

2.5 Banking with Inside Money

2.5.1 One-period inside money

Now assume the private issue of money is permitted. The bank can finance consumptions through allocations of private money. In this section, I study the banking arrangement where outside money is not valued and the bank issues a particular kind of inside money, one-period inside money (OPIM). Namely, it is issued at the beginning of each \( t \geq 0 \) and expires at the end of \( t \) after the current-period settlements are done.\(^\text{15}\) As before, let \( \alpha \) be the equilibrium measure of bankers.

The contract specifies that (i) at the beginning of all \( t \geq 0 \), the bank pays the non-banker \( m_t \in \mathbb{R}^+ \) units of inside money to finance his/her date-\( t \) consumption of type \( b \) goods; (ii) at \( \tau = 3 \) of all \( t \geq 0 \), the non-banker must sell the entire endowment \( y_t \) for inside money and then contribute to the bank the money income \( p_t^a y_t \), where \( p_t^a \) is now the market price of type \( a \) goods for inside money.

The same notations are used as in the previous section. In particular, let \( c_t \) denote a non-banker’s date-\( t \) consumption of type \( b \) goods financed by the contract. That is, \( c_t = \frac{m_t}{p_t^b} \)

\(^{15}\) The expiration of inside money can be thought of as the object deteriorates after a certain amount of time. Or we can interpret it as an electronic account whose balance automatically becomes zero at the prescribed point of time. Accordingly, a new issue of money is simply an amount newly transferred into the account by bankers.
where $p_t^b$ is the price of type $b$ goods for inside money. Denote $h_t$ as a non-banker’s history of reported endowments up to period $t$ and $P_t$ as the price sequence of settlement stages up to period $t$. The contract and the banking equilibrium are still defined by Definition 1 and Definition 2, respectively. The objective of the recursive contract design problem by implementing OPIM is given by (2.13) subject to the same constraints as (2.16)–(2.21). However, the incentive compatibility constraints are now different from (2.14) and (2.15):

$$u \left[ c_{t+1} (y, s, v_{t+1}) \right] + \beta v_{t+2} (y, s, v_{t+1})$$

$$\geq u \left[ c_{t+1} (\bar{y}, s, v_{t+1}) + \varepsilon (y - \bar{y}) \right] + \beta v_{t+2} (\bar{y}, s, v_{t+1})$$

$$\forall s, v_{t+1}, \forall \bar{y} < y$$

(2.25)

$$u \left[ c_{t+1}^B (y, s) \right]$$

$$\geq u \left[ c_{t+1}^B (\bar{y}, s) + \varepsilon (y - \bar{y}) \right]$$

$$\forall s, \forall \bar{y} < y.$$  (2.26)

Constraint (2.25) is the incentive compatibility constraint for a representative non-banker and (2.26) is the constraint for a representative banker. The right-hand sides of the constraints are the payoffs of default. As required by the contract, type $a$ agents must sell all endowments for inside money. As a result, outside money is not valued by type $b$ or $c$ agents. Moreover, it is not beneficial for a non-banker or a banker to sell any misreported endowment for inside money because it will expire before period $t+1$ comes. Thus the only profitable way to default is to save the hidden endowments for next-period consumption.
Since the initial allocation of inside money does not depend on any report of endowments, naturally $m_0 = 1$ and $c_0 = \bar{y}$. Again, $v_0 = u(\bar{y}) + E \sum_{t=1}^{\infty} \beta^t u(c_t)$. The equilibrium contract must be the one that offers the highest achievable $v^*_0$. Index values of banking with outside money by superscript $o$ and values of banking with one-period inside money by superscript $I$. Provided that $\varepsilon < E(s)$, we have the following propositions:

**Proposition 14** $W^o(\mu) < W^I(\mu)$ for any given $\mu$.

**Proposition 15** $W^o_0(v_0; \alpha) < W^I_0(v_0; \alpha)$ for any given $v_0$ and $\alpha$.

**Proposition 16** $v^*_0 < v^I_0$. Moreover, $v^I_0 \to \bar{V}$ as $\varepsilon \to 0$ while $v^*_0$ is independent of $\varepsilon$.

Propositions 14–15 establish that all else being equal, bankers can always achieve higher utility by offering contracts with one-period inside money rather than with outside money. Accordingly, the bank will choose to implement the former contract. This result is driven by the fact that the incentives to default are weaker with one-period inside money than with outside money. When outside money is valued, agents expect it to carry value into the future. On evaluating the options to default, agents find it more profitable to sell endowments for outside money than to save them for consumption in the following period (given that $\varepsilon < E[s]$). One-period inside money, however, expires right after settlements. Thus, type $a$ agents cannot benefit from selling the hidden endowments for inside money. The only benefit from default now is to save the endowments for next-period consumption, which is associated with a much lower utility gain. Thus it is less costly to induce truthful revelation with OPIM. This allows the bank to achieve more efficient risk-sharing and to
offer higher equilibrium promised values, which is established by Proposition 16. As a result, welfare of type \(a\) agents is improved by the contract that requires settlement be made with one-period inside money. The overall welfare of the economy is also improved because the expected lifetime utility of a type \(b\) or type \(c\) agent is \(\frac{\bar{y}}{1-\beta}\) regardless of his/her optimal decisions to trade.

Furthermore, the advantage of the OPIM contract gets stronger as type \(a\) agents value less of their own endowments. As \(\varepsilon \to 0\), the utility gain of consuming their own endowments becomes negligibly small. With one-period inside money, the incentives to default diminish because neither saving endowments nor trading endowments for money is profitable. The result approaches the allocations achieved by the unconstrained first-best contract. That is, \(c_t(y_{t-1}, s_{t-1}, v_t) = \bar{y}\), \(c_t^B(y_{t-1}, s_{t-1}) = \bar{y}\) and \(v_{t+1}(y_{t-1}, s_{t-1}, v_t) = \frac{u(\bar{y})}{1-\beta}\) for all \((y_{t-1}, s_{t-1}, v_t)\). However, these policy functions obviously do not satisfy the constraints (2.22)–(2.23) of the contract with outside money. With outside money, the incentives to default are merely driven by the gain of carrying outside money into the following period. These incentives do not go away even if one does not value one’s own endowments. Therefore, there is no way the contract with outside money can implement perfect risk-sharing, not even when \(\varepsilon = 0\).

### 2.5.2 Inside money with longer durations

The previous section studies a special kind of inside money, i.e., one-period inside money. Welfare is higher with one-period inside money than it is with outside money. Now I turn to inside money of more generalized forms and investigate the associated welfare
2.5 Banking with Inside Money

implications. The bank issues inside money that has a duration of \( \kappa \) periods, where \( \kappa \) is an integer and \( 2 \leq \kappa < \infty \). (Note that if \( \kappa = \infty \), inside money never expires, which is equivalent to outside money in this environment.) That is, each issuance of inside money is made at the beginning of period \( t = 0, \kappa, 2\kappa, \cdots \), and expires at the end of period \( t = \kappa - 1, 2\kappa - 1, \cdots \). Other than that, the bank functions in the same way as in Section 4. Definition 1 and Definition 2 still apply.

The objective of the recursive contract design problem with \( \kappa \)-period inside money is given by

\[
(TW_{+1}^\kappa) (\mu_{+1}, t) = \max_{(c_{+1}^B,c_{+1}^v,v_{+2})} \int_0^1 \int_0^1 \{ u [c_{+1}^B (y, s)] + \beta W_{+2}^\kappa (\mu_{+2}, t_{+1}) \} g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) f (s) ds
\]

and is subject to the same constraints as (2.16)–(2.21). However, instead of constraints (2.14) and (2.15), here the incentive compatibility constraints are formulated by the following:

\[
u [c_{+1} (y, s, v_{+1})] + \beta v_{+2} (y, s, v_{+1}) \\
\geq \nu [c_{+1} (\bar{y}, s, v_{+1}) + \nu \Phi_1 + (1 - \nu) \Phi_2] + \beta v_{+2} (\bar{y}, s, v_{+1}) \\
\forall s, v_{+1}, \forall \bar{y} < y
\]
2.5 Banking with Inside Money

\[
u \left[ c_{i+1}^B (y, s) \right] \\
g \geq u \left[ c_{i+1}^B (\tilde{y}, s) + \nu \Phi_1 + (1 - \nu) \Phi_2 \right]\\n\forall s, \forall \tilde{y} < y
\]

(2.29)

where

\[
\nu_t = \begin{cases} 
1, & \text{if } t = \kappa - 1, 2\kappa - 1, \cdots \\
0, & \text{otherwise}
\end{cases}
\]

\[
\Phi_1 = \varepsilon (y - \tilde{y})
\]

\[
\Phi_2 = \max_{\gamma \in [0, 1]} \left\{ \varepsilon \gamma (y - \tilde{y}) + (1 - \gamma) (y - \tilde{y}) \frac{p^\kappa}{p_{i+1}^\kappa} \right\}.
\]

(2.30)

Let \( v_0^* \) and \( W_0^\kappa \) be the initial values of a representative non-banker and a representative banker, respectively. The banker’s recursive contract design problem now differs in periods with and without expiration of money. In periods with expiration of money, that is, \( t = \kappa - 1, 2\kappa - 1, \cdots \), the banker’s problem is similar to the case with one-period inside money. Since the current issue of money expires at the end of the period, the only profitable way for type \( a \) agents to default is to save the endowments for next-period consumption. The incentive compatibility constraints are equivalent to (2.25)–(2.26). In periods without expiration of money, the problem is similar to the case with outside money. Agents would prefer to default by carrying money into the next period. Accordingly, the incentive compatibility (IC) constraints are equivalent to (2.14)–(2.15). Let \( v_0^{\kappa*} \) be the equilibrium initial promised value with \( \kappa \)-period inside money.

**Proposition 17** \( v_0^{\kappa*} > v_0^* < v_0^{\kappa*} \).
Proposition 17 establishes that welfare is the highest with one-period inside money. Banking with \( \kappa \)-period inside money comes second while the outside money arrangement ranks last. With \( \kappa \)-period inside money, incentives to default in periods without expiration of money are as strong as with outside money. It does provide more stringent discipline when there is expiration of money at the end of a period. However, overall agents are not always as disciplined as with one-period inside money. Not surprisingly, incentive compatibility is still more costly with \( \kappa \)-period inside money than with one-period inside money. Hence Proposition 17.

### 2.6 Co-circulation of Inside and Outside Money

In this section I study co-circulation of inside money and outside money. It has been previously established that one-period inside money is the best of all kinds of inside money in that it helps the banking contract achieve the highest welfare level. Therefore, it makes sense here to focus on the co-circulation of outside money and one-period inside money.

The contract specifies that (i) at the beginning of all \( t \geq 0 \), the bank pays the non-banker a portfolio of \( \{m^I_t, m^O_t\} \) to finance his date-\( t \) consumption of type \( b \) goods, where \( m^I_t \in \mathbb{R}^+ \) is the amount of current period inside money and \( m^O_t \in \mathbb{R}^+ \) is the amount of outside money; (ii) at \( \tau = 3 \) of all \( t \), the non-banker must sell \( (1 - \phi) y_t \) units of endowment for current-period inside money and \( \phi y_t \) units of endowment for outside money, where \( \phi \in [0, 1] \) is a constant. Then the portfolio of money incomes \[ p_t^a, (1 - \phi) y_t, p_t^{a, \phi} y_t \] must be contributed to the bank, where \( p_t^a \) is the market price of type \( a \) goods for date-\( t \).
inside money and $p_t^{a,o}$ is the market price of type $a$ goods for outside money. Trivially, $m_0^o = 1$.

Note that if $\phi = 0$, the contract reduces to one with only one-period inside money; if $\phi = 1$, the contract becomes one with only outside money. In this section I focus on $\phi \in (0, 1)$. Now define $P_t = \left( \left( p_0^{a,I}, p_0^{a,o} \right), \left( p_1^{a,I}, p_1^{a,o} \right), \ldots, \left( p_t^{a,I}, p_t^{a,o} \right) \right) \in (\mathbb{R}_+ \times \mathbb{R}_+)^{t+1}$ as the price sequences of settlement stages up to period $t$. Definition 1 still applies. Let $p_t^{b,o}$ and $p_t^{b,I}$ be the market prices of type $b$ goods for outside money and date-$t$ inside money, respectively. Then $c_t = \frac{m_t^I}{p_t^I} + \frac{m_t^o}{p_t^o}$ for all $t$. Let $\left\{ m_t^{B,I}, m_t^{B,o} \right\}$ denote a banker’s beginning-of-date-$t$ portfolio, where $m_t^{B,I}, m_t^{B,o} \in \mathbb{R}_+$ and $m_0^{B,o} = 1$. It follows that $c_t^B = \frac{m_t^{B,I}}{p_t^I} + \frac{m_t^{B,o}}{p_t^o}$ for all $t$.

**Definition 6** A banking equilibrium with co-circulation of inside money and outside money consists of a contract with the initial promised value $v_0$ to a representative non-banker and the associated initial value $W_0$ to a representative banker, an aggregate measure $\alpha$, allocations $\left\{ C_{t,e}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c \right\}_{t=0}^{\infty}$ and market prices $\left\{ p_t^{a,I}, p_t^{a,o}, p_t^{b,I}, p_t^{b,o}, p_t^{c,I}, p_t^{c,o} \right\}_{t=0}^{\infty}$ such that: (i) given $v_0$ and $\alpha$, the contract maximizes $W_0$ while delivering the promised $v_0$; (ii) $\alpha$ clears the market of contracts, that is, $W_0 = v_0$; (iii) given prices and the contract, allocations $\left\{ C_{t,e}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c \right\}_{t=0}^{\infty}$ maximize type $b$ and type $c$ agents’ utilities; (iv) prices clear goods markets for all $t \geq 0$.

In equilibrium, $p_t^{a,I} = 1/ \left[ (1 - \phi) s_t \int_0^1 \theta_t g (\theta_t) \, d\theta_t \right]$ and $p_t^{a,o} = 1/ \left[ \phi s_t \int_0^1 \theta_t g (\theta_t) \, d\theta_t \right]$ for all $t$. Obviously in equilibrium,
2.6 Co-circulation of Inside and Outside Money

\[
\frac{p_{t}^{o,o}}{p_{t}^{a,I}} = \frac{p_{t}^{b,o}}{p_{t}^{b,I}} = \frac{p_{t}^{c,o}}{p_{t}^{c,I}} = \frac{1 - \phi}{\phi}, \forall t.
\]

That is, the value of outside money relative to inside money on island \( a \) is given by the ratio of the amounts of goods required to sell in the respective markets. Expecting this, type \( b \) and type \( c \) agents value inside and outside monies by the same ratio.

The objective of the recursive contract design problem now is given by (2.13) subject to the same constraints as (2.16)–(2.21). Instead of constraints (2.14)–(2.15), here the incentive compatibility constraints are formulated by

\[
\begin{align*}
&u_c^{c+1}(y, s, v_{+1}) + \beta v_{+2}(y, s, v_{+1}) \\
&\geq u_c^{c+1}(\tilde{y}, s, v_{+1}) + \max_{\gamma \in [0, 1]} \left\{ \varepsilon \gamma (y - \tilde{y}) + (1 - \gamma) (y - \tilde{y}) \frac{p_{t}^{o,o}(s)}{p_{t+1}} \right\} \\
&+ \beta v_{+2}(\tilde{y}, s, v_{+1}) \quad \forall s, v_{+1}, \forall \tilde{y} < y \quad (2.31)
\end{align*}
\]

\[
\begin{align*}
&u_c^{B}(y, s) \\
&\geq u_c^{B}(\tilde{y}, s) + \max_{\gamma^B \in [0, 1]} \left\{ \varepsilon \gamma^B (y - \tilde{y}) + (1 - \gamma^B) (y - \tilde{y}) \frac{p_{t}^{o,o}(s)}{p_{t+1}} \right\} \\
&\forall s, \forall \tilde{y} < y \quad (2.32)
\end{align*}
\]
2.6 Co-circulation of Inside and Outside Money

In fact, the above constraints are equivalent to (2.14)–(2.15) because $\frac{p_{t+1}}{p_{t+1}} = \frac{\phi y}{\phi y} = \frac{E(s)}{\phi} = \frac{p_{t+1}}{p_{t+1}}$. Similar to the case with exclusive circulation of outside money, here agents can default by selling endowments for outside money. The extra outside money obtained is used to purchase more type $b$ goods. Each unit of hidden endowment can be converted into $\frac{p_{t+1}}{p_{t+1}}$ units of next-period type $b$ goods. Given prices, an agent optimally chooses $\gamma^s = 0$ (or $\gamma^{B_1} = 0$) if $\frac{p_{t+1}}{p_{t+1}} > \varepsilon$. Therefore, provided that $\varepsilon < E(s)$, we have $\gamma^s = \gamma^{B_1} = 0$ for all equilibrium prices. This is exactly the same result as in the case with outside money only. Let $v_{0}^{CO}$ denote the equilibrium initial promised value associated with co-circulation of one-period inside money and outside money, i.e., $\phi \in (0,1)$. Hence the following proposition:

**Proposition 18**  $v_{0}^{CO} = v_{0}^{O}$. 

According to Proposition 18, co-circulation of one-period inside money and outside money generates the same outcome as the sole circulation of outside money. The incorporation of inside money into the outside money system, $\phi \in (0,1)$, has no impact on welfare at all. As long as outside money is valued, agents’ incentives to default are just as high with or without inside money. The reason is that the profitability of carrying the misreported outside money to the succeeding period depends on the ratio of the prices of goods for outside money, $p_{t+1}^{o} / p_{t+1}^{o}$. With a constant supply of outside money, the price ratio $p_{t+1}^{o} / p_{t+1}^{b}$ only depends on the ratio of aggregate market supplies of goods, $y_{t}/y_{t}$. The parameter $\phi$, however, only affects the relative value of outside money to inside money. Therefore, the incentives are as strong as ever unless outside money is not valued ($\phi = 0$).
2.6 Co-circulation of Inside and Outside Money

2.6.1 Inflation and incentives

Thus far a constant money supply has been assumed. Now I relax this assumption and explore the effect of changes in the money supply on incentive compatibility and welfare. According to previous results, the incentives to default are high when outside money is valued. Moreover, incorporation of inside money into the outside money system does not weaken the incentives in any way. The value of carrying misreported outside money crucially depends on the ratio of the price of date-$t$ type $a$ goods for outside money relative to the price of date-$t+1$ type $b$ goods for outside money, i.e., $p_{a, t}^{o, o} / p_{t+1}^{b, o}$. A change in outside money supply can affect $p_{a, t}^{o, o} / p_{t+1}^{b, o}$, and hence the equilibrium outcomes. In contrast, any change of the stock of inside money has no impact on $p_{a, t}^{o, o} / p_{t+1}^{b, o}$. Without loss of generality, the supply of inside money is still assumed to be constant and normalized to one.

Let $M_t$ be the outside money supply at date $t$. Assume $M_t = (1 + \pi) M_{t-1}$, where $\pi$ is a constant. New money is injected as lump-sum transfers to type $a$ agents at the beginning of $t \geq 1$. Now $\phi \in (0, 1]$. Analogously, $c_t = \frac{m_t^I}{p_t^I} + \frac{m_t^{I+T_t}}{p_t^{o, o}}$ and $c_t^{B} = \frac{m_t^{B, I}}{p_t^{I}} + \frac{m_t^{B, o+T_t}}{p_t^{o, o}}$ for all $t$, where $T_t \in \mathbb{R}$ are the money transfers and taken as given by agents. Moreover, $m_t^I, m_t^{B, I} \in \mathbb{R}^+$ and $m_t^o, m_t^{B, o} \geq -T_t$. Note that now $m_t^o$ and $m_t^{B, o}$ can be negative, which is interpreted as payments from a non-banker to a banker ($m_t^o$) or reallocation of money among bankers ($m_t^{B, o}$), right after the money transfer is received.

The equilibrium prices are $p_t^{a, o} = \frac{M_t}{\phi y_t}$ and $p_{t+1}^{b, o} = \frac{M_{t+1}}{\phi y_t}$. Revisiting constraints (2.31)–(2.32), recall that an agent optimally chooses
2.6 Co-circulation of Inside and Outside Money

\[
\begin{align*}
\gamma_t^*, \gamma_t^{B*} &\begin{cases}
0, & \text{if } \frac{p_t^a, o}{p_{t+1}^o} > \varepsilon \\
[0, 1], & \text{if } \frac{p_t^a, o}{p_{t+1}^o} = \varepsilon \\
1, & \text{otherwise}
\end{cases} 
\end{align*}
\tag{2.33}
\]

Let \( v_0^{inf*} \) denote the equilibrium initial promised value with inflation of outside money.

Hence follows the proposition:

**Proposition 19** If \( \pi \leq E(s)/\varepsilon - 1 \), \( v_0^{inf*} \) is constant in \( \pi \) and \( v_0^{inf*} = v_0^{cos*} = v_0^{os*} \); if \( \pi > E(s)/\varepsilon - 1 \), \( v_0^{inf*} \) is strictly increasing in \( \pi \). Also, \( v_0^{inf*} \to v_0^{f*} \) as \( \pi \to +\infty \).

Provided that \( \varepsilon < E(s) \), Proposition 19 implies that a high enough positive inflation rate can correct incentives to some extent. As a result, outside money is getting less valuable as time goes on. If the aggregate endowment on island \( a \) is high, outside money is more costly to obtain. It is even more so considering that it will not be as valuable tomorrow as it is today. Therefore, for aggregate states above a certain threshold, i.e., \( s_t > \frac{E(s)}{\varepsilon(1+\varepsilon)} \), type \( a \) agents would choose to save endowments for next-period consumption if they were to default. Otherwise, they prefer to default by holding outside money across periods.

To sum up, with a positive inflation rate, from time to time type \( a \) agents may find it more profitable to default by saving endowments than carrying outside money across periods. In this case, agents get better disciplined as inflation goes higher.
2.7 Existence and Uniqueness of Equilibrium

Now it has been established that it is optimal for the bank to implement the contract with one-period inside money. This section studies the existence and uniqueness of a banking equilibrium. In the banking equilibrium, the bank makes allocations of money to finance a type $a$ agent’s consumptions according to $c_0 = \bar{y}$ and the optimal policy functions $c^*_t (y_{t-1}, s_{t-1}, v_t; \mu_t)$, $v^*_t (y_{t-1}, s_{t-1}, v_t; \mu_t)$, $c^B_t (y_{t-1}, s_{t-1}; \mu_t)$ for all $t \geq 1$ that solve (2.13) subject to constraints (2.16)–(2.21) and (2.25)–(2.26). The aggregate measure $\alpha$ clears the market of contracts such that no one can offer a contract that achieves a higher initial value $v_0 > v^*_0$ that satisfies $W_0 (v_0) = v_0$.

**Proposition 20** There exists a unique equilibrium initial value $v^*_0$.

Proposition 20 shows that the banking equilibrium exists and is unique. Note that the equilibrium outcome is not the constrained efficient (i.e., second-best) outcome unless $\alpha^* = 0$ in the equilibrium. When $\alpha^* = 0$, the size of the bank is negligibly small. In this case, the bank’s contract design problem is analogous to the efficiency problem addressed by Atkeson and Lucas (1992) and others, in which a planner endeavours to minimize the expected value of the total resources he/she allocates. The reason why constrained efficiency is not necessarily achieved here is because in general the minimum resources needed to attain a given distribution of promised values may not exhaust all the resources available. In this model, there is no planner as the residual claimant. A utility-maximizing private banker can profit by retaining any positive residual. The competition in banking reaches equilibrium until the expected value of being a banker equals the expected value of being a
non-banker. The equilibrium outcome is not the second-best if the equilibrium measure of bankers is not negligible relative to that of non-bankers ($\alpha^* > 0$).

However, as established by Propositions 14–16, the main result of this paper is robust to any banking contract: one-period inside money can help the banking contract achieve better allocations for any $\alpha$. That is to say, if the second-best allocations can be achieved in the banking equilibrium, it can only be achieved if the contract requires payments to be made with one-period inside money.

2.8 Conclusion

This paper has developed an integrated theory of money, banking and dynamic contracts. The theory is used to evaluate inside money and outside money as alternative instruments for settling debts. The model has micro-founded roles for both banks and a medium of exchange. A banking sector arises endogenously and offers dynamic contracts to help agents smooth consumption over time. According to the contract, bankers lend money to agents at the beginning of a period and agents settle the current debt with bankers as they receive endowments at the end of the period. In each period, the amount of loan entitlement of an agent depends on the individual’s history of past settlements (that is, history of reported endowments) and the sequence of prices at settlement stages. The environment is characterized by a two-sided incentive problem. At the individual level, agents have private information about their random endowments. Contracts must be incentive compatible for individuals to report the true endowments. On the aggregate level, bankers have private information about the uncertain aggregate endowments. This incentive problem on the
bank’s side gives rise to a role for the market to generate information-revealing prices so that the bank cannot lie about the aggregate states.

I have shown that the optimal instrument for settlements is the kind of inside money that expires immediately after each settlement. With such one-period inside money, fewer resources are needed to reward truthful revelation and agents are better insured against idiosyncratic risks. Agents cannot benefit from holding one-period inside money across periods because it expires right after a settlement (which happens at the end of a period). As a result, the only profitable way for one to default is to save endowments for one’s own consumption. However, when outside money is valued, an agent finds it more profitable to default by carrying outside money across periods than saving endowments. That is, the gain of default is higher with outside money than with one-period inside money. The same argument applies to inside money of longer durations. Longer-termed inside money functions similarly to outside money and involves higher incentives to misrepresent in periods when the current issue of money does not expire. Therefore, induction of truthful revelation is the least costly with one-period inside money, which helps the optimal dynamic contract implement better allocations. In equilibrium, more efficient risk-sharing is achieved and welfare is improved.

The key to the above result is the timing of the expiration of inside money, which is exactly when each settlement of current debts is done. Once an agent obtains such inside money for the settlement, making the payments to the bank is nothing but giving up some worthless objects. However, this is not true if outside money is required for settlements. Outside money will still be valuable to the agent after the settlement stage. Hence the
incentives to default are much stronger with outside money. Not surprisingly, inflation of outside money can be used to correct incentives. As outside money gets less valuable with time, induction of truthful revelation tends to get less costly.
Chapter 3
Private Money and Bank Runs

3.1 Introduction

This paper examines bank runs in a model with coexistence of fiat money and private money. One of the most influential theories about bank runs, following Diamond and Dybvig (1983), is that banks are inherently unstable institutions. In particular, the sequential service rule of the demand deposits is considered a key element that causes potential banking panics. The goal of this paper is to study whether allowing demand deposits to circulate can become a natural mechanism to prevent bank runs. We argue that it is the lack of liquidity instruments (such as private money, i.e., circulating demand deposits), not the sequential service rule per se, that makes banks vulnerable to runs. We show the following results: when fiat money is the only medium of exchange, a bank run equilibrium coexists with a banking equilibrium that achieves optimal risk-sharing. In contrast, when banknotes are also allowed to circulate, there exists a unique banking equilibrium which achieves the first-best outcome. Therefore, once private money is permitted, the sequential service rule of demand deposits no longer generates potential banking panics.

The model features overlapping generations of agents. There is heterogeneity within each generation in that agents face private, idiosyncratic liquidity shocks. Moreover, there is uncertainty regarding the aggregate liquidity needs. Banks offer demand deposit con-

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16 Chapter 3 is jointly written with Xiuhua Huangfu, University of Sydney.
tracts and invest deposits of fiat money in nominal bonds. After depositing in the bank, agents observe liquidity shocks and choose when to withdraw fiat money from the bank. Withdrawal demand is served on a sequential basis. Early withdrawal of fiat money is costly in that the bank must liquidate investments before maturity and end up with a zero net return. We consider scenarios when private money is and is not allowed to be used as a medium of exchange. We examine the implications of these alternative mechanisms on the banking equilibrium.

When fiat money is the only medium of exchange, optimal risk-sharing requires a gross rate of return $r > 1$ on early withdrawals. As a result, agents who need liquidity can have more fiat money to spend on consumption goods by depositing in the bank ex ante. Nevertheless, the mechanism is vulnerable to bank runs in that the bank does not have enough assets to honour $r > 1$ if all agents decide to withdraw early. The pessimistic belief of bank runs is self-fulfilling.

When private money is also allowed to circulate, the banking equilibrium is unique and it achieves the first-best outcome. A critical feature of the equilibrium demand deposit contract is that it offers $r < 1$ on early withdrawals, i.e., redemption of private money for fiat money. It is this particular offer that eliminates the bank run equilibrium. With $r < 1$, the bank is actually charging transaction fees on early redemption. This has two direct implications: first, agents do not panic under any circumstances because the rate $r < 1$ guarantees a positive amount of residual bank assets after any volume of early redemption; second, depositors who need liquidity prefer to use private money rather than fiat money.
3.1 Introduction

to buy goods because now early redemption is costly. Sellers are willing to accept private money because it is backed up by the bank’s assets.

If the bank offers any $r > 1$, however, a bank run equilibrium always exists even though private money is allowed. With $r > 1$, the bank’s assets will be depleted if all depositors try to redeem early. As a result, no one is willing to accept private money in trades because they expect zero assets backing up private money. A depositor can only use fiat money to buy goods. Thus it is the dominant strategy for a depositor to demand early redemption if all other depositors do so. The belief of bank runs is indeed self-fulfilling. *Ex ante*, since agents are aware of the potential bank runs associated with any offer of $r > 1$, they will choose to accept a contract with $r < 1$. Thus it is optimal for a bank to offer $r < 1$ by competitive banking.

As a result, in the unique banking equilibrium with private money, no one demands early withdrawals of fiat money and agents in need of liquidity only use private money to finance consumption. In effect, the bank manages to promote the use of private money as a medium of exchange by imposing costs on early redemption. The equilibrium is immune to bank runs and achieves the first-best outcome without having to resort to any government intervention. This result is robust to aggregate uncertainty.

Our model is based on the seminal work of Diamond and Dybvig (1983) [DD]. In contrast to DD, our paper presents a dynamic general equilibrium model of banking with serious roles for money. The key insight is that the form of money matters for bank runs. When fiat money is the only medium of exchange, the demand deposit contract suffers from inherent instability. On one hand, the mechanism is designed to provide liquidity for indi-
3.1 Introduction

Individual agents. On the other hand, the mechanism itself has inherent liquidity problems in that it does not have enough assets to serve if all depositors demand early redemption. Essentially, the demand deposit contract relies on fiat money to perform two potentially conflicting roles: the medium of exchange (i.e., a liquid asset) and the instrument for illiquid investments. It provides liquidity at the expense of terminating profitable illiquid investments.

However, when private money is allowed, the bank can modify the demand deposit contract to avoid any inherent instability. The modified contract promotes the use of private money as a medium of exchange by imposing costs on early withdrawals of fiat money. The key is to assign the two roles to different monies: private money as the medium of exchange and fiat money as the instrument for investment. Accordingly, the contract manages to provide liquidity without liquidating any existing profitable investments.

There have been previous papers that examine bank runs in a monetary context. For example, Champ, Smith and Williamson (1996) also show that bank runs occur if banks are restricted from issuing notes, but do not occur if banks are not restricted. Our paper differs from theirs in two ways. First, Champ et al. study fundamental-driven bank runs, i.e., banking panics triggered by shocks on information regarding fundamentals. In contrast, we focus on expectations-driven bank runs of the DD type. To date, it is by no means clear exactly what kind of shocks causes banking panics. Therefore, it is worth investigating both types. Second, we show that allowing private money changes the demand deposit contract significantly: the bank imposes an explicit cost on early redemption to discourage the use
of fiat money. Accordingly, in equilibrium, depositors in need of liquidity choose to use only private money as a medium of exchange.

Our paper is also closely related to the recent research that studies banking and the co-circulation of fiat money and private money. For example, Sun (2007a,b) focuses on the roles of alternative media of exchange in a banking environment with aggregate uncertainty. Both papers show that private money improves the efficiency of banking and helps achieve higher welfare than fiat money does. Our paper follows the same agenda, but targets the issue of bank runs.

The remainder of the paper is organized as follows. Section 3.2 describes the environment of the model. Section 3.3 studies banking when fiat money is the only medium of exchange. Section 3.4 examines banking when private money is allowed. Section 3.5 compares the results of Sections 3.3–3.4 and Section 3.6 concludes the paper.

### 3.2 The Environment

Time is discrete and infinite. Each period \( t = 0, 1, \cdots, \infty \) consists of two sub-periods, *morning* and *afternoon*. The economy is populated by overlapping generations of agents. At the beginning of period \( t \), a continuum of agents with mass one is born. Each generation of agents lives for two periods and is indexed by time of birth, \( t \). We call those who are in the first period of life *young* agents and those in the second period of life *old* agents.

Agents are endowed with one unit of divisible goods when young and can costlessly harvest endowment goods. However, goods are perishable between sub-periods. Therefore, agents need to decide the amounts of goods to be harvested in the morning and in the
afternoon, respectively. An agent only derives utility from consumption in old age. Moreover, in the second period of life, each agent receives a privately observed preference shock. With probability \( s \), the agent is of type \( H \) and only consumes goods in the morning. With probability \( 1 - s \), the agent is of type \( L \) and only consumes goods in the afternoon. Let \( c^H_t \) represent a type \( H \) agent’s consumption in period \( t \) and \( c^L_t \) a type \( L \) agent’s consumption in period \( t \). An agent born in period \( t \) has the following lifetime utility function:

\[
E_t \left[ \theta_{t+1} U \left( c^H_{t+1} \right) + (1 - \theta_{t+1}) U \left( c^L_{t+1} \right) \right],
\]

where \( U(\cdot) \) is twice continuously differentiable, strictly increasing and strictly concave. Also,

\[
\theta_{t+1} = \begin{cases} 
1, & w.p. \ s_{t+1} \\
0, & w.p. \ 1 - s_{t+1}
\end{cases}
\]

The variable \( s_{t+1} \) is also stochastic. The expectation is taken over \( \theta_{t+1} \). It has support on \([s, \bar{s}]\) and is i.i.d. across time. Here \( 0 < s < \bar{s} < 1 \). Note that \( s_{t+1} \) is common to all agents of generation \( t \). Thus, \( s_t \) also denotes the aggregate measure of type \( H \) old agents in period \( t \).

### 3.2.1 The Planner’s Problem

Suppose \( \theta \) is costlessly observable to a benevolent planner. The planner assigns equal weights to all agents of the same generation and discounts across periods with factor \( \beta \). Let \( E_t \) represent the aggregate amount of goods consumed in the morning of period \( t \). Then, the planner chooses \( E_t \) to maximize social welfare:

\[
W = \sum_{t=0}^{\infty} \beta^t \left[ s_t U \left( \frac{E_t}{s_t} \right) + (1 - s_t)U \left( \frac{1 - E_t}{1 - s_t} \right) \right]. \tag{3.34}
\]
The measure of type $H$ old agents is $s_t$. They share the amount $E_t$ of goods in the morning. Similarly for type $L$ agents, they have measure $1 - s_t$ and share the amount $1 - E_t$ in the afternoon. It is straightforward to show that the optimal solution to the planner’s problem is $E_t^* = s_t$. Therefore, the planner allocates the following consumption levels: $C_t^H = C_t^L = 1$. This is the first-best outcome, which achieves the full-information optimal risk-sharing.

3.3 Fiat Money and Bank Runs

3.3.1 The demand deposit contract

At the beginning of time, there lives a continuum of old agents with mass one, each of whom is endowed with $M$ units of intrinsically worthless and durable objects, called fiat money. Throughout time, there also exists a large number of financial intermediaries, called banks. Banks have access to the tradings of nominal bonds, which are supplied by the government. The duration of bonds is one sub-period. In particular, bonds are issued at the end of period $t$ and they mature at the onset of the afternoon of $t + 1$. Nominal bonds are sold at par for fiat money. For each unit of nominal bonds, the government commits to a gross return of $R > 1$ units of fiat money. Moreover, bonds can be liquidated early, i.e., in the morning of $t + 1$. Nevertheless, early liquidation is costly in that it gives zero net interests. The interest payments on bonds are financed by a lump-sum tax $T_t$ on generation-$t$ agents’ incomes in the first period of life. Let $M_t$ denote the aggregate after-tax fiat money holdings of young agents at the end of period $t$. 
At the end of period $t$, banks offer demand deposit contracts to the young agents of generation $t$. By accepting the contract, agents agree to deposit fiat money in a bank and the bank invests fiat money in nominal bonds. Assume the bank is mutually owned by depositors. A depositor has the liberty of withdrawing fiat money from the bank either in the morning or in the afternoon of $t + 1$. Withdrawal demand is served according to the sequential service rule, i.e., on a first-come-first-served basis.

Upon withdrawal in the morning of $t + 1$, a depositor is entitled to $r$ units of fiat money for each unit of fiat money deposited in $t$. Upon withdrawal in the afternoon of $t + 1$, depositors are entitled to the residual assets (returns to matured bonds) of the bank according to their relative shares of remaining deposits. If a bank goes bankrupt, i.e., does not enough assets to satisfy all withdrawal demand, the bank dies and will be replaced by a new-born bank. By competition, banks offer the same optimal contract in equilibrium. Without loss of generality, from now on we consider the banking sector as one representative bank.

The demand deposit contract is similar to the one studied by DD, except that here we have nominal deposits rather than real deposits. For now, we assume that $s_t$ is not random and is publicly known. We will focus on the banking equilibrium where young agents choose to deposit the after-tax fiat money holdings in the bank. Let $f_t^i$ denote the amount of withdrawals served before agent $i$ as a fraction of the total demand deposits. Moreover, $f_t$ denotes the total amount of withdrawals in the morning of $t$ as a fraction of the total demand deposits. For any $0 \leq r \neq 1$, the rates of return to each unit of deposit withdrawn
are defined as:

\[ R_{1,t} = \begin{cases} r, & \text{if } f^i_t < \frac{1}{r} \\ 0, & \text{if } f^i_t \geq \frac{1}{r} \end{cases} \]  

(3.35)

for morning withdrawals and

\[ R_{2,t} = \max \left\{ 0, \frac{1 - r f^i_t}{1 - f^i_t} \right\} \]  

(3.36)

for afternoon withdrawals. For \( r = 1 \), define \( R_{1,t} = 1 \) and \( R_{2,t} = \bar{R} \). \textit{Ex ante}, at the end of period \( t - 1 \), a young agent has three options: (i) deposit fiat money in the bank; (ii) invest fiat money in nominal bonds (through the bank);\(^{17}\) (iii) simply carry fiat money into period \( t \). If \( r = 1 \), for an individual agent, the demand deposit contract is offering the same payoff schedule as the investment in nominal bonds. Without loss of generality, in this paper we focus on contracts with \( r \neq 1 \).

### 3.3.2 Timing

The following describes the detailed timing of events:

1. At the start of period \( t \), young agents are born; they choose the amounts of goods to supply in the morning and in the afternoon, respectively.

2. By the end of \( t \), young agents pay taxes \( T^i_t \) and decide whether to deposit fiat money income in the bank.

3. At the start of \( t + 1 \), old agents of generation \( t \) observe their type \( i = H, L \); old agents choose the fraction \( d^i_t \) of deposits to be withdrawn in the morning, depending on type.

\(^{17}\) Here bonds are illiquid in the sense that individuals are not allowed to trade bonds for goods.
In the meantime, the morning market opens. Old agents trade fiat money for young agents’ goods.

4. In the afternoon of \( t + 1 \), nominal bonds mature. The bank liquidates assets. Old agents withdraw the remaining deposits, if any.

5. The afternoon market opens. Old agents trade fiat money for young agents’ goods.

### 3.3.3 Equilibrium

In this section, the only medium of exchange is fiat money. Young agents supply goods to competitive markets to trade for fiat money. Type \( H \) old agents trade fiat money for young agents’ goods in the morning. In the afternoon, type \( L \) old agents trade with young agents. Note that type \( L \) old agents do not trade for goods in the morning because goods are perishable.

Let \( P_{1,t} \) denote the price of fiat money for goods in the morning market and \( P_{2,t} \) the price in the afternoon market. Denote \( e_t \in [0, 1] \) as a young agent’s supply of goods in the morning. We will focus on symmetric banking equilibria where agents of the same age and type apply the same strategy. Let \( m_t \) denote the amount of fiat money deposited in the bank by a young agent of generation-\( t \). Let \( d^i_t \in [0, 1] \) denote the fraction of a type \( i = H, L \) agent’s deposits withdrawn in the morning of period \( t \). Denote \( D_t \in [0, 1] \) as the expectation of the aggregate fraction of deposits to be withdrawn in the morning of \( t \). In equilibrium, expectations are consistent with outcomes, i.e., \( D_t = s_t d^H_t + (1 - s_t) d^L_t \).
3.4 Private Money and Bank Runs

Proposition 21  Provided that fiat money is the unique medium of exchange, there exists a banking equilibrium where there is no bank run and

(i) young agents deposit all after-tax holdings of fiat money in the bank;

(ii) $d_t^H = 1$ for all type $H$ old agents;

(iii) $d_t^L = 0$ for all type $L$ old agents.

Proposition 22  Given $r^* = \frac{R}{1 + (R-1)s}$, suppose all other depositors withdraw fiat money in the morning, then it is optimal for a type $i = H, L$ agent to withdraw in the morning.

Propositions 21 – 22 show that a good equilibrium that achieves the first-best outcome coexists with a bad (Pareto-inferior) equilibrium with bank runs. Beliefs about these equilibria are self-fulfilling. Which equilibrium arises will depend on the confidence level of the economy.

3.4 Private Money and Bank Runs

In this section, we consider the same environment as before, only that now demand deposits are also allowed to circulate as a medium of exchange. When an agent deposits a unit of fiat money in period $t - 1$, the bank issues him a banknote to be redeemed in period $t$. Any bearer of the banknote can redeem it at the bank for fiat money at the promised rates $(R_{1,t}, R_{2,t})$ given by (3.35) and (3.36). To finance consumption, old agents can redeem private money for fiat money and use fiat money to buy goods. Alternatively, they can use private money to buy goods directly. Therefore, in the morning of $t$, there will be two markets, where goods are respectively traded for fiat money and private money. Note
that a private money market does not arise in the afternoon because the bank redeems its outstanding private money before any afternoon market opens. This will become clear in the following section, where detailed timing is described. As before, we focus on symmetric equilibria where (i) all agents choose to accept the demand deposit contract; (ii) agents of the same type apply the same strategies; (iii) markets are all competitive. Note that now the aggregate state $s_t$ is stochastic.

### 3.4.1 Timing

Let $Q_t$ represent the price of private money for goods in the morning of $t$. The timing of events is summarized as follows for a representative generation $t$ of agents:

1. At the start of period $t$, young agents are born. They choose the amounts of goods to supply to the morning markets (fiat money and private money, respectively) and to the afternoon market (fiat money). Young agents also choose the fraction of private money holdings (if any) to be redeemed in the morning.

2. The bank starts morning redemption and markets open in the meantime.

3. In the afternoon of $t$, young agents redeem the remaining private money (if any) at the bank; young agents sell goods in the afternoon market for fiat money.

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18 Unredeemed (i.e., expired) private money does not circulate in future periods. That is, in any period $t$, private money issued prior to period $t - 1$ will not be accepted as a medium of exchange. This is because fiat money dominates expired private money for two reasons. On one hand, fiat money can be used to purchase bonds, which offers a non-negative net return. On the other hand, banks accept fiat money as deposits.
4. By the end of $t$, young agents pay taxes $T_t$ and decide whether to deposit fiat money income (from redemption of private money and from proceeds of selling goods) in the bank.

5. At the start of $t + 1$, old agents of generation $t$ observe their type $i = H, L$; old agents choose the fraction $d_i^t$ of private money holdings to be redeemed in the morning, depending on type. In the meantime, morning markets open. Old agents trade fiat money and unredeemed private money respectively for young agents’ goods.

6. In the afternoon of $t + 1$, nominal bonds mature. The bank liquidates assets. Young and old agents who are holding private money demand redemption at the bank.

7. The afternoon market opens where old agents trade fiat money for young agents’ goods.

### 3.4.2 Equilibrium

We will focus on the banking equilibrium where young agents deposit all the after-tax fiat money holdings in the bank. Later we will prove that there is no profitable individual deviation from such a strategy.

#### The type $H$ old agent’s problem

Let $m_t$ represent a young agent’s after-tax fiat money holdings at the end of $t$. By depositing all fiat money in the bank, $m_t$ is also an old agent’s total private money holdings at the beginning of morning $t + 1$. As mentioned before, there are two ways for a type $H$
old agent to benefit from private money. First, the agent can use private money to purchase goods directly. Second, the agent can redeem private money for fiat money and use the latter to purchase goods.

At the beginning of period $t$, given prices $(P_{1,t}, P_{2,t}, Q_t)$, contracts $(R_{1,t}, R_{2,t})$ and expectation $D_t$, a type $H$ old agent chooses the fraction of private money holdings to be redeemed in the morning, $d_t^H$, to maximize his expected utility:

$$W^H(m_{t-1}) = \max_{d_t^H} \left\{ \Delta_t U \left[ \frac{rd_t^H m_{t-1}}{P_{1,t}} + \frac{(1-d_t^H)m_{t-1}}{Q_t} \right] + (1 - \Delta_t) U \left( \frac{m_{t-1}}{Q_t} \right) \right\} ,$$

(3.37)

where $\Delta_t$ is the probability of the agent being served for redemption. If the agent successfully gets served for redemption, he spends fiat money $rd_t^H m_{t-1}$ and the remaining private money holdings $(1 - d_t^H) m_{t-1}$ on goods. If the agent is not served for redemption, he uses all his private money $m_{t-1}$ to buy goods. The first-order condition to the above maximization problem is given by:

$$\Delta_t U' \left[ \frac{rd_t^H m_{t-1}}{P_{1,t}} + \frac{(1-d_t^H)m_{t-1}}{Q_t} \right] m_{t-1} \left( \frac{r}{P_{1,t}} - \frac{1}{Q_t} \right) > 0, \quad \text{if} \quad d_t^H = 1 \quad \text{if} \quad d_t^H \in [0,1] \quad \text{if} \quad d_t^H = 0 .$$

(3.38)

**The type $L$ old agent’s problem**

To maximize expected utility, a type $L$ agent of generation $t - 1$ chooses the fraction of private money to be redeemed in the morning, $d_t^L$. With probability $\Delta_t$, he will be served for redemption. Then he redeems the remaining $(1 - d_t^L)m_{t-1}$ units of private money in the afternoon, which yields him $R_{2,t}(1 - d_t^L)m_{t-1}$ units of fiat money. The agent spends all his fiat money (received in the morning and the afternoon) in the afternoon goods market.
With probability \(1 - \Delta_t\), the agent will not be served for morning redemption. In this case, the agent redeems all his private money in the afternoon. Note that the type \(L\) old agents do not participate in morning markets because goods are non-storable.

Taking prices \((P_{1,t}, P_{2,t}, Q_t)\), contracts \((R_{1,t}, R_{2,t})\) and expectation \(D_t\) as given, a type \(L\) old agent’s problem is

\[
W^L_t(m_{t-1}) = \max_{d_t} \left\{ \Delta_t U \left[ \frac{rd_t^L m_{t-1} + R_{2,t}(1 - d_t^L)m_{t-1}}{P_{2,t}} \right] + (1 - \Delta_t) U \left( \frac{R_{2,t}m_{t-1}}{P_{2,t}} \right) \right\}.
\]

The first-order condition to the above maximization problem is given by

\[
\Delta_t U' \left[ \frac{rd_t^L m_{t-1} + R_{2,t}(1 - d_t^L)m_{t-1}}{P_{2,t}} \right] m_{t-1} = 0, \quad \text{if} \quad d^L_t = 1;
\]

\[
\Delta_t U' \left[ \frac{rd_t^L m_{t-1} + R_{2,t}(1 - d_t^L)m_{t-1}}{P_{2,t}} \right] m_{t-1} = 0, \quad \text{if} \quad d^L_t \in [0, 1];
\]

\[
\Delta_t U' \left[ \frac{rd_t^L m_{t-1} + R_{2,t}(1 - d_t^L)m_{t-1}}{P_{2,t}} \right] m_{t-1} < 0, \quad \text{if} \quad d^L_t = 0.
\]

**The young agent’s problem**

Let \(e_t^f \in [0, e_t]\) denote a young agent’s supply of goods to the morning fiat money market. Denote \(\pi_t \in [0, 1]\) as the fraction of private money holdings that the agent redeems in the morning. Taking prices \((P_{1,t}, P_{2,t}, Q_t)\), contracts \((R_{1,t}, R_{2,t})\) and expectation \(D_t\) as given, a young agent of generation \(t\) seeks to maximize the expected lifetime utility:

\[
\max_{(e_t^f, e_t, \pi)} E_t \left[ s_{t+1} W^H_t (m_t) + (1 - s_{t+1}) W^L_t (m_t) \right].
\]

The expectation is taken over \(s\) and

\[
m_t = P_{1,t} e_t^f + Q_t \left( e_t - e_t^f \right) \left[ \pi_t \Delta_t r + (1 - \pi_t \Delta_t) R_{2,t} \right] + P_{2,t} (1 - e_t) - T_t,
\]
where \( R_{2,t} = \max \left\{ 0, \frac{1-\lambda D_t}{1-\lambda} \right\} \). Recall that \( \Delta_t \) is the probability of the agent being served for redemption. The young agent sells \( e^f_t \) units of goods in the morning fiat money market, \( e_t - e^f_t \) in the morning private money market and the rest \( 1 - e_t \) in the afternoon fiat money market. Young agents can only redeem private money either in the morning or in the afternoon, which earns an expected rate of return \( \pi_t \Delta_t r + (1 - \pi_t \Delta_t) R_{2,t} \). If the total demand of redemption is no more than the maximal liquidation value of the bank, i.e., \( r D_t \leq 1 \), then the agent will be served with probability one and receive redemption rate \( r \). Otherwise, the agent is served with probability \( \frac{1}{r D_t} \). Thus

\[
\Delta_t = \min \left\{ 1, \frac{1}{r D_t} \right\}.
\] (3.42)

By (3.37) and (3.39), it is straightforward that both \( W^H \) and \( W^L \) are strictly increasing in \( m_t \). Therefore, the problem in (3.41) is simplified to maximize the expected money income:

\[
\max_{(e^f_t, e_t, \pi)} P_{1,t} e^f_t + Q_t \left( e_t - e^f_t \right) \left[ \pi_t \Delta_t r + (1 - \pi_t \Delta_t) R_{2,t} \right] + P_{2,t} (1 - e_t).
\]

\footnote{When production is costly, agents face a trade-off between marginal cost and marginal benefit of holding money. Agents choose the output level to maximize the expected value of money holdings. Since production is costless in this model, the young agent’s utility-maximizing problem is equivalent to maximizing money holdings. Nevertheless, it can be shown that the main results of this paper also hold when costly production is introduced.}
The optimal solutions to the above problem are given by:

\[
\pi_1^*\begin{cases} 
= 1, & \text{if } r > R_{2,t} \\
\in [0, 1], & \text{if } r = R_{2,t} \\
= 0, & \text{if } r < R_{2,t}
\end{cases}
\]

(3.43)

\[
e_1^*\begin{cases} 
= e_t, & \text{if } P_{1,t} > Q_t R_t \\
\in [0, e_t], & \text{if } P_{1,t} = Q_t R_t \\
= 0, & \text{if } P_{1,t} < Q_t R_t
\end{cases}
\]

(3.44)

\[
e_2^*\begin{cases} 
= 1, & \text{if } Q_t R_t > P_{2,t} \\
\in [0, 1], & \text{if } Q_t R_t = P_{2,t} \\
= 0, & \text{if } Q_t R_t < P_{2,t}
\end{cases}
\]

(3.45)

where

\[
\tilde{R}_t \equiv \pi_1^* \Delta_t r + (1 - \pi_1^* \Delta_t) R_{2,t}.
\]

(3.46)

**Equilibrium Definition**

**Definition 7** A symmetric banking equilibrium with coexistence of fiat money and private money consists of the demand deposit contract \((R_{1,t}(r), R_{2,t}(r))\), prices \((P_{1,t}, P_{2,t}, Q_t)\), individual choices \((e_1^f, e_t, \pi_t, m_t, d_t^H, d_t^L)\) and aggregate variables \((E_t^f, E_t, D_t, M_t, s_t)\) such that

(i) given \((R_{1,t}, R_{2,t}, P_{1,t}, P_{2,t}, Q_t, D_t)\), all individuals choose quantities and strategies to maximize expected utility;

(ii) the bank chooses \(r\) to maximize the expected utility of a depositor;

(iii) consistency: \(e_t = E_t^f, e_t = E_t, D_t = s_t d_t^H + (1 - d_t^H) s_t \pi_t + (1 - s_t) d_t^L, m_t = M_t;\)

(iv) \(T_t = M_{t-1} \max [0, (\bar{R} - 1)(1 - r D_t)];\)

(v) all markets clear.
3.4 Private Money and Bank Runs

The above definition is mostly self-explanatory. Condition (iii) characterizes the consistency between individual choices and their aggregate counterparts. Note particularly that the aggregate fraction of withdrawals in the morning, $D_t$, is equal to the sum of the fractions of withdrawals by type $H$ old agents $s_t d_t^H$, by young agents who have traded goods for private money $\left(1 - d_t^H\right) s_t\pi_t$ and by type $L$ old agents $\left(1 - s_t\right) d_t^L$. Condition (iv) specifies that the government aims to maintain a constant end-of-period money supply. The government collects no tax if the demand for morning redemption forces the bank to liquidate all bonds early, i.e., $rD_t \geq 1$. Otherwise, the amount of the lump-sum tax is equal to the positive net returns on bonds, $(R - 1) (1 - rD_t) M_{t-1}$.

**Characterization of Equilibrium**

**Proposition 23** Provided that $0 \leq r < 1$, there exists a unique banking equilibrium where no one redeems private money for fiat money in the morning, i.e., $d_i = \pi_t = 0$ for all young agents and type $i = H, L$ old agents. The unique equilibrium achieves the first-best outcome, i.e., $c^i = 1$ for all type $i = H, L$ agents.

**Proposition 24** Provided that $r > 1$, there always exists a self-fulfilling bank run equilibrium.

According to Proposition 23, when $r^* < 1$ there exists a unique equilibrium and it achieves the first-best outcome. In this case, bank runs never occur. In contrast, Proposition 24 shows that there always exists a self-fulfilling bank run equilibrium provided that $r^* > 1$. With pessimistic beliefs ($D_t = 1$), young agents will not value private money because they
expect that the bank will have no remaining assets for redemption. In the meantime, type \(L\) old agents redeem private money for fiat money even if they do not have pressing needs for consumption. By doing so, they try to get hold of fiat money before the bank runs completely out of assets.

If \(r^* < 1\), however, type \(L\) agents do not panic over any given belief of \(D_t\) in any aggregate state \(s_t\). With \(r^* < 1\), the bank is essentially imposing transaction fees on withdrawal demands. This guarantees that the bank’s assets will never be depleted, i.e., \(r^* D_t < 1\) for any given \(D_t\). Therefore, agents do not panic over any volume of redemption demand. Furthermore, since \(r^* < 1\), redemption in the morning offers a strictly lower return than redemption in the afternoon, i.e., \(R_{1,t} = r^* < \overline{R} \leq R_{2,t}\). Therefore, a type \(L\) old agent, or any young agent who has sold goods for private money, has no incentive to redeem private money in the morning. Moreover, none of the type \(H\) agents has the incentive to redeem private money in the morning because redemption is costly by \(r^* < 1\). They are better off buying goods with private money, which is fully supported by the demand deposit contract \((R_{2,t} \geq \overline{R})\). As a result, only private money is traded in the morning.

Since Proposition 23 applies to any aggregate state \(s_t\), a contract with \(r^* < 1\) guarantees the first-best outcome for any realization of \(s_t\). Therefore, it dominates any contract with \(r^* > 1\), which is plagued by potential bank runs. Through competitive banking, a bank optimally offers the contract that maximizes agents’ expected utilities. Hence follows the corollary:
3.5 Private Money and Fiat Money

Recall that \( r < 1 \) cannot be an equilibrium bank offer if agents are not allowed to trade demand deposits for goods. The equilibrium bank offer is given by 

\[
r^* = \frac{\pi}{1-s+t-R} > 1,
\]

which provides the optimal risk-sharing. Intuitively, this offer helps buffer the liquidity shock by injecting more liquidity (money) when needed. Nevertheless, it is also \( r^* > 1 \) that creates the vulnerability to bank runs. The demand deposit contract becomes inherently unstable. On one hand, the mechanism is designed to provide liquidity for individual agents. On the other hand, the mechanism itself has inherent liquidity problems in that it does not have enough assets to serve if all depositors demand early redemption.

The inherent instability disappears if private money is allowed. The bank offers \( r^* < 1 \), essentially charging transaction fees to discourage early redemption. This effectively prevents the depletion of assets due to panicking withdrawals. Furthermore, offering \( r^* < 1 \) does not compromise risk-sharing as it does when fiat money is the only medium of exchange. Now the bank can conveniently provide liquidity through circulation of private money. As a result, agents no longer rely on fiat money to purchase consumption goods in that private money is just as good a medium of exchange.

**Corollary 25**  With coexistence of fiat money and private money, the banking equilibrium is unique and the optimal demand deposit contract offers \( r^* \in [0,1) \). The unique equilibrium achieves the first-best outcome.
Another striking feature of the equilibrium with private money is that it is robust to aggregate uncertainty. The demand deposit contract delivers the first-best outcome for any realization of the stochastic aggregate state. Recall that when private money is restricted, preventing bank runs requires government intervention. With aggregate uncertainty, it becomes more problematic in that intervention itself may be costly. In contrast, once private money is allowed in trades, bank runs are no longer an issue. Achieving the first-best outcome requires no intervention whatsoever.

3.6 Conclusion

We have built a simple banking model with micro-foundations of money. There are overlapping generations of agents with idiosyncratic liquidity risks. Banks offer demand deposit contracts and invest deposits of fiat money in nominal bonds. When fiat money is the only medium of exchange permitted, a bank run equilibrium exists along with a "good" equilibrium where the first-best outcome is achieved. The optimal risk-sharing requires a gross rate of return $r > 1$ on early withdrawals. As a result, agents who face liquidity shocks can have more money to spend on consumption goods by depositing in the bank ex ante. Nevertheless, the mechanism is vulnerable to bank runs in that the bank does not have enough assets to honour $r > 1$ should all agents decide to withdraw early.

In contrast, when private money (i.e., banknotes) is allowed to circulate, banks no longer rely on $r > 1$ to provide liquidity in need. Agents in need of liquidity can simply use private money to buy goods instead of redeeming private money for fiat money at the bank. Moreover, an offer of $r < 1$ can help prevent bank runs. The bank is essentially
charging transaction fees on early withdrawals of fiat money. This guarantees a positive amount of residual bank assets for any volume of early demand of withdrawals. In effect, agents do not form any panic beliefs. *Ex ante*, to choose from the demand deposit contracts, agents are aware of the potential bank runs associated with any contract offering \( r > 1 \). Thus agents would accept any contract with \( r < 1 \) rather than those with \( r > 1 \). Through competitive banking, it is optimal for a bank to offer \( r < 1 \).

Consequently, in the unique banking equilibrium with private money, no one demands early withdrawals of fiat money and agents in need of liquidity use private money to finance consumption. This result is robust to aggregate uncertainty. The economy manages to eliminate bank runs and achieves the first-best outcome without having to resort to any government intervention.

Finally, our model is a simple mechanism of money and banking. For future research, it will be interesting to embed the mechanism in environments with more sophisticated banking activities or alternative monetary environments with trading frictions (e.g., Shi [1997] and Lagos and Wright [2005]), so as to study other relevant issues on money and banking.
References


Sun, H., 2007, "Banking, Inside Money and Outside Money," Queen’s University, manuscript.


Appendix A
Appendix to Chapter 1

Proof of Proposition 2  Consider the borrower reports $\bar{y} \not\in \Omega$ so that monitoring does not occur. It follows that the repayment level must be the same for all $y \in \Omega$. Otherwise, suppose $z(y_1) < z(y_2)$ for any $y_1, y_2 \in \Omega$ and $y_1 \neq y_2$. Then the borrower will always report $y_1$ instead of $y_2$. Hence, incentive compatibility requires that $z(y) = x$ for $y \in \Omega$ and $z(y) < x$ for $y \in \Omega$, where $x$ is a constant. Moreover, repayment schedule must be feasible, that is, $0 \leq z(y) \leq y$. Together, we need $0 \leq z(y) \leq y < x$. Now it remains to prove that it is optimal to have $z(y) = y$ for $y \in \Omega$. Define $\Omega_1 = \{y : 0 \leq y < x\}$ and $\Omega_2 = \{y : 0 \leq z(y) < x\}$ for any function $z : [0, 1] \rightarrow [0, y]$. Obviously, $\Omega_1 \subseteq \Omega_2$. Substituting the constraint into the objective, (1.2) becomes

$$V_1 = \max_{c} \left\{ \frac{\bar{y}R}{1 + R} + \varepsilon E(y) - \varepsilon R - \varepsilon \mu \int_{\Omega} h(y) \, dy \right\}.$$ 

For any given $R$ that the bank chooses to offer on banknotes, the bank’s problem remains to minimize $\int_{\Omega_1} h(y) \, dy$. By $\Omega_1 \subseteq \Omega_2$, it follows that $\int_{\Omega_1} h(y) \, dy \leq \int_{\Omega_2} h(y) \, dy$. Therefore, it is optimal to have $\Omega = \{y : 0 \leq y < x\}$ and $z(y) = y$ for $y \in \Omega$.

Proof of Propositions 3-4  Condition (i) of Proposition 3 is by Proposition 1. Note that $t(y) \geq 0$ implies that $T(s) \geq 0$ for all $s$. Define $\phi = \left[\int_0^1 D(s) f(s) \, ds\right] / \left[\int_0^1 T(s) f(s) \, ds\right]$. Suppose $\frac{D(s)}{T(s)} \leq \phi$, with strict inequality for some $s$. It follows that

$$\phi = \frac{\int_0^1 D(s) f(s) \, ds}{\int_0^1 T(s) f(s) \, ds} < \frac{\phi \int_0^1 T(s) f(s) \, ds}{\int_0^1 T(s) f(s) \, ds} = \phi,$$
which is an obvious contradiction. Therefore, (1.5) implies that \( \frac{D(s)}{T(s)} = \phi \) for all \( s \), which proved Proposition 4. By condition (i) of Proposition 3, it follows that \( D(s) \neq D(s') \) for any given \( s \neq s' \in [0,1] \), which guarantees that condition (ii) of Proposition 1 holds. To guarantee the expected returns on banknotes are \( R(x) \) for given \( x \),\[ \int_{0}^{1} [D(s) + T(s)] f(s) \, ds = R(x) \]. Thus condition (i) of Proposition 3 holds by Proposition 4.

**Proof of Proposition 5**  
Assuming interior solutions, the first-order condition of (1.3) is:

\[
\frac{\bar{y} R'(x)}{[1 + R(x)]^2} - \varepsilon [1 - H(x)] = 0. \tag{A.1}
\]

Note that the left-hand side is exactly \( \Psi_1(x) \). Derive \( \Psi_1'(x) = \bar{y} \frac{R'(x)[1 + R(x) - 2|R(x)|^2]}{[1 + R(x)]^3} + \varepsilon h(x) \). By \( h(y) = \int_{y}^{1} f(s) g \left( \frac{y}{s} \right) \frac{1}{s} \, ds \), we have \( h'(y) = -f(y) g(1) \frac{1}{y} + \int_{y}^{1} f(s) g' \left( \frac{y}{s} \right) \frac{1}{s} \, ds \).

It follows that \( h(0) = g(0) \int_{0}^{1} f(s) \frac{1}{s} \, ds \), \( h(1) = 0 \) and \( h'(1) = -f(1) g(1) \). Then it is straightforward to derive from (1.4) that \( R(0) = 0 \), \( R'(0) = 1 - \mu h(0) \), \( R(1) = E(y) - \mu \), \( R'(1) = 0 \) and \( R''(1) = \mu f(1) g(1) \). Therefore, \( \Psi_1(0) = \bar{y} \left[ 1 - \mu h(0) \right] - \varepsilon \), \( \Psi_1(1) = 0 \), \( \Psi_1'(1) = \frac{\bar{y} \mu f(1) g(1)}{[1 + E(y) - \mu]^2} > 0 \). By continuity, if \( \Psi_1(0) > 0 \), there exist at least one \( x \in (0,1) \) such that \( \Psi_1(x) = 0 \) and \( \Psi_1'(x) < 0 \). Assumption 1 ensures that \( \Psi_1(0) > 0 \) holds. Since \( \Psi_1(0) > 0, \Psi_1(1) = 0, \Psi_1'(1) > 0 \) and \( \Psi_1''(x) > 0 \) for all \( x \in [0,1] \) by Assumption 2, there exists a unique \( x^*_1 \in (0,1) \) that satisfies \( \Psi_1(x) = 0 \) and \( \Psi_1'(x) < 0 \).

**Proof of Proposition 7**  
The proof is analogous to the proof of Proposition 2. For any given \( R_j \) that bank \( j \) chooses to offer, the problem remains to minimize \( \int_{\Omega_j} h(y_j) \, dy_j \). Refer to the proof of Proposition 2 for details.
Appendix A  Appendix to Chapter 1

Proof of Propositions 8-9  Define the left-hand side of (1.9) as $LHS(x)$. Then

$$LHS'(x) = \frac{(n-1)(1+R)-2[n+(n-1)R]}{n(1+R)^2} \bar{y}R'^2 + \frac{n+(n-1)R}{n(1+R)^2} \bar{y}R'' + \varepsilon h(x).$$

Previously in the proof of Proposition 4, it has been derived that $R(0) = 0$, $R'(0) = 1 - \mu h(0)$, $R(1) = E(y) - \mu$, $R'(1) = 0$ and $R''(1) = \mu f(1) g(1)$. It follows that $LHS(0) = \bar{y} [1 - \mu h(0)] - \varepsilon$, $LHS(1) = 0$, and $LHS'(1) = \frac{n+(n-1)[E(y)-\mu]}{n(1+[E(y)-\mu])^2} \bar{y} \mu f(1) g(1) > 0$ if $\mu < E(y) + \frac{n}{n-1}$. By continuity, if $LHS(0) > 0$, there exist at least one $x \in (0,1)$ such that $LHS(x) = 0$ and $LHS'(x) < 0$. Note that $LHS(0) = \Psi_1(0)$. Assumption 1 ensures $LHS(0) > 0$. Therefore, $x^*_n \in (0,1)$. Define set $X = \{ x \in (0,1) : R(x) > 0 \text{ and } R(x) > R(x') \text{ for all } x' \in [0, x) \}$. The following proves that $x^*_n \in X$ must be true. Obviously, $R(x^*_n) > 0$. Now suppose $R(x^*_n) \leq R(x')$ for some $x' \in [0, x^*_n)$. (i) If $R(x^*) = R(x')$, then $V_n|_{x=x'} - V_n|_{x=x^*_n} > 0$ because $\varepsilon \int_x^{1} (y_j - x) h(y_j) dy_j$ is strictly decreasing in $x$. Hence $x^*_n$ cannot be the optimal choice, which is a contradiction. (ii) Consider $R(x^*_n) < R(x')$. Since $R(0) = 0$ and $x' < x^*_n$, continuity implies that there must exist some $x'' \in [0, x')$ such that $R(x'') = R(x^*_n)$.

By the same argument, $x^*_n$ cannot be the optimal choice, which is another contradiction. Therefore, it must be true that $R(x^*_n) > R(x')$ for all $x' \in [0, x^*_n]$ and that $x^*_n \in X$. Since $R(x_1) < R(x_2)$ for any $x_1 < x_2 \in X$, $x^*_n$ must be unique. This proved Proposition 8.

Note that it can be proven analogously for the one-bank case that $x^*_1 \in X$.

From (1.7),

$$V^*_n = \frac{\bar{y} R(x^*_n)}{1 + R(x^*_n)} + \varepsilon \int_{x^*_n}^{1} (y_j - x^*_n) h(y_j) dy_j.$$

The right-hand side is exactly the objective of $V_1$. Recall that $x^*_1$ is the maximizer of $V_1$ and
it satisfies (A.1), that is,
\[ \frac{\overline{y}R'(x)}{[1 + R(x)]^2} = \varepsilon [1 - H(x)]. \] (A.2)

By (1.9), \( x_n^* \) satisfies
\[ \frac{\overline{y}R'(x)}{[1 + R(x)]^2} = \frac{\varepsilon [1 - H(x)]}{1 + \frac{n-1}{n}R(x)} < \varepsilon [1 - H(x)]. \] (A.3)

The inequality holds for \( n > 1 \) because \( R(x_n^*) > 0 \). Therefore, \( x_n^* \neq x^* \), which implies \( V_n^* < V_1^* \). By (A.3), \( \Psi_1(x_n^*) < 0 \) where \( \Psi_1(x) \) is the left-hand side of (A.1). It has been shown in the proof of Proposition 4 that \( \Psi_1(0) > 0 \), \( \Psi_1(1) = 0 \), \( \Psi_1'(1) > 0 \) and \( \Psi_1''(x) > 0 \) for all \( x \in [0,1] \). It follows that \( x_n^* > x_1^* \). Hence, \( R(x_n^*) > R(x_1^*) \) because \( x_1^* \in X \) and \( x_n^* \in X \), which proved Proposition 9.

**Proof of Propositions 11-12**  The proof of Proposition 11 is analogous to the proof of Proposition 8. The main steps are to show (i) \( x_n^{**} \in (0,1) \) if \( \varepsilon < \overline{y} \left[ 1 - \mu g(0) \int_{0}^{1} f(s) \frac{1}{s} ds \right] \) and (ii) \( x_n^{**} \in X \) and \( x_n^{**} \) is unique, where \( X = \{ x \in (0,1) : R(x) > 0 \) and \( R(x) > R(x') \) for all \( x' \in [0,x) \} \). (Details of the proof are omitted for brevity.) From (1.10),
\[ V_n^{**} = \frac{\overline{y}R(x_n^{**})}{1 + R(x_n^{**})} + \varepsilon \int_{x_n^{**}}^{1} (y_j - x_n^{**}) h(y_j) dy_j. \]

Note that the right-hand side is exactly the objective of \( V_1 \). Recall that \( x_1^* \) maximizes \( V_1 \) and it satisfies (A.1), that is,
\[ \frac{\overline{y}R'(x)}{[1 + R(x)]^2} = \varepsilon [1 - H(x)]. \] (A.4)

By (1.11), \( x_n^{**} \) satisfies
\[ \frac{\overline{y}R'(x)}{[1 + R(x)]^2} = n \varepsilon [1 - H(x)] > \varepsilon [1 - H(x)]. \] (A.5)
The inequality holds for all \( n \geq 2 \). Therefore, \( x_n^{*} \neq x_1^{*} \) and hence \( V''_n < V'_1 \). By (A.5), \( \Psi_1(x_n^{*}) > 0 \) where \( \Psi_1(x) \) is the left-hand side of (A.1). It has been shown in the proof of Proposition 4 that \( \Psi_1(0) > 0, \Psi_1(1) = 0, \Psi'_1(1) > 0 \) and \( \Psi''_1(x) > 0 \) for all \( x \in [0,1] \). It follows that \( x_n^{*} < x_1^{*} \) and thus \( R(x_n^{*}) < R(x_1^{*}) \) because \( x_1^{*} \in X, x_n^{*} \in X \) and \( R(x_1) < R(x_2) \) for any \( x_1 < x_2 \in X \). By Proposition 9, \( x_n^{*} < x_1^{*} < x_n^{*} \) and \( R(x_n^{*}) < R(x_1^{*}) < R(x_n^{*}) \), which proved Proposition 12.

**Proof of Proposition 13** In the symmetric equilibrium, \( c_B^1 = c_C^2 = \frac{\gamma}{1 + R(x)} \). Hence,

\[
W(x) = \frac{\gamma R(x)}{1 + R(x)} + \epsilon \int_x^1 (y - x) h(y) \, dy + \frac{\gamma R(x)}{1 + R(x)} + \ln \frac{\gamma}{1 + R(x)} + R(x) = \frac{\gamma R(x)}{1 + R(x)} + \epsilon \int_x^1 (y - x) h(y) \, dy + \ln \gamma - \ln [1 + R(x)] + R(x).
\]

Suppose \( W(x) \) is maximized at some \( x \in (0,1) \). Then it must satisfy \( W'(x) = 0 \) and \( W''(x) < 0 \). Then the first-order condition is given by

\[
\left\{ \frac{\gamma}{[1 + R(x)]^2} + \frac{\gamma}{1 + R(x)} \right\} R'(x) - \epsilon [1 - H(x)] = 0. \tag{A.6}
\]

The left-hand side of the above equation is exactly \( \Psi_2(x) \). Analogous to the proof of Proposition 4, by Assumptions 1-2 it is straightforward to show that there exists a unique \( \hat{x} \in (0,1) \) such that \( W'(\hat{x}) = 0 \) and \( W''(\hat{x}) < 0 \). (A.6) implies that

\[
R'(\hat{x}) = \frac{\epsilon [1 - H(\hat{x})]}{\gamma + R(\hat{x}) [1 + R(\hat{x})]}.
\]

By (1.9),

\[
R'(x_n^{*}) = \frac{\epsilon [1 - H(x_n^{*})]}{\gamma [1 + \frac{n-1}{n} R(x_n^{*})]} > \frac{\epsilon [1 - H(x_n^{*})]}{\gamma + R(x_n^{*}) [1 + R(x_n^{*})]}.
\]
The inequality holds because $R(x_n^*) > 0$ and $0 < \bar{y} \leq 1$. The above implies $\Psi_2(x_n^*) > 0$.

By Assumption 1-2, $\Psi_2(0) > 0$, $\Psi_2(1) = 0$ and $\Psi''_2(x) > 0$ for all $x \in [0, 1]$. Therefore, it must be true that $x_n^* < \hat{x}$. Proposition 12 implies $x_n'^* < x_1^* < x_n^* < \hat{x}$. It follows that $W(x_n'^*) < W(x_1^*) < W(x_n^*) < W(\hat{x})$, which proved Proposition 13.
Appendix B
Appendix to Chapter 2

Proof of Propositions 14-15  Consider the contract problem (2.13) subject to constraints (2.16)-(2.21) and (2.25)-(2.26). Since $u$ is strictly increasing in consumption, constraint (2.26) implies that

$$c_{+1}^B (y, s) > c_{+1}^B (\bar{y}, s), \quad \forall \ s, \ \forall \ \bar{y} < y. \quad \text{(B.1)}$$

Given $\mu_{+1}$, let $\{\widehat{c}_{+1} (y, s, v_{+1}; \mu_{+1}), \widehat{v}_{+2} (y, s, v_{+1}; \mu_{+1}), \widehat{c}_{+1} (y, s; \mu_{+1})\}$ be the optimal policy functions for the banking contract with outside money. That is, they maximize the objective of (2.13) subject to constraints (2.16)-(2.23). Note that $\{\widehat{c}_{+1}, \widehat{v}_{+2}, \widehat{c}_{+1}^B\}$ also satisfy constraints (2.25)-(2.26). That is,

$$u [\widehat{c}_{+1} (y, s, v_{+1})] + \beta \widehat{v}_{+2} (y, s, v_{+1})$$

$$\geq u [\widehat{c}_{+1} (\bar{y}, s, v_{+1}) + (y - \bar{y}) \frac{p^0 (s)}{p_{+1}^b}] + \beta \widehat{v}_{+2} (\bar{y}, s, v_{+1})$$

$$> u [\widehat{c}_{+1} (\bar{y}, s, v_{+1}) + s (y - \bar{y})] + \beta \widehat{v}_{+2} (\bar{y}, s, v_{+1}), \quad \forall \ s, v_{+1}, \ \forall \ \bar{y} < y$$

$$\text{B.2}$$

$$u [\widehat{c}_{+1}^B (y, s)]$$

$$\geq u [\widehat{c}_{+1}^B (\bar{y}, s) + (y - \bar{y}) \frac{p^0 (s)}{p_{+1}^b}]$$

$$> u [\widehat{c}_{+1}^B (\bar{y}, s) + s (y - \bar{y})], \quad \forall \ s, \ \forall \ \bar{y} < y \quad \text{B.3}$$

The above two strict inequalities hold because $\varepsilon < E (s)$ and $s \in [0, 1]$. 
Appendix B  Appendix to Chapter 2

Now construct the following policy function such that (2.26) holds:

$$\tilde{c}_{+1}^B (y, s; \mu_{+1}) = \begin{cases} 
\tilde{c}_{+1}^B (y, s; \mu_{+1}) + \delta_y, & \text{if } y \leq \frac{1}{2}s \\
\tilde{c}_{+1}^B (y, s; \mu_{+1}) - \Delta_y, & \text{if } y > \frac{1}{2}s 
\end{cases}, \quad (B.4)$$

where $\delta_y$ and $\Delta_y$ are infinitely small positive numbers and satisfy $\delta_y g \left( \frac{y}{s} \right) = \Delta_{s-y} g \left( 1 - \frac{y}{s} \right)$ for $\frac{y}{s} \leq \frac{1}{2}$ and all $s$. Values of $\delta_y$ and $\Delta_{s-y}$ exist by the strict inequality of (B.3). In what follows, it will be proven that $\{\tilde{c}_{+1}^B (y, s; v_{+1}; \mu_{+1}), \tilde{v}_{+2} (y, s, v_{+1}; \mu_{+1}), \tilde{c}_{+1}^B (y, s; \mu_{+1})\}$ achieve a higher value of $W_{+1}^I (\mu_{+1})$ than $\{\tilde{c}_{+1}^B (y, s; v_{+1}; \mu_{+1}), \tilde{v}_{+2} (y, s, v_{+1}; \mu_{+1}), \tilde{c}_{+1}^B (y, s; \mu_{+1})\}$ do. First note that given $s$

$$\begin{align*}
\alpha \int_0^1 \tilde{c}_{+1}^B (y, s; \mu_{+1}) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\
= \alpha \int_0^{1/2} \left[ \tilde{c}_{+1}^B (y, s; \mu_{+1}) + \delta_y \right] g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\
+ \alpha \int_{1/2}^1 \left[ \tilde{c}_{+1}^B (y, s; \mu_{+1}) - \Delta_y \right] g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\

= \alpha \int_0^1 \tilde{c}_{+1}^B (y, s; \mu_{+1}) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) + \alpha \int_0^{1/2} \delta_y g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\
- \alpha \int_0^{1/2} \Delta_{s-y} g \left( \frac{s-y}{s} \right) d \left( \frac{y}{s} \right) \\

= \alpha \int_0^1 \tilde{c}_{+1}^B (y, s; \mu_{+1}) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) + \alpha \int_0^{1/2} \delta_y g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\
- \alpha \int_0^{1/2} \delta_y g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) \\
= \alpha \int_0^1 \tilde{c}_{+1}^B (y, s; \mu_{+1}) g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) 
\end{align*}$$
Therefore, $\widehat{c}_{i+1}(y, s; \mu_{i+1})$ and $\widehat{c}_{i+1}(y, s, v_{i+1}; \mu_{i+1})$ satisfy constraint (2.18). Rewrite the objective of (2.13) as the following:

$$W_{i+1}(\mu_{i+1}) = \max_{(c_{i+1}^{0, c_{i+1}})^{(u')}} \left\{ \int_0^1 \int_0^1 u \left[ \frac{\partial}{\partial s} \left( \frac{y}{s} \right) \right] \frac{y}{s} g \left( \frac{y}{s} \right) f(s) ds \right\}$$

(B.5)

Apply the first-order Taylor expansion on the first term of the above with $\widehat{c}_{i+1}(y, s; \mu_{i+1})$:

$$\int_0^1 \int_0^1 u \left[ \frac{\partial}{\partial s} \left( \frac{y}{s} \right) \right] \frac{y}{s} g \left( \frac{y}{s} \right) f(s) ds$$

$$= \int_0^1 \left\{ \int_0^{1/2} u \left[ \frac{\partial}{\partial s} \left( \frac{y}{s} \right) \right] \frac{y}{s} g \left( \frac{y}{s} \right) f(s) ds \right\}$$

$$= \int_0^1 \left\{ \int_0^{1/2} u \left[ \frac{\partial}{\partial s} \left( \frac{y}{s} \right) \right] \frac{y}{s} g \left( \frac{y}{s} \right) f(s) ds \right\}$$

(B.6)

The strict inequality holds because (B.1) implies that $u' \left[ \frac{\partial}{\partial s} \left( \frac{y}{s} \right) \right] - u' \left[ \frac{\partial}{\partial s} \left( \frac{s-y}{s} \right) \right] \geq 0$ for all $\frac{y}{s} \in \left[ 0, \frac{1}{2} \right]$, with an equality if and only if $\frac{y}{s} = \frac{1}{2}$. As has been established, $\{\widehat{c}_{i+1}, \widehat{v}_{i+2}, \widehat{c}_{i+1}^{B}\}$ satisfy constraints (2.16)-(2.21) and (2.25)-(2.26). Note that $W_{i+2}(\mu_{i+2})$
takes the same value for \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) and \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) because of the same policy function \( \tilde{v}_{+2} \). Thus the second term in (B.5) also takes the same value for \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) and \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \). The strict inequality in (B.6) shows that \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) achieve a higher value of \( W^I_{+1}(\mu+1) \) than \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) do. Therefore, \( \{ \tilde{c}_{+1}, \tilde{v}_{+2}, \tilde{c}_{+1}^B \} \) cannot be the optimal policy functions for the contract problem of banking with one-period inside money. It follows that \( W^I(\mu) > W^0(\mu) \) for any given \( \mu \) By (2.24), \( W^I_0 > W^0_0 \) for any given \( v_0 \) and \( \alpha \).

**Proof of Proposition 16** Given \( \alpha \), consider two optimal contracts with associated initial values and consumption streams of \( \{ \tilde{v}_0, \{ \tilde{c}_t \}_{t=0}^{\infty}, \tilde{W}_0, \{ \tilde{c}_t^B \}_{t=0}^{\infty} \} \) and \( \{ \tilde{v}_0, \{ \tilde{c}_t \}_{t=0}^{\infty}, \tilde{W}_0, \{ \tilde{c}_t^B \}_{t=0}^{\infty} \} \), respectively. Suppose \( \tilde{W}_0 \geq \tilde{W}_0 \) for any \( \tilde{v}_0 > \tilde{v}_0 \). This means the optimal contract \( \{ \tilde{v}_0, \{ \tilde{c}_t \}_{t=0}^{\infty}, \tilde{W}_0, \{ \tilde{c}_t^B \}_{t=0}^{\infty} \} \) achieves higher lifetime utilities for both bankers and non-bankers than \( \{ \tilde{v}_0, \{ \tilde{c}_t \}_{t=0}^{\infty}, \tilde{W}_0, \{ \tilde{c}_t^B \}_{t=0}^{\infty} \} \) does. Hence the latter cannot be an optimal contract given \( \alpha \), which is a contradiction. Therefore, it must be true that \( W_0 \) is strictly decreasing in \( v_0 \) given \( \alpha \). Given \( \alpha \), let \( v^o_0 \) and \( v^I_0 \) be the solutions to (2.24), respectively for the outside money arrangement and the one-period inside money arrangement. By Proposition 15, we have \( v^o_0 = W^o_0(v^o_0; \alpha) < W^I_0(v^I_0; \alpha) \). Obviously, \( v^o_0 \neq v^I_0 \). Suppose \( v^o_0 > v^I_0 \), then it follows that \( W^o_0(v^o_0; \alpha) = v^o_0 > v^I_0 = W^I_0(v^I_0; \alpha) \). This is a contradiction because \( W_0 \) is strictly decreasing in \( v_0 \). Thus, \( v^o_0 < v^I_0 \) all given \( \alpha \). Then it must be that \( v^{os}_0 < v^{is}_0 \) because \( v^{os}_0 = \max_{\alpha \in [0,1]} v_0(\alpha) \).

With outside money, the contract design problem is given by (2.13) subject to constraints (2.16)-(2.23). It is obvious that \( \varepsilon \) does not enter into the problem at all. Therefore,
$v_0^{\alpha}$ is independent of $\varepsilon$. With OPIM, $\varepsilon$ only enters into the incentive compatibility constraints. Consider any $\varepsilon_1 < \varepsilon_2$. The values of the right-hand sides of the IC constraints are smaller with $\varepsilon_1$ than with $\varepsilon_2$. Analogous to the proof of Proposition 14, one can construct alternative policy functions that achieve a higher value for the problem with $\varepsilon_1$ than the optimal policy functions for the problem with $\varepsilon_2$. (Details are omitted for brevity.) Then it follows that $v_0^{I_{\alpha}}$ strictly increases as $\varepsilon$ decreases. When $\varepsilon \to 0$, the incentives to default diminish and the optimal contract approaches the unconstrained first-best contract.

That is, $c_t (y_{t-1}, s_{t-1}, v_t) \to \bar{y}$, $c_t^B (y_{t-1}, s_{t-1}) \to \bar{y}$ and $v_{t+1} (y_{t-1}, s_{t-1}, v_t) \to \frac{u(\bar{y})}{1-\beta}$ for all $(y_{t-1}, s_{t-1}, v_t)$, which concludes the proof.

**Proof of Proposition 17** It is straightforward that $W_{t+1}^I (\mu_{t+1})$ is equivalent to $W_{t+1}^\kappa (\mu_{t+1}, t_t)$ with $t_t = 1$ for all $t$. Given $\mu_{t+1}$, let \( \{ \bar{c}_{+1} (y, s, v_{+1}; \mu_{+1}, t) ; \bar{v}_{+2} (y, s, v_{+1}; \mu_{+1}, t) ; \bar{c}_{+1}^B (y, s; \mu_{+1}, t) \} \) be the optimal policy functions for the banking contract with $\kappa$-period inside money. As the case with outside money, we have $\gamma^* = 0$ for the problem given by (2.30) and hence $\Phi_2 > \Phi_1$. It follows that when $t = 0$,

\[
\begin{align*}
&u [\bar{c}_{+1} (y, s, v_{+1})] + \beta \bar{v}_{+2} (y, s, v_{+1}) \\
\geq &\ u [\bar{c}_{+1} (\bar{y}, s, v_{+1}) + \Phi_2] + \beta \bar{v}_{+2} (\bar{y}, s, v_{+1}) \\
> &\ u [\bar{c}_{+1} (\bar{y}, s, v_{+1}) + \Phi_1] + \beta \bar{v}_{+2} (\bar{y}, s, v_{+1}) \\
&\forall s, v_{+1}, \forall \bar{y} < y
\end{align*}
\]
Therefore, \( \{ \bar{\tau}_{t+1}, \bar{\nu}_{t+1}, \bar{\tau}^B_{t+1} \} \) satisfy all constraints (2.16)-(2.21) and (2.25)-(2.26). It follows that \( W^I_{t+1} (\mu_{t+1}) \geq W^\kappa_{t+1} (\mu_{t+1}, \iota_{t}) \) for any given \( (\mu_{t+1}, \iota_{t}) \). Analogous to the construction in (B.4), one can find other policy functions that achieve a higher value for \( W^I_{t+1} (\mu_{t+1}) \) than \( \{ \bar{c}_{t+1}, \bar{\nu}_{t+2}, \bar{\tau}^B_{t+1} \} \) do, which implies \( W^I_{t+1} (\mu_{t+1}) > W^\kappa_{t+1} (\mu_{t+1}, \iota_{t}) \) for any given \( (\mu_{t+1}, \iota_{t}) \). (Details are omitted for brevity.) This in turn implies that \( W^0_I > W^\kappa_0 \) for any given \( v_0 \) and \( \alpha \). Analogous to the proof of Proposition 3, one can show that \( v^*_0 < v^{I*}_0 \).

By the same token, \( W^0_{t+1} (\mu_{t+1}) \) is equivalent to \( W^\kappa_{t+1} (\mu_{t+1}, \iota_{t}) \) where \( \iota_{t} = 0 \) for all \( t \). Similarly, one can prove that \( W^0_{t+1} (\mu_{t+1}) < W^\kappa_{t+1} (\mu_{t+1}, \iota_{t}) \) for any given \( (\mu_{t+1}, \iota_{t}) \) and hence \( W^0_0 \leq W^\kappa_0 \) for any given \( v_0 \) and \( \alpha \). It follows that \( v^*_0 < v^{I*}_0 \).

**Proof of Proposition 18** Since constraints (2.14)-(2.15) are equivalent to constraints (2.31)-(2.32), the contract problem under co-circulation of inside money and outside money is exactly the same as under exclusive circulation of outside money. Hence \( v^{co*}_0 = v^{oa*}_0 \).

**Proof of Proposition 19** Plugging in the equilibrium prices, (2.33) becomes

\[
\begin{align*}
\gamma^*_t, \gamma^{B*}_t & \quad \left\{ \begin{array}{ll}
0, & \text{if } \pi < \frac{E(s)}{E(s_{t})} - 1 \\
[0,1], & \text{if } \pi = \frac{E(s)}{E(s_{t})} - 1 \\
1, & \text{if } \pi > \frac{E(s)}{E(s_{t})} - 1
\end{array} \right.
\end{align*}
\]
It is straightforward to show that $\gamma^*, \gamma^B = 0$ for any $s_t \in [0, 1]$ if $\pi \leq E(s)/\varepsilon - 1$. The contract design problem is the same as given by (2.13) subject to the same constraints as (2.16)-(2.21) and (2.31)-(2.32). Hence $v^0_{inf}$ is constant in $\pi$ and $v^0_{inf} = v^{co}_{0} = v^{0*}$ by Proposition 18. Provided that $\pi > E(s)/\varepsilon - 1$, then $\gamma_t^*, \gamma_t^B = 1$ if $s_t > E(s)/[\varepsilon (1 + \pi)]$ and $\gamma_t^*, \gamma_t^B = 0$ if $s_t \leq E(s)/[\varepsilon (1 + \pi)]$ for given $s_t$. Thus for high enough aggregate state, it becomes less profitable to default and carry outside money into the future. Now the recursive contract design problem is given by (2.13) subject to the same constraints as (2.16)-(2.21). Nevertheless, instead of constraints (2.14) and (2.15), the IC constraints become the following: for $s > E(s)/[\varepsilon (1 + \pi)]$,

$$
u \left[ c_{+1}(y, s, v_{+1}) \right] + \beta v_{+2}(y, s, v_{+1})$$

$$\geq \nu \left[ c_{+1}(\bar{y}, s, v_{+1}) + \Psi_1 \right] + \beta v_{+2}(\bar{y}, s, v_{+1}) \quad (B.7)$$

$$\forall \; s, v_{+1}, \; \forall \; \bar{y} < y$$

$$\nu \left[ c^B_{+1}(y, s) \right]$$

$$\geq \nu \left[ c^B_{+1}(\bar{y}, s) + \Psi_1 \right], \; \forall \; s, \; \forall \; \bar{y} < y \quad (B.8)$$

and for $s \leq E(s)/[\varepsilon (1 + \pi)]$,

$$\nu \left[ c_{+1}(y, s, v_{+1}) \right] + \beta v_{+2}(y, s, v_{+1})$$

$$\geq \nu \left[ c_{+1}(\bar{y}, s, v_{+1}) + \Psi_2 \right] + \beta v_{+2}(\bar{y}, s, v_{+1}) \quad (B.9)$$

$$\forall \; s, v_{+1}, \; \forall \; \bar{y} < y$$

$$\nu \left[ c^B_{+1}(y, s) \right]$$

$$\geq \nu \left[ c^B_{+1}(\bar{y}, s) + \Psi_2 \right], \; \forall \; s, \; \forall \; \bar{y} < y \quad (B.10)$$
where $\Psi_1 = \varepsilon (y - \bar{y})$ and $\Psi_2 = (y - \bar{y}) \frac{P_{t+1}}{P_{t+1}}$. Let $\pi = E(\varepsilon (1 + \pi))$. Then the objective of (2.13) can be rewritten as:

$$W_{inf+1}^{r} (\mu+1) = \max \left\{ \int_0^\pi \int_0^1 \{ u [c_{t+1}^R (y, s)] + \beta W_{inf+2}^r (\mu+2) \} g (\frac{y}{s}) d (\frac{y}{s}) f (s) ds \right\}.$$  

The above problem can be further decomposed into:

$$W_{inf+1}^{r} (\mu+1) = W_{inf+1}^{r} (\mu+1; \Psi_1) + W_{inf+1}^{r} (\mu+1; \Psi_2)$$  \hspace{1cm} (B.11)

where

$$W_{inf+1}^{r} (\mu+1; \Psi_1)$$  \hspace{1cm} (B.12)

$$= \max \left\{ \int_0^\pi \int_0^1 \{ u [c_{t+1}^R (y, s)] + \beta W_{inf+2}^r (\mu+2) \} g (\frac{y}{s}) d (\frac{y}{s}) f (s) ds \right\}$$  \hspace{1cm} (B.13)

s.t. (2.16)-(2.21) and (B.7)-(B.8)

and

$$W_{inf+1}^{r} (\mu+1; \Psi_2)$$  \hspace{1cm} (B.14)

$$= \max \left\{ \int_0^\pi \int_0^1 \{ u [c_{t+1}^R (y, s)] + \beta W_{inf+2}^r (\mu+2) \} g (\frac{y}{s}) d (\frac{y}{s}) f (s) ds \right\}$$  \hspace{1cm} (B.15)

s.t. (2.16)-(2.21) and (B.9)-(B.10)

Note that the problem in (B.12) is equivalent to the problem for the exclusive circulation of one-period inside money, except that the lower bound for $s$ is now $\pi$ instead of zero. Similarly, the problem of (B.14) is equivalent to the problem for circulation of outside money with a constant money supply, with the upper bound of $s$ being $\pi$ instead of one.
It is obvious that \( W_{+1}^{\inf} (\mu_{+1}) \rightarrow W_{+1}^I (\mu_{+1}) \) as \( \overline{s} \rightarrow 0 \) and \( W_{+1}^{\inf} (\mu_{+1}) \rightarrow W_{+1}^o (\mu_{+1}) = W_{+1}^{co} (\mu_{+1}) \) as \( \overline{s} \rightarrow 1 \) given any \( \mu_{+1} \).

Consider any \( s_1 < s_2 \). Given \( \mu_{+1} \), let \( \{ \tilde{c}_{+1} (y, s, v_{+1}; \mu_{+1}) , \tilde{v}_{+2} (y, s, v_{+1}; \mu_{+1}) , \tilde{c}_{+1}^B (y, s; \mu_{+1}) \} \) be the optimal policy functions for (B.11) given \( \overline{s} = s_2 \). It is straightforward to see that \( \{ \tilde{c}_{+1} , \tilde{v}_{+2} , \tilde{c}_{+1}^B \} \) also satisfy all the constraints for (B.11) given \( \overline{s} = s_1 \). Now construct the following policy function \( \tilde{c}_{+1}^B (y, s; \mu_{+1}) \) such that (B.8) and (B.10) hold:

\[
\tilde{c}_{+1}^B (y, s; \mu_{+1}) = \begin{cases} 
\tilde{c}_{+1}^B (y, s; \mu_{+1}) & \text{for } y \in [0, s], \\
\tilde{c}_{+1}^B (y, s; \mu_{+1}) + \delta_y & \text{for } y \in [0, \frac{1}{2}s], \\
\tilde{c}_{+1}^B (y, s; \mu_{+1}) - \Delta_y & \text{for } y \in (\frac{1}{2}s, s],
\end{cases}
\]

if \( s \in [0, s_1) \cup (s_2, 1] \)

where \( \delta_y \) and \( \Delta_y \) are infinitely small positive numbers and satisfy \( \delta_y g \left( \frac{y}{s} \right) = \Delta_s g \left( 1 - \frac{y}{s} \right) \) for \( y \in [0, \frac{1}{2}s] \) and \( s \in [s_1, s_2] \). Values of \( \delta_y \) and \( \Delta_y \) exist by the strict inequality of (B.8) given \( \tilde{c}_{+1}^B (y, s; \mu_{+1}) \) for \( s \in [s_1, s_2] \). Analogous to the proof of Proposition 14, it can be shown that given \( \overline{s} = s_1 \), \( \{ \tilde{c}_{+1} (y, s, v_{+1}; \mu_{+1}) , \tilde{v}_{+2} (y, s, v_{+1}; \mu_{+1}) , \tilde{c}_{+1}^B (y, s; \mu_{+1}) \} \) achieve a higher value of \( W_{+1}^{\inf} (\mu_{+1}) \) than \( \{ \tilde{c}_{+1} (y, s, v_{+1}; \mu_{+1}) , \tilde{v}_{+2} (y, s, v_{+1}; \mu_{+1}) , \tilde{c}_{+1}^B (y, s; \mu_{+1}) \} \) do. This implies that \( W_{+1}^{\inf} (\mu_{+1}) (\overline{s}) \) is strictly decreasing in \( \overline{s} \) for any given \( \mu_{+1} \). Hence by the same argument of the proof of Proposition 17, \( v_0^{\inf*} \) is strictly decreasing in \( \overline{s} \). Note that \( \overline{s} \) is strictly decreasing in \( \pi \). Thus \( v_0^{\inf*} \) is strictly increasing in \( \pi \).

**Proof of Proposition 20** Consider the following policy functions: \( \underline{c}_{+1} (y, s, v_{+1}; \mu_{+1}) = \underline{c}_{+1}^B (y, s; \mu_{+1}) \) for any given \( v_{+1} \) and \( \mu_{+1} \). Trivially, \( \underline{v}_{+2} (y, s, v_{+1}; \mu_{+1}) = \frac{1}{1 - \beta} E \left\{ u \left[ \underline{c}_{+1}^B (y, s; \mu_{+1}) \right] \right\} \).

Given \( \underline{c}_{+1} \) and \( \underline{v}_{+1} \), \( \underline{c}_{+1}^B \) solves the following maximization problem:

\[
W_{+1} = \max_{c_{+1}^B} \left\{ \frac{1}{1 - \beta} \int_0^1 \int_0^1 u \left[ c_{+1}^B (y, s) \right] g \left( \frac{y}{s} \right) d \left( \frac{y}{s} \right) f (s) ds \right\}
\]
subject to
\[ u \left[ c_{+1}^B (y, s) \right] \geq u \left[ c_{+1}^B (\tilde{y}, s) + \varepsilon (y - \tilde{y}) \right], \quad \forall \ s, \ \forall \ \tilde{y} < y \]
\[ \int_0^1 c_{+1}^B (y, s) \ g \left( \frac{y}{s} \right) \ d \left( \frac{y}{s} \right) = \tilde{y}, \quad \forall \ s \]

Since \( y = s\theta \), the above problem can be rewritten as
\[ W_{+1} = \max_{c_{+1}^B} \left\{ \frac{1}{1 - \beta} \int_0^1 \int_0^1 u \left[ c_{+1}^B (\theta, s) \right] \ g (\theta) \ d (\theta) \ f (s) \ ds \right\} \]
subject to
\[ u \left[ c_{+1}^B (\theta, s) \right] \geq u \left[ c_{+1}^B (\tilde{\theta}, s) + \varepsilon s (\theta - \tilde{\theta}) \right], \quad \forall \ s, \ \forall \ \tilde{\theta} < \theta \]
\[ \tilde{y} = \int_0^1 c_{+1}^B (\theta, s) \ g (\theta) \ d (\theta), \quad \forall \ s \]

It is straightforward to show that the solution to this problem \( c_{+1}^B \) exists and is unique.

The policy functions \( \{ c_{+1}, v_{+2}, c_{+1}^B \} \) imply that \( v_0 = W_0 (v_0) \). This is true for any \( \alpha \in [0, 1] \). By definition, the policy function \( c_{+1}^B \) is optimal given \( \{ c_{+1}, v_{+2} \} \) and hence \( v_0 \). But \( \{ c_{+1}, v_{+2} \} \) may not be the optimal policy functions to achieve \( v_0 \). If they are optimal, then it is trivial that the banking equilibrium exists and is unique. Suppose they are not optimal.

Let \( \{ c_t \}_{t=0}^{\infty} \) be the sequence of consumptions achieved by \( c_0 = \overline{y} \) and policy functions \( \{ c_{+1}, v_{+2} \} \) for all \( t \geq 0 \). Since the goal of the bank is to maximize the lifetime expected utility of a banker, it chooses functions \( c_{+1} \) and \( v_{+2} \) to minimize the expected value of the total resources it allocates to the non-bankers for any promised value \( v_0 \). Since \( \{ c_{+1}, v_{+2} \} \) are not optimal by assumption, there must be a less costly sequence of allocations other than \( \{ c_t \}_{t=0}^{\infty} \) that achieves \( v_0 \). Put it another way, there must be allocations that achieves a higher value than \( W_0 \) for a representative banker while delivering the promised \( v_0 \). Formally, it must be true that \( W_0 (v_0') > W_0 \) and \( W_0 (v_0') = W_0 \) for some \( v_0' > v_0 \). This holds for
any $\alpha$. Let $\varphi(v_0) = W_0(v_0)$. The Theorem of the Maximum delivers $\varphi$ as a continuous function on $[\underline{v}_0, \overline{v}]$, where $\overline{v} = \frac{\alpha(\beta)}{1-\beta}$ is the value achieved by the first-best contract. Recall from the proof of Proposition 16 that given $\alpha$, the function $W_0(v_0)$ is strictly decreasing in $v_0$. Therefore, it must be true that there exists a unique $v_0 \in [\underline{v}_0, \overline{v}]$ that satisfies $W_0(v_0) = v_0$ for any given $\alpha$. The uniqueness of the equilibrium value $v_0^*$ follows because $v_0^* = \max_{\alpha \in [0,1]} v_0(\alpha)$. 
Proof of Proposition 21  Suppose (i)-(iii) are all true. It is obvious that \( d_H^t = 1 \) is the dominant strategy for type \( H \) agents as long as they choose to deposit fiat money in the bank. By (i), each young agent of generation \( t \) deposits \( M_t \) units of fiat money in the bank. By (ii) and (iii), \( f_t = s_t \). Suppose the bank chooses \( r \) such that \( rs_t < 1 \). By (3.35) and (3.36),

\[
R_{1,t} = r \\
R_{2,t} = \frac{1 - rs_t}{1 - s_t} R.
\]

Given the demand deposit contract \((R_{1,t}, R_{2,t})\) and prices \((P_{1,t}, P_{2,t})\), an old agent’s expected utility before observing his type is given by

\[
V(r) \equiv s_t U \left( \frac{r M_{t-1}}{P_{1,t}} \right) + (1 - s_t) U \left( \frac{1 - rs_t}{1 - s_t} \frac{M_{t-1}}{P_{2,t}} \right).
\]

In equilibrium,

\[
P_{1,t} = \frac{rs_t M_{t-1}}{E_t} \quad \text{(C.1)} \\
P_{2,t} = \frac{(1 - rs_t) RM_{t-1}}{1 - E_t}. \quad \text{(C.2)}
\]

Recall that \( E_t \) is the aggregate supply of goods in the morning. Hence the above becomes

\[
V(r) = s_t U \left( \frac{E_t}{s_t} \right) + (1 - s_t) U \left( \frac{1 - E_t}{1 - s_t} \right),
\]

which is exactly the same as the objective of the planner in (3.34). Hence we know that \( V(r) \) is maximized by a unique solution \( E_t^* = s_t \). Equilibrium requires no arbitrage oppor-
tunities across morning and afternoon markets, that is, \( P_{1,t} = P_{2,t} \). Thus

\[
\frac{rs_t M_{t-1}}{E_t} = \frac{(1 - rs_t)RM_{t-1}}{1 - E_t}.
\]

The above yields

\[
E_t = \frac{rs_t}{rs_t + (1 - rs_t)R}.
\]  \hspace{1cm} (C.3)

Plugging \( E_t^* = s_t \) into the above yields

\[
s_t = \frac{s_t R}{s_t R + (1 - s_t R)R}.
\]

Therefore, to maximize a young agent’s expected utility of accepting the contract, the bank optimally chooses

\[
r^* = \frac{\bar{R}}{1 + (\bar{R} - 1) s_t}.
\]  \hspace{1cm} (C.4)

Notice that \( 1 < r^* < \bar{R} \), and indeed \( r^* s_t < 1 \).

As mentioned before, it is optimal for a type \( H \) agent to choose \( d_t^H = 1 \) because he does not care about consumption in the afternoon. By (C.4), \( R_{1,t} = R_{2,t} = r^* \). Therefore, given \( r^* \) and expectation \( D_t = s_t \), a type \( L \) agent is indifferent between withdrawing in the morning and in the afternoon. Thus it is also optimal for the type \( L \) agent to choose \( d_t^L = 0 \).

\textit{Ex ante}, at the end of period \( t - 1 \), a young agent has three options: (i) deposit fiat money in the bank; (ii) invest fiat money in nominal bonds (through the bank); (iii) simply carry fiat money into period \( t \). It follows immediately that option (ii) strictly dominates option (iii). For each unit of fiat money, if the agent invests in bonds, he will have one unit of fiat money at disposal if he is a type \( H \), and \( \bar{R} > 1 \) units of fiat money if type \( L \).

The agent chooses between option (i) and option (ii), given \( (R_{1,t}(r^*), R_{2,t}(r^*)) \) and the expectation that all other young agents deposit all their fiat money in the bank. Let


\( \delta \) denote the fraction of the agent’s fiat money to be deposited in the bank. The agent’s utility-maximizing problem is given by

\[
\max_{\delta} \left\{ s_t U \left[ \frac{r^* \delta M_{t-1} + (1 - \delta) M_{t-1}}{P_{1,t}} \right] + (1 - s_t) U \left[ \frac{r^* \delta M_{t-1} + \overline{R} (1 - \delta) M_{t-1}}{P_{2,t}} \right] \right\}.
\]

For each unit of fiat money, the demand deposit offers a payoff profile of \( \{r^*, r^*\} \) for morning and afternoon, whereas the direct investment yields a profile of \( \{1, \overline{R}\} \). Hence the above gives the expected utility of the young agent. By (C.1), (C.2), (C.4) and \( E_t = s_t \), the above becomes

\[
\max_{\delta} \left\{ s_t U \left[ \frac{(r^* - 1) \delta + 1}{r^*} \right] + (1 - s_t) U \left[ \frac{(r^* - \overline{R}) \delta + \overline{R}}{(1 - r^* s_t) \overline{R}} (1 - s_t) \right] \right\}.
\]

The solution to the above problem is \( \delta^* = 1 \). Therefore, the agent optimally deposits all his fiat money in the bank.

Therefore, given contract \( (R_{1,t} (r^*), R_{2,t} (r^*)) \) there exists an equilibrium where (i)-(iii) are all satisfied. The belief of no type \( L \) agents withdrawing early is self-fulfilling. The equilibrium delivers the first-best outcome, \( c_t^{H*} = c_t^{L*} = 1 \).

**Proof of Proposition 22** It is obvious that a type \( H \) agent always withdraws all holdings of fiat money in the morning. Consider a type \( L \) agent. Given the belief that \( D_t = 1 \), (i) if the agent waits till the afternoon, he receives \( R_{2,t} = 0 \) because \( r^* f_t = r^* > 1 \). Since all other agents withdraw in the morning, all the investments in bonds are liquidated early to meet the demand. There is no asset left in the bank by the end of the morning;

(ii) if the agent chooses to withdraw in the morning, with probability \( \frac{1}{r^*} \) he can successfully receive \( r^* M_{t-1} \) units of fiat money; otherwise, he receives zero money back.
Hence for the type $L$ agent, the strategy $d^L_t = 1$ is stochastically dominant. That is, it is optimal to "run" along with other agents.

**Proof of Proposition 23**  
Given $r \in [0, 1)$, we have $rD_t < 1$ for any given $D_t$. It follows that $\Delta_t = 1$ and $R_{1,t} = r < \tilde{R} \leq R_{2,t}$. The first-order condition (3.40) implies that $d^L_t = 0$, because the left-hand side of (3.40) is strictly negative. By (3.43), $\pi^*_t = 0$ and $\tilde{R}_t = R_{2,t}$. It also follows that $rQ_t \leq \tilde{R}_t Q_t$, where the equality holds if and only if $Q_t = 0$. (i) Suppose $Q_t > 0$,

- if $rQ_t < \tilde{R}_t Q_t < P_{1,t}$, then $rQ_t < P_{1,t}$ implies $d^H_t = 0$ by (3.38). Moreover, $\tilde{R}_t Q_t < P_{1,t}$ implies $e^I_t = e_t$ by (3.44). This means a type $H$ old agent does not redeem private money for fiat money yet young agents do not sell goods for private money in the morning. It follows that type $H$ agents do not consume in the morning, which cannot be optimal. Thus $rQ_t < \tilde{R}_t Q_t < P_{1,t}$ cannot be an equilibrium outcome;

- if $rQ_t < \tilde{R}_t Q_t = P_{1,t}$, then (3.38) implies $d^H_t = 0$ and (3.44) implies $e^I_t \in [0, e_t]$. Therefore, the only symmetric equilibrium is $d^L_t = d^H_t = \pi_t = 0$, and $e^I_t = 0$;

- if $P_{1,t} < rQ_t < \tilde{R}_t Q_t$, then (3.38) implies $d^H_t = 1$ and (3.44) implies $e^I_t = 0$. This means young agents do not sell goods for fiat money yet all type $H$ old agents redeem private money for fiat money. It follows that type $H$ agents do not consume in the morning. Thus $P_{1,t} < rQ_t < \tilde{R}_t Q_t$ cannot be an equilibrium outcome;

Again, the only symmetric equilibrium is $d^L_t = d^H_t = \pi_t = e^I_t = 0$;

- if $rQ_t < P_{1,t} < \tilde{R}_t Q_t$, then (3.38) implies $d^H_t = 0$ and (3.44) implies $e^I_t = 0$. Again, the symmetric equilibrium is $d^L_t = d^H_t = \pi_t = e^I_t = 0$;
(ii) Suppose \( Q_t = 0 \). Then it must be true that \( d_t^H = 0 \) for all type \( H \) old agents by (3.38) and \( e_t^L = e_t = 0 \) by (3.44) and (3.45). This means all type \( H \) old agents hope to trade private money yet young agents supply no goods to the private money market. Obviously, this cannot be an equilibrium outcome. The price \( Q_t \) must be adjusting upwards till demand equals supply.

To summarize, when \( r \in [0, 1) \), any symmetric equilibrium must have \( d_t^L = d_t^H = \pi_t = e_t^L = 0 \). Note that in the morning no one demands redemption and only private money is traded for goods. Accordingly, the equilibrium prices are \( P_{2,t} = \frac{(1-s_t)M_{t-1}}{1-E_t} \), \( Q_t = \frac{s_t M_{t-1}}{E_t} \) and \( P_{1,t} \geq r Q_t \). No arbitrage condition requires that \( P_{1,t} = \tilde{R}_t Q_t = P_{2,t} \). Thus \( e_t = s_t \) and \( e_t^L = 1 \) for all type \( i = H, L \) agents, which coincide with the planner’s optimal choices. Equilibrium consistency is satisfied: \( D_t = s_t d_t^H + (1-s_t) d_t^L + (1 - d_t^H) s_t \pi_t = 0 \).

Therefore, given \( r \in [0, 1) \) there exists a unique symmetric equilibrium with \( d_t^* = 0 \) for all type \( i = H, L \) old agents, and \( \left( \pi_t = e_t^L = 0, e_t^* = s_t \right) \) for all young agents.

The last step is to prove that there is no profitable deviation for a young agent from depositing all after-tax fiat money holdings in the bank. Given the contract \((R_1(r), R_2(r))_{r<1}\) and the expectation that all other young agents deposit all their fiat money in the bank, a young agent of generation-\( t \) decides how to invest in bonds and in banking. (Same as in the proof of Proposition 21, investment in bonds strictly dominates no investment whatsoever.) Let \( \delta \) denote the fraction of the agent’s fiat money to be deposited in the bank. The agent’s utility-maximizing problem is given by

\[
\max_{\delta} E \left\{ s_{t+1} U \left[ \frac{\delta M_t}{Q_{t+1}} + \frac{(1-\delta) M_t}{P_{1,t+1}} \right] + (1-s_{t+1}) U \left[ \frac{\tilde{R} M_t}{P_{2,t+1}} \right] \right\}.
\]
The expectation is taken over \( s \). If the agent turns out to be of type \( H \), he trades for goods using the private money holdings of \( \delta M_{t-1} \) units and the fiat money holdings of \( \delta M_t \) units (obtained from early liquidation of bonds). If the agent is of type \( L \), he redeems private money for fiat money, which provides a gross rate of return \( \bar{R} \). Also, government bonds mature and yield the same return \( \bar{R} \). Altogether, the agent will have \( \bar{R} M_t \) units of fiat money to spend on goods. As mentioned before, no arbitrage implies that \( P_{1,t+1} = \bar{R}_{t+1} Q_{t+1} = \bar{R} Q_{t+1} > Q_{t+1} \). It follows immediately that it is optimal to choose \( \delta^* = 1 \). In other words, it is optimal for a young agent to deposit all fiat money holdings in the bank.

**Proof of Proposition 24**  
Given the expectation that \( D_t = 1 \), we have \( R_{2,t} = 0 \). Thus \( d_t^L = 1 \) by (3.40), \( \pi_t = 1 \) by (3.43). It follows that indeed \( D_t = s_t d_t^H + (1 - s_t) d_t^L + (1 - d_t^H) s_t \pi_t = 1 \) for any \( (s_t, d_t^H) \). Therefore, \( \Delta_t = \frac{1}{r} \) by (3.42) and \( \bar{R}_t = 1 \) by (3.46). No arbitrage requires that \( P_{1,t} = P_{2,t} = \bar{R}_t Q_t = Q_t \). By (3.38), we have \( d_t^H = 1 \) as \( r Q_t > P_{1,t} \). Therefore, there exists an equilibrium where \( d_t^H = d_t^L = \pi_t = 1 \) and \( e_t^f = e_{t-1} = 1 \). Indeed, the expectation of bank runs is self-fulfilling. Due to the pessimistic belief of \( D_t = 1 \), type \( H \) and \( L \) agents choose to redeem all private money in the morning. Accordingly, young agents do not value private money because they expect there is zero asset to back it up (i.e., available for redemption).