Analysis of QAM Schemes in a CCI-Limited Environment

by

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A thesis submitted in conformity with the requirements
for the degree of Master of Applied Science,
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Abstract

We consider the performance analysis of uncoded and coded QAM schemes in a cochannel interference (CCI) limited environment. Constellation and pulse shaping properties that are important for QAM schemes operating in a CCI-limited environment are studied. The modified peak-to-minimum-distance ratio, \(d_m(p)/d_{\text{min}}\), of the constellation determines the tolerable carrier to interference ratio (CIR), below which an error rate of \(p\) cannot be achieved. This parameter is independent of constellation expansion and has been termed the interference margin. When coding is used at low CIRs, the dominant error event is not the one with the minimum distance but the one with the minimum distance to code length ratio.
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Chapter 1

Introduction

A communication system can in general be represented as shown in Fig. 1.1. The information source is in the form of an electrical signal. This signal is subjected to many disturbances in the course of its path from the source to the destination. The challenge is therefore to recover the signal as reliably as possible at the destination. For this, the signal is conditioned to get rid of some of the effects of the undesired but inherent disturbances. The conditioning is usually in the form of encoders, modulators, filters, etc. The particular type of conditioning will depend upon the disturbances that the signal encounters, which in turn depends on the type of channel or the medium that is used to transmit the signal. The channel could be telephone lines, microwave links, satellite links, fiber optic links and so on. Each of these channels pose a different type of disturbance to the signal. Some of the disturbances that are common to most communication systems are listed below.
1. *Thermal Noise*: This occurs due to the random motion of electrons, and so wherever there is a conductor there is thermal noise. Telephone lines are subjected to thermal noise. Further, irrespective of the channel, all receivers introduce thermal noise. Thermal noise is Gaussian distributed. The effects of Gaussian noise on the communication system is well studied, and efficient modulation schemes have been designed to give good performance in such a channel.

2. *Impulse noise*: Impulse noise is a transient voltage disturbance and its amplitude is greater than the background noise. It can be caused by relay opening and closing in telephone lines, or coupled from adjacent power lines during switching, or in the case of mobile radio, it could be coupled from other electrical equipment operating in the vicinity of the receiver. Studies have been done to characterize the impulse noise and also to determine the error rate performance in the presence of impulse noise (e.g., [1]).

3. *Intersymbol interference*: The amplitude and delay distortion caused by a non-ideal channel frequency response, causes a succession of pulses transmitted through the channel at rates comparable to the bandwidth, to be smeared to the extent that they are no longer distinguishable as well defined pulses at the receiving terminal. These pulses overlap and cause intersymbol interference.

4. *Adjacent channel interference*: This occurs when geographically contiguous areas use adjacent bands of frequencies. Due to the increasing demand for spectral efficiency (capacity), the signal spectrum has to be made as broad as the channel bandwidth allows. But the severe adjacent channel interference places constraints on the signal spectrum. These constraints warrant very little out of band energy.
5. **Cochannel interference:** This occurs when two or more independent communication circuits use the same band of frequencies simultaneously to transmit information. This scenario arises in cellular radio communications, where frequencies are reused in cells that are physically apart in order to achieve high bandwidth efficiencies. Another area where CCI poses a serious limitation is in the form of crosstalk in telephone lines [2, 3, 4]. Crosstalk is caused by electromagnetic coupling of multiple signals that share adjacent twisted pair cables in a cable bundle. There are several different types of crosstalk, but the one that is cochannel is called intelligible crosstalk interference [6, Chapter 11], where intelligible means that the crosstalk interference is of the same type as the desired signal and therefore could be amplified and decoded if the desired signal were absent. Crosstalk can also be unintelligible when a frequency conversion occurs during the crosstalk process.

6. **Fading:** In mobile cellular environment the signal transmitted between the base station and the mobile is reflected off many fixed and/or stationary objects. As a result, the receiver receives signals along multiple paths, with different delays. This causes the signal to be constructively added in some locations and almost completely cancelled in others. This phenomenon is called fading [7].

Apart from the above mentioned noises there are other impairments like quantisation noise which are present in any digital transmission system.

### 1.1 Crosstalk/CCI in telephone lines

The types of crosstalk of most concern are the near end crosstalk (NEXT) and the far end crosstalk (FEXT) [4]. The sources of these two crosstalks can be conceptually seen from Fig. 1.2. NEXT occurs when unwanted energy from a transmitter couples
into the received signal at a receiver co-located with the transmitter. FEXT occurs due to coupling between signals transmitted in the same direction, in adjacent cable pairs. This kind of coupling is controlled by controlling the level difference of signals transmitted in the same direction. Since the level difference between the transmit and the receive signals at the same point on the cable is significant, NEXT loss is more prevalent than FEXT. Crosstalk cannot be eliminated, but it can be reduced by twisting the different conductors of a pair and using different twist lengths in a pair group, by keeping capacitance unbalances to a minimum during cable manufacture and so on [5, Chapter 5]. Various studies have modelled the NEXT coupling loss (e.g., [2, 3]) and it is seen that the NEXT loss or signal to interference ratio increases with frequency. In digital loop plants the twisted pair cable is used up to a frequency of 1.5MHz [5, p. 7], at which point the NEXT loss is 40dB. The twisted pair cable is used to transmit much higher data rates, of the order of 16Mbps [5, p. 544] in office buildings at which point the NEXT loss is about 23dB. The effects of the different

Figure 1.2: Types of coupling in a single cable.

types of impairments on the twisted pair cable has been summarised in Fig. 1.3 by obtaining the limitations on repeater spacing that each impairment imposes and it is seen that NEXT and impulse noise pose much more of a problem than thermal noise [6, p. 645].
1.2 CCI in cellular radio systems

In the cellular radio system, a given metropolitan area is divided into a number of cells, with a base station in the center of each cell. The base station acts as an interface between the mobile subscribers and the cellular radio system. Due to the increased demand on spectral efficiency, the same set of frequencies is reused in cells which are geographically separated. Although frequency reuse increases spectral efficiency, it also poses the problem of cochannel interference [9, 10, 8].

The CCI in cellular systems is a function of the ratio $D/R$ [12, 10, 11], where $D$ is the distance between two adjacent cochannel cells, and $R$ is the radius of one cell. The smaller the $D/R$ ratio, the larger the interference. We desire a small $D/R$ ratio from the point of view of spectral efficiency, but this is limited by the carrier to interference ratio (CIR) that can be tolerated at the cell site. A system that can
tolerate more CCI, can allow a smaller $D/R$ ratio.

1.3 Thesis objectives

The subject of this thesis is the design of good signal transmission schemes in the CCI limited environments discussed above. System design for a CCI limited channel differs from that of the additive white Gaussian noise (AWGN) channel in one important aspect: in the AWGN channel, the choice of the input signal does not influence the noise that is encountered in the medium, while in the CCI channel, the interference depends on the choice of the input. A modulation format with a larger peak will also give rise to a larger CCI. Codes must be designed to compensate for the increased CCI. The specific objectives of the thesis are as follows:

1. To characterize cochannel interference in order to gain more insight into the interfering process. Past work has focussed on error performance analysis of modulation schemes in CCI limited channels (e.g., [14, 15, 16, 17, 18, 19, 20]). Error rates are determined by using bounds (e.g., [14, 15, 16]) or in some cases by simulation (e.g., [18]) or in others by using numerical methods (e.g., [19, 20]). By characterising the CCI we can better understand the interference process and will enable us to determine important coding and modulation parameters that affect performance.

2. To identify the important constellation and signal parameters that determines the CCI immunity of a system. In previous works on cochannel interference, the peak of the constellation has been identified as limiting the performance of a CCI limited system (e.g., [14, 18]). The peak of large multidimensional constellations may affect performance only at very low error rates, much below the range of practical interest, owing to the very low probability of occurrence of
such peaks. Thus further investigation is to be done to identify parameters that can help in designing realistic systems. Since the shape of the pulse used for transmission can affect the peak of the interference signal [18, 21], it is necessary to study the effect of pulse shaping on the interference immunity of the system.

3. *To design coded modulation schemes that give good interference immunity while maintaining a good signal to noise ratio (SNR) gain over uncoded modulation schemes.* It has been shown that coded modulation schemes achieve good coding gain over uncoded modulation schemes, without bandwidth expansion or sacrificing data rate [22, 23]. But coded modulation always comes with constellation expansion, which appear to reduce the interference immunity. The performance of trellis coded modulation schemes that were optimised for the AWGN channel, are analysed in the presence of CCI in [24]. But in general a code that has been optimised for the AWGN channel need not perform well in other channels (e.g., [25, 26]). Good codes that can overcome the effect of constellation expansion while maintaining a good coding gain, have to be developed. Further constellation shaping which constrain the peak of the constellation (see e.g., [27, 28]) will also be studied.

### 1.4 Thesis organization

This thesis is organised as follows. In Chapter 2 we characterize the CCI and examine the constellation properties that are important for QAM schemes that are operating in CCI-limited channels. We find that the interference margin or the CIR cutoff, below which an error rate of $p$ cannot be achieved is determined by the parameter $d_m(p)/d_{\text{min}}$. In Chapter 3, the effects of pulse shaping on the interference immunity of a system is studied. We find that with non-rectangular pulses, the interference margin
is reduced by a factor called the shaping factor. Shaping factor depends on the rate of decay of the pulse. In Chapter 4 the analysis is extended to coded QAM schemes. We show that at low CIR, the dominant error event is not one with the minimum squared distance but one with the minimum $d^2/l_c$ ratio. Using this parameter, coded modulation schemes that show good interference immunity can be designed. Some conclusions are offered in Chapter 5.
Chapter 2

Constellation Properties

2.1 Introduction

It is known that the minimum Euclidean distance is the single most important parameter in determining the performance of modulation schemes in additive white Gaussian noise. However, as discussed in Chapter 1, in many communications systems, the dominant performance impediment is cochannel interference (CCI) and not noise. In this chapter we analyze QAM schemes in the presence of cochannel interference, to identify a performance-determining parameter in CCI limited communication. We observe that the peak to minimum distance ratio of the constellation determines the cutoff carrier to interference ratio (CIR), which is the value of CIR below which error free transmission is not possible using conventional detection methods. However at low to moderate error rates, we show that the CIR cutoff below which a probability of error of \( p \) cannot be achieved is determined by a value less than the peak, called the modified peak.
2.2 The system model

We analyze a system where only Additive White Gaussian Noise (AWGN) and Co-Channel Interference (CCI) impair the received signal. As mentioned in Section 2.1, in this chapter we are concerned with determining the constellation properties that affect performance in a CCI-limited environment. The signal constellation does not depend on the shape of the pulse that is used for transmission. It only depends on the energy of the pulses and the phase difference between the pulses that are transmitted. Therefore, in this analysis we assume that the input signal $s(t)$ is rectangular. The channel is assumed to have a bandwidth that is wide enough to accommodate the transmission of the modulated signal $s(t)$ with negligible distortion. In the channel, the signal is perturbed by

1. zero mean additive white Gaussian noise $n(t)$ and

2. an interfering signal $i(t)$, that is independent of the noise and occupying the same band of frequencies as the required signal. This is Co-channel Interference (CCI).

CCI is assumed to be of the same modulation format as the desired signal, but statistically independent. Although the carrier frequency of the CCI is the same as that of the required signal, it occurs at a random phase and time delay.

![Figure 2.1: The model of a cochannel system](image)
Accordingly, for the purpose of analysis, we represent the system as in Fig. 2.1. Mathematically, the system accepts the input waveform \( s(t) \) at the transmitter, and produces a waveform \( r(t) \) at the receiver that is the superposition of the input waveform \( s(t) \), the interfering signal \( i(t) \), and the Gaussian noise \( n(t) \), i.e.,

\[
r(t) = s(t) + i(t) + n(t),
\]

where the signal \( s(t) \) is

\[
s(t) = \sum_{j=-\infty}^{\infty} s_{cj} \text{rect} \frac{t - jT}{T} \cos \omega_c t + s_{sj} \text{rect} \frac{t - jT}{T} \sin \omega_c t.
\]

Here \( \omega_c \) is the carrier frequency, \( T \) the signaling interval, and \( \text{rect}(x) \) is the rectangular window function, defined as

\[
\text{rect}(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}
\]

The two independent and identically distributed data streams \( \{s_{cj}\} \) and \( \{s_{sj}\} \) modulate the envelopes of a cosine and sine wave with values from the set

\[
\Lambda = \{\pm d, \pm 3d, \ldots, \pm (M-1)d\}.
\]

where \( M \) is even, giving an \( M^2 \) QAM modulation scheme. The interfering signal \( i(t) \) is an attenuated time delayed and phase shifted version of the same modulation format as the desired signal, but statistically independent and can be written as

\[
i(t) = \sum_{j=-\infty}^{\infty} K [(s'_{cj} \text{rect} \frac{t - jT + \tau}{T} \cos(\omega_c t + \phi) + s'_{sj} \text{rect} \frac{t - jT + \tau}{T} \sin(\omega_c t + \phi)],
\]

where \( \{s'_{cj}\} \) and \( \{s'_{sj}\} \) are data streams that modulate the envelopes of cosine and sine waves of the interfering signal with values from the set

\[
\Lambda = \{\pm d, \pm 3d, \ldots, \pm (M-1)d\},
\]
\( \tau \) is the random time delay which is assumed to be uniformly distributed on \([0, T]\), and \( \phi \) is the random phase shift between the interference carrier and the signal carrier, which is assumed to be uniformly distributed in the interval \([0, 2\pi]\). The carrier to interference ratio (CIR) is defined as the ratio of the powers of the desired signal to the interfering signal, which in our case is

\[
\text{CIR} = \frac{1}{K^2}.
\]

The signal is also corrupted by AWGN \( n(t) \), with two-sided power spectral density \( N_0/2 \).

### 2.2.1 Coherent detection of signals in noise and CCI

Given the received signal \( r(t) \), the receiver must make a decision as to which signal was transmitted, ideally in a way that minimizes the probability of error. This is done in two stages. First the received signal \( r(t) \) is fed to a bank of two product integrators as shown in Fig. 2.2, to produce an observation vector \( r \). The receiver is assumed to be phase locked and time synchronous with the main transmitter. The observation vector can be written as

\[
r = s + i + n.
\]

or

\[
\begin{bmatrix}
  r_{cj} \\
  r_{sj}
\end{bmatrix}
= \begin{bmatrix}
  s_{cj} \\
  s_{sj}
\end{bmatrix}
+ \begin{bmatrix}
  i_{cj} \\
  i_{sj}
\end{bmatrix}
+ \begin{bmatrix}
  n_{cj} \\
  n_{sj}
\end{bmatrix},
\]

(2.5)
where

\[
\begin{align*}
  r_{cj} &= \frac{1}{T} \int_{jT}^{(j+1)T} r(t) \cos \omega_c t \, dt \\
  &= s_{cj} + i_{cj} + n_{cj}, \quad (2.6) \\
  r_{sj} &= \frac{1}{T} \int_{jT}^{(j+1)T} r(t) \sin \omega_c t \, dt \\
  &= s_{sj} + i_{sj} + n_{sj}, \quad (2.7) \\
  i_{cj} &= \int_{jT}^{(j+1)T} i(t) \cos \omega_c t \, dt, \\
  &= \frac{1}{T} \left[ (s'_{cj} \tau + s'_{c(j+1)}(T - \tau)) \cos \phi + (s'_{sj} \tau + s'_{s(j+1)}(T - \tau)) \sin \phi \right] \\
  &= \frac{1}{T} \sqrt{(c^2 + s'^2)} \cos (\phi + \theta), \quad (2.8) \\
  c' &= s'_{cj} \tau + s'_{c(i+1)}(T - \tau), \quad (2.9) \\
  s' &= s'_{s(i+1)}(T - \tau), \\
  \theta &= \arctan \frac{s'}{c'}, \\
  i_{sj} &= \int_{jT}^{(j+1)T} i(t) \sin \omega_c t \, dt, \\
  &= \frac{1}{T} \sqrt{(c^2 + s'^2)} \sin (\phi + \theta), \quad (2.10) \\
  n_{sj} &= \int_{jT}^{(j+1)T} n(t) \sin \omega_c t \, dt, \quad (2.11) \\
  n_{cj} &= \int_{jT}^{(j+1)T} n(t) \cos \omega_c t \, dt. \quad (2.12)
\end{align*}
\]

Here \(n_{sj}\) and \(n_{cj}\) have zero mean and variance equal to \(\frac{N_0}{2}\). Given the observation vector \(r\), the decoder has the job of estimating the transmitted signal \(s(t)\) in a way that minimizes the probability of error in the decision making process. This is the second stage of the decoder. For the case where the transmitted signals are equally likely, this is achieved by a maximum likelihood decoder. The decision rule of the
maximum likelihood decoder is as follows. The estimated signal

$$\hat{s} = s_i \quad \text{if} \quad \ln f_{r|s_k}(r) \quad \text{is max for} \quad k = i,$$

where $f_{r|s_k}(r)$ is the probability density function of the observation vector given $s_k$ was sent. This is also called the likelihood function. The likelihood function is given by

$$f_{r|s_k}(r) = f_{i+n}(r - s_k). \quad (2.13)$$

![Diagram](image)

Figure 2.2: Demodulation scheme for the interference channel.

Since $i$ and $n$ are independent, the probability density function of their sum is the convolution of the individual probability density functions and is given by

$$f_{i+n}(r_1, r_2) = \int_{x_2 = -\infty}^{\infty} \int_{x_1 = -\infty}^{\infty} f_i(x_1, x_2) f_n(r_1 - x_1, r_2 - x_2) dx_1 \, dx_2. \quad (2.14)$$

The components of $n$ are uncorrelated and jointly Gaussian. We need to develop the statistical characterization of the interference component $i$. 

14
2.2.2 Characterization of the interference component

The interference component \( i \) of the observation vector \( r \) was defined in (2.5) as

\[
i = \begin{bmatrix} i_{c,j} \\ i_{s,j} \end{bmatrix},
\]

where \( i_{c,j} \) and \( i_{s,j} \) are defined by (2.8) and (2.10) respectively. To obtain the statistical characteristics of the interference component, it is beneficial to look at the graphical interpretation of the above equation.

![Interference constellation graph](image)

Figure 2.3: Interference constellation; the effect of random phase shift \( \phi \).

The effect of the interference component being at a phase shift \( \phi \) with respect to the reference, is a rotation of the whole constellation by an angle \( \phi \). This is graphically represented in Fig. 2.3. Since the interference component occurs with a time delay with respect to the reference, the component \( i \) of the correlator output will be the linear combination of the interference signal sent in two successive bit intervals. Let
the interference signal sent in two successive bit intervals be \( s'_j \) and \( s'_{j+1} \). Graphically the vector \( i \) divides the line joining the two interference signal points \( s'_j \) and \( s'_{j+1} \), in the ratio \( \tau : (T - \tau) \). As \( \phi \) changes, the point moves along a circle centered at the origin. Since \( \phi \) is uniformly distributed in the interval \([0, 2\pi)\), the distribution of \( i \) given \( \tau \) is uniform along the circumference of the circle. Now as \( \tau \) varies uniformly from 0 to \( T \), \( i \) goes uniformly from \( s'_j \) to \( s'_{j+1} \). Thus the distribution of \( i \) given \( \phi \) is uniform along the line joining \( s'_j \) and \( s'_{j+1} \).

It should be noted here that, the above conclusion that the component \( i \) of the correlator output is the linear combination of the interference signal sent in two successive bit intervals, is particular to the rectangular pulse assumption. This is not the case for a general pulse shape. In the case where the tail of one pulse spills into adjacent bit intervals, the value of the correlator output at any time \( \tau \), will depend on pulses sent in many other intervals. This can be seen from the eye pattern, which is formed by superimposing the received waveform over a 1 symbol interval. The eye pattern of a 2-PAM scheme using rectangular and raised cosine pulse shape is given in Fig. 2.4 and Fig. 2.5 respectively. As seen from the figures, the value of the correlator output at any instant \( \tau \), depends only on the present and the past symbols in the case of rectangular pulses. This is not the case with raised cosine pulses.

In what follows an expression for the pdf of the interference component will be derived for a two point constellation, with the points having arbitrary orientation. The signals are assumed to have rectangular pulse shape. The effect of arbitrary pulse shape is considered in Chapter 3. For a many point constellation, the pdf is nothing but the sum of the distributions of all possible pairs of points weighted by their respective probabilities. Since the angle \( \phi \) is uniform in the interval \([0, 2\pi)\), the pdf is independent of the initial phase of the two points. What matters is the relative phase difference between the two points. Therefore any pair of points can in general be represented as \( s_1 = r_1 \exp^{j\phi} \) and \( s_2 = r_2 \exp^{j\theta} \), \( r_1 \leq r_2, -\pi \leq \theta \leq \pi \). Here \( r_1 \) and
Figure 2.4: Eye diagram of a 2-PAM scheme using rectangular pulse shape.

Figure 2.5: Eye diagram of a 2-PAM scheme using raised cosine pulse shape with roll off factor 0.5.
$r_2$ are the distances of the two points from the origin and $\theta$ is the relative phase shift between the two points. Depending on the relative position of the pairs of points, the interference component takes on values in two different regions.

Figure 2.6: Illustrating the region where the interference pdf exists for points with angular separation $\leq \pi/2$.

Case 1

$$|\psi = \angle(s_2 - s_1)| \leq \pi/2$$

If the points $s_1$ and $s_2$ are such that the angle $\psi$ of the line joining $s_1$ to $s_2$ is between $-\pi/2$ and $\pi/2$, then the pdf of the CCI takes on values between concentric circles with radii $r_1$ and $r_2$. This is illustrated in Fig. 2.6. When $\phi = 0$, the interference component is distributed uniformly along the line joining $s_1$ and $s_2$. This is shown in Fig. 2.6(a). As $\phi$ is varied this line moves as shown in Fig. 2.6(b), covering the region between the concentric circles with radius $r_1$ and $r_2$.

Case 2

$$|\psi = \angle(s_2 - s_1)| > \pi/2$$
Figure 2.7: Illustrating the region where the interference pdf exists for points with angular separation $> \pi/2$.

The region where the interference pdf exists for points belonging to Case 2. is shown in Fig. 2.7. Here we can see that the pdf exists even within the circle with radius $r_1$. It exists between concentric circles of radius $r' < r_1$ and $r_2$. So we consider this case as having two pairs of points each belonging to case 1. The first pair is $s_1 = r' \exp^{i\theta}$, $s_2 = r_2 \exp^{i\theta - \theta'}$ and the second pair is $s_1 = r' \exp^{i\theta}$, $s_2 = r_1 \exp^{i\theta'}$. Notice that both these pairs belong to case 1, because both have $|\angle(s_2 - s_1)| = \pi/2$, with respect to the $\exp^{i\theta'}$ axis. This is illustrated in Fig. 2.7(a). Thus we analyze points belonging to case 1 only, as any point belonging to case 2 can be split into two pairs belonging to case 1.

Let $d$ denote any point along the line joining $s_1$ to $s_2$, and $D$ the distance between $s_1$ and $s_2$. Then, referring to Fig. 2.8, the pdf of $i$ for a particular value of $\phi$ is uniform along the line joining $s_1$ and $s_2$. It is convenient to have the pdf as a function
of the radius $r$. From Fig. (2.8) we have,

$$r = g(d) = \sqrt{(r_1 + d \cos \psi)^2 + (d \sin \psi)^2}$$

$$= \sqrt{r_1^2 + 2dr_1 \cos \psi + d^2}. \quad (2.15)$$

Writing $d$ in terms of $r$, we have from (2.15)

$$d = g^{-1}(r) = -r_1 \cos \psi + \sqrt{(r_1 \cos \psi)^2 + (r^2 - r_1^2)}. \quad (2.16)$$

Differentiating (2.15) we get

$$2r dr = 2r_1 \cos \psi dd + 2d dd$$

$$\frac{dr}{dd} = \frac{r_1 \cos \psi + d}{r}. \quad (2.17)$$

The interference component $i$ can be written in polar coordinates as

$$i = \begin{bmatrix} i_r \\ i_\phi \end{bmatrix}.$$ 

Thus the pdf of $i_r$ can be written as

$$f_{i_r}(r) = \left. \frac{f_{i_d}(d)}{dr} \right|_{d=g^{-1}(r)} \bigg|_{d=g^{-1}(r)}$$

$$= \frac{1}{2r} \frac{r}{\sqrt{(r_1 \cos \psi)^2 + (r^2 - r_1^2)}} \quad r_1 \leq r \leq r_2. \quad (2.19)$$
The joint probability density function is given by,

\[ f_{i_1, i_2}(r, \phi) = f_i(r) f_{i\mid i, r}\phi(r) \]

Therefore using (2.19) and noting that given \( r \), the interference component is uniform along a circle of circumference \( 2\pi r \), we have

\[ f_{i_1, i_2}(r, \phi) = \frac{1}{D} \frac{1}{2\pi} \frac{1}{\sqrt{(r_1 \cos \psi)^2 + (r^2 - r_1^2)}} \quad r_1 \leq r \leq r_2. \tag{2.20} \]

This is the pdf of the interference component as a function of \( r \) and \( \phi \), for pairs of points with angular separation less than or equal to \( \pi/2 \). As discussed earlier, pairs of points with angular separation greater than \( \pi/2 \) can be split into two pairs, each having angular separation equal to \( \pm\pi/2 \).

We see from (2.20) that the pdf does not depend on the value of \( \phi \), which is to say that the interference component has circular symmetry. The pdf of the interference component for a 16-ary QAM is plotted in Fig. 2.9. The impulse functions correspond to those pairs, where a constellation point is transmitted in two consecutive symbol intervals.

### 2.3 Decision regions and the evaluation of error rate

In Section 2.2.1 it was shown that the decision regions for the case under consideration, are defined by those points where the likelihood function is maximum. The likelihood function is given by (2.13) and (2.14). The effect of convolving the spectrum of \( i \) with a Gaussian distribution, would be to smear the distribution of \( i \). Also since both \( i \) and \( n \) have circular symmetry, the likelihood function, which is the convolution sum, also has circular symmetry. Beyond the distance equal to the
peak of the interference component $i_{\text{peak}}$ (since we have circular symmetry, $i_{\text{peak}}$ is along the circle of radius $r_{\text{peak}}$), the likelihood function is a monotonically decreasing function of distance. i.e., for $\Delta r > 0$,

$$f(r) \geq f(r + \Delta r), \quad r > r_{\text{peak}}.$$  \hfill (2.21)

This is because beyond $r_{\text{peak}}$, the likelihood function is the sum of the tails of monotonically decreasing Gaussian distributions. Also, because of the Gaussian distribution of noise we can say that the likelihood function

$$f(r) > 0 \text{ for } r \leq r_{\text{peak}}.$$  \hfill (2.22)

Thus we can always find a point $\hat{r}$ such that

$$f(r) \geq f(2\hat{r} - r) \text{ for } r < \hat{r}.$$  \hfill (2.23)
Figure 2.10: Illustrating the condition to be satisfied by the likelihood function to obtain rectangular decision regions.

This relation is shown in Fig. 2.10. The position of the point \( \hat{r} \), will depend on the pdf of \( i \), and on the signal to noise ratio (SNR) of the system. From (2.23), we can conclude that if the minimum half distance \( d_{\text{min}} \) of the constellation is such that

\[
d_{\text{min}} \geq \hat{r},
\]

(2.24)

then Euclidean distance can be the basis of comparison for two likelihood functions, evaluated at a received point. Thus if the conditions given by (2.23) and (2.24) are satisfied, the decision region of a particular transmitted signal point, is defined by all points closest in Euclidean distance to the given point. Hence in the case of a QAM constellation, the decision regions would be rectangular.

The above conclusion was arrived at, subject to the condition stated in (2.23). To see what limitations this condition imposes on our analysis of CCI limited systems, we need to look at the values of \( \hat{r} \). Asymptotically, as the signal to noise ratio (SNR) approaches infinity, the noise pdf approaches a delta function. This means that the
likelihood function approaches the pdf of $i$. Since $f_i(i) = 0$, for $i > i_{\text{peak}}$, $\hat{r}$ is given by

$$\hat{r} = r_{\text{peak}}.$$

Thus using (2.24) we can say that, asymptotically, the decision regions derived above applies if $d_{\text{min}} \geq r_{\text{peak}}$. Later in the chapter we see that $d_{\text{min}} < r_{\text{peak}}$ is not within the carrier-to-interference ratio of interest. For a QAM scheme with distance between the points equal to $2d$, the decision regions are rectangular if

$$d_{\text{min}} = d \geq r_{\text{peak}} = \frac{d_{\text{peak}}}{\sqrt{\text{CIR}}}.$$  \hfill (2.25)

Based on this decision region we can evaluate the probability of error of a QAM scheme. Given a transmitted signal $s_i$, the receiver makes a correct decision if the received vector $r$ falls inside the decision region corresponding to $s_i$. The probability of error given that $s_i$ is sent is

$$P(e|s_i) = 1 - P(c|s_i),$$

where $P(c|s_i)$ is the probability of correct decision given $s_i$. Referring to Fig. 2.11, the probability of correct decision given that the interference component $i = r \exp^{j\phi}$ is, $P(c|s_i, i) =

$$P\left((-d + r \cos \phi) \leq n_c \leq (d - r \cos \phi)\right) \cdot P\left((-d + r \sin \phi) \leq n_s \leq (d - r \sin \phi)\right)$$

$$= \left(1 - \frac{(P(n_c > d - r \cos \phi) + P(n_c > d + r \cos \phi))}{p_1}\right) \left(1 - \frac{(P(n_s > d - r \sin \phi) + P(n_s > d + r \sin \phi))}{p_2}\right).$$
Figure 2.11: Illustrating the relation between the interference and noise components.

Therefore the probability of error given that the interference component is $r \exp^{i\phi}$ can be written as

$$P(e|s_1, i) = p_1 + p_2 - p_1p_2.$$ 

Integrating over all possible values of $r$ and $\phi$ we have,

$$P(e|s_1) = \int_{r=0}^{r_{\text{peak}}} \int_{\phi=0}^{2\pi} (p_1 + p_2 - p_1p_2) f(r) r \, d\phi \, dr$$

$$= \int_{r=0}^{r_{\text{peak}}} \left( \int_{\phi=0}^{2\pi} (p_1 + p_2) d\phi \right) r \, f(r) \, dr,$$

where the product term $p_1p_2$ can be neglected at high values of SNR. Also

$$\int_{\phi=0}^{2\pi} p_1 \, d\phi = \int_{\phi=0}^{2\pi} p_2 \, d\phi,$$

as $p_1$ and $p_2$ are the same function shifted in phase. Further

$$\int_{\phi=0}^{2\pi} p_1 \, d\phi = 2 \int_{\phi=0}^{2\pi} p_1' \, d\phi,$$

where

$$p_1' = p(n_c > d - r \cos \phi).$$

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This is because $d - r \cos \phi$ and $d - r \sin \phi$ take on the same set of values as $\phi$ goes from 0 to $2\pi$. Using the above results the probability of error can be written as

$$P(e|s_i) \approx \int_{r=0}^{r_{\text{peak}}} 4 \left( \int_{\phi=0}^{2\pi} p(n_c > d - r \cos \phi) d\phi \right) r f(r) dr$$

$$= 8 \int_{r=0}^{r_{\text{peak}}} \left( \int_{\phi=0}^{\pi} p(n_c > d - r \cos \phi) d\phi \right) r f(r) dr. \quad (2.26)$$

This is the probability of error given $s_i$, where $s_i$ is a signal point whose decision region is a rectangle. For points along the edges, the decision regions are three sided, and for those on the corners, it is two sided. For these two cases, the expression for the probability of error are the same except for the constant coefficients. The conditional probability of error given $s_i$ is given by (2.26). The unconditional probability of error is obtained by taking the summation over all possible signal points weighted by their respective probabilities. Therefore we can write,

$$P(e) = k_1 \int_{r=0}^{r_{\text{peak}}} \left( \int_{\phi=0}^{\pi} (P(n_c > d - r \cos \phi) d\phi \right) r f(r) dr, \quad (2.27)$$

where $k_1$ is a constant that depends on the constellation. Since we know that $n_c$ is Gaussian distributed,

$$P(n_c > d - r \cos \phi) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{(d - r \cos \phi)^2}{N_0}} \right). \quad (2.28)$$

where $N_0/2$ is the power spectral density of the noise spectrum. Considering that at large values of CIR it is the peak of the interference $r_{\text{peak}}$ that will limit performance. we can upper bound the probability of error expression of (2.27) as

$$P(e) \leq k_1 \text{erfc} \left( \sqrt{\frac{(d - r_{\text{peak}})^2}{N_0}} \right). \quad (2.29)$$

The peak of the interference component $r_{\text{peak}}$ can be written in terms of the peak of the constellation $d_{\text{peak}}$, as

$$r_{\text{peak}} = d_{\text{peak}}/\sqrt{\text{CIR}}.$$
Further if $E$ is the average energy per two dimensions of the $M^2$ QAM constellation, with the distance between the points equal to $2d$, then we can write

$$d^2 = \frac{6E}{M^2 - 1}.$$  

Using the above two equations, (2.29) can be rewritten as

$$P(e) \leq k_1 \text{erfc} \left( \sqrt{\frac{(d - \frac{d_{\text{peak}}}{\sqrt{\text{CIR}}})^2}{N_0}} \right)$$

$$= k_1 \text{erfc} \left( \sqrt{\frac{d^2}{N_0} \left(1 - \frac{d_{\text{peak}}}{d^2 \sqrt{\text{CIR}}} \right)^2} \right)$$

$$= k_1 \text{erfc} \left( \sqrt{\frac{6E}{(M^2 - 1)N_0} \left(1 - \frac{d_{\text{peak}}}{d^2 \sqrt{\text{CIR}}} \right)^2} \right)$$

$$= k_1 \text{erfc} \left( \sqrt{k \text{SNR} \left(1 - \frac{d_{\text{peak}}}{d^2 \sqrt{\text{CIR}}} \right)^2} \right). \quad (2.30)$$

where $k$ is some constant. It should be noted that (2.30) holds only for the range of values of CIR that satisfy (2.25). Thus we see from (2.30) that as long as $\left(1 - \frac{d_{\text{peak}}}{d^2 \sqrt{\text{CIR}}} \right) > 0$, the probability of error decreases exponentially, as SNR is increased and ultimately reliable communication is possible. But for a given SNR, if $d_{\text{peak}}/d$ and CIR are such that $\left(1 - \frac{d_{\text{peak}}}{d^2 \sqrt{\text{CIR}}} \right) \leq 0$, then error free transmission is impossible. Thus it is the parameter $d_{\text{peak}}/d$, that determines the amount of CIR below which error free transmission is not possible and we call this parameter the interference margin. The CIR corresponding to the interference margin is called the CIR$_{\text{cutoff}}$.

Thus CIR$_{\text{cutoff}}$ gives the value of CIR below which arbitrarily low probabilities of error are not achievable. Above this value the probability of error tends to zero as the signal power tends to infinity. However, often we are not interested in such arbitrarily low error rates. The parameter $d_{\text{peak}}/d_{\text{min}}$ fails to give a true CIR$_{\text{cutoff}}$ in
such a problem. This is especially true in the case of multidimensional constellations which are discussed in Chapter 4. For example we might be interested in a probability of error of $10^{-4}$. At this rate the peak CCI with a probability of occurrence of $10^{-8}$ will not affect performance. In such cases the value of CCI that determines the CIR_{cutoff} will be less than $d_{\text{peak}}$ and we call it the modified peak $d_m(p)$, where $p$ is the error rate that we are interested in.

The value of $d_m(p)$ can be determined from the cumulative distribution function (cdf) of the cosine component of the CCI, where for our analysis the cdf ($F_X(x)$) is defined as

$$F_X(x) = p(X \geq x).$$

Here $p(x)$ is the pdf of the cosine component of the interference spectrum which is the pdf of (2.20) in rectangular coordinates. If we are interested in an error rate of the order of $p$, then the modified peak $d_m(p)$ is given by

$$F_x(d_m) = p. \quad (2.31)$$

Any value of CCI greater than $d_m(p)$ occurs with a probability less than $p$ and therefore does not affect performance at an error rate of the order of $p$. Thus it is the parameter $d_m(p)/d$ that defines the CIR_{cutoff}.

The pdf and cdf of the cosine component of a 16-QAM CCI is shown in Fig. 2.12 and Fig. 2.13 respectively. The pdf of the cosine component is a continuous distribution as opposed to the mixed type distribution shown in Fig. 2.9. This is because when the same point is transmitted in two consecutive intervals, the cosine component has an arccosine distribution.

From (2.31), we see that the modified peak is given by the inverse of the distribution function. Therefore from Fig. 2.13 it can be seen that the modified peak at an error rate of $10^{-4}$ is equal to $d_{\text{peak}}$. The error rate curves for 16-ary QAM are plotted in Fig. 2.14. The exact error rate curves are obtained by using (2.27) and
Figure 2.12: The pdf of the cosine component of a 16-QAM CCI.

Figure 2.13: The cdf of the cosine component of 16-QAM CCI.
(2.28), where the integration is performed numerically on the computer. The upper bound is obtained using (2.29). The curves show the probability of error as a function of SNR, for various values of CIR. From the equations derived above we find the CIR cutoff in dB, for reliable communication.

\[
\text{CIR}_{\text{cutoff}} = 20 \log(d_{\text{peak}}/d) = 20 \log(4.24) = 12.54 \text{dB}
\]

This is seen from the plot. For all values of CIR \( \geq 13 \text{dB} \), the probability of error falls to zero as SNR is increased. Below this value the probability of error does not fall.

### 2.4 Conclusion

In this chapter, the constellation properties that are important for QAM schemes operating in a CCI limited channel are studied. The threshold CIR below which
error free transmission is not possible is determined by the peak of the constellation. However, at low to moderate error rates that might be of interest to us, the CIR cutoff is not determined by the peak. We show that the threshold CIR below which a probability of error of $p$ cannot be achieved is given by a parameter called the modified peak $d_m$. The value of $d_m$ can be obtained from the interference spectrum, for any error rate that we might be interested. Thus given any modulation scheme, the interference margin or the CIR cutoff, below which an error rate of $p$ cannot be achieved is defined by $d_m(p)/d_{\text{min}}$. The analysis in this chapter depends on the rectangular pulse assumption. So the interference margin as defined by $d_m(p)/d_{\text{min}}$ gives accurate results for rectangular pulses. In Chapter 3, we look at how the interference margin can be modified for non-rectangular pulses.
Chapter 3

Pulse shaping properties

3.1 Introduction

In Chapter 2, it was shown that the property of the constellation that is of importance in designing a communication system in a CCI limited environment is the modified peak to minimum distance ratio. It was shown that the lower the peak to minimum distance ratio of the constellation, the larger the interference margin, or the larger the interference power that can be allowed before reliable communication fails. Since the properties of the constellation are independent of the shape of the pulse, a rectangular pulse shape is assumed. However, a rectangular pulse shape is not often preferred due to bandwidth considerations. Pulse shaping is done so that the resulting signal occupies a suitable bandwidth. In this chapter the effect of pulse shaping on the interference margin is explored.
In Chapter 2 a rectangular pulse shape was assumed to study the constellation properties. However in practice rectangular pulses are not used due to the large bandwidth they occupy. Pulse shaping is done so that the resulting pulses occupy a reasonable bandwidth. When pulse shaping is done the tail of one pulse spills into the adjacent bit intervals and therefore causes intersymbol interference (ISI). Nyquist showed that the minimum bandwidth required to transmit $R$ symbols/s, with zero ISI at the sampling time, is $R/2$ Hz. This occurs when the overall system transfer function $P(f)$ is rectangular and the corresponding pulse shape is

$$p(t) = \frac{\sin(\pi R t)}{\pi R t}.$$

Note that $P(f)$ refers to the overall system transfer function, which incorporates the transmit filter, the channel and the receive filter as shown in Fig. 3.1. Although this solves the problem of zero ISI with minimum bandwidth, the amplitude characteristic of $P(f)$ is physically unrealizable, because the rectangular shape requires abrupt transitions at the band edges. Further the pulses are slowly decaying and therefore a small timing error in sampling can result in a large ISI.
One way to overcome this problem is to have the overall system transfer function as raised cosine pulses, which solves the problem of zero ISI, but extends the bandwidth from the minimum of $R/2$ Hz to anywhere between $R/2$ and $R$ Hz. Raised cosine pulses are given by

$$P(f) = \begin{cases} \frac{1}{R}, & 0 \leq |f| < f_1; \\ \frac{1}{2R} \left(1 - \sin \left[ \frac{\pi (f-f_1)}{R} \right] \right), & f_1 \leq |f| < R - f_1; \\ 0, & |f| \geq R - f_1. \end{cases}$$

The frequency parameter $f_1$ can be anywhere between 0 and $R/2$ Hz, and defines the transmission bandwidth $B_T$ as

$$B_T = R - f_1.$$ 

A parameter $\alpha$, called the rolloff factor, is defined as the ratio of the bandwidth in excess of the minimum to the minimum bandwidth required for zero ISI transmission, i.e.,

$$\alpha = \frac{R/2 - f_1}{R/2}.$$ 

The time domain pulse $p(t)$ is the inverse Fourier transform of the function $P(f)$ and is given by ([29])

$$p(t) = \left( \text{sinc}(Rt) \right) \left( \frac{\cos (\pi \alpha Rt)}{1 - 4\alpha^2 R^2 t^2} \right).$$ (3.1)

### 3.3 Effect of pulse shaping on interference margin

In Chapter 2, it was shown that the interference margin parameter determines the amount of CCI that can be allowed before reliable communication fails. Interference margin measures by what factor the peak of the interference component reduces the minimum distance between the signal points. In order to see how pulse shaping affects the interference margin, it is convenient to define the concept of the eye pattern [29].
An eye pattern for a random input pulse train is formed by superimposing the received waveform before the sampler in Fig. 3.1, over a 1 symbol interval. The eye pattern for a 4-PAM in the absence of noise and CCI is shown in Fig. 3.2, over two symbol intervals.

It can be seen from the figure that there are points on the eye pattern where there is no ISI. The eye pattern shows only 4 levels at this point, corresponding to the points in a 4-PAM scheme. This zero ISI point is chosen as the sampling instant. The effect of having noise and CCI would be to blur these points. To see how the noise and CCI affects the eye pattern, we look at the demodulation scheme of a CCI limited system, shown in Fig. 3.3. For an $M^2$QAM modulation scheme, the eye pattern taken at the input of the sampler on the cosine and sine components of the demodulator (Fig. 3.3), will be that of an M-PAM. In the absence of noise and CCI the eye pattern will have exactly $M$ equally spaced points at the sampling instant $t = T_s$. When CCI is considered the eye pattern of the CCI will be superimposed at each of these points.
The CCI is assumed to be of the same modulation format as the desired signal, but with a random phase and time delay with respect to the required signal. The effect of having the interfering signal at a phase shift of \( \phi \), is to rotate the constellation of the CCI at an angle \( \phi \) with respect to the required signal. Thus if the eye pattern of the interfering signal is considered just before the decision device in Fig. 3.3, the points at the zero ISI time instant will correspond to the projection of the constellation points on the cosine and sine basis as in Fig. 3.4. Depending on the angle \( \phi \), the number of levels can be anywhere between \( M \) and \( M^2 \), for an \( M^2 \) QAM modulation scheme. The effect of the random time delay \( \tau \) is to shift the eye pattern by \( \tau s \) along the time axis. Thus the eye pattern of the CCI is similar to that of Fig. 3.2, but with unequally spaced levels of varying amplitudes at the zero ISI point, and shifted along the time axis. This is attenuated, depending on the CIR, and superimposed on each of the levels of the eye pattern of the original signal, shown in Fig. 3.2.

Thus it can be seen that the presence of CCI reduces the distance between the points at the sampling instant. The peak value by which the CCI reduces the distance between the points, defines the interference margin. As can be seen from Fig. 3.2, for a given eye pattern the peak value occurs at \( \tau = \pm T/2 \), approximately, where \( \tau \)
is the time offset from the sampling instant. At the angle $\phi = n\pi/4$, where $n$ is any odd integer, the farthest point of the QAM constellation is along the cosine or sine basis. Therefore the peak level at the zero ISI time instant occurs at this angle of phase. Thus the eye pattern of the CCI attains its peak amplitude at $\tau \approx \pm T/2$ and $\phi = n\pi/4$, where $n$ is any odd integer. In [30], the peak intersymbol interference is computed for the case of raised cosine pulses. Intersymbol interference arises from the overlapping tails of the other pulses at the sampling instant, other than the one sent during that particular bit interval where sampling occurs. In the case of CCI, we have to find the peak over the entire bit interval, and not just the sampling instant. We also have to consider the pulse sent in the particular sampling interval where sampling occurs. Thus the problem of computing the peak intersymbol interference is different from the problem of computing the peak CCI.

Figure 3.4: Projection of the interference constellation on the signal basis.
3.3.1 Computation of $\text{eye}_{\text{peak}}$

When shaping is considered, we redefine the interference margin (im) as,

$$im = \frac{\text{eye}_{\text{peak}}}{d},$$

$$= \left(\frac{d_{\text{peak}}}{d}\right) \left(\frac{\text{eye}_{\text{peak}}}{d_{\text{peak}}}\right),$$

where $d$ is the minimum half distance, $d_{\text{peak}}$ is the peak of the constellation and $\text{eye}_{\text{peak}}$ is the peak of the eye pattern which is defined as

$$\text{eye}_{\text{peak}} = \text{peak} \left[ \sum_i s_i p(t - iT) \right] \quad mT < t \leq (m + 1)T.$$  

where $\{s_i\}$ takes values in the interval $[0, d_{\text{peak}}]$, depending on the value of $\phi$ and $\tau$. In the expression for $\text{eye}_{\text{peak}}$ as defined above, the peak value is calculated by taking the sum of the tails of all possible pulses. In practice we need consider only a finite number of pulses to compute $\text{eye}_{\text{peak}}$ within a certain percentage of error. Thus we write (3.4) as

$$\text{eye}_{\text{peak}} \approx \text{peak} \left[ \sum_{m-n}^{m+n} (s_i p(t - iT)) \right] \quad mT \leq t \leq (m + 1)T.$$  

There are different possible sequences of $s_i$ of which $\text{eye}_{\text{peak}}$ will correspond to the sequence with $s_i = d_{\text{peak}}$ for all $i$. Thus (3.5) can be written as

$$\text{eye}_{\text{peak}} \approx \text{peak} \left[ \sum_{i=-n}^{+n} d_{\text{peak}} p(t - iT) \right].$$  

To compute the peak of the above summation we make some simplifying assumptions. First we observe that the peak of the eye diagram occurs at $\tau \approx T/2$. This however will not give accurate results for certain values of $\alpha$. For example, for $\alpha = 1$ the value of the sidelobes are zero at $t = mT/2$, $|m| > 1$. This can be seen from Fig. 3.5. Infact for $\alpha$ between 0.75 and 1, the first sidelobe passes through zero between $T$ and
2T. The zero crossing occurs at \( t = \frac{3T}{2\alpha} \), which ranges from 1.5T to 2T for values of \( \alpha \) between 0.75 and 1. For \( 0 \leq \alpha \leq 0.75 \), the duration of the first sidelobe is T's. Hence it is more accurate to assume that the peak of the eye pattern occurs at \( \tau \) equal to half the duration of the first sidelobe. Thus

\[
\tau = \frac{T}{2}, \quad 0 \leq \alpha \leq 0.75 \text{ and }
\]

\[
\tau = \frac{\left(\frac{3}{2\alpha} - 1\right) T}{2}, \quad 0.75 < \alpha \leq 1.
\]

Thus we can write (3.6) as

\[
\text{eye}_{\text{peak}} \approx d_{\text{peak}} \sum_{i=-n}^{n} |p(t - iT)|_{t=\tau},
\]

or

\[
\frac{\text{eye}_{\text{peak}}}{d_{\text{peak}}} = \sum_{i=-n}^{n} |p(t - iT)|_{t=\tau}, \quad (3.9)
\]

where \( n \) is chosen such that

\[
p(\tau - iT) \leq 0.001 \quad \text{for } |i| > n.
\]
3.3.2 Shaping factor

We see that the parameter \( \frac{\text{eye}_{\text{peak}}}{d_{\text{peak}}} \), is purely a function of the pulse \( p(t) \). It depends on the rate of decay of the pulse and the amplitude of the side lobes which for raised cosine pulses depends on the spectral rolloff factor \( \alpha \). We call the parameter \( \frac{\text{eye}_{\text{peak}}}{d_{\text{peak}}} \) the shaping factor and denote it by \( s(\alpha) \); i.e.,

\[
s(\alpha) = \frac{\text{eye}_{\text{peak}}}{d_{\text{peak}}}. \tag{3.10}
\]

The shaping factor for the raised cosine family of pulses is shown in Fig. 3.6, as a function of the rolloff factor \( \alpha \). As can be seen from Fig. 3.6, the shaping factor decreases as a function of \( \alpha \). The number of pulses used to compute the shaping factor is tabulated in Table 3.1.

It should be noted here that \( \text{eye}_{\text{peak}} \) corresponds to the sequence of \( 2n \) symbols, all taking the value \( d_{\text{peak}} \). As discussed in Section 3.3, \( d_{\text{peak}} \) is obtained when the angle \( \phi \) is such that the peak of the constellation aligns with the cosine and sine basis. For
Table 3.1: The number of pulses considered in the computation of shaping factor as a function of $\alpha$.

The simplest case of a 2 point constellation, the probability that the correlator output is within 5% of $d_{\text{peak}}$ is the probability that $\phi$ is within $35^\circ$ of the desired angle, which is equal to $35/360 = .097$. Thus the probability of getting a sequence of $2n$ symbols with amplitude $d_{\text{peak}}$ is $\approx .1/2^{2n}$. For values of $n$ shown in Table 3.1, this probability is very low and may not affect performance if we are interested in a probability of, say $10^{-4}$. For a probability of error of $10^{-4}$, the value of $\text{eye}_{\text{peak}}$ that would affect performance would be given by a sequence of 10 pulses with amplitude $d_{\text{peak}}$. The value of $\text{eye}_{\text{peak}}$ that would affect performance at an error rate of $10^{-4}$, is obtained both by using (3.9) with $2n = 10$ and by simulation. The results are plotted in Fig. 3.3.

### 3.3.3 Interference margin

The interference margin given by (3.2) can be rewritten using (3.10) as

$$\text{im} = \frac{d_{\text{peak}}(m)}{d} s(\alpha),$$

(3.11)
where $d_{\text{peak}}(m)$ is the peak of a $2^m$-ary constellation. Higher values of $\alpha$ are preferable from the point of view of interference margin but they come at the expense of bandwidth. The higher the value of $\alpha$, the higher the excess bandwidth and therefore the lower the throughput (bits/s/Hz).

Thus for a given throughput, an interesting optimisation problem is to determine if a small constellation with a small excess bandwidth, or a larger constellation with a larger excess bandwidth has better interference margin. We consider the family of $M^2$ QAM constellations, where $M^2 = 2^m$, $m$ any integer, and determine the best constellation in the sense of having the largest interference margin, as a function of the bit rate. The shaping factor is calculated using (3.9) with $2n = 10$ pulses. The constellations for $m = 1, 2, 3$ and 4, are shown in Fig. 3.8. We compare the interference margin by keeping the minimum distance the same in all constellations, i.e.,

$$d_{\text{min}} = 2d,$$
for all constellations. The peak of a $2^m$-ary QAM constellation $d_{\text{peak}}(m)$ is given by

$$d_{\text{peak}}(m) = \frac{(2^m/2 - 1)}{\sqrt{2}} \quad \text{for } m \text{ even},$$

and

$$d_{\text{peak}}(m) = \frac{(2^{(m+1)/2} - 1)}{2} \quad \text{for } m \text{ odd}.$$

A $2^m$-point signal constellation transmitting at an excess band width of $\alpha$, is referred to as an $(m, \alpha)$ signaling scheme, and achieves the transmission rate $x$ given by,

$$x = \frac{m}{1 + \alpha} \text{ bits/s/Hz}$$

(3.12)

Figure 3.8: $2^m$-ary QAM constellations for $m = 1, 2, 3, 4, 5$.

The same rate $x$ as obtained in (3.12), can also be obtained by an $(m + 1, \alpha')$ signaling scheme, or an $(m + 2, \alpha'')$ signaling scheme and so on, where $\alpha$, $\alpha'$ and $\alpha''$ are related by the equation

$$x = \frac{m}{1 + \alpha} = \frac{m + 1}{1 + \alpha'} = \frac{m + 2}{1 + \alpha''} \ldots$$

(3.13)

where, $\alpha$, $\alpha'$ and $\alpha''$ are constrained to be between 0 and 1. The signaling scheme which gives the best interference margin as computed by (3.11), is more immune to interference than others and is chosen for transmission on a CCI limited channel. The
signaling schemes that give the best interference margin is computed as a function of the bit rate and plotted in Fig. 3.3.3.

As can be seen, at the rate of 1.5 bits/s/Hz, a 4 point QAM constellation gives better noise margin than an 8 point constellation. However at the rate of 2.5 bits/s/Hz, a larger constellation (16 point QAM) is better than a smaller constellation (8 point QAM) with smaller excess bandwidth.

3.4 Conclusion

In Chapter 3, the effect of the pulse shape on the interference immunity of a system is considered. Raised cosine pulses of different excess bandwidths are studied, for their interference immunity. It is shown that with non-rectangular pulses, the interference margin is reduced by a factor called the shaping factor. Pulses with faster rate of decay and smaller amplitude of sidelobes give better interference margin. This comes at the expense of large excess bandwidth. The optimisation problem that is considered is to determine if, for a given throughput, a small constellation with a small excess bandwidth, or a larger constellation with a larger excess bandwidth had better interference margin. Different QAM constellations are analyzed and it is found that for every bit rate there is one particular QAM constellation and the corresponding excess bandwidth that has better noise margin than all other QAM constellations.
Chapter 4

Performance analysis of coded QAM signals

4.1 Introduction

It is well known that for a Gaussian noise channel, the minimum squared Euclidean distance between the transmitted symbol sequences determines the probability of error at large values of signal to noise ratio [31, 32]. In the case of an interference channel, it was shown that the minimum distance is reduced by the peak of the interference constellation. When considering the decision distance of a multi-dimensional constellation it turns out that the dominant error event for the Gaussian channel is not always the dominant error event for the interference channel. This means that an optimal code designed for a Gaussian channel is not always optimal for the interference channel. In this Chapter the error event probability bounds are evaluated for coded QAM signals using the bounds developed in Section 2.3.
4.2 Asymptotic error event probability for high SNR

We base our discussion on 2-dimensional QAM signals. Most of our analysis will be centered around coded 16-QAM signals.

In Section 2.3 it was shown that, in the case of a two point constellation in a CCI limited channel, the probability of error can be upper bounded by

\[ p_e(a, c) \leq \text{erfc} \left( \sqrt{\frac{d - d_{\text{peak}}}{2N_0}} \right), \]  

(4.1)

where the distance between the points \( d \), is reduced by the peak of the CCI \( d_{\text{peak}} \) that can interfere between the points. Now if we consider two distinct data sequences \( a \) and \( c \), then the pairwise probability of error can be written as

\[ p_e(a, c) \leq \text{erfc} \left( \sqrt{\frac{(d(a, c) - d_{\text{peak}}(a, c))^2}{2N_0}} \right), \]  

(4.2)

where \( d(a, c) \) is the Euclidean distance between \( a \) and \( c \), given by

\[ d^2(a, c) = \sum_k d^2(a_k, c_k), \]

where \( d^2(a_k, c_k) \) is the squared Euclidean distance between the symbols at time \( k \). The distance between the data sequences is reduced by \( d_{\text{peak}}(a, c) \), which is the peak of the cochannel signal that can interfere between \( a \) and \( c \). Let the sequences \( a \) and \( c \) differ in \( l \) symbol intervals as shown in Fig. 4.1. At every symbol interval the interference signal can take on a peak square value of \( d_{\text{peak}}^2 \), which is equal to the peak of the two dimensional constellation. Hence over a length of \( l \) symbol intervals the interference signal can take on a possible peak of \( ld_{\text{peak}}^2 \). This is the maximum
possible value of the signal that can interfere between a and c. But whether this peak value will limit performance will depend on the order of the probability of error we are interested in, and also the probability of occurrence of the peak $ld_{2\text{peak}}^2$. For instance, if we are interested in a probability of error of the order of $10^{-4}$, then a peak CCI of $ld_{2\text{peak}}^2$ with a probability of occurrence of $10^{-8}$ will not limit performance.

![Figure 4.1: Two data sequences differing in l symbol durations.](image)

As an example, we consider the case where the 2-dimensional signal points are chosen from a 16-QAM constellation. An expression for the probability density function of the interference component was derived in Chapter 2. The cumulative distribution function $F_X(x)$ of the cosine component of a 16-QAM CCI is plotted in Fig. 4.2, where for this analysis cdf is defined as

$$F_X(x) = P(X \geq x). \quad (4.3)$$

The probability distribution of CCI over higher symbol intervals can be obtained by the convolution of the pdf over lower symbol intervals. In Fig. 4.3 is shown the cdf of CCI over 1, 2, 4, 8, 16 and 32 symbol intervals. The plot is shown on a different scale in Fig. 4.4 to show the distribution of CCI at an error rate of $10^{-4}$. As can be seen from the plot, at probabilities of the order of $10^{-4}$, the peak squared CCI is not proportional to $l$. 

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Figure 4.2: The cdf of the cosine component of a 16-QAM CCI.

Figure 4.3: The cdf of 16-QAM CCI over many symbol intervals.
Thus we define a parameter called the corrected length $l_c(l, p)$ as

$$l_c(l, p) = \frac{d_{\text{peak}}(l, p)}{d_{\text{2peak}}}, \quad (4.4)$$

where $d_{\text{peak}}(l, p)$ is the peak CCI of duration $l$ symbol intervals that can affect performance at an error rate of $p$, and is given by

$$F_{X,l}(d_{\text{peak}}(l, p)) = p,$$

where $F_{X,l}(x)$ is the probability distribution of CCI over $l$ symbol intervals, which is obtained by the convolution of the distribution over smaller intervals. The corrected length for a 16-QAM is plotted as a function of $l$ in Fig. 4.5, for $p = 10^{-4}$.

Based on the above discussion, we can rewrite the probability of error expression...
of (4.2) as

$$p_e(a, c) \leq \text{erfc} \left( \sqrt{\frac{\left( \frac{d(a,c)}{2} - \frac{\sqrt{l_c(l_p)d_{\text{peak}}}}{\sqrt{\text{CIR}}} \right)^2}{N_0}} \right).$$

An overall upper bound on the average event error probability can be obtained by the union bound summing over all possible error events. This is given by,

$$p_e \leq \sum_{\{(a, c)\}} \text{erfc} \left( \sqrt{\frac{\left( \frac{d(a,c)}{2} - \frac{\sqrt{l_c(l_p)d_{\text{peak}}}}{\sqrt{\text{CIR}}} \right)^2}{N_0}} \right).$$

where the summation is over all pairwise distinct sequences \{(a, c)\}. From the above equation it can be seen that the dominating term at low values of \(N_0\) is given by the smallest \( \left( \frac{d(a,c)}{2} - \frac{\sqrt{l_c(l_p)d_{\text{peak}}}}{\sqrt{\text{CIR}}} \right)^2 \). At high values of CIR, the upper bound is still dominated by terms with minimum \(d^2(a,c)\). As CIR becomes low, the dominant
term is determined by the combination of $d(a, c)$, $d_{peak}$ and $l_c$. If the CIR is such that $\frac{d(a, c)}{2} > \frac{\sqrt{l_c d_{peak}}}{\sqrt{\text{CIR}}}$, then the probability of error of $p$ can be achieved ultimately as SNR is increased. If the CIR is such that $\frac{d(a, c)}{2} \leq \frac{\sqrt{l_c d_{peak}}}{\sqrt{\text{CIR}}}$, then the probability of error of $p$ cannot be achieved even for high SNR s.

To see why this is so we look at the graphical interpretation of this inequality. The decision distance between the data sequences $a$ and $c$ is given by $\frac{d(a, c)}{2}$. From the definition of the corrected length $l_c$, we can say that, with a probability greater than $p$, the interference component will take on a value greater than $\frac{\sqrt{l_c d_{peak}}}{\sqrt{\text{CIR}}}$. Thus if the CIR is such that $\frac{d(a, c)}{2} \leq \frac{\sqrt{l_c d_{peak}}}{\sqrt{\text{CIR}}}$, then with a probability greater than $p$, the interference component will be greater than the decision distance even when Gaussian noise is not present. Therefore a probability of error of $p$ cannot be achieved even for high SNR s.

Thus the interference margin is determined by the term in the upper bound with the smallest $\frac{d(a, c)}{l_c}$. This determines the cutoff CIR, below which a probability of error of $p$ cannot be achieved. Notice that as CIR approaches cutoff, the dominant error event is also the one with minimum $\frac{d(a, c)}{l_c}$ ratio. Thus the design of coded modulation schemes in a CCI limited channel should be aimed at obtaining the best minimum $\frac{d}{l_c}$ ratio. In the following section we look at the performance of some coded modulation schemes with examples.

4.3 Performance of coded modulation schemes

4.3.1 Block coded modulation

We consider the rate 3/4 block code, represented by the trellis in Fig. 4.6. We begin with a 16-QAM constellation partitioned into four subsets [35], as shown in Fig. 4.7. This coded modulation scheme achieves a rate of 3 bits/2D (see [36]). Referring to
Table 4.1: Distance profile of the rate 3/4 block coded modulation scheme.

<table>
<thead>
<tr>
<th>l</th>
<th>(d_{\text{min}}^2(l))</th>
<th>(l_c)</th>
<th>(\frac{d^2}{l_c})</th>
<th>CIR(_{\text{cutoff}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1.9</td>
<td>2.1053</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2.45</td>
<td>1.6327</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>1.33</td>
<td>11.3142 dB</td>
</tr>
</tbody>
</table>

The trellis of Fig. 4.6, the error events are of lengths 1, 2, 3 and 4. Corresponding to each length we find the minimum distance error event. In Table 4.1, we list the minimum distance, the corrected length, and the parameter \(\frac{d^2}{l_c}\), for each error event length.

As can be seen, although the minimum distance corresponding to each length is 4, the peak value of interference that can affect performance at the error rate of \(10^{-4}\) is 3 for \(l = 4\), while it is less for \(l = 1, 2\) and 3. Therefore the dominant error event
at low values of CIR are the ones with length 4. Referring to Fig. 4.6, we see that if the all zeros path is transmitted, it is more likely to be mistaken for any path in the $B$ sub trellis which have a length of 4 symbol intervals, than any path in the $A$ sub trellis, even though they are at the same squared distance from the all zeros path. The CIR cutoff of this scheme is also shown in the table.

In contrast the uncoded scheme that achieves the same bit rate of 3 bits/2D, gives a CIR cutoff of 9.54dB which is almost 2dB less than the coded scheme mentioned above. Yet it is not the best choice for CCI limited communication, because at low to moderate CIR s, the performance is below that of the coded scheme. This is because the uncoded scheme has a lower minimum free distance than the coded scheme. This is seen in the simulation results shown in Fig. 4.8. This figure gives the error rates of uncoded and the rate 3/4 block coded modulation schemes. As can be seen, for CIR $\geq 14$dB, the coded scheme performs better. Below this value the uncoded scheme is better. The CIR cutoff for the coded scheme is seen to be 12dB which is more than we had predicted through the model. This is because simulations were done using raised cosine pulses with a roll off factor of 0.3, while the results of Table 4.1 are for
rectangular pulses. The effect of pulse shaping was discussed in Chapter 3.

4.3.2 Trellis coded modulation

Now we look at the rate 1/2, 4 state convolutional code with the best $\frac{d^2}{I_c}$ ratio. The trellis for this code is shown in Fig. 4.9. Signals are chosen from the 16-QAM constellation partitioned as in Fig. 4.7, to give an overall rate of 3 bits/2D. The code with the best $\frac{d^2}{I_c}$ ratio was found by computer search to have the generator polynomials $g^{(1)} = 101$ and $g^{(2)} = 111$. A bidirectional search procedure [41, 42] is used, and the details are given in Appendix A. The distance profile of this code is shown in Table 4.2.

The CIR cutoff in this case is about a dB better than the block coded modulation scheme with the same number of states and same rate of 3 bits/2D. It also has the
\[ g^{(1)} = 101; \quad g^{(2)} = 111 \]

<table>
<thead>
<tr>
<th>( l )</th>
<th>( d_{\text{min}}^2(l) )</th>
<th>( \ell_c )</th>
<th>( \frac{d^2}{\ell_c} )</th>
<th>CIR_{\text{cutoff}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>2.45</td>
<td>2.041</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>3</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3.375</td>
<td>1.778</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>3.75</td>
<td>1.867</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>4.125</td>
<td>1.697</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>4.5</td>
<td>1.778</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>4.75</td>
<td>1.684</td>
<td>10.3 dB</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>5</td>
<td>1.8</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>9</td>
<td>5.25</td>
<td>1.7143</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>5.5</td>
<td>1.8182</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>5.75</td>
<td>1.7391</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2: The distance profile of the best rate 1/2 convolutional code with 4-states.
same minimum free distance as the block code and therefore will perform well even at low to moderate values of CIR.

We also analyse rate $R = 1/2$ convolutional codes with higher number of states. All the codes select signals from the 16-QAM constellation partitioned as shown in Fig. 4.7 and therefore achieve a rate of 3bits/2D. The results are summarised in Table 4.3. The distance profile of these codes are given in Appendix B. The memory order of the convolutional code is given by $m$ and the generator polynomials are given by $g^{(1)}$ and $g^{(2)}$ [33, 34].

### 4.4 Shaping

Thus far we were concerned with how to design codes that give a good interference margin and we found that the best codes gave modest gains over codes which were not optimised for the CCI channel. The shape of the constellation affects the peak CCI
that can interfere and therefore can affect the interference margin. As an example we consider the rate 2/3 block code given by the trellis of Fig. 4.6. This can be considered as an 8 dimensional code. By choosing signals from a 16–QAM constellation for each branch of the trellis we are actually considering a constellation shaped as a hyper cube in 8-D. We consider the case where the constellation is shaped as an 8-D sphere [37, 38]. We assume that regions with the same volume have the same number of points in them, and therefore we choose the radius of the sphere such that the volume of the 8-D sphere and the hyper cube are the same. Thus if $d_{\text{peak}}$ is the peak of the 2-D QAM, the volume of the 8-D hyper cube, $V_{8\Delta}$ is

$$V_{8\Delta} = \left( \frac{2d_{\text{peak}}}{\sqrt{2}} \right)^8.$$  (4.7)

The volume of an 8-D sphere of radius $R$ is

$$V_{8\Theta} = \frac{(\pi R^2)^4}{4!}. \quad (4.8)$$

From (4.7) and (4.8) we find that the radius $R$, of the sphere with the same volume as the hyper cube is given by,

$$R^2 = \frac{2d_{\text{peak}}^2 (4!)^{1/4}}{\pi} \quad (4.9)$$

$$= 1.4d_{\text{peak}}^2. \quad (4.10)$$

Table 4.3: The best rate 1/2 convolutional codes.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$g^{(1)}$</th>
<th>$g^{(2)}$</th>
<th>$d^2/l_c$</th>
<th>CIR$_{\text{cutoff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>101</td>
<td>111</td>
<td>1.68</td>
<td>10.3</td>
</tr>
<tr>
<td>3</td>
<td>1101</td>
<td>1111</td>
<td>1.6842</td>
<td>10.3</td>
</tr>
<tr>
<td>4</td>
<td>11111</td>
<td>01011</td>
<td>1.6</td>
<td>10.52</td>
</tr>
<tr>
<td>5</td>
<td>100011</td>
<td>101101</td>
<td>1.7143</td>
<td>10.22</td>
</tr>
<tr>
<td>6</td>
<td>1101011</td>
<td>1000101</td>
<td>1.7391</td>
<td>10.16</td>
</tr>
</tbody>
</table>
Table 4.4: Distance profile of rate 3/4 block coded modulation with spherical shaping.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$d_{\text{min}}^2$</th>
<th>$l_c$</th>
<th>$\frac{d^2}{l_c}$</th>
<th>CIR_{cutoff}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1.4</td>
<td>2.857</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1.4</td>
<td>2.857</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1.4</td>
<td>2.857</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1.4</td>
<td>2.857</td>
<td>7.99 dB</td>
</tr>
</tbody>
</table>

The distance profile of the resulting scheme is shown in Table 4.4. The peak CCI that can interfere with any length of error event is the same because the projection of a sphere on any number of dimensions is the same. Comparing Table 4.4 with Table 4.1, we see that the $d^2/l_c$ ratio has decreased for $l = 1$, but for all other lengths it has increased. This is because, although the hyper sphere has a smaller overall peak, the peak in smaller dimensions may be larger than the corresponding peak of a hyper cube. Nevertheless constellation shaping has helped increase the $d^2/l_c$ ratio of the dominant error event, thus decreasing the CIR_{cutoff} from 11.3dB to 7.99dB.

Choosing the optimal shaping region in the sense of obtaining the best interference margin, will depend on the distance profile of the code used. For the convolutional code of Section 4.3.2, with distance profile as in Table 4.2, the best shaping region would be one that projects least into 18 dimensions, and then 14 dimensions and so on. This is because, the dominant error event (one with least $d^2/l_c$ ratio) is of length 9 symbol intervals and the next dominant error event is of length 7 intervals and so on. The problem of designing the best shaping region has not been studied in this thesis.
4.5 Conclusion

In this chapter we consider the performance of coded modulation schemes in a CCI limited environment. In particular, we consider coded QAM schemes. It turns out that at low CIR, the dominant error event is not one with the minimum squared distance but one with the minimum $d^2/l_c$ ratio. Using this parameter, coded modulation schemes that show good interference immunity can be designed. We present rate $1/2$ convolutional codes with the best interference margin for memory orders 2 to 6, which were obtained through a search procedure.

We also show that constellation shaping affects the interference immunity of a system. In general, the shape that is optimal in the sense of providing a good interference margin will depend on the distance profile of the code. Spherically shaped constellations used with a rate 2/3 block code gave a CIR cutoff gain of 3dB.
Chapter 5

Conclusion

The design of a communication system depends on the characteristics of the channel that is used for transmission, and in many cases the performance of the system can be improved by successfully exploiting these characteristics. The common impairments that are present in any channel are listed in Chapter 1, of which the effects of additive white Gaussian noise are most studied. However in many communication systems the dominant interference is cochannel interference and not noise. In general, a system that is designed to perform well in the AWGN channel need not perform well in the CCI limited channel. This thesis is concerned with designing modulation and coding techniques that exhibit robust performance in a CCI limited channel.

In Chapter 2, the constellation properties that are important for QAM schemes operating in a CCI limited channel are studied. The threshold CIR below which error free transmission is not possible is determined by the peak of the constellation. However, at low to moderate error rates that might be of interest to us, the CIR cutoff is not determined by the peak. In Chapter 2 we show that the threshold CIR below which a probability of error of \( p \) cannot be achieved is given by a parameter called the modified peak \( d_m(p) \). The value of \( d_m \) can be obtained from the interference...
spectrum, for any error rate that we might be interested. We have shown that given any modulation scheme, the interference margin or the CIR cutoff, below which an error rate of \( p \) cannot be achieved is determined by \( d_m(p)/d_{\min} \).

In Chapter 3, the effect of the pulse shape on the interference immunity of a system is considered. Raised cosine pulses of different excess bandwidths are studied, for their interference immunity. We show that the interference margin depends on the rate of decay of the pulses. QAM constellations are analyzed to determine if, for a given throughput, a small constellation with a small excess bandwidth or a large constellation with a large excess bandwidth had better interference margin.

In Chapter 4, analysis is extended to coded QAM schemes. We have shown that in a CCI limited system the dominant error event is not one with the minimum squared distance but one with the minimum \( d^2/l_c \) ratio, where \( l_c \) is the corrected length. The corrected length can be determined from the interference spectrum as a function of error rate. Rate 1/2 convolutional codes with the best interference margin are presented in Chapter 4, for memory orders 2 to 6. We also show that constellation shaping affects the interference immunity of a system. In general, the shape that is optimal in the sense of providing a good interference margin will depend on the distance profile of the code. Spherically shaped constellation used with a rate 2/3 block code gave a CIR cutoff gain of 3dB.

Thus this thesis analyses some aspects of coding and modulation in a CCI limited channel, and offers some solutions to design systems with better interference immunity. The problem of designing optimum detectors that make use of the statistical knowledge of CCI has not been considered in this work. In fact using multi-user detection techniques have shown to give good interference suppression capability [39, 40]. More work needs to be in this area to obtain increased interference immunity.
Appendix A

An algorithm to find the minimum $d^2/l_c$ ratio of a code

The parameter $d^2/l_c$ of a code has been shown to be a good indicator of the interference immunity of the coding scheme (Chapter 4). The code with the best $d^2/l_c$ ratio is found using search procedures. In order to do this we first determine the minimum free distance, $d_{\text{min}}^2(l)$, corresponding to each length $l$, up to a maximum length $l_{\text{max}}$. Corresponding to each of these lengths, we compute the parameter $d^2/l_c$, where $l_c$ is the corrected length and can be determined using the procedure outlined in Chapter 4. It was also shown in Chapter 4 that the rate of increase of $l_c$ with $l$, flattens as $l$ increases and therefore the parameter $d^2/l_c$ should exhibit an increasing trend as $l$ is increased large enough. The value of $l_{\text{max}}$ is chosen large enough to get an increasing trend of $d^2/l_c$ with $l$.

The algorithm we propose searches through the trellis diagram of the encoder. The trellis diagram is considered as a weighted graph, whose nodes are the states of the encoder and whose branches are the allowable state transitions. Each branch in the trellis corresponds to a subset of a constellation rather than to a single signal.
point. For example if we consider a constellation partitioned into 4 as in Fig. A.1, and a 4 state rate 1/2 trellis as shown in Fig. A.2, then corresponding to each of the outputs of the transition we assign a constellation subset. The weights of the branches are the minimum inter subset squared distances with reference to the subset corresponding to the all zeros path.

\[
\begin{array}{cccc}
  a_0 & b_0 & a_0 & b_0 \\
  x & x & x & x \\
  b_1 & a_1 & b_1 & a_1 \\
  x & x & x & x \\
  a_0 & b_0 & a_0 & b_0 \\
  x & x & x & x \\
  b_1 & a_1 & b_1 & a_1 \\
  x & x & x & x \\
\end{array}
\]

Figure A.1: Set partitioning of 16-QAM

In order to find the minimum weight path corresponding to each length we use a bidirectional search algorithm. We define \( f(i, j) \) as the minimum weight of the paths of length \( i \) that start with state 0 and end at state \( j \). Also \( b(i, j) \) is defined as the minimum weight of the paths of length \( i \) that start at state \( j \) and end at state 0. Therefore the minimum weight of the paths corresponding to length \( l = i + k \) will be given by

\[
d_{\text{min}}^2(i + k) = \min [f(i, j) + b(k, j)] \quad 0 \leq j \leq 2^m,
\]

where \( m \) is the memory order and \( 2^m \) is the number of states. In order to compute \( f(.) \) and \( b(.) \), we define another square matrix \( m(i, j) \), called the state transition matrix, and it gives the weight of the branch from state \( i \) to \( j \). An \( x \) denotes the position
where a transition from state $i$ to $j$ is not possible. We now describe the algorithm in detail. We find $f$ and $b$ recursively. The first 4 steps are for initialising $f$ and $b$ and also for setting a counter for $l$.

1. $i = 1$.

2. $f(1, j) = m(1, j) \quad 0 \leq j \leq 2^m - 1$,

3. $f(1, j) = m(1, j) \quad 0 \leq j \leq 2^m - 1$,

4. $d_{\min}^2(2) = \min [f(1, j) + b(1, j)] \quad 0 \leq j \leq 2^m - 1$.

5. $k = 1$.

6. $f(i + 1, k) = \min [f(i, j) + m(j, k)] \quad 0 \leq j \leq 2^m - 1$,

7. $b(i + 1, k) = \min [b(i, j) + m(k, j)] \quad 0 \leq j \leq 2^m - 1$,

8. If $k < 2^m$ then $k = k + 1$, go to 6.
9. \( d_{min}^2(2i + 1) = \min \{ f(i, j) + b(i + 1, j) \} \quad 0 \leq j \leq 2^m - 1 \),

10. \( d_{min}^2(2i + 2) = \min \{ f(i + 1, j) + b(i + 1, j) \} \quad 0 \leq j \leq 2^m - 1 \).

11. If \( i < \lceil m/2 \rceil \) then \( i = i + 1 \), go to 6.

12. Stop.
Appendix B

Distance profiles of the best rate 1/2 convolutional codes of memory orders 3, 4, 5 and 6

The distance profiles for the best rate 1/2 codes that have been found using the procedure outlined in Appendix A is given here.
$g^{(1)} = 1101; \ g^{(2)} = 1111$

<table>
<thead>
<tr>
<th>$l$</th>
<th>$d_{\min}^2(l)$</th>
<th>$\frac{e^2}{I_e}$</th>
<th>CIR$_{\text{cutoff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>2.33</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1.778</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>1.867</td>
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<tr>
<td>7</td>
<td>7</td>
<td>1.697</td>
<td></td>
</tr>
<tr>
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<td>8</td>
<td>1.778</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>8</td>
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<td>10.3dB</td>
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<td>10</td>
<td>9</td>
<td>1.8</td>
<td></td>
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<td>11</td>
<td>9</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>1.818</td>
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<tr>
<td>13</td>
<td>10</td>
<td>1.73</td>
<td></td>
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</table>

Table B.1: The distance profile of the best rate 1/2 convolutional code with 8 states.
Table B.2: The distance profile of the best rate 1/2 convolutional code with 16 states.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$d_{\text{min}}^2(l)$</th>
<th>$\frac{d^2}{J_c}$</th>
<th>CIR$_{\text{cutoff}}$</th>
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<tr>
<td>1</td>
<td>4</td>
<td>4</td>
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<td>5</td>
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</tr>
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<td>8</td>
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<td>1.778</td>
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</tr>
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<td>8</td>
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</tr>
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<td>11</td>
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<td>1.9</td>
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</tr>
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<td></td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>1.66</td>
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</tr>
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<td>15</td>
<td>10</td>
<td>1.6</td>
<td>10.52dB</td>
</tr>
<tr>
<td>16</td>
<td>12</td>
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<td></td>
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<td>17</td>
<td>12</td>
<td>1.804</td>
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<tr>
<td>18</td>
<td>12</td>
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</table>

$g^{(1)} = 01011; \quad g^{(2)} = 11111$
Table B.3: The distance profile of the best rate $1/2$ convolutional code with 32 states.

\[
g^{(1)} = 100011; \quad g^{(2)} = 101101
\]

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<tr>
<th>(l)</th>
<th>(d_{\text{min}}^2(l))</th>
<th>(\frac{\Delta^2}{I_c})</th>
<th>CTR_{cutoff}</th>
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Table B.4: The distance profile of the best rate 1/2 convolutional code with 64 states.

\[ g^{(1)} = 1101011; \quad g^{(2)} = 1000101 \]

<table>
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<tr>
<th>( l )</th>
<th>( \mathbf{d}_{\text{min}}^2(l) )</th>
<th>( \frac{d}{l_c} )</th>
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Bibliography


