Robust $\mathcal{H}_\infty$ – Based Control of Flexible Joint Robots with Harmonic Drive Transmission

by

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A thesis submitted in conformity with the requirements for the Degree of Doctor of Philosophy

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To my wife
and
to our parents
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Abstract

The problem of the control of flexible joint robots is considered. Motion and torque control of robot joints, especially those equipped with harmonic drive (HD) transmission, is a challenging task due to the inherent nonlinear characteristics and joint flexibility of such systems. Two issues related to the control of flexible joint robots are investigated. The first is the development of a systematic scheme for selecting uncertainty bounds of robot joint nonlinearity and flexibility for control design purposes. The second is the development of motion and torque control schemes that guarantee robust performance in the presence of model uncertainties.

We propose a twofold robust control design for flexible joint robots with HD: an actuator-level torque control and a link-level motion control. We utilize multivariable $\mathcal{H}_\infty$—based optimal control laws supported entirely by frequency domain measures at both levels. The proposed method provides a unified framework for achieving the desired performance requirements and for preserving robust stability in the presence of model uncertainties. Using simulation it is shown that the proposed
method is more robust than conventional methods because the uncertainties due to
the actuator-transmission nonlinearity are explicitly considered in the control design
procedure.

In the design of the actuator level torque control, we analyze nonlinear har-
monic drive phenomena that have been widely observed in experiments. The focus is
on incorporating into the control design process a knowledge of the mismatch between
the physical system and its mathematical models. The describing function and conic-
sector-bounded nonlinearity methods are used to build into the control design process
the effects of mismatch between hysteresis, friction and nonlinear stiffness of HD
transmission and their mathematical models. In the design of the link level motion
control, a nonlinear compensator based on the computed torque technique is de-

erived. It is shown that the closed-loop system achieves robust performance using the
proposed control design technique.

Using the Small Gain theorem and the Lyapunov Function method, the sta-
bility of the proposed control scheme is verified. Finally, in order to illustrate the
proposed technique two control designs are presented for the IRIS-facility experi-
mental testbed (a versatile, modular, and reconfigurable prototype robot developed
at the Robotics and Automation Laboratory of the University of Toronto) together
with simulation and experimental results.
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Nomenclature

Fields, Norms and Spaces

- \( \mathbb{C} \): Complex numbers
- \( \| f \|_\infty := \text{ess sup}_{t \in \mathbb{R}^+} \| f(t) \|_\infty \), \( \mathbb{L}_\infty \) norm of \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^r \)
- \( \mathcal{H}_2 \): Space of Fourier transform of signals in \( \mathcal{L}_{2+} \)
- \( \mathcal{H}^\perp \): Space of Fourier transforms of signals in \( \mathcal{L}_{2-} \)
- \( \mathcal{H}_\infty \): Hardy space of functions analytic in \( \mathbb{C}_+ \) and bounded on the imaginary axis, with norm \( \| Q \|_\infty := \sup_{\omega \in \mathbb{R}} \| Q(j\omega) \| \)
- \( K \): Linear \( \mathcal{H}_\infty \) controller transfer function
- \( \mathcal{L}^p \): Normed linear space
- \( \mathcal{L}^p_\times \): An extended linear space
- \( \mathcal{L}_2 \): Space of Lebesgue measurable functions on \( \mathbb{R}_+ \) with finite \( \mathcal{L}_2 \) norm
- \( \mathcal{L}_\infty \): Space of Lebesgue measurable functions on \( \mathbb{R}_+ \) with finite \( \mathcal{L}_\infty \) norm
- \( \mathcal{L}_{2+} \): Space of signals defined for positive time and zero for negative time
- \( \mathcal{L}_{2-} \): Space of signals defined for negative time and zero for positive time
- \( \mathbb{R} \): Real numbers
- \( \mathcal{R} \mathcal{H}_\times \): Space of real-rational functions in \( \mathcal{H}_\infty \)
- \( \hat{x} \): Fourier transform of \( x \)
- \( \sigma_{\max}[.] \): Maximum singular value of a complex matrix
- \( \| x \| := (x^*x)^{\frac{1}{2}} \), Euclidean norm of \( x \in \mathbb{C}^r \)
- \( \| x \|_\infty := \max_{i=1,...,r} |x_i| \), \( \infty \)-norm of \( x \in \mathbb{R}^r \)
- \( \lambda \): Maximum eigenvalue of the matrix

Parameters

- \( a_{f_1} \) and \( a_{f_2} \): Constant coefficients
- \( \dot{B}_w(\dot{\phi}_w) \): Total friction torque generated at the bearings
- \( \dot{B}_w(\dot{\phi}_w + \dot{\phi}_f) \): Friction term of the wave-generator bearing
- \( e \): Angular position and velocity errors
\( I_s \) upper-bound of the current defined by the amplifier electronics
\( i_m \) Motor current
\( J_{mw} \) combined inertia of the rotor, shaft and the wave-generator
\( J_m \) \( n \times n \) diagonal matrix of actuator inertia
\( K_{sw} \) Soft windup correction factor
\( k_{sw} \) and \( a_{sw} \) Constants
\( k_b \) Back e.m.f. constant of the motor
\( k_t \) Motor torque constant
\( k_s \) Amplifier gain
\( M(q_c) \) \( n \times n \) link inertia matrix
\( M \) Output signal amplitude
\( \hat{M} \) Mathematical model of the \( M \)
\( \hat{N} \) Mathematical model of the \( N \)
\( N(q_c, \dot{q}_c) \) \( n \times 1 \) vector of centrifugal, gravity, and Coriolis (generalized) forces
\( N \) Gear ratio
\( N_t \) Number of teeth on the flex-spline outer circumference
\( N(s) \) Describing function
\( P(s) \) Nominal plant model
\( \hat{P}(s) \) Perturbed plant transfer function
\( \Delta P \) An unknown perturbation
\( \dot{q}_m \) Motor angular velocity
\( \dot{q}_f \) Flex-spline angular velocity
\( q_c \) \( C\)-end angular displacement
\( \delta(q_f) \) an unknown nonlinearity
\( q_c \) \( n \times 1 \) link position
\( q_c^* \) \( n \times 1 \) vector of desired link position trajectory
Motor armature resistance
C-end load torque
Driving torque
Motor Torque
Torque sensor output located at the link side
\( \tau_m \) n \times 1 vector of control torque input
Torque exerted on the wave-generator
New control input
Applied torque
Amplifier input command
\( u_l \) n \times 1 vector of fictitious control input
Maximum output voltage of the amplifier
Frequency points
A real-rational, stable minimum phase transfer function
Frequency dependent weighting matrices
Noise weighting function
Slope at the transition from loading to unloading
A norm bounded complex number
Phase shift
An arbitrary signal
Chapter 1

Introduction

1.1 Motivation

The use of gearing systems is a well established means for lighter manipulators to handle heavier payloads. However, gear systems suffer from nonlinearities such as backlash, nonlinear stiffness, and various types of friction. Correspondingly, the inherent compliance between the input and output shafts of a transmission system introduces undesirable resonant behavior during robot motion. The design of control laws that overcome transmission elasticity is quite challenging from both practical and theoretical standpoints. The highly nonlinear nature of robot dynamic models, coupled with the joint flexibility problem, is indeed an open control problem. As the operating bandwidth of the robot increases, the joint flexibility becomes more important [47, 46, 52]. The joint flexibility introduces a fast dynamic characteristic in robots [40]. The inherent interaction of the fast dynamics (joint flexibility dynamics) and the slow varying robot dynamics (rigid body dynamics) can destabilize the system. Moreover, high frequency present in the input signal may excite the fast dynamics introduced by the joint flexibility.

Most of the developed control schemes for flexible joint robots assume that torque commands can be arbitrarily applied to each robot joint. In reality, these
torques are applied through the use of joint actuators. Unfortunately, the actuators possess non-negligible mechanical dynamics such as inertia, friction and compliance. For the design of high performance motion and torque control of robotic applications it is necessary to model accurately the drives involved. Motion and torque control of manipulators relies mostly on the ability of the actuation systems to provide desired joint torques. But, for robotic manipulators, particularly those equipped with harmonic drives (HD), the ability of the actuation system to provide desired joint torques is considerably restricted by the inherent nonlinearity, friction and flexibility of the drive involved. Therefore, more advanced controllers must be employed to guarantee robust stability and performance of such robotic systems.

There are various approaches in the literature dealing with the control of flexible joint robots. Such approaches suffer from two major drawbacks. First, the joint flexibility is usually modeled as a pure linear spring, whereas the joint stiffness is nonlinear and there is hysteresis and nonlinear friction in the joints. Second, the performance is sensitive to model uncertainty and measurement noise. The existing control methods fail to integrate into a common framework explicit knowledge of model errors, robustness bounds and performance specifications. These deficiencies often degrade the effectiveness of the control schemes available in the literature.

To overcome these deficiencies, this thesis formulates a robust control scheme based on the \( \mathcal{H}_\infty \) - optimal control approach. The \( \mathcal{H}_\infty \) - optimal control design technique, based entirely on frequency domain measures, has been developed recently [8, 11]. It provides a unified framework for addressing simultaneously uncertainty measures and performance requirements. It has also the advantage of allowing the designer to use experimental results to yield uncertainty descriptions and corresponding levels (i.e., weights) for control design purposes. We propose a twofold robust control design of flexible joint robots with HD: an actuator-level torque control and a link-level motion control. We propose multivariable \( \mathcal{H}_\infty \) - optimal control laws at both levels. This approach allows us to focus on link and actuator control problems.
It is shown that the closed-loop system achieves robust performance using the proposed control design technique. Experiments are also performed to test the proposed control design scheme.

1.2 Background on Flexible Joint Robot Control

As the operating bandwidth of a robot increases, joint flexibility becomes more important. High frequency present in the input signal may excite the fast dynamics introduced by the flexibility of the drive and may destabilize the system. The source of joint flexibility may be torque transducers [57, 3], drive shaft stiffness [2], or harmonic drives [47]. The problem of robot manipulation in experiments, and sources of the performance limitation of a robot were investigated by Sweet and Good [47]. Rivin [41] determined many sources of joint flexibility such as harmonic drives, belt chains, torsional shafts, and gear reducers.

Much effort has been devoted to control of flexible joint robots. Different techniques have been proposed in the literature to deal with modeling and control issues. In general, the control schemes suitable for flexible joint robots can be categorized into the following techniques: simple PID, feedback linearization, integral manifold, singular perturbation, passivity approach, torque feedback and adaptive control designs. A simple PD controller for robots with elastic joints was introduced by P. Tomei [49]. A singular perturbation technique to control flexible joint robots was proposed by Marino and Nicosia [30]. Khorasani [1] constructed an adaptive controller based on the integral manifold and singular perturbation techniques, but his design assumes high joint flexibility. Several other researchers proposed the use of joint torque feedback for control of robot manipulators [15, 22, 27, 29]. Spong introduced a feedback linearization approach to the control of elastic joint robots [46]. A class of output feedback globally stabilizing controller for flexible joint robots was proposed by R. Ortega et al. [38]. Readman [39] showed that when the actuator drive inertia matrix is sufficiently small, there exists a decentralized velocity control law.
for flexible joint robots which asymptotically stabilizes the flexible joint dynamics. An adaptive controller for flexible joint robots is designed in [44]. J. Yuan and Y. Stepanenko proposed a composite adaptive control of flexible joint robots [58]. An adaptive control of robot manipulators with flexible joints was also presented by R. Lozano and B. Brogliato [26]. Furthermore, since the control law contained variable structure-like terms, the control input exhibited an undesirable chattering phenomenon.

A comprehensive study of adaptive control of flexible joint robots was given in [46]. Ghorbel and Spong showed that an asymptotic link trajectory could be achieved by modifying an adaptation law, assuming that the desired trajectory tracking error approaches zero as time approaches infinity [12]. In [4], Benallegue and M’Sirdi designed an adaptive controller based on the passivity approach. However, their approach required measurements of link acceleration and jerk. In [23], Chen and Fu used an adaptive approach which required measurement of link acceleration. Dawson et al. designed a hybrid adaptive flexible joint robot controller that achieved GAS position tracking [6]. Using joint torque feedback, Lin and Goldenberg designed an adaptive control of flexible joint robot [25]. However, it was required to compute up to the second derivative of the torque signal. A global dynamic output feedback tracking controller for robots having elastic joints was presented in [37]. Observers for robots with elastic joints were also reported in [36, 48]. In [48] an observer with global convergence property was obtained assuming that the link position and speed were available from measurements. In [36] only the link position was assumed to be needed. Tracking control of flexible joint robots with uncertain parameters and disturbances were considered in [50]. Control of robots with elastic joints via nonlinear dynamic feedback was also reported in [28].

At the same time, extensive research was carried out to deal with the uncertainty compensation of robots. Adaptive and robust control schemes are major areas of this research work. However, there is very limited amount of published work in the
area of flexible joint robot control that made use of \( \mathcal{H}_\infty \) - based robust controllers.

The feasibility of \( \mathcal{H}_\infty \) - based control of a rigid robot manipulator was investigated in [56], but no assumption regarding the existence of joint flexibility was made. It was assumed that the parametric uncertainty was the only type of uncertainty to be considered. Then, a very simple uncertainty model was proposed for the robot under study. Also, there was no experimental validation of this work.

A robust control scheme based on \( \mathcal{H}_\infty \) theory was presented in [10]. The controller consisted of a model-based controller (inverse-dynamics) followed by a linear dynamic controller (designed by using \( \mathcal{H}_\infty \) - control theory). There was no consideration of joint flexibility or of any form of friction in the design. Uncertainty models based on unmodeled dynamics assumptions were proposed. Using the experimental result of the resolved acceleration control implemented on a two D.O.F. robot, weighting functions were chosen. Then, the suggested controller performance was compared with the aforementioned resolved acceleration controller implemented on the same robot. This method did not guarantee a consistent performance in practice.

A control strategy which consisted of feedforward and feedback compensation loops for a single-link manipulator with joint flexibility was investigated in [54]. A state-space form of \( \mathcal{H}_\infty \) - controller was used. The theoretical development was validated by implementing the proposed controller on a single-link robot. Then, it was argued that the proposed SISO method could be extended to the MIMO case too. However, further investigation is necessary to conclude whether or not this assertion is feasible.

A nonlinear \( \mathcal{H}_\infty \) control design in robotic systems under parameter perturbation and external disturbance was proposed by B.S. Chen et al. [5].

1.3 Contributions of this Research Work

The contributions of this research work are as follows:
Flexible Joint Robots Model and Model/Data Consistency Problem

It was observed that the HD flex-spline subsystem model could not represent the experimental data adequately. Hence, the model was further extended and compared with existing models. An extended model that estimates the hysteresis phenomenon in more detail was proposed. The extended model along with other subsystem models were used and a set of parameters were identified in order to verify the model/data consistency. This issue was addressed in the context of experimental data obtained by running open-loop experiments on the system. It was shown that with the extended model, the open loop time and frequency domain responses based on the mathematical model are consistent with the experimental data.

Robust Control Design of Flexible Joint Robots with HD

A twofold robust control design scheme of flexible joint robots with HD is proposed: an actuator-level torque control and a link-level motion control. Using $\mathcal{H}_\infty$ – optimal control, a systematic method of designing robust control laws at each level is introduced. Two multivariable $\mathcal{H}_\infty$ – control designs based on the model of the IRIS (RAL) testbed are presented together with simulation results to illustrate the technique. It is shown that, because the uncertainties due to the actuator-transmission nonlinearity are explicitly considered in the control design procedure, the proposed method is more robust than conventional methods.

Uncertainty Description of Flexible Joint Robots

In the proposed method the selection of uncertainty bounds plays a major role in the trade-offs between robustness and performance requirements. A systematic approach to select uncertainty bounds of flexible joint robots with HD is presented. The describing function method is introduced to select friction and hysteresis uncertainty weighting functions (bounds). The conic-sector-bounded nonlinearity method is proposed to select the nonlinear stiffness uncertainty weights.
Experimental Implementation of the Proposed Controller

A successful implementation of the proposed control design technique is presented. The control laws are synthesized for different uncertainty bounds. It is experimentally verified that the performance of the closed loop system is sensitive to uncertainty descriptions and bounds, and high performance can be achieved by tuning the bounds. The experimental results have also shown that the stability of the control system is maintained. Moreover, the robustness of the proposed control scheme to model uncertainty and measurement noise is demonstrated experimentally.

1.4 Outline of the Thesis

In the previous sections the motivation, background on flexible joint robot control and the contribution of the research have been presented. The remainder of the thesis is organized as follows. Chapter 2 considers flexible joint robot models. Actuator and HD transmission system models are presented. Chapter 2 also discusses the motivation for the need of uncertainty modeling of flexible joint robots with HD transmission. Different approaches to uncertainty modeling are then presented. Chapter 3 details robust motion and torque control design of flexible joint robots. Chapter 4 explains in detail the experimental setup (IRIS facility) developed at the Robotics and Automation Laboratory (RAL) of the University of Toronto. Robustness and performance trade-offs due to uncertainty modeling are also investigated. Chapter 5 addresses the stability analysis of the proposed control design of flexible joint robots. A summary of results and a discussion of future research directions are presented in Chapter 6. A review of $\mathcal{H}_\infty$ design and structured singular value ($\mu$) framework is given in Appendix A.
Chapter 2

Modeling and Uncertainty

Description of Robots with Harmonic Drive Transmission

Due to its unique gearing configuration, which offers compactness, light weight, high ratio, efficiency and virtually zero backlash, harmonic drive transmission has gained wide acceptance in industrial applications. However, the sources of some nonlinear phenomena and their impact on the actual dynamic response have not been fully addressed thus far. The design of high performance motion and torque controllers for robotic applications requires an accurate knowledge of the characteristics of the drive involved. Previously published models characterize the nonlinear kinematic and kinetic behavior of harmonic drives. The transmission compliance, resulting from gear-tooth interaction and wave-generator deformation due to high radial forces, is commonly approximated by a nonlinear piecewise linear stiffness curve [17, 24, 21, 43]. However, hysteresis losses caused by meshing gears are mostly ignored. Extensive experiments focusing on kinematic error and frictional effects have been conducted by Tuttle [51].
We focus on the aspects of both model development and parameter identifications [31, 33, 34, 32]. We review the previously published models of HD [21]. Experimental observations illustrate that the torque transmission characteristics show hysteresis depending on the operating conditions. Also, the deviation in the output rotation due to hysteresis effects cannot be ignored. In [21] these effects are represented by a simple Coulomb friction that does not match the HD behavior completely. These memory-dependent properties and hysteresis effects are captured in our new model. We then identify a set of parameters to complete the model development, and the proposed models are used for motion and torque control designs.

In practice, there is a trade-off between model complexity and performance, in that a simpler model is easier to simulate and use in the design. At the same time, uncertainties due to unmodeled dynamics, neglected nonlinearities, the compliance effect of gear transmission and measurement noise must be considered.

This chapter is organized as follows. First, we present the flexible joint robot model. Second, we propose time and frequency domain experiments to verify the consistency of the proposed model against the experimental data. Then, we propose frequency domain-based uncertainty descriptions to account for unmodeled high frequency dynamics. Finally, we introduce the describing function and conic-sector-bounded-description methods to derive flexible joint robot uncertainty bounds.

2.1 Actuator and Harmonic Drive Transmission Models

In this section, a brief introduction of the mathematical model of the IRIS-facility joint is provided. The entire system including the actuator and HD can be divided into five subsystems: 1) motor-amplifier, 2) wave-generator, 3) flex-spline, 4) circular-spline and 5) load (Fig. 2.1). The detailed description and mathematical formulation of each subsystem can be found in [21].
2.1.1 Motor-Amplifier Subsystem

The motor shaft is driven by the motor input torque $\tau_m$ that is proportional to the input current $i_m$ in the non-saturated region of the motor amplifier subsystem, i.e.,

$$\tau_m = k_t i_m$$

with $k_t$ being the motor torque constant. On the other hand, the relationship between the amplifier input command $u$, motor current $i_m$ and rotor velocity $\dot{q}_w$ can be established as:

$$i_m = k_u u, \quad |i_m| \leq \min(I_s, (U_s - k_b |\dot{q}_w|)/r_m)$$

where $k_u$ is the amplifier gain, $I_s$ is the upper-bound of the current defined by the amplifier electronics, $U_s$ is the maximum output voltage of the amplifier, $k_b$ is the back e.m.f. constant of the motor, and $r_m$ is the motor armature resistance.

2.1.2 Wave-Generator Subsystem

The torque $\tau_m$, developed by the DC motor, drives the motor armature and the harmonic-drive wave-generator. The torque exerted on the wave-generator will be
considered as the subsystem output. The following model describes the wave-generator subsystem model. It involves Coulomb and viscous friction at the bearings, and the wave-generator ball bearing:

\[ \tau_n = J_{nw} \dot{q}_w + \dot{B}(q_w) + \dot{B}_w(q_w, \dot{q}_f) + \tau_w \]  

(2.3)

where \( \dot{q}_f \) is the flex-spline angular velocity with the reference direction opposite to the wave-generator velocity \( \dot{q}_w \). \( J_{nw} \) is the combined inertia of the rotor, shaft and the wave-generator, \( \dot{B}(q_w) \) is the total friction torque generated at the bearings and \( \dot{B}_w(q_w, \dot{q}_f) \) is the friction term of the wave-generator bearing, and \( \tau_w \) is the torque exerted on the wave-generator.

### 2.1.3 Flex-spline Subsystem

The experimental observations reveal three physical phenomena related to the behavior of the flex-spline: nonlinear stiffness, hysteresis, and quasi-backlash due to a soft windup effect. The transmission compliance acting between input and output shaft is approximated by a cubic plus a linear function of the torsion angle, and a correction term, i.e.,

\[ \tau_c(q) = a_{f1} q^3 + a_{f2} q + K_{sw} \]  

(2.4)

where \( a_{f1} \) and \( a_{f2} \) are constant coefficients, \( q \) is the angular displacement of the flex-spline and \( K_{sw} \) is the soft windup correction factor that can be modeled as a saddle-shape function

\[ K_{sw} = -k_{sw} e^{-a_{sw}q^2} \cdot q \]  

(2.5)

where \( k_{sw} \) and \( a_{sw} \) are constants. Furthermore, the flex-spline hysteresis is modeled as a simple Coulomb friction (i.e., \( \dot{B}q \)) [21]. However, it does not completely represent this phenomenon when the system is in motion. An extended model of the hysteresis is presented. The shape of the hysteresis is estimated as a combination of Coulomb friction and a weighted friction function, which is represented by a hyperbolic function.
where \( \tau_c \) has been defined in (2.4) and the Coulomb friction parameter \( B_f \) defines the limits \( \tau_c - B_f \leq \tau_h \leq \tau_c + B_f \) of the torque transmission function. Whereas the factor \( \gamma \) determines the slope at the transition from loading to unloading, i.e., \( \dot{\theta} \leq 0 \rightarrow \dot{\theta} > 0 \), or vice versa. The hyperbolic function in (2.6) implies that the friction torque asymptotically approaches a constant friction torque \( B_f \) for large arguments \( \gamma (q - q^*) \), so that the parameter \( \gamma \) determines the contribution of the actual friction, as shown in Fig. 2.2. The superscript \( (\cdot)^* \) in (2.6) relates to quantities at the reversal point from loading to unloading, that can occur at any point \( (q^*, T^*_h) \), so that the shape of the torque function (2.6) also depends on the load history.

Using (2.6), the corresponding parameters (i.e., \( \gamma, B_f \)) must be estimated linking experimental phenomena to modeling assumptions. The estimated parameters for \( \gamma \) and \( B_f \) as functions of the load history \( \tau^*_h \) of the test-joint are shown in Fig. 2.3. Using the estimated parameters \( \gamma \) and \( B_f \) values, the identified hysteresis has been shown in Fig. 2.4, which shows good agreement with the measured hysteresis.
Figure 2.3: Identified parameters $\gamma$ and $B_f$ vs. load history $\tau_h$.

Figure 2.4: Identified and measured hysteresis. $\tau_h$ versus $q_m$. 
2.1.4 Circular-Spline Subsystem

The output link to the joint can be attached either to the flex-spline or to the circular spline. Therefore, the following two equations will be used to describe C-end/F-end link dynamics:

\[ \tau_c = J_c \ddot{q}_c + \ddot{B}_c (\dot{q}_c) + \tau_{Ct} \]  
\[ \tau = J_F \ddot{q}_F + \ddot{B}_F (\dot{q}_F) + \tau_{Ft} \]  

where \( q_c \) is the C-end angular displacement, \( \tau_{Ct} \) is the C-end load torque, while \( \tau_c \) is the driving torque transmitted to the circular spline at the engagement zone cross section. The parameters \( J_c \) and those in \( \ddot{B}_c \) are inertia and friction parameters at the C-end. The F-end parameters are also analogous to the C-end (2.9).

2.1.5 Harmonic Drive

The flex-spline has two fewer teeth than the circular spline, and thus each full turn of the wave generator moves the flex-spline two teeth in the opposite direction relative to the circular spline. This directly implies the kinematic constraint

\[ q_w = N \dot{q}_F + (N + 1) \dot{q}_c \]  

where \( N \) is the gear ratio defined as \( N = N_t / 2 \), and \( N_t \) is the number of teeth on the flex-spline outer circumference. Applying the power conservation law on the 3-port device described by kinematic constraint (2.10), or more precisely on its derivative form:

\[ \dot{q}_w = N \dot{q}_F + (N + 1) \dot{q}_c \]  

yields the input/output torque relationships:

\[ \tau_f = N \tau_w \]  
\[ \tau_c = (N + 1) \tau_w \]
2.1.6 Flexible Joint Robot Model

With reference to [21] and section 2.1.3, the model of a n-DOF flexible joint robot with HD and an actuator can be written as follows:

\[ M(q_c)\ddot{q}_c + N(q_c, \dot{q}_c) = \frac{r + 1}{r} \tau_h \]

\[ J_m \ddot{q}_m + B_{m w}(q_m, \dot{q}_m, q_c) + (B_{m w}^+ \cdot B_{m w}^-) Sgn(q_m) + \frac{1}{r} \tau_h = \tau_m \]

\[ \tau_h(q, \dot{q}) = \tau_c(q) + \Delta \tau^*_h + [B_f - \Delta \tau^*_h \cdot \text{sgn}(\dot{q})] \cdot \tanh(\gamma (q - q^*)) \]

with \( \Delta \tau^*_h = \tau^*_h - \tau^*_c \)

where \( q_c \) and \( q_m \) are the \( n \times 1 \) vectors of shaft displacement on the joint side and actuator side respectively, \( M(q_c) \) is a \( n \times n \) link inertia matrix, \( N(q_c, \dot{q}_c) \) is a \( n \times 1 \) vector of centrifugal, gravity, and Coriolis(generalized) forces. \( \tau_s \) is the torque sensor output located at the link side, \( r \) is the harmonic drive gear ratio, \( J_m \) is the \( n \times n \) diagonal matrix of actuator inertia, \( B_{m w} \) is a \( n \times 1 \) vector of damping and \( (B_{m w}^+, B_{m w}^-) \) are friction terms associated with the actuator and H.D. bearings. \( \tau_m \) is the \( n \times 1 \) vector of control torque input. The rest of the parameters have been previously defined. It is obvious that in the following model if we ignore viscous and Coulomb frictions and represent the joint flexibility by a simple linear spring, then the proposed model will reduce to a commonly used model of flexible joint robots, for example in [46].

2.2 Model Verification Based on Experimental Results

In the previous section a mathematical model of a flexible joint robot was introduced. This section addresses the problem of checking the consistency of the model with open-loop experimental time and frequency response data for the experimental setup.
(IRIS facility) developed at the Robotics and Automation Laboratory (RAL) of the University of Toronto. The time domain experiments are performed with the output link of the joint being allowed to move without any constraint. By contrast, the link is constrained against the ground in the frequency domain experiments. This prevents the interaction of the fast dynamic motion of the joint and the slow dynamic motion of the output link so that the joint is not damaged.

2.2.1 Time Response Characteristics

In order to verify the model and its parameters, a set of tests was performed that involved the dynamic excitation of the system and the comparison of actual output signals with numerically derived data. Fig. 2.5 and Fig. 2.6 contrast experimental wave-generator angle and velocity responses for five different current amplitudes with results obtained by simulation. The current command amplitude covers the full range of motor current from 0 to 3 amps. Fig. 2.5 shows the theoretical and experimental wave-generator angle response to these sets of current command inputs. Obvious from the figure, there is a good consistency between the model and the real system responses. Fig. 2.6 shows the wave-generator velocity response for the model and the experimental setup. A comparison of the experimental and simulated time responses shows good agreement between experiment and simulation for different current input amplitudes. For an input amplitude of 3 amp, there is a slight increase in the simulation response compared with the experimental response which suggests a variation in the motor armature and amplifier parameters.

2.2.2 Frequency Response Characteristics

The frequency response was predicted via simulation and compared with the response of the physical system. During testing, the output shaft was restrained against the ground and the motor was driven by a sinusoidal current input with varying frequencies. Joint torques in simulation and experiment were measured for a sinusoidal motor
Figure 2.5: Theoretical and experimental wave-generator angle for step input of current

Figure 2.6: Theoretical and experimental wave-generator velocity
current with a sweeping frequency ranging from 0 to 500Hz. The Bode magnitude and phase plots of the input/output torque transfer functions are shown in Fig. 2.7 and Fig. 2.8. A comparison of experimental and simulated transfer functions shows a slight reduction in the breakpoint frequency, indicating a small variation of the effective stiffness. For excitation frequencies of $f < 30$ Hz, the system response matches the model response perfectly. However, due to the inaccuracy of the torque sensor readings and measurement noise for $f > 100$ Hz, the recorded data were unreliable. For control design purposes, the working range of the system is below 30 Hz, so the model is applicable.

We use the models introduced for control design purposes. The following sections describe how to approximate the model uncertainty bounds between the experimental data and the model. The frequency domain-based uncertainty description is proposed to model the effect of unmodeled high frequency dynamics. The describing function-based uncertainty description method is used to approximate the effects of hysteresis and friction at robot joints. The conic-sector-bounded nonlinearity method is proposed to approximate the drive nonlinear stiffness bounds.
2.3 Uncertainty Description: The Frequency Domain-Based Method

The frequency domain-based method is proposed to describe frequency dependent variations between the experimental data and the model via uncertainty bounds (weighting functions). These allow us to account for the variation in experimental data at specific frequency points. A frequency response experiment is performed to establish upper and lower bounds on both the magnitude and phase of the real system as a function of frequency. Variations in the data are then approximated by disk shaped regions in the complex plane, leading to either a multiplicative or additive uncertainty description of the bounds [8].

The plant transfer function can be described by $P(s) + \Delta P(s)$, where $P(s)$ is the nominal plant model and $\Delta P(s)$ is an unknown perturbation [35]. Consider a SISO system with $\Delta P(s)$ bounded across frequency by a weighting function $W_a$, a real-rational, stable minimum phase transfer function, and a norm bounded $\Delta$, where $|\Delta| \leq 1$, such that

$$|\Delta P(j\omega)| < |W_a(j\omega)||\Delta(j\omega)| \quad \text{for all} \quad 0 \leq \omega \leq \infty,$$

(2.17)
where \( w \) represents individual frequency points. Equation (2.17) is referred to as an additive uncertainty description and defines the bound on the allowable additive uncertainty.

As shown in Fig. 2.9, the additive uncertainty weighting is wrapped around the plant and is often used to account for additive plant errors and uncertain right half plane zeros. A variety of additive uncertainty weights can be developed for a MIMO system, each adding states to the control problem. For instance, the additive uncertainty weights can be wrapped around the plant from each input channel to each output channel with different weights. Low order weighting functions are usually employed to limit the number of states added to the problem formulation. Another reason for low order weights is that knowledge of the exact size of the uncertainty is often limited. Therefore describing the variation by a complex, high-order weight can not be justified.

![Block diagram of additive uncertainty](image)

Figure 2.9: Block diagram of additive uncertainty

Consider the following SISO system with additive plant uncertainty:

\[
y = (P + W_a \Delta)u = \left( \frac{22}{s + 20} + 0.1 \Delta \right)u, \quad \|\Delta\|_\infty \leq 1.
\]  

(2.18)

This describes a set of plant models within which the "real" system lies. A Nyquist plot of this uncertain system is shown in Fig. 2.10. The plant is described at each frequency point \( w \) by a circle centered at \( P(jw) \) of radius \( |W_a(jw)| \).
Another approach to modeling errors involves multiplicative uncertainty descriptions. Multiplicative uncertainty descriptions are used to account for relative variations in input or output signals. Input multiplicative uncertainty is useful in describing actuator errors at high frequency and unmodeled actuator dynamics. Output multiplicative uncertainty is used to model similar quantities on output signals and time delays. Sensor noise attenuation and output response to output commands are performance measures that can be specified with such weighting. Typically, testing of actuators and sensors involves inputting signals into the components and measuring their response (i.e., force, displacement, torque). The output response is measured with a percentage error from a nominal plant model that may vary across frequency. Fig. 2.11 shows the block diagram of an output multiplicative uncertainty, represented by

\[
\dot{P}(s) = P(s)(1 + W_a(s)\Delta(s)).
\]  

(2.19)
2.4 Uncertainty Description: The Describing Function-Based Method

In uncertainty modeling, one needs a methodology for dealing with nonlinear phenomena in the system (e.g., friction, hysteresis). For some nonlinear systems and under certain conditions, an extended version of the frequency response method, the *describing function method* [45, 53], can be used to analyze and predict nonlinear behavior approximately. The main use of the *describing function method* is the prediction of limit cycles in nonlinear systems, although the method has a number of other applications. We propose to use it for uncertainty weight description.

Before addressing uncertainty weight description, let us briefly discuss how to represent a nonlinear component using a describing function, which is critical for our purposes [45]. As shown in Fig. 2.12a, let us consider a sinusoidal input \( c(t) = A\sin(\omega t) \) to the nonlinear element of amplitude \( A \) and frequency \( \omega \). The output of the nonlinear component \( c(t) \) is often a periodic function though generally non-sinusoidal. Note that this is always the case if the nonlinearity \( f(c) \) is single-valued because the output is \( f[A\sin(\omega(t + 2\pi/\omega))] \). Using a Fourier series, this
periodic function can be expanded as

\[ c(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]. \] (2.20)

where the Fourier coefficients \( a_n \)'s and \( b_n \)'s are generally functions of \( A \) and \( \omega \), determined by

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) d(\omega t) \] (2.21)
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \cos(n\omega t) d(\omega t) \] (2.22)
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \sin(n\omega t) d(\omega t) \] (2.23)

If we assume the nonlinearity is odd, one has \( a_0 = 0 \). Furthermore, if we consider the fundamental component \( c_1(t) \) in the output \( c(t) \) then,

\[ c_1(t) = M \sin(\omega t + \phi) \] (2.24)

where,

\[ M(A, \omega) = \sqrt{a_1^2 + b_1^2} \quad \text{and} \quad \phi(A, \omega) = \arctan(a_1/b_1) \] (2.25)

where \( M \) is the output signal amplitude, and \( \phi \) is the phase shift. Expression (2.24) indicates that the fundamental component corresponding to a sinusoidal input is a sinusoids at the same frequency. We define the describing function of the nonlinear element to be the complex ratio of the fundamental component of the nonlinear element by the input sinusoids, i.e.,

\[ N(A, \omega) = \frac{Me^{j(\omega t + \phi)}}{Ae^{j\omega t}} = \frac{M}{A} e^{j\phi} = \frac{1}{A} (b_1 + ja_1) \] (2.26)

With a describing function representing the nonlinear component, the nonlinear element, in the presence of sinusoidal input, can be treated as if it were a linear element with a frequency response function \( N(A, \omega) \), as shown in Fig. 2.12b.
2.4.1 Computing Describing Function (DF)

In this section, we take a closer look at the nonlinearities found in HD transmission systems. The describing functions for a few common nonlinearities are computed.

Saturation (Friction) DF:

The input-output relationship for a saturation nonlinearity is plotted in Fig. 2.13. with \( a \) and \( k \) denoting the range and slope of the linearity. Since this nonlinearity is single-valued, we expect the describing function to be a real function of the input amplitude.

Consider the input \( e(t) = A \sin(\omega t) \). If \( A < a \), then the input remains in the linear range, and therefore the output is \( y(t) = kA \sin(\omega t) \). Hence, the describing function is simply a constant \( k \).

Now consider the case \( A > a \). The input and the output are plotted in Fig. 2.13. The output can be expressed as

\[
c(t) = \begin{cases} 
  kA \sin(\omega t) & 0 \leq \omega t \leq \omega t_1 \\
  ka & \omega t_1 < \omega t \leq \frac{\pi}{2} 
\end{cases} 
\]  

(2.27)

where \( \omega t_1 = \arcsin(a/A) \). The odd nature of \( c(t) \) implies that \( a_1 = 0 \), and the symmetry over the four quarters of a period implies that

\[
b_1 = \frac{2kA}{\pi} [\omega t_1 + \frac{a}{A} \sqrt{1 - \frac{a^2}{A^2}}] 
\]  

(2.28)

Therefore, the describing function is

\[
N(A) = \frac{2k}{\pi} [\arcsin \frac{a}{A} + \frac{a}{A} \sqrt{1 - \frac{a^2}{A^2}}] 
\]  

(2.29)
Figure 2.13: Saturation nonlinearity and corresponding input-output relationship

As a special case, one can obtain the describing function for the relay-type (Coulomb friction) nonlinearity shown in Fig. 2.14. This case corresponds to shrinking the linearity range in the saturation function to zero, i.e., \( a \rightarrow 0, k \rightarrow \infty \) but \( ka = M \). Though \( b_1 \) can be obtained from (2.28) by taking the limit, it is more easily obtained directly as

\[
b_1 = \frac{4}{\pi} \int_0^{\pi/2} M \sin(\omega t) d(\omega t) = \frac{4}{\pi} M \tag{2.30}
\]

Therefore, the describing function of the Coulomb friction nonlinearity is

\[
N(A) = \frac{4M}{\pi A} \tag{2.31}
\]

**Hysteresis DF:**

Consider the hysteresis nonlinear operator \( N \) shown in Fig. 2.15. In the steady-state, the output \((Nx)(t)\) follows the upper straight line when the input is increasing i.e.,
Figure 2.14: Coulomb friction type nonlinearity

$\dot{x}(t) > 0$ and the lower straight line when the input is decreasing. The value $a$ in Fig. 2.15 depends on the amplitude of the input and is not a characteristic of $N$ itself. Thus

$$(N x)(t) = \begin{cases} 
mx(t) + b & \text{if } \dot{x}(t) > 0 \\
mx(t) - b & \text{if } \dot{x}(t) < 0 
\end{cases}$$

and "jumps" when $\dot{x}(t)$ goes through zero.

Suppose a sinusoidal input $r$ is applied to $N$. The resulting steady-state output is shown in Fig. 2.15. It is clear that the first harmonic of the steady-state part of $Nr$ is

$$c(t) = ma \sin(\omega t) + \frac{4b}{\pi} \cos(\omega t).$$

Hence

$$N(a) = m + \frac{4b}{\pi a}$$

(2.34)

Note that $N$ is independent of $\omega$, because time scaling does not affect the output of $N$. 26
Figure 2.15: Hysteresis type nonlinearity

2.5 Uncertainty Weights Selection

Once the describing function representation of a nonlinear system is known, it can be used to characterize the uncertainty involved in the system. The describing functions of the Coulomb friction and the hysteresis phenomenon were introduced in the previous sections. This section addresses the problem of finding uncertainty weights of HD friction, hysteresis and nonlinear stiffness. The describing function method is used to find the norm bounded weighting function for motor friction and HD hysteresis.

2.5.1 Motor Friction Uncertainty Weight Selection

Fig. 2.16 shows the complete nonlinear block diagram of flexible joint robots with HD transmission. The Coulomb friction at the motor side can be replaced by its equivalent DF introduced in (2.31).
Figure 2.16: Complete nonlinear model of the robot HD drive system

Figure 2.17: Motor and DF of Coulomb friction
Let the nominal model of the motor in Fig. 2.17 be as

\[ P(s) = \frac{1}{J_m s + B_m} \]  

(2.35)

where there is no Coulomb friction and where \( J_m \) is the motor inertia and \( B_m \) is the viscous damping coefficient of motor. Also assume the perturbed model of the motor be

\[ \dot{P}(s) = \frac{1}{J_m s + B_m + N(s)} \]  

(2.36)

where \( N(s) \) is the DF of the Coulomb friction (Fig. 2.17). Now if a multiplicative uncertainty description is used to account for relative variations between nominal and perturbed plants, one can write,

\[ \dot{P}(s) = [1 + W(s)\Delta(s)]P(s) \]  

(2.37)

where \( P(s) \) is the nominal plant transfer function (2.35), \( \dot{P}(s) \) is the perturbed plant transfer function (2.36), \( W(s) \) is the weighting function to be found and \( \Delta(s) \) is the norm bounded complex uncertainty transfer function such that \( \|\Delta\|_\infty \leq 1 \). Taking the \( \infty \)-norm of both sides of (2.37), and using the commutative property of \( \infty \)-norm, we can write

\[ \left\| \frac{\dot{P}}{P} - 1 \right\|_\infty \leq \|W\|_\infty, \quad \|\Delta\|_\infty \leq 1 \]  

(2.38)

and,

\[ \left| \frac{\dot{P}(jw)}{P(jw)} - 1 \right| \leq |W(jw)|, \quad \forall w \]  

(2.39)

that is,

\[ \left| \frac{-N(A)}{J_m(jw) + B_m + N(A)} \right| \leq |W(jw)|, \quad \forall w \]  

(2.40)
Figure 2.18: Frequency response of the motor-friction model for different amplitude of inputs

Fig. 2.18 shows the Bode magnitude frequency response plot of the $\frac{\hat{P}(s)}{P(s) - P(s)}$ for different input signal amplitudes. Using MATLAB [18], one can find a suitable weighting function $W(s)$ to satisfy (2.40). The multiplicative uncertainty model of the motor-friction is shown in Fig. 2.19.

### 2.5.2 Hysteresis Uncertainty Weight Selection

The HD hysteresis uncertainty weight can be selected following the same procedure as in the previous section. The nominal HD model is considered as a pure linear spring (i.e., no hysteresis). The perturbed HD model is selected when the hysteresis effect is present. A multiplicative uncertainty weight is used to describe the variation between the nominal and perturbed models. Moreover, the DF of the hysteresis introduced in (2.34) is used for uncertainty weight selection. The frequency domain hysteresis uncertainty weight is obtained using a set of swept sine input signals with varying amplitudes. Fig. 2.20 shows the frequency response of the variation between the
nominal and perturbed models (i.e., \( \Delta \)). and the corresponding weighing function. In the following section, the so called method of the \textit{conic-sector-bounded nonlinearity} is used to represent nonlinear stiffness nonlinearity \cite{59}.

### 2.5.3 Nonlinear Stiffness Uncertainty Weight Selection

In this thesis, we use the so called \textit{conic-sector-bounded nonlinearity} method, whose definition is given below, to find an uncertainty bound for the nonlinear stiffness \cite{45}.

**Definition:** A continuous function \( \phi(y) \) is said to belong to the sector \([k_1, k_2]\), if there exist two non-negative numbers \( k_1 \) and \( k_2 \) such that

\[
y \neq 0 \Rightarrow k_1 \leq \frac{\phi(y)}{y} \leq k_2
\]

Geometrically, condition (2.41) implies that the nonlinearity function always lies between two straight lines \( k_1 y \) and \( k_2 y \), as shown in Fig. 2.21.

The uncertainty weight description for the nonlinear stiffness is derived using this method. In section 2.1.3, the torque-torsion relation is defined as follows:

\[
\tau_c(q) = a_{f1}q^3 + a_{f2}q - k_{sw}e^{-a_{sw}q^2}q
\]
Figure 2.20: Frequency response of the variation between the nominal and perturbed model of hysteresis and the corresponding weighting function.

Figure 2.21: Conic-sector representation of the nonlinear stiffness.
Figure 2.22: Linearization and conic-sector bound

where the parameters are defined in section 2.1.3. This nonlinear function can be linearized about an operating point of the system:

\[ \tau_l = r_l \cdot q_l \]  

(2.43)

where \( r = \left( \frac{d\tau}{dq} \right)_0, \tau_l = (\tau_c - \tau_0), \) and \( q_l = (q - q_0). \) \( \tau_0, \) \( q_0 \) denote the values of variables at the operating point. If we wish to have bounds on the error involved in the linearization, we can employ a "conic-sector" description of the nonlinearity by writing

\[ |\tau_l - r \cdot q_l| \leq \kappa |q_l| \]  

(2.44)

This procedure is depicted in Fig. 2.22. Equation (2.44) implies the characteristic falls in the cone drawn in Fig. 2.22 (this of course is valid for a limited range of \( \tau_l, q_l \)).

The previous bound is a static constraint, but we can generalize this by writing

\[ \tau_l = r \cdot q_l + \kappa \delta(q_l) \]  

(2.45)

where \( \delta(q_l) \) is an unknown nonlinearity operator such that \( \|\delta(q_l)\|_\infty \leq 1. \) We might think of it "covering" dynamic effects which are not described in our static equations.
Fig. 2.23 shows the additive type uncertainty representation of (2.45). $\kappa$ is the weighting function which can be selected using experimental results (e.g. Fig. 2.3).

2.6 Summary and Discussion

In this chapter, the nonlinear properties of HD transmission are analyzed, and a model is presented that takes into account hysteresis, friction and nonlinear stiffness (Section 2.1). Moreover, the HD flex-spline hysteresis model is further developed (Section 2.1.3). Using the proposed model, a complete model of an n-DOF flexible joint robot with HD is introduced (Section 2.1.3). The model and its parameters are verified experimentally against the actual data (Section 2.2). The model is then used as a basis for uncertainty bound selection. Frequency-based domain uncertainty description is proposed to describe frequency dependent variations between the experimental data and the model. In addition, a systematic approach is introduced for selecting uncertainty bounds for the actuator-transmission stiffness nonlinearity, friction, and hysteresis (Section 2.3). The describing function and the conic-sector bounded nonlinearity methods are used to incorporate the effects of hysteresis, friction and nonlinear stiffness into the control design. Using these methods, appropriate uncertainty weighting functions are selected. In the following chapter, a control design technique is proposed which, in conjunction with the proposed uncertainty bound descriptions, yields robust performance for flexible joint robots.
Chapter 3

A New Robust Motion and Torque Control Design Method

This chapter proposes a new motion and torque control design scheme to achieve robust performance in robots with harmonic drive transmission. It describes the steps one needs to follow for control design using the $\mathcal{H}_\infty$ - optimal control and $\mu$-analysis and synthesis method outlined in Appendix A.

The design procedure proposed involves the following steps:

- Generating the error model of the rigid body dynamics of a flexible joint robot (Section 3.1)

- Defining performance specifications and uncertainty bounds of the error model (Section 3.2)

- Constructing open-loop interconnections

- Designing a controller for the error model to satisfy the performance requirements (Section 3.2)

- Closing the feedback loop with the robust motion controller and examining the behavior of the system (Section 3.2.2)
Generating uncertainty model of the actuation system

- Defining performance specifications and uncertainty bounds (Section 3.3.1)

- Constructing the open-loop interconnection of the overall system

- Designing an actuator-level torque control law based on $\mathcal{H}_\infty$ (Section 3.3)

- Performing a variety of tests on the closed-loop system, and exploring the impact of the uncertainty models on the robust stability and robust performance requirements (Section 3.3)

### 3.1 Rigid Body Error Model

The first step in motion control design of flexible joint robots is the formulation of the error system. Let the link position error be defined as

$$\epsilon = q^d - q_c$$

(3.1)

where $q_c$ is the $n \times 1$ link position and $q^d_c$ is the $n \times 1$ vector of desired link position trajectory. We assume that $q^d$ and its derivatives up to the third order are bounded.

Now we can write rigid-body dynamics (2.14) in terms of (3.1) as:

$$M(q_c)\ddot{q}_c^d - M(q_c)\dot{\epsilon} + N(q_c, \dot{q}_c) = \frac{r+1}{r} \tau_h$$

(3.2)

In the state-space form (3.2) can be written as

$$\dot{\mathbf{e}} = A_0 \mathbf{e} + B_0 [\dot{\mathbf{q}}_c^d + M^{-1}N - M^{-1}\frac{r+1}{r} \tau_h]$$

(3.3)

where $A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 \\ I \end{bmatrix}$, and $\mathbf{e} = \begin{bmatrix} \epsilon \\ \dot{\epsilon} \end{bmatrix}$.

Since there is no control input in (3.3), let add and subtract the $B_0M^{-1}u_l$ on the RHS of (3.3) yield

$$\dot{\mathbf{e}} = A_0 \mathbf{e} + B_0 [\dot{\mathbf{q}}_c^d + M^{-1}N - M^{-1}u_l] + B_0M^{-1}[u_l - \frac{r+1}{r} \tau_h]$$

(3.4)
where \( u_l \) is a \( n \times 1 \) vector of fictitious control input defined later. \( u_l \) can be designed so that the tracking error \( e \) approaches zero in spite of external disturbances. Herein, we design \( u_l \) as

\[
u_l = \hat{M}(\bar{q}_c^d - u_\infty) + \hat{N}
\]

where \( \hat{M} \) and \( \hat{N} \) are the mathematical models of the \( M \) and \( N \) respectively, and \( u_\infty \) is the new control input designed using \( H_\infty \) and \( \mu \)-synthesis design methods defined in Appendix A. Substituting (3.5) for only the first \( u_l \) in RHS of (3.4) yields.

\[
\dot{e} = A_0 e + B_0 (\eta + u_\infty) + B_0 M^{-1} [u_l - \frac{r + 1}{r} \tau_h] \tag{3.6}
\]

where

\[
\eta = \Delta (\bar{q}_c^d + u_\infty) + \delta, \quad \Delta = I - M^{-1} \hat{M}, \quad \delta = M^{-1} (N - \hat{N})
\]

let define

\[
\eta_f = [u_l - \frac{r + 1}{r} \tau_h] \tag{3.7}
\]

then, we can write (3.6) as:

\[
\dot{e} = A_0 e + B_0 (\eta + u_\infty) + B_0 M^{-1} C \eta_f \tag{3.8}
\]

where

\[
C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \eta_f = \begin{bmatrix} \eta_f \n \hat{\eta}_f \end{bmatrix} = \begin{bmatrix} \dot{u}_l - \frac{r + 1}{r} \tau_h \\
\hat{\dot{u}}_l - \frac{r + 1}{r} \tau_h \end{bmatrix} \tag{3.9}
\]

Now in (3.8) if we make the last term \( \eta_f \) disappear, then it reduces to a simple state-space error equation where by appropriate design of \( u_\infty \) the error stably vanishes. Hence, our goal is to force \( \eta_f \) to zero. This requirement can be satisfied if dynamics of the actuator and transmission system is known. In other words, an actuator-level torque control law is required to be designed to provide the desired
torque of the error-model motion control (3.5). The proposed method in this work requires only the measurement of link position, velocity and output torque. Section 3.2 presents the robust motion control design procedure. Then, in section 3.2.2 a numerical example of the motion control design without considering the actuation system dynamics is given. Section 3.3 presents the design procedure for the actuator-level torque feedback control law. It starts with uncertainty weights selection (section 3.3.1) and continues with the robust $H_\infty$ control design procedure. Two different controllers are designed, one ignoring the effects of actuator uncertainty ($K_h$), and the next one considering uncertainty weight levels ($K_u$). The nominal performance, robust stability and robust performance characteristics of the closed-loop system with two controllers are compared. Finally, the step response characteristics of the system are investigated.

3.2 Robust Motion Control Design

3.2.1 Design Method

Let us represent the transfer matrix of the error dynamics in (3.8) (without $\eta_f$ term) by $P_e(s)$. The system can be represented in Fig. 3.1, where $U = \eta + u_\infty$ is the applied torque and $e$ is the resultant angular position and velocity errors.

Consider $H_\infty$ -- optimal design of a controller for the error dynamics. Let $K_r$ denote the motion controller for the error dynamics model. The design specifications are taken to be as follows:

1. The arm position and velocity errors should vanish ($e \to 0$).
2. The control torque, $u_\infty$, should not exceed a pre-specified saturation limit.

![Figure 3.1: Error dynamics block diagram.](image-url)
3. The unmodeled high-frequency dynamics should be compensated for.
4. The effects of measurement noises should be cancel out.

Therefore, the $\mathcal{H}_\infty$ design of $K_e$ may be carried out with reference to Fig. 3.2, where $W_{a1}, W_{a2}, W_{a3}$ and $W_n$ are frequency dependent weighting matrices, used to reflect aforementioned performance specifications. The multiplicative uncertainty weight ($W_m$) accounts for the neglected high frequency modes and some low frequency errors. It is modeled as an unstructured full block uncertainty, $\Delta_m$, after the error dynamics transfer matrix ($P_e$). To limit the actuator power in control design $W_{a2}$ has been considered. Also, to reflect the noise propagation in different frequency range another weighting matrix has been included ($W_{a3}$).

The block diagram is reformulated into the LFT framework to design control laws using the $\mu$-synthesis methodology (Fig. 3.3). A series of control law syntheses for the error model using the $\mu$ synthesis approach are designed. Robustness and performance of the control designs are traded off in the design process, as one is increased the other is decreased. Each design is iterated on until it achieves a $\mu$ value of approximately 1. A control law with a $\mu$ value of 2.0 indicates that for the uncertainty and performance criteria prescribed, the control laws achieves $\frac{1}{2}$ or 50%
of the performance for $\frac{1}{2}$ or 50% of the uncertainty level.

3.2.2 Design Example

This section demonstrates a design example of the control design procedure introduced in previous section. It consists of the selection of nominal plant parameters, the weighting matrices, and clarification of the control design objectives.

A one DOF flexible joint robot has been considered for simplicity. The nominal parameters of the robot and the actuator+H.D. are given in Table 3.1. For this example, the design specifications are:

1. settling time $\leq 10$ sec.
2. $\|u_{\infty}\|_{\infty} \leq 2$ N.m.

The weighting matrices are chosen as

$$W_{a1} = 10 \frac{0.001S + 1}{0.005S + 1}$$

for the weight on the error signal.

$$W_{a2} = 0.1 \frac{2S + 1}{S + 1}$$
Table 3.1: Flexible joint robot parameters.

<table>
<thead>
<tr>
<th>manipulator parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>2.81 Kg</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m/s^2</td>
</tr>
<tr>
<td>$l_1$</td>
<td>0.25 m</td>
</tr>
<tr>
<td>$l_{cx}$</td>
<td>0.108 m</td>
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<tr>
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<td>0 m</td>
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<tr>
<td>$l_{cz}$</td>
<td>0.02 m</td>
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<tr>
<td>$I_{cxy}$</td>
<td>0.0076 Kg m^2</td>
</tr>
<tr>
<td>$I_{cyy}$</td>
<td>0.0388 Kg m^2</td>
</tr>
<tr>
<td>$I_{czz}$</td>
<td>0.0340 Kg m^2</td>
</tr>
<tr>
<td>$I_{czz}$</td>
<td>0.0059 Kg m^2</td>
</tr>
</tbody>
</table>

for the weight on the control signal $u_{\infty}$.

\[
W_{\delta 3} = 0.3 \frac{0.1S + 10}{0.01S + 100}
\]

to model the noise and

\[
W_m = 0.1 \frac{0.1S + 1}{1S + 100}
\]

for the weight on the unmodeled high-frequency dynamics.

Figure 3.4 shows the frequency response of these weighting matrices. Using MATLAB $\mu$-Analysis and Synthesis Toolbox a 4th order controller is obtained (after model reduction) as.

\[
K = \begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix}
\]

where

\[
A_k = \begin{bmatrix}
-1.2581 & 2.3098 & -3.7667 & 1.4677 \\
1.5790 & -10.8848 & 51.9426 & -14.8740 \\
3.3786 & -51.1799 & -57.8835 & 46.4295 \\
-1.3607 & 9.1579 & -1.0842 & -13.7679
\end{bmatrix}
\]
Using this controller for the closed loop system, one can examine the response of the system. The closed loop error response for the step input ($q_c^d = 1$ rad) are illustrated in Fig. 3.5.
3.3 Actuator-Level Torque Control Design

The motion control design procedure for the error dynamics of the flexible joint robots was presented in section 3.2. This section presents a procedure to design an actuator-level torque control law. It consists of selecting uncertainty bounds for the actuator and HD transmission. To design uncertainty weighting functions of nonlinear stiffness, hysteresis and friction, the established method in Chapter 2 is used. Fig. 2.16 shows the open-loop interconnection of the overall system including the actuation system, HD, and the arm. A new model replacing the nonlinearities by their equivalent linear models plus uncertainty weights is presented in the next section. The model is then used for the actuator-level torque control design purpose.

3.3.1 Design Method

Fig. 3.6 shows the closed-loop interconnection structure of flexible joint robots with weighting functions. In Fig. 3.6, $W_m$ is the weighting function of the motor friction. $W_s$ is the nonlinear stiffness weighting function, and $W_h$ is the hysteresis weighting function. $\Delta$'s are unknown disturbances of the aforementioned nonlinearities respectively.
Figure 3.6: Open-loop interconnection of the complete flexible joint robots with weighting functions.
Sensor Noise:

Each measurement is corrupted with sensor noise which becomes more severe with increasing frequency. Since $\tau_h$ is measured with torque sensors, their sensor noise weights can be modeled as

$$W_{noise} = 0.025 \frac{1 + s/5}{1 + s/10}$$

(3.10)

This weighting function implies a low frequency measurement error in $\tau_h$ of 0.025 N.M., and a high frequency error of 0.05 N.m. The model of the measured value of $\tau_h$, denoted $\tau_h^{meas}$ is given by

$$\tau_h^{meas} = \tau_h + W_{noise} \eta_p$$

(3.11)

where $\eta_p$ is an arbitrary signal, with $\|\eta_p\|_2 \leq 1$. The noise weighting functions are denoted by $W_{noise}$ in the control block diagram.

Errors

There are several variables which are to be kept "small" in the face of the exogenous signals. In this context, these variables will be considered errors.

- **Actuator signal levels**: the amplifier current command ($i_m$) should remain reasonably "small" in the face of the exogenous signals. The signals are weighted to give a desired actuator level using $W_{act}$.

$$W_{act} = 0.001$$

(3.12)

- **Performance variables**: the actuator torque response ($\tau_h$) should follow the desired torque ($\frac{r}{r+1}u$). In other words, the torque error signal should vanish as times goes to infinity. These can be ensured if they are weighted by frequency dependent weights, to give required performance. In the open-loop interconnection (Fig. 3.6) $W_{per}$ is used for this purpose.

$$W_{per} = 0.5 \frac{1 + s/500}{1 + s/0.5}$$

(3.13)
Uncertainty Weights

Appropriate weighting functions can be selected following the methods described in Chapter 2 for the nonlinear stiffness, friction, and hysteresis effects. It results,

\[ W_s = \frac{1}{s + 1} \]

for the nonlinear stiffness weighting function

\[ W_h = \frac{0.01}{s + 1} \]

for the hysteresis weighting function

\[ W_f = \frac{1.2}{s + 1} \]

for the friction weighting function. The frequency response of the weighting functions are shown in Fig. 3.7.

3.3.2 Design Example

In this section, the robustness properties of two different controllers for a one-DOF flexible joint robot are analyzed using \( \mu \). The controllers receive 1 sensor measurement (\( \tau_h \)) along with the \( q^d \) command signal which are required for feedback and produce one control signal for the actuator (\( i_m \)). In this section, each controller has different characteristics:

- \( K_h \) is designed to optimize \( H_\infty \) performance, under the assumption that there is no model uncertainty;

- \( K_\mu \) is designed with the help of \( \mu \)-synthesis, and it is assumed that model uncertainty is present.
Figure 3.7: Frequency response of weighing transfer functions for actuator side control design.

**Frequency Response**

**Nominal Performance**

In the closed-loop system, there are 4 exogenous signals (1 disturbance signal (η), 1 sensor noise, and 2 command signals), and 2 errors (weighted performance error, weighted actuator error). The nominal performance objective is that the transfer function matrix from exogenous signals to errors should have an $\mathcal{H}_\infty$ norm less than 1. Using, $\mu$-Analysis and Synthesis Toolbox it is easy to evaluate this performance criterion. Fig. 3.8 shows the frequency response of the closed loop system for two different controllers ($K_h, K_\mu$). Note that the best nominal performance is achieved by controller $k_\mu$. 

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Robust Stability

Using $\mu$, the robust stability characteristics of each of closed-loop system can be evaluated. The uncertain parameters can be treated as complex. According to Fig. 3.9, the $K_\mu$ controller has the best robust stability properties, when the perturbations are treated as complex (dynamic). The peak of the $\mu$, 0.98, implies that there is a diagonal, complex perturbation of size $\frac{1}{0.98}$ that causes instability. Similar interpretation is possible for the closed-loop system with controller $K_h$, though the $\mu$ plots have larger peaks, the bounds on allowable perturbations is smaller. Hence the closed-loop system with the controller $K_\mu$ achieves robust stability to complex perturbations, whereas the other controller does not.

Robust Performance

Using $\mu$, the robust performance characteristics of each of closed-loop systems can be evaluated.
The appropriate block structure for the robust performance test is

$$ p := \{ \text{diag}[\delta_1, \delta_2, \delta_3, \Delta_4] : \delta_i \in \mathbb{C}^{1 \times 1}, \Delta_4 \in \mathbb{C}^{4 \times 2} \} $$ \hspace{1cm} (3.14)

which is simply an augmentation of the original real robust stability uncertainty set with a complex $4 \times 2$ full block to include the performance objectives.

The $\mu$ calculations are performed on the entire $8 \times 6$ closed-loop matrix, which include the perturbation channels and the exogenous signals and errors (Fig. 3.10).

Fig. 3.11 shows the closed-loop response of the system for two different controllers. At very low frequency, the closed-loop robust performance with both of the controllers achieved, but at medium and high frequency with $K_h$ implemented the $\mu$ gets as bad as 1.8. The closed-loop system achieves a robust performance $\mu$ value of 0.8 with controller $K_\mu$. 

\[ \text{Figure 3.9: Robust stability plots of controllers: } K_h, K_\mu. \]
Figure 3.10: LFT block diagram of robust performance control design.

Figure 3.11: Robust performance plots of controllers: $K_h$, $K_{\mu}$. 
Figure 3.12: Nominal $\eta_f$ step response. $K_k$: solid, $K_\mu$: dashed

**Time Response**

In this subsection nominal and perturbed time-domain simulations will be demonstrated. For each of the closed-loop controllers a simulation model is produced to demonstrate the time response characteristics of the closed-loop system.

**Nominal Step Responses**

The main performance objective is link angle tracking, so the response to a 1 radian step input of $q^d$ is investigated. The output signals of interest are $\eta_f$, $\varepsilon_c$, and $\dot{\varepsilon}_c$. Fig. 3.12 shows the $\eta_f$ response for nominal system without any perturbation present. Fig. 3.13 also shows the link angle and angular error velocity. It is obvious that the performance of the error closed-loop system with $K_\mu$ is much superior than controller $K_k$.

As it can be seen from the nominal step response, the controller $K_\mu$ performs much better than $K_k$. 
Figure 3.13: Nominal $\epsilon_c$ and $\dot{\epsilon}_c$ step responses. $K_k$:solid, $K_u$:dashed

**Perturbed Step Responses**

The time-domain analysis is repeated on the perturbed models. Fig. 3.14 and 3.15 show the response of the system for this case. As expected, the time domain simulations reinforce the conclusions that were reached in the frequency domain analysis.

Figure 3.14: Perturbed $\epsilon_{taf}$ step response. $K_k$:solid, $K_u$:dashed
Figure 3.15: Perturbed $\epsilon_c$ and $\dot{\epsilon}_c$ step responses. $K_h$: solid, $K_d$: dashed

3.4 Summary

A new technique is proposed for designing motion and torque control of flexible joint robots. It is based on a careful study of the characteristics of HD transmission in robotic systems. A closed-loop flexible joint robot built according to this method is expected to be more robust, because the uncertainties due to the actuator system nonlinearities are explicitly considered in the control design procedure. The technique uses the perturbed model of flexible joint robots introduced in Chapter 2. It involves two steps: a robust arm motion control is designed; and the desired torque for the motion control design is provided by an actuator-level torque feedback control law. This allows the designer to focus on arm and actuator control problems separately. A complete example demonstrates the systematic procedure for designing control laws. Different controllers are designed to demonstrate the effect of uncertainty bounds on control design. It is shown that considering uncertainty bounds in control problem formulation will create closed-loop systems which perform robustly in the face of uncertainties.
Chapter 4

Experimental Evaluation and Robustness and Performance Trade-offs in Uncertainty Modeling

In the previous chapter, a twofold control design strategy was presented. In this chapter, the focus is on the experimental evaluation of the proposed control design. But because the experimental setup (IRIS-facility) was itself under development, and because of the lack of link side measurement instrumentation, we were not able to evaluate the entire control strategy experimentally. Therefore, we limited our experiments to only the actuator-level torque control. In spite of this, the results were quite promising, and it is expected that the proposed theoretical control design can be implemented successfully when the setup is ready. For the purpose of torque control design, a nominal linear model of the joint is identified from the input-output experimental tests. Subsequently, by varying the input signal amplitude level, a set of models incorporating the effect of nonlinearities is extracted. The differences
between the nominal model and the set are formulated as uncertainty bounds for control design purposes. Using the uncertainty bounds, an \( \mathcal{H}_\infty \) - based optimal controller is designed. Experiments are performed for different uncertainty levels on the IRIS robot facility testbed.

4.1 Experimental Setup Facility

This section briefly presents the IRIS-facility experimental setup introduced in [16, 20, 21] (Fig. 4.1). The IRIS-facility is a versatile and reconfigurable robot arm. It is designed to be easily disassembled and assembled as required. It provides a multitude of configurations. Each joint-module is composed of a frameless DC-motor, an HD gear, an optical rotary encoder to measure the motor displacement, and a custom-designed torque sensor to measure the load torque.

For the purpose of the experimental tests, the joint-module (this joint will be referred to as the "test-joint" in the text to follow) is constituted by an RBE-01202 motor components set (Inland Motor Corp.), capable of delivering up to 1.12 Nm. The motor is coupled to a harmonic drive with 100:1 speed reduction [17]. The rated torque for this unit is 40 Nm, the maximum average torque is 49Nm, and the momentary peak torque is 108.4 Nm. The custom-designed torque sensor, which has a stiffness coefficient 10 times higher than that of the harmonic drive itself, is used to measure the load torque.

For the purpose of torque-feedback control law designs, the stator of the joint-test motor is fixed to a stationary frame to prevent it from rotation. Hence, the measured torque is proportional to the torque experienced by the rotor. (Fig. 4.2).

The IRIS-facility is controlled by a distributed computer system based on RISC processor nodes able to provide up to 80 MFLOPS per node, and a fast I/O system associated with each node allowing up to 5 KHz sampling rate.

The control designs are implemented on the IRIS-joint via a computer node built around a 50 MHz EISA bus-based IBM-PC compatible host computer. A RISC
coprocessor board (ATM-29050) is attached to the PC bus. The RISC board represents a whole computer system except for I/O peripherals. It has a powerful RISC processor with a built-in floating-point unit and its own memory (several MBytes). The memory access time that is very critical to the execution speed of standard computers (typically 70ns), preventing them from crossing over the speed of a couple of MFLOPS, is effectively shortened by a factor of 2 in this RISC board. The details about the RISC board can be found in the YARC Systems reference manual. The
control system hardware includes the following I/O boards attached to the Host bus:
a. ADC boards, b. DAC boards, and c. Digital I/O boards.

The I/O boards are connected to the low-level control system that includes interfaces to the optical encoders, signal-conditioning circuits and the power amplifiers. The IRIS hardware organization is shown in Fig. 4.3.

![Figure 4.3: Real time control implementation organization](image)

4.1.1 Experimental Identification of Transfer Function

The frequency response method is used experimentally to provide an input-output torque model of the test-joint. Such experiments can be dangerous if the arm is unrestrained, due to the elastic coupling between high-frequency "internal" (joint) and low-frequency "external" (arm) dynamics. In contrast, the experiments can be performed successfully if the arm is restrained against a stiff environment. The stiff environment prevents the external dynamics from being excited such that no effect on the internal dynamics takes place. The experiments were carried out for the test-joint using an input signal frequency range of 0-150 Hz. The same set of experiments were repeated for varying levels of input signal amplitude. Fig. 4.4 and Fig. 4.5 show the magnitude and phase plot of input-output torque for three different amplitudes of sinusoidal signal.

Since the highest working bandwidth of the system is below 100 Hz, a trans-
Figure 4.4: Input-output torque magnitude plot of IRIS-joint for three different input signal amplitude

Figure 4.5: Input-output torque phase plot of IRIS-joint for three different input signal amplitude
Figure 4.6: Estimated and measured input-output torque transfer function of the IRIS-joint

The transfer function of a model of the input-output torque based on the experimental data between 0-100Hz is identified. More precise analysis using Signal Processing and Identification Toolbox of MATLAB [18] shows that the transfer function is better approximated by $P_{nom}(s) = \frac{b(s)}{a(s)}$, where $b(s)$ and $a(s)$ are second and fourth order polynomials respectively given as:

\begin{align*}
  b(s) &= 0.001 \times (-0.0022s^2 - 0.1411s + 0.1258) \\
  a(s) &= s^4 - 3.9607s^3 + 5.9049s^2 - 3.9276s + 0.9834
\end{align*}

The comparison of the real and the identified input-output torque signals in Fig. 4.6 shows very good match between the measured and the estimated signals for the operating range of 0-100Hz.

Utilizing the $H_\infty$ control design framework, the aforementioned identified nominal model and uncertainty levels are used for control design purposes.
4.2 Robustness and Performance Trade-offs

The selection of uncertainty description plays a major role in the trade-off between robustness and performance requirements in the control design process. Uncertainty descriptions are introduced to account for variations between models and the actual system and provide a quantitative measure of the differences.

This section investigates this trade-off in the selection of uncertainty descriptions and levels. A set of control laws is designed by changing the uncertainty levels for the test-joint experiment. A frequency domain uncertainty description of the variation between the model and the "real" system is used for the design of control laws. These designs make use of a multiplicative uncertainty model to account for high-frequency unmodeled dynamics (Fig. 4.8). It is shown that the closed loop system may maintain the performance or even become unstable as the uncertainty levels are changed. These clearly indicate the importance of uncertainty descriptions in the control design process.

4.2.1 Control Objectives

The control objective is to track a desired command torque in spite of nonlinearities, friction and flexibility in the actuator-transmission system. This is formulated as minimizing the $||.||_\infty$ norm between the input disturbances and sensor outputs.

4.2.2 Uncertainty Descriptions

A frequency domain description of uncertainty is employed to account for the variation between the model and the "real" system. A multiplicative uncertainty weight is used for the low frequency (below 0.2 Hz) and for the unmodeled high frequency dynamics (above 50Hz). The nominal linear model of the joint (4.1-4.2) is used. Subsequently, by varying the input signal amplitude level, a set of models incorporating the effect of nonlinearities in the system are extracted. A plot of the differences between the nominal model and the set along with the multiplicative uncertainty
Figure 4.7: Magnitude plot of error torque and multiplicative uncertainty weight

weight are shown in Fig. 4.7. The magnitude of the multiplicative uncertainty weight at high frequency is selected to envelop the unmodeled modes of the system (Fig. 4.7).

Additive uncertainty weight is also included to represent the torque sensor noise measurement (Fig. 4.8).

4.2.3 Control Problem Formulation

The identified SISO nominal model of the test-joint, $P_{nom}$, is used to describe the test-joint experiment. It serves as a baseline model to which uncertainty models are appended. A block diagram of the problem formulation is shown in Fig. 4.8. The multiplicative uncertainty weight accounts for the neglected high-frequency modes and some low frequency error. It is modeled as an unstructured full block uncertainty, $\Delta_{mult}$, after the test-joint nominal model as seen in the block diagram. Performance weight, $W_p$, and actuator-saturation limit weight, $W_a$, are the parameters varied to examine trade-off between the robustness and performance of the control designs.

The block diagram is reformulated into the LFT general framework to design
control laws using the \( \mu \)-synthesis methodology. The block diagram is shown in Fig. 4.9. The dimensions of the \( \Delta \) blocks are: \( 1 \times 1 \) for \( \Delta_1 \), and \( 3 \times 2 \) for \( \Delta_2 \). \( \Delta_1 \) is associated with the multiplicative uncertainty, \( \Delta_2 \) with the performance block.

A pure \( H_\infty \) control design would synthesize a control law for one full block of size \( 4 \times 3 \), neglecting the inherent structure associated with the two blocks. Ignoring the structure of the uncertainty block leads to overly conservative control laws. The \( \mu \)-synthesis methodology incorporates knowledge of this structure in the control design process, reducing the conservatism.

### 4.3 Experimental Results

#### 4.3.1 Model Uncertainty Weight

Three control laws are synthesized based on the block diagram in Fig. 4.8 with varying levels of model uncertainty weight (i.e. \( W_{\text{mult}} \) varied), while the rest of the
The set of weighting functions that are used for control design purposes are as follows:

1. Model uncertainty weight:
\[ W_{\text{mult}}(s) = r \frac{500s + 5000}{s + 10000} \]

2. Noise weight:
\[ W_{n}(s) = n \frac{2s + 2.56}{s + 320} \]

3. Performance weight
\[ W_{p}(s) = p \frac{0.005s + 200}{0.01s + 0.00518} \]

4. Actuator saturation weight
\[ W_{a}(s) = a \frac{0.01s + 0.01}{0.01s + 0.001} \]

Table 1 contains a list of the control parameters used in the design and the results of implementation on the test-joint experiment.
Table 4.1: Parameters for control design with fixed performance weight

<table>
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<th>control</th>
<th>input</th>
<th>$a$</th>
<th>$n$</th>
<th>$p$</th>
<th>$r$</th>
<th>$\mu$</th>
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<td>MU-K1</td>
<td>step</td>
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<td>0.1</td>
<td>1.3</td>
<td>Fig. 4.14</td>
</tr>
<tr>
<td>MU-K3</td>
<td>sin</td>
<td>1.0</td>
<td>$10^{-4}$</td>
<td>$10^{-3}$</td>
<td>1.0</td>
<td>2.7</td>
<td>Fig. 4.15</td>
</tr>
</tbody>
</table>

The closed-loop torque responses of the test-joint experiment for implementing these controllers are shown in Fig. 4.10 to Fig. 4.15. MU-K1 achieves a $\mu$ value of 9.4, and the step response settles down in about 1 second. But the step response shows an overshoot as high as two times the desired step command. MU-K2 is designed by increasing the model uncertainty weight by a factor of 10 compared to the previous case. However, the step response doesn’t show any overshoot, but a steady-state error of 15% can be observed in Fig. 4.11. The level of uncertainty weight is increased to 1 in the MU-K3 controller and it can be observed that the step response of the system gets better compared to the two previous cases.

Fig. 4.13 to Fig. 4.15 show the sinusoidal response of the system for the same sets of controllers. In this set of experiments, controller MU-K3 shows superior performance compared to the two other controllers.

4.3.2 Actuator-Saturation Limit

A series of control laws is synthesized using varying levels of actuator-saturation limits (i.e., $W_a$ varied), while multiplicative uncertainty and noise weights remain fixed. Table 2. shows the control parameters used in the design and the results of implementation on the test-joint experiment.

Fig. 4.16 shows the step torque responses of the test-joint for controller AC-K1. The actuator-saturation level is chosen as 10 in this case. It shows an steady-
Table 4.2: Parameters for control design with varying performance weight.

<table>
<thead>
<tr>
<th>control</th>
<th>input</th>
<th>a</th>
<th>n</th>
<th>p</th>
<th>r</th>
<th>μ</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC-K1</td>
<td>step</td>
<td>10</td>
<td>10^{-4}</td>
<td>10^{-5}</td>
<td>1</td>
<td>2.26</td>
<td>Fig. 4.16</td>
</tr>
<tr>
<td>AC-K2</td>
<td>step</td>
<td>2.0</td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>1.0</td>
<td>2.27</td>
<td>Fig. 4.17</td>
</tr>
<tr>
<td>AC-K3</td>
<td>step</td>
<td>1.0</td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>1.0</td>
<td>2.7</td>
<td>Fig. 4.18</td>
</tr>
</tbody>
</table>

Table 4.3: Parameters for control design with varying performance and uncertainty weight.

<table>
<thead>
<tr>
<th>control</th>
<th>input</th>
<th>a</th>
<th>n</th>
<th>p</th>
<th>r</th>
<th>μ</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC-K4</td>
<td>sin</td>
<td>2.0</td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>10^{-3}</td>
<td>1.55</td>
<td>Fig. 4.19</td>
</tr>
<tr>
<td>AC-K5</td>
<td>sin</td>
<td>1.0</td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>0.01</td>
<td>9.4</td>
<td>Fig. 4.20</td>
</tr>
<tr>
<td>AC-K6</td>
<td>sin</td>
<td>0.5</td>
<td>10^{-4}</td>
<td>10^{-3}</td>
<td>0.001</td>
<td>0.73</td>
<td>Fig. 4.21</td>
</tr>
</tbody>
</table>

state error of up to 15% which continues to increase. If the actuator-saturation limit decreases to 2 (i.e. AC-K2 controller), the performance of the system does not change compared to the first case (Fig. 4.17. But if the actuator-saturation limit decreases to 1, the steady-state error gets smaller values (Fig. 4.18).

Three new control laws are synthesized, using sinusoidal input torque command. Table 3 shows the parameters used in the design as well as the result of implementation on the test-joint experiment. As can be observed, the performance of the system for the case of AC-K4 controller is considerably better than two other controllers for the best selection of uncertainty and actuator-saturation levels.

### 4.4 Summary

Representing the actuator-transmission system in flexible joint robots with a nominal model and an uncertainty description provides an excellent design model for use in robust torque control techniques. The theoretical and experimental results indicate that uncertainty modeling plays a major role in the trade-off between the
requirements and the robustness properties of synthesized control laws. A series of control laws are synthesized with varying levels of model uncertainty and different actuator-saturation limits. It is experimentally verified that the performance of the system is sensitive to the uncertainty description and the actuator saturation level in the control design. Increasing the level of multiplicative uncertainty while the actuator-saturation level is fixed leads to better performance of the system when the control law is applied experimentally. On the other hand, choosing a high actuator-saturation limit while the model uncertainty weight remains fixed causes performance to deteriorate and large steady-state errors to appear in the step response. Finally, it is verified that simultaneously varying the actuator-saturation level and the model uncertainty weights in control design leads to high performance closed-loop systems in experimental implementation.
Figure 4.10: Step response of the IRIS-joint for controller MU-K1

Figure 4.11: Step response of the IRIS-joint for controller MU-K2
Figure 4.12: Step response of the IRIS-joint for controller MU-K3

Figure 4.13: Sinusoidal response of the IRIS-joint for controller MU-K1
Figure 4.14: Sinusoidal response of the IRIS-joint for controller MU-K2

Figure 4.15: Sinusoidal response of the IRIS-joint for controller MU-K3
Figure 4.16: Step response of the IRIS-joint for controller AC-K1

Figure 4.17: Step response of the IRIS-joint for controller AC-K2
Figure 3.18: Step response of the IRIS-joint for controller AC-K3

Figure 4.19: Sinusoidal response of the IRIS-joint for controller AC-K4
Figure 4.20: Sinusoidal response of the IRIS-joint for controller AC-K5

Figure 4.21: Sinusoidal response of the IRIS-joint for controller AC-K6
Chapter 5

Stability Analysis of the Proposed Robust Torque Control Design

The problem of absolute stability of a feedback system which has a linear time-invariant part and a nonlinear element in the feedback loop is known as the classical Lur'e's Problem [53, 7]. Many systems which are primarily characterized as linear time-invariant systems except for a few nonlinear components such as saturating actuators can be represented as Lure systems. Therefore, traditionally, there has been significant interest in the stability of such systems. The stability results are available primarily for sector bounded nonlinearity [53, 7].

This chapter presents a stability analysis of the feedback control of flexible joint robots with sector-bounded perturbations, as shown in Fig. 5.1. The controller is designed following the procedure mentioned in Chapter 3. The focus is on the stability of the closed-loop torque feedback control of flexible joint robots with harmonic drive transmission. It is assumed that the transmission system exhibits nonlinear stiffness. We transform the closed-loop system into a Linear Time Invariant (LTI) part in the feedforward loop and a nonlinear element (i.e., nonlinear stiffness) which is inside sector \([a, b]\) (i.e., having bounded \(\infty\)-norm) in the feedback loop. Utilizing the small-gain theorem, we demonstrate that this guarantees the closed-loop system
stability provided that the $\infty$-norm of the interconnection of the LTI system and the nonlinearity is less than one. Linear fractional transformation and loop transformation techniques are employed to prove the main results of our stability analysis. Finally, as an alternative to the small-gain-theorem approach, a Lyapunov function is derived that guarantees that the feedback interconnection of the LTI and the nonlinearity are stable.

A review of a sector bounded perturbation condition and a linear fractional transformation of the system is given in the next two sections. The small-gain theorem is then presented, and the problem of robust stability of torque feedback control of a flexible joint robot is formulated. Finally, the stability of the system is investigated using the small-gain theorem and the Lyapunov function.

### 5.1 Sector Bounded Conditions

Sector conditions for the stability of the feedback interconnection of general input-output systems were introduced in [59], and further expanded in [55, 42].
A memoryless, time-varying nonlinearity, $\psi(y, t)$, is said to be inside sector $[a, b]$ if

$$
(\psi - by)^T (\psi - ay) \leq 0
$$

for all $y \in \mathbb{R}^m$ [19, 14]. Geometrically, these sector conditions imply that the graph of the nonlinearity lies within a conical region in the $\mathbb{R}^m \times \mathbb{R}^m$ input-output space for all time, $t$. For $m = 1$, Fig. 5.2 shows a nonlinearity, $\psi(y, t)$, inside sector $[a, b]$; the graph of $\psi(y, t)$ must lie in the shaded region within the two lines of slopes $a$ and $b$.

![Figure 5.2: Sector-bounded nonlinearity](image)

The Nyquist plot of an LTI system, inside a sector $[a, b]$, $b > a$, lies within a circle in the frequency plane, whose center is at $[(a + b)/2, j0]$ and has a radius of $(b - a)/2$. Note that a square, bounded, real system, that is, a system satisfying $\|G(s)\|_\infty \leq 1$, is inside sector $[-1, 1]$; and its Nyquist plot lies within a unit circle centered at the origin. For example, $G(s) = 10/(s+2)(s+5)$ is inside sector $[-0.4, 1.0]$ and its Nyquist plot lies within the corresponding circle, shown in Fig. 5.3. Note that $\|G(s)\|_\infty \leq 1$, so that its Nyquist plot also lies within the unit circle centered at the origin, as shown in Fig. 5.3.
5.2 Linear Fractional Transformation

Given a complex variable matrix $M$, partitioned as

$$
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix} \in C^{(p_1+p_2)\times(q_1+q_2)}
$$

and let $\Delta_l$ be another complex variable matrix:

$$
\Delta_l \in C^{(q_2 \times p_2)}
$$

Then we can formally define a lower Linear Fractional Transformation (LFT) of $M$ with respect to $\Delta_l$ as the map

$$
F_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}
$$

provided that the inverse $(I - M_{22}\Delta_l)^{-1}$ exists. The terminologies of lower LFT should be clear from Fig. 5.4.

We can define the closed loop transfer function from $w$ to $z$ in Fig. 5.4 (or Lower Linear Fractional Transformation [8]) as follows:

$$
F_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}
$$
This transformation exists provided that the closed-loop is well posed, i.e. \((I - M_{22} \Delta_1)^{-1}\) exists.

### 5.3 Input-Output Stability and Small Gain Theorem

The formalism of input-output stability is useful in studying the stability of interconnections of dynamical systems. This is particularly so for the feedback connection of Fig. 5.5.

To prepare for the small-gain theorem, we first give the following definitions.

**Definition 1:** Let \(\mathcal{L}^p\) denotes the normed linear space where \(p \in [1, \infty)\).

Also, define \(\mathcal{L}_e^p\) as an extended linear space defined by

\[
\mathcal{L}_e^p = \{ u|u_e \in \mathcal{L}^p, \forall \tau \geq 0 \} \tag{5.2}
\]

where \(u\) belongs to \(\mathcal{L}^p\), and \(u_e\) is the truncation of \(u\), defined by

\[
u_e(t) = \begin{cases} 
t & \text{if } 0 \leq t \leq \tau \\
0 & \text{if } \tau < t \end{cases} \tag{5.3}
\]
The extended space $\mathcal{L}_e$ is a linear space that contains the unextended space $\mathcal{L}_p$ as a subset.

**Definition 2**: A mapping $H : \mathcal{L}_p \to \mathcal{L}_e$ is $L_r$-stable ($r \in [1, \infty]$), if there exist finite nonnegative constants $\gamma$ and $\beta$ such that

$$
\|(Hu)_\tau\| \leq \gamma \|u_\tau\| + \beta
$$

for all $u \in \mathcal{L}_p$ and $\tau \in [0, \infty)$. $\mathcal{L}_p$ is a space of signals $u : [0, \infty) \to \mathbb{R}^p$. $\mathcal{L}_e$ is also an extended space of signals.

Now, we are ready to introduce the small-gain theorem [7]. The small-gain theorem gives a sufficient condition for the $L$-stability of the feedback connection in Fig. 5.5.

**Theorem 1** Consider the system in Fig. 5.5. Let $H_1, H_2 : \mathcal{L}_p \to \mathcal{L}_e, p \in [1, \infty]$, be two $\mathcal{L}_p$ stable operators with finite gains $\gamma_1, \gamma_2$ and associated constants $\beta_1, \beta_2$. Let the operator $H_1 H_2$ be strictly causal. If

$$
\gamma_1 \gamma_2 < 1
$$

(5.5)
then for all $v_1 \in L^p$ and $v_2 \in L^p$

\[
\|u_1\|_p \leq \frac{1}{1 - \gamma_1 \gamma_2} (\|v_1\|_p + \gamma_2 \|v_2\|_p + \beta_2 + \gamma_2 \beta_1)
\] (5.6)

\[
\|u_2\|_p \leq \frac{1}{1 - \gamma_1 \gamma_2} (\|v_2\|_p + \gamma_1 \|v_1\|_p + \beta_1 + \gamma_1 \beta_2)
\] (5.7)

for all $\tau \in [0, \infty)$. If $v_1 \in L^p$ and $v_2 \in L^p$, then $u_1, y_2 \in L^p$, and $u_2, y_1 \in L^q$ and the norms of $u_1$ and $u_2$ are bounded by the right-hand side above with non-truncated functions.

The reader may refer to [7] for the proof of the previous theorem. Paraphrasing the small-gain theorem proves that the feedback connection of two input-output stable systems, as in Fig. 5.5, will be input-output stable provided that the product of the system gain is less than one.

Now, for the interconnection of an LTI system in forward loop and a non-linearity in the feedback, the following theorem can also be applied. Consider the system of Fig. 5.6. Let the LTI system be represented by

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\] (5.8)

Suppose that $G(s)$ is Hurwitz. Let

\[
\gamma_1 = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)] = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2
\] (5.9)

where $\sigma_{\max}[.]$ denotes the maximum singular value of a complex matrix. The constant $\gamma_1$ is finite since $G(s)$ is Hurwitz. Suppose that the nonlinearity $\psi(., .)$ satisfies the inequality

\[
\|\psi(t, y)\|_2 \leq \gamma_2 \|y\|_2, \forall t \geq 0, \forall y \in \mathbb{R}^p
\] (5.10)

then it satisfies the sector condition (5.1) with

\[
b = \gamma_2; a = -\gamma_2
\] (5.11)
We can apply the *small-gain theorem* and conclude that the system is absolutely stable if

\[ \gamma_1 \gamma_2 < 1 \]  \hspace{1cm} (5.12)

This a a robustness result which shows that closing the loop around a Hurwitz transfer function with a nonlinearity satisfying (5.10), with a sufficiently small \( \gamma_2 \), does not destroy the stability of the system.

### 5.4 Loop Transformation

By making a suitable transformation of the system in Fig. 5.5, one can significantly expand the range of applicability of theorem 1. The following theorem presents the generalization of theorem 1.

**Theorem 2** [19] **Consider the system shown in Fig. 5.5, and suppose \( p \in [1, \infty) \) is specified. Suppose \( H_2 \) is causal and \( \mathcal{L}_p \)-stable. Under these conditions, the system is**
\( \mathcal{L}_p \)-stable if there exists a causal linear operator \( K \) which is \( \mathcal{L}_p \)-stable such that:

(i) \( H_1(I + K H_1)^{-1} \) is causal and \( \mathcal{L}_p \)-stable, and

(ii) \( \gamma_p[H_1(I + K H_1)^{-1}]\gamma_p(H_2 - K) < 1 \).

Fig. 5.7 shows an equivalent representation of the system in Fig. 5.5 where a constant-gain negative feedback \( K \) is applied around the \( H_2 \) of the system. The proof of this theorem can be found in [19], [53].

5.5 Stability Results

5.5.1 Small-gain Theorem-Based Stability Analysis

The stability problem of the torque-feedback control of robots with sector-bounded stiffness nonlinearity is introduced in this chapter. The closed loop torque-feedback control law is transformed into an LTI and a nonlinearity which is inside sector \([a, b]\).
Figure 5.8: Linear fractional transformation

to prove the stability problem. The linear fractional transformed (LFT) form of the
closed loop system introduced in Section 5.2 is primarily used for this purpose.

The closed loop system of Fig. 5.1 is transformed into an LFT form as
introduced in Fig. 5.8. In Fig. 5.8, $G_{ij}, (i, j = 1, 2)$ are all known transfer functions.
For instance, $G_{11}$ can be found to be as,

$$G_{11} = 1$$

and

$$G_{12} = \frac{-r - P_m S}{r + P_m S + r S P_m K_{\infty}}$$

where $r$ is the gear ratio, $P_m$ is the actuator transfer function, $S$ the stiffness coefficient, and $K_{\infty}$ is the linear $H_{\infty}$ controller transfer function shown in Fig. 5.1. The
remaining of $G_{ij}$’s can be found similarly from Fig. 5.1. $w$ is the norm bounded
exogenous inputs and $z$ is the signal to be kept small.
The transfer function from \( y \) to \( u \) can be defined as,

\[
u = \Delta_r y
\] (5.15)

\[
z = G_{11} w + G_{12} u
\] (5.16)

\[
y = G_{21} w + G_{22} u
\] (5.17)

Substituting (5.15) in (5.17) we obtain,

\[
y = v + G_{22} \Delta_r y
\] (5.18)

where

\[
v = G_{21} w
\] (5.19)

Fig. 5.9 shows the block-diagram of the above formulation.

In Fig. 5.9, \( G_{22} \) is the LTI \( \mathcal{L}_\infty \)-stable transfer matrix, by design, and \( \Delta_r \) is the stiffness nonlinearity designed to be in sector \([a, b]\). To state our main results, the following lemma is needed[14].

_Lemma 1_: Given a stable LTI system \( \Sigma: \dot{x} = Ax + Bu, y = Cx + Du \), where the quadruple \([A, B, C, D]\) is a minimal realization of the transfer function matrix, \( G(s) = C(sI - A)^{-1}B + D \), the following statements are equivalent.
(i) Σ is inside sector [a, b].

(ii) There exist real matrices $P = P^T > 0$, $L$ and $W$ which satisfy

$$PA + A^TP = -C^TC - L^TL, \quad PB = C^T(\alpha I - D) - L^TW,$$

$$D^TD - \alpha(D + D^T) + abI = -W^TW \quad (5.20)$$

where $\alpha = (a + b)/2$. The proof of this lemma can be found in [14]. Now, we are ready to state our theorem.

**Theorem 3** Consider the negative feedback interconnection of a stable LTI system $G_{22} : \dot{x} = Ax + Bu, y = Cx + Du$, where the quadruple $[A, B, C, D]$ is a minimal realization, and a memoryless, time-varying, nonlinearity, $\Delta_r$ (as shown in Fig. 5.9). If the nonlinearity, $\Delta_r$, belongs to an arbitrary sector not necessarily centered at origin, then the origin is a Lyapunov stable equilibrium point of the closed-loop system if $G_{22}$ is inside sector $[-\frac{1}{b}, -\frac{1}{a}]$ (i.e., the Nyquist plot of $G_{22}$ lies within a circle in the frequency plane, whose center is at $[\frac{-\frac{1}{b} + \frac{1}{a}}{2}, j0]$ and has a radius of $(\frac{-\frac{1}{b} + \frac{1}{a}}{2})$).

**Proof:** Since $\Delta_r$ does not necessarily belong to sector $[a, b]$ where $b > 0 > a$, we employ the loop-transformation technique introduced in section 5.4 for the problem in hand. Let us define

$$k = \frac{a + b}{2}, r = \frac{b - a}{2} \quad (5.21)$$

Fig. 5.10 shows the definitions of $k$ and $r$ graphically. Then, the new nonlinearity is given by

$$\psi = \Delta_r - ky \quad (5.22)$$

It can easily be verified that if $\Delta_r$ satisfies the sector condition (5.1), then $\psi$ satisfies the sector condition

$$(\psi - ry)^T(\psi + ry) \leq 0 \quad (5.23)$$
Fig. 5.1 shows the transformed block diagram of the system. Now, in Fig. 5.1, the new linear system, $G_T$, represented by

$$G_T = G_{22}[I + kG_{22}]^{-1}$$

is a Hurwitz transfer function by design. Utilizing the small-gain theorem, we can conclude that the system will be stable if

$$\gamma(G_T) \gamma(\nu) \leq 1.$$  \hspace{1cm} (5.25)

where $\gamma$ is defined in (5.9). Since $\nu$ belongs to sector $[-r, r]$, then by definition $\gamma(\nu) \leq r$. Therefore, from (5.25) $\gamma(G_T) \leq 1/r$. Thus, $G_T$ should belong to sector $[-1/r, 1/r]$. It can be easily verified that if $\nu \in [-r, r]$ then from (5.22), $\Delta r \in [a, b]$ and also if $G_T \in [-\frac{1}{r}, \frac{1}{r}]$, then from (5.24) $G_{22}[I + kG_{22}]^{-1} \in [-\frac{1}{r}, \frac{1}{r}]$ that results in $G_{22} \in [-\frac{1}{b}, -\frac{1}{a}]$. This proves the theorem.

Alternatively, the above theorem can be also proved utilizing Lemma 1 and Lyapunov stability criteria. This is shown in the next section.

**5.5.2 Lyapunov-Based Stability Analysis**

*Proof:* For notational convenience, assume that in the state-space form, the $G_T$ transfer function is presented as:

$$\dot{x} = Ax + Bu$$
and the nonlinearity $\psi$ belongs to sector $[a, b]$. Then the following Lyapunov function candidate can be chosen to prove the system stability. Let $V(x) = x^TPx$ be the Lyapunov function with $P = P^T > 0$ being a positive definite matrix which satisfies the sector boundedness in lemma 1. Differentiating the Lyapunov function we get.

$$\dot{V}(x) = x^T(AP + PA)x + u^TB^TPx + zPBu$$

(5.27)

Using the relations of (5.20) in (5.27) leads to

$$\dot{V}(x) = x^T(-C^TC + L^T)L)x - x^T(\alpha C^T - C^TD - L^TW)\psi$$
$$- \psi^T(\alpha C^T - D^TC^T - W^TL)x$$

(5.28)

but,

$$C^Tx = y - Du \quad \text{and} \quad x^TC^T = y^T - u^TD^T.$$  

(5.29)
Using the fact that the nonlinearity, \( \psi \), is inside sector \([a, b]\), \((\psi - by)^T(\psi - ay) \leq 0\) for all \(y\) and \(u = -\psi\), results in,

\[
C^{'}x = y - D\psi \quad \text{and} \quad z^T C^{'}T = y^T - \psi^T D^T.
\] (5.30)

Hence, using the last relationship in (5.20) and some algebraic manipulations results in

\[
\dot{V}(x) = -[Lx - Wu]^T[Lx - Wu] - (y + \alpha \psi)^T(y + \alpha \psi) - \frac{(a - b)^2}{4}
\] (5.31)

Thus, from (5.31) we have \( \dot{V}(x) \leq 0 \), and the origin is Lyapunov stable.

5.6 Summary

Sufficient conditions are presented for the stability of torque-feedback control laws of flexible-joint robots subjected to nonlinear stiffness perturbation. The closed-loop system is rearranged to include an LTI part and a sector-bounded nonlinearity. The linear fractional transformation technique is employed to study the stability of the interconnection system of the LTI part and the nonlinearity. Moreover, conditions are presented for a nonlinearity to be inside a given sector. The small-gain theorem and loop-transformation technique are employed to prove the stability of a closed-loop system. Finally, the Lyapunov-stability theorem is used as a complementary approach to the small-gain theorem to prove the stability of the feedback interconnection of the LTI system and the nonlinearity.
Chapter 6

Conclusions and Future Directions

6.1 Conclusions

New results are presented concerning motion and torque control problems of flexible joint robots. A complete model of flexible joint robots with HD transmission is proposed and then used for motion and torque control design purposes. The thesis focuses on the selection of uncertainty bounds and their incorporation into the control design process. A systematic approach is introduced in order to incorporate actuator-transmission uncertainties into the control design. The describing function and conic-sector-bounded nonlinearity methods are used to model the effect of hysteresis, friction and nonlinear stiffness on the control design. Numerical examples of uncertainty bound selection are provided. This research shows the need to incorporate accurate descriptions of model errors into the control problem formulation.

A twofold robust control design of flexible joint robots with HD is introduced: it includes an actuator-level torque control and a link-level motion control. The design method is based on $\mathcal{H}_\infty$—optimal control and $\mu$-synthesis techniques. The control design developed allows us to focus on link and actuator control problems separately. Moreover, the performance specifications and stability requirements can be satisfied simultaneously. Accurately describing the flexible joint robot models and
correctly identifying model errors guarantee high performance control laws. The robustness of the control laws is shown to be directly tied to the selection of uncertainty bounds. Inaccurately modeling uncertainty bounds leads to unstable control laws in experimental implementation. The $H_\infty$ - optimal control and $\mu$-synthesis framework proved to be very suitable for applications to the motion and torque control problems of flexible joint robots. This is mainly because performance and robustness specifications can be incorporated into the control design process. The results presented will aid control engineers in the design of control laws using $H_\infty$ - optimal control and $\mu$-synthesis methods.

This research verifies the closed-loop stability of the proposed control design with nonlinear stiffness using the Small-Gain Theorem and the Lyapunov Function method. The actuator-level torque control design was implemented experimentally. Robustness and performance trade-offs in torque control design were investigated. Experiments were performed for different uncertainty levels on the IRIS-facility testbed. It was verified that varying the actuator-saturation level and the model uncertainty weights in control design simultaneously guaranteed high performance closed-loop responses and maintained the stability of the system. Moreover, the robustness of the proposed control scheme in modeling uncertainty and measurement noises was demonstrated experimentally.

6.2 Suggestions for Future Research

The following research directions are suggested:

a) The role of uncertainty descriptions should be quantified in the design of motion and torque control laws for robots with harmonic drive transmission such as those introduced in this work. It is important to understand how the accuracy of the design model and uncertainty descriptions affect the robustness and performance properties of the control laws.

b) Systematic methods must be further developed for identifying both nom-
nal models and uncertainty descriptions. Currently, uncertainty descriptions are
developed based on the experience of the control engineer. A systematic approach
to system identification, which provides both a plant and uncertainty models of the
system, would be a major step forward in the control design process.

c) All robust control design methods require explicit worst-case bounds on
the existing plant uncertainty. This research established some worst case bounds:
however, it is desirable that new system identification methods be introduced to
identify explicit bounds for the plant uncertainty.

d) The proposed control scheme in this work should be extended to the case
where a robot end-effector is in contact with an environment (force control).

e) In this work, a continuous time controller for a continuous plant model is
designed. Then, a bilinear transformation technique is used to discretize the control-
er and to implement it experimentally. Further investigation is required to evalu-
ate the sampling effects in control design. Moreover, other common digital control
design methods like “discrete-time control design for the discretized plant” and “dir-
ect sampled-data control design for the continuous plant” should be compared with
the method proposed in this work.
References


[34] Majid M. Moghaddam and Andrew A. Goldenberg. Robustness and performance tradeoffs in torque control of robots with harmonic drive transmission. *to


Appendix A

$\mathcal{H}_\infty$— Design and Structured Singular Value ($\mu$) Framework

A.1 Mathematical Preliminaries

A.1.1 The Signal Spaces $\mathcal{L}_2$ and $\mathcal{H}_2$

In the context of this dissertation, $\mathcal{L}_2$ is the space of all signals, or vectors of signals, with bounded energy. This is a Hilbert space with inner product

$$< x, y > = \int_{-\infty}^{\infty} x^*(t)y(t)dt \quad \text{and norm} \quad \|x\|_{\mathcal{L}_2} = \sqrt{< x, x >} \quad (A.1)$$

i.e. the square root of the total energy. Alternatively, $\mathcal{L}_2$ may be thought of as a frequency domain space, with inner product

$$< \hat{x}, \hat{y} > = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}^*(jw)\hat{y}(jw)dw \quad \text{and norm} \quad \|\hat{x}\|_{\mathcal{L}_2} = \sqrt{< \hat{x}, \hat{x} >} \quad (A.2)$$

where $\hat{x}$ is the Fourier transform of $x$. Parseval’s theorem states that $\|\hat{x}\|_{\mathcal{L}_2} = \|x\|_{\mathcal{L}_1}$.

In fact, we shall think of these spaces as being one and the same and not use a separate notation for signals in the time and frequency domains. Thus, when we say that $x$ is in $\mathcal{L}_2$ we mean that $x$ in a signal of bounded energy, which can be thought of either as a function of time or frequency.
In the time domain, $L_2$ can be decomposed into $L_{2+}$ and $L_{2-}$, where $L_{2+}$ is the space of signals defined for positive time and zero for negative time. Similarly, $L_{2-}$ is the space of signals defined for negative time, and zero for positive time. In the frequency domain, $L_2$ can be similarly decomposed into $H_2$ and $H_2^\perp$, where $H_2$ is the space of Fourier transform of signals in $L_{2+}$ and $H_2^\perp$ is the space of Fourier transforms of signals in $L_{2-}$. Thus, when we say that $x$ is in $H_2$ we mean that $x$ is a signal of bounded energy which, when considered in the time domain, is zero for negative time. For such signals, the Fourier transform is identical to the Laplace transform with the Laplace variable $s$ replaced by $jw$. $R^H_2$ is the subspace of $H_2$ consisting of only rational functions, and may be equivalently thought of the rational functions of the complex variable $s$ with denominator of strictly greater order than numerator and with no poles in the closed $rhp$. Similarly, the Hardy space $H_2$ can be defined as the space of functions of the complex variable $s$ that are analytic for all $s$ in the open right half plane (i.e., $s : \mathcal{C}(s) > 0$), and for which the integrals

$$
\frac{1}{2\pi} \int_{\sigma - \infty}^{\sigma + \infty} \bar{\mathbf{x}}(s) \mathbf{x}(s) s \quad \text{and norm } \|\mathbf{x}\|_{L_2} = \sqrt{<\mathbf{x}, \mathbf{x}>} \quad (A.3)
$$

are uniformly bounded for all $\sigma > 0$, with norm as in (2.14).

**A.1.2 The Function Spaces $L_\infty$ and $H_\infty$**

We shall often think of systems as being operators on $H_2$. A system $P$ will be called stable if, for any input in $H_2$, the output is also in $H_2$. That is, a stable system maps bounded energy inputs onto bounded energy outputs. If a system is unstable then its output can have infinity energy in response to a finite energy input. $H_\infty$ is the space of transfer functions of stable linear, time-invariant, continuous time systems. This is a Hardy space with norm

$$
\|P\|_{H_\infty} := \sup_{u \in H_2: u \neq 0} \frac{\|P u\|_{H_2}}{\|u\|_{H_2}} \quad (A.4)
$$

Thus stable systems have a finite $H_\infty$ norm, and this norm measures the maximum energy gain of a system. For a constant matrix $A$, its maximum singular value, $\sigma(A)$
is defined as the maximum "gain" of the matrix when the "inputs" and "outputs" are constant vectors equipped with the standard Euclidean norm $\|x\|_2 := \sqrt{x^T x}$. Or

$$\tilde{\sigma}(A) := \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_2}$$  \hspace{1cm} (A.5)

The maximum singular value is more easily calculated as $\tilde{\lambda}^\frac{1}{2}(A^*A)$, where $\tilde{\lambda}$ denotes the maximum eigenvalue of the matrix. An input that comes arbitrarily close to attaining the $\mathcal{H}_\infty$ norm concentrates its energy at the frequency where the gain of the system, as measured by the maximum singular value of its frequency response matrix, is largest. Thus,

$$\|P\|_{\mathcal{H}_\infty} = \sup_{s \in \mathbb{R}(s) > 0} \tilde{\sigma}(P(s)).$$  \hspace{1cm} (A.6)

Formally, $\mathcal{H}_\infty$ is defined as the space of functions of the complex variable $s$ analytic for all $s$ in the open right half plane with finite norm as defined in (A.6). $\mathcal{RH}_\infty$ is defined as the subspace of $\mathcal{H}_\infty$ whose elements are rational functions of $s$ and arise as the transfer functions of finite dimensional systems, i.e., those systems described by ordinary differential equations. For such systems, the supremum is attained on the boundary $s = jw$, (for possibly infinite $w$) i.e.,

$$\|P\|_{\mathcal{RH}_\infty} = \max_{w \in \mathbb{R}} \sum_{\infty} \tilde{\sigma}(P(jw)).$$  \hspace{1cm} (A.7)

provided $P(s)$ is analytic in the right half plane. $\mathcal{L}_\infty$ is the (Lebesgue) space of all functions essentially bounded on the imaginary axis with norm

$$\|P\|_{\mathcal{L}_\infty} := \text{esssup}_w \tilde{\sigma}(P(jw)).$$  \hspace{1cm} (A.8)

Similarly $\mathcal{RH}_\infty$ is the subspace of $\mathcal{L}_\infty$ whose elements are rational functions of $s$, for which the norm simplifies to

$$\|P\|_{\mathcal{RH}_\infty} := \max_{w \in \mathbb{R}} \sum_{\infty} \tilde{\sigma}(P(jw)).$$  \hspace{1cm} (A.9)

We shall regard $\mathcal{H}_\infty$ as a subspace of $\mathcal{L}_\infty$. Any proper rational transfer function matrix is in $\mathcal{L}_\infty$ provided it has no imaginary axis poles.

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A.2 $\mathcal{H}_\infty$ Control Theory

This section briefly reviews frequency domain methods for analyzing and synthesizing the performance and robustness properties of feedback systems using the structured singular value ($\mu$) [8]. The general framework, shown in Fig. A.1, is based on linear fractional transformations (LFTs). Any linear interconnection of inputs, outputs, and commands along with perturbations and a controller can be viewed in this context and rearranged to match this diagram. $G$ represents the system interconnection structure, $\Delta$ the uncertainties, and $K$ the control law. $v$ is a vector of exogenous inputs and disturbances. $\epsilon$ is a vector of errors to be kept small, $y$ is a vector of measurement signals provided to the control design. $u$ is a vector of inputs from the control law. $z$ and $\hat{u}$ are outputs to and from the uncertainty block.

$\mathcal{H}_\infty$ control design is concerned with meeting frequency-domain performance criteria [11]. The Hardy Space $\mathcal{H}_\infty$ consists of all complex-valued function $F(s)$ of a complex variable $s$ which are analytic and bounded in the open right half plane (rhp). For real rational functions, $F \in \mathbb{RH}_\infty$, the infinity norm is given by

$$\|F\|_\infty := \sup_{\omega} \sigma[F(j\omega)] \quad \forall w \in \mathbb{RH}_\infty$$

(A.10)
where \( \hat{\sigma} \) denotes the largest singular value [9]. In single-input/single-output (SISO) (A.10) states that \( \|F\|_\infty \) is the distance from the origin to the farthest point on the Nyquist plot of \( F(s) \).

**Definitions**

Following is a list of terms used extensively in this chapter.

**Nominal Stability.** The nominal plant model achieves the nominal stability if it is stabilized by the controller. This is a minimum requirement.

**Nominal Performance.** In addition to nominal stability, the nominal closed-loop response should satisfy some performance requirements.

**Robust Stability.** The closed-loop system must remain stable for all possible plants as defined by the uncertainty descriptions.

**Robust Performance.** The closed-loop system must satisfy the performance requirements for all possible plants as defined by the uncertainty descriptions.

Most modern control design methods only address the problem of nominal stability and nominal performance. Stability margins used in classical frequency domain methods attempt to address the robust stability problem, but they may be misleading for multivariable systems. These margins neglect the interaction and cross coupling present in multivariable systems. One method that deals with the robust performance question, in a multivariable framework, is \( \mu \)-based analysis and synthesis techniques.

**A.2.1 Nominal Performance**

The \( \mathcal{H}_\infty \) norm appears as a performance measure in many input/output systems. Consider the performance in terms of bounds on the output \( e \) in the presence of uncertain bounded input \( v \). Bounds for both \( v \) and \( e \) can be expressed in terms of power, energy, or magnitude norms.
Figure A.2: $\mathcal{H}_\infty$ disturbance attenuation problem.

Frequency dependent weighting functions can be used to shape the spectral content of signals and performance specifications. Often the physical process is modeled as composed of inputs of bounded power rather than perfectly known signals of fixed power spectrum, leading to the $\|\cdot\|_\infty$ norm. Modeling of uncertainties with the $\|\cdot\|_\infty$ norm is motivated by the types of model errors. The benefit of transforming performance measure into the $\|\cdot\|_\infty$ norm becomes apparent when robustness and performance objectives are included in the control problem formulation [59, 8].

The system shown in Fig. A.2 is an example of the application of the $\mathcal{H}_\infty$ norm to the disturbance attenuation problem. The transfer function from $v$ to $e$ is the sensitivity function, $S$. Suppose $z$ is any signal and $v = Wz$. The disturbance attenuation objective is to minimize the energy of $e$ for the worst input signal $v$, or equivalently, the $\mathcal{H}_\infty$ norm of the weighted sensitivity function, $\|WS\|_\infty$ is to be minimized. In the $\mathcal{H}_\infty$ synthesis problem, $P$ and $W$ would be given and $K$ would be chosen to minimize $\|WS\|_\infty$ subject to internal stability of the closed-loop system. Without any uncertainty in the model, the control design optimizes the nominal
A.2.2 Robust Stability

The infinity norm, \( \| \cdot \|_\infty \), lends itself to analyzing systems for stability in the presence of uncertainty. Uncertainty is often described as norm-bounded variations from a nominal model. The uncertainty can vary across frequency, which is reflected in a weighting function associated with the norm-bound. Plants described by a nominal model and a perturbation of the model are very different from systems described by a nominal model and an additive noise process. The difference lies in the fact that plant perturbations can destabilize a nominally stable plant whereas an additive noise process can not destabilize a plant. Consider the block diagram in Fig. A.3. In Fig. A.3, treat \( \Delta P \) as a norm-bounded perturbation and \( d \) as the disturbance to the system. Depending on the size of the norm bound, \( \Delta P \) can destabilize the system \( P \). It is even more apparent when feedback is introduced.

Models of real systems are never exact, because there is always some variation between the physical system and the mathematical model. Describing the model by a nominal plant \( P \) and norm-bounded perturbations or uncertainties, \( \Delta P \), one is able to define a rich class of models. This type of description allows the inclusion
of destabilizing perturbations, therefore, the issue of robust stability must be addressed. That is, we desire a set of plant models, defined by \( P + \Delta P \), to be stabilized by a controller \( K \), in the presence of the perturbation \( \Delta P \).

From the physics of the problem, the perturbation of the plant model, \( \Delta P \), can be bounded across frequency by a function \( W \in \mathcal{RH}_\infty \), such that for all \( w \in [0, \infty) \)

\[
|\Delta P(jw)| < |W(jw)| \tag{A.11}
\]

\( W \) is a weighting function describing the uncertainty in the model as a function of frequency. This uncertainty description is unstructured, because the only assumption made about the model error is that it is magnitude bounded.

The \( \mathcal{H}_\infty \) control methodology provides a common framework in which one is able to incorporate knowledge of the modeling limitations, in terms of frequency response data, into the control analysis and design problem. A shortcoming of the \( \mathcal{H}_\infty \) methodology is that it uses singular value tests for stability and performance analysis, which often is inappropriate and leads to conservative results. The structured singular value (\( \mu \)) was developed to address this limitation.

### A.3 \( \mu \)-Analysis Methods

The structured singular value, \( \mu \), is used as a measure of the robustness of systems to structured and unstructured uncertainties. \( \mu \) is a generalization of \( \mathcal{H}_\infty \) analysis methods limited to unstructured uncertainty. \( \mathcal{H}_\infty \) control design and \( \mu \) analysis methods are combined to form the \( \mu \)-synthesis technique for control design.

\( \mu \)-analysis methods are used in the analysis of systems with structured and unstructured uncertainties. For the purpose of analysis, the controller may be thought of as just another system component. The inclusion of the controller into the plant reduces the diagram in Fig. A.1 to that in Fig. A.4. The analysis problem involves determining whether the error \( e \) remains in a desired bound for bounded input \( u \) and bounded perturbation \( \Delta \).
Figure A.4: General analysis framework.

$G$ can be partitioned so that the input-output map from $v$ to $e$ is expressed as the following linear fractional transformation

$$e = F_u(G, \Delta)v$$  \hspace{1cm} (A.12)

where

$$F_u(G, \Delta) = G_{22} + G_{21} \Delta (I - G_{11} \Delta)^{-1} G_{12}$$

$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$

The nominal performance is simply

$$\|G_{22}\|_\infty = \sup_{\omega} \tilde{\sigma}(G_{22}(j\omega)) \leq 1$$  \hspace{1cm} (A.13)

This is the transfer function from $v$ to $e$ with the uncertainty, $\Delta$, set to zero. Robust stability for unstructured uncertainty (assuming $\tilde{\sigma}(\Delta) \leq 1$ known) depends only on $\|G_{11}\|_\infty$. Unfortunately, norm bounds are inadequate for dealing with robust performance and realistic models of structured plant uncertainty. To handle these questions, the structured singular value, $\mu$ is used. $\mu$ analyzes linear fractional transformations when $\Delta$ has structure. A more complete background on $\mu$ is found in [8].
Figure A.5: General framework.

We define the structured singular value, a matrix function denoted by $\mu(\cdot)$, defined as follows. Let $M \in \mathcal{C}^{n \times n}$, $S$ and $F$ be two non-negative integers and $r_1, \ldots, r_S; m_1, \ldots, m_F$ be positive integers where

$$\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$$

Define a family $\Delta \subset \mathcal{C}^{n \times n}$ of block-diagonal matrices as

$$\Delta := \{ \text{diag} [\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_1, \ldots, \Delta_F] \}$$

$$\delta_i \in \mathcal{C}, \Delta_j \in \mathcal{C}^{m_j \times m_j}$$

Thus, a matrix in $\Delta$ is block diagonal with $S$ scalar blocks and $F$ full blocks. Then $\mu(M)$ is defined as

$$\mu(M) := [\min \{ \tilde{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0 \}]^{-1}$$

unless no $\Delta \in \Delta$ makes $I - M\Delta$ singular, in which case $\mu_\Delta(M) := 0$.

We may interpret the above definition in the following way. Suppose $M \in \mathcal{C}^{n \times n}$ and consider the loop shown in Figure A.5. Then $\mu(M)^{-1}$ is a measure of the smallest structured perturbation, $\Delta$, that causes instability of the feedback loop. Obviously, $\mu$ is a function of $M$, which depends on the structure of $\Delta$. The importance of $\mu$ in studying robustness of feedback system is due to two theorems that characterize
in terms of $\mu$, the robust stability and robust performance of a system in the presence of structured uncertainty.

**Theorem [8]: Robust Stability.**

$F_u(G, \Delta)$ is stable $\forall \Delta \in \Delta$ iff $\sup_\omega \mu(G_{11}(j\omega)) \leq 1$, $0 \leq \omega \leq \infty$

This reduces to $\sup_\omega \sigma(G_{11}(j\omega)) \leq 1$ when $\Delta$ is a full block, unstructured uncertainty. Hence the interconnection with the $\mathcal{H}_\infty$ norm. When $\Delta$ has structure, the $\|\cdot\|_\infty$ provides an upper bound which is a more conservative measure of robustness.

For control design, one is really interested in *robust performance*. That is, achieving the performance required in the presence of uncertainty. We will characterize performance in terms of the $\|\cdot\|_\infty$ of the transfer function from disturbance ($v$) to error ($e$) in Fig. A.5. The robust performance question can be formulated as a robust stability question, by associating a full block uncertainty, $\Delta_{k+1}$, with the performance norm. $\Delta_{k+1}$ is of size(number of disturbances $v$), by (number of error outputs $e$). Thus, robust performance is equivalent to robust stability with respect to a different block structure. Formally stated:

**Theorem[8]: Robust Performance.**

$F_u(G, \Delta)$ is stable and $\|F_u(G, \Delta)\|_\infty \leq 1 \forall \Delta \in \Delta$ iff $\sup_\omega \mu(G(j\omega)) \leq 1$ $0 \leq \omega \leq \infty$, where $\mu$ is computed with respect to the structure $\Delta = \{\text{diag}(\Delta, \Delta_{k+1})\}$. $\Delta_{k+1}$ is the performance block and $\Delta \in \Delta$.

**A.4 $\mathcal{H}_\infty$ Synthesis**

For the purpose of synthesis, the maximum singular value of the perturbation $\Delta$ can be assumed to be bounded in magnitude by 1. This results in the synthesis problem presented in Fig. A.6. Hence, the synthesis problem involves finding a stabilizing controller $K$ such that the performance requirement are satisfied with the inclusion of
uncertainties. The interconnection structure $P$ is partitioned so that the input-output map from $v'$ to $e$ is expressed as the following linear fractional transformation

$$
e' = F_i(P, K)v' \quad \text{where} \quad F_i(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \quad (A.14)$$

For a $\mathcal{H}_\infty$ optimal control problem, the objective is to find a stabilizing controller $K$, which minimizes $\|F_i(P, K)\|_\infty$.

The $\mathcal{H}_\infty$ optimization problem has been the subject of an enormous amount of research in the past 10 years. New state-space formulas have become available, which make this problem numerically tractable [9, 13]. These algorithms involve a search method and the solution of two Ricatti equations. For control design, a value $\gamma$ is selected and checked as to whether a controller, $K$, can be generated which satisfies $\|F_i(P, K)\|_\infty < \gamma$ and the closed-loop system is internally stable. If either of these tests fails, the $\gamma$ value is increased and control design is reformulated. In the limit as $\gamma \to \infty$, the control law approaches the $\|.\|_2$ optimal solution. Assuming the weighting functions have been selected to normalize the desired $\|.\|_\infty$ to 1, then, $\|F_i(P, k)\|_\infty < 1$ indicates the formulation of a control law $K$ that satisfies the specified criteria. The $\mathcal{H}_\infty$ control design methods are used in the $\mu$-synthesis design methodology.