EXEMPLARY MATHEMATICS TEACHERS:
SUBJECT CONCEPTIONS AND INSTRUCTIONAL PRACTICES

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Education
Department of Curriculum, Teaching & Learning
Ontario Institute for Studies in Education of the University of Toronto

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ABSTRACT

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Theoretical links between conceptions of mathematics and teaching practice have been postulated and research exploring these conjectures has been conducted at the elementary school level and to a limited extent with selected secondary school mathematics topics. Studies have shown that instrumentalist images of mathematics are carried into transmissive modes of instruction, but the few teachers with social constructivist views have troubles bringing their visions to practice. This study involving two exemplary secondary school mathematics teachers, that is teachers who are working to implement mathematics education reform, extends this line of research, exploring: their conceptions of mathematics, teaching practices, and the struggles they experience in bringing subject images to the classroom.

Qualitative methods were used to explore conceptions of mathematics and teaching practice. The teachers' subject images were examined through personal writing on the nature of mathematics, repertory grid technique, and the construction of concept maps, and interviews analysing these products. Participant
observation was employed to gather data on mathematics teaching and lessons were analysed from a sociological and epistemological perspective. In two case studies, using alternating sections presenting subject image themes and narratives of related teaching, links are built between practice and epistemology.

In this study, one teacher with a well developed social constructivist image of mathematics managed to struggle against pupil and administrative opposition and put his personal subject philosophy into practice. The second teacher, with a more mixed subject conception, one that was undergoing transition and reconstruction, had difficulties translating his epistemology into practice and when confronted with opposition turned to traditional transmissive modes of instruction. His emerging constructivist views showed through in activities provided beyond the core of his lessons.

Obstacles that the two teachers faced in attempting to implement their personal philosophies are examined and sources of support are identified. For both teachers, links to university faculty that provided opportunities for mathematical explorations beyond the school curriculum were valuable. The development of the stronger teacher's more complete and consistent image was helped by a professional network that provided opportunities for discussion of philosophical matters.
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Chapter 1: Introduction

Problem Statement

The most commonly used methods for teaching ... mathematics are presentation of information to the class by chalkboard or overhead projector and assignment of individual work (Ontario Ministry of Education, 1991a, 1991b, p. 1)

During April of 1990 the Ontario Ministry of Education conducted Provincial Reviews of two secondary school mathematics courses: Grade 12 (Advanced Level), and Grade 10 (General Level). The above statement, appearing in the public reports for both reviews, summarizes the data collected from questionnaires completed by a randomly selected set of school principals, teachers, and students. A more recent study by the Scarborough Board of Education (Colgan & Harrison, 1997), examining the teaching of Grade 12 mathematics at all academic levels, shows the continuing use of the rather limited range of activities described above. "On any given day in a Grade 12 mathematics classroom the routine is comprised of taking up the homework, teacher-centred delivery of a new concept, students' practice of that concept, and the completion of assigned work including homework" (p. 15).

This picture of mathematics teaching and learning in Ontario high schools contrasts with that presented in the guidelines under which these courses are offered (Ontario Ministry of Education, 1985). Here problem solving is presented as a major theme for the curriculum. The recommended program takes an
experiential approach, with students employing manipulative materials and simulations to explore concepts and discussing their emerging understandings with each other. Mathematics is to be developed through applications and modelling supported by the use of calculators and computers.

Studies finding the rather unimaginative mathematics instruction described in the quotes above are not restricted to the province of Ontario. In fact, "the nature of classroom teaching is quite similar in all countries" (Anderson, 1987, p. 81), and is remarkably stable over time. Fey (1979), summarizing the results of three nationwide American studies of the mid-1970's, states, "the profile of mathematics classes emerging from the survey data is a pattern in which extensive teacher-directed explanation and questioning is followed by student seatwork on pencil-and-paper assignments" (p. 494). The US data from the 1980 Second International Mathematics Study [SIMS] confirmed that "students at the twelfth-grade level spent the major portion of their class time listening to teacher presentations. Doing seatwork and taking tests and quizzes accounted for other large blocks of time" (Crosswhite, 1987, pp. 75-76). Internationally, the teachers surveyed in the twenty-two countries (including Canada) that participated in SIMS reported that "most of their time was used in whole-class instruction" (McLean, Wolfe & Wahlstrom, 1987, p. 7). The student attitudinal data collected in the SIMS survey show the results of the dominant teacher-directed style of instruction. The majority of students saw mathematics as a set of rules rather than a
discipline involving creativity, speculation and conjecture (Miwa, 1987; Rogers, 1990).

More recent international studies have replicated the SIMS findings. From 1981 to 83, the Classroom Environment Study conducted by the International Association for the Evaluation of Educational Achievement [IEA] made repeated observations of teaching in grades 5 and 8 mathematics classes in nine countries (Anderson, 1987). Again, teacher presentations and individual student work consumed the majority of class time. A decade later, in 1991, data from grade 8 classes in 20 countries, collected in an International Assessment of Educational Progress [IAEP] study, showed that "students in many countries regularly spent their instructional time listening to mathematics lessons" and that "another common classroom activity is to require students to work mathematics exercises on their own" (Lapointe, Mead & Askew, 1992, p. 48). The most recent international research, the Third International Mathematics and Science Study [TIMSS], found that, at the Grade 8 level, "the most frequent approaches used across countries involved students working individually with assistance from the teacher, and working as a class with the teacher leading" (Robitaille, Taylor & Orpwood, 1996, p. 5-6).

The United States, through its National Assessment of Educational Progress [NAEP] program, regularly samples mathematics instruction. NAEP surveys from 1973 to 1990 showed that teacher lectures were the predominant mode for high school mathematics lessons (Mullis, 1992) and the 1992 results were summarized with, "reports from students and their teachers show
that no real progress has been made in shifting the instructional atmosphere in the nation's mathematics classrooms to one in which active learning and in depth problem solving are emphasized" (Lindquist, Dossey & Mullis, 1995, p. 49).

That effective teaching and learning, especially in mathematics, is more than simply the transfer of facts and must actively involve the pupil is not a new idea. In the third century B.C., Plato (1953) described a geometry lesson where the interaction between Socrates, the teacher, and a slave boy pupil is significantly greater than that suggested by the Provincial Review summary given at the beginning of this chapter. Recently, professional organizations for mathematics educators have produced documents (National Council of Teachers of Mathematics [NCTM], 1989; Ontario Association for Mathematics Education [OAME]/Ontario Mathematics Coordinators Association [OMCA], 1993) that call on teachers to develop mathematics curricula in which pupils actively construct their own personal mathematical understandings through investigating, conjecturing, testing hypothesis, and the sharing and discussing of ideas.

The curriculum initiatives taken by the professional associations have had an impact on government policies. In the US, a 1992 survey (Blank & Pechman, 1995) showed that 41 states had developed or were in the process of developing new mathematics frameworks modelled after the NCTM (1989) *Curriculum and Evaluation Standards for School Mathematics* (hereafter referred to as *Standards*). In Canada the Atlantic Provinces Education Foundation (1996) has adopted the NCTM's statements as the guiding principles for its new mathematics curriculum. The
Ontario curriculum guide (Ontario Ministry of Education, 1985) for secondary school mathematics in many ways foreshadowed the present mathematics education reform movement, presenting in the introductory pages a vision of teaching and learning similar to that developed in the NCTM Standards.

Although the leaders of the teaching profession are calling for change in mathematics curricula and pedagogy and government policies are reflecting this thinking, "many high school teachers do not endorse the instructional strategies recommended" (Weiss, 1994, p. 5). Only a minority of mathematics classes are engaged in extended problem solving activities and mathematical exploration, conjecturing and communication. Why is progress towards reformed mathematics programs so slow? Why has implementation of the curriculum and teaching methodologies advocated by both government and the professional leadership been so limited?

Thomas Romberg (1992a) attributes the gulf between the program advocated by the NCTM Commission on Standards for School Mathematics, of which he was the chair, and the predominant school practices, to differences in subject conceptions and notes that "the single most compelling issue in improving school mathematics is to change the epistemology of mathematics in schools, the sense on the part of teachers and students of what the mathematical enterprise is all about" (p. 433). In this view, teachers' pedagogical choices are seen to flow from their personal conceptions of mathematics. Ernest (1989a, 1989b, 1991) has developed a theoretical scheme linking personally held philosophies of mathematics to instructional practices. Teaching
styles that encourage students to personally and collaboratively construct mathematical concepts are linked to images of the discipline as a living, creative human endeavour, while teachers who see mathematics as a fixed set of rules and procedures are expected to adopt a predominantly transmissive mode of instruction. The slow pace of change in mathematics teaching and learning is linked, by those leading the present mathematics reform movement, to "the prevailing view of educators ... that mathematics consists of a set of procedures and that teaching means telling students how to perform those procedures" (Battista, 1994, p. 463). Thus the problem of teacher reluctance to embrace the mathematics education reform proposals is seen to originate in a conflict between disciplinary images.

While classroom implementation of the mathematics education reform proposals has generally been slow, some teachers are employing curricula and practices that capture the spirit of the NCTM (1989) Standards. The picture of mathematics instruction produced by the Ontario Provincial Reviews and quoted above is not universal. There exist classrooms where one can observe in action the teaching and learning processes described in the Ontario Ministry of Education (1985) guidelines. Are these exemplary teachers' instructional practices, as the theory suggests, expressions of conceptions of mathematics that are different from the predominant views?

In summary, a problem concerning curriculum implementation, that is, the slow paced and limited adoption of the process policies articulated by the Ontario Ministry of Education (1985) course guidelines and more recently expanded upon by professional
associations for mathematics education (NCTM, 1989; OAME/OMCA, 1993), has been recast as a clash between differing conceptions of the nature of mathematics. Is this in fact the case? Looking at the general teacher reluctance to embrace the mathematics reform proposals as an opportunity to learn and noting that some teachers have adopted many of the instructional methods advocated by the curriculum guideline and the current reform movement, we can turn this problem into a set of questions.

What are the conceptions of mathematics held by teachers who are attempting to implement the reform proposals?

How are these teachers' images of their subject connected to their classroom practices?

and

What are the struggles involved in these teachers' efforts to translate subject images into classroom practice?

This study addresses the above questions, through an in-depth examination of the subject conceptions of two exemplary mathematics teachers, that is teachers who are working to implement mathematics education reform, detailed observation of their instructional practices, and exploration for connections between subject visions and pedagogy.

**Overview of the Study**

Different philosophies of mathematics have widely differing outcomes in terms of educational practice. However the link is not straightforward. (Ernest, 1991, p. 111)

The research for this thesis was conducted as separate case studies involving two purposely selected teachers, with
invitations to participate based upon my personal observations that they were in fact attempting to incorporate aspects of the mathematics education reform proposals in their classroom activities. The case study approach permitted the extended, close contact with the participants necessary for an extensive examination of their teaching and conceptions of mathematics. Spending multiple full working days with each teacher allowed me to experience and record the contexts of their professional lives.

Collaborating teachers participated in three activities to help surface their views of the nature of mathematics: the writing of a brief personal description of the nature of mathematics and the discipline's historical development, the building of a repertory grid (Beail, 1985) comparing mathematics to other school subjects, and the drawing of a concept map (Novak & Gowin, 1984) presenting their image of the structure of mathematical knowledge. The products of these exercises were used, with each teacher separately, as the foci of subsequent interviews (Hewson & Hewson, 1989; Seidman, 1991) to collaboratively build a picture of their personal philosophy of mathematics. In addition, interviews were conducted to ascertain the teachers' motivations and intentions for observed teaching activities and to attempt to relate planning choices to the emerging pictures of their conceptions of mathematics.

Over the span of one school semester, a period of approximately four months, a minimum of 20 visits were made to each teacher's classroom. Observations over short blocks of consecutive days permitted the study to record the development of
individual concepts. Visits were spread out and arranged to allow observations of lessons addressing different curriculum topics developed at a variety of grade levels.

A sociological and epistemological view (Koehler & Grouws, 1992) of mathematics teaching, one that examines and records intellectual processes as well as overt behaviours, was employed for the collection and analysis of data concerning the participating teachers' classroom practices. It was not the intention of the study to measure quantitatively the frequency of various teaching methods, but to build profiles of the practices of two exemplary teachers. As such, the observed teaching is reported in the form of narratives of illustrative classroom episodes.

The extended time spent with each participant, a minimum of 8 full teaching days, afforded many opportunities for observation of their professional environment and their interactions with fellow teachers, recording of resource materials employed in lesson planning and the products generated by students, and informal discussions concerning the nature of mathematics and its teaching and learning. All these were recorded in field notes.

Although the two teachers have in common the employment of classroom practices that incorporate some of the themes of the ongoing mathematics education reform, they are distinct, unique individuals, with differing personalities, life histories, and working contexts. To preserve this uniqueness, the data for each case was analysed separately with comparisons drawn only in the concluding chapter of this study. For each participant the data concerning visions of mathematics: writing on the nature of
mathematics, school subjects repertory grid, concept map for the discipline, and transcripts of related interviews; were examined for recurring themes. With the dominant components of the subject conceptions identified, analysis efforts were focussed on the classroom observation record: field notes, audio tapes of lessons, materials employed, and student products. For each teacher, classroom practice was searched for incidents where mathematics was portrayed or presented in a style that captured one or more features of the participant's personal mathematical epistemology. In the case studies that follow, the analysed and reordered data is presented in sections alternating between mathematical image themes and classroom incidents putting these into practice.

As the full case studies that follow will show, the strengths and stabilities of the two teachers' subject images varied and in turn their successes in carrying their personal philosophy of mathematics into the classroom differed. During the school semester of the study, for both teachers, the use of non-traditional instruction was met with opposition from pupils, parents and school administrations. The individual responses to these challenges and the abilities to persevere with alternative teaching approaches was most revealing of the unity and strength of subject visions.

In the concluding chapter the struggles involved in these teachers' efforts to translate their visions of mathematics into classroom action are examined. The Perry (1981) theory of epistemological development is employed to analyse the two teachers' personal philosophies of mathematics and the stances
taken in the face of conflict. Factors in teachers' professional environments that appear to play a role in the development of a strong mathematical epistemology are identified.
Chapter 2: Related Literature:
Beliefs, Subject Conceptions, and Teaching Practice

This study explores the conceptions of mathematics held by exemplary secondary school mathematics teachers, their instructional practices, and the links between subject images and teaching. The background literature related to such a study is wide ranging and follows a number of themes: beliefs, teachers' personal knowledge, teacher thinking, and subject images, in particular as these relate to mathematics; teaching practices, and work examining relations between the foregoing. The following literature survey examines research related to these themes and is organized into two main sections addressing subject conceptions and teaching practice.

A teacher's subject conception resides in their belief system and comprises part of their personal knowledge. The translation of subject beliefs into instructional action is initiated by teacher thinking. Thus the literature review begins with a brief examination of research dealing with beliefs, personal knowledge and thinking in general. This is followed by a section addressing philosophies of mathematics and contrasting the various conceptions of the discipline found in the literature.

Those studies most closely related to the questions addressed by this thesis are surveyed in the third and fourth sections of the review. Research linking teacher conceptions of academic disciplines to instructional practices have been largely
conducted with elementary school teachers or have involved subjects other than mathematics. When secondary school mathematics has been the object of study, the research has focussed on very specific topics within the overall curriculum. Thus there remains a need for further research in this field.

In addition, the studies reported in the literature have been able to address only one-half of the question concerning links between teaching practice and images of mathematics. The mathematics teachers participating in the small number of studies at the secondary school level have been found to employ traditional transmissive modes of instruction. This style of teaching has been linked to an instrumentalist conception of mathematics; an image of the discipline as a collection of fixed rules and procedures. Questions concerning the images of mathematics held by teachers employing non-traditional reform oriented instruction are still open.

The second unit of the chapter, addressing teaching practice, begins with an examination of the professional literature from the NCTM and OAME and describes the instruction advocated by the mathematics education reform movement. Adopting these reforms means a shift from teacher-centred to student-centred lessons and this in turn necessitates a change in methods for observational studies. Literature addressing the observation of mathematics instruction is surveyed and an emerging sociological and epistemological perspective, appropriate for the study of non-traditional classrooms, is identified.
Conceptions of the Nature of Mathematics

The issue, then, is not, What is the best way to teach? but, What is mathematics really all about? (Hersh, 1979, p. 33)

Thompson (1992) defines a teacher's conceptions of the nature of mathematics as their "conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline", which, "constitute the rudiments of a philosophy of mathematics" (p. 132). As a component of teachers' belief systems, conceptions of subject are closely allied with knowledge and thought. Distinctions between teacher beliefs, knowledge and thinking are not clear in the literature with the three concepts being employed interchangeably under a variety of labels (Clandinin & Connelly, 1987). In this, authors are following the path set by Dewey (1910) who virtually equated the three ideas in noting that in one sense, "thought denotes belief resting upon some basis, that is, real or supposed knowledge" (p. 4). A precise and universally agreed upon separation between the three concepts is probably not possible, but for the purposes of this study there is a need to set out the relationships between the three ideas and identify the focus of the research. This is the task of the following section.

Beliefs, Personal Knowledge, and Teacher Thinking

"When I use a word," Humpty Dumpty said in a rather scornful tone, "it means just what I chose it to mean - neither more nor less."
"The question is," said Alice, "whether you can make words mean different things."
"The question is," said Humpty Dumpty, "which is to be master - that's all." (Carroll, 1954, p. 185)

Beliefs are personal principles, constructed from experience, that an individual employs, often unconsciously, to interpret new experiences and information and to guide action (Pajares, 1992). As personally held mental constructs, beliefs, unlike knowledge, do not require community consensus or agreement to establish their validity (Nespor, 1987). Knowledge is taken to be built up through intellectual activity: experimentation, debate and reasoning, and is stored in the form of propositions that are open to further evaluation and change. Beliefs on the other hand are not developed through rational thought, but are mental summaries of significant past episodes. An individual's beliefs may fail to exhibit logical consistency, conflicting with each other and with knowledge or the observed world. The power of beliefs to filter new information and colour one's comprehension of events means that once established they are not likely to change (Nespor, 1987) even in the presence of contradictory evidence. Despite their looser structure and potential inconsistencies, beliefs are much more influential than knowledge in defining individual behaviour.

The many dimensions and experiences of an individual teacher's life generate a vast number of beliefs which are linked in a belief system, a network of belief clusters each focused on a particular situation or facet of life. "When researchers speak of teachers' beliefs, however, they seldom refer to the teachers' broader belief system of which educational beliefs are but a part, but to teachers' educational beliefs" (Pajares, 1992, p. 316). Pajares goes on to note that the educational belief
cluster, while only part of a teacher's broader belief system, is still too large and complex for research purposes. A further sub-division into clusters of "educational beliefs about" is required -

beliefs about confidence to affect students' performance (teacher efficacy), about the nature of knowledge (epistemological beliefs), about causes of teachers' or students' performance (attribution, locus of control, motivation, writing apprehension, math anxiety), about perceptions of self and feelings of self-worth (self-concept, self-esteem), about confidence to perform specific tasks (self-efficacy) (Pajares, 1992, p. 316).

Epistemological beliefs or beliefs about the nature of knowledge, in particular mathematical knowledge, are the focus of this study. Epistemological beliefs play a major role in the organization and interpretation of one's knowledge (Abelson, 1979; Kitchener, 1986). Thus it is possible for two teachers, possessing similar mathematical knowledge, to, in expressing different epistemological beliefs, present distinctly different interpretations of content through their teaching (Ernest, 1989a). Models of teachers' knowledge, such as those of Grossman, Wilson and Shulman (1989) for school subjects in general or Fennema and Franke (1992) for mathematics in particular, indicate this guiding role of disciplinary image by placing beliefs about the nature of subject above content knowledge.

Thus as a first stage, distinctions may be made between beliefs and knowledge by making reference to the evidence bases upon which they rest and their relative strengths. Knowledge must meet certain cannons of evidence, that is there are publicly recognized criteria for the acceptance or rejection of knowledge
claims, while beliefs may exist without supporting, or in fact in the presence of contradictory, evidence. Beliefs often take precedence over knowledge, shaping the interpretation of presently held knowledge and selectively admitting or rejecting new knowledge claims. The introduction of the term "personal knowledge" into the research literature has complicated the belief-knowledge picture and made separation of the concepts more difficult. Connelly and Clandinin, in describing personal knowledge as, "that body of convictions and meanings, conscious or unconscious, which have arisen from experience" (1984, p. 137), significantly blur the knowledge-belief boundary.

In a similar manner Elbaz (1981) appears to extend the use of the term "knowledge" to include concepts that might by others be labelled as beliefs. Image, Elbaz's third level in the structure of practical knowledge, like belief, guides teacher action in an intuitive rather than logically reasoned manner. Subject matter image is a composite of the teacher's visions, metaphors, and values concerning the subject content to be taught.

In this study the terms employed by those researching personal practical knowledge: conviction, view, vision, and image are used interchangeably with Thompson's (1992) term, conception, as labels for the teacher's epistemological beliefs about mathematics. In each case these terms are taken to refer to beliefs rather than knowledge. That is, a teacher's vision of subject is an intuitive sense or belief built unconsciously through experience rather than a rationally constructed picture developed through the purposeful acquisition of knowledge in
formal or informal study. Of course, a teacher may have a significant knowledge base for their epistemological beliefs and, in fact, Shulman (1986), presents such knowledge, knowledge of the substantive and syntactic structures of subject, as necessary for effective teaching.

Substantive structure refers to the organization of a discipline's basic concepts and syntactic structure is the set of principles and processes by which new knowledge is generated and proven within the subject (Schwab, 1964). For Shulman (1986) and Grossman, Wilson and Shulman (1989) substantive and syntactic knowledge is a necessary component of a teacher's content understandings. Substantive and syntactic knowledge, if present, are likely to contribute to a teacher's conception of their discipline, but such knowledge would not be necessary for the development of an image of what a subject is about and how its "facts" are developed. Schooling, professional preparation, teaching and general life, all provide considerable subject experience for mathematics teachers. Beliefs concerning the nature of mathematics develop from this experience without opportunities to explore more formally the subject's substantive or syntactic structures.

Perry (1981) provides a theory that describes an individual's progress in the development of beliefs concerning knowledge; their progress through epistemological positions. In the Perry scheme one ideally moves through four stages of development: Dualism, Multiplicity, Relativism, and Commitment; but may, in fact, become fixed any one level. These stages
correspond to attitudes or beliefs about knowledge that may be briefly described as follows:

**Dualism** - All knowledge claims are either true or false and authorities in any discipline can determine validity.

**Multiplicity** - Diversity of opinion is legitimate and any one view is as valid as another.

**Relativism** - Diversity of opinion is legitimate, but some positions have more validity than others. There is a need to employ evidence and reasoning in the evaluation of positions.

**Commitment** - The adopting of a position from among those seen to be potentially valid.

The stages of the Perry scheme and the transitions between them provide a framework within which teachers' images of subject may be analysed and this approach will be taken in the concluding chapter of this study.

Teaching may be viewed as transformation and reflection; transformation of knowledge and beliefs into pedagogical acts followed by reflection on the outcomes of instruction. Thinking is the process through which these complementary processes proceed.

As we have come to view teaching, it begins with an act of reason, continues with a process of reasoning, culminates in performances of imparting, eliciting, involving, or enticing, and is then thought about some more until the process can begin again. ... We ... emphasize teaching as comprehension and reasoning, as transformation and reflection. ... Sound reasoning requires both a process of thinking about what they are doing and an adequate base of facts, principles, and experiences from which to reason. (Shulman, 1987a, p. 13)
Thus thinking is the cognitive activity through which the mental structures of belief and knowledge are revealed in action and through which reflection on action in turn may modify cognitive and belief structures. Thought also links knowledge and belief. Thinking is involved when beliefs influence the interpretation of knowledge and reflective thought on knowledge can on occasion lead to modified beliefs.

As with belief and knowledge, some research usage of the concept of thinking expands its meaning and generates possible confusion. For Clandinin and Connelly (1987) teacher thought includes both prior experience stored as knowledge and belief and teacher action, along with the mental processes linking these end points. In this study to avoid the overlap in terms, teacher thinking is used in the narrower sense of the mental activity that links experience, beliefs, knowledge and practice.

Images of Mathematics

Philosophy [Nature] is written in that great book which ever lies before our eyes - I mean the universe - but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth. (Galileo, 1610)

Insofar as the propositions of mathematics give an account of reality they are not certain; and insofar as they are certain they do not describe reality. ... But it is, on the other hand, certain that mathematics in general and geometry in particular owe their existence to our need to learn something about the properties of real objects. (Einstein, 1921)
Research in philosophy of mathematics can be roughly divided into two streams (Van Bendegem, 1993). One branch, the study of the foundations of mathematics, has become formalized and absorbed into the discipline to become part of the mainstream of mathematical research (Rav, 1993). Being imbedded in the discipline, foundational studies can not fully address the question of, What is mathematics? A second, more recently initiated but growing (Rav, 1993) research direction is the analysis of mathematical practice. This field of enquiry is concerned with questions such as: What are the origins of mathematics?, What does it mean to do mathematics?, When and why are mathematical statements accepted as true?, Is it possible to gather evidence of the correctness of a mathematical statement?, and How is it possible for flaws and errors to arise in mathematics? Personal and collective answers to these questions have significance for mathematics education and it is this second orientation to the philosophy of mathematics that is of concern in this study. "If you wish to study problems related to the educational aspects of mathematics ... you will obviously need a theory or at least a model of what mathematical practice is about" (Van Bendegem, 1993, p. 22).

One central question concerning the nature of mathematics and the discipline's development can be simply stated as, Do humans "discover" or "construct" mathematics? Do we live in a world that is governed by fixed mathematical rules which, over the centuries, we have discovered and recorded as mathematical theorems, or is mathematics a human construct that we project
onto our world whenever we find patterns of events that appear to support our creations?

Platonism takes the former view. For Platonists, mathematics is a body of facts existing independent of human knowledge. These facts, as part of the laws of nature, have held from the beginning of time and will not change in the future. For any mathematical question the universe's basic mathematical rules determine a definite answer. Knowledge in the discipline develops as the natural mathematical laws are identified and open puzzles persist only because we have not yet discovered the appropriate procedures for solving them. The Platonist image of mathematics as a static and unified body of truths is an example of what Ernest (1991) labels the "absolutist" view of mathematics, the belief that mathematical knowledge is certain and without flaw.

Deductive methods make it possible to maintain an absolutist position while mathematical knowledge expands with the discovery of new facts that lack direct reference to the physical world. Initial postulates, assumed to be true due to their roots in observations of the world, form the bases for mathematical systems and strict application of the laws of logic ensures that validity is preserved in any statements proven from these axioms. For centuries the geometry of Euclid, as expressed in the Elements (300 B.C./1956), was the model for this program designed to ensure mathematical truth. Geometry was also the source of the absolutist view's first major crisis when in the mid 1800's Lobachevsky and Riemann produced examples of geometries that revealed contradictions in the Euclidean axioms (Davis & Hersh,
1981). Platonists now needed to find a more secure footing for mathematics if the absolutist view of the discipline was to be preserved. Arithmetic as formalized in set theory and logic appeared to have potential as a solid starting point for mathematics, but here again contradictions were found and the search for concrete foundations lost favour. The image of mathematics as the embodiment of nature's truths holds strong attraction for mathematicians and thus Platonist views persist despite the failures to axiomatize the discipline.

The Realist [Platonist] position is probably the one which most mathematicians would prefer to take. It is not until he becomes aware of the difficulties in set theory that he would even begin to question it. If these difficulties particularly upset him, he will rush to the shelter of Formalism. (Cohen, 1971, p. 11)

Formalists, admitting that mathematics might not truly correspond to the experienced world, seek to preserve the consistency of the discipline by denying all claims of representation. The "theory is not properly a theory at all as we formerly understood the term, but a system of meaningless objects like the moves in chess" (Kleene, 1950, p. 62). With statements encoded in precisely defined symbols and explicit proof procedures, formal systems appear to allow mathematics to claim to be the model for truth and certitude, but even with formalism the absolutist position is not secure. Gödel, in 1930, established that formal systems lack completeness in that there still exist statements, the truth of which can not be determined (Penrose, 1989).

Formalism may describe research in some branches of mathematics, but in doing so it separates the discipline as an academic activity from mathematics in practice as a tool in
commerce and industry. For users of mathematics and some teachers of the subject formalism translates into an instrumentalist (Ernest, 1989b) view of the subject. Instrumentalists, while not directly involved in the logical deduction of new mathematical knowledge, have faith in the formalists' program and are willing to employ the procedures developed to accomplish their everyday mathematical tasks. Thus mathematics becomes a collection of often unrelated facts, rules and skills to be used in the pursuit of solutions to problems external to the subject.

Despite the problems of Platonism and formalism in their attempts to provide firm foundations for mathematics, absolutist conceptions of the subject still find considerable favour with both mathematicians (Mura, 1993) and the general public. Witness the common use of the arithmetic statement, "1+1=2", as the prototypical example of absolute truth. With the contradictions existing on the periphery of the discipline, many everyday users of mathematics are not aware of or concerned about the potential flaws in the subject. Their experience of mathematics, as a set of rules that if employed accurately produce single correct answers, supports instrumentalist and Platonists personal philosophies. "For many educated persons mathematics is a discipline characterized by accurate results and infallible procedures, whose basic elements are arithmetic operations, algebraic procedures, and geometric terms and theorems." (Thompson, 1992, August, p. 2).

Mura (1993) in a survey of Canadian university based research mathematicians identified a tendency towards the
formalist emphasis on abstraction, logic and rigour. A second study of the views held by university teachers of mathematics education (Mura, 1995) showed only a slight reduction in the dominance of the formalist view, with mathematics educators also giving inductive processes and the study of patterns a place in mathematics. While Platonist and instrumentalist views were not prevalent among university faculty, Roulet (1995) found that such images were common among university students preparing to teach secondary school mathematics.

With a growing acceptance that "the concept of a universally accepted, infallible body of reasoning - the majestic mathematics of 1800 and the pride of man - is a grand illusion" (Kline, 1980, p. 6) alternative conceptions of the nature of mathematics are taking form. In contrast to the absolutist view of mathematics, Ernest (1991) proposes a fallibilist position, one that accepts that the statements of mathematics are potentially flawed and must be held open to revision and correction. But if the development of mathematics is not the discovery of universal truths, how is knowledge in the discipline generated and warranted? To address this issue a social constructivist (Bishop, 1985; Ernest, 1991, 1992) or problem-solving (Ernest, 1989b; Lerman, 1983) philosophy of mathematics is proposed. This conception of mathematics begins with "the assumption that the concepts, structures, methods, results and rules that make up mathematics are the inventions of humankind" (Ernest, 1992, p. 93). Social constructivism is concerned with the nature of mathematical knowledge and the process of its development both in individuals and societies. In locating mathematics in a larger
social context this philosophical approach avoids the separation between the discipline of the research mathematician and mathematics as employed in the community.

Mathematics is seen as an extension of natural language and, as a language, is acquired and developed through social interaction. Individuals, through observations of patterns in the physical world and reflection on previously constructed concepts, develop new mathematical ideas. This individual or subjective knowledge is communicated to a wider audience where it undergoes critique, debate and modification. If, through this social interaction, the new concept is found to fit with previously accepted mathematical knowledge it becomes warranted by the social group and attains the status of objective knowledge. At any time the truth of a statement is a matter of its fit with other accepted mathematical concepts and human experience with the world.

Mathematics is a branch of knowledge which is indissolubly connected with other knowledge, through the web of language. Language functions by facilitating the formations of theories about social situations and physical reality. Dialogue with other persons and interactions with the physical world play a key role in refining these theories, which consequently are continually being revised to improve "fit". As a part of the web of language, mathematics thus maintains contact with the theories describing social and physical reality. (Ernest, 1992, p. 94)

Most often new concepts originate from analysis of previously accepted theories and as such, if accepted, result in incremental additions to the body of mathematical knowledge. More dramatically, it is possible for mathematicians, by linking previously separate theories or by drawing from knowledge beyond the discipline, to challenge existing truths and structures. Old
ideas may be found to be flawed and require modification or be abandoned and mathematical knowledge may undergo restructuring. This image of the construction and refinement of mathematical knowledge builds on Lakatos' (1976) earlier quasi-empirical mechanism for the development of mathematical ideas and Popper's (1959) description of the growth of scientific knowledge.

Mathematics teachers, many of whom may not have formally studied philosophy of mathematics, might not describe their personal conceptions of the nature of the subject in terms of the Platonist, formalist (instrumentalist), or social constructivist (problem-solving) positions, but such categories may be employed in the analysis of teachers' less formally defined images of the subject. Through their experiences with mathematics, teachers construct images of the discipline, and this rudimentary philosophy is important for "controversies about high-school teaching cannot be resolved without confronting problems about the nature of mathematics" (Hersh, 1979, p. 33). In particular the social constructivist position in presenting "a view of mathematics as a way of knowing and as a social construct can powerfully affect the aims, content, teaching approaches, implicit values, and assessment of the mathematics curriculum" (Ernest, 1992, p. 89). The validity of the foregoing statement is evident in the social constructivist themes running through the recent mathematics education reform proposals such as those presented in the NCTM's (1989) Curriculum and Evaluation Standards for School Mathematics. In fact, Thomas Romberg (1992a), the chair of the Commission on Standards for School Mathematics, cites Ernest's (1991) work in arguing for the
adoption of the school mathematics program projected by the standards. The instructional practices suggested by the mathematics education reform movement and their connection to the social constructivist philosophy will be explored in later sections focussing on new visions of teaching.

Conceptions of Subject and Teaching Practice: Mathematics

In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, p. 204)

There is considerable agreement that beliefs influence action (Abelson, 1979) and in particular that teachers' educational beliefs (Pajares, 1992) guide teaching practice (Campbell, 1985; Clark, 1988; Crocker, 1983; Munby, 1982; Nespor, 1987). More specifically still, there is evidence that epistemological beliefs or subject images affect teachers' interpretations of content knowledge (Kitchener, 1986) and influence their instructional approaches (Elbaz, 1981; Pope & Scott, 1984).

Ernest (1989b, 1991) describes anticipated instructional styles associated with the three philosophical positions, instrumentalism, Platonism, and social constructivism. Teachers holding either of the two absolutist views of mathematics are expected to adopt teacher-centred transmissive modes of instruction but the content focus of lessons would vary with subject image. For instrumentalists pupil mastery of mathematical skills is the key objective and thus good instruction entails clear presentation of the steps in any
procedure followed by extensive drill to ensure memorization.

Platonists, taking mathematics to have an underlying structure, see a need to demonstrate the logical nature of the subject and explain the reasons for rules and procedures. In both cases mathematical authority in the classroom resides with the teacher and textbook who together determine the truth or correctness of any student answer or solution method. Teachers holding problem-solving or social constructivist views of mathematics would see their classroom role as a facilitator, providing stimulating problems for investigation and building an environment in which pupils may discuss their emerging understandings. Research studies have explored these potential links between epistemological beliefs and teaching methods. While constraints such as: fixed curricula, time pressures, external examinations, and school or departmental rules and standards, provide competing incentives for the guidance of practice, there is evidence that teachers' instructional practices do reflect their visions of subject.

Thompson (1984) studied the beliefs and teaching of three junior high school (grades 7 and 8) mathematics teachers. While Ernest's labels for the various conceptions of mathematics were not employed by Thompson, descriptions of the teachers' professed views allow her subjects' positions to be identified with the three positions: Platonism, instrumentalism, and a mild form of problem-solving. Jeanne (Thompson's assigned pseudonym) held that mathematics, originating from ideas present in the physical world, is fixed and predetermined. Mathematics is certain and consistent, with a logical and coherent underlying structure.
Thompson's report of Jeanne's teaching shows the influence of her absolutist philosophy.

Although Jeanne conducted class in a question-and-answer fashion, there were no observable signs that she was making an effort to encourage discussions among the students or between them and herself. The students' participation typically was limited to elicit short, simple answers, and she had a tendency to disregard the students' suggestions and not to follow through with their ideas. (p. 112)

Thus we see the transmissive mode of instruction expected from a teacher holding Platonist views, but there is little evidence of an effort to build student understanding as predicted by Ernest.

Lynn, the teacher with an instrumentalist conception in Thompson's study, viewed mathematics as an exact discipline, producing procedures that guarantee correct answers. Her comparison of mathematical activity to "mental calisthenics" (p. 116) echoes the formalist position. As expected, Lynn's teaching followed a highly transmissive mode aimed exclusively at skill development.

She conducted class in a way that allowed as little interaction as possible. Her explanations were brief and aimed at demonstrating the procedures that students were to use in working out the day's assignment. The bulk of the remaining class time was given to independent seat work during which the students practiced the procedures taught. (p. 117)

For Thompson's third research participant (Kay) teaching practice follows from belief, but her conception of mathematics does not fit exclusively into any of the identified philosophical categories. Kay pictures the subject as continuously growing and changing in response to practical problem-solving needs, but she sees this expansion as proceeding by rigorous axiomatic methods and thus producing propositions of guaranteed validity. Kay's teaching, in which she frequently encourages students to guess,
conjecture and reason out their own solutions to problems was compatible with a problem-solving image of mathematics, while her emphasis on formal geometric proofs reflects her formalist conceptions of the discipline.

McGalliard (1983), employing the Perry (1981) scheme of ethical and intellectual development to classify conceptions of geometry, found that the four senior high school mathematics teachers in his study held dualistic images of the subject. This absolutist view, that every question has a single correct answer that can be defended by reference to authorities, was consistent with the observed instructional practice which focused on the preparation of pupils for the next mathematics course by careful compliance with the syllabus and the teaching of rules without explanations. In emphasizing the taking of notes in class and the memorization of answers, the teachers communicated their beliefs in external authority as the source of mathematical truth.

The Perry (1981) scheme was also used by Kesler (1985) (as reported in Thompson, 1992) to analyse the conceptions of mathematics held by senior secondary school teachers. Here the two teachers with dualistic views were observed to teach in a transmissive authoritarian style consistent with their epistemological beliefs. On the other hand two participants with multiplistic views, acknowledging a variety of approaches and answers to questions, did not translate their subject images into any one consistent teaching method, using both inquiry and authoritarian modes of instruction.
Studies investigating links between teaching practices and instructors' conceptions of the nature of mathematics have been conducted in programs both following and preceding junior and senior high school. Ferrell (1995) looked at instruction in a community college level trigonometry course. Despite the college leadership's promotion of teaching that actively involved students, all three instructors in the study rejected the role of facilitator and presented themselves as dispensers of knowledge. This transmissive style of teaching was compatible with their instrumentalist images of mathematics as a body of useful facts and procedures. Raymond (1993) found similar agreement between conceptions of mathematics and instructional styles with six beginning elementary school teachers. Where practice was not entirely consistent with beliefs, teachers possessing problem-solving images explained deviations from their desired approaches as accommodations to time constraints and lack of resources. In a further analysis of her data, Raymond (1997) confirmed the strong influence of beliefs about the nature of mathematics on teaching practice. The six study participants expressed non-traditional images of learning and teaching mathematics, ideas that they had met in their recent teacher education program, but the practices implied by these views did not appear in their lessons. Their traditional absolutist conceptions of the discipline dominated thinking and encouraged a teacher-centred transmissive style of instruction.

In research with close parallels to the study conducted for this thesis, Philipp, Flores, Sowder, and Schappelle (1994) examined the conceptions of mathematics and its teaching and
learning, classroom practices, and the mathematical preparation of four teachers working in grades 3 to 7. As in this present inquiry, these four individuals were invited to participate in the study due to their reputations as exemplary teachers. In previous contacts with these teachers, those conducting the research had observed classroom practices that reflected ideas expressed in the Standards, the guideline for reform issued by the National Council of Teachers of Mathematics (1989). These teachers did not appear to have fully developed social constructivist conceptions of mathematics but all rejected an image of the subject as a collection of disconnected algorithms. Mathematics as language and its development out of problem solving were central to their visions of the discipline. Classroom activities that regularly included explorations using manipulatives, student discussions demonstrating, explaining and justifying answers, and journal writing, reflected the teachers' problem-solving conceptions of mathematics. The researchers report that they did not observe any extensive teacher explanations of procedures or the use of exercise sets for drill.

The success of these teachers in putting their problem-solving images of mathematics into practice could be partially attributed to the professional support they had experienced in graduate level courses, national research conferences, and participation in local and state-level curriculum and assessment reform projects. Here the four teachers possessed images of mathematics compatible with teaching reform proposals and were heavily involved in collaborative efforts for educational change. In contrast to this situation,
four extensive case studies (Heaton, 1992; Prawat, 1992; Putnam, 1992; Remillard, 1992) examined the results when fifth-grade teachers with instrumentalist views of mathematics were expected to implement a reformed mathematics program with only a new textbook for guidance.

The four participants in these studies, selected to be representative of the professional population, were found to have rather restricted images of mathematics.

The teachers in our cases believe that the computational algorithms that pervade the traditional elementary school curriculum constitute the core of mathematics. The teachers have differing views on what it means to understand those algorithms and how important that understanding is, but it is the algorithms of arithmetic that define their mathematics. (Putnam, Heaton, Prawat & Remillard, 1992, p. 223)

For these teachers mathematics is a collection of disjoint, precisely defined tools and techniques that prove useful, once mastered, in solving questions arising in daily life. When confronted with new textbooks giving increased emphasis to: estimation processes, the use of manipulatives for student exploration of concepts, and rich problem situations designed to encourage student reflection and discussion, these teachers made major adjustments to the intended program to maintain teaching practices compatible with their epistemological beliefs. Textbook units on problem solving and activities requiring the use of calculators were skipped while supplementary drills for the practice of computational skills were added. Even when textbook units were used for lessons, the teachers' interpretations of the content and instructional emphases portrayed their images of mathematics rather than that intended by the new program.
In the textbook lessons that she did follow fairly closely, Valerie highlighted procedural aspects of content and downplayed opportunities for students to reflect on and discuss mathematical ideas. We saw this procedural emphasis in the textbook-centred lesson on averages. Valerie omitted a number of questions in the student text that required reflection on the adequacy of averages for various purposes and the kinds of information lost with any average (e.g., change over time). Instead, she spent the lesson time carrying out the computational steps for getting an average. (Putnam, 1992, p. 176)

Dorgan (1994) noted similar interactions between teachers' conceptions of mathematics and the implementation of a revised fifth and sixth grade mathematics curriculum designed to reflect the NCTM Standards (1989). Teachers with instrumentalist and Platonist views maintained their traditional teaching styles with an emphasis on basic skills and computation. Lacking knowledge of how to enact a Standards oriented program on a daily basis, the teacher with an emerging problem solving conception of mathematics struggled with change and made little progress towards the desired instructional practice.

Ernest's (1989b) conceptual model of the relationships between a teacher's beliefs about the nature of mathematics and his or her instructional practice acknowledges the "constraints and opportunities provided by the social context of teaching" (p. 252). From the studies cited it would appear that the long standing, dominant tradition of teacher-directed transmissive instruction in school mathematics makes it easier for those holding absolutist conceptions to translate their subject images into teaching practice. Teachers possessing instrumentalist or Platonist conceptions of the subject continue to employ instructional styles compatible with these beliefs even in the presence of official curriculum reform programs based upon
alternative problem solving images. Teachers with social constructivist views of mathematics must employ newer and less popular (Black & Atkin, 1996) styles of teaching if they are to put their epistemological beliefs into practice. In the face of time pressures that discourage the provision of opportunities for student discussion of new approaches to the subject and meeting opposition from parents, whose only experience is with traditional mathematics programs, these teachers compromise and employ a mix of traditional teacher-centred lessons and activities promoted by the mathematics education reform movement.

Not surprisingly the connections of beliefs to intended or preferred teaching practices are stronger or more consistent than those to actual observed teaching. Teachers and teacher candidates can imagine employing less popular instructional methods free of the constraints provided by the general school milieu. Day (1993), working with preservice secondary mathematics teachers, found that three students who saw mathematics as a growing, flexible and changing discipline with problem solving as a core activity stressed the importance of their future pupils' active involvement in learning and favoured the use of guided-discovery activities and student communication. Lerman (1990) and Roulet (1995) have also connected student-teachers' beliefs about mathematics and preferred teaching practices.

Through the use of a questionnaire, Lerman (1990) identified two pairs of mathematics teacher candidates that represented the extremes of the absolutist and fallibilist perspectives. After the presentation, via video recording, of a short mathematics
lesson, interviews were conducted to elicit these four students' assessments of the teaching observed. The observations "that the two student teachers who were the most 'absolutist' felt that the teacher in the extract was not directing the students enough and was too open, whereas the most 'fallibilist' thought she was not open enough, and was too directed" (p. 59) shows the link between views of mathematics and orientations to teaching practice.

In the initial days of a preservice mathematics methods course Roulet (1995) had students write position papers setting out their personal views of the nature of mathematics and their descriptions of "best" mathematics lessons (Roulet, 1995). Analysis of the student writing showed a predominant "toolkit" view of mathematics.

For two-thirds of the class "mathematics in its simplest form is only a set of rules", or "methodologies of how to work with numbers and symbols" which "demands from you patience and logical thinking to follow step by step procedures". These students find comfort in a system that "is rigorously precise" with "no grey areas or exceptions" and where one can "get the right answer just by following rules and methods". (p. 133)

This instrumentalist conception of mathematics was reflected in the students' descriptions of good teaching which involved the careful and organized presentation of clear examples for students to emulate followed by time for student practice of the modelled procedure.

For a smaller portion of the class, mathematics was a language for expressing patterns and relationships. A more fallibilist view of the discipline was shown in noting that there are often multiple ways to solve a problem and in some cases, starting from different assumptions, a variety of valid answers.
may be found. These teacher candidates saw good lessons as involving pupils in personal discovery of concepts and they suggested that investigations could effectively begin with a problem or application context. In this study, as in the previously cited research linking beliefs and observed practice, it was possible to identify definitive instrumentalist images of mathematics while social constructivist conceptions were less fully formed. Links between instrumentalist views and teacher-centred transmissive instruction were evident while teachers with emerging social constructivist images were not as adamant in their support for compatible teaching practices.

A number of published reports of programs designed to help teachers change their mathematics teaching practices have noted a link between instructional choices and epistemological beliefs. After working with more than 500 elementary school teachers at summer workshops exploring alternative ways for teaching mathematics, Schifter and Twomey Fosnot (1993) conclude that "constructing a practice consistent with the new paradigm entails a new conception of the nature of mathematics" (p. 193).

Evidence of the necessity for epistemological change is given in the results of a study by Arsac, Balacheff and Mante (1992). Two grade 8 teachers were extensively prepared for implementing instructional scenarios designed from a social constructivist perspective. Students, grouped in pairs, were to develop and validate solutions to a geometric problem and then the expected multiple solution techniques would be debated by the whole class. Teachers were instructed to provide only organizational support and intervene only to manage the flow of debate. During actual
implementation of the lessons it was found that, "precautions consisting of carefully presenting the situation, and the theoretical ideas behind it, before the class session, to the teacher were not enough to avoid difficulties" (p. 26). In each case, beliefs that mathematics consists of fixed true statements conflicted with the proposed instructional methods. The teachers' images of the discipline led them to intervene in the pupils' explorations and debates with the purpose of directing progress towards the "discovery" of an acceptable solution.

A similar failure to effect changes in teaching without altering beliefs was observed in a study involving a preservice secondary teacher (Wilson, 1994). Here a course presenting content knowledge and instructional methods related to the function concept did not have an effect on the teacher candidate's instrumentalist image of mathematics. Despite increased knowledge, of both content and alternative methods of instruction the student teacher's views of mathematics and mathematics teaching remained narrow and traditional.

Wood, Cobb, and Yackel (1991) describe work with a second-grade teacher during her efforts to move from a traditional transmissive mode of instruction to a style similar to that attempted in the previously described study by Arsac, Balacheff and Mante (1992). The target instruction envisioned pupils collaboratively constructing shared understandings through exploration of open-ended problem situations, first in pairs, and then in whole-class discussions. As the teacher struggled with change, attempting new practices, reflecting on lessons given and discussing issues with the research team, the authors noted that
parallel to progress towards the intended new practice was a complimentary change in conceptions of mathematics. As teaching moved from transmission of information to the initiation and guidance of students' collaborative development of knowledge, images of mathematics changed from that of a collection of rules and procedures to a meaningful human activity. A similar shift in teachers' beliefs about mathematics was observed (O'Brien, 1995) in a larger scale ten-year program designed to bring about change in the instructional styles employed by the teachers in one elementary school. Although growth was uneven among teachers, all experienced some movement away from an instrumentalist image as instructional roles changed from lecturer to facilitator.

Although the studies identified in the preceding survey provide evidence of the effects of conceptions of mathematics on teaching practice, in a number of ways the link has not been fully explored. Only two of the reports found in the literature dealt with secondary school mathematics teaching and one of these focused on a single content area, geometry. Other research addressing high school mathematics involved preservice teachers and their intended instructional methods. There are many constraints that can lie between envisioning certain practices and effectively employing them in the classroom. The translation is especially difficult for those teachers holding social constructivist views of mathematics.

The cited studies were able to identify definitive examples of Platonist and instrumentalist images of mathematics but clear and fully developed examples of social constructivist or problem
solving conceptions were not located. The tentative nature of the problem solving images exhibited in the research and the strong legacy of opposing teaching traditions combine to make it difficult to establish clear links between beliefs and practices at this end of the scale. The study reported here was designed to address the need for research that explores for social constructivist views of mathematics at the high school level, and, if teachers possessing such images are found, investigates the extent to which their subject images are translated into practice and follows the struggles involved in efforts to make this translation.

Conceptions of Subject and Teaching Practice: Other Disciplines

A teacher is a member of a scholarly community. He or she must understand the structures of subject matter, the principles of conceptual organization, and the principles of inquiry that help answer two kinds of questions in each field: What are the important ideas and skills in this domain? and How are new ideas added and deficient ones dropped by those who produce knowledge in this area? That is, what are the rules and procedures of good scholarship or inquiry? (Shulman, 1987a, p. 9)

The impact of beliefs about subject matter on teachers' choices of content and instructional approaches is not restricted to mathematics. Ball and McDiarmid (1990) assert that generally, "teachers' conceptions of knowledge shape their practice - the kinds of questions they ask, the ideas they reinforce, the sorts of tasks they assign" (p. 438). At the secondary level, where schools are most often organized into subject based departments, "the nature of the parent discipline and features of the school subject, as well as teachers' beliefs regarding the subject, help
create a conceptual context within which teachers work" (Grossman & Stodolsky, 1995, p. 5).

Pope and Scott (1984), in an study of the epistemologies of science held by science teachers, employed Kelly's (1955) categories of: "constructive alternativism", which takes knowledge to be constructed by individuals, and "accumulative fragmentalism", the vision of knowledge as a collection of substantiated facts. These terms closely parallel those employed by Ernest (1991), fallibilist and absolutist, to describe conceptions of the nature of mathematical knowledge. Pope and Scott noted that generally science teachers held accumulative fragmentalist or absolutist views of science knowledge, and in turn, the manner in which students were taught placed little value on pupils' own conceptions and did not encourage active student participation.

In a study similar to Roulet's (1995) research with mathematics teacher-candidates, Aquirre, Haggerty, and Linder (1990) analysed the written responses given by beginning science student-teachers to questions concerning: the nature of science, how scientific knowledge is produced, effective ways for teaching the subject, and the manner in which high school pupils learn science. The results of this study were also parallel to those of Roulet (1995), with the responses showing that "over the course of at least 16 years of formal education, students had grasped only some of the characteristics of the 'whys' and 'hows' of science." "The most crucial aspects of the ways in which scientific knowledge is produced (the most important of which is the concept that scientific knowledge is a creation of the human
mind) are not being assimilated by undergraduate science students" (p. 388). Again, as in the mathematics study, these teacher-candidates adopted a transmissive approach to teaching. The teacher was seen as the source of knowledge and the teaching-learning process, as viewed by these students, involved the transfer of this knowledge through logically clear explanations and demonstrations.

Experienced (minimum of five years and average of 15.8 years of teaching) senior high school biology teachers were the subjects of a study by Lederman and Zeidler (1987). Here, completion of a survey form, the Nature of Scientific Knowledge Scale, was employed to measure the teachers' understandings of the nature of science. Data from subsequent observations of teaching were analyzed to determine differences in classroom atmosphere and instructional procedures used by the participating teachers. When the conceptions of science and teaching practices data were compared, the results of the study did, "not support the prevalent assumption that a teacher's classroom behavior is directly influenced by his/her conception of the nature of science" (p. 729). The fact that the outcomes of this project differ from those of the previously cited mathematics and science studies and from those of the research to be described in the following paragraphs, may rest on the fact that here, to generate quantitative data, the authors employed a questionnaire. Through courses in the history or philosophy of their disciplines or independent reading, teachers may have met academic presentations of the nature of their teaching subjects. Unless these formal images fit with teachers' more extensive experience learning
subject matter content, this information is likely to be held strictly as surface knowledge and not be incorporated into belief systems. Such knowledge may show up on a survey while relatively unrestricted writing activities such as those employed by Roulet (1995) with mathematics teacher-candidates and Aquirre, Haggerty, and Linder (1990) with science student-teachers and also in this present study are more likely to accurately reveal beliefs.

Wilson and Wineburg (1988) studied the subject images and instructional approaches taken in secondary school American history courses by four teachers with a variety of university academic backgrounds: anthropology, political science, American Studies, and American history, and found that "their disciplinary backgrounds wielded a strong - and often decisive - influence on their instructional decision making" (p. 526). For the political science and American Studies graduates, history consisted of facts, while the teacher with a history background saw her subject as an exciting narrative. For the anthropologist, history involved collecting archeological evidence and from this identifying strings of discrete events. Both the history and American Studies graduates identified unifying themes that linked events and periods. These disparate images of history resulted in distinctly different subject presentations. In the classrooms led by the political science and American Studies graduates, the study of history became, essentially, debates around current events with references back to related facts concerning past human experience. The single historian among the four teachers incorporated music, literature, the study of old photographs, and
debates into lessons to paint pictures of social, cultural, political and economic issues of past periods.

In a study similar to that of Wilson and Wineburg (1988), Grossman (1991) examined the disciplinary orientations and instructional practices of two teachers of English. At university Colleen majored in English literature with the intention of becoming a teacher while Martha followed an interdisciplinary route, examining literature from a number of language sources. Colleen's orientation towards literature centred on the text itself, while Martha took a more personal interpretive or reader-response theory approach, looking at overall themes and linking the story to personal experience. In turn these teachers held differing goals for instruction and planned and presented different lessons. Colleen's focus on the text translated into a goal of encouraging students to pay close attention to the author's words when interpreting literature and her questions to students, to a large extent, concerned eliciting definitions for unfamiliar words. Martha was much less concerned with students' understanding of particular texts and class time was consumed with pupils writing personal responses to the stories and poetry studied.

It would appear from the above studies, with the exception of Lederman and Zeidler (1987), that the subject orientations developed by teachers during their university years play a significant role in their subsequent instructional decisions made while teaching. It is also interesting that, while considerable research related to conceptions of subject and teaching practices has been conducted with secondary school teachers of subjects
other than mathematics, most of the mathematics related studies have been situated in the primary and junior or middle school grades.

A number of studies (Grossman & Stodolsky, 1995; Siskin, 1994; Stodolsky, 1993) have looked at differing views of knowledge across high school subjects. Although teachers are somewhat removed from their past discipline based university studies, "the discipline's language and epistemology are interwoven in the ways teachers - as subject-matter specialists - conceptualize the world, their roles within it, and the nature of knowledge, teaching, and learning" (Siskin, 1994, p. 152). In these studies mathematics teachers described their subject as well defined, having clear boundaries, highly structured with a definite sequence of topics, and relatively static. Science and modern languages teachers also saw their subjects as structured, with required sequences of study, but did not see their disciplines as being as static as did mathematics teachers. On the other hand, teachers of English and social studies saw their subjects as being less well defined and having less structure. As a result, members of English and social studies departments were relatively free to personally select course topics and instructional methods, since the outcomes of each individual course would have little direct impact on subsequent study. In mathematics departments there was more collegial knowledge of what instructors did in other courses, and more uniformity in terms of content coverage, teaching approaches, assessment standards, and curricular materials. Here there were significant departmental pressures to conform to the majority, traditional
ways of organizing courses and lessons. "Subject subcultures may be characterized by both beliefs about subject matter that bind teachers together and by norms regarding teaching practice, curricular autonomy, and coordination" (Grossman & Stodolsky, 1995, p. 8).

Mathematics Teaching Practice

I think a reading of both the Curriculum and Evaluation Standards and the Professional Teaching Standards will show that both documents were heavily influenced by contemporary thinking on students building meaning and constructing their own knowledge. (Black & Atkin, 1996, p. 81)

While the terms, constructivism and social constructivism, are not employed in the NCTM's (1989, 1991, 1995) mathematics education reform documents, the leaders of this project have, as in the quote above, and elsewhere (McLeod, Stake, Schappelle, Mellissinos & Gierl, 1996; Romberg, 1992a, 1992b), made it clear that such a philosophy of knowledge informed their efforts. "The term that we did not use in writing up the Standards (but we certainly talked about) is what might be called the social constructivist's notion of learning" (McLeod et al., 1996, p. 38). This conception of mathematics that underlies the reforms proposed by both the NCTM and OAME leads to a call for the implementation of specific standards (NCTM, 1989, 1991, 1995; OAME/OMCA, 1993, 1995) of practice. An expanded view of teaching is described by these documents, as the development of students' mathematical understandings is no longer seen as flowing exclusively from teacher explanations. If the classroom development of mathematics is to reflect the discipline's social
constructive epistemology, then an environment in which students interact with each other and the teacher while pursuing meaningful and stimulating mathematical tasks (NCTM, 1991) is required. The following section of the literature review will first present an overview of the expected teaching practices described by the NCTM (1989, 1991, 1995) and OAME/OMCA (1993, 1995). This will be followed by a brief description of the research paradigms through which mathematics teaching practice has been examined and evaluated and a discussion of these methodologies in relation to the pedagogical practices encouraged by the present reforms. Finally an epistemological perspective (Koehler & Grouws, 1992) for the examination of teaching will be examined and identified as appropriate for the proposed study.

New Visions of Practice

Just as our vision of the twenty-first century differs dramatically from that of the twentieth century, so the vision of mathematics education described in this document [Focus on Renewal of Mathematics Education] differs significantly from the traditional vision of the subject. (OAME/OMCA, 1993, p. 2)

Calls for change in the teaching of mathematics are heard in many parts of the world (Black & Atkin, 1996) and proposals for reform share common themes across national boundaries (Romberg, 1992b). The National Council of Teachers of Mathematics [NCTM], through its publication and promotion of the Curriculum and Evaluation Standards for School Mathematics (1989) and related documents, has been a major player in this international movement. The NCTM publications and the parallel but less ambitious Focus on Renewal of Mathematics Education, produced by
the Ontario Association for Mathematics Education [OAME] and Ontario Mathematics Coordinators Association [OMCA] (1993), are the major sources of guidance for Ontario teachers wishing to change their mathematics curricula and instruction. The observation and analysis of teaching practice conducted within this study were pursued from the perspective of the proposals made in these documents.

In 1986 the NCTM, in an effort to stimulate improvements in the quality of elementary and secondary school mathematics curricula and instruction, established the Commission on Standards for School Mathematics (Romberg, 1992b). The Commission and its various Working Groups, composed of a cross section of the mathematics education community: classroom teachers, supervisors, educational researchers, teacher educators, and research mathematicians, produced a draft set of standards by the fall of 1987. After addressing feedback gathered from the educational community during the 1987-88 academic year, the Commission and NCTM in 1989 issued the Curriculum and Evaluation Standards for School Mathematics. The work of the NCTM did not end with the publication of this report, for companion documents: Professional Standards for Teaching Mathematics (1991) and Assessment Standards for School Mathematics (1995), and grade specific support materials have subsequently been developed. In recent studies the NCTM (1996) and others have begun to refer collectively to the three "standards" documents (NCTM, 1989, 1991, 1995) as the Standards. This approach will be followed here with year of publication
given when reference is to one of the three documents in particular.

The first four standards issued by the NCTM (1989) present goals for instructional practice that address the full curriculum and apply to the teaching of all topics. The proposals set out in these standards and their elaboration in subsequent NCTM documents have been employed in this study as a background when analyzing teaching practice. The images of mathematics teaching embodied in these standards will be described next.

Standard 1: Mathematics as Problem Solving

Teaching in mathematics is expected to move beyond the traditional sequence of presentation of specific techniques followed by the solution of word problems to which the recently developed procedures apply. Exploration for problem solutions is to become the vehicle for the development of course content. "Problems and applications should be used to introduce new mathematical content, to help students develop both understanding of concepts and facility with procedures, and to apply and review processes they have already learned" (NCTM, 1989, p. 137). The problems presented by the teacher should serve to link the course content to the world outside the classroom and to make connections within mathematics itself. The use of calculator and computer technology, to support a shift in focus from computational details to the construction of mathematical models, is encouraged.

Standard 2: Mathematics as Communication

Students should be actively involved in their learning through whole-class and small-group explorations that provide
opportunities for discussion, questioning, listening, and summarizing. Teacher questions should call for more than a report of an answer or a listing of solution steps. Students should describe in their own words how they arrived at a problem solution, elaborating their thinking and the difficulties they encountered. Classroom discussion should go beyond a focus on mathematical procedures and address social issues making the connection between the course content and society at large.

Standard 3: Mathematics as Reasoning

Both inductive and deductive reasoning should be employed in mathematics lessons. Through the examination of patterns students should be encouraged to make generalizations and formulate conjectures. In individual work or group debate the validity of these hypothesis may be explored, with students encouraged to develop deductive arguments supporting the conjecture or to construct counter examples revealing its flaws. Thus mathematics is created within the classroom rather than delivered by the teacher as a complete formal package.

Standard 4: Mathematical Connections

Instruction should illustrate the connections between mathematics and other disciplines and within the subject itself. Students are to experience the process of constructing mathematical models for problem situations occurring in the natural and social sciences and arising in commercial and industrial contexts. The parallels between alternative representations of mathematical concepts, such as the numeric, algebraic and graphical views of functions, should be explored.
The United States has in fact lagged behind other jurisdictions in addressing mathematics education reform and the themes developed in the *Standards* (1989), as calls for policy change, are found in the official curricula of Australia, France, Germany, Japan, the Netherlands, Norway, Spain, the United Kingdom. Although content emphases and the terms employed differ, "the four standards for mathematics teaching and learning in the NCTM's *Standards*, problem solving, communication, reasoning, and connections, are reflected in all eight national curricula, not just in the reform curricula" (Romberg, 1992b, p. 229). Similarly these themes are part of the official policy for Ontario, the setting of this study, appearing in the guidelines issued in 1985 (Ontario Ministry of Education).

The NCTM's image of mathematical instruction can not be captured in a prescriptive list of distinct teacher behaviours, but must be seen as a set of complex interactions between teacher, students, mathematical tasks, and materials. The picture of this learning environment is further developed by the NCTM in the *Professional Standards for Teaching Mathematics* (1991). The presentation of appropriate and worthwhile mathematical tasks is the teacher's first step in building an environment for mathematical learning. "Teachers should choose and develop tasks that are likely to promote the development of students' understandings of concepts and procedures in a way that also fosters their ability to solve problems and to reason and communicate mathematically" (NCTM, 1991, p. 25). Problems that capture students' curiosity and invite them to pose conjectures for further exploration are required to address the *Standards*.
call for mathematical discourse and reasoning. Tasks that may be approached in multiple ways and those that may have more than one reasonable solution are valuable in generating the student debate that can lead to deeper understanding.

Having posed a mathematical task for exploration the teacher must encourage, support and orchestrate the discussion that ensues. Here the traditional teacher processes of presenting examples and explaining procedures must give way to careful listening and prompting of students to clarify and justify their ideas. The teacher must be constantly analyzing the ideas put forth and making decisions concerning productive leads to follow, the balance between free flowing discussion in common language and the use of formal mathematical language and notation, when to intervene to highlight an idea or clarify an issue and when to let students struggle towards their own solutions. The flow of discussion must be carefully monitored to ensure that all pupils are participating, both in making contributions and listening and responding to the ideas presented by others. Classroom norms of conversation, in which students critique each other's hypothesis and suggestions with civility and respect, must be encouraged.

The physical organization of the classroom and appropriate student groupings can encourage communication. The teacher must decide when they wish students to develop ideas individually, possibly writing journal entries for later sharing, and when it is appropriate for pupils to exchange ideas in pairs, small groups, and in a whole class format. Any one lesson is likely to employ a variety of such groupings and thus class time must be effectively sub-divided to provide the opportunities for students
to complete significant explorations and fully exchange their thoughts arising out of the activity.

Concrete materials may serve as models for mathematical concepts and exploration with these can provide patterns for student observation and generalization. Calculator and computer supported explorations can similarly promote discussion and learning.

This expanded vision of what may occur in a mathematics classroom and what it means to teach and learn within the subject requires a parallel expansion of the activities subsumed under the term "teaching practice". Teaching is no-longer restricted to mean the activities of a teacher during a lesson but now includes the preparation tasks of designing or selecting and modifying the learning activities, arranging student groupings, and choosing and organizing the supporting resources that will be employed in the learning environment.

Sharing the same concerns for mathematics education as the NCTM, the Ontario Association for Mathematics Education [OAME] and the Ontario Mathematics Coordinators Association [OMCA] in 1993 issued Focus on Renewal of Mathematics Education. As an Ontario interpretation of the Standards (NCTM, 1989) the first four teaching practices identified as key components of all mathematics programs: communication, reasoning, problem solving and connections, echo the NCTM vision described above. Reflecting the rapid advances in technology and its later publication date the Focus on Renewal of Mathematics Education (OAME/OMCA, 1993) identifies a fifth key component of effective mathematics teaching, the use of computers and calculators.
"Teachers and students should use computers as tools to assist with the exploration and discovery of concepts, with the transition from concrete experiences to abstract mathematical ideas, with the practice of skills, and with the process of problem solving" (p. 5).

The vision of mathematics instruction expressed by the NCTM (1989, 1991) and OAME/OMCA (1993) was, in many ways, not a new one for the high school mathematics teachers of Ontario. Although the data from the provincial reviews of 1990 (Ministry of Education, 1992a, 1992b) suggests that, in Ontario, mathematics teaching practice was much different from that advocated in the reform documents, the basic themes outlined in the new approaches had been, to a great extent, presented to the province's teachers five years earlier in the introductory pages of new curriculum guidelines (Ontario Ministry of Education, 1985). Here, in a less complete and elaborate form, the curriculum developers presented a program that took problem solving as a major goal and encouraged the development of concepts and skills through the investigation of applications located in other disciplines. Student understandings were to be developed through the generalization of patterns observed in explorations employing manipulative materials, computers and calculators. Just as for the NCTM (1989), the development of mathematical reasoning through student communication was key.

Students should be consistently asked "Why?" and helped to use language that expresses their answer to such a question and conveys their meaning to others. Opportunities should be provided for students to engage in writing activities in which they explore their own perceptions of mathematical concepts and report on their attempts to apply mathematics to problem solving. Students should also be provided with opportunities to
engage in speaking and listening activities in which they test their thinking and reasoning against those of their peers. It is through these types of activities that students modify, verify, and consolidate their learning. (Ontario Ministry of Education, 1985, p. 17)

Although the developers of this new curriculum wished to see mathematics instruction in Ontario move beyond its traditional teacher-centred transmissive style, the structure of the course guideline documents did not encourage change. Instructional issues in the program were given just five pages at the beginning of the guideline. The individual course descriptions that followed this severely limited discussion of process, consisted of lists of topics with no references back to the teaching ideas given in the introductory pages. Teachers could ignore the calls for change, turn to the content lists for their assigned courses, and continue their practice of past years. That this was in fact the route taken by many of Ontario's secondary school mathematics teachers is indicated by the predominance of teacher presentations followed by individual student practice recorded in the provincial reviews (Ontario Ministry of Education, 1991a, 1991b).

Recognizing that "testing drives curriculum", in 1995, both the NCTM and OAME/OMCA, to compliment their previous reform efforts, released documents addressing issues concerning student assessment. Here assessment is seen as more than the task of assigning grades and is presented as a tool to enhance learning. As such, assessment tasks should be linked to and expand upon instructional activities. While pencil-and-paper quizzes, tests, and examinations may still be employed, the reform movement sees this list of assessment instruments significantly expanded to
include: projects, student writing activities, oral presentations, teacher observations of student action, pupil-teacher conferences exploring problem solving processes, and self, peer, and group evaluations. Such activities can provide data on student thought processes and attitudes as well as measure pupils' abilities with mathematical procedures. Such data may be used by: students to identify strengths and weaknesses and adjust efforts and educational plans accordingly, parents to help guide their support and encouragement efforts, teachers to help plan changes in instructional processes, and administrators to assess program effectiveness. Emphasis has shifted from measuring student performance on discrete mathematical skills to determining their understanding of mathematical processes and ability to combine concepts and procedures to tackle larger scale and un-rehearsed problems.

Mathematics teaching, learning, and assessment as described in the reform proposals of the professional associations and in the official Ontario curriculum documents is a complex interaction among task, resources to support exploration, students and teacher. Instruction is no longer just the teacher's performance at the front of the classroom. Research projects that wish to record practice in this new style must take a broad view of the classroom, recording the organization of the learning environment, materials used, problems posed, and the actions and conversations of the pupils and teacher.
Even when researchers find reasonable variables such as "lesson development", which are demonstrably effective, they are only effective within the traditional conception of teaching. They can only make current teaching more efficient or effective, but they cannot make it radically different. (Romberg & Carpenter, 1986, p. 865)

Instructional methods in mathematics have been objects of study for the past hundred years. As the 19th century closed, those participating in the emerging discipline of educational research began to adopt scientific methods for the study of teaching technique. "The result of the wish to employ scientific methods was a rather single-minded focus on quantification and 'experimentation' - the kind of experimentation that gave rise to data that could be analysed statistically" (Schoenfeld, 1994, p. 699). The need for larger student samples and a desire for control of variables made mathematics or more specifically arithmetic, being a dominant subject and having fairly consistent curricula across teachers and schools, a favourite topic for study. "No other school subject has even approached the level and frequency of studies conducted in the area of arithmetic" (Shulman, 1970, p. 25). Only in the past two decades has the dominance of the empirical-analytical tradition for research into mathematics teaching and learning been challenged (Kilpatrick, 1992).

The development of pencil-and-paper tests at the beginning of this century provided a means for researchers to measure in a standardized fashion the outcomes of instruction and thus gave birth to "process-product" studies of teaching. Here selected
teachers, or often student-teachers, conduct lessons using specific contrasting styles of instruction and the success of each experimental group is measured by a common test. Alternatively, the practice of teachers in existing classrooms is observed and coded so that contrasting approaches may be identified. Again pupil success is determined by testing upon the completion of instruction. In either form of the research, correlation coefficients linking teaching practice (process) to student test scores (product) allow the identification of more and less effective instructional methods.

In many of these process-product studies the measured performance is of low-level procedural skills (Schoenfeld, 1994). Most of the mathematics studies cited in Brophy and Good's (1986) extensive survey of research linking teacher behaviour and student achievement, involved elementary school arithmetic. In general, process-product studies deal with discrete teacher actions or particular lesson structures and do not analyse in any depth the mathematics being addressed. Brophy and Good (1986) are able to provide lists of effective and ineffective processes that are independent, not just of mathematical topic, but also of school subject. The superficial measurement of outcomes and the traditional selection of topics addressed by standardized tests mean that these studies have little connection with the new curricula and deeper understandings called for by the NCTM Standards (1989, 1991, 1995).

Process-product research in identifying distinct instructional actions, such as, thinking level of questions posed, wait time for student responses, and the giving of praise,
addresses the teacher-centred classroom. The image of instruction (Fennema & Peterson, 1986) is of the traditional Socratic lesson where concepts are transmitted from teacher to pupil via sequences of well crafted questions. In a study analysing the use of a reward system to encourage student accuracy in correcting mathematics homework, Miller, Duffy and Zane (1993) describe the introductory phase of the experimental lessons with,

The teacher then asked the students to take out their homework, and called on individual students to verbally provide the answer to each question and problem. When a student gave a correct answer, the teacher indicated so by repeating the answer one time. If a student responded incorrectly, the teacher either prompted the student or called on another student until the correct response was elicited (p. 185).

There is no sense here of the student to student communication of mathematical ideas called for by the NCTM (1989, 1991) and OAME/OMCA (1993) reform documents. The process-product research paradigm may be well suited to the study of the traditional transmissive mode of mathematics instruction but it can not capture the complexities of life in a classroom where the guiding principles for instruction are the NCTM (1989, 1991) standards.

Wishing to capture the complex cognitive processes of teaching, researchers have expanded the process-product research paradigm to studies of expertise in teaching. In this perspective, "one stands figuratively behind the shoulder of the teacher and watches as the teacher juggles the multiple goals of script completion, tactical information processing, decision making, problem solving, and planning" (Leinhardt, 1989, p. 52). These studies often also include observations of novice teachers, with the expert-novice contrasts being used to highlight the
expert's instructional skills. In this research perspective, as in the process-product studies, the teacher is seen as the principal player in the teaching-learning process. Teaching mathematics is viewed as explaining (Leinhardt, 1989) and skill in elaboration of subject matter is taken as a measure of expertise.

Bromme and Steinbring (1994) selected expert mathematics teachers for their study using measures of skill in control of instructional flow, clarity of teacher statements, and clearness of presentations on blackboard and transparencies. Berliner (1986) describes an expert's lesson-opening homework review with,

The expert teacher was found to be brief, taking about one-third less time than a novice. She was able to pick up information about attendance, about who did or did not do homework, and to identify who was going to need help in the subsequent lesson. She was able to get all the homework corrected and elicited mostly correct answers throughout the activity. And she did so at a brisk pace and without ever losing control of the lesson (p. 5).

But is this "good" teaching in terms of the present reform proposals? Such "well-taught" lessons can be disasters when it comes to giving students some sense of mathematics as an evolving discipline (Schoenfeld, 1988). This model of teaching in no way reflects the spirit of the Standards (NCTM, 1989, 1991) and pupils leave such lessons believing that mathematics is just a collection of precisely defined procedures such as those that have been so carefully explained and illustrated by the expert pedagogue. Research methods that identify as expert teaching, the use of drills, games, and individual written work to address at least 40 problems per lesson (McKinney, 1986), can not be effectively employed to search for classrooms exhibiting the
mathematics learning environments imagined by the present reform movement.

Capturing a Fuller Picture:  
A Sociological and Epistemological Perspective

We believe that teaching involves reasoning as well as acting: it is an intellectual and imaginative process, not merely a behavioral one. (Shulman, 1987b, p. 41)

There has been a growing feeling in parts of the mathematics education research community that, while process-product and novice-expert studies have identified important teaching actions, "a much-needed emphasis on the central role of the students in their own learning is missing" (Anderson & Postlethwaite, 1989, p. 84). There is a need to examine the interactions among students as well as exchanges between teachers and pupils. The entire culture or environment of the classroom and the ways that children construct knowledge must become objects of study. In this way the full range of teacher responsibilities: setting mathematical tasks, structuring investigations, focussing pupil attention on critical points, managing student discourse, helping students identify and resolve contradictions, and encouraging metacognitive processes and generalization (Bishop, 1985; Hoyles, 1988), become decisions and actions to be recorded and analysed.

Combining this wider view of teaching with a social constructivist epistemology of mathematics, Lampert (1990) examines instructional practice for its congruency "with ideas about what it means to do mathematics in the discipline" (p. 33).

The analysis of practice draws on both familiar social science approaches to educational research and the
epistemological arguments that characterize the subject being taught. Mathematical notions about what constitutes knowledge are considered in concert with frameworks drawn from the sort of generic study of knowledge acquisition that has characterized research on teaching and learning. (p. 36)

Acknowledging the two components of Lampert's program, Koehler and Grouws (1992), in their survey of research on mathematics teaching practices, title this approach the "sociological and epistemological" view.

In Lampert's work the richness of the mathematical experience is conveyed in the form of detailed descriptions of the tasks posed, and narratives capturing the classroom interactions. These are then analysed, examining the teacher's planning and decision making and the children's construction of meaning, all against a social constructivist image of the discipline. In this work, access to the teacher's thinking is direct since, in her teaching experiments, Lampert acts as both teacher and researcher.

Using a similar research program, Simon (1995) presents and examines his teaching of mathematics at the college level in an experimental elementary school teacher preparation program. As in Lampert's research, "for each situation, a description is provided of the challenge that faced the teacher as construed by the teacher, the decision that he made to respond to that challenge, and subsequent classroom interaction that was constituted by the students and teacher" (p. 122). Again analysis is conducted against a social constructivist perspective of teaching and learning.

While research reports have not always labelled the methodology employed as sociological and epistemological, a
number of recent studies concerning classroom mathematics teaching and learning have taken this approach. Through narrative presentation of classroom episodes and sociological and epistemological analyses of the recorded vignettes, studies have examined children's construction of knowledge through explorations and mathematical discourse in: third, fifth, and seventh-grade classes working with fractions (Flores, Sowder, Philipp & Schappelle, 1995; Kieren, 1995), grade three pupils exploring addition of quantities greater than ten (Lo & Wheatley, 1994), a second grade class looking at number patterns (Wood, Cobb & Yackel, 1991), grade two pupils working in pairs on number problems (Yackel, Cobb & Wood, 1991), and high school students using computer support for exploration in geometry (Schwartz, 1989). In each case the focus is on the teachers' actions in establishing learning environments and students' making and exploring conjectures, resolving conflicts and negotiating meanings.

The establishment of taken-as-shared meanings for students' personal constructions enables mathematical ideas to be established that are accepted by members of the class. It is through this process of negotiation of meaning that students become acculturated to the mathematical truths established by society. (Wood, Cobb & Yackel, 1991, p. 594)

Cobb (1988) points out that translating a constructivist view of learning into a theory of instruction is a difficult task. Even when a teacher provides rich mathematical contexts, materials to support student experimentation, and an environment that encourages flowing discourse, miscommunication can occur and students fail to develop the intended understandings. "Because teachers and students each construct their own meanings for words
and events in the context of ongoing interaction, it is readily apparent why communication often breaks down, why teachers and students frequently talk past each other" (p. 88). When working with secondary school pupils, who may have experienced eight or more years of transmissive mathematics instruction, the opportunities for miscommunication and tension increase. Thus, in the present study, the focus of classroom observations is on the participants' efforts to engage in teaching that implements features of the reform proposals and on the struggles involved in this exercise.

In the studies cited above, only those of Simon (1995) and Schwartz (1989) involved classes beyond the middle grades, and in Simon's research, although the students were adults, the topic, an introduction to the area concept, is from the middle school curriculum. While research employing a sociological and epistemological perspective to inquiry into social constructivist approaches to mathematics instruction appears to be increasing in popularity, studies have been concentrated at the primary and middle school levels. The study described in the following pages contributes to the expansion of this program to the secondary school level.

**Summary: Subject Conceptions and Teaching Practice**

'Curiouser and curiouser!' cried Alice. (Carroll, 1954, p. 9)

This study was undertaken to explore the questions:

What are the conceptions of mathematics held by exemplary secondary school teachers, those who are attempting to implement the reform proposals?
How are these teachers' images of mathematics connected to their classroom practices?

or more specifically:

To what extent are these teachers' instructional practices expressions of their subject images?

and

What are the struggles involved in these teachers' efforts to translate subject images into classroom practice?

The research literature reviewed above provides some starting points for constructing answers to these questions, but there remain some important missing components. The conceptions of mathematics held by primary and middle school teachers have been quite extensively investigated. At the secondary level only teachers' images of selected topics have been explored and these in just a small number of studies. The research reported here builds a fuller picture by looking at the discipline of mathematics more generally. Teachers' translation of conceptions of mathematics into classroom practice has been reported in the published literature. Again this work has been mainly at the primary and middle school levels with secondary school based studies focussing on particular topic areas. As well, most of the study participants in published research have been teachers employing traditional transmissive modes of instruction and have been found to possess absolutist images of mathematics. Thus, at the secondary school level, connections between absolutist conceptions of mathematics and traditional transmissive modes of instruction have been made, but the practices employed by teachers possessing fallibilist philosophies have not been explored.
The present study involves two secondary school teachers who have been observed to employ teaching methods representative of those proposed by the mathematics education reform documents from the National Council of Teachers of Mathematics and the Ontario Association for Mathematics Education. While these teachers, as will be shown, do not both hold fully developed social constructivist views of mathematics, their conceptions of the nature of mathematics are definitely not absolutist. Thus they provide an opportunity to examine connections between subject images and instructional practices not yet reported.

The past research found in the literature provides an extensive background for this study but leaves the above questions open for exploration.
Chapter 3: Method

The research examined in the previous chapter clearly shows the complexity of the interactions between beliefs and practice. Teachers possessing specific sets of beliefs may be found to usually employ specific sets of practices, but these apparently coincident occurrences of events and qualities are not the result of deterministic laws. These are examples of "mutual simultaneous shaping" (Lincoln & Guba, 1985)

Everything influences everything else, in the here and now. Many elements are implicated in any given action, and each element interacts with all of the others in ways that change them all while simultaneously resulting in something that we, as outside observers, label as outcomes or effects. (p. 151)

A study examining phenomena in such an environment and wishing to make any significant headway in addressing the questions posed at the close of the previous chapter needs to focus on particular cases and explore these in detail. Thus a case study approach has been employed for the research reported here.

The case study offers a means of investigating complex social units consisting of multiple variables of potential importance in understanding the phenomenon. Anchored in real-life situations, the case study results in a rich and holistic account of a phenomenon. (Merriam, 1988, p. 32)

The methodologies employed within the two cases of this study, like the preceding literature review, come from the two domains of research into: teachers' beliefs or thinking and classroom practice.
Orientation and Conceptual Framework

Reason ... must approach nature in order to be taught by it. It must not, however, do so in the character of a pupil who listens to everything that the teacher chooses to say, but of an appointed judge who compels the witnesses to answer questions which he has himself formulated ... It is thus that the study of nature has entered on the secure path of science, after having for so many centuries been nothing but a process of merely random groping. (Kant, 1787)

The present mathematics education reform efforts, designed to emphasize the inductive aspects of mathematical development and to reduce the subject's portrayal as a pure deductive science, and teachers' struggles to come to grips with this change in philosophical orientation, are paralleled in the selection of methods for this inquiry. I, like the two teachers who agreed to participate in this study, am a product of a mathematics education that focused upon deduction. Textbook chapters and mathematics lessons began with a statement of a mathematical 'truth', the theorem to be proved or the algorithm to be illustrated in the following pages of text or the teacher's blackboard presentation. The result or theory was already set. All that remained to do, was to provide a proof or explanation. In a similar fashion, from my own experiences as a mathematics student, mathematician, and mathematics teacher and from the reading of the literature previously cited, I had, prior to beginning my research, developed an image of what the results of this study might say. There was thus a temptation to approach the research with a predetermined thesis and to employ the study as a proof of this theory. This I did not want to do. It was intended that the summary thesis of this study, if one was in
fact found, emerge from the data. To hopefully avoid the selective observation and interpretation that may follow from a previously determined theory, or at least to identify and acknowledge such selectivity if and when it occurs, a brief overview of the research perspective is provided.

The research literature previously cited suggests that, within the constraints provided by context, teachers employ practices that are warranted by their beliefs. In particular, studies have found that teachers holding absolutist beliefs concerning mathematics adopt transmissive modes of instruction in the subject. Turning this around, it might be postulated that teachers in whose classrooms pupils are encouraged to investigate mathematical situations, to make and test hypotheses, and to discuss and defend conjectures hold fallibilist images of the subject, possibly social constructivist views. Is this the case? For one who has come to take a social constructivist perspective on mathematics and who supports the reform efforts of the professional associations, a positive answer to this question would be desirable.

While efforts can be made to reduce the effects of researcher bias, it is probably undesirable and likely impossible for a researcher to initiate a study without some prior view of the potential results. As Kant asserts in the quote above, to enter the research field (nature) without prior questions, constructed from some initial tentative theory, is likely to lead to random groping. The researcher can not focus his powers of observation and details that might answer his questions are likely to be lost in a overwhelming mass of data. "Anticipatory
data reduction" (Miles & Huberman, 1984a, p. 24) guided by an initial conceptual framework can help focus a study. Thus the present study acknowledges the above "hypotheses" or speculations and looks for evidence of these while also being open to other interpretations of the data.

**Participants**

The fact that lawfulness and individuality are considered antitheses has two sorts of effect on actual research. It signifies in the first place a limitation of research. It makes it appear hopeless to try to understand the real, unique, course of an emotion or the actual structure of a particular individual's personality. (Lewin, 1935, p. 18).

Lewin argues that knowledge and theory might be advanced more rapidly by abandoning the common search for repetition and frequency in our world and by examining fortuitous individual special cases (p. 13). This study takes this approach, employing "purposeful sampling" (Bogdan & Biklen, 1992, p. 71; Lincoln & Guba, 1985, p. 199) in the selection of participants.

The literature reviewed in the previous chapter presents examples where teachers' absolutist conceptions of mathematics were enacted in the classroom through traditional transmissive modes of instruction. The prevalence of such situations in research reports and survey data describing the most common mathematics instructional processes suggest that small scale random sampling of secondary school mathematics teachers would be likely to generate case studies that again featured teachers with limited views of mathematics and traditional teaching styles. Such a study would not contribute much that was new to our
understandings of mathematical beliefs and teaching. Case studies that focus on less representative special examples from a population can be enlightening. These purposefully selected cases, by their contrast to the more general, may cast light on the larger situation and "reveal the properties of the class to which the instance being studied belongs" (Guba & Lincoln, 1981, p. 371). "The importance of a case, and its validity as proof, cannot be evaluated by the frequency of its occurrence" (Lewin, 1935, p. 42).

This research consists of two case studies involving exemplary teachers of secondary school mathematics. Here the adjective "exemplary" is employed with the meaning given in The Random House Dictionary (Stein et al., 1967, p. 498), "serving as an illustration or specimen". These teachers are examples of those professionals who are attempting to implement classroom practices consistent with the reforms outlined in the documents from the NCTM and OAME. Following the hypothesized connection between teachers' conceptions of mathematics and instructional practice (Ernest, 1989; Lerman, 1983) it was anticipated that, if teachers holding fallibilist images of mathematics are to be found at the secondary school level, they are most likely to be among those employing non-traditional teaching methods. Working with exemplary mathematics teachers, as the study results will show, does not guarantee the finding of social constructivist subject images, but the chances are likely to be increased.

These teachers' purposeful selection for this research was based upon their reputations as users of non-traditional instructional methods. Limited prior contact and observation had
revealed that in their classrooms students developed concepts through investigations, and that mathematical reasoning was encouraged through student discussion. The competence of both teachers had been recognized by their school and board administrations and peers. Both at times had acted as mentor teachers for pre-service teacher education programs.

These two teachers were approached and after extensive discussions concerning the purposes and nature of the project agreed to participate. Since part of this study involved observing classes taught by the participating teachers, permission to conduct such research was obtained from the administrations of the boards and schools in which the participants were employed.

Jonathan Ode (all names are pseudonyms) is the more senior of the two teachers, having considerable work and educational experience prior to beginning secondary school teaching. Jonathan began university studies in engineering but left after two years to enter the military where he received training in electronics technology. After five years of military service, Jonathan returned to university completing a B.Sc. degree in mathematics and physics and continuing with four years of graduate school, receiving masters degrees in both philosophy and mathematics. Jonathan completed education courses through summer programs during his first three years of teaching. He has taught mathematics for 23 years and held the position of Assistant Department Head in large secondary schools (1500 plus students) in a suburban region of a large metropolitan centre. During his teaching career Jonathan has written articles in professional
journals, made presentations at teachers' conferences, and participated in curriculum development activities.

Randy Walker has taught mathematics, physics and general science for 17 years and been the Assistant Head of the mathematics department in small (200 pupils) and medium sized (1000 pupils) secondary schools located in an industrial city with a population of 160,000. Randy began university studies in engineering but after two years switched to science, completing a degree with a major in mathematics. This was followed by five years employment in a variety of jobs, the longest lasting (3 years) as a management trainee in banking. Deciding that he wanted to be a teacher, Randy returned to university for a year and received a B.Ed. degree in mathematics and physics teaching. Recent personally directed exploration and study has lead to the development of materials for the high school teaching of the mathematical topics of fractal geometry and chaos. Using his teaching materials, Randy has conducted teacher workshops and made conference presentations in a variety of Canadian and US cities.

Data Collection

Qualitative research is essentially an investigative process, not unlike detective work. (Miles & Huberman, 1984b, p. 37)

Explorations in the two strands of the research: beliefs and teaching practice, required use of the full range of data collection techniques associated with qualitative research. These included: semi-structured interviews to explore beliefs,
with the structure provided by participants' previously developed writings, repertory grids and concept maps, 'participant observation' of lessons, and the collection of documents employed in teaching and the products of student work. The materials collected or generated during these activities were stored in the form of: document files, audio tapes of interviews and lessons, and field notes recorded in log books. Extensive contact with the teachers involved also provided opportunities for informal discussions of issues related to the teaching of mathematics and schooling in general. Field notes recording the substance of such discussions were kept. Involvement in these teachers' daily school routines provided opportunities to experience their interactions with other professionals during departmental meetings and in less formal social gatherings in staffrooms and lunch rooms. While intrusive data collection techniques, such as audio taping, were not employed during the participants' contacts with other teachers, field notes of such occasions were kept.

To keep track of the data, all contacts with participants and materials collected or generated were assigned codes (Miles & Huberman, 1984b) and recorded in ongoing data logs (see Appendix A for data coding scheme and data logs).

The research consumed considerable amounts of the participating teachers' time and was rather invasive of their lives. To avoid creating undue pressures that could lead to hurried casual treatment of the research activities, or possibly create participant resentment of the project, visits to the schools were arranged at the teachers' convenience. Classroom practice data was collected over a four month period while the
activities related to gathering information of the participants' images of mathematics were spread over 10 months.

Until the concluding data analysis the two cases were addressed separately but data collection in both proceeded simultaneously. Thus, while breathing spaces were provided for the teachers involved, data collection and recording required the full-time attention of the researcher. While the two studies proceeded in parallel, there was no need to have activities follow a common timetable. Scheduling decisions were made with the primary purpose of generating maximum participant comfort.

The extended data collection period had benefits beyond participant convenience. Prolonged engagement at a research site and the opportunity to repeatedly sample over an extended time increased the internal validity and reliability of the study (Lincoln & Guba, 1985).

The following sections provide descriptions of the procedures employed in the data collection within the study's two strands: conceptions of mathematics and teaching practice.

Conceptions of the Nature of Mathematics

There is no clear window into the inner life of a person, for any window is always filtered through the glaze of language, signs, and the process of signification. And language, in both its written and spoken forms, is always inherently unstable, in flux, and made up of the traces of other signs and symbolic statements. Hence there can never be clear, unambiguous statement of anything, including an intention or meaning. (Denzin, 1989, p. 14)

Conceptions of the nature of a discipline, being a component of one's belief system, are subjective and difficult to access.
The language potentially employed by research participants to articulate their personal subject images may use common words, but with differences in personal meanings, the researcher may misinterpret the communication (Rose, 1962). In addition, participants themselves may not be fully aware of their personal points of view or may not have readily at hand the words or 'significant symbols' (Mead, 1934) to present their meanings. Thus they may require assistance in exploring and articulating their personal knowledge, but any aids provided have the potential to impose external meanings. While the possibility of misinterpreting research participants' words and symbols can never be totally eliminated, there exist procedures to support the exploration of personal meanings that can reduce this danger. Three of these: reflective writing, repertory grid technique, and concept mapping were employed in this study.

Researchers (Aguirre, Haggerty & Linder, 1990; Beyerbach, 1988; Easterby-Smith, 1981; Morine-Dershimer, 1991) in other studies have employed these techniques in a single step manner, taking the participants' meanings directly from their writings, repertory grids, or concept maps. Such an approach substitutes new possibilities for misinterpretation in place of potential confusion over spoken words. By combining the interpretation of personal writings, repertory grids, and concept maps with interviews, the danger of false interpretations can be reduced. In this study, interviews were scheduled with each participant after they had completed each task: writing, constructing a repertory grid, and drawing a subject concept map. Their products from the respective tasks were used as the focal points
of these interviews where I and the teacher collaboratively analysed their writing, grid, or map. In this manner the interviews were given structure, but this structure was one provided by participant rather than researcher. Following the advice of Seidman (1991) all interviews were kept to less than 90 minutes in length and, in fact, no session exceeded an hour (55 minutes maximum). Interviews were audio taped and later transcribed.

The use of four data sources: personal writing, repertory grid, concept map, and the accompanying interviews, provided triangulation in this study: triangulation, not to just provide verification but in the sense of Mathison (1988) as a means of lighting a phenomenon from multiple angles and thus revealing more and better detailed evidence.

**Personal statements: Images of mathematics.**

During the first meeting, each participant was given a request to write a brief personal statement concerning the nature of mathematics, the sources of mathematical knowledge, and the measures of mathematical truth (see Appendix B for a copy of the instructions to participants). To avoid putting pressure on the participating teachers no deadline for completion of this task was set. Once a participant's response to this task had been received he and I set a time for an interview to examine his statement. The intent of this interview was not to test or challenge the teacher, but to collaboratively explore the meanings of his words to ensure that my interpretation was
essentially correct and to give the participant an opportunity to expand upon any of the ideas expressed in his paper.

**Repertory grids.**

Personal writing provided an opportunity for each teacher to set out his views on mathematics, but without probes to initiate deeper investigation it was possible that underlying meanings would be missed. Repertory grid technique can serve as such a probe to, "elicit, systematise and exhibit personal meanings" (Thomas & Harri-Augstein, 1985, p. 260). In this study participants constructed grids that focused on their conceptions of school subjects. By having each participant compare and contrast subjects, including mathematics, features of their image of mathematics were revealed.

The development of a grid begins with the identification of a set of elements, in this case 8 school subjects, including: mathematics, English, physical education, a science, a modern language, an art, a subject from the social sciences or humanities, and a commercial or technical subject (see Appendix C for instructions to participants concerning the selection of subjects and examples of displays of repertory grids).

Elicitation of constructs proceeded by presenting to the teacher random selections of three elements and having him describe some way in which two are similar and different from the third. This description and its opposite were used to generate a scale upon which the elements were arranged. Elicitation
continued until the constructs revealed began to overlap and became redundant.

The computer program, RepGrid (Shaw, 1990), was employed to provide an analysis of the arrangements of elements and constructs. Of the various displays generated by this software (see Appendix C), that of the principle components (PrinCom) is the easiest to read and was used as the focus of subsequent interviews.

On the PrinCom display the elements (subjects) are arranged according to the descriptors provided by the constructs. Subjects that have descriptions that are similar are plotted in close proximity. The constructs appear as lines with the degree of collinearity between two lines being a measure of the constructs' dependence. Thus perpendicular constructs may be regarded as independent. In addition, the length of a line is an indication of the polarity of the construct. Those constructs for which subjects were clustered at the extremes of the scale are displayed with extended lines. The PrinCom display identifies those subjects that carry similar descriptions and records the qualities attributed to each discipline.

In the subsequent interviews, participants were asked to react to their grids, and to elaborate on any information with which they agreed or disagreed. In this conversation new and hidden features of their images of mathematics surfaced.
**Concept maps.**

In constructing concept maps one externalizes personal frameworks of knowledge. In this process, developed by Novak and Gowin (1984), one begins with the title of a knowledge area (mathematics) set on the mapping surface and around this are placed the labels of concepts that are components of this area. A network of lines is drawn between these concept labels to indicate the relationships between them and their hierarchy (see Appendix D for an example of a map and the instructions given to the study participants).

This study employed concept mapping in its least restricted form, asking participants to generate both the concepts to be mapped and the connections to be shown. Novak and Gowin (1984) note that users often have difficulties finding appropriate labels for the connecting lines. Anticipating that such a problem might arise in this study, participants were encouraged to label connections, but this detail was not considered essential. Mapping a large knowledge domain such as mathematics, especially when one has considerable experience with the subject, is likely to be a time consuming task. Participants were encouraged to tackle the problem in parts and to take as long as required. No deadline was set for the completion of the map.

Once a participant's map was, in their terms, complete, an interview, in which he and I would explore its content and organization, was arranged. The maps and subsequent interviews provided pictures of what within the discipline was important to
each teacher and give some sense of how each saw the subject growing.

Mathematics Teaching Practice

The task of accounting for successful instruction is not one of explaining how students take in and process information transmitted by the teacher. Instead, it is to explain how students actively construct knowledge in ways that satisfy constraints inherent in instruction. (Cobb, 1988, p. 87)

A sociological and epistemological perspective, as described in Chapter 2, guided the gathering of data on the teachers' instructional practices. That is, the focus was on the processes by which pupils constructed mathematical knowledge and the teacher's role in this, in: planning, creating supportive environments, setting tasks, grouping students, and orchestrating discourse.

During whole class activity the interactions between students and between the teacher and students are observable from a position at the back of the room. This is not true when students are engaged in individual or small group work. In these situations it is necessary for a researcher to move into the class to closely observe student and teacher activity and to potentially interact with pupils. This has the potential to disrupt the regular lesson flow unless pupils see the researcher's presence as natural. Efforts were made to ensure that this was the case.

I was introduced to each class by the participating teachers, as a high school mathematics teacher who now works at a university, one who is interested in how students learn
mathematics in this course, and as a person who at times may provide assistance in the class. As such, my role approached that of the participant observer (Spradley, 1980) common to much qualitative research. I had previously visited the classrooms of both participants and had discussed my potential actions with them. Both teachers appeared to be comfortable with the planned approach.

A minimum of 20 lessons led by each teacher were observed during the course of the semester. This provided opportunities to participate in classes addressing a variety of topics and grade levels. Whole class instruction portions of lessons were unobtrusively audio taped and illustrative segments later transcribed. Extensive field notes recording: classroom arrangements, questions posed, responses obtained, locations of interacting pupils, and interval times for various activities were kept. In addition, handouts used during the lesson and copies of representative student products were gathered.

Two further research activities served to link images of mathematics to practice. As often as possible, within the busy schedules of the participating teachers, short interviews were conducted to obtain the reasons behind their instructional decisions: Why did they organize the lesson in this way? Why was a particular task employed? Why were certain materials and resources used?. In addition, during the time I spent with the teachers, observations were made of the resource materials they consulted and conversations they had with fellow professionals and students outside regular class time.
Data Analysis and Reporting

'It seems very pretty,' she said when she had finished it, 'but it's rather hard to understand!' (You see she didn't like to confess even to herself, that she couldn't make it out at all.) 'Somehow it seems to fill my head with ideas - only I don't exactly know what they are!' (Carroll, 1954, p. 130)

Although Jonathan and Randy have in common their reputations as teachers contributing to mathematics education reform, they are still distinctly unique individuals. Anticipating that their conceptions of mathematics and the instructional practices they employed would be equally unique, the initial data analysis was conducted on the two cases separately.

Once the cases were developed, a second level analysis was conducted to search for common themes. These are reported in a concluding chapter of the thesis along with implications derived from common trends.

Analysis

Most conclusion-drawing tactics amount to doing two things: reducing the bulk of data and bringing a pattern to them. Such tactics are sometimes rationally trackable, sometimes not. (Miles & Huberman, 1984a, p. 27)

A "rationally trackable" tactic for data analysis, one following the methods described by Bogdan and Biklen (1992) and Miles and Huberman (1984b) was employed. Initial data analysis was conducted as materials were collected.

Beginning with the data connected to a single participant's image of mathematics, all items were examined for the existence of frequently occurring ideas or themes. Theme labels were
developed and attached to the various 'conceptions of mathematics' articles in the data. Once the themes apparent in the participant's conceptions of the discipline had been identified, the data related to practice was examined for incidents of the same ideas. Although analysis began with beliefs, if a new re-occurring idea was located in the practices material a new label was created and the conceptions of mathematics data re-examined for incidents of the same theme.

Through repeated cycles of the analysis process it was possible to locate themes that existed in both data strands: subject image and instructional practice. Themes that did not fully find development in both strands were also noted since they marked cases where beliefs were not appearing in practice.

**Reporting**

One starting point of almost all research on teacher thinking has been the concern for the tacit aspect of teachers' knowledge and for the paradox implied by this quality; but while knowledge must be made explicit if the teacher's voice is to be heard, we thereby risk turning teachers' knowledge into researchers' knowledge, colonizing it, and thus silencing the voice of the teacher. (Elbaz, 1991, p. 11)

The results of the data analysis are reported in two chapters, each dealing with just one case. For each case, the report consists of sections alternately presenting a theme from the teacher's conception of mathematics and a narrative of illustrating classroom practice. In each case this pair is related but the link may be either an example of belief in action or of the teacher's struggles and incomplete transfer of image to pedagogy. As much as possible the participants' own words are
used to present their visions of mathematics. In the concluding chapter themes that cross both cases are addressed.

**Corroboration**

We can argue for negotiated outcomes on the ground that such negotiation is essential if the criteria of trustworthiness are to be met adequately. We shall see that a major trustworthiness criterion is *credibility* in the eyes of the information sources, for without such credibility the findings and conclusions as a whole cannot be found credible by the consumer of the inquiry report. (Lincoln & Guba, 1985, p. 213)

Within the limits of the participants' schedules, efforts were made to collaboratively develop, meanings and interpretations of the data. This process, by the research design, took place within the investigation of subject image with interviews to explore and elaborate on the participant's writing, repertory grid and concept map. In these interviews there were opportunities for the teacher to point out problems with my interpretations and to provide alternative meanings. Vignettes illustrating classroom interactions were provided to each teacher for his feedback. Such participant commentary not only helped verify the account, but also addressed issues of confidentiality.

Walker (1980) notes that in case studies, confidentiality can not be guaranteed. In delivering 'thick' descriptions of participants and locations, the study makes cases identifiable even when pseudonyms are employed. Walker suggests that this issue be addressed, not by promising the impossible, but by admitting that anonymity can not be ensured. An alternative form of protection was provided for the participants by continually involving them in negotiations of meaning and data release.
Let us...
Expatriate free o'er all this scene of man;
A mighty maze! but not without a plan;
(Pope, 1733/1846, p. 10)

Having completed, as a minimum, university honours degree programs in mathematics and physics, both Jonathan and Randy have participated in significant formal studies of mathematics. This focussed academic experience with the discipline has been, for both, augmented by periods of employment in military, industrial or business settings requiring the use of mathematical skills and knowledge. These opportunities, coupled with self-initiated reading, personal independent study, and reflection on teaching during at least 17 years in the classroom, have contributed to the development of multidimensional conceptions of mathematics. Similarly, reading of the mathematics education professional literature and regular attempts to put new ideas into practice mean that both teachers have developed multifaceted styles of teaching. The structure of the cases set out in the following two chapters was developed to show this complexity and highlight both connections and missing connections between conceptions and practice. This section is designed as an orientation for a reading of the two cases and provides an outline of their organization.

Multiple readings of the data related to the teachers' images of mathematics led in both cases to the identification of a number of re-occurring themes. With these themes in mind the field notes, gathered materials, and audio tapes of the observed
lessons were examined and teaching incidents that conveyed compatible or contrary messages were noted.

The two case chapters are organized in alternating subject conception-teaching practice units. That is, each case, after opening with an introductory view of classroom practice, follows the pattern of a section presenting one aspect of the teacher’s image of mathematics followed by a unit giving a picture of instruction that appears to either support or disagree with the identified belief.

**Composing a Picture of Subject Vision**

The participants, through their writing, subjects repertory grid, concept map for mathematics and interviews focussed on these, articulated their images of mathematics. Although the sequence of particular statements has been altered to permit the thematic grouping of evidence, it is the teacher's voice, their words and products, that is used to paint a picture of their conception of mathematics.

Individual image themes arose repeatedly in interviews, informal discussions, and in the teacher products: writing, repertory grid, and concept map. These are gathered together to develop a particular aspect of the teacher's vision of mathematics, but in each case the original source is given by a data code appearing at the end of the quote or description (see Appendix A for the coding scheme and data logs).
**Telling the Story of Teaching Practice**

Teaching practice is described through narratives of selected classroom events, those most closely related, either positively or negatively, to subject images. To give a sense of "being there" in the classroom, these stories are told in the present tense. In this way the reader can see the event unfolding and gain a sense of the mathematical epistemology being displayed by the teacher's actions and words. Images and text displayed on the classroom blackboards or overhead projector, or appearing in student handouts are reproduced in the narratives whenever essential to the understanding of the story. In the class dialogues the teachers are identified as Mr. Ode and Mr. Walker, while the generic terms "pupil", "student" and "class" are employed to identify other speakers. Student names (pseudonyms) are used only when there is a continuing conversation with or about a particular pupil.

Teaching involves much more than conducting lessons. It is the choices made during course and lesson planning that determine the activities and thus the epistemology revealed through individual classroom events. In the cases that follow, resource materials consulted by the teachers are identified and their thoughts in planning for and reflecting on lessons, as revealed in interviews and discussions, are given. In activities and conversations outside the classroom the participants are identified by their given names, Jonathan and Randy.
Chapter 4: Case 1: Jonathan Ode: Forging Ahead on the Reform Path

For Jonathan Ode, mathematics is an intellectual activity; one that involves higher order thinking processes rather than the routine application of procedures (JO-DIS-N-31). Doing mathematics means reasoning, communicating and problem solving, and Jonathan's lessons are designed to promote these activities. In all but two (JO-LES-N-18, JO-LES-N-19) of the 20 lessons I observed there was evidence of at least one of the NCTM's first four standards: problem solving, communication, reasoning, and connections. In most classes, students, in pairs or small groups, worked collaboratively on problems or mathematical investigations. While Jonathan played a strong leading role in five teacher centred lessons, he still managed to present reasoning as the core of mathematics. Here in a highly Socratic fashion, concepts were developed through the observation and generalization of patterns.

Graduate work in both mathematics and philosophy has provided Jonathan with opportunities to formally study logic and the foundations of mathematics. This rich academic experience showed through in his ability to articulate his conception of the nature of mathematics. But, Jonathan's mathematical background is not all within the domain of pure mathematics. His more practical problem solving experience in engineering studies and as an electronics technician were also evident in his descriptions of the subject.
During the school semester of this study, Jonathan taught three classes per day: one at Grade 9, the first year of secondary school, and two in the Finite Mathematics, Ontario Academic Course (OAC), for graduating-year students who are intending to pursue university based post-secondary study. The following vignettes of classroom interaction describe segments of lessons from both of Jonathan's courses.

Class participation was observed to be high during the 20 lessons attended. Most pupils appeared to be ready to tackle the mathematically rich problems and projects that Jonathan provided, but there was opposition from some graduating-year students who objected to the intellectual demands made by Jonathan's teaching style. During the course of the semester this dissention resulted in parental complaints and a caution from the school administration. Nevertheless, Jonathan forged on in the face of this opposition and continued to deliver lessons that captured the spirit of the mathematics education reform program.

*Guiding from the Back of the Room*

My first visit to Jonathan's classroom revealed a theme that was to run through the twenty lessons that I observed. One word answers are not acceptable in mathematics. In Jonathan's view a complete answer involves a carefully reasoned explanation and his classroom practices are deliberately planned to encourage such student input.

The Grade 9 students are arriving for mathematics class as Mr. Ode writes a question on the board.
Johnny says, "Integers are easy as long as you remember the rule: two negatives make a positive." Give an example of what Johnny means and explain why it is true.

Mr. Ode points out the question to each boisterous and noisy group of arrivals and invites their consideration of the problem. Finally all are in their seats and the lesson can begin. Mr. Ode, rather than assuming the usual teacher location at the front of the room, moves to an empty desk at the back of the middle row.

Mr. Ode: "What did we talk about yesterday?"
The class responds collectively with a variety of answers: "Plus and minus, positive and negative numbers, integers, adding and subtracting."

Mr. Ode: "Johnny is suggesting a strategy. What does he mean?"
One pupil provides an answer, using money, bank balances and debts as a model for situations involving the addition and subtraction of integers. This explanation is made orally to the whole class with nothing written on the board. Mr. Ode makes no comment on the answer, just calls for others to expand upon the ideas presented or to ask questions. This process continues until three students have collectively built up a strong explanation and the lack of further suggestions or questions suggests that all are satisfied.

Now it is on to the usual first task of traditional mathematics lessons, taking up the homework, but here again the style departs from the norm. Mr. Ode, still seated at the back of the room, calls for volunteers to put solutions for six homework questions (JO-LES-D-01, Bober et al., 1988, p. 44) on
the board. Students are eager to contribute. Solutions are quickly written on the board and the writers return to their seats. Now Mr. Ode calls for a second wave of volunteers to check the answer and explain how a problem was solved. As the solution to each question is explained, those at their seats ask questions or offer help whenever a presenter is stuck. Through all this, Mr. Ode's only words are repeated reminders, "Be polite", "Help out". With much cooperation and mutual support all six questions are completed and the class breaks into spontaneous applause for the presenters. (JO-LES-N-01)

Students can contribute to the collective construction of mathematical meaning only if teachers provide spaces for their input. When an instructor stands at the front of a class, delivering examples and explanations, there is little encouragement of student mathematical conversation. As in the lesson just described, Jonathan regularly steps out of the spotlight and creates opportunities for the pupils' suggestions.

*Mathematics as Reasoning and Communicating*

Jonathan does not abandon his authority as a mathematician and teacher. He has definite plans for his lessons and knows the mathematics that he wishes to develop. His choices of instructional approach are based upon an image of his subject, one aspect of which is that mathematics involves reasoned communication.
Reflecting back on the above lesson, Jonathan provides the thinking behind his approach to the perennial problem of correcting the homework.

Taking up homework is - can be one of the most useless things we do. I think in most cases it becomes a spectator sport, you know, students sit and they watch and at best they hope that they never get called on. So to make homework, putting homework on the board, of any value, what I try to do is to get at different aspects of learning. And of course one of them is communication. Putting the homework on the board itself is communication, written communication, and a lot of what I talk about when the homework goes out, is the form of that communication so that it can be more effective. But again just putting it on the board is only one dimensional because all we're really doing is looking at it. We're not peering [vocal emphasis] at it. By calling on another student, not the student that put it up on the board, to go up to the board and explain the other student's problem, we're creating a whole new environment. First of all, a student has to read another student's work and interpret that, and then describe that interpretation so that other people can understand what that student thought the first person meant. So we're actually on a couple of different dimensions at one time when we do that. (JO-INT-D-01)

Jonathan's writing about the nature of the discipline makes it clear that mathematics is communication; communication between individuals, communication of reasons along with answers. "In other words, a necessary condition for an activity being called mathematics is the ability to explain the processes used [bold in original]" (JO-DOC-D-03). Answers alone are not sufficient. "It doesn't take on any significance until they can explain how they got the answer" (JO-DOC-D-03). This theme is further developed in interviews exploring Jonathan's writing. "Mathematics has to be an active process ... Um, doing with understanding. By understanding I mean more than a passive understanding. That is, it has to be an understanding that can turn around and be communicated back" (JO-INT-D-03b).
In Jonathan's conception of the discipline, reasoning and communicating one's thinking are central activities of mathematics. His teaching strategies, as illustrated above and shown in the fifth lesson I observed, described below, are designed to develop students' abilities in these areas.

**Challenging Students to Think**

I and other experienced mathematics teachers have found that senior, university-bound secondary school students, responding to the demands of high grades, often lose their interest in working to understand concepts and demand direct failsafe routes to correct answers. Jonathan's senior classes follow this trend, but he is not about to abandon his goal of developing mathematical reasoning.

Mr. Ode's OAC-Finite Mathematics class has been studying combinatorial analysis and last night had a selection of textbook (JO-LES-D-05, Stewart, Davison, Hamilton, Laxton & Lenz, 1988) word problems for homework. Today the students unanimously report that they could not solve question 9.

In the binary number system which is used in computer operations, there are only two digits allowed: 0 and 1.

(a) How many different binary numbers can be formed using at most four binary digits (for example, 0110)?

(b) If eight binary digits are used (for example, 11001101), how many different binary numbers can be formed?

(c) A **binary code** is a system of binary numbers with a fixed number of digits that are used to represent letters, numbers, and symbols. To produce enough binary numbers to represent all of the letters of our alphabet (both upper- and lower-case), how many binary digits must be used? (p. 53)
Mr. Ode, after reading the question out loud, exclaims, "Wow! That's a difficult question. I don't know why I assigned that problem. I'm not sure that I can do it!" There is a slight pause for dramatic effect and then Mr. Ode adds, "You all worked on it last night; right? So we should have lots of ideas. We should be able to come up with a solution if we work together." With this introduction Mr. Ode sits at an empty desk, approximately in the middle of the room. "Okay, who will get us started?" When there are no immediate volunteers, Mr. Ode adds, "Mary, how about you going up to the board and recording our thoughts?" Mary moves to the front of the room and takes up the chalk. With Mr. Ode serving as chairperson, calling for more suggestions and identifying speakers, a solution is slowly built.

Actually the students do not have major problems with the computations, but they are not getting the answers given in the text. Following some of the text's worked examples, they have been led into misinterpreting the problem and seeing the binary number 0010 as different from 10. Thus for part (a) they have four cases: one, two, three, and four digit numbers, for a total of 30 binary numbers. The students protest that this does not match with the "official" answer of 16.

Mr. Ode is not willing to provide a solution. He thanks Mary for her work as class scribe and, moving to the front of the class to begin a new topic, provides the suggestion, "Remember we are talking about a number system, like our base ten system, only here we have just two digits. Think about how we write our numbers, say three hundred and twenty-five, and you should be
able to come up with a different interpretation of the problem."

(JO-LES-N-05)

As we can see Jonathan does not give in to student demands for precise algorithms to generate test and examination answers. This is a course in mathematics and thus thinking is required.

**Mathematics as Problem Solving**

Problem solving has a long history in the mathematics curriculum (Stanic & Kilpatrick, 1989) and its importance has recently been re-confirmed in the reform documents from the NCTM (1989) and OAME/OMCA (1993). For Jonathan, problem solving is not just a curricular issue. As he informs us in his words and concept map, the process of solving problems is central to the mathematical discipline.

"In my opinion there are two prominent characteristics [of mathematics]; patterning and problem solving" (JO-DOC-D-03). Multiple connecting lines in Jonathan's concept map (JO-INT-D-12a, see Appendix E) are labelled "problem solving". In discussing his map, Jonathan elaborates upon this emphasis.

I see that [problem solving] as more of what we do in mathematics as opposed to just, you know, going out and using formulas. ... To make that work you have to move into - higher level thinking processes and that's where problem solving comes in because it's not a matter of getting to an answer. It's a matter of, of creating a structure that provides answers and then looking at the reasonableness of those answers and the structure itself to the problem. (JO-INT-D-12a)

Thus mathematical problem solving involves more than generating answers. "Mathematics by its very nature is a higher level thinking process. Mathematics requires some contact with, not
just the operations used in getting the desired result, but with the process itself" (JO-DOC-D-03).

With a bit of work, I could teach one of these [Grade 7 or 8] students to perform the power rule as used in Calculus. Utilizing rote learning and a lot of repetitive examples I could train a student to answer the question; "What is the derivative of $x$ cubed?" and that student would supply the correct answer "three $x$ squared". The question is: has any mathematics occurred? I believe the answer is clearly no. The student was simply performing in a preconditioned manner. No mathematics here! (JO-DOC-D-03)

Furthermore, this understanding "must be active, that is, it must be followed by some form of communication" (JO-DOC-D-03).

Jonathan constructs lessons that are specifically designed to address mathematical understanding, communication and "the ability to solve problems",

and by that I mean genuine problem solving when they are given a problem which is sufficiently different from previously learned problems. One that they cannot immediately determine the answer to without first investigating the problem, exploring various alternatives and deciding on the best strategy. (JO-DOC-D-03)

In translating his conception of mathematics as problem solving into teaching practice, Jonathan has, over the years, read the professional literature in search of problem solving activities and ways to provide instruction in the processes involved. The techniques developed by Whimbey and Lochhead (1979), which Jonathan read about almost a decade ago, play an important role in his planning for the problem solving lesson described next.
PS News: A Sharing of Ideas About Problem Solving,
distributed by the Department of Chemical Engineering at a local university, is a bi-monthly list of resources and digest of journal articles and books addressing the development of problem solving skills. Jonathan has been receiving this publication for a number of years, filing away those articles with application to his teaching. He shows me Issue 36 (Woods, 1958 Jan.-Feb.), focussing on an article examining the work of Whimbey and Lochhead (1979) (JO-DIS-D-02a). This is the source of the Whimbey-pair method to be used during Jonathan's next OAC-Finite Mathematics class, about to begin in a few minutes.

Class starts with Mr. Ode posing the question, "Why do we go to school?" This initiates a series of Socratic exchanges through which Mr. Ode develops the idea that we need school graduates who can think and solve problems in general, not just repeat a set of fixed facts or procedures. Mr. Ode is obviously committed to this mission, developing the theme at an upbeat pace from his favourite teaching location, the middle of the classroom. The students, while politely participating in the discussion, seem to have less commitment and many appear content to learn the rules and get their credit. A brief story, about Mr. Ode's "friend", a chemical engineer who struggled with design problems until he realized that textbook solutions could not be applied directly to industrial processes, completes the lesson introduction.
Whimbey and Lochhead (1979) identify as a major source of errors, the tendency of pupils to rush through problem solving activities, performing computations with the given data without prior thinking or planning. They call upon problem solvers to vocalize their thinking, to think aloud, and encourage this by having students work in doer-listener pairs. Mr. Ode adopts this strategy to encourage attention to the meta-cognitive level.

The class is divided into pairs and assigned four textbook questions (JO-LES-D-03, Stewart, Davison, Hamilton, Laxton & Lenz, 1988, pp. 52-53). Mr. Ode outlines the approach to be used. One member of the pair, the doer, will solve the problem, telling the listener what is being done and why at each step. The listener will encourage thinking aloud by constantly asking for detailed explanations. After each question is completed the doer-listener roles will switch.

The class eagerly starts into the task. There is considerable interaction within the pairs, but often the listeners take on a coaching role rather than probing for details at the meta-cognitive level. Many still see the task as getting the correct answer and this is brought out in the summary discussion were students agree that they appreciated the help from their partners but objected to how the thinking aloud requirement slowed them down. (JO-LES-N-03)

Although the students' desire for direct procedures to generate answers works against total success for problem solving lessons, Jonathan persists. Problem solving, being a central theme of mathematics, must be addressed. As we will see in
subsequent observed lessons, Jonathan provides other structured activities exploring a variety of problem solving strategies.

**Mathematics as the Study of Patterns**

For Jonathan, the study of patterns is at the root of mathematics. When humans organize their observational data, note patterns and formulate these into a language, mathematics is born.

Mathematics is the outgrowth of a number of human characteristics. We seem to have a need to simplify things of a complex nature, we need to organize things. Another characteristic is the desire to think beyond the immediate empirical data of his perceptual world to attempt to gain an understanding of a more global nature.... It is a mistake to separate mathematics from our perceptual world. There is a rudimentary connection between doing things and thinking about things. (JO-DOC-D-04)

Looking at the history of mathematics, Jonathan sees this human desire to summarize and extend observed patterns as the driving force behind the development of the subject. "I think most people need to have some sort of motivation for doing mathematics, so they probably would be involved in some kind of a process that would be experimental in nature and that would give them incentives to go beyond [the data]" (JO-INT-D-12b). The mathematics that we develop through observation of patterns allows us to move to a more conceptual level and extend our world.

You're taking what you have and you're looking, kind of pushing it together to see what's there, to find the patterns and then the patterns allow you to extend beyond what you have as - as the given data. And, I think that's a lot of what we do is - is we want to say well what if, and that's kind of our - part of our creative drive. We want to extend beyond what we have
right in front of us and say well, if this situation varied what would happen. (JO-INT-D-08b)

In Jonathan's vision of mathematics the development of the discipline is collaborative, with the subject falling in the cooperative half of the principal components (PrinCom) display of his school subjects repertory grid (JO-INT-D-04, see Appendix F). As we have seen in the previously described lessons, Jonathan translates his vision of mathematics as an interpersonal activity into whole class discussions and collaborative pair work. This enacting of subject image continues in the next reported classroom experiences where the Grade 9 students work in pairs and groups of four to explore patterns and solve problems.

**Organizing Data for Patterns and Problem Solving**

Jonathan's commitment to an image of mathematics as the study of patterns and problem solving extends beyond his own classroom. He has taken on leadership positions in his school system and in these roles encourages other mathematics teachers to provide students with collaborative investigation experiences.

On my second visit to Western Secondary, Jonathan is running around, gathering materials for his next period Grade 9 class, and performing his Assistant Department Head duties, encouraging his fellow Grade 9 teachers to adopt some non-traditional instructional approaches. He is frustrated. There are so many good ideas and resources in the journals, especially the NCTM publications, but he can not keep up with the reading and his filing system is falling apart (JO-DIS-N-02). Finally Jonathan locates what he is looking for, an article from the *Mathematics*
Teacher (Krulik & Rudnick, 1985) that provides patterning and problem solving activities. We make copies of the student materials for class and copies of the full article to distribute to the other teachers. (JO-DIS-D-02b)

Krulik and Rudnick's activities illustrate how recording data in charts or tables can aid the search for patterns. The authors supply ready made charts for each problem, but this is too much guidance for Mr. Ode's liking. He would like the students to come up with the chart idea, so before the student worksheet pages are distributed the class explores question number one. The problem scenario is acted out. Five game chips, labelled 2, 1, 5, 3, and 6 are put into a brown paper bag, with Mr. Ode showing each in turn to the class. "Now we will draw out three chips without looking." A volunteer draws the chips 1, 5, and 3 and writes these three numbers on the board. "We will add these together and call it the score. What is the score this time?" The student records 9 beside his three numbers. The chips are returned to the bag, another volunteer selects three, and records the numbers and the score on the board near their seat. "How many different scores can we get?" A variety of answers are given. "We have two draws and scores. How might we record our work to help us figure out the possible scores?" A couple more draws and recordings are made before one student suggests that the results all be gathered on one board. Mr. Ode copies the second, third and fourth selection data below the first. "Any other suggestions about how we might keep a record?" A couple more draws are made and recorded below the first four but there is no suggestion to make a chart and look for
duplications. Some students are getting restless. Sensing that
the whole-class lesson is taking too long, Mr. Ode cuts short the
exploration and presents an organizing scheme by adding lines and
column labels to produce a chart. Mr. Ode explains how recording
information in chart form can make patterns more obvious and
possibly help us solve problems. This is the central idea in the
problems for today.

Student pairings are determined by drawing cards and
matching: 2 of hearts with 2 of diamonds, 5 of spades with 5 of
clubs. The worksheet pages are distributed and the pairs get to
work. There is strong interaction within the pairs, sharing
solution ideas, conjecturing and justifying answers. With ten
minutes left in the class, Mr. Ode breaks into the activity and
initiates a whole class summary. "What did you do in these
problems? How did it help?" Mr. Ode wants to focus on general
strategies, but the lively discussion soon turns to a debate over
the answer for the first problem on the second sheet.

Helen Chen wants to seed her front lawn. Grass seed
can be bought in 3-pound boxes that cost $4.50, or
5-pound boxes that cost $6.58. She needs exactly 17
pounds of seed. How many boxes of each size should she
purchase to get the best buy? (Krulik & Rudnick, 1985,
p. 696)

A chart, with columns for: numbers and cost of each size of box,
total cost and total weight of seed, is put on the board. Five
pairs contribute entries to complete the table but there is still
debate as to the best buy. Would this mean the lowest price or
the least amount of wasted seed? Mr. Ode encourages the debate,
asking for reasons for each position, and lets the class end
without a definitive answer. (JO-LES-N-04)
During the term, mathematics as the study of patterns and the value of patterning in mathematical problem solving appear as themes for several lessons. Two months after the above lesson the grade 9 class experiences another problem solving activity that re-emphasizes the need for an organized approach and the value of charts and tables. This time Jonathan selects work from the "Number Patterns" section of *Get It Together* (Erickson, 1989, pp. 132-137), a book of problems designed for groups of four.

Here cooperation and sharing are encouraged by providing the essential information on four cards, one for each group member. Each person is responsible for communicating their information to the group and ensuring that any suggested solutions fit their data. The group task in each case (JO-LES-D-17b, see Appendix G) is to develop a process for generating the numbers and to project it forward or back in time. (JO-LES-N-17)

As we can observe, students are expected to be active in Jonathan's classroom. His instructional methods call for pupils to explore, conjecture about, and discuss mathematical patterns. In Jonathan's image of the discipline, mathematics has historically emerged from such human activity and he provides parallel experiences from which his students can develop their mathematical understandings.

**Conflict: Just Give Us the Rules**

While the student activities in Jonathan's classroom: investigations, mathematical reasoning and debate, solving original problems, and collaborative group work, are all found in
the recent reform literature (NCTM, 1989; OAME/OMCA, 1993), not all those involved in the educational enterprise support such teaching (Jackson, 1997; Mathematically Correct, 1996). Students, especially those in the senior years, more interested in high grades than mathematical understanding, desire precise instruction and closely prescribed tasks (Colgan & Harrison, 1997). They wish for assurances that studying will generate good test and examination scores. Parents, with high expectations for their children and memories of their own mathematics education, expect instruction in well defined techniques not open-ended mathematical exploration. Past experience in mathematics class has taught many of these students and their parents that mathematics is essentially just a set of rules and thus direct transmissive teaching makes sense. Jonathan, in bringing his different image of mathematics to the classroom, is swimming against a strong current.

When I arrive for my fourth visit at Western Secondary, Jonathan is obviously frustrated with recent events. There have been complaints concerning his teaching. Some of the OAC-Finite Mathematics students, having received rather low marks on a recent test, have been explaining their poor performance to parents. Rationalisations have focussed on the style of Jonathan's lessons. Jonathan has not talked directly with parents, but using information from the school administration, he constructs his version of the student complaints. "He expects us to do all the work and will not tell us exactly how to do the questions." "He does not tell us the correct answers." Citing the previously described (JO-LES-N-05) examination of question 9
from page 53 of the text (Stewart, Davison, Hamilton, Laxton & Lenz, 1988), some students have suggested to parents that "Even Mr. Ode does not know how to do the questions!"

The vice-principal, having received some phone calls, has talked to Jonathan. Yes, he has complete faith in Jonathan's abilities to do the problems. His reputation as a strong mathematician ensures that. And yes, Jonathan is probably using an excellent approach to get students to think and really understand, but please tone it down. Parents, students, and the public expect mathematics to be delivered as a set of rules.

Publicly, Jonathan displays a casual disregard for the complaints and the administration's requests. Such things are to be expected and they don't bother him. After all, with 23 years of service his job is secure and he is not trying to gain points for a climb up the administrative ladder. Still, our talk of teaching and learning that fills the rest of his first period spare are more strained than usual. Jonathan provides defences for his style, citing his work with other risk-taking teachers and contacts beyond the school system with those teaching university mathematics. (JO-DIS-N-03)

The second period OAC-Finite Mathematics class begins with Mr. Ode writing on the board, in very neat printing, a list of objectives for the next unit on binomial expansion; dealing with $(a+b)^n$. The recent complaints are obviously having an effect on planning and teaching style. A particular example is developed through a rather mechanical Socratic lesson. The class is attentive, but participation has narrowed, with most answers coming from one single student. The rule for the $r$-plus-first
term is produced, but still Jonathan strives for understanding. "Don't memorize this. - There is a pattern here. - You know this already and can always figure it out." (JO-LES-N-06)

The criticism of his classroom practice is obviously a setback, but I get the sense that Jonathan will be back to his former teaching style in little time. Mathematical puzzling is too much a part of his own life for him to let it go. As we wait for his next class, the Grade 9's, to arrive, Jonathan shows me an interesting question that he presented to them a few days ago.

What would the value be if we continued this fraction for ever?

\[
\frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}}
\]

Jonathan tells me, "I left the puzzle with them for a few days and then suggested that they look for some patterns and make a prediction. Some of them are still working in it. I want to see what they come up with" (JO-DIS-N-04).

Mathematical Growth: Experiments and Induction

As we have seen, mathematics, according to Jonathan, is constructed by humans as they solve problems and explore patterns, problems and patterns that exist in their experienced world. Thus mathematics is rooted in human experience. From his years in graduate school, where he worked with highly abstract
aspects of the discipline, Jonathan appreciates the power of mathematics to formally envision other worlds. Still, as he tells us in the following paragraphs, the starting point for mathematics is personal experimentation.

There are two different sides to math. One is the analytical side and that's where the mathematics is produced out of its own ideas and the other side is the uh, I guess you could call synthetic math, which is the mathematics that's used in the production of certain things or in, in the quantification or prediction of certain kinds of events. (JO-INT-D-08b)

When constructing his school subjects repertory grid Jonathan attached the attributes, conceptual, scientific and uses experiments, to mathematics. The principal components (PrinCom) display of his grid (Appendix F) places mathematics between these constructs and when discussing this display, Jonathan notes a tension in his image of the subject. The discipline is rooted in and built from experience, that is experimental, but it can also be extended through purely mental activity.

Maybe what it's [the PrinCom display] saying though is that there, there are two, two of my images of mathematics, which are - maybe are in fact in conflict with each other. [pointing to the label, 'uses experiments'] More of the, the constructivist approach in terms of concrete. You work from experiments - scientific, and up here [pointing to the label, 'conceptual'] we have the, maybe the more pure mathematics approach. (JO-INT-D-09b)

The conceptual and experimental aspects of mathematics are also highlighted in Jonathan's concept map for the subject (Appendix E). Here, these labels along with "visual" appear in large arrows indicating the routes by which mathematics grows. Jonathan reports that "I tried to make it [the concept map] in terms of what I'm doing as a high school teacher." While the map pictures how an individual's mathematical knowledge grows it also
represents the historical development of the discipline. "I think historically it happens in the same way that one may think of the learning development process. You work your way up through the concrete into the conceptual level, so I think that it's natural to do things in a concrete way first" (JO-INT-D-12b).

I have seen lots of teachers who, for instance, will leave out the experimental part. They get up there. they have well-designed, well-planned lessons with well thought out responses in advance and they teach the lesson and pull out the responses that they want to get and then they leave. And they've left out an important part of the process and they lose a lot of students because of that. (JO-INT-D-12b)

In many lessons, including the next to be related, Jonathan follows his image of the development of mathematics and employs an experimental approach, building abstractions from examples.

Working from the Specific to the General

Jonathan's personal philosophy of mathematics is strong enough to support curriculum planning that runs counter to the sequence suggested by officially approved course textbooks. As we shall observe, he rejects a text's formal abstract approach and, implementing his view that mathematics is constructed inductively, helps his students develop a formula through sequences of concrete examples.

When I arrive for my seventh visit to Western Secondary School, Jonathan is getting ready for his next lesson, OAC-Finite Mathematics. The class has been studying probability and the next topic is hypergeometric distributions. Jonathan shows me
the section in the text (JO-DIS-D-11, Stewart, Davison, Hamilton, Laxton & Lenz, 1988, pp. 246-247) where the authors begin the presentation with,

Suppose we have a outcomes that would be classified as successful outcomes and b outcomes that would be failure outcomes. If a stochastic process of n trials involving sampling without replacement took place, then the probability of x successes in the n trials is given by

The Hypergeometric Distribution

\[ P(X = x) = \binom{a}{x} \binom{b}{n-x} / \binom{a+b}{n} \]

"I could do it by this way, but it does not seem to make much sense. I have never taken a formal course in stats or probability and it's easier for me to understand if the idea is developed from patterns" (JO-DIS-N-11).

As the class begins Mr. Ode explains that they will not be following the textbook exactly for this next section. "We could do it the way the text does, but in the long run I think that you will find it easier to do it this other way. We will come back to the textbook examples later." A month has passed since the parental complaints incident, and Mr. Ode is moving back into his preferred teaching style. Still he sees a need to justify deviating from a "give the rule" approach.

The Socratic lesson that follows is strongly teacher led, but it does demand considerable student attention and thought.
Mr. Ode draws a picture on the board; a jar holding five red blocks and three blue. "We are going to draw out blocks, one at a time. We can model this here with this big beaker I borrowed from the science department and these blocks." Mr. Ode holds up a large beaker and adds the eight blocks. "The first thing you do is you reach in and randomly select one of these. How big is your sample space?" The answer "eight" is given. "Let's say we are talking about the probability of selecting a red one." Mr. Ode takes out a red block. "How big is your event space?" "Five", comes back as an answer. On the board, Mr. Ode begins the construction of a probability tree diagram, with two branches labelled R and B. "What is the probability here [pointing to the R]?" Mr. Ode writes 5/8 on the red branch in response to the answer "five out of eight". The red block is returned to the beaker. "Now instead we are going to try to get a blue one. What is the event space, how big?" A student gives the answer "three" and Mr. Ode writes the probability 3/8 on the B branch of the tree. "And there's your first experiment." The blue block is returned to the beaker and Mr. Ode moves around the class with students taking out and retaining blocks. "What happens to the sample space as we perform this experiment?" Some of the students have noted the change from the situations examined during the past few lessons and one responds with, "It decreases because you don't replace what you take out." "Exactly!", replies Mr. Ode.
Mr. Ode leads the extension of the probability tree through three draws, asking for sample and event space sizes and probabilities for each branch. Eventually the board contains the following diagram.

Now the stage is set for a look at a particular case. Mr. Ode continues, "First we need to define a random variable. What do you want red or blue?" The popular answer is red. On the board Mr. Ode writes "$X =$ the number of red blocks drawn in 3 trials". This is the first appearance of a variable. "What is the probability that $X$ is equal to three? What question did I just ask?" A chorus of answers gives, "That there are three red blocks."

Mr. Ode: "Good, we are going to take three red blocks and not replace any as we go along. Which branch are we referring to here [pointing to the tree diagram]?" Again there is a chorus of correct answers; "The top one."
After a brief pause to let everyone catch up with the development so far, Mr. Ode begins a sequence of numerical manipulations. The writing of each line on the board is accompanied by an explanation.

\[
P(X = 3) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5 \times 4 \times 3}{8 \times 7 \times 6} = \frac{3 \times 2 \times 1}{8 \times 7 \times 6} = \frac{5 \binom{3}{3}}{8 \binom{8}{3}}
\]

By the end of this sequence the students can see where Mr. Ode is going and help him with the last line. "That's the same as five choose three and eight choose three."

Some students protest that this type of expression will not work in every case and possible problem situations are given; "Not if you have ten trials." "What if we're not looking at all red ones?" This is exactly the lead that Mr. Ode is looking for. He takes up the last suggestion. "What if we are looking at two red ones and one blue one in three draws? Let's look at this question next." With the students supplying the details of possible routes through the tree and the probabilities on each
branch, Mr. Ode creates a sequence parallel to the development in the first question. This ends in the expression,

\[ P(X = 2) = \frac{\binom{3}{1} \binom{5}{2}}{\binom{8}{3}} \]

Together the class identifies the origin of each number in the expression and Mr. Ode adds the labels: number of reds, number of red chosen, number of blues, number of blues chosen, total number, and total number of trials.

A pattern has been noted here, but does it work in the first case? Mr. Ode raises the question. "What we are trying to do is develop a pattern that would work here [pointing to the work for the second question] and here [pointing to the first question]. But something looks wrong here [again pointing to the expression developed for the initial question]." The numbers in this expression are labelled: total number of blocks, number of trials, five red blocks, number of red blocks selected. "But what happened to the blue blocks?" "We did not choose any" is the response from one student.

Mr. Ode continues, "So in terms of choices that's -.." This generates a chorus of, "Three choose zero."

Mr. Ode: "And what is three choose zero?" "One", comes back from the class.

With this information, Mr. Ode makes one final alteration to the expression for the first question, and the probability of three red blocks in three draws becomes;
"Oh!" "Wow!", collectively from the class. Despite the rather fast pace and the complexity of the problems, the students see the pattern. With a short summary discussion they are ready to tackle the questions in the text. (JO-LES-N-12)

With his senior pupils, Jonathan's teaching has shifted slightly as he takes more control of the class and moves away from the use of open-ended investigations, but his lesson still demands much student thought. Rather than receiving the text's magic formula along with an explanation of its use, the class has been involved in the reasoned construction of a generalized procedure.

**Applications: Mathematics as Modelling**

Since the solution of problems is the driving force behind mathematical development, it is not surprising that the resulting concepts and techniques have wide ranging application. Jonathan acknowledges the utilitarian aspects of his subject, both in its history and instruction. His words, concept map, and repertory grid all give applications an important place, but as he tells us in the following paragraphs, the sequence is important. Mathematics is not just a set of algorithms to be applied in other fields, but is a collection of models and processes that
have been developed in the solution of problems. Applications, while important, come after the initial problem solving.

On Jonathan's concept map (Appendix E) all the linking lines terminate in a node labelled "Applications of Mathematics". For Jonathan, mathematics, both as a domain of knowledge and a school subject, can not exist as a purely abstract mental enterprise. It must have some ultimate connection with other human activity. "I think if you took this [the "Applications of Mathematics" node of the concept map] off - if we remove the applications for mathematics you would almost devastate the whole process. So I think we're looking at kind of - a necessary aspect of teaching and learning of mathematics" (JO-INT-D-12b). This link between mathematics and other dimensions of human action and thought is also displayed in Jonathan's school subjects repertory grid (Appendix F). The FOCUS analysis of the grid shows that the subjects most closely allied with mathematics are physics and electronics. In fact, on the PrinCom display, the mathematical end of Jonathan's "mathematical-non-mathematical" construct points directly toward these two subjects. But here "mathematical" is being used in a limited sense. "When I was using the word mathematical in this context I was thinking more in terms of mathematical algorithms rather than mathematics" (JO-INT-D-09b). Jonathan notes that he has produced a second construct, "answer oriented-open ended", almost parallel to "mathematical-non-mathematical". "Maybe those are the words I should have used ... because the algorithmic fits in with the electronics and the scientific and the physics in terms of the mathematics" (JO-INT-D-09b).
And, in my opinion, that's the short-sighted end. It's the focusing in on - on this one aspect, getting an answer. It - it takes away the creative aspect. And so many teachers, most of the teachers that teach mathematics, fit right in there, in my experience, and in many ways they devastate mathematics because the, the kids who - who want to soar can't with these people. (JO-INT-D-09)

Of more interest to Jonathan, is a second meaning for the label "applications", given by "Mathematical Modeling", his sub-title for the "Applications of Mathematics" node of the concept map.

When I talk about applications for mathematics I'm talking about real problems or problems where you can ... create a model ... and then try it and see whether it works or not. If it doesn't work then talk about it. Well how come it didn't work? What can we change to make it better, and so on. That's what I see as the modelling. (JO-INT-D-12b)

This other sense of applications, as stimuli for mathematical development, is important, for "our experiences are the seeds - that's where our mathematics comes from" (JO-INT-D-12b).

Jonathan provides an example of how an application can lead rather than follow mathematics.

If you were teaching probability or statistics, where you get some kids and say okay we're going to go outside and you're going to clock cars and we're going to talk about queuing, but your problem is to design a stop light that is going to make traffic flow better. We're going to go out and make some measurements, and then we're going to come back in and we're going to look at creating a mathematical model that will help us do that problem. And then maybe we'll go out and measure again and see if we were to change this light to work on the basis of your model, what happens to the traffic. (JO-INT-D-12)

Looking for sources of mathematics in problem contexts, means that the boundaries of the subject become ill-defined. A concern for some teachers but not for Jonathan.

I've had that experience in teaching calculus where people question me, "So what are you doing here,
teaching physics or teaching math?" And, and that's a mistake to ask that question. The richness of mathematics comes out when you, when you stop worrying about those barriers and you start looking at it as an interdisciplinary thing. (JO-INT-D-08b)

In fact, for Jonathan, good teaching requires that mathematics arise out of experience and problem contexts.

As a teacher I would say it's a mistake to try to teach an abstract concept without having formed a good foundation in experience. Even though kids can learn that way all they do is they learn a bunch of rules and algorithms and they apply them without really understanding what's going on. And it's really not a very meaningful experience for them. And, without that meaningfulness then they'll never really go on and use that in any kind of a meaningful way. So I think in that sense it's very important to ground everything in experience. (JO-INT-D-08b)

Thus, in Jonathan's vision, mathematics does involve concepts and techniques that are useful in areas beyond the subject, but for him this is not surprising given the discipline's origins in the solution of problems. In his teaching, Jonathan provides opportunities for students to construct mathematical models that, while developing new concepts, also link the subject matter to the physical world and to potential applications.

Algebra as Modelling: A Meaning for $N$

As the school term progresses the Grade 9 work becomes more abstract and by my ninth visit to Jonathan's classroom the students are studying algebra. The passage from arithmetic to algebra can be difficult. What do all those $x$, $y$, and $n$'s mean? Jonathan wants to help his students grow into algebra through experiences that give the symbols meaning. The next lesson to be
described, shows how he employs manipulative materials, patterning activities, and collaborative work (Bennett, Maire & Nelson, 1988) to help the students build algebraic models for physical objects. Of course, with the blocks, groups, and independent work, there are some risks. Jonathan takes the time to coach his class of eager 14 year-olds, both on the mathematics and appropriate behaviours.

"Today we will be constructing geometric figures from blocks, and then using the construction to generate numbers, and then analysing the numbers to find patterns. Let's look at an example; a nice easy one. I don't want to make it too hard for me!" Mr Ode places an overhead acetate of an example (JO-LES-D-20a) on the projector.

![Image](image_url)

Mr. Ode continues, "A word of caution - I think that it's nice when we can have fun. Just as long as the fun does not get in the way of the learning, I'm all for it. Today we are going to be playing with blocks.

This announcement is greeted with a class chorus of, "Yea!", "Ho Ho", "Yahoo!"
Over the roar Mr. Ode continues his coaching. "We all understand the difference between progression and regression, Right?" Puzzled silence is the class response. "No?" A chorus of "No's" comes back from the students.

Mr. Ode: "Progression - means going forward - regression means going ..." - "backwards". The pupils complete Mr. Ode's statement.

Mr. Ode: "We've all been in kindergarten already - don't want to go back, Okay?"

Mr. Ode returns to the sample problem on the overhead projector. "All of the problems are designed so that you can not answer all of the parts by doing the construction. In other words, you are going to be able to build levels - level one, level two, level three - and at each of those levels you will be able to count numbers. Then it's going to ask you for level ten, and I'm pretty sure that in every case there aren't enough blocks to construct level ten. Let's begin."

With Mr. Ode building the figures and focussing attention, the students provide numbers for the surface area and volume for levels 1 and 2. When multiple answers are given for the level 3 surface area, Mr. Ode switches to more direct questioning.

"Let's take a look one at a time. How much on the top?" The answer, "Two", comes back. "And on the bottom?" "Two." "How much here [pointing to one side]?" "Three." "And [pointing to the other side]?" "Three." "All together so far?" "Ten." "Good! Now how much on this face [pointing to the front surface]?" "Six." "And here [pointing to the back]?" "Six." "Good! So the number all together?" "Twenty-two."
With some debate the students supply the answers 28 and 8 for level 4.

Mr. Ode: "What's the surface area at level five?"
Students: "Thirty-four."
Mr. Ode: "Are you sure?"
Students: "Yes!"
Mr. Ode: "Why did you say thirty-four?"
Students: "Because you went up by six. There's a pattern."
"Great!" Mr. Ode is obviously happy. The students have noted the pattern in the numbers, but he wants to take this one step further. Holding up the solid model for level 4 and adding a pair of joined blocks he announces, "Notice that you can see it here too. If we add two blocks like this, how many sides do we add? One, two, three ..." The class joins in, "Four, five, six, seven, eight, nine, ten." As some students complete the count others protest. "But you are covering some. You only add six."
Mr. Ode likes this line of reasoning and links the solution paths. "So you can look at the patterns geometrically, and you can look at them numerically."

Mr. Ode continues, "Now what is the answer at level ten?"
"Ooh", collectively from the class. A number of answers are suggested but the popular opinion is sixty-four. Discussion leads to the conclusion that 9 sixes must be added to the first level, so the answer must be 9 times 6 plus 10 or 64.

Mr. Ode: "Now - let's go one step further. The next one is interesting, because now instead of saying the tenth level, we are going to say what about the Nth level? That is, for any number N how could I get there?" Several pupils suggest, "Six
times N plus ten."', which Mr. Ode records on the board as $6 \times N + 10$. "Okay, let's check it out and see if it works. If N were one, that would be what level?"

Students: "The first level."

Mr. Ode: "Is the first level sixteen?"

Students: "No."

Mr. Ode: "If N is 2, what does this [pointing at the formula] give?"

Students: "Twenty-two."

Mr. Ode: "Is that the second level?"

Students: "No."

One student suggests, "We need N minus one.", to which Mr. Ode replies, "N minus one, I'm going to write that in a bracket.", and the expression on the board is changed to $6 \times (N - 1) + 10$.

Mr. Ode continues, "Now maybe it will work. Let's check the fifth level." The expression now generates the correct answer of thirty-four and all appear to agree that it will work at all levels.

Mr. Ode: "What we have done is develop a formula that allows us to find the area at any level you want to go to. If somebody asks what's the area at the hundredth level, it's easy. How much?"

Students: "Ninety-six."

The steps are quickly repeated with the volume results and then the class divides into groups of four that will work on similar activities to build algebraic expressions for block patterns. (JO-LES-D-20b, See Appendix H) (JO-LES-N-20)
In this lesson the students have followed the concrete to conceptual sequence of Jonathan's vision of mathematical development. Thus algebra does not appear as magic. Both in historical terms and within this lesson, algebra has been constructed for a purpose; the task of providing general models for patterns.

Mathematics as Ill-defined Truth

Jonathan rejects the popular image of mathematics as the embodiment of truth. As he tells us in the following paragraphs, mathematics is an evolving discipline with the validity of concepts in flux.

I'm not a Platonist. I don't believe that there's a mathematics that we discover. I believe that we create the mathematics. I mean certainly, for me, if there is a God he's a mathematician, there's no two ways about that. But, I don't believe that it's something that we simply look for and discover.... I think we create it as we go along. (JO-INT-D-08b)

Jonathan rejects an absolutist view of mathematics. On the PrinCom display of his subjects repertory grid (Appendix F) the labels "convergent" and "factual" are found on construct rays oriented 180° away from mathematics. For Jonathan, mathematics is not a fixed body of facts and when describing truth he tells us, "I'm trying to get away from that concept of it [truth] being out there.... It's part of the process that the person doing mathematics is involved in" (JO-INT-D-03b). Jonathan finds that his expanded image of mathematical proof and truth differs from that of many of his fellow teachers.

If you were to ask one of the teachers in my Department, they would probably say that the best place
to look for mathematical truth would be Euclidean geometry. After all, Euclidean geometry was all about "truth". The road to mathematical truth was called deductive reasoning. Deductive proof techniques were the only sure-fire ways for determining whether a statement was true or not. The concept of mathematical truth has changed drastically over the past fifty years. The correctness of the answer is now something that must be negotiated between players. (JO-DOC-D-03)

The lack of absolute certainty in mathematics is not a problem for Jonathan. In fact, he finds the "divergent" nature of the discipline, as presented in his subjects repertory grid (Appendix F), exciting. "As soon as you fix the concept of truth you've actually put blinders on because - you've now excluded a whole host of alternatives as being of any interest" (JO-INT-D-03b).

As we see, for Jonathan, the openness of mathematics is a source of excitement. The direction of the subject is not determined by some fixed set of rules. There are choices and interpretations to be made.

**Exploring Multiple Correct Answers**

Jonathan sees a need to provide students with experiences that present an open view of mathematical truth.

If we restrict the questions to those with specific answers we run the risk of giving the student the impression that there is nothing new or exciting to do in math because everything has already been determined to be true or false.... What we need to do is introduce some experimentation into math, allow the students to play with the unknown and not worry about the correctness of the answer. (JO-DOC-D-03)

In his classes the validity of mathematical statements is determined through reasoned discussion, not through appeals to mathematical authorities. During my fifth visit to Western
Secondary School I observed a lesson in which Jonathan encouraged multiple interpretations and a variety of answers.

Mr. Ode's Grade 9 class is reviewing recent work with powers. The questions on the blackboard begin with, "1) Simplify $2^{13} \times 2^6 \times 2^3$." After writing out three more questions, Mr. Ode returns to the first and casually adds beside it the message, "Hint: the answer is not $2^{20}$." Now there is some confusion and neighbouring pupils begin sharing their thoughts on the solution to question 1. "Don't the exponents add up to twenty?" "Does he mean to write it in some form other than two to the twenty?" "Are there other ways?" Clusters of students exchange suggestions and soon there are multiple possibilities.

Mr. Ode begins the discussion of question 1 with an admission, "Yes, this does simplify to $2^{20}$, but are there other ways of saying this?"

"$1024^2$", is given as a possible answer.

Mr. Ode: "Good, any other way of writing this?"

Another student suggests, "$(2^{10})^2$." Mr. Ode writes these answers on the board with the word "or" between them.

Mr. Ode continues, "Any others?"

"$4^{10}$" and "$(2^2)^{10}$" are suggested and these are added below the first two answers.

The next response of, "$8^5$", causes a murmur of concern to ripple through the class.

Mr. Ode: "Oh! Some of you don't agree with this last one. Why?"
One of the objectors presents an argument. "We can write eight as two to the three, and so that answer, eight to the fifth, is only two to the fifteenth."

Mr. Ode writes \((2^3)^5 = 2^{15}\) beside the original \(8^5\), and when all seem to agree with the equalities, a line is drawn through this suggested representation for \(2^{20}\).

Mr. Ode: "So if this eight [pointing to the 8 in the line just scratched out] is not correct, what should it be?"

A chorus of "sixteens" comes back from the class.

Mr. Ode: "Any other way of writing sixteen to the fifth?"

When "two to the four to the fifth" is given, Mr. Ode writes, "16^5 or \((2^4)^5\)" below the previous answers. The students are invited to look for patterns in the list and this prompt results in the suggestion, "We can always switch around the order of the exponents and get two to the fifth to the fourth or thirty-two to the fourth." "32^4 or \((2^5)^4\)", is added to the list.

Mr. Ode pushes the thinking further. "Is that all the possible answers? Are there any others?"

There is a short pause for thinking before a pupil resumes the exchange. "Make more brackets."

Mr. Ode: "How could we do that?"

Mr. Ode invites the student to put his expression on the board and "\(((2^2)^5)^2\)", is written.

Mr. Ode: "Great! How many variations can we make on this one?"

Two more students come up and write answers of, "\(((2^5)^2)^2\)" and "\(((2^2)^2)^5\)", beside the first.
"So we can write two to the twenty in many ways", and with this summary by Mr. Ode the class moves on to look at solutions to the second question (JO-LES-N-10a).

After the lesson, as we walk back to the Mathematics Department office, I comment upon the extensive investigation that came out from question 1. Jonathan responds with, "Yeah, and all that out of my mistake. I meant to give the hint, that the answer is not two to the nineteenth. I expected that some would forget that two is two to the one and would get nineteen as the sum of the exponents, but when I wrote the hint I focussed on the correct answer. I saw the error as soon as I began working with individual students, but since it was sending everyone off to look for alternative answers I decided to take this direction" (JO-LES-N-10b).

The Multiple Personalities of Mathematics

Mathematics, for Jonathan, is multi-facetted, and this complexity shows through in his subjects repertory grid and the efforts in its construction. "It was difficult to find things that for me were clear enough that I could say okay everything for this concept fits these subjects and the opposite fits the other ones well" (JO-INT-D-09b). Despite these struggles, the PrinCom display of the grid (see Appendix F) and Jonathan's subsequent analysis reveal a split personality for mathematics.

On the graphical PrinCom display of Jonathan's school subjects repertory grid, mathematics sits midway between visual art, which Jonathan labels as being "creative", and the physics-
electronics pair of subjects, where Jonathan sees mathematics as "answer oriented". When mathematics is employed as a tool in other disciplines it has the rather limited purpose of algorithmically generating solutions to their questions. "The algorithmic fits in with the electronics and the scientific and the physics in terms of the mathematics. You know, you have formulas and you use them" (JO-INT-D-09b). In this sense, "the fact that, 'mathematical', 'uses experiments', and 'scientific' are all clustered, I think for me is a natural. I mean, that certainly brings out one of my attitudes towards mathematics" (JO-INT-D-09b). On the other hand, "mathematics is aligned with 'conceptual' and 'divergent', and 'creative' up here is not too far off. I do believe that there is a connection between all of these elements right here [pointing to mathematics, conceptual, divergent, visual art, and creative] and that is one of the ways I look at mathematics" (JO-INT-D-09b).

I guess what I'm saying is that there are two different pictures, and maybe that's the way it should be because in some situations, certainly, you want to teach it from an experimental hands-on approach and then move from this area [answer oriented] into this area [creative]. Once you get up to here [creative] then you're free. It's not divorced [from the algorithmic] but it's kind of like a quantum jump, and then you jump right back down again. I know that's interesting, very interesting.... I noticed it [the relative position of mathematics] right off the bat and thought that's very interesting and very odd. And I didn't really get a chance to put it together. Now that I look at it, it's not that it's odd, it's correct. (JO-INT-D-09b)

This interplay, between creativity and more restricted algorithmic activity, that Jonathan suggests as a teaching approach, is for him a reflection of the history of mathematics.

In a sense, science drives mathematics but on the other hand, there are those ideas which are extended beyond the experiential, where people are developing patterns
themselves. For instance as in group theory new ideas were developed and then they ended up using them in chemistry. (JO-INT-D-08b)

Thus mathematics has two dimensions. On the one side, "it's convergent, when you've got it neatly bundled up, you have a bunch of facts, and it's all there. And the other way it's divergent, it's open-ended, there's no limitations and you're extending constantly" (JO-INT-D-09b).

Jonathan's concept map for mathematics (Appendix E), further amplifies these multiple personalities of the subject. Here mathematical knowledge, both for the individual and collectively for a society, is pictured as progressing via three modes of inquiry, "experimental, conceptual, and visual". The experimental mode, which is linked with "numeracy, where we play with numbers and we try to go from one step to the next" (JO-INT-D-12b), identifies the core relationships and processes of mathematics. The conceptual dimension, through the use of symbols and abstraction, takes us beyond the basic number patterns and facts.

I guess you could define conceptual as being non-factual. Conceptual means that what you're doing is you're either investigating or learning or discussing something about something that has factual content. So you're looking at connections between the facts, and you're looking at overlying patterns, and maybe even a hierarchy of that depending on where you were to take it. (JO-INT-D-09b)

The adjective pairs, "factual-conceptual" and "convergent-divergent" are essentially synonymous, with their construct rays being collinear on the subjects repertory grid. The "divergent" and "conceptual" ends of these constructs pointing directly at mathematics, show how, for Jonathan, an essential component of mathematics, one that separates the discipline from science
(physics) and technology (electronics), is its ability to move beyond the experimental data.

Effective mathematical progress integrates all three of Jonathan's inquiry modes.

If you're taking a look at a problem, it would be well worth it to look at it in all three ways: as an experiment, look at a visual counterpart to the problem, and then try to conceptualize this and abstract from it.

... Calculus, for instance, employs the visual geometric and moves into the algebraic conceptual area, although a lot of classical calculus tends to be centred or people attempt to centre it entirely in the conceptual realm, more as an algebraic process, and look at the visual part as being secondary. I don't see that. I see those two as, as being equal. (JO-INT-D-12b)

"Often the algebra symbols just get in the way when a picture might show you some symmetry that solves your problem" (JO-DIS-N-06), "but we see all those algebraic rules in the books, without anything else around them, and you don't see any visual support" (JO-INT-D-12b). For Jonathan, pictures are important tools in the development of mathematics, both for his students and for the history of the subject. He laments the de-emphasis of geometry in the school curriculum but, recognizing the cyclic nature of educational trends, hopes for its return. "Some day, some one is going to come up with the revolutionary idea of doing Euclidean geometry in high school" (JO-DIS-N-08).

While Jonathan's conception of mathematics is of a multifaceted discipline, there is no real conflict between the subject's variety of personalities. A concept begins its life tied to and restricted by the nature of its experiential source. Once sufficient abstraction has occurred, human thought and creativity are able to extend a new idea beyond the bounds of its
original context. Mathematics becomes a product of the human mind.

Being Visually Creative

Jonathan's teaching provides opportunities for his students to experience creativity within mathematics. "Periodic Pictures" (Nowak, 1987), an extended project with the Grade 9 class, provides an example.

I think the assignment has a number of features that attracts me to it. First of all, it's an extension of mathematics in the area of art, because what they're doing is they're using the math to create a pretty picture or design, and I think that's a nice thing for kids to do. It's something that provides them a lot of pleasure.... what they have to do is to take their number sequence and create a picture, and here's where I encourage as much creativity as possible. (JO-INT-D-02)

The class has been studying rational numbers, in particular how to convert from the fractional form to a decimal expression. Last night's home work, question number 3 of the project, asked for the decimal representation of 1/7, but with a new twist. How could you do this with a calculator that displayed only three digits? Now Mr. Ode wants to use the ideas suggested on the project pages to tackle more complex examples.

"We need to know how to write a fraction as a repeating decimal when the repeating part has a period that is longer than the number of digits in the display of the calculator."

"I'm going to be assigning some neat [vocal emphasis] fractions like one over thirty-one, or one twenty-ninth, or maybe two twenty-thirds."
The enthusiasm and excitement communicated by Mr. Ode's emphasis of the "neat fractions" is contagious and there is a general buzz of anticipation in the class. "Neat!, All right!"

Mr. Ode continues. "You are going to be working out what these repeating decimals are and then you are going to be graphing them!" "You're not going to believe how great these graphs are going to look."

After a pause to let the class settle down and get ready for work Mr. Ode proceeds. "If we took the fraction 1/31, what is the maximum number of digits we could get before it repeats?" This question generates a variety of answers, but with some debate the class agrees that 30 is the maximum number.

Mr. Ode: "Let's check and see exactly what it turns out to be." "We will take 1 and divided it by 31. Right? Now you tell me what I do next."

For the next ten minutes, Mr Ode, using a TI-81 calculator with an overhead projector display, leads a highly active Socratic lesson that develops a recursive technique for generating any number of decimal places. After three iterations the class has a decimal expansion of .03225806451612903225806 and has identified the repeating segment for this periodic decimal.

Now the class sets to work repeating the process for the fractions 1/17, 1/19, 1/21, 1/23, and 1/29. In ad hoc groups of two and three they support each other, providing help, debating next steps, and comparing answers, while Mr. Ode circulates through the class providing assistance to any groups that appear to be lost.
After approximately 10 minutes, Mr. Ode calls the class back together so that they can look at the next steps in the project, how to create pictures of the decimals. Using a rectangular coordinate system drawn on the black board, Mr. Ode expands upon the example and instructions provided in the assignment guide.

Consider the periodic decimal \( \frac{1}{7} = 0.\overline{142857} \). The digits of the period can be grouped sequentially to form ordered pairs, as follows:

\[
(1, 4), (4, 2), (2, 8), (8, 5), (5, 7), (7, 1)
\]

Each of these points can be plotted on graph paper and joined sequentially to produce a "picture" of the period, as shown. (The last point is joined to the first point to close the figure.) Check each point and segment on the graph to make sure you understand the method of plotting this picture.

"What can you tell me about this, that's interesting about that picture?"

"Mr. Ode's question generates a number of suggestions. "It's the same." "It's like a mirror." "You could flip it."

With a little probing by Mr. Ode these ideas are put together to identify the symmetry in the pattern.

Mr. Ode: "That's what you are going to be doing with different fractions that I'm going to assign to you." "But I want you to do more to show the patterns and the symmetry." "Let me show you some examples."

Mr. Ode puts up some student work from past years. Colour, and patterns have been added to emphasize the symmetry. In some cases the axes have been skewed or bent to introduce new patterns.
A chorus of "Wow! Awesome! Cool! Neat!" indicates the class' approval of the pictures. They are ready to tackle the project themselves.

Mr. Ode: "We are not going to be able to do all this in class but I want to make sure you can get started, so today we will work on ordinary graph paper."

Mr. Ode comes around the room and assigns particular fractions to pairs of students and they set to work calculating the decimal equivalent and planning their picture. (JO-LES-N-07)

Later, back in the mathematics department work room, Jonathan reflects upon the assignment.

It has a very direct connection to patterning which I feel very strongly about. That's exactly what you're doing is you're recognizing patterns in the decimal expansion, the sequence of numbers, and you're using those patterns and you're translating that from a numerical pattern into a geometric pattern which means that you're moving from one realm into another one and you're making a connection which is an important thing for these kids to be able to do. Historically I like it because much of what we do here has strong connections to geometry that have been lost over the years, and any opportunity I get to kind of re-establish some of those connections I like to jump at that. (JO-INT-D-02)

Twelve days later when I am again visiting the Grade 9 class, Mr. Ode is collecting the students' period pictures. As each pair hands in their work he has enthusiastic comments and questions. "How did you do it?" "Explain it to me." "Show me the symmetry." Mr. Ode listens carefully to the explanations, always asking for more details. The results are impressive and next day the hall outside the classroom is decorated with colourful intriguing patterns (JO-LES-D-07e,j,m,n, see Appendix I).
Jonathan is pleased with the students' work. He has met his goals, to provide an opportunity for open-ended exploration and creativity, but still keep the independent activity within the course outline (JO-DIS-N-07).

Of course there are restrictions as there are on any creative process. It has to demonstrate the symmetry that is inherent in the fractional picture. And they have to exemplify that in, in the colouring and how they present it. So, they're learning how to, how to create something and then produce it in a form which is presentable, and the presentation of your work, I think, is very important. (JO-INT-D-02)

Case Summary: Subject Conceptions and Teaching Practice

Ontario mathematics teachers, on provincial surveys (Ontario Ministry of Education, 1991c, 1991d), claim almost unanimously that problem solving is an important component of their subject, but the strength of this belief must be questioned in the face of data from the same studies showing the predominance of teacher-centred instruction. Jonathan Ode is an exception to this pattern. Mathematics as problem solving is a central theme of all his conversations concerning the discipline, and this image is carried into his lessons where pupils are regularly provided with opportunities to experience mathematics as a problem solving enterprise. Of the various images of mathematics identified in the literature, Jonathan's conception of the subject fits most closely the problem-solving or social constructivist position. Jonathan himself applies the "constructivist" label to his vision (JO-INT-D-09b) when analysing his school subjects repertory grid (JO-INT-D-04, Appendix F) and noting his belief that mathematics originates in experiments. This statement is not just simple
rhetoric, for repeatedly in his writing, repertory grid, concept map and interviews he reveals a problem-solving, social constructivist stance.

Jonathan states directly that he rejects the Platonist position (JO-INT-D-08b) and, despite locating nodes labelled "Basic Mathematical Facts" and "Algorithmic Skills" at the centre of his concept map (JO-INT-D-12a, Appendix E), he appears to also dismiss the instrumentalist view, writing that "a necessary condition for an activity being called mathematics is the ability to explain the processes used [bold in original]" (JO-DOC-D-03). While acknowledging that mathematics can be algorithmic when associated with electronics and other applications (JO-INT-D-09b), Jonathan argues that a focus on generating answers limits the subject and such a teaching approach destroys school mathematics (JO-INT-D-09). Mathematics is strongly linked to its applications and school programs should develop this connection; not in a instrumentalist manner by teaching procedures and then applying them for practice, but in the reverse direction, starting with an application context and building mathematics from that (JO-INT-D-12b).

Despite his extensive graduate school experience with formal logics, Jonathan appears to also disagree with the formalist view. He acknowledges the ability of mathematics to exist independent of physical referents but still gives the activities of formal abstract mathematics a constructivist tone, stating that "a lot of the work in universities; combinatorics, graph theory and things like that; still take an experimental mode" (JO-INT-D-12b). He sees the ideas of group theory growing in an
inductive manner through the observation of patterns within the formal symbols of the subject (JO-INT-D-08b). Both Jonathan's repertory grid (JO-INT-D-04, Appendix F) and concept map (JO-INT-D-12a, Appendix E) present two aspects of mathematics: "conceptual" and "experimental". The conceptual dimension Jonathan links to abstract ideas and symbols, allowing mathematics to be "creative" and "divergent", but he notes that the development of the discipline follows a balanced combination of these two paths (JO-INT-D-09b). The abstract conceptual stage builds on the experimental (JO-INT-D-12b).

Jonathan understands the formal proof processes of pure mathematics, but when looking at mathematics in the larger world, puts himself firmly in the social constructivist camp, asserting that deductive reasoning is not the only route to truth and that experimental investigations can lead to general agreement on the validity of processes and answers (JO-DOC-D-03). Mathematics is collaboratively (JO-INT-D-04) constructed by humans (JO-INT-D-08b), both at the present time and through history. Humans have a need to analyse and attempt to explain the world and out of this desire is born the language of mathematics (JO-DOC-D-04).

As the previous eight narratives of teaching episodes show, Jonathan's social constructivist view of mathematics is carried into the classroom. In each lesson the class structure, tasks, and activities can be seen to reflect at least one aspect of Jonathan's conception of the discipline. Although selected for their illustrative value, these classes were not significantly different from others observed. In only two cases did lessons appear to run counter to Jonathan's subject vision. These, a
pair of OAC-Finite Mathematics classes, took place immediately
after illness had forced Jonathan to be absent for a day. In an
effort to cover two days work and get back on his course
schedule, Jonathan delivered full periods of direct instruction
(JO-LES-N-18, JO-LES-N-19). Students remained attentive during
these lessons but the large number of questions arising in the
closing minutes suggested that there was considerable confusion.
Jonathan, recognizing the lack of success, ended the afternoon
lesson with an apology for the volume of work and delivery style
(JO-LES-N-19). In all other lessons the story of mathematics
portrayed by Jonathan's instruction was one compatible with his
social constructivist philosophy.

Students, either in a whole class format or in smaller
groups, collaboratively built solutions to problems.
Communication and discussion of ideas presented mathematics as a
reasoned activity. Students conducted investigations or
experiments, gathered data, noted patterns and through
generalizations constructed their mathematics. Jonathan took
advantage of opportunities to encourage alternative
interpretations of questions and the development of multiple
different but correct solutions. Through a link to visual art,
mathematics was experienced as an open creative activity. Of the
themes identified in Jonathan's subject conception, only that of
mathematics as applications failed to find a significant place in
the observed teaching.

Jonathan's vision of mathematics originating in application
situations did have expression in the Grade 9 activity where
students built structures from blocks and developed expressions
for volumes and surface areas. Here mathematics was presented as a modelling tool, but the "application" did not match the more realistic situations that he described during interviews (JO-INT-D-12). During OAC-Finite Mathematics classes Jonathan justified the exploration of ill-defined problems and efforts to develop general heuristics, with references to the work of engineers and the design process (JO-LES-N-02, JO-LES-N-03), but problems originating in these settings were not addressed.
Chapter 5: Case 2: Randy Walker: Tentative Steps along the Reform Path

In 1983, at a local teachers' conference, Randy Walker was introduced to the subject of fractal geometry. Reading, independent study, and mathematical investigations in this topic have consumed much of his spare time since then and have contributed to an expanded vision of mathematics. The open-ended questions he has met and the experiments that he has performed in the search for answers have excited Randy and convinced him that school mathematics could be much more enjoyable if it broke from tradition. Now he has as a goal a less formal and more student centred program where pupils are engaged in investigations rather than routine symbol manipulation (RW-INT-D-05b). Randy has, to a limited extent, put his plans in place, but during the period of this study there remained many ideas, in his head and on paper, that did not reach the classroom.

When given the course time to pursue his new mathematical interests, Randy engages his pupils in projects and investigations, often computer supported, that involve random events, chaos, and dramatic mathematical art. Students meet significant questions concerning probabilities, make conjectures, debate their reasoning, conduct experiments, and through the observation of patterns build toward, often tentative, solutions. These activities are frequently linked to practical problems such as those of industrial quality control (RW-LES-N-25) or the study of nature's repeating patterns (RW-DIS-N-25). In this environment the mathematics education reform themes of problem
solving, communication, reasoning, connections and the use of technology are in play. Assessment activities are expanded beyond the traditional tests and examinations, with students receiving grades for their reports on open-ended investigations.

Randy has focused his efforts in developing an alternative mathematics curriculum and teaching style on the new topics of fractal geometry and chaos. When it comes to teaching the regular school mathematics content his approach is much more traditional. During the time of this study, Randy was working in a school that did not value new mathematical topics and confined itself to the narrow curriculum as described in the content listings of the official guidelines (Ontario Ministry of Education, 1985). Lacking administrative support for the teaching of his new interests, Randy's opportunities to address these topics were pushed to the edges of his program: optional bonus projects for students to complete outside of class, short compressed lessons when some class time remained after the completion of traditional lessons, classes in a loosely organized experimental science course, and extracurricular activities with interested students.

Eighteen of the twenty-three lessons observed during this study were with two Grade 12 classes in the university preparation Academic stream. Here, for the well established topics of trigonometry and exponents, Randy employed almost exclusively a teacher centred mode of instruction. For parts of three of these lessons group activities were organized, but for the most part instruction took the form of Socratic lessons followed by student practice, either individually or in ad hoc
groups. With Randy's very detailed explanations and his insistence that pupils give fully reasoned answers, mathematics was presented as a logical activity, but the final outcome of most lessons appeared to be well understood, rules, facts, and procedures.

When listening to Randy outline his conception of mathematics the reader will find that this direct instruction style of teaching is not incompatible with his vision. When addressing newer mathematical topics such as fractal geometry, dynamical systems, and chaos theory, Randy's position has fallibilist trends but generally he views the bulk of the discipline in absolutist terms. Here he presents a blend of Platonist, formalist, and to a limited extent instrumentalist philosophies.

Three Coins

During the academic year prior to the commencement of my research project I visited Golden District Secondary School to meet with Randy and explore the possibilities of his joining the study. My brief time in Randy's classroom that day provided a vivid illustration of a key theme in his conception of mathematics; mathematical knowledge can be advanced through inductive processes. Sometimes mathematics is an experimental science.

Period one in the day's schedule is already half over when I arrive at Golden District Secondary School, so I proceed directly to Randy's classroom. This is the scheduled time for his special
course, "Fractal Geometry and Chaos Theory", a non-guideline course that he has just this year been granted permission to teach. The students are busy working in groups and Randy is circulating around the room, dropping in on each cluster of pupils, asking for reports of their work and posing questions. There is a computer at one side of the room and it is obviously performing some repeated calculations as the same message with different numbers keeps flashing on the screen.

<table>
<thead>
<tr>
<th>Life of latest coin trio</th>
<th>Average trio life (tosses)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 toss(es)</td>
<td>527248/167700 = 3.143995</td>
</tr>
<tr>
<td>1 toss(es)</td>
<td>527568/167800 = 3.143981</td>
</tr>
<tr>
<td>3 toss(es)</td>
<td>527885/167900 = 3.144044</td>
</tr>
</tbody>
</table>

When he sees me at the door, Randy strides over, gives me his usual strong hand shake, asks one quick question about my trip, and then turns the conversation to mathematics. He pulls three pennies from his pocket and announces, "We are going to do an experiment." Randy tosses the three coins into the air so that they fall on the top of the bookshelf near the computer. The coins come up two tails and one heads. "We remove the heads", Randy continues as he sets the two coins aside and picks up the remaining tails. He tosses the coin and it comes up tails again. Randy gives it another toss. Heads results and Randy puts the last coin with the first two. "All three are gone now." "This time it took three tosses for all the coins to come up heads. On average how long will it take?"

Randy's puzzles are always interesting but never easy. I think about the problem but can only give a range for one trial. "It could happen on the first toss but it could also take an
infinite number of tosses." "Yes", Randy replies, "But on average?"

Randy tells me how the class has been working on coin toss probability problems and how he modified a situation presented in one of his reference books (RW-LES-D-i, Paulos, 1991) to get this new puzzle. "We have worked on it experimentally and then I wrote this computer simulation - look at the numbers coming up for the average."

Life of latest coin trio = 7 toss(es)
Average trio life = \frac{529404}{168400} = 3.143729 tosses

Life of latest coin trio = 1 toss(es)
Average trio life = \frac{529703}{168500} = 3.143638 tosses

Life of latest coin trio = 4 toss(es)
Average trio life = \frac{529994}{168600} = 3.143499 tosses

"Notice anything interesting?", Randy asks. I note the results are close to 3, actually close to the decimal expansion for \pi. Randy is obviously excited. "Yes! We keep getting answers close to \pi, but I can not see any reason for this. I'm working on a way to calculate an answer directly, but have not figured it out yet." (RW-DOC-D-02a)

For Randy, mathematics can be explored through experiments. He has no difficulties with his present lack of a definitive answer. This is just another challenge on which to work. I wander around the room and talk to the students as they puzzle over related coin toss problems. They seem to share their teacher's view that mathematics is an experimental science and appear happy to be involved in open-ended explorations. (RW-LES-N-i)
Recently Randy has been studying the new mathematical sub-discipline of fractal geometry. Ideas from his reading and courses that he has taken at a local university appear throughout the materials he produced in this study's structured activities: the writing on the nature of mathematics, repertory grids, and his concept map for mathematics; and resurface regularly in subsequent interviews examining these works.

In Randy's eyes the nature of mathematics is changing and in discussing the sources of mathematical knowledge he writes, "Computer experiments now drive much of mathematics research" (RW-DOC-D-07). Later, in an interview, Randy acknowledges the source of his new vision of mathematics. "I probably never would have said that if I hadn't been involved in fractals over the last few years. I never would have had a notion of mathematics as being very experimental" (RW-INT-D-01b).

Randy recognizes that his view of mathematics as an empirical discipline is not universally popular within the mathematics community, but he sees progress on this point.

Some [mathematicians] are converts. Many of them didn't want to have anything to do with computers and considered that type of mathematical exploration to be inferior mathematics but, I think, the power of the computer, especially the computer graphics, has now forced some of those people to change their minds. (RW-INT-D-05b)

Traditions change slowly and, while experimental activity can suggest productive paths for research, it is still formal work, with its emphasis on proof, that legitimatizes new developments in the subject. In this debate Randy takes the side of the
formalists, but he envisions a productive union between computers, experimentation and formal mathematics.

I think even many people in the field that are now using the computer as a tool know that's not enough.... Many of those, maybe even all, I can't speak for those people right now, know that ultimately to validate their work, their discoveries and computer experiments have to be expressed algebraically to fit into existing mathematics and that proof is still a necessity ultimately. But it's driving mathematics. Mathematics is now trying very hard to keep up with new discoveries made by computer. So, you know, they compliment each other in the end. It seemed like there was perhaps conflict at the beginning but it's a very happy marriage right now, I think. (RW-INT-D-05b)

Randy has personally internalized the mathematics community's continuing tension and sometimes hostile conflict between open-ended exploration and formal methods. In his teaching and visions of mathematics we will see an ongoing struggle between these two dimensions.

The alternating features of Randy's vision of mathematics show up in his first repertory grid for school subjects (RW-INT-D-01a, see Appendix J). Here physics and chemistry, the two representatives for science selected by Randy, are the subjects that come closest to mathematics, but the match is weak. The construct label, "experimental", that he provides for science also applies to mathematics, but not as strongly. For the descriptor, "investigative", Randy places mathematics midway between the sciences and history, English and French. When, for a second subjects repertory grid, mathematics is replaced by three sub-disciplines: calculus, algebra and fractal geometry; a different picture emerges (RW-INT-D-07, see Appendix K). Fractal geometry is closely matched with physics, Randy's example for science, while calculus and algebra retain the separation from
science shown by mathematics on the original grid. Fractal geometry and physics are described with, "allows for expression and exploration", while calculus and algebra are "rigid". Thus, for Randy, some branches of mathematics, particularly those that are new, are experimental in nature, but some areas of the discipline retain their formal traditions.

Randy is personally excited by mathematical experiments and sees their classroom use as contributing to his goals, "for students to have understanding and procedures that are general, that is, apply in many situations" (RW-DIS-N-10).

I think the whole pursuit of mathematics can be much more enjoyable than it is and I'm trying to find some ways to make it so in my regular courses.... What I have a mind to do is to try and create a mathematics course where it's somewhat student driven, where we would do much experimentation, much less formal mathematics in, in the usual sense of the proofs and applications that require a lot of algebra and the learning of techniques. (RW-INT-D-05b)

But Randy goes on to somewhat qualify this aim by attaching it to certain mathematical domains. "The new fractal geometry and chaos topics offer us just virtually unlimited access to those kinds of activities" (RW-INT-D-05b).

Iteration of Functions and Pixel Rainbows

After the initial classroom visit that I have described above, Randy and I made plans for the research project to begin the next school year. As the Winter-Spring term wound down and we set in place a schedule for the next Fall's semester, the North City School Board announced the closing of Golden District Secondary School and transferred Randy to Northern High School.
Despite losing the opportunity to teach his special course, "Fractal Geometry and Chaos Theory", Randy was determined to find a way to fit some of his favourite topics and investigations into the courses he had been assigned. He started with the obvious choice, his 12E course, a Grade 12 program that had been designated as "enriched". Here, early in the school year Randy expanded the regular curriculum to include experiments exploring the iteration of functions. Reflecting back on his planning, Randy provides his rationale for this intellectual excursion.

I wanted to present the students with some modern mathematics. I thought that there is a lot of stuff in the area of dynamical systems that's fairly easy to do. It's really very interesting, has a lot of tie-ins, especially visual ones with concepts of fractal geometry and chaos. And which really could also be thought of as an extension of one of the topics we do in the Grade 12 course anyway in composition of functions. Often the composition of functions doesn't seem to have any purpose and we certainly don't do any applications.... So I knew it was important and I also knew it was easy. We had graphing calculators and a computer now available in the classroom so we could really actually have an entire unit on it, time permitting. We started off with something simple, iteration of the squaring function - start with any number and keep pressing the square function until perhaps something interesting might show up on the calculator display. (RW-INT-D-03)

It's Monday morning and the students are involved in animated chatter concerning the weekend's social events. Mr. Walker brings the class to order by writing the lesson title on the board,

**Iteration of Functions**  
(Dynamical Systems)

"On Friday we finished with a question that I asked you to think about over the weekend". Mr. Walker repeats the question, "Can a function be composed with itself?", and writes it on the board. "What do you think?" A number of students contribute opinions
and the general consensus appears to be that the answer is yes.

Mr. Walker has some sample functions for the class to consider, to ensure that all understand the concept and are ready to move on. The functions, \( f(x) = 2x - 3 \) and \( g(x) = x^2 - 1 \), are written on the board and the students supply expressions for \( f \circ f(x) \), \( g \circ g(x) \), \( f \circ f \circ f(x) \), and \( g \circ g \circ g(x) \). The powers of \( x \) involved are noted and it is observed that, while repeated composition of \( f \) always produces a linear function, the results for \( g \) become increasingly complex. Writing on the blackboard, Mr. Walker summarizes the work so far with a note which the pupils copy into their books.

Repeated composition of a function with itself is called iteration (from the Latin "iter" meaning "journey"). Because the algebra of repeated composition (iteration) quickly becomes very messy for non-linear functions, iteration is usually done numerically, often with a computer.

The stage is now set for an experiment which Mr. Walker introduces with the title,

**Investigation A: Pixel Rainbows**

"We are going to use the function, \( f \) of \( x \) equals \( x \)-squared and repeat the composition over and over for different \( x \)-values."

Using mathematical notation, Mr. Walker repeats this information on the blackboard.

\[
f(x) = x^2 \quad \text{find } f \circ f \circ f \circ f \circ \ldots \cdot (x) \]
\[
\text{for } x = 5, 1.2, .2, .8, 1.8
\]

The pupils are instructed to get out their calculators and to think about how the iterations could be done easily. After a brief pause one student suggests, "Just put in the starting number and press the \( x \)-squared key over and over." This algorithm satisfies Mr. Walker, and responding with, "Okay, try
that with each of the x's and record what happens." he sets the class to work. The students begin these simple experiments and share their answers with each other, noting that eventually the calculators give either an error message or 0.

Mr. Walker: "What does the error mean?"

Student: "The number is getting too large"

Mr. Walker: "So what might we say the numbers are heading towards?"

Students collectively: "Infinity"

With the potential iteration results of 0 and infinity noted, Mr. Walker expands the task slightly. "Good, now I want you to repeat the experiments but this time to count how many steps it takes before the error or zero." With various students reporting their counts the following information is recorded on the board.

\[
\begin{align*}
  x &= 5, \text{ 8 steps to error (\(\infty\))} \\
  x &= 1.2, \text{ 12 steps to } \infty \\
  x &= .2, \text{ 8 steps to 0} \\
  x &= .8, \text{ 10 steps to 0} \\
  x &= 1.8, \text{ 9 steps to } \infty \\
\end{align*}
\]

Mr. Walker announces, "Now we are ready for the pixel rainbow part of this example.", and draws a chart on the blackboard.

| 0 | .2 | .4 | .6 | .8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | ..... | 5 |

"Suppose we colour each number or block to show how fast it goes to zero or infinity. For example let's use red for numbers that go quickly." Mr. Walker shades in the squares for .2 and 5 with red chalk. "The colour choice does not really matter. So what colour do you want for 1.2 which took 12 steps?" One student suggests green and Mr. Walker shades in the square. Blue is
given as a colour for 10 steps and yellow is assigned to 9. Mr. Walker shades in the squares for .8 and 1.8 with the appropriate colours. "There are a few more pixels to colour. Test these numbers and find out what colours we should use." The students perform the iteration on the numbers not yet shaded and report colours for .4, .6, 1.4 and 1.6.

Mr. Walker: "What happens at 0 and 1.0?"

Student: "Nothing happens. They don't change."

Mr. Walker: "Right! These are called fixed points. Let's leave them black."

Pointing to a large picture of colourful fractal images that is posted on the back bulletin board, Mr. Walker continues, "You have all seen these pictures before. The computer that drew those pictures worked in the same way we have here. It used a special function, tested each pixel or point on the screen, and then picked a colour from a table provided in the program. In the next few days we will look at this more closely."

The investigation of \( f(x) = x^2 \) continues with Mr. Walker identifying examples of orbits and attracting, repelling and Julia points. With the work just completed as a model, Mr. Walker asks the class to analyse the behaviour of \( f(x) = \sqrt{x} \) and for homework to investigate the iteration of three, more complicated quadratic functions of the form

\[ f(x) = kx(1-x). \]

The investigations of the iteration of functions continue over the next two days with the students using graphing calculators and the single classroom computer to support their
experiments. The fast pace of the lessons and the punch in his voice show Mr. Walker's enthusiasm for this topic. Looking back he reports his satisfaction with the outcomes of this sequence of mathematical experiments.

Very harmlessly we encountered notions of fixed points attracting and repelling fixed points, bases of attraction, neutral points or Julia points, things like that where we can develop a vocabulary. The students often find vocabulary a stumbling block, so early on in a very simple non-threatening situation we start applying names to all these things that students can see very, very easily.... And because it was all done visually I have to say that the students were engaged. Going by the number of enquiries about things they were seeing, and the ease with which I got a list of student responses to any question about what was happening I could tell they were right into it, so to speak. So that went very well and I was pleased to take them that far. (RW-INT-D-03)

**Conflict and Retreat**

Despite the mathematics curriculum and teaching proposals published by the NCTM (1989) and OAME/OMCA (1993) and the support provided in the introductory pages of Ontario's Curriculum Guideline (Ontario Ministry of Education, 1985), students experimentally investigating significant mathematical questions is not a common occurrence, provincially or internationally (Ontario Ministry of Education, 1991c, 1991d; Robitaille, Taylor & Orpwood, 1996). Years of teacher-centred mathematics instruction have lulled senior students to sleep and the vast majority "are content and comfortable assuming a passive role in the mathematics classroom" (Colgan & Harrison, 1997, p.7). Not long after the Grade 12E class' brief digression from the regular curriculum to explore the iteration of functions, Randy
discovered that not all students and parents share his enthusiasm for modern mathematical topics.

When I phone Randy to arrange my next classroom observation session he expresses his frustrations with the Grade 12 Enriched course. Some students and parents have been objecting that the marks in his course are low. Presently, on tests, these pupils are earning grades in the 70s, while in past years their marks had been in the 90s. Randy protests, "Yes we have been looking at interesting but difficult questions. But, if a course is supposed to be enriched, challenging to pupils and providing extra opportunities for problem solving, how can the class mean be expected to be 90 plus?" (RW-DIS-N-01).

A few days later when I visit the school, Randy fills in the conflict's details.

I thought things were going well but apparently there were some students who were feeling a little uneasy, and that showed up. And I got called aside by the Department Head of Mathematics who said that, the guidance people were finding that there were three people in my class who were thinking of dropping the course because they're having a little bit of a hard time with the new stuff and because they knew that their friends in the regular math course were ahead of us. So they were starting to think now that maybe there was something negative about this enrichment course they'd gotten themselves into. And, so I found out a little bit about the politics of introducing new curriculum. (RW-INT-D-03)

The pupils in Randy's 12E class are generally from upper middle-class homes and have families that are very success oriented. They and their parents have future career plans that involve specialized university courses. Entrance requirements are high, so marks rather than real learning are the primary concern. School policies have been set to address these
realities, with essentially nothing extra expected of students in enriched courses. (RW-DIS-N-01, RW-DIS-N-03a).

Students are very, very mark conscious. They're aware of what everybody else is doing, and they're pretty quick to feel uneasy when put in a new setting sometimes, although I wouldn't say that was generally true, but it did happen. And then I was reminded that the students in the enriched course in this school had, up until my coming, always done exactly the same curriculum as the others, but they did a few extra questions here and there. (RW-INT-D-03)

Randy has been going against past practices and official school policies, expanding the curriculum, and more significantly, expecting the students to learn this extra material.

It sort of was imposed upon me that I wasn't to test these students any differently than the others, which meant that the unit I had just completed on fractal geometry and chaos couldn't really be assigned a mark as if it were a test. Whereas to me it was simply covered under the topic of composition of functions and it was my way of doing that with them that I thought would be beneficial to them. (RW-INT-D-03)

As I spend more time at Northern High School it becomes increasingly obvious that Randy's teaching approach differs from that of other teachers in his department. At one point the Department Head tells me that there is no research to support the proposals made by the OAME and NCTM, and any studies that purport to favour student investigations, group work, and mathematical discussion must be flawed. All present in the department office agree. They all know that group work and investigations are a waste of time (RW-DIS-N-22).

Randy feels his isolation, but he does not attribute blame. The dominant practices in mathematics instruction have a long tradition and the system just rolls on.
The idea of throwing problem solving into a course seems to muddy the course a little bit. Some students really take that as almost extra. They feel there are ways to pass without solving problems—doing something they don't like, it's different. Um, and that impression, I think is, is the fault of the system because we try to focus on mathematical content and we never really get to the essence of the subject. Our courses always demand of us to cover so much material and we do it in the most efficient way, which is just disseminating the information. (RW-INT-D-05b)

Given the almost universal rejection of change by students, parents and teaching colleagues it is not surprising that Randy can also be observed to use the dominant transmissive mode of instruction. This is especially so when the content is traditional, such as in a 12E lesson that took place three weeks after the troubles over student grades.

The class has been studying trigonometry and for homework, using the values generated by their calculators, they were to draw graphs of the six trig functions. After working his way around the room, checking each student's work, Mr. Walker returns to the front of the class, turns on the overhead projector and displays a graph of one of the trigonometric functions. "What graph would that be?"

Student: "Sine"

Mr. Walker: "That would be sine. You can tell that it's sine by the way that it varies. It goes through the origin. It hits a maximum of 1 at 90 degrees, comes back down to 0 at 180, hits a minimum value of -1 at 270, then back up to 0 at 360, and then after that? —"

Student: "It does the same thing."
Mr. Walker, picking up on the student's answer, continues. "Does the same thing - continues in the same way. All right, that is in fact the sine."

"Now this one you would also recognize." Mr. Walker switches overhead slides to now show the cosine curve. "The point I'm getting at here is this - you must have noticed that the page with the sine and cosecant didn't look very much different from the page with the cosine and secant. If you had looked at the tables that generated these graphs, you would find that you've got all the same numbers in both tables, except that they just don't occur in the same place. I'm going to put my sine graph and cosine graph on the same axes." Mr. Walker places one overhead acetate on top of the other and lines up the two sets of axes.

Mr. Walker: "Now we are going to start shifting the one on top, the cosine, over to the right a little bit. - Now it disappears. - What does that mean?"

Student: "Horizontal shift"

Mr. Walker: "Horizontal shift - so the two graphs are identical in the sense that a horizontal shift - how much? -"

Student: "Ninety degrees"

Mr. Walker: "Ninety degrees - so we shift the cosine graph 90 degrees to the right and it becomes the sine curve. And you know from the numbers we've had that occurs all the way along."

Mr. Walker: "Any questions about either of these two curves?"

After a pause in which the class raises no questions, Mr. Walker continues. "You see these are curves that you have to get
awfully familiar with. - We have to keep track of these properties that we've discovered and get evermore familiar with them. We will study lots of details in these curves."

Writing the title, "Graphs of Periodic Functions" on the board, Mr. Walker introduces the next portion of his lesson. "I'd like you to list with me some observations."

The lesson continues with Mr. Walker providing a series of short notes summarizing the horizontal shift information noted earlier and continuing with seven other facts concerning the graphs of trigonometric functions (RW-LES-N-01, RW-LES-T-01).

In the face of critical comments, from students, parents, school administration, and fellow department members, mathematical experiments have disappeared from Randy's classroom practice. Still, as we shall see, the teacher-centred direct instruction of this trigonometry lesson is not incompatible with some aspects of Randy's image of mathematics.

**Mathematics as a Technical Language**

The previous lesson's emphasis on precisely noting the properties of trigonometric curves and carefully recording these observations in correct mathematical words and symbols captures Randy's view that "math is like a language" and students "need to develop the ability to converse and follow an argument" (RW-DIS-N-07). This theme of mathematics as a formal language with precise rules arises often in Randy's writing, repertory grids, and interviews.
In justifying the existence of mathematics as a secondary school subject Randy writes, "Mathematics is the language of many technical subjects, the language of problem-solving" (RW-DOC-D-07). When developing his first school subjects repertory grid, Randy provided the pair of opposite descriptors, "languages" and "science", to separate the disciplines. Although on most constructs mathematics is closest to physics and chemistry, along the "languages-science" axis Randy places mathematics nearer English and French (RW-INT-D-01a, see Appendix J). While English, French and mathematics are "languages" and "mental" activities, there is a distinction; mathematics is "quantitative" while the two more conversational languages are "qualitative" (RW-INT-D-01a). All three languages: mathematics, English and French; are described as "traditional" (RW-INT-D-01a), that is, they have a history to respect.

When, on a second school subjects repertory grid, mathematics was divided into separate strands, Randy described the traditional school content of algebra and calculus as being a "technical language" and "concise" "symbolic expression" (RW-INT-D-07, see Appendix K). Although not as strongly, these labels are also attached to the newer mathematical strand of fractal geometry. Randy sees a need to help students develop a precise language in this new sub-discipline, for "some of the vocabulary of dynamical systems will be encountered by many students in their future careers" (RW-INT-D-03).

As his opening sentence in response to the question, "What is mathematics?", Randy writes, "Mathematics is a body of knowledge, a set of rules, tools and techniques" (RW-DOC-D-07).
Mathematics, this collection of rules and techniques, deals "with things" rather than "with people" (RW-INT-D-01a) and is "objective" and "rigid" (RW-INT-D-07). "It's not just all fun and games and say unstructured play, but it's a discipline that has certain measurable aspects that you can count on" (RW-INT-D-03).

Reflecting his view of mathematics as a structured discipline, Randy's lessons are carefully planned and organized. During an interview I comment on Randy's extensive lesson preparation and structured approach and ask if he sees this as a quality of good teaching. Randy's response is mixed.

I think it's somewhat ambivalent. I don't think it's necessarily good or bad, although it tends to the good.... The danger is you don't let any students get in your way of completing it exactly as planned. So it's not necessarily a quality all the time.

This mixed view is characteristic of both Randy's conception of mathematics and teaching practice. Apparently opposing components of his subject image, such as the discipline's growth through open ended experiments and the view of mathematics as a rigid formal language, are reflected in instructional methods that appear to come from opposite ends of the spectrum.

**Mixed Practices**

Randy's image of mathematics as a precise formal language comes through most clearly in his teaching when he is introducing new content from the traditional core curriculum. Here his lessons follow the North American tendency to emphasize

As the previously introduced trigonometric graphs lesson continues and Mr. Walker provides concise blackboard notes for the students to copy, he turns to the topic of "Reciprocal Pairs".

Mr. Walker: "When \( \sin\theta \) equals 0 what about \( \csc\)?"

Student: "Undefined"

Mr. Walker: "Undefined - There's a gap there. What we call a discontinuity. It's discontinuous. And where does this occur?"

Student: "Zero, 180, and 360."

Mr. Walker: "Zero degrees, 180°, 360° - whenever the sine graph is 0."

The first line of the next note is put on the blackboard, "when \( \sin\theta = 0 \), \( \csc\theta = \text{undefined 'discontinuity'} \) (at 0°, 180°, ...)", and Mr. Walker continues. "What does the graph look like near where it would be undefined. I can't ask you what it looks like here (pointing to 0° on the projected graph), because it doesn't look like anything. Right? - It doesn't have a point at 0°. What does it look like near there?"

Student: "It is going to go straight up. - It follows an asymptote."

Mr. Walker: "Okay - so \( \csc\theta \) approaches - what kind of asymptote? -"

Student: "Vertical."

Mr. Walker completes the note with the second line, "\( \csc\theta \) approaches vertical asymptotes when \( \sin\theta = 0 \)."
Mr. Walker: "You've got to be really careful in mathematics. We are so picky about the way we use words" (RW-LES-N-01, RW-LES-T-01).

Although almost the full 75 minutes of this lesson featured such precise examination of the trigonometric graphs, there was an aside at about the mid-point in the period that both underscored Randy's focus on mathematical language and illustrated his potential to shift direction and engage students in rather open ended explorations.

It is coming up to the middle of the semester and the date when mid-term grades are released. At a pause in the lesson Mr. Walker takes time to outline the process for grade calculation and to comment upon and return recent work that he has marked.

Mr. Walker: "I was very pleased with the notebooks that I marked over the weekend. Most of them are going to end up with a mark of ten out of ten. Almost all the others are very close to that. There were just one or two that were low. - To keep a notebook all you have to do is to copy the notes off the board, have them dated, titled. I am going to start asking though that you underline the titles. A few of you are very economical with your paper, and one note starts right in the middle of another. There is no spacing so underlining will make them a lot easier to use, for yourself especially. I only get to see them occasionally but you get to see them everyday."

After a short pause to let this message be absorbed, the announcements about assignments and grades are continued. "Notebook mark was good. Notebook counts out of ten. - And then I thought that a really very good effort was made on bonus
assignments. I know that it took me forever to mark them. That's just because I was reading them all the way through. There was really a lot of good stuff there. Not only are most of you doing a good job on it, but some of you were doing unbelievably good work in exploring the question even more deeply than I asked you to, and coming up with some new mathematics. I'm impressed and the marks will reflect this effort" (RW-LES-N-01, RW-LES-T-01).

As the file of bonus assignments is passed around the class, the students retrieve their work and the focus of the lesson returns to trigonometric graphs. I ask Stephen, the student who sits in the desk beside me, if I can take a look at his work. He searches through his pile of assignments and quickly selects an example with the title Circles Squared and the "Superegg". His eagerness to share and his smile suggests that he is rather proud of his work. A quick read of the assignment shows that this pride is fully justified.

Stephen's introductory paragraph provides a good summary to his project.

I noticed that the equation of a circle, \( y = \pm \sqrt{r^2-x^2} \), can be slightly altered to form almost a dozen different shapes. If we use \( y = \pm \sqrt{r^2-x^2} \), and give \( e \) different values, 10 distinct shapes can be found.

The rest of the opening page contains ten sketches of graphs that resulted when \( e \) was assigned a single value such as 1, 0.5 or -0.5, or more generally came from subsets of the real numbers such as "\( e = \) odd integers and \( e>1 \)". Stephen's work goes on to conduct the same explorations using the equation for an ellipse as the starting point (RW-DOC-D-04).
In an interview, conducted a few days after the trigonometry graphs lesson, Randy talked about the assignment of bonus questions and his experience with them so far this term.

There is usually a spontaneous nature to the whole affair. Sometimes, something related to the current work comes up that seems a little bit more interesting or a little bit too deep to pursue in the class within the regular schedule, so I try to devise some way of getting the students to explore it on their own. Usually there'll be a bit of an introduction to the question given in class, then they're assigned to try to follow that up. Other times the question arises spontaneously from a student question or, or a student discovery. There's a few of the students in that course who will actually go home and come up with some fairly creative questions. (RW-INT-D-03)

School policies, having denied him the opportunity to take class time for mathematical explorations that extend the regular curriculum, Randy has turned to a strategy of bonus questions and marks. Students are expected to submit work, but there is little risk, as the marks received can not reduce their regular course grade.

They're obligated to hand in something. That's the commitment they make when they sign up for the course as far as I'm concerned. And some of the responses you'd have to describe as feeble. They're going through the motions. There actually turns out to be a few students that don't seem to have been interested in any of the questions. But from the amount of time they've spent on them, it looks like they wouldn't know enough about the question to know if it's interesting. They just don't feel like doing it. And a number of them will do fairly well on anything quite mechanical, but only a few of them are risk takers enough to actually give responses that you can call original, and those are usually much better rewarded. (RW-INT-D-03)

Randy appears ready to accept the fact that some students will not respond to his invitations to mathematical adventure and he is happy to focus on the successes. The interview continues with his description of these.
There's often an investigative aspect to it and, again much of the time it comes spontaneously from the students. They just see more in the question than I expected and I've been getting some stuff that is just absolutely excellent, that I end up copying so that the next time I ask that question I'll realize just how much it involves. (RW-INT-D-03)

Randy brings out a copy of a sample assignment, Stephen's work that I examined earlier in the week.

As far as I can remember, I gave him no kind of prompting at all for that question. That was completely his own.... I was very intrigued by it, especially by the level with which he pursued it and the different dimensions of it that he looked at. (RW-INT-D-03)

On issues of mathematics pedagogy Randy appears to be of two minds. When addressing the traditional curriculum, the formal mathematical language, he is at home using a teacher-centred approach. His lessons are well prepared and through the use of direct instruction he manages to execute his plans. On the other hand, when extending the curriculum or introducing new non-traditional topics, Randy's preferred approach is the use of rather open-ended independent student investigations. In many ways Randy's mixture of teaching practices and somewhat fractured discipline image parallels the debates presently taking place within the mathematics teaching profession.

A Personal Philosophy Under Construction

Randy's conception of the nature of mathematics appears to be in transition and does not fit conveniently under any of the labels provided in schemes for categorizing philosophies of mathematics (Carnap, Heyting & Von Neumann, 1983; Davis & Hersh, 1981; Ernest, 1991; MacLane, 1986). His struggles with some of
the tasks involved in this study and comments made during the subsequent interviews show that Randy is in the process of reformulating his personal philosophy of mathematics. Recent study and reading have raised new questions for Randy, but on a number of these issues he indicates that he is not yet ready to take a stand. "Well there are a lot of ideas in philosophy that I don't find I have to take sides with. - It's just interesting to listen to the whole thing. They're probably unanswerable anyway" (RW-INT-D-05).

As Randy works on his school-subjects repertory grid, searching for descriptive labels for the disciplines, he reports, "I can not believe how hard I find this" (RW-INT-N-01). He often goes back and revisits the indexing he gave on past constructs, making minor adjustments in the rankings provided for subjects. At times Randy becomes frustrated and while working on one particularly difficult construct asks, "How long should I puzzle at this at any stage? I change my mind back and forth as I think about each description" (RW-INT-N-01).

Even after the grid is complete, Randy is not sure of his description for mathematics. Reflecting on his choice of "deals with things" as opposed to "deals with people" as a feature of mathematics, he rethinks his position and suggests that it could change.

I'm not sure what I'll think tomorrow about dealing with people and dealing with things. Maybe there is a mark of a human aspect in mathematics as we're solving people's problems. We may, you know, get into a social issue or whatever, but that's an idea I really have to develop a little more before I want to say much about it. (RW-INT-D-01b)
In the end, after considerable careful thought, Randy is able to attach definite labels to most of the disciplines and the result is a dichotomous arrangement on the PrinCom display (RW-INT-D-01a, see Appendix J), with two separate clusters of subjects: the sciences on one side and the arts, languages and humanities on the opposite. Mathematics is more difficult to categorize and on six of the ten constructs produced, Randy places mathematics in the grey, middle ground between the opposing descriptors. Looking at the display, Randy acknowledges his mixed view of the subject.

Well, it looks like a group - the arts and sciences at opposite poles - and mathematics, although it's on the side of the sciences, it certainly seems to be off by itself.... It's harder to classify is what it seems to be saying. It's harder to say where mathematics exactly fits in. It's kind of between things, rather than being easily pinned down. (RW-INT-D-01b)

In particular, Randy places mathematics in the middle of the "truth determinable-truth indeterminable" scale and thus does not commit to either the absolutist or fallibilist position (Ernest, 1991). His recent studies in fractal geometry and chaos theory have shown Randy that some branches of mathematics are "much nearer to rapidly evolving and experimental and investigative" (RW-INT-D-01b) than traditional school mathematics, but this does not really alter his overall image of the discipline. Moving mathematics on the PrinCom display and thus, "putting it closer to some of those other things though really wouldn't be consistent with what I'm thinking. You know what I mean, truth indeterminable, I don't think it's any closer to that, or belongs any closer to that" (RW-INT-D-01b).
Randy is not ready to take a side in the debate between Platonism and constructivism and responds to my questions concerning the sources of mathematical knowledge with:

The discovering versus the inventing of mathematics - well I don't know where I stand on that. Um, it's probably a little bit of both. I think some days I could give you a better answer than other days. But I know it's an interesting debate among the experts themselves as to whether or not mathematics is invented or discovered... If the universe is put together mathematically then certainly we're discovering some of its workings and that becomes part of mathematics. But, it seems like the universe is not any longer a giant clock, at least in a dynamical system, and that affects all the basics that we seem to understand about the universe and mathematics. (RW-INT-D-05b)

The outer ring of Randy's concept map for mathematics (RW-INT-D-09a, see Appendix L), with the label, "Creators of Mathematics", appears to suggest a constructivist position, a view of mathematics as developing out of human thought. But, this is not completely Randy's intent, for he reserves the title "creator" for only a few key mathematicians.

What I had in mind as I was writing that was certain very prominent people in the history of mathematics who made enormous individual contributions to the subject .... It seems to me that, although there are many mathematicians making contributions, the contributions of some were just huge. They were almost super human in their abilities to grasp mathematics, to relate it to solve difficult problems quickly. (RW-INT-D-09c)

It appears that Randy does not see mathematics as part of general human culture, something to which all contribute.

Without a strong social constructivist conception of mathematics, the epistemology that underlies the NCTM reform proposals (McLeod, Stake, Schappelle, Mellissinos & Gierl, 1996), Randy is not ready to fully adopt teaching practices as outlined in the Standards (1989). These he reserves for new, non-traditional topics. "When studying fractals it's largely
student activity. The students sit down with a question basically, and then they do some exploring, gathering data or somehow investigating the problem" (RW-INT-D-06). Given the constraints provided by the present school system, Randy argues for a less adventurous approach for the core curriculum.

I'm sure other topics could be taught in the same way, but teachers have a lot of time pressure on them. In order to build a course like that you have to sit down and think about it with lots of time to do it. And another thing is that, you don't cover as much content in as much detail, and you don't master skills such as you have to do in algebra to the same degree. What you may do is, I think, make an awful lot more connections in math. And the concepts may become clearer but you may be sacrificing certain skills that some people might want to see in the course. So somehow you find a balance. (RW-INT-D-06)

**Mathematics as Problem Solving**

Randy believes that his subject is an important component of the secondary school curriculum, since "mathematics not only enriches one's general education, but continued study in the program will keep many career doors open" (RW-DOC-D-07).

I think the that the reason for mathematics being a prominent subject in education is the number of ways that it can be applied. If you ask the average parent for one of the most important subjects that the students take in high school, I'm sure that mathematics would be your number one or number two and I think that's because of the perception that almost every subject requires mathematics, requires you to use mathematical thinking or mathematical language or mathematical problem solving. (RW-INT-D-09c)

Concern for his pupils' future employability leads Randy to personally explore applications of mathematics and look for examples to bring to the classroom. But, his reasons for interest in the uses of mathematics extend beyond student needs.
For Randy, the discipline and its applications are intertwined. "I can't really justify the existence of mathematics all by itself. There has to be a reason why the system exists" (RW-INT-D-14d). As Randy will explain in the following paragraphs, mathematics and its applications can not really be separated.

In writing about the nature of mathematics, Randy states that the discipline "has developed, originally, out of necessity in areas such as land surveying, navigation, commerce and war" (RW-DOC-D-07). This theme of mathematics arising out of human efforts to solve practical problems is further developed in Randy's concept map for the subject (RW-INT-D-09a, see Appendix L). Here he shows a ring labelled "APPLICATIONS" drawn around the cluster of nodes that represent various branches of mathematics. Arrows run from a "Creators of Mathematics" (mathematicians) node through this ring into the body of the discipline. Randy explains the meaning of this picture.

Applications along the periphery are not meant to indicate any kind of secondary importance it's just a way of putting them together. I tried to indicate that I think in the history of mathematics problems, real needs, led to the mathematics. (RW-INT-D-09c)

Nineteen of the thirty-six nodes in the concept map represent activities, such as medicine, music, art, psychology and sociology, that others might not view as mathematics. Randy asserts that these belong in a picture of the discipline since they represent "real world problems of some kind that are solved using mathematical techniques or mathematical knowledge" (RW-INT-D-09c).
Along with applications, Randy's concept map shows another motivating force for the development of mathematics, "Desire for Understanding (Philosophy)". This human drive has led to "much modern mathematics that derives from pure research interests" and generally "runs ahead of applications development" (RW-DOC-D-07). Randy sees a synergistic relationship between pure mathematics and other disciplines.

On the other side of the coin certainly there are pure mathematicians who may not have any interest in solving any practical problems but still produce mathematics that someone else sees in a different light. There are some fairly well-known instances of physicists who have gone to some pure mathematical concept to fill in some blank in the structure of matter that they were exploring and in some cases had even such confidence in the mathematics that predictions were made as to say things such as the existence of particles that had never been detected before. And whole experiments sometimes, expensive I assume, were constructed just on hypotheses that were based on purely mathematical considerations....So there are instances of course, of the mathematics preceding the application. I think it works both ways. I think it's very much a human endeavour and there's no straight-line path from problem to solution. (RW-INT-D-05b)

The significance of applications in the history of mathematics implies, for Randy, an importance in the teaching and learning of the subject. "As I view my job as teacher today the applications have to be there. The applications are the reasons for the mathematics....I don't think we could justify the existence of the mathematical infrastructure in the education system without that" (RW-INT-D-09c). Unfortunately mathematics education, according to Randy, fails in this task. "It is somewhat doubtful, in my view, that university and probably even high school mathematics finds many direct applications in the lives of most people, beyond simple arithmetic. As of now we have not determined how to deliver useful, meaningful,
mathematics to the majority of our students" (RW-DOC-D-07). In a subsequent interview Randy expands on this idea.

I realize that when I'm expressing that thought, I'm revealing some of my own ignorance about how some of the current mathematics might actually have more application than I'm aware of. Notwithstanding that, I've come to believe that a large amount of it doesn't have any relevance for most of the students and maybe some of it has no relevance for any of the students. That we may not even know why we're still teaching it except that we always have....I think that our mathematics courses seem to be designed to make students into images of math teachers or people who have gone before and we haven't put enough thought into what valuable skills or topics we could include in our mathematics courses. (RW-INT-D-05b)

Randy realizes that in the past he has been part of the problem, but he has recently begun looking more closely at applications.

I was always the teacher who delighted in the mathematical ideas and knew there were applications out there somewhere, but the students would meet these in an appropriate time when they began to specialize and choose their careers. And so I felt it was my job just to make the ideas as clear and interesting as possible. But the applications didn't seem to be very many at the high school level nor was I particularly inclined towards that kind of thing....I am beginning to realize the problems people do in physics or in industry are really important and they're quite interesting areas. Before I'd say, I hadn't paid much attention to them and didn't consider them to be as interesting as the pure mathematical ideas. (RW-INT-D-09c)

Randy also objects to the common textbook approach, where any application is "pared down until it comes out looking like all the other questions" at the end of the chapter (RW-INT-D-09c).

I don't think it works....You may get there but I don't think that works well at all. I don't think that's the way people learn....They can learn mathematical algorithms but the understanding won't come about until they use it and can see, hands-on in some way, what it is they're doing.... Mathematics is the abstract, you don't start with the abstract. By the time the students and the teacher get to the end of that
mathematics topic, at least their interest is very much
exhausted so that the applications often just get paid
lip service. (RW-INT-D-14d)

In his recent studies, Randy has found that he personally learns
best when he starts with a problem and moves to the mathematics.

I really get excited about mathematics when I have a
problem in front of me that I don't know how to solve
and I start to explore and I occasionally look up
something that I don't know or don't remember.
(RW-INT-D-09c)

He sees this problems-to-mathematics order as best for his
students. He explains using the topic of graphing as an example.

Start from the concrete and move towards the abstract,
so the science should come first and the learning of
the graphing done with the data and with the motivation
to try to come to an understanding of the data, and
then the mathematician might refine that. Because
after all, I think that's what mathematics is, it's
just the refinement of some of the abstract ideas,
taken out of applications. It may be the history of
the subject but it also makes a lot of sense from the
point of view in the way people learn. (RW-INT-D-14d)

But, as Randy has found with other innovative teaching ideas, "to
use stuff like that you really need lots more time than you
have...You've got to work it within your time constraints and
those things take up much more time than I think most people
would imagine" (RW-INT-D-09c). As we shall see, the constraints
of the system mean that sometimes Randy has trouble putting his
plans into action.

**Connecting Science and Mathematics**

Along with his mathematics classes Randy also teaches a
Grade 9 science course, part of Northern High School's special
program linking science and technology. Over the past three
years the school has been experimenting with the curriculum for
this program and as a result there is no fixed prepared outline for Randy's course. He is frustrated with this "seat of the pants planning", for it conflicts with his desire for organization and careful preparation of lessons. Still, the lack of a course outline has some benefits, and Randy uses this as a justification for taking his class off in directions that he wishes to explore (RW-DIS-N-02).

During a unit on the particle model of matter, Mr. Walker provides a demonstration, dropping a piece of potassium permanganate into a beaker of water. The solid disappears and a purple colour spreads out through the liquid. Why? The class postulates that the potassium permanganate particles must separate from each other and become suspended in the spaces between the water particles. But why does the colour move through the water solvent? The pupils suggest that the potassium permanganate particles are bumped by the water particles and are thus pushed along. Mr. Walker, pretending to point to one individual solute particle in the middle of the beaker, inquires, "On what side will the bumps occur?" "All sides. From all directions, equally", is the popular student opinion. Mr. Walker continues, "So, why does the particle move? Do you think the particle should move or stay in one place?" After a brief debate the class holds a vote. Despite the evidence provided by the spreading purple colour, the majority believe that the potassium permanganate particles should stay in one place. Mr. Walker takes this science puzzle as an opportunity to develop some mathematical ideas, random events and probability, two concepts that he has observed are problematic for many people.
There are difficult concepts there. There are things that you sort of don't want to touch, might not be able to answer, or complete usually. So I think they've kind of just been filtered out of the curriculum whereas I consider them to be basic....We take it [randomness] for granted and there are things to know about random processes. So, I think that's something important that I could address, something that university students don't have a good grasp of at all. Students flounder in probability courses originally because they don't have any simple mental constructs to deal with some of the concepts. I think that the students learn how to do the problems, but I'm not sure they ever learn very much about probability. That's what I have in mind. That's what drives me to do it.

(RW-INT-D-02)

Mr. Walker suggests that the class imagine looking at just one solute particle and focus on its progress along one line across the beaker.

Mr. Walker: "Along this one line, in how many directions can the particle move?"

Students: "Two, left and right."

Mr. Walker: "According to our theory why might the particle move?"

Students: "It gets bumped by water particles."

Mr. Walker: "How do the chances of a bump from the right compare to the chances from the left?"

After some debate, the class concludes that the chances of a collision from either side are equal or each 50 percent.

Mr. Walker: "We are going to do an experiment or simulation of this situation. We need some way of deciding if a bump will come from the right or the left. We need something that is not predictable but has only two possibilities. Any suggestions of what we could use to tell us the direction of a bump?"

A number of pupils suggest tossing a coin. Mr. Walker is ready for this, pulls a penny from his pocket and tosses it onto
the lab bench. It comes up heads. "What do you want to call that, moving left or right?"

The popular response is "right", so Mr. Walker writes "Heads - moves right" on the blackboard and puts "Tails - moves left" below this. Two more tosses of the coin give another head followed by a tail. On the blackboard Mr. Walker charts the progress of the particle, showing that it is now one step to the right of its starting point. Mr. Walker continues, "Of course that is just one possible example of three bumps out of thousands of collisions. We need to look at more."

Mr. Walker describes what he wants the class to do for homework. Each student is to toss a coin 100 times, record the results, heads or tails, and graph the progress of a solute particle. To finish they must note how far the particle is away from the origin after the 100 bumps (RW-LES-N-04).

Next day the class begins with Mr. Walker asking for the results from the homework simulation activity.

Student: "You won sir!"

Mr. Walker: "How? I did not say anything about what might happen. I just asked questions."

Student: "Our theory was that it would stay in the same place. Mine moved."

Mr. Walker: "Who else found that their particle had moved by the end of 100 collisions?"

All but two of the nineteen students in the class report some total migration of the solute particle.

Mr. Walker: "We will not worry about left or right, just how far away the particle is from its starting point."
The results from the class are collected on the blackboard and an average distance of 11.1 is computed. With this new data in mind Mr. Walker encourages the class to re-examine their hypothesis. After a short debate a re-vote is taken. Again a majority vote for the solute particle staying in one place, but the margin is slightly reduced.

Mr. Walker: "What if we tossed the coin more times, a thousand times, a million times?"

As the debate continues a variety of opinions and arguments are offered, the majority of which focus on the idea that there should be equal numbers of heads and tails and thus they should cancel out.

Later, after the completion of this unit and Randy and I discuss the sequence of lessons, he reflects back on the class debates and votes.

It seems that kids have enthusiasm for that. They like the idea of talking about things that take you to the edge and I think they appreciate the idea that there are a lot of things that people can't answer. I mean, we have discussions on randomness so they get to express themselves. They get to take some chances in the discussions that they might not otherwise do when we're dealing with topics of certainty. (RW-INT-D-02)

As the lesson continues, Mr. Walker leads the class in the development of formal notes recording the coin toss experiment and the resulting data. As this is done the vocabulary and concepts of random events, Brownian motion, and random walks are introduced. Finally, returning to his question of larger samples, Mr. Walker turns to the computer that he has brought to class and runs a coin toss simulation program that reproduces the students' work for a sample of 2000 tosses. A distance of 44 steps from the origin results (RW-LES-N-08).
The initial question, "Why does the purple potassium permanganate colour spread?", originating in a science experiment, motivates two more class periods of mathematical exploration. Together, Mr. Walker and his class develop an argument as to why a solute particle could be expected to migrate away from its starting position and establish a function linking the number of bumps or coin tosses to the expected distance of travel. Freed from the constraints of a packed rigid curriculum, Mr. Walker takes the opportunity to expand the results of a simple science demonstration into a deep exploration of probability and random events.

A different picture emerges when, facing the pressures of a packed Grade 12 mathematics curriculum, Randy addresses the traditional topic of trigonometric functions. In previous pages I have described the interaction between Mr. Walker and his Grade 12E class as they looked at trigonometric curves. Later in this lesson, when introducing the words, amplitude and period, Mr. Walker made a brief digression to provide context and justify the topic.

Mr. Walker: "Does that [amplitude] sound like a word from physics? - You can produce graphs like this in physics. You could hook up a microphone to an oscilloscope. A microphone will convert sounds from your mouth to an electrical pulse. The oscilloscope then will display the pulse as a vertical displacement, but it will also change with time. In other words, time would be the horizontal axis. If you whistled into a microphone you would get a sine curve. And if you blew harder how would it affect the amplitude?"
Student: "It would go higher."

Mr. Walker: "So that's a word right out of physics. - And if you blew a higher note, what else would change?"

Student: "The period."

Mr. Walker whistles two notes, the second with a higher pitch, and pointing to the peaks of the sine curve graph continues, "You get more of these over the same amount of space, so the period is shorter. So we are getting words out of physics. There are a tremendous number of applications of periodic functions in physics and elsewhere. The mathematics of these simple curves is a frequent model for a number of physical systems. We will have a look at some of them, but first we want to get some of the mathematics down."

Here, although Randy acknowledges that the mathematics being studied can be used to model real concrete phenomena, the beginning point is the mathematical abstraction of trigonometric functions and the accompanying vocabulary details. As the trigonometry unit progressed, time pressures restricted applications to brief asides such as the above.

Proofs and Mathematical Truth

In his initial writing about the nature of mathematics, Randy expresses a strong absolutist position, stating that "mathematics, more than any other field of knowledge attempts to deal with truth....Mathematics is equally concerned with truth and with number" (RW-DOC-D-07). But, when exploring his vision of mathematics more deeply through the construction of the school
subjects repertory grid, Randy reveals that his image of mathematics as truth might be less strong. Here on the PrinCom display (RW-INT-D-01a, see Appendix J) mathematics falls some distance away from the labels "consistency required" and "truth determinable" and Randy, while expressing some surprise at this arrangement, does not reject it outright.

It's not as close to consistency required. I'm wondering what pulls it away from those things that we often associate with mathematics. Truth indeterminable almost seems to be in the wrong end of that line. Truths are very determinable in mathematics if we have the proper assumptions to begin with....It seems quite strange. (RW-INT-D-01b)

When discussing the links between mathematics and the natural world, Randy further questions the discipline's claim on truth. Although "mathematics has helped humans gain some understanding and control over nature" (RW-DOC-D-07), Randy rejects "the Laplacian reductionist view" (RW-DOC-D-07) and holds that this understanding is neither complete nor absolute.

It was largely believed by philosophers, mathematicians, and scientists that we almost knew everything there was to know. There were just a few details to fill in about how nature worked. And then, of course, quantum mechanics comes along and we realize the probabilistic nature of some truths. And now in the age of dynamical systems we're learning more about what we don't know or can't know, about how things are unsolvable....That's an idea that we hadn't accepted for a long time in western culture. We thought we were on the verge of conquering nature and now we find out that we can't even understand nature, let alone be the controller of it. So it's a pretty deep and important, I think, fundamental limitation of human beings that we're learning about. Now, who would have thought that mathematics would teach us that? Mathematics was supposed to be about finding truths and answers and the knowable. (RW-INT-D-01b)

Randy is also in the process of re-thinking his position on the nature of mathematical proof. During Grade 12 lessons examining the trigonometric solution of triangles Randy rejects
his students' loose arguments for the equality of lengths and angles and insists on the use of formal Euclidean geometry. He justifies this stance in a brief aside on the nature of mathematics.

There is a danger in geometry in trusting one's eye to give equal angles or lengths. We need to prove everything. That's how mathematics works. We start with things that all of us agree are true and argue from there. Anything we prove today can be used tomorrow to continue our work. We always need to refer to past proofs. (RW-LES-N-15)

In an interview, Randy further develops this view of mathematics as a formal system and links this to learning within the subject.

Of course that's how mathematics grows. That's how Euclid developed his geometry it seems, or at least that's the way he presented it to us. And that's the way we learn, period. We form concepts that help us to understand other concepts. It's the building upward sure and then building outward as well. And mathematics pays an awful lot more attention to that than other disciplines do, I guess. We try to make sure that our mathematics is consistent every which way. So we're always checking it to see that it doesn't contradict the earlier things that we assumed, that were fairly obvious to everybody. It's one of the essences in math, I'm sure. (RW-INT-D-05b)

As the interview progresses Randy softens his stance and admits some doubts.

Mathematical things are agreeable to anybody who can understand what is being said. At least, that's what I was told and came to believe for quite a long time. I'm aware now of some mathematical proofs that are so difficult that there might be only a few people who understand them. In which case, it might be challenging the notion of what a proof is and hence making it much more difficult for me to answer questions of truth. (RW-INT-D-05b)

In Randy's emerging, less formal image of mathematics there are different levels of proof and indisputability.

I think that what makes truth, truth, is indisputability at its very highest and maybe lower than that there are some graduations of convincibility where there seems to be overwhelming evidence; be it by
virtue of a large number of examples, a complete lack of counter examples, and otherwise strong evidence either in the form of examples or logical construction. But, I'm sure the new proofs in mathematics are making everybody rethink. The proof just done by Wiles, of Fermat's Last Theorem is much too difficult for a large number of people who are not in certain specialized fields to understand. So that's getting away from the original idea of proof. I may not really appreciate what mathematical proof is anymore because I used to think it was a lot simpler. (RW-INT-D-05b)

Randy's recent consideration of alternative definitions for mathematical proof interacts with his teaching experiences and his students' struggles with the more formal aspects of mathematics. Although "simplicity and clarity are the essence [of mathematics], the mathematics student will not always agree that such demonstrations are successful" (RW-DOC-D-07). Even when proofs "use only relatively elementary algebra students have troubles following the logic. They can follow the proof, but it's not very convincing. Experimental approaches are stronger. I have found that the algebra excludes a number of people" (RW-DIS-N-08b). "They know that it's right, but they still don't always feel that it's right in their system. Their intuition just isn't along the same lines as our logic. They construct their world differently and we're imposing ours on them" (RW-INT-D-05b). Randy realizes that formal logic, while mathematically correct, does not always produce a convincing argument for students. "They really have to have more than just their logic satisfied it's got to feel right too. We don't function on a logical level most of the time so that's not satisfactory" (RW-INT-D-05b). He believes that in mathematics teaching, "we just overlook the tremendous power of intuition which drives students"
(RW-INT-D-05b), and in his lessons Randy employs visual proofs that appeal to his pupils' intuition.

**Visual Proofs: Helping Students See Mathematics**

Randy's Grade 12 class is continuing its study of trigonometric functions and there is a request to look at one of the homework questions; "Show \( \sin(180^\circ - \alpha) = \sin \alpha \)." On the board, Mr. Walker draws a sketch of the sine function for a full cycle from 0° to 360° and turns to the class. "Any suggestions?"

Student: "Could you pick a particular angle? I tried ninety degrees and then they're equal. You get sine ninety on both sides so they are equal."

Mr. Walker: "That's true, but it does not prove that the equation is true. We need to show it for any angle. That's what we mean by a proof in mathematics."

Mr. Walker calls for a proof, but in fact, he sees a general demonstration as sufficient and proceeds to develop one using his sketch of the sine curve. "Suppose we measure out an angle of size alpha along the x-axis." Mr. Walker draws an arrow along the horizontal axis, starting at the origin and pointing right. He labels this \( \alpha \). "Now where is a hundred and eighty degrees on my graph?"

Student: "Where the curve crosses the axis."

Mr. Walker marks the 180° point and continues. "On the graph what would it mean to take away alpha? What move do I need to make to subtract alpha from one hundred and eighty?"

Student: "Move left."
Mr. Walker: "Yes, but how far?"

Student: "Same distance you moved before on the left for alpha."

Mr. Walker, satisfied with this answer, draws an arrow of length $\alpha$ along the horizontal axis, left from $180^\circ$, and marks the distance as $-\alpha$. He points to the tip of the first arrow (\(\alpha\)) and continues. "This is the angle alpha. How do we find the sine value here?"

Student: "Go up to the curve."

Mr. Walker responds with "Right", and draws an arrow up to the sine curve. "How would we find the sine at one hundred and eighty minus alpha?"

Student: "Measure up to the curve at the point of the other arrow."

Mr. Walker, responding with "Good", draws a second vertical arrow up to the sine curve.

Mr. Walker: "What do you notice about the lengths of these two vertical arrows?"

Student: "They're equal."

Mr. Walker draws a dashed horizontal line across the tips of the two arrows and over to the vertical axis.
Mr. Walker: "What does that tell us about the sine values then at alpha and a hundred and eighty minus alpha?"

Student: "They're equal."

Mr. Walker: "There is nothing special about our example for alpha. We could repeat all this for other alphas. So we have a proof" (RW-LES-N-11).

On this occasion, the above simple, but effective visual demonstration serves as a proof, but Randy is not always willing to be this non-rigorous. Two weeks later, when introducing the topic of trigonometric identities, Randy insists that the students go beyond graphical demonstrations of equalities.

The pupils enter the equation, \( y = (\sin \theta)^2 + (\cos \theta)^2 \), into the graphing calculators that Mr. Walker has brought to class. Graphing the equation they get the horizontal line, \( y = 1 \). Mr. Walker acknowledges the students' "discovery" that \( \sin^2 \theta + \cos^2 \theta = 1 \), and writes the equation on the board. But, the graphical evidence is not sufficient and Mr. Walker insists, "We need to prove it. We can see it on the calculator and it's pretty convincing, but we need a proof." The lesson continues with Mr. Walker, using the basic definitions of sine and cosine, leading the class through a formal proof of the Pythagorean identity (RW-LES-N-19).

In the previous section, while discussing the nature of mathematical truth, Randy showed that he is in the process of developing a flexible definition of proof. Randy, as he says, "may not really appreciate what mathematical proof is anymore" (RW-INT-D-05b), but he is not willing to completely abandon formal algebraic methods in favour of visual demonstrations.
Randy sees strong connections between mathematics and the arts, and indicates this in his concept map for the subject (RW-INT-D-09a, see Appendix L), where nodes labelled "Art" and "Music" are directly linked to another for "Geometry". When later explaining his map, Randy expands on this connection. "I just think in a person's brain, if you had studied geometry and art both at some time, then connections would be made. You know, perspective, pattern, and organization, spatial organization are all geometric attributes" (RW-INT-D-9a). In fact, for Randy, mathematics, or at least some branches of the discipline, is art.

It's obvious to anybody that there's beauty in mathematics, and fractals in particular, but in much of mathematics, especially geometry. And whether or not someone wants to define it as art, that's a matter of opinion I guess. But as the old saying goes, "I may not know art but I know what I like". Fractals are beautiful and they are visual, so we can't escape the connection to art. (RW-INT-D-08b)

Among the books on Randy's desk in the mathematics department office is a large format volume of computer generated fractal images (Peitgen & Richter, 1986). Often, for relaxation during spare moments, Randy opens this text and carefully studies a picture. Sharing classrooms with teachers of French and history means that Randy has limited bulletin board space available, but what he has is covered with commercial posters of fractal art.

The artistic dimensions of Randy's image of mathematics are further developed in his subject repertory grids (RW-INT-D-01a, RW-INT-D-07, see Appendices J and K) and in interviews where he reflects upon the grid PrinCom displays. In the first grid
developed (RW-INT-D-01a), Randy's labelling of the subjects produced two clusters; the sciences, chemistry and physics, side by side on the right and music, languages and history together on the left. Mathematics sits alone, some distance from all other subjects. Randy responds positively to the message contained in this arrangement.

I think it's kind of nice to see mathematics out there a little bit by itself, because we tend to get labelled. People outside of our discipline kind of dismiss mathematics rather quickly and probably lump us in with technical subjects much more than we belong. I think mathematics is still considerably an art.

On the second repertory grid, when mathematics is replaced by three of its sub-disciplines: fractal geometry, calculus and algebra; the multi-dimensional nature of fractal geometry is further accentuated. Calculus and algebra separate from fractal geometry which falls almost exactly in the centre of the grid. It captures aspects of all the other subjects and is, at the same time, rigid and expressive, subjective and objective, and both a descriptive and problem solving tool. Randy confirms this picture with, "I think it's kind of found, what I'd like to think of as its niche, right there where it is" (RW-INT-D-08b).

Randy's love for fractal geometry is not because he sees it as non-mathematical. In fact,

there's lots and lots of technical mathematics associated with fractal geometry. You can open books full of equations on fractals associated with differential equations, all in symbolic form, difficult to read for the, the non-initiated. It could look as mathematical as anything we normally think of as mathematics. (RW-INT-D-08b)

But, within this technical mathematics there are opportunities for self-expression.
Just because you might have some basic structure in mind doesn't mean that we can't improvise on it and be creative.... There is a sense of excitement that just rubs off of it, feeds on itself, and works on both parties, the teacher and the student. (RW-INT-D-07)

Randy wants to introduce his students to this excitement and struggles to find opportunities within his packed curriculum to build toward an introductory understanding of fractals, especially the Mandelbrot Set.

Mandelbrot Art

Since his first introduction to fractal geometry over ten years ago, Randy has been fascinated by the complex details of the Mandelbrot set. The intricate patterns that make up its boundary continually excite him. From past work he knows that the mathematics involved in this subject is accessible to Grade 12 students. Their course contains the key ideas, composition of functions and complex numbers, but getting to the Mandelbrot set does require some digression from the official curriculum. Randy would like his Grade 12 Enriched class to explore this topic, but he feels pressured to cover the regular course content. Looking back on the project, Randy describes his dilemma and choices.

For weeks since I first decided that I wanted to do it with them I couldn't find the right time. We were into mid-term exams and rushing to complete curriculum and always feeling a little bit behind in the Grade 12 course. So I tried to work it in, in a way I'd never tried before, in little bits over a period of about seven classes. In five of these we actually did something on the Mandelbrot set, and in two we did nothing at all.

On day one, with his lesson on transformations of trigonometric functions completed and 11 minutes remaining in the
class period, Mr. Walker switches topics to look at solving quadratic equations. Using the examples, \( x^2 - 9 = 0 \) and \( x^2 + 9 = 0 \), and a highly teacher directed approach, the roots, \( x = \pm \sqrt{3} \) and \( x = \pm i \sqrt{3} \), are identified. Mr. Walker responds to student comments that \( \sqrt{-3} \) is undefined with the suggestion that they can still talk about the quantity, keeping in mind that it is not a real number. The root, \( \sqrt{-3} \), is rewritten as \( 3i \) and then \( 3i \), with the introduction of the definition, \( i^2 = -1 \). As the class ends, Mr. Walker assigns for homework the task of calculating values for \( i^1, i^2, i^3, \ldots i^{12} \). (RW-LES-N-06)

Day two brings another rushed mini-lesson on complex numbers. The students provide the solutions for the powers of \( i \) calculated for homework and note that these cycle through four values: \( i, -1, -i, \) and \( 1 \). At the board, Mr. Walker plots these as vectors on a complex plane, and the counter-clockwise rotation around the unit circle is noted (RW-LES-N-09).

Mr. Walker begins his third compressed lesson by asking the class to recall their earlier work with the iteration of \( f(x) = x^2 \) and pixel rainbows. These ideas will be extended for the case of complex numbers. The class notes that all complex numbers with modulus less than 1 will, when iterated in the function, \( f(z) = z^2 \), spiral into the origin of the complex plane, while those with modulus greater than 1 shoot off to infinity. Using ideas and vocabulary from their previous pixel rainbow experience, students identify the basin of attraction as a unit disk centred around the origin of the complex plane and the Julia set as the unit circle. As class ends Mr. Walker assigns as
homework the task, "trace the path of $0+0i$ when it is used as the seed value for iteration of the function, $f(z) = z^2 + 3+2i$".

Although the 10 minute lesson is very rushed, with Mr. Walker essentially lecturing, the students remain attentive. There is a rising sense of anticipation. Mr. Walker is obviously excited by this mathematics and for some of the pupils his enthusiasm is contagious. Stephen, Mr. Walker's strongest pupil, stays after class to talk further, concerning the paths of points within the Julia set (RW-LES-N-12a, RW-LES-N-12b).

When the fourth short lesson begins, Mr. Walker finds that the students are confused and were not able to do the calculations left for homework. Acknowledging the messy nature of the arithmetic, Mr. Walker announces that they will now use a computer program that he has written to perform the calculations and plot the progress of the iterations. For the remaining 5 minutes of class time, functions of the form, $f(z) = z^2 + c$, with $c$ taking on a variety of complex values, are input to the computer and the iteration path of $0+0i$ plotted. Some traces spiral into attractors while others disappear off the screen on their way to infinity. Class ends with the announcement, "Tomorrow we will put all these ideas together and take a look at some beautiful pictures" (RW-DIS-N-16).

Randy is excited as he and I collect the media equipment for the last class in the Mandelbrot set sequence. We are going to watch the video, Nothing But Zooms! (Hubbard, Smith & Staller, 1988) which shows an animated exploration of the details of the Mandelbrot set. Randy finds these images intensely beautiful and tells me, "Every time I watch it I get shivers down my spine"
(RW-DOC-N-06). While Randy's pupils do not appear to be quite as passionate about the pictures, they watch the video intently and, when class ends before the program has finished, several students remain behind to catch the final images.

Looking back on the sequence of rushed lesson, Randy is not happy with the results and reports, "I'd say it was entirely unsatisfactory and I was wishing I could do it over again in one or two sessions" (RW-INT-D-06). Despite this negative appraisal Randy has provided an opportunity for his students to experience the art in mathematics.

**Mathematics as a Creative Activity**

Randy has "been picking up *Scientific America* sporadically since high school but hardly ever ran into anything in the Mathematical Recreations that [he] could appreciate". "I couldn't see what mathematics was in that, or how it would be important because I thought you solved problems with algebra. What was all this other stuff about?" (RW-INT-D-11b). All this changed during Randy's year of teacher education when he was "exposed to some neat stuff that really wasn't so terribly difficult" and discovered "that you really could be creative in mathematics" (RW-INT-D-11b).

Today Randy believes that "mathematics can be an enjoyable recreation for anyone interested in puzzles, games, problems or computer investigations....One might call such pursuits mental exercise, which, like physical exercise, has it pleasures and benefits" (RW-DOC-D-07). The pleasure is in the exploration
rather then the final product since, "it would seem that the
creative element cannot be focussed on practical problems alone"
(RW-DOC-D-07).

It is important for students to see the creative side of
mathematics for:

It probably seems to everybody in high school, it was
my experience, that everything's been done, that
mathematics is a finished body of knowledge....And so
the students, I think, get the feeling that they just
have to catch up. They just have to learn what
everybody else has learned and then once they have the
knowledge then they're finished. (RW-INT-D-05b)

Randy argues that this image is false. Mathematics is a changing
and growing discipline.

The new mathematics, the computer experimentation, and
things related to dynamical systems show us a whole
different view of mathematics, if not of the world
itself, where the questions aren't all answerable. The
situation is a lot fuzzier and uncertain and it
underlines something that's always been true about math
anyway, which is that mathematics is itself dynamic, it
is always being discovered, discovered and
reinterpreted and investigated. There are lots of
unsolved problems and I think it's important to try to
give students some of that flavour. Otherwise
mathematics is a dead subject and most of them don't
see any future in it. (RW-INT-D-05b)

Randy wishes to have students "see there's more to it....You
don't just do a page full of problems and you're finished. There
are always more questions to ask and more avenues to explore"
(RW-INT-D-03). "But it's very difficult, within the constraints
of both the system and our own experience, to actually accomplish
anything more than just the regular course of study" (RW-INT-D-
05b).
Randy's personal experience with creative and entertaining mathematical explorations has led him to provide his pupils with open-ended investigations, but such activity has been generally restricted to the special "Fractal Geometry and Chaos Theory" course that he taught while at Golden District Secondary School.

In my fractals course it's largely that type [open ended investigations] of activity....The course is not nearly as structured because it's not so content oriented. The students enjoy that tremendously, they tell me. In fact, that has to be one of the reasons why I do it this way, not so much that I have an objective that this meets so well as to just have the students enjoy themselves a little bit more. It's a better way to learn mathematics....There's an awful lot of direct involvement, it's not nearly so passive. (RW-INT-D-06)

Since transferring to Northern High School, Randy has been feeling increased pressure to complete the specified curriculum and reports, "I've managed to get myself into a little bit of a time line difficulty this year, so we have to always be forging ahead and there's no time to relax and let things develop at their natural pace" (RW-INT-D-06). During my recent visits to Randy's classes I have not seen his students involved in any major independent mathematical explorations. While there have been group and individual work in class, the tasks have been well defined questions set by the teacher. For the time being, until he feels released from the demands to cover all course content, Randy restricts his promotion of mathematical investigations to the Grade 12 bonus questions and independent extracurricular activities with interested pupils.
After a long drive I arrive in the late afternoon at Northern High School for my sixth visit with Randy, but he is not in the mathematics department office where we had arranged to meet. Looking through the classrooms I find him in the computer lab listening intently to Gary, a Grade 10 student, excitedly explain the workings of a computer program that he has written. Randy apologizes for keeping me waiting, but he just has to see the work that Gary has been doing. I join him for the demonstration.

Although Gary is not in any of Randy's classes I have observed the two of them interact briefly on each of my visits to Northern High School. Randy describes their relationship.

Gary is a Grade 10 student who comes into a French class as I'm leaving my math class in the same room and he is always taking note of either what's on the computer when he comes in, or diagrams or equations that are on the blackboard, and he wants instant lessons on all these things. He finds them fascinating and inevitably the conversation ends with his commenting how cool, or fantastic, or great this stuff is. Words that most of your students don't share with you even if they ever do have those feelings. He's developed an interest in fractals from a few of those things. Asks me endless numbers of questions, but takes some of those ideas and goes a fairly long way with them himself. (RW-INT-D-06)

Over the past weekend Gary has used his home computer to write a program that simulates the Chaos Game and generates a fractal called the Sierpinski Triangle. It is this software that Gary demonstrates for Randy and I. Although Randy has written an equivalent program himself and has used it to perform numerous experiments, he joins into Gary's investigations with the enthusiasm of a beginner. With his past experience he knows the questions to ask to get Gary thinking along new routes. "I wonder what would happen if the jump ratio was something other
than one to two?" Gary excitedly replies, "Oh! I can change the program to do that." He immediately makes an adjustment to the code to get a jump ratio of 1:3 and has the computer producing a new plot. The triangle fills in with no apparent pattern emerging. "That's strange!", says Randy as if seeing this result for the first time, "I wonder why that happens?" Gary makes a couple of attempts at an explanation but stops each time when he realizes that he is confused and has made some errors. Randy listens carefully, offering only simple encouragement and coaching, "Yes", "Sounds good to me.", "Why?", and "Are you sure?". When Gary finally comes to a full stop, Randy has a suggestion. "Seems to me it might be a good idea to try out lots of jump ratios and see the types of patterns you get. Then you might be able to figure out why some work better than others." Gary likes this plan and packing his books away announces, "I'll try that tonight and tell you what I get tomorrow" (RW-DIS-N-17).

As Randy and I walk back to the mathematics department office, his words make it clear that Gary is not the only one who has benefitted from the past half-hour.

Gary has been showing an amazing amount of energy and interest in the whole thing, almost overwhelmingly so....It is their enthusiasm that forces me to do things like learn how to program a computer just so I can talk to them....Somebody like that stimulates you. You want to think up good questions for them to work on. (RW-INT-D-06)
Case Summary: Subject Conceptions and Teaching Practice

For Randy Walker, "in mathematics there's still an essence of the undisputable" (RW-INT-D-05b). This tone of absolutism runs through his conversations about the nature of the discipline and at various times takes on Platonist, formalist, and instrumentalist shapes. Randy presents a Platonist image of an activity that is about finding truth; a body of objective knowledge that we can count on (RW-DOC-D-07, RW-INT-D-03, RW-INT-D-07). Platonism is melded with an instrumentalist view in Randy's understanding that humans have discovered some of the workings of the universe and fashioned this knowledge into a set of rigid rules, tools and techniques (RW-INT-D-05b, RW-INT-D-07, RW-DOC-D-07). Mathematics is a technical language that concisely, through the use of symbolic expressions, deals with the idea of quantity (RW-INT-D-01a, RW-INT-D-07). If we begin with the proper assumptions, Randy has faith in the formalist program to maintain truth and consistency through the careful use of deductive proofs, following the model originally provided in Euclid's geometry (RW-INT-D-01b, RW-INT-D-05b).

The absolutist dimension of Randy's subject conception can be seen in action in a number of the teaching episodes presented in the previous narratives and observed throughout the duration of the study. In the Grade 12 course: observations about trigonometric functions and their graphs were distilled down and precisely recorded as a list of facts (RW-LES-N-01,02,03,05,06,07), strategies for the solution of trigonometric equations were given as well defined algorithms (RW-LES-N-12,13,20), and
properties of the exponential operation were described as the "Laws of Exponents" (RW-LES-N-22). Formal proof methods were demonstrated to and required of students when establishing the validity of trigonometric identities (RW-LES-N-19,21). But, absolutism was not the only theme present in Randy's teaching or conception of mathematics. In both we see the beginnings of a fallibilist view.

Randy is not sure where he stands in the debate between Platonism and constructivism. His recent work and reading in the area of dynamical systems suggests that the world may not run in a mechanical clockwork fashion and that possibly our mathematics does not perfectly reflect truths encoded in the universe (RW-INT-D-05b). The beginnings of a problem solving, social constructivist view are revealed with his comments that mathematics is "very much a human endeavour" (RW-INT-D-05b) which has its origins in efforts to address human needs and problems (RW-INT-D-09c). Reflecting on his own probability experiments, Randy is modifying his formalist position as he expands his definition of mathematical proof and adopts the view that there may be validity in arguments based on the collection of overwhelming evidence (RW-INT-D-05b).

We see these newer, emerging views in action in the teaching episodes related to fractal geometry and probability. Here students conducted mathematical experiments and voiced conjectures based upon the data collected. A more tentative image of mathematical "facts" was presented when pupils were allowed to debate and vote on their choice of answers (RW-LES-N-04,08,10). Mathematics was portrayed as an open, growing
discipline with creative and artistic dimensions (RW-LES-N-I, RW-DOC-N-06, RW-DIS-N-17). Randy's problem solving conception of mathematics and more flexible view of proof also made limited appearances in his instruction concerning traditional course content. He expanded the topic of composition of functions to include an exploration of iteration (RW-LES-N-ii). Pupils were encouraged to pursue open-ended investigations in the form of bonus assignments (RW-DOC-D-04, RW-INT-D-03). In several classes the use of small groups promoted student mathematical conversations (RW-LES-N-09,11,12,13,15,16) and intuitive visual proofs were employed in trigonometry (RW-LES-N-11).

While the fallibilist image that Randy is developing with work in fractal geometry is to some extent being carried over into his view of more traditional mathematical topics, the reverse is also occurring; some of his absolutist views are projected onto the newer topics. Randy acknowledges the power of Monte Carlo methods for addressing probability questions, but still holds the view that a problem is not solved until an algebraic proof has been developed (RW-INT-D-05b). When experimenting with the iteration of functions Randy puts an emphasis on terminology as he sees a need for students to develop a precise vocabulary (RW-LES-N-ii, RW-INT-D-03).

In his philosophy of mathematics Randy stands with his feet in both the absolutist and fallibilist camps. Similarly his teaching practice is a mix of traditional direct instruction and features from the mathematics education reform program.
Understanding is not a matter of \textit{appreciating} the "real" causal links prescribed by Nature but of \textit{imposing} a purposive structure that emerges from the \textit{interaction} between investigator and phenomenon....What emerges from the interaction is a \textit{constructed} reality that is shaped in equal proportion by the investigator's purpose and the phenomenon's presentational aspect [emphasis in original]. (Lincoln & Guba, 1985, p. 152)

The study reported here has three central questions:

What are the conceptions of mathematics held by exemplary secondary school teachers, those who are attempting to implement the reform proposals?

To what extent are these teachers' instructional practices expressions of their subject images?

and

What are the struggles involved in these teachers' efforts to translate subject images into classroom practice?

To a large extent the first two questions have been addressed within the case studies related in the two preceding chapters. Through the purposeful selection, as participants, of exemplary secondary school mathematics teachers, this study has been able to fill-in some gaps left by previous research concerning teachers' conceptions of mathematics and the links between these beliefs and instructional practices.

In the studies cited in the literature review, participants were generally teachers who employed traditional teacher-centred transmissive lessons; the kind of instruction that provincial (Colgan & Harrison, 1997; Ontario Ministry of Education, 1992a, 1992b), and international (Crosswhite, 1987; Lapointe, Mead & Askew, 1992; McLean, Wolfe & Wahlstrom, 1987; Robitaille, Taylor
& Orpwood, 1996) surveys have shown to be predominant at all levels of mathematics teaching. Most of these teachers were also found to hold absolutist conceptions of mathematics; images that matched their teaching styles. On the other hand, teachers holding distinctly problem solving conceptions of mathematics were found only in studies (Philipp, Flores, Sowder & Schappelle, 1994; Wood, Cobb & Yackel, 1991) at the elementary level and where there was extensive coaching and support provided by the researchers. In these cases the teachers' developing fallibilist conceptions of mathematics were compatible with their new styles of instruction. The present study with Jonathan and Randy extends these results in a number of directions.

Jonathan, in Chapter 4, through his writing, repertory grid, and concept map, and the interviews analysing these products displays a social constructivist conception of mathematics. The classroom episodes related in Jonathan's case show the main themes of his mathematical philosophy at work in his planning and instruction. Thus in this case we find a teacher, exemplary in his use of instruction advocated in the reform documents, possessing a compatible social constructivist subject conception.

Randy's instruction does not stray as far from tradition, but links may still be drawn between his subject conception and teaching. In Chapter 5, Randy presents a mixed image of mathematics; one that has both absolutist and fallibilist dimensions. In a similar fashion his teaching is mixed, being quite traditional when addressing standard school mathematics topics and becoming more adventurous whenever newer content such as fractal geometry or probability is the focus. We see
parallels in Randy's emerging fallibilist philosophy of mathematics and his changing instructional practice. The development of Jonathan's social constructivist subject image and non-traditional teaching style, and the emergence of Randy's new fallibilist conceptions of mathematics and class use of mathematical experiments have occurred, unlike the situation in previous studies, in the absence of officially sanctioned reform programs.

The research did not set out to establish a causal link between subject conceptions and instructional practices, and the two case studies do not suggest that one exists. Although Jonathan possesses a social constructivist image of mathematics and his observed lessons predominantly display this philosophy, he is still capable, on occasion, of employing a transmissive style of instruction (JO-LES-N-06). Randy's vision of fractal geometry as an experimental discipline suggests that learning should progress through student investigations, but, in forcing his lessons, building to the Mandelbrot set (RW-LES-N-06, 09, 12a), to fit within the very limited available time, he uses a highly teacher-centred style. What can be said is, that these teachers' conceptions of mathematics appear to encourage them to make particular choices for instruction and, in the absence of significant competing factors, practice is compatible with beliefs. The translation from beliefs to instruction is not easy or smooth and involves considerable effort. My extended contact with Jonathan and Randy, observing their full teaching days, provided opportunities to record the obstacles placed in their way and the struggles to bring their visions of mathematics to
the classroom. The visions into practice translation process and the struggles involved are the focus of the following section.

Visions Into Practice

Teachers do enter into dialogue with innovation. The new practices and the old interact in complex ways. We can picture the new and the old overlapping to create a zone of turbulence and challenge. (Black & Atkin, 1996, p. 148)

Jonathan Ode and Randy Walker are unique individuals with different personal histories and professional contexts. Their conceptions of mathematics and teaching practices, while having some common themes, are quite different. Jonathan's philosophy of mathematics can be described as social constructivist while Randy presents a mixture of absolutist and fallibilist views. In their teaching practices, Jonathan more consistently than Randy displays features of the mathematics education reform ideas presented by the NCTM (1989, 1991, 1995) and the OAME/OMCA (1993). When facing opposition to their use of non-traditional instruction, Jonathan, after a short detour, continues along the reform path while Randy retreats.

Nevertheless, there are some parallels between the two cases. Both Jonathan and Randy experience challenge and turbulence in their efforts to put problem solving visions of mathematics into classroom practice. This challenge is partly external, in the opposition from students, parents, and school administrators, but it is also internal, as Jonathan and Randy work to construct arguments for instructional choices; arguments not for presentation to those in opposition but for themselves.
In examining these teachers' thinking and action within the zone of turbulence, questions may be asked and partially answered. What is the nature of the struggles experienced by teachers as they work to put subject visions into practice? What characteristics of a philosophy of mathematics appear to influence the extent to which a teacher translates vision into action? What features in a teacher's professional environment contribute to the construction of a subject conception that promotes and supports changed instructional practices? In the following sub-sections these questions will be explored, using the data from the two cases previously developed and field notes recording features of the teachers' professional lives.

**Visions into Practice: The Struggle**

To open up one's class so that students are pursuing problems whose outcomes cannot be easily foreseen is a hazardous business....What happens when students refuse to work independently? When they balk at taking risks in their learning? What if one's colleagues, the students' parents, or the school administrators decline to take their own gamble of supporting...'adventurous teaching'? (Black & Atkin, 1996, p. 132)

Ernest (1991) acknowledges that the impact on practice of a teacher's beliefs concerning the nature of mathematics, "is mediated by the constraints and opportunities provided by the social context of teaching" (p. 290). Teachers may be supported or hindered in the classroom expression of their images of mathematics, by the expectations of students, parents, school administrations, and colleagues. The results of studies (Cooney, 1985; Dorgan, 1994; Ferrell, 1995; Heaton, 1992; Kesler, 1985; McGalliard, 1983; Philipp, Flores, Sowder & Schappelle, 1994;
Prawat, 1992; Putnam, 1992; Raymond, 1993; Remillard, 1992; Thompson, 1984) examining the link between teachers' views of mathematics and their instructional practices reveals the constraints provided by the teaching context. In all cases, those teachers holding an absolutist philosophy of mathematics were also observed to employ a compatible transmissive style of teaching. Such agreement should not be surprising since in both their images of mathematics and their teaching practice these teachers fit the predominant mode. Neither their thoughts about mathematics nor their instructional methods would meet much opposition or challenge. On the other hand, of those teachers with a distinctly problem-solving image of mathematics, only the elementary school teachers in the studies by Wood, Cobb and Yackel (1991) and Philipp, Flores, Sowder and Schappelle (1994) translated their views into practice. Here the teachers had received extensive support through participation in graduate level courses and curriculum reform projects. In all other situations, teachers were found to modify their teaching practices to bring them into line with the teacher-centred norm.

In the present study, the teaching contexts of both Jonathan and Randy posed constraints to their translation of subject conception into teaching practice. They met opposition from pupils which, through the intervention of parents, resulted in challenges from their schools' administrations. In the face of these difficulties both teachers adjusted their teaching, but for Jonathan this was only a short detour. A month after being cautioned about student and parent complaints, Jonathan was observed taking a problem-solving approach that portrayed
mathematics as an inductively developed human construct (JO-LES-N-12). After his brush with authority, Randy's problem solving approach was consigned to the edges of his program, appearing in: optional bonus assignments, lesson extensions whenever a few minutes could be spared, a loosely organized experimental course, and in extra-curricular explorations with committed pupils.

Randy realizes that, "there are always more questions to ask and more avenues to explore" (RW-INT-D-03), but notes that "it's very difficult, within the constraints of both the system and our own experience, to actually accomplish anything more than just the regular course of study" (RW-INT-D-05b). The two teachers of this study found that there are risks, conflicts and challenges that teachers seeking change must struggle against.

As noted above by Black and Atkin (1996), there are intellectual and administrative risks in setting open-ended tasks for students. Jonathan has experienced such situations. Reflecting on the Grade 9 periodic decimal art project (JO-LES-N-07), he reports,

That study actually can construct multiple 'what-ifs' and students are likely to ask why. There are some funny patterns. I predict, you can predict, and I mean it's natural that the students are going to then want to know why, which I think is great. But for some teachers that's threatening. It's, I guess, because they just haven't done it themselves. I mean it's tough. (JO-INT-D-15c)

Teachers generally are not ready to take such risks and thus stick to the "binomial theorem because they know how to do that and it's so much more difficult to do something else" (JO-INT-D-15c). Randy also recognizes teachers' reluctance to tackle difficult topics. Discussing open questions in probability, he notes, "There are difficult concepts there. There are things
that you sort of don't want to touch, might not be able to answer, or complete usually. So I think they've kind of just been filtered out of the curriculum" (RW-INT-D-02). Jonathan sees teachers' adjustment to such risk situations as a growth process and notes that time and experience are required.

I think what has to happen is these teachers have to be exposed to that [open-ended problem solving] and at first they're going to get scared. After a while they'll get used to it and they'll begin to learn how to respond and their scope will widen. I think that the end result is we're going to have better teachers and better students. (JO-INT-D-15c)

The intellectual demands that come from a problems centred program do not all fall on the teacher. Open-ended investigations, reasoning, and mathematical discussions make intellectual demands of students. Secondary school pupils have experienced eight or more years of study in mathematics. This background defines the 'rules' of the discipline, what it consists of, how it is taught, and what the expectations of students will be. Schoenfeld's (1989) surveys of high school students' attitudes towards mathematics show that all were unwilling to commit any more than 20 minutes to the solution of any one problem. Colgan and Harrison (1997) found that Grade 12 students had come to accept teacher centred mathematics instruction with 81% reporting that they "liked to learn mathematics in school by listening to the teacher" (p. 7). Randy has observed such lack of intellectual commitment from students and notes that marks and not knowledge appear to be their goal.

The students just take the courses to get marks without any objective beyond that as to how they'd apply any of their knowledge. They just need a course that's a prerequisite for another course that's a prerequisite for a piece of paper that allows them into a profession. (RW-INT-D-14)
Changes in teaching style, especially those that create ambiguity and call for increased pupil effort, are likely to lead to student opposition. Such conflict reduces the range of possible classroom activities and effectively, pupils become covert curriculum decision makers.

The lesson does not simply belong to the teacher, children can and do make it their own. They put so much on the agenda of the lesson, to a point where, they are the curriculum decision-makers. They make a major contribution to the social construction of classroom knowledge. (Riseborough, 1985, p. 214)

Pupils' curriculum defining roles have been recorded in studies involving mathematics teaching. Cooney (1985) working with a beginning high school mathematics teacher, holding the view that problem solving was the core of mathematics, observed that "the means by which he had decided to teach mathematics conflicted with the expectations of many students, particularly the less able ones, about what constituted mathematics and how it should be taught" (p. 333). Within ten weeks of beginning teaching, while still maintaining that problem solving was important, the teacher had separated this from the content material of the course, the mathematical procedures that pupils were to learn. Problem solving was no longer the core of mathematics but an add-on to be addressed if time and student motivation would permit. Similarly Brown and Borko (1992) report how a beginning secondary school mathematics teacher's instructional intentions were adjusted in the face of what he believed his students were capable of or willing to do.

In this study Randy adjusted his demands of students to fit the efforts they were willing to give and converted his open-ended investigation projects into bonus assignments. He came to
accept that only a few students are risk takers and willing to make the effort to come up with original ideas (RW-INT-D-03). Both Jonathan and Randy value intellectual commitment and are willing to put in considerable time with students who wish to explore mathematics beyond the confines of the curriculum. Randy worked one-on-one with Gary, a Grade 10 student, encouraging his computer supported explorations of fractal geometry. Jonathan founded a school club, the "Math Senior Scholars", which met in his classroom after school to look at problems that extended their course curricula (JO-INT-N-06).

Students often enlist the aid of parents in mounting opposition to changed teaching practices and increased work requirements. For most parents the image of mathematics is one of fixed rules and procedures that they developed while experiencing the traditional mathematics instruction that teachers such as Jonathan and Randy are working to change.

Parents in middle class neighbourhoods, such as the locations of Western Secondary School and Northern High School, may have experience of mathematics study beyond the high school level, but that in many cases does not promote a more open view of the discipline. "Many educated persons, especially scientists and engineers, harbor an image of mathematics as akin to a tree of knowledge: formulas, theorems, and results hang like ripe fruits to be plucked by passing scientists to nourish their theories" (Steen, 1988, p. 611).

Parents may bring their concerns directly to teachers but more often they make their representations to school, board or government officials where they may find individuals with equally
traditional views of mathematics teaching. During Jonathan's most highly teacher centred lesson, the school vice-principal visited his classroom to collect some student data for a board survey. On exiting the room, as an aside to Jonathan, he commented, "I'm very impressed, everyone is very quiet and working diligently" (JO-LES-N-19). When such definitions of "good" teaching and learning are held by school administrators it is not surprising that adventurous teachers receive little support.

In fact, often times it's, it's just the other way around. The teacher who is creative and wants to try something new and gets in there and takes a chance and has a student who reacts negatively to that, maybe sees the guidance counsellor or something like that, that's the teacher who's going to get reprimanded because they're doing something out of the ordinary. They might be cautioned about, you know, you should keep everything a little bit more in line with what's done traditionally. So it's almost there in a negative way to hold back the good teachers as opposed to help along the others. (JO-INT-D-10b)

In this study both Jonathan and Randy experienced the results of parental complaints. Jonathan was cautioned by the vice-principal about concerns that his problem solving approach had raised (JO-DIS-N-03) and Randy was reminded that assessing his Grade 12 students' work on extended open-ended investigations was against school policy (RW-INT-D-03). They discovered that "being a teacher is not merely a matter of subject knowledge and methodological skills. Teaching also implies operating as a member of an organization, that is 'loosely coupled' and highly political" (Kelchtermans & Vandenbergh, 1995, August, p. 9). Neither Randy nor Jonathan appear to have developed or be interested in developing the skills needed to operate in the political domain. Neither directly challenged the school
authorities. Jonathan ignored the criticism and warning while Randy adjusted his practice to fit within the school's rules.

In secondary schools the strong departmental subcultures define what teaching approaches are acceptable and unacceptable (Grossman & Stodolsky, 1995; Siskin, 1991) thus reducing dissent and opportunities for intellectual development. "They constitute social communities of common thinking, feeling, and belief in relation to the nature of knowledge and learning, desirable and undesirable approaches to pedagogy, attitudes to student grouping, student discipline, and so on" (Hargreaves, Wignall & Macmillan, 1992, p. 7). Adopting innovative pedagogical methods can bring a teacher into conflict with colleagues. When techniques that violate the unwritten rules are employed, those following traditional patterns are threatened. Those opposed to change may organize covert interference, discontinue cordial social interactions, or indulge in openly hostile actions (Beynon, 1985).

Conversations in the Mathematics Department office of Northern High School (RW-DIS-N-07, 22) focused on issues of student discipline and how to get pupils to learn the rules and procedures of mathematics. All department members voiced opposition to alternative teaching practices such as collaborative group work. They "knew" it did not work and were not interested in research on the issue or experimenting with it themselves. While Randy accepted his colleagues' highly traditional views of teaching (RW-INT-D-05b, RW-INT-D-13), he felt very isolated by their collective lack of interest in mathematics beyond the school curriculum (RW-DIS-N-32). At
Golden District Secondary School, his former workplace, he had found one "kindred spirit" with whom he was able to discuss his explorations in fractal geometry, but at Northern High School, without anyone with which to share ideas he was not sure that he could continue his work (RW-DIS-N-32).

Jonathan was not as isolated at Western Secondary School, for there he had colleagues who to some extent shared his interests in mathematics and also experimented with alternative teaching styles. With their desks clustered together in one corner of the departmental workroom these three were able to participate in supportive conversations concerning mathematics and its teaching (JO-DIS-N-08,09,12). The rest of Western Secondary's mathematics department appeared to employ quite traditional styles of teaching and Jonathan was frustrated in his efforts to bring about change. In preparation for an official school board-wide "Math Day", a day set aside for alternative exciting mathematics activities, Jonathan provided his fellow Grade 9 teachers with copies of collaborative problem solving activities from Get It Together (Erickson, 1989, pp. 132-137, JO-LES-D-17b, see Appendix G). Later, while using these activities in class (JO-LES-N-17), he confided to me that, despite the aims and plans for Math Day, he suspected that in most classes lessons were proceeding in their regular direct instruction manner (JO-DIS-N-13).

Recent studies (Barnes, 1995; Edwards, 1994; McGlamery, 1994; Secada & Byrd, 1993) examining the processes of change in secondary school mathematics teaching have highlighted the need for administrative support and department-wide commitment;
features that were missing in both Jonathan's and Randy's school settings. Both participants in this study talked regularly of a lack of support, but put this in terms of the teaching population in general rather than relating it directly to their particular situations.

I see huge problems in education today. Ones that need to be solved before teachers are going to be able to go on and do what they need to do. It's too often the case that all of these other problems are at the root of what's not happening. What really happens is everybody comes in and says well what's wrong with the teachers? You're not teaching today's mathematics you're teaching a hundred years old mathematics. And it's not necessarily their fault. (JO-INT-D-15c)

"What we're learning in the teaching profession is that if change is to come about, then it has to be more than just policy. We don't have the kind of support that goes with a commitment to the change" (RW-INT-D-14).

For Jonathan and Randy the major problems lie in a lack of time; time within the classroom to address topics in depth:

If we were ever to set out to take any effective approach to improve mathematics education, I think we'd do less and we'd do it better. So many curriculum attempts have involved just adding more and taking nothing away. And then we're forced to try and be efficient in our presentation and we end up with the lecture formats that we have now, rather than much of a discovery and exploration approach. (RW-INT-D-14)

and preparation time to develop interesting and meaningful student activities:

I don't have time to go to Hydro and ask them, can you give me some unique applications that I can use in my classroom. Neither do any other math teachers. It happens because once in a while a teacher does something which goes above and beyond, not just the call of duty, but above and beyond what would normally be expected. That happens. But it happens at a cost to that individual too and sometimes their classroom. (JO-INT-D-15c)
Not all problems or impediments to change originate out in the teachers environment; some come from within. Randy's image of mathematics as a technical language (RW-INT-D-01a, RW-DOC-D-07, RW-DIS-N-07) that his students will someday employ in their future careers, leads him to provide lessons from which students can learn the details of that language: the terminology, definitions, and algorithms of use. Jonathan, although holding a social constructivist philosophy, still on occasions expressed concerns for students' learning that echoed an instrumentalist 'back-to-the-basics' view. While waiting for his OAC-Finite Mathematics class to complete a quiz (JO-LES-N-12), Jonathan remarked to me, "It's taking longer and longer each year for students to do simple work. They lack so many basic skills. This is making me think, maybe we do need a database of knowledge; things such as arithmetic skills" (JO-DIS-N-10).

Kilbourn (1992), in relating the struggles of a history teacher, who like Jonathan and Randy was attempting to change to an inquiry mode of instruction that reflected his new understandings of subject, notes, "In some teaching situations part of the skill is to honour the demands of one agenda without sacrificing the integrity of another. How can interaction be promoted while maintaining a degree of rigour with regard to the subject?" (p. 83). Teachers seeking change must work in a zone of turbulence that involves their classroom, teaching milieu and personal thoughts.
Visions into Practice: The Commitment

The fundamental issue from which mathematics teachers cannot escape is that a commitment to a theory of mathematical knowledge logically implies a particular choice of syllabus content and teaching style. (Lerman, 1983, p. 65)

It can be argued that there is a logical connection between a teacher's conception of mathematics and their choices for instructional practice, but, as discussed in the previous section, serious impediments can lie in the way of this logical progression. Teachers face considerable struggle in the classroom realization of images that motivate teaching in non-traditional styles. Studies (Cooney, 1985; Dorgan, 1994; Kesler, 1985; Raymond, 1993) have reported cases where the connection was not made and teachers employed methods that did not match their professed subject images, but were compatible with school norms and fit student expectations. The teachers had been socialized into adopting the school and department patterns. But socialization to the norm is not inevitable. The process involves "a constant interplay between choice and constraint, between individual and institutional factors. Individual teachers are not merely passive recipients in the process of socialization" (Brown & Borko, 1992, p. 221). They may resist normative forces and make their own independent instructional choices.

Both Jonathan and Randy do not accept their school and department patterns. They struggle against the norm to bring different views of mathematics into practice. They possess conceptions of mathematics that are strong enough to provide the
motivation for efforts against the constraints provided by their school contexts. In particular, Jonathan chooses to silently ignore the school's administration and continue to make a problem solving approach the core of his teaching.

Senior teachers such as Jonathan and Randy are in no professional danger from the mild rebukes that they received. Their competence has not been and can not justifiably be questioned. Still, for experienced teachers, criticism of classroom practice generates a sense of vulnerability (Kelchtermans & Vandenberghe, 1995, August).

Conceptions of subject are just one cluster of a teacher's educational beliefs (Pajares, 1992). Teachers also possess perceptions of self and their effectiveness as a teacher. A criticism of teaching approach is first and foremost a negative comment on teacher efficacy and a challenge to self-esteem. There is a need for the teacher to rebuild self-confidence and self-concept through a reasoned defence of their actions. The arguing of such a defence is essentially internal and the teacher, as in this study, may not bother presenting the case to the challenging school officials, students or parents. Thus the argument must be satisfying to the individual and follow their rules for logic and proof.

For Jonathan and Randy the models for reasoning, logic, and proof lie in their knowledge of and experience with mathematics and physics. They project the requirements of mathematical logic onto their own thinking and thus an argument is not valid if it violates the rules of the predicate calculus. Basic postulates must be assembled and a deductive sequence developed from these
to desired consequences. Consistency in the defence is of prime importance, for in logic the existence of one contradiction means that all statements may be taken as true. In turn, these high standards mean that the base from which action is argued must be strong.

Jonathan presents a complex, integrated and consistent set of beliefs about mathematics. There is a problem solving theme that runs throughout the writing, repertory grid and concept map generated for this study and the interviews that further explored these products. He acknowledges that for some users of mathematics, the discipline may be a set of algorithms, but argues that these procedures were originally constructed by humans through investigations in the pursuit of solutions to problems. Formal research mathematics, in Jonathan's view, is also built on a constructivist platform. Although mathematicians may work with systems that do not model the physical world, their actions continue the original experimental processes for building mathematics. They develop their understandings inductively, building from patterns observed in abstract symbol systems.

Jonathan's strong conception of mathematics provides him with a solid base for personally arguing for his classroom use of group work, student discussions, inductive reasoning from patterns, collaborative problem solving, and open-ended creative investigations. In the interviews and discussions exploring the reasons for his pedagogical choices (JO-INT-D-01,02,11b,15d; JO-DIS-N-06,07,08,11,14) Jonathan linked his decisions back to his image of mathematics. When his teaching approach was questioned, Jonathan was able to construct a personally
satisfying argument that allowed him, with reason, to persist in his efforts to reform mathematics education.

Randy's conception of mathematics, although complex, is less integrated and complete than Jonathan's. His philosophy is a mixture of absolutist and fallibilist beliefs. On a number of points, such as the origins of mathematical ideas and the nature of proof, Randy reports that he has not yet made up his mind (RW-INT-D-05b). Randy's teaching practices are also mixed. The episodes related in the case study show both examples of teacher centred direct instruction and situations where pupils are involved in open-ended investigations. Of Randy's variety of teaching approaches, only those that require students to participate in significant problem solving activities are challenged and thus require a defence. With his more fractured image of the discipline, Randy has difficulties completing an argument that would support pushing ahead with his plans and he adjusts his program to fit the school rules and student wishes.

Randy does hold one strong, tightly linked cluster of mathematical beliefs; those related to his recent work in fractal geometry. He consistently presents this sub-discipline as experimental, inductive, open, growing, and creative. His teaching efforts in this area, although reduced to extras in his curriculum, carry his image into practice. As Randy explores the subject of fractal geometry and its teaching, new ideas emerge that challenge both his more general conceptions of mathematics and his traditional teaching style.

The relative success of Jonathan and Randy in putting their subject conceptions into practice is similar to the results of
Thompson's (1984) research. Of her three research participants, the teacher with the most integrated belief system was most successful in translating subject image into teaching action. Moreover when practice did not match subject beliefs this participant was aware of the conflict and reflected upon it. Shealy (1995) found a similar pattern in a recent study examining the translation of beliefs about mathematics teaching into practice. This study involved two beginning secondary school teachers, recently graduated from an education program that emphasized open-ended investigations. Both were excited about such student activities and had intentions of bringing them to their classrooms. The participant whose beliefs about teaching were well-developed and integrated was able to translate these into practice while the other who held isolated and competing beliefs retreated from the use of open-ended investigations.

The Perry (1981) scheme of cognitive growth can be employed to compare Jonathan's and Randy's positions concerning mathematical epistemologies and teaching. Their statements in multiple interviews indicate that both are at least at the third stage in the scheme, Relativism. They realize that there exist multiple images of mathematics and a variety of approaches to teaching. Moreover, as Jonathan indicates, they understand that one can establish reasons for each position.

This teacher approached everything from an algorithmic point of view because that's where the teacher obtained the security to teach. Because they, the teacher, would essentially learn an algorithm and then do it in class. And the students were successful in that teacher's class because they always knew that the teacher was going to test them on something just like they had been shown how to do. (JO-INT-D-10b)
Acknowledging other teachers' reasons for traditional approaches, Jonathan and Randy are not judgemental, but they do believe that their positions have greater validity. Randy, speaking of his fellow department members makes the argument,

I could hardly criticize them in some way. I really believe that they intend to do a very good job at what they do, but their view of their job is just different from mine. It is a little bit narrower. They stick to doing things exactly the same way all the time. They're not the slightest bit interested in mathematical problems or anything like that. It is just getting the job done every day. (RW-INT-D-13)

Jonathan has in fact progressed beyond Relativism and has reached the stage of Commitment. He has made a choice of epistemologies and through reasoned thought come to a social constructivist position. From this stage of commitment he is able to take action and adopt practices that embody his philosophy. Randy appears to be in transition between Relativism and Commitment, and has not yet settled on an epistemological stance. Perry notes that periods of transition between stages, when one is trying to find a path, are unsettling. Arguments are not clear and there are choices to be made. It is difficult to select definitive actions and one may act in contradictory manners. Randy would appear to be at this point in forming his philosophy of mathematics. During periods of cognitive turmoil there may be a temporary return to Dualism and a desire for authorities to make decisions and select a course of action. At times Randy takes this approach to mathematics education reform, calling for new official guidelines explicitly mandating the change.

It has to come down from the top that certain changes are necessary, and maybe the teachers are given some flexibility to learn how to interpret those changes.
But if there aren't prescribed changes in the curriculum and support materials to go with them, nothing happens... There won't be positive results until teachers really know in a day-to-day sort of way what the changes mean. (RW-INT-D-14b)

Looking Above and Beyond the Curriculum

Each project began with the premise that developing collegiality among professional mathematicians and teachers can reduce teachers' sense of isolation, foster their professional enthusiasm, expose them to a vast array of new developments and trends in mathematics, and encourage innovation in classroom teaching. (Black & Atkin, 1996, pp. 145-146)

Studies (Underhill, 1988) have found that mathematics teachers' subject conceptions show little change with years of professional experience. Randy and Jonathan are exceptions to this pattern. In interviews Randy talks of his changing image of mathematics: that experimentation can be a source of mathematical ideas (RW-INT-D-01b), that mathematics may not be a perfect representation of the universe (RW-INT-D-05b), and that there may be methods of proof beyond the Euclidean deductive model (RW-INT-D-05b). In the case study we witness Randy working on these new ideas and fitting them with his former conceptions of mathematics. Jonathan, with his graduate degrees in mathematics and philosophy, entered the teaching profession with an expanded conception of the discipline. Still he reports that, although he has retained his core philosophy, there have been changes in details, and he sees that his classroom practice has "changed considerably since [his] first year of teaching" (JO-INT-D-14b).

There are features in these two teachers' professional lives that appear to have contributed to expanding subject images;
conditions that may be missing from the experience of others. Jonathan and Randy have not directly sought out alternative visions of mathematics, but they have pursued mathematical knowledge. In recent years Randy has spent much time studying fractal geometry and for Jonathan a "primary motivation has always been to learn" (JO-INT-D-14b).

"Distinguishing knowledge from belief is a daunting undertaking" (Pajares, 1992, p. 309) and in fact beliefs are often taken to have a cognitive component. Ernest (1989a), when examining teachers' knowledge and beliefs about mathematics, links the two with the view that teacher thought has a cognitive outcome, knowledge, and an affective outcome, belief. Thus as teachers pursue mathematical knowledge they also meet opportunities for change in their beliefs about the discipline. In particular, when teachers work on their own personal mathematical investigations and become "creators" of mathematical knowledge they gain firsthand experience with the substantive and syntactic structures of the discipline, the critical elements in Shulman's (1986, 1987a) subject knowledge base for teaching. In the search for such knowledge, teachers must look outside their classrooms and schools.

Increasing classroom experience brings greater instructional mastery and comfort, but generally this is due to developing instructional craft knowledge and does not involve intellectual growth within the teacher's subject. In fact, for many teachers, as the years since formal subject study pass, interest in their discipline fades (Sikes, 1985). Similarly department and school collegial relationships, although important, may not support
intellectual development. Department based collaborative activities involve exchanging "tricks of the trade" rather than discussions concerning the nature of disciplines or underlying principles of instruction (Hargreaves, 1992). Staffroom conversations focus on classroom anecdotes, comments on individual pupils, and issues on which participants are unlikely to have professional disagreements.

Educational theory, long-term plans, discussions about basic purposes and underlying assumptions are virtually absent features of teacher talk. Sharing is confined to stories, tips and news - to things that will not intrude upon or challenge the autonomous judgement of the classroom-isolated teacher. (Hargreaves, 1992, p. 221)

Thus relying on classroom experience and relationships that arise by chance within a school placement may not provide opportunities for growth. There is a need to seek opportunities beyond the immediate professional environment.

The study by Philipp, Flores, Sowder, and Schappelle (1994), the one case where teachers were found to translate problem solving conceptions of mathematics into classroom practice, shows the power of extended collegial networks. In this case the four elementary school teachers involved had national and state level opportunities to meet with other teachers interested in mathematics education reform and they had developed ongoing collaborative links with university faculty. Similarly a US study comparing the professional interactions of presidential award-winning mathematics teachers to those of other mathematics teachers (Noddings, 1992) showed that awardees had more collegial contacts outside their schools, not only with other teachers but
also with university professors and district and state administrators.

Furthermore there is a need for such extended professional interaction to address more than just issues of teaching and learning mathematics. Raymond's (1997) study with beginning elementary school teachers shows the dominance of absolutist conceptions of the nature of mathematics over non-traditional pedagogical beliefs. Here, although the teachers had been introduced to the mathematics teaching reform messages of the NCTM (1989, 1991, 1995) and professed agreement with such non-traditional instructional methods, in their classrooms they employed teacher-centred transmissive lessons. In the struggle between deeply held instrumentalist images of mathematics and the goals of the NCTM, the conceptions of discipline won out, suggesting that "beliefs about the nature of mathematics are more strongly linked to actual teacher practice than are pedagogical beliefs" (Raymond, 1997, p. 573). Teachers require experiences that encourage change and growth in beliefs about the nature of mathematics.

Jonathan and Randy have sought out opportunities to interact with university faculty and fellow teachers and in doing so have looked at issues that lie above and beyond their course curricula and related teaching methods. Randy has established a link with a professor at a local university; one who has published extensively in the field of fractal geometries. It is here that he has received guidance and encouragement in his study of the subject and his development of related classroom activities (RW-INT-D-13). Randy's changing conception of mathematics and
the investigations that he provides for his pupils are evidence of the impact of this liaison.

Jonathan also has contacts with university faculty, but in his case the links are more numerous and eclectic. Through these relationships he has explored: new interesting problems in calculus and differential equations, fractal geometry, chaos theory, and the history and philosophy of mathematics (JO-DIS-N-03,14,20,29). These activities and curriculum development projects have put Jonathan in contact with other like-minded teachers who are also seeking new mathematical knowledge. Jonathan values this network of colleagues and the intellectual stimulation their interchange provides.

There's a large number of people that I've met and have established an intellectual bond with. They always affect me when I listen to them, and I'm sure it's reciprocal when I talk to them about things. I mean that's what the intellectual community is all about as far as I'm concerned. You stimulate each other with new thoughts. (JO-INT-D-14b)

In this intellectual community, conversations are not about school mathematics or possible methods of instruction, but focus on philosophical issues and interesting mathematical investigations that lie above and beyond the school curriculum. The opportunities provided by these conversations for Jonathan to analyse, re-work, strengthen, and extend his conception of mathematics is evidenced in the complex integrated subject image that he presented through his writings, repertory grid, concept map, and interviews.
Reflections on the Study: Internal and External Validity

Studies of open classrooms, free schools, or other radical educational innovations are often conducted using case studies or ethnographic methods. In these studies the researcher is attempting to portray the workings of circumstances that differ dramatically from what typically presents itself in the "natural" functioning of our society and our educational systems. It is as if the researcher is attempting to document with vivid characterizations that nature need not be the way it typically is. (Shulman, 1988, p. 14)

This study has developed pictures of two unique teachers' conceptions of mathematics and classroom practices. Links between the participants' subject beliefs and their instructional methods have been made, and in particular their struggles in translating subject images into action have been related. The use of a case study approach has permitted the development of vivid characterizations, but also raises questions of internal and external validity (Merriam, 1988). There is a need to ask, "How faithfully do the descriptions and stories related here capture the realities of the participants' beliefs about mathematics and their teaching practices?", and "To what extent are the findings concerning Jonathan and Randy applicable to other teachers and educational settings?". The two issues of "trustworthiness" (Lincoln & Guba, 1985, p. 281) and "transferability" (Lincoln & Guba, 1985, p. 124) are addressed in this section.

Thompson (1992), in a survey of the research concerning teachers' beliefs about mathematics, notes that:

Any serious attempt to characterize a teacher's conception of the discipline he or she teaches should not be limited to an analysis of the teacher's professed views. It should also include an examination of the instructional setting, the practices
characteristic of that teacher, and the relationship between the teacher's professed views and actual practice. (p. 134)

The present study goes beyond Thompson's call for "triangulation" (Lincoln & Guba, 1985, p. 283). Along with addressing the issues that Thompson identifies, the study also employed "methodological triangulation" (Mathison, 1988, p. 14) when exploring the participants' conceptions of mathematics. Writing in response to questions concerning the nature, history and foundations of mathematics, repertory grids comparing mathematics to other school subjects, and concept maps displaying the teachers' structures of the discipline were all used to provide data on the study participants' beliefs. Each of these techniques was followed by an interview in which the teacher and I explored ideas that surfaced in their earlier work. Through this use of multiple methods, the study went beyond an examination of the teachers' "professed views" of mathematics.

In exploring teachers' subject conceptions through a variety of techniques, this study has gone beyond previous research investigating beliefs about mathematics. All but two of the studies found in a search of the literature and cited in Chapter 2 report using only structured interviews or surveys or a combination of these two research instruments. McGalliard (1983) and Raymond (1993, 1997) employed, as did the present study, written responses to open-ended questions concerning the nature of mathematics, but only Raymond and the research reported here used the teachers' responses as the focus of subsequent interviews. No study reported using instruments such as repertory grid or concept mapping; techniques that require
participants to reflect on and reveal deeply held images of mathematics. The application of three research instruments and follow-up interviews illuminated the participants' conceptions of mathematics from a variety of angles and allowed the study to explore the depth and complexity of Jonathan's and Randy's subject beliefs. It is this complexity and connectedness that supported Jonathan's efforts to carry visions into everyday instructional practice. Previous studies (Cooney, 1985; Dorgan, 1994; Kesler, 1985; Thompson, 1984) found that teachers with a problem-solving orientation to mathematics failed to fully translate their subject images into classroom practice. It is possible that these research projects' methods only explored the surface level of the teachers' belief systems and failed to locate underlying inconsistencies and instrumentalist views. Raymond's (1997) close analysis of her data showed conflicts between deeply held subject beliefs and more surface opinions about how a subject should be taught. The teachers' instrumentalist discipline images were dominant and encouraged transmissive styles of instruction.

The classroom observation schedule employed in this study was also more extensive than that of most reported research. Only Thompson (1984), Wood, Cobb, and Yackel (1991), and Ferrell (1995) watched 20 or more individual lessons by single teachers, the minimum number of observations in this project. This study, in a style similar to that employed by Wood, Cobb and Yackel, spread classroom visits out over an extended time period and thus observed teaching of a variety of topics. This comprehensive classroom data, more extensive than that recorded in other
secondary school based studies (Kesler, 1985; McGalliard, 1983; Wilson, 1994), permitted the identification of links between beliefs and practice.

In addition, the full teaching days spent with the study participants and observation of regular school routines allowed me to describe their teaching contexts. Observation of the struggles involved in bringing beliefs to practice and identification of features in these teachers' professional environments that aided or hindered the translation process came from this extended contact and opportunities to indulge in spontaneous conversations.

A study wishing to develop understanding of individuals' actions must go beyond objective observation and ask for the "facts of the case" as perceived by the research participants. "The case study worker constantly attempts to capture and portray the world as it appears to the people in it. In a sense for the case study worker what seems true is more important than what is true [italic in original]" (Walker, 1980, p. 45). For a study to be trustworthy the report must first be credible to the participants (Lincoln & Guba, 1985) and this necessitates negotiation of data interpretation. This approach was taken in the present study. Interviews focussing on the writing, repertory grids, and concept maps provided the teachers with opportunities to flesh out the ideas emerging through these research instruments and to interpret them in their own words. It is these words that comprise the core of those sections presenting the participants' images of mathematics. In interviews and less formal discussions, Jonathan and Randy
provided their reasons for selecting lesson approaches and their understandings of the resulting action. They were provided with the lesson descriptions selected for this report and invited to give their interpretations. The problems and struggles encountered in translating subject visions into teaching practice are reported here as related by Jonathan and Randy.

Although it can be argued that the conceptions of mathematics and narratives of classroom experience presented in previous chapters faithfully capture what was said and happened, there remains the question of whether these beliefs and practices were altered by my presence. That is, what was the "observer effect" (Bogdan & Biklen, 1992, p. 47) in this study?

I believe that my presence had little impact on Randy's and Jonathan's teaching. Visits to Western Secondary and Northern High School did not follow any fixed schedule and the only days avoided were those that involved extensive paper and pencil testing. Thus I frequently arrived while Randy and Jonathan were in the middle of lesson sequences and were not in a position to adjust activities or styles. Both teachers planned their lessons without consulting me and did not make adjustments as a result of our discussions. This is in contrast to those projects (Philipp, Flores, Sowder & Schappelle, 1994; Wood, Cobb & Yackel, 1991) where, in the presence of considerable coaching, teachers' emerging social constructivist subject images did translate into compatible instruction. In the research reported here, Randy and Jonathan independently selected their approaches to lessons.

On the other hand, this study, in encouraging focused reflection on personal mathematical philosophies, may have had
some impact on the participants' subject images. Clarke (1997), in a project with two Grade 6 teachers implementing a nonroutine problems unit, studied the participants' instructional practices and conceptions of their teaching roles. He noted that the one teacher, with whom he had numerous opportunities to meet and collaboratively reflect upon lessons and roles, developed expanded beliefs concerning mathematics teaching and learning. A similar effect on subject conceptions may have occurred in the present study. In fact, Jonathan, in describing his intellectual community and its effects on his thinking about mathematics, included myself and the recent research activities (JO-INT-D-14b). I believe that the project activities did not suggest alternative subject conceptions, but they likely provided opportunities for Jonathan and Randy to strengthen their images of mathematics.

The research reported here employed "purposeful sampling" (Bogdan & Biklen, 1992, p. 71; Lincoln & Guba, 1985, p. 199) in the selection of participants. That is, sampling "based on the assumption that one wants to discover, understand, gain insight; therefore one needs to select a sample from which one can learn the most" (Merriam, 1988, p. 48).

Previous studies examining teachers' beliefs about mathematics and instructional practices had, except for two cases (Philipp, Flores, Sowder & Schappelle, 1994; Thompson, 1984), involved teachers exhibiting traditional transmissive instructional styles. The present study, in an effort to expand our knowledge of the interplay between subject beliefs and practice, took the opposite tack and involved two teachers who
were moving beyond teacher centred traditional instruction. Jonathan, and Randy to a somewhat lesser degree, are atypical in their teaching practices. Surveys, both in Ontario (Colgan & Harrison, 1997; Ontario Ministry of Education, 1992a, 1992b), and internationally (Crosswhite, 1987; Lapointe, Mead & Askew, 1992; McLean, Wolfe & Wahlstrom, 1987; Robitaille, Taylor & Orpwood, 1996), show that engaging students in mathematical investigations, conjecturing and discussion are practices that are absent in most mathematics classrooms.

Although focusing on teachers who are not representative of the typical case produces problems of generalizability, for this research as an example of "studying what could be" (Schofield, 1990, p. 203) such "uniqueness is an asset rather than a liability" (Donmoyer, 1990, p. 194). It allows us to look for exactly those conditions that make the case exceptional; in this study, the conceptions of mathematics held by teachers who are pursuing mathematics education reform. For such research, the aim should not be universal generalizability, but to provide thick description to permit others to assess the potential for transfer to their sites of interest (Schofield, 1990).

Lincoln and Guba (1985) argue that, in qualitative research, we should look for transferability rather than generalizability. The degree to which the findings of any particular study apply to another case is a function of the similarity between contexts. Thus the chief research requirement is not to provide a general case, but to describe the particular cases studied in detail sufficient to allow others to assess the degree of fit. The two cases developed in this report contain extensive descriptions of
Jonathan's and Randy's conceptions of mathematics and teaching practices. The teachers' reports of the struggles to translate beliefs into action give information concerning their teaching environments. It is left to the reader "to ask, what is there in this study that I can apply to my situation, and what clearly does not apply?" (Walker, 1980, p. 34).

On the other hand, this study when combined with the parallel research cited earlier in the literature survey, does support the development of a "working hypothesis" (Cronbach, 1975; Lincoln & Guba, 1985). Instrumentalist views of mathematics appear to translate easily into the predominant transmissive modes of instruction. Teachers with less absolutist discipline images, unless provided with considerable coaching and support, also adopt traditional teaching practices. With this present study, we see that social constructivist conceptions of mathematics encourage alternative styles of teaching, and a well developed social constructivist philosophy can, in the face of considerable opposition, support a teacher's efforts to bring mathematics education reform to their classroom.

Implications and Questions for Future Study

Understanding how teachers, individually and collectively, think, act, develop professionally and change during their careers might provide new insights as to how one might approach the reform, change and improvements in education that are necessary to equip our students for a desirable future within a context that is rapidly altering the nature of teachers' work. (Butt, Raymond, McCue & Yamagishi, 1992, p. 51)

The extensive time spent with Jonathan and Randy, studying their images of mathematics and classroom practices and observing
their professional lives, has raised, for me, a number of questions not directly addressed by this present study. In addition, the working hypothesis that flows from this and related research, that a strong social constructivist image of mathematics can support a teachers' struggle to adopt and implement non-traditional instruction, has implications for those who promote mathematics education reform.

This study and parallel research suggest that teachers' images of mathematics influence their choices of instructional methods. Thus teachers' conceptions of mathematics are important, and a question that flows from this is:

What factors, events and experiences contribute to the development of a teacher's conception of mathematics? In particular, what experiences encourage the development of a social constructivist image of the discipline?

Discussions with Jonathan and Randy have identified some recent experiences that have influenced their mathematical beliefs. Interactions with university faculty and fellow professionals that encourage the exploration of mathematics beyond the school curriculum and raise philosophical questions appear to be productive. But, teachers construct their personal knowledge and meanings from a wide range of experiences (Kelly, 1955; Polanyi, 1958) including those prior to and beyond their professional lives. Research (Butt, Raymond, McCue & Yamagishi, 1992) has shown the influence of childhood experiences on teachers' general philosophies of education and drawn links between family life and teacher-candidates' images of mathematics (Roulet, 1995). Elementary and secondary schooling provide opportunities for future teachers to develop subject matter
beliefs (Ball & McDiarmid, 1990; Roulet, 1995; Schoenfeld, 1988, 1989). We might thus ask a more focused question of Jonathan and Randy, and of teachers with similar views of mathematics.

Are there childhood, elementary school, and secondary school experiences that encourage the development of social constructivist images of mathematics?

Educational systems are not in a position to influence teachers' childhood and adolescent experiences, but they can, to some extent, specify the formal university programs of those students who plan to enter the teaching profession. There is evidence that the present policy of mandating numbers of course credits has little positive effect on prospective teachers' understandings of the nature of individual disciplines (Ball & McDiarmid, 1990; Roulet, 1995). In fact, undergraduate study in mathematics may promote a narrowing of discipline conceptions and a reduction in enjoyment of the subject (Galbraith, 1984). On the other hand, research has shown that graduate level study in mathematics increases a teacher's syntactic knowledge of the discipline and inclination to lead school lessons that present a larger picture of mathematics (Grossman, Wilson & Shulman, 1989). There is a need to ask:

What university level mathematics courses or topics and what styles of university teaching promote prospective teachers' understanding of the syntactic and substantive knowledge of the discipline?

The present work with Jonathan and Randy provides some hints as to processes by which teachers may alter and expand their conceptions of mathematics. Following these leads might prove to be fruitful research.

This study and related research indicate that subject beliefs can have an effect on practice. The reverse may also be
true. Clarke (1997) and Cobb, Wood and Yackel (1990), in projects with elementary school teachers who were moving to non-traditional teaching styles, found that reflection on teaching resulted in a shift in beliefs about mathematics. Randy's and Jonathan's cases also suggest that "beliefs and practices are dialectically related" (Clarke, 1997, p. 279).

Randy's observations that many students can not follow and really do not believe formal proofs, has led him to explore more visual presentations and suggested an extended image of proof, one based upon the idea of "convincibility" (RW-INT-D-O5b). Jonathan has found the seeds for some of his mathematical investigations within the school curriculum. Early in his career he became interested in the topic of geometric transformations, and over a five year period explored higher dimensional extensions of the high school work (JO-INT-D-14b).

Are there secondary school topics and classroom activities through which teachers may reflect on and alter mathematical beliefs? Could such activities be employed for in-service work with teachers?

There is considerable agreement that collaborative activities and supporting professional networks contribute to teacher growth (Black & Atkin, 1996; Butt, Raymond & Townsend, 1990, April; Hargreaves, Davis, Fullan, Wignall, Stager & Macmillan, 1992). This professional development is usually pictured in terms of changed or expanded classroom practice, but the experiences of Jonathan and Randy show that altered subject images are also possible outcomes.

What types of professional networks and activities are supportive of change in subject beliefs? What roles could university based mathematicians play in these networks? Can effective networks be provided for large numbers of teachers?
The collaborative activities experienced by Randy and Jonathan were available to other teachers in their schools, but in general these people did not make the efforts to seek them out.

How can teachers be encouraged to take advantage of professional networks and collaborative activities?

This study also raises some research and educational policy questions that involve participants or audiences beyond mathematics teachers. The opposition to Jonathan's and Randy's teaching approaches expressed by parents and the school administrations appears to be rooted in differing conceptions of mathematics. The instrumentalist image of mathematics, prevalent within the mathematics education community, has even greater popularity in the general population. Reform oriented teachers will continue to experience conflicts unless major efforts are made to alter public images of what it means to do mathematics.

What strategies might the professional organizations and allies from the university based mathematics community employ to counteract the dominant instrumentalist public conception of mathematics?

Both Jonathan and Randy felt constrained by the official mandated curriculum. It is obvious that these teachers, if freed from curricular restrictions, would provide stimulating and challenging mathematics courses; programs that would be unique. Such diversity would be likely to clash with the government's perceived need for uniformity.

What would be the results of giving adventurous reform minded mathematics teachers freedom to design their own curricula? What is the appropriate balance between freedom for individual teachers and Ministry control of curriculum? Can guidelines be designed to provide directions for the majority of teachers while providing sufficient options for reform minded educators?
The focus of this study was teachers; their beliefs about mathematics and teaching practices. Although students appear as characters in the narratives of classroom experience, they are there primarily as foils for the teacher. There is a need for studies that shift this focus and examine the learning of students in classrooms lead by teachers such as Jonathan.

What are the conceptions of mathematics held by students who have experienced secondary school courses designed around a social constructivist philosophy of the discipline? Do students receive and internalize the messages concerning the nature of mathematics delivered by such courses and teaching? Are students who have experienced social constructivist oriented mathematics instruction more or less inclined than the general population to pursue tertiary education or careers that involve mathematics? What is the experience of such students in post secondary mathematics courses?

Such questions are important, for in the final analysis the goal of mathematics education reform is increased student knowledge and appreciation of mathematics.

Conclusion

In this study innovative instructional practice, activities that put into place ideas expressed in the mathematics education reform literature (NCTM, 1989, 1991, 1995; OAME/OMCA, 1993), were found to be reflections of the conceptions of mathematics held by the teachers. The teachers struggled against considerable opposition in their efforts to express their subject visions in classroom practice. Jonathan, who possessed a well developed social constructivist philosophy, persevered and managed to regularly bring to his classroom lessons that captured his view of mathematics. Randy's mixed and changing conception of
mathematics did not support continuing reform efforts within the core mathematics program, but did motivate the provision of non-traditional activities in a style that would not invite student complaints or official notice. Randy found no intellectual companions within the staff of his school and received no support in his efforts to alter mathematics education. Jonathan had the company of two department members for conversations concerning mathematics and its teaching, but generally the school was not welcoming of his style of mathematics teaching. Both teachers sought out and found the intellectual stimulation that contributed to their images of mathematics in collegial relationships that went beyond the school. Both found stimulation in their contacts with university faculty, and Jonathan was able to build a supportive intellectual network within his professional community.

Intellectually live mathematics teachers, those who are natural knowledge seekers, can, with effort, find collegial links that support their interests in furthering their mathematical understanding. The resulting explorations can help develop conceptions of mathematics that support changed instructional practices. The potential for such sequences exist, but the teacher effort involved is considerable. In the present environment of little or no support for intellectually adventurous teachers, the population of mathematics educators such as Jonathan and Randy is not likely to rapidly grow.
References


Connelly, F. M., & Clandinin, D. J. (1984). Personal practical knowledge at Bay Street School: Ritual, personal philosophy


Images of complex dynamical systems (pp. 175-180). Berlin: Springer-Verlag.


Jackson, A. (1997). The math wars: California battles it out over mathematics education reform (Part I). *Notices of the
American Mathematical Society, 44(6), 695-702.


Abstracts International, 44, 1363A.


Appendix A: Data Coding Scheme and Data Logs

All contacts (face-to-face, mail, telephone, e-mail) with each participant, all observations of teaching or interaction between each participant and students or teaching colleagues, and all materials gathered are recorded in the two data logs that follow. For organizational purposes and referencing within the text of this thesis each item has been assigned a code that records: participant, event type, record type, and order in the data collection sequence. The coding scheme is given on the following page. In addition, the data log entries record: form and storage location of the item, date obtained, place where it was obtained, type of event, and a brief description of the contents.
### Exemplary Mathematics Teachers: Subject Conceptions and Instructional Practices

#### Data Log Codes

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<td>Jonathan Ode</td>
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<td>RW</td>
<td>Randy Walker</td>
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<td>Document Collection</td>
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<td>Interview</td>
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<td>Lesson</td>
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<td>MTG</td>
<td>Meeting</td>
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<td>Tape</td>
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| NN-T-#-A/B-ccc-ccc | Participant-Tape-# - side - count - count |
| NN-F-#        | Participant-File-Folder # |
| NN-INT##     | Participant-INTnumber computer files of interview transcriptions |
## Exemplary Mathematics Teachers: Subject Conceptions and Instructional Practices

### Data Log - Jonathan Ode

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<td>JO-DIS-N-01</td>
<td>JO-L-1-01</td>
<td>15-08-93</td>
<td>home</td>
<td>phone call</td>
<td>contact to ask for JO's participation, JO's efforts on behalf of visually impaired student</td>
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<td>JO-LES-N-01</td>
<td>JO-L-1-01-03</td>
<td>30-09-93</td>
<td>classroom</td>
<td>9E lesson</td>
<td>quiz on integer arithmetic, homework take up with high student involvement</td>
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<td>JO-LES-D-01</td>
<td>JO-F-1</td>
<td>30-09-93</td>
<td>math prep room</td>
<td>after school</td>
<td>JO's timetable</td>
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<tr>
<td>JO-DIS-N-02</td>
<td>JO-L-1-03-04</td>
<td>07-10-93</td>
<td>math prep room</td>
<td>spare before classes</td>
<td>article from PS News, used in OAC Finite</td>
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<td>JO-DIS-D-02a</td>
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<td>10-10-93</td>
<td>math prep room</td>
<td>spare before classes</td>
<td>article from NCTM Mathematics Teacher, problem solving, copies for rest of dept. teaching Grade 9</td>
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<td>JO-F-1</td>
<td>10-10-93</td>
<td>math prep room</td>
<td>spare before classes</td>
<td>non-text books on JO's desk</td>
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<td>JO-DOC-N-02</td>
<td>JO-L-1-03-04</td>
<td>07-10-93</td>
<td>math prep room</td>
<td>admin re project, quiz permutations and combinations, problem solving via Whimbey Pairs</td>
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<td>JO-LES-N-02</td>
<td>JO-L-1-05-08</td>
<td>07-10-93</td>
<td>classroom</td>
<td>OAC Finite lesson (class 1)</td>
<td>same as LES-02 with expanded input by JO at intro to problem solving activity</td>
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<td>JO-L-1-08-10</td>
<td>07-10-93</td>
<td>classroom</td>
<td>OAC Finite lesson (class 2)</td>
<td>pages from Finite Mathematics text with problems that were investigated during Whimbey Pairs activity</td>
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<td>10-10-93</td>
<td>classroom</td>
<td>9E lesson</td>
<td>problem solving by organizing data in charts, work in pairs</td>
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<tr>
<td>JO-LES-D-04</td>
<td>JO-F-1</td>
<td>10-10-93</td>
<td>classroom</td>
<td>OAC Finite Meth lesson</td>
<td>sheets of problems used in LES-04</td>
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<tr>
<td>JO-LES-D-05</td>
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<td>OAC Finite Meth lesson</td>
<td>taking up homework, page 53 from text, students struggling with # 9</td>
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<td>JO-LES-D-05</td>
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<td>classroom</td>
<td>OAC Finite Meth lesson</td>
<td>pages from Finite Mathematics text</td>
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*Note: The text above is a data log from a study on exemplary mathematics teachers, detailing the events and content related to instruction and student participation.*
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<td>JO-L-1-13-14</td>
<td>22-10-93</td>
<td>math prep room</td>
<td>spare 1st period</td>
<td>problems re complaints by some Finite Math students/parents, intellectual contexts and support, commitment to intellectual life</td>
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<td>22-10-93</td>
<td>math prep room</td>
<td>spare 1st period</td>
<td>given writing re images of Math, not complete, Question 1, What is Math &amp; Question 3, Mathematical Truth addressed</td>
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<td>JO-LES-N-08</td>
<td>JO-L-1-19</td>
<td>22-10-93</td>
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<td>OAC Finite Math lesson</td>
<td>begin with quiz, binomial expansion, more direct instruction but still cases of problem solving and pattern recognition</td>
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<td>JO-LES-D-06</td>
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<td>quiz binomial expansion</td>
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<td>JO-DIS-N-04</td>
<td>JO-L-1-20-21</td>
<td>22-10-93</td>
<td>classroom</td>
<td>as students enter for</td>
<td>JO excited about &quot;great&quot; class, examples of problem solving</td>
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<td>JO-LES-N-07</td>
<td>JO-L-1-20-24</td>
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<td>9E lesson</td>
<td>repeating fractions, preparation for project, calculators, independent student work, flexible grouping</td>
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<td>03-11-93</td>
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<td>after school</td>
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<td>JO-F-1</td>
<td>28-11-93</td>
<td>math prep room</td>
<td>after school</td>
<td>photographs of a selection of the best periodic fraction projects, retained by JO for future use in teacher workshops</td>
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<td>questions prepared for INT-09, not all required as JO voluntarily</td>
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<td>observations of JO's efforts re interpretation of grid</td>
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<td>copy of OAC-Calculus examination from June 1992, divergence</td>
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<td>collaborative learning, and deep understanding carried over into</td>
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<td>21-12-93</td>
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<td>late afternoon</td>
<td>interview re mechanisms of and reasons for approach to OAC-</td>
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<td>Just after INT-11</td>
<td>concerns re the potential imposition of standardized testing in Ontario</td>
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<td>Annoucement from OAME'94 conference program for Session 137, CHAOS in the Grade 9 Classroom by JO and April Gibbons</td>
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<td>14-05-94</td>
<td>Jeffrey Hall, Queen's University</td>
<td>Saturday 9:00am presentation at OAME'94</td>
<td>Handout prepared by JO and April Gibbons for presentation CHAOS in the Grade 9 Classroom given at OAME'94, Kingston, May 12-14</td>
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<td>JO-DOC-N-11</td>
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<td>14-05-94</td>
<td>Roulet's home</td>
<td>BBQ after OAME'94</td>
<td>original copy of Participant Biographical Information form, completed but then lost, recently found by JO, data complements that of JO-DOC-D-09a</td>
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<td>JO-DOC-N-09b</td>
<td>JO-L-1-80</td>
<td>14-05-94</td>
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<td>JO’s concern for research project and his problems with time and organization</td>
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<td>JO-LES-D-07m-p</td>
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<td>14-05-94</td>
<td>Roulet's home</td>
<td>BBQ after OAME'94</td>
<td>JO brings photographs of repeating decimal art taken by AV technician at school, better quality than my originals, misplaced by AV technician and then found and given to JO</td>
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<td>JO-DIS-N-16</td>
<td>JO-L-1-83-84</td>
<td>14-05-94</td>
<td>Roulet's home</td>
<td>BBQ after OAME'94</td>
<td>Discussion between JO, FZ, and RW and others re mathematics and teaching</td>
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<td>JO-DIS-N-17</td>
<td>JO-L-1-85-87</td>
<td>21-08-94</td>
<td>restaurant near school</td>
<td>lunch 12:30pm, just after arrival at school</td>
<td>Discussion of next years teaching at Adult Centre, career plan, course material writing</td>
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<td>JO-DOC-D-12</td>
<td>JO-F-1</td>
<td>21-06-94</td>
<td>school</td>
<td>on return from lunch</td>
<td>Graph that JO received from First Brande re. freezing point vs. concentration of Prestone, JO excited as may show example of Chaos, makes copy for me and RW</td>
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<td>JO-DOC-N-12</td>
<td>JO-L-1-87</td>
<td>21-06-94</td>
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<td>Notes re. graph and its impact for JO</td>
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<td>JO-INT-N-12</td>
<td>JO-L-1-84-85</td>
<td>21-08-94</td>
<td>yearbook office, school</td>
<td>2:30-4:00 pm</td>
<td>Interview re. concept map, trying to get JO to express views of mathematics beyond school subject, largely focused on mathematics as in school curriculum</td>
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<td>JO-INT-D-12a</td>
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<td>15-09-94</td>
<td>Fac of Ed</td>
<td>Article in OAME Gazette</td>
<td>Article in <em>Ontario Mathematics Gazette</em>, 33(1), 4-8, September 1994; &quot;Calculus, A Student-Centred Approach&quot;; discussion of his more investigative approach to OAC Calculus, with reference to Taylor's work and book, example of activity, &quot;Estimating the Mile Record&quot;, interesting observations as to the calculus of the &quot;real world&quot;.</td>
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<td>JO-DIS-N-18</td>
<td>JO-L-1-88-89</td>
<td>29-09-94</td>
<td>university office</td>
<td>phone call to JO at work</td>
<td>Plans for thesis, in particular JO's interviews re Mathematical Life History, JO's new position at independent learning centre, very enthusiastic</td>
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<td>JO-DIS-N-19</td>
<td>JO-L-1-89-91</td>
<td>26-10-94</td>
<td>from home</td>
<td>phone call to JO at home</td>
<td>Call to set up meeting next week, discussion of his new teaching, in particular the Grade 11 data program course. Discussion of presentation to be made at WEST-MATH mini-conference, activity following and building from the Gazette article. Plans to generate data for a cooling curve with glass of water and thermocouple while making initial portion of talk. Analysis resulting curve. Discussion of features of new version of IBM's MET that JO has just received. Talk of programming TI calculators. Discussion of teaching, plans for future, needs of pupils.</td>
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<tr>
<td>JO-DOC-D-14</td>
<td>JO-F-1</td>
<td>02-11-94</td>
<td>Hotel room, Toronto</td>
<td>start of INT-13</td>
<td>JO's Mathematical Life History Time-Line, completed with factual info, re places and dates but little re influences on mathematical development</td>
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<tr>
<td>JO-DOC-D-16</td>
<td>JO-F-1</td>
<td>02-11-94</td>
<td>Hotel room, Toronto</td>
<td>start of INT-13</td>
<td>Resume for JO, given to help flesh out Mathematical Life History Time-line</td>
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<tr>
<td>JO-DOC-D-18</td>
<td>JO-F-1</td>
<td>02-11-94</td>
<td>Hotel room, Toronto</td>
<td>just before start of INT-13</td>
<td>JO brings two packages relating art to mathematics, using mathematical patterns to generate visual patterns or art, in response to our phone conversation, JO-DIS-19, when I mentioned the art and mathematics group. JO interested in such activities (Periodic decimal project) and had these rather old packages in his file, lends them to me for copying if interested, Front pages here in file, rest in Art and Mathematics file. Appears that JO has used as he knows the contents and takes time to show me some of the ideas. Expects me to recognize the material as has been around Toronto for some time. (Bob Alexander)</td>
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<td>JO-DOC-D-17</td>
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<td>02-11-94</td>
<td>Hotel room, Toronto</td>
<td>just before start of INT-13</td>
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<td>JO-INT-N-13</td>
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<td>02-11-94</td>
<td>4:00pm-5:30pm</td>
<td>Initial interview re Mathematical Life History, early childhood to</td>
<td>Toronto</td>
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<td>arrival at University of Waterloo for graduate work, very full and</td>
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<td>rich life, little formal mathematical influences but many life</td>
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<td>use of TI-82 and CBL, MST integration, staying intellectually</td>
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<td>Discourse between JO, FZ and others, JO's new teaching job,</td>
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<td>JO-DIS-N-20</td>
<td>JO-L-1-92, 94-95</td>
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<td>WEST-MATH fall conference, presenting idea of &quot;Patterns &amp;</td>
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<td>Structures&quot;, OAC student involvement in determining associations</td>
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<td>JO-DIS-N-22</td>
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<td>6:00pm-9:00pm</td>
<td>WEST-MATH Mini-Conference, presentation of &quot;Patterns &amp; Structures&quot;,</td>
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<td>JO-DIS-N-23</td>
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<td>6:00pm-9:00pm</td>
<td>Dinner between sessions</td>
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<td>Weston-MATH Mini-Conference, presentation of &quot;Patterns &amp; Structures&quot;,</td>
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<td>Page from CBL manual, Newton's Law of Cooling experiment directions, model for JO's presentation</td>
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<td>JO-DIS-N-24</td>
<td>JO-L-1-107</td>
<td>03-11-94</td>
<td>classroom</td>
<td>after JO's session as packing up</td>
<td>Discussion JO, Ron Newton, Jim Ranger, potential for curricular reform, pessimistic, sense of isolation from main stream</td>
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<td>JO-DIS-N-25</td>
<td>JO-L-1-107-108</td>
<td>03-11-94</td>
<td>walk to car</td>
<td>after packing up equipment</td>
<td>Talk with JO, JO's negative feelings towards administrative tasks and restrictive structures</td>
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<td>JO-DIS-N-26</td>
<td>JO-L-1-109-111</td>
<td>22-11-94</td>
<td>from Kingston</td>
<td>evening</td>
<td>Phone call to JO to set up next interview, discussions concerning Math, Sci, Tech integration and JO's negative observations, not enough structure to promote mathematical modelling, new text for Calculus using the TI-82, art and mathematics, computing</td>
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<tr>
<td>JO-DIS-N-27</td>
<td>JO-L-1-111-112</td>
<td>23-11-94</td>
<td>from Kingston</td>
<td>evening</td>
<td>Phone call to JO to confirm time for next interview, continued discussion of integration, support for others (Albert North, mathematics coordinator) doing hard work for mathematics</td>
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<td>JO-DIS-N-28</td>
<td>JO-L-1-112-113</td>
<td>08-12-94</td>
<td>classroom at Adult Centre</td>
<td>11:15am, JO scheduled time without students</td>
<td>Discussion prior to starting interview, JO showing new teaching facilities and resources, new text Calculus using TI graphing calculators</td>
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<tr>
<td>JO-INT-N-14</td>
<td>JO-L-1-114</td>
<td>08-12-94</td>
<td>JO's work area</td>
<td>11:45am</td>
<td>Continuation of interview re Mathematical Life History, Waterloo, teacher training, professional life, relationships with other teachers, support and conflicts, mathematical explorations</td>
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<td>JO-INT-D-14a</td>
<td>JO-F-1</td>
<td>08-12-94</td>
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<td>Questions prepared for INT-14, many answered indirectly without explicit asking</td>
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<td>JO-INT-T-14</td>
<td>JO-T-8A-000-880, JO-T-8B-000-175</td>
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<td>Tape of JO-INT-14</td>
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<td>JO-INT-D-14b</td>
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<td>JO-DIS-N-29</td>
<td>JO-L-1-115, 116-117</td>
<td>08-12-94</td>
<td>JO's work area</td>
<td>just after completion of interview</td>
<td>Discussion of thesis progress and analysis of data, JO relates to Chaos Theory and Fuzzy Logic, reference to recent reading of Fuzzy Thinking</td>
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<td>JO-DIS-D-29</td>
<td>JO-F-1</td>
<td>08-12-94</td>
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<td>Copies of pages from book Fuzzy Thinking, contain source of ideas expressed by JO</td>
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<td>JO-LES-N-22</td>
<td>JO-L-1-116</td>
<td>06-12-94</td>
<td>JO's work area</td>
<td>after interview, while writing notes</td>
<td>Observations of JO giving personal lessons to two students, JO’s personal interactions with students, focus on general solutions strategies</td>
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<tr>
<td>JO-DOC-D-18</td>
<td>JO-F-1</td>
<td>13-04-95</td>
<td>Ontario Mathematics Gazette</td>
<td>vol. 33, #3, April 1995</td>
<td>Letter to the editor by JO commenting on a previous article by Hugh Allen (vol. 33, #2, December 1994). JO making much of restrictions on teachers due to curriculum and available time for planning.</td>
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<td>JO-DIS-N-30</td>
<td>JO-L-1-117-118</td>
<td>19-05-95</td>
<td>Phone call to JO from Kingston</td>
<td>evening</td>
<td>Phone call to JO to set up interview time, discussion for plans for summer and recent activity, writing for D C Heath, presentations at OISE</td>
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<tr>
<td>JO-DIS-N-31</td>
<td>JO-L-1-119-120</td>
<td>28-05-95</td>
<td>JO's work/office area at Adult Centre</td>
<td>late morning after drive to Toronto</td>
<td>relationships with AC students, JO's concerns re education system, unqualified teachers, restricted images of mathematics held by some, mathematics as an intellectual activity, school as social engineering, world view re equity and education</td>
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<td>JO-INT-N-15</td>
<td>JO-L-1-121</td>
<td>26-05-95</td>
<td>JO's work/office area at Adult Centre</td>
<td>noon</td>
<td>JO given thought to “my” questions re technology and mathematics but interpreted as use of computers in mathematics, gives me a “prepared” statement, discussion re teacher support for computer use, role of Ministry, boards, schools and teachers. Direction of discussion changed by introduction of my specific questions, supportive of MST integration when done in parallel, i.e. collect science data and build model not develop formulas and just stick in values</td>
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<td>JO-INT-D-15a</td>
<td>JO-F-1</td>
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<td>Questions prepared for JO-INT-15 and selected materials from writing on the nature of mathematics, RepGrid of school subjects, and concept map and related interviews</td>
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<td>JO-INT-D-15b</td>
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<td>JO's prepared page for JO-INT-15, developed while thinking about the use of computers in mathematics teaching/learning</td>
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<td>JO-INT-T-15a</td>
<td>JO-T-9A-000-247</td>
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<td>Tape of First half of JO-INT-15 with JO giving his position on computer use in teaching/learning of mathematics, discussion of impediments to the implementation of computers</td>
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<td>JO-INT-T-15b</td>
<td>JO-T-9A-258-687</td>
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<td>Second half of JO-INT-15 with JO responding to questions concerning MST Integration, 258-355 introduction by Roulet, presenting statements previously made by JO and suggesting a conflict. 355-418 initial response to suggestion of conflict setting out how MST Integration could fit with JO's image of mathematics as involving the observation of patterns and the developing of models</td>
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<td>JO-DIS-N-32</td>
<td>JO-L-1-122-124</td>
<td>26-05-95</td>
<td>JO's work/office area at Adult Centre</td>
<td>after interview over lunch</td>
<td>Discussions concerning MST Integration, JO's attempts, experiences, and resources. PD required by teachers to expand image of mathematics, JO's attempts and experiences. New version of Mathematics Exploration Toolkit</td>
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<td>JO-DIS-N-33</td>
<td>JO-L-1-124-126</td>
<td>13-06-95</td>
<td>phone cell from Kingston</td>
<td>evening</td>
<td>Phone cell to JO to set up meeting time to return materials borrowed from JO. Talk of summer plans for writing and conference presentations, support provided by board Mathematics Coordinator</td>
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<tr>
<td>JO-DIS-N-34</td>
<td>JO-L-1-128-127</td>
<td>10-08-95</td>
<td>Queen's University</td>
<td>Queen's-Gage National Mathematics Institute</td>
<td>JO at Institute for one day to give topic group session in afternoon, use of TI-82 to link physics and mathematics, gives overview of his planned session during first session of the working group on Technology in Mathematics Education - desire to build excitement or &quot;awe&quot; in mathematics</td>
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<td>JO-DIS-N-35</td>
<td>JO-L-1-128-129</td>
<td>10-08-95</td>
<td>Queen's University</td>
<td>Queen's-Gage National Mathematics Institute</td>
<td>Interaction while JO and I work together on problem posed by FZ to initiate the first session of the working group on Technology in Mathematics Education</td>
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<td>JO-DIS-N-36</td>
<td>JO-L-1-130</td>
<td>10-08-95</td>
<td>Kingston Brew Pub</td>
<td>evening before JO drives home</td>
<td>JO talks of his work on his geometry text, excitement for project and mathematics</td>
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Exemplary Mathematics Teachers:  
Subject Conceptions and Instructional Practices

Data Log - Randy Walker

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<td>RW-LES-N-I</td>
<td>RW-F-1</td>
<td>25-11-92</td>
<td>Golden District SS</td>
<td>Fractal Geometry lesson</td>
<td>Visit to RW at school to explore possibility of his joining my research project, observation of new Fractal Geometry course in action, 3-coins problem</td>
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<td>RW-LES-D-I</td>
<td>RW-F-1</td>
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<td>Pages from Paulos (1991), Beyond Numeracy that suggested 3-coin problem</td>
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<td>RW-F-1</td>
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<td>classroom</td>
<td>12E lesson</td>
<td>Beginning lesson on Iteration, Pixel Rainbow</td>
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<td>RW-DIS-N-01</td>
<td>RW-L-1-01</td>
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<td>home</td>
<td>phone call</td>
<td>frustrations with expectations and commitment of enriched (12E) class students</td>
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<td>RW-DIS-N-02</td>
<td>RW-L-1-01-02</td>
<td>15-11-93</td>
<td>school</td>
<td>welcome to school before first class</td>
<td>welcome to school and math office, RW's classroom for enriched grade 12, frustrations with grade 9 science program</td>
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<td>graphs of trig f'ns, period, amplitude, intro to radian</td>
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<td>15-11-93</td>
<td>math prep room</td>
<td>talk during spare &amp; lunch</td>
<td>past work in 12E, beginning iteration activity, recursion from composition of f'ns, problems re evaluation and dept. policy</td>
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<td>rough sketches by RW showing past work done in 12E</td>
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<td>math prep room</td>
<td>talk during lunch</td>
<td>present state of Fractal Geometry course, lack of interest and support</td>
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<td>RW-L-1-09-12</td>
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<td>math prep room</td>
<td>compiled during spares</td>
<td>list of books on RW's desk, mathematics, fractal geo, chaos</td>
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<td>RW-L-1-15-16</td>
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<td>math prep room</td>
<td>brief talk after classes</td>
<td>schedule for week, outline of Grade 9 science class just posted, Brownian motion, random walks, coin tossing activity</td>
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<td>RW-LES-N-03</td>
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<td>18-11-93</td>
<td>classroom</td>
<td>12E lesson</td>
<td>homework, drill-and-practice of deg-rad conversions, graphing trig f'ns with altered amplitude and period</td>
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<td>classroom</td>
<td>end of 12E lesson</td>
<td>my brief discussion with 2 students, obviously understand transformations and could employ with trig f's</td>
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<td>RW-L-1-21</td>
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<td>math prep room</td>
<td>discussion during spare with other math dept member</td>
<td>conservative/negative re destreaming view of math as content to be learned, best way to teach in by telling &quot;what and how&quot;; group work does not work, RW's comments</td>
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<td>math prep room</td>
<td>info providing</td>
<td>given set of 12E student notes to examine</td>
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<td>copy of part of 12E student notes, lessons of Sept 27-29, 3 class aside on iteration of functions arising out of composition of functions work Sept 21-22</td>
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<td>copy of R's notes for 12E lesson introducing iteration of functions and orbits</td>
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<td>info seeking</td>
<td>questions for RW re my visit to Golden District SS, November 1992, coin tossing experiment</td>
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<td>info</td>
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<td>written during class notes</td>
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<td>notes re first meeting with RW, phone call</td>
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<td>RW-LES-N-04</td>
<td>RW-L-1-23-28</td>
<td>16-11-93</td>
<td>STEP lab</td>
<td>9 STEP lesson</td>
<td>results of coin tossing experiment, combined results, computer generated example, random walks</td>
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<td>RW-T-2A-000-853</td>
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<td>12A lesson</td>
<td>home work, drill-and-practice re deg-rad conversions, graphing cos f's with stretched amplitude</td>
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<td>RW-L-1-29</td>
<td>16-11-93</td>
<td>math prep room</td>
<td>talk after classes</td>
<td>discussion re 9 STEP activity, potential for discovery of square root rule for coin toss experiment, multiple smaller samples from data of 100 tosses</td>
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<tr>
<td>RW-DIS-N-19</td>
<td>RW-L-1-90</td>
<td>01-12-93</td>
<td>math prep room</td>
<td>before first class</td>
<td>question to RW re his reference to the &quot;Laplacian reductionist world view&quot; in his Images of Mathematics writing and RW's response</td>
</tr>
<tr>
<td>RW-DIS-D-19</td>
<td>RW-F-1</td>
<td>01-12-93</td>
<td>staff room</td>
<td></td>
<td>copy of article by Ellenberger from Peitgen &amp; Richter (1988) with paraphrase of Laplace's argument. RW's reference for Laplacian view</td>
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<tr>
<td>RW-LES-N-17</td>
<td>RW-L-1-90-91</td>
<td>01-12-93</td>
<td>classroom</td>
<td>12E lesson</td>
<td>test, triangle applications of trig., whole period used, Bonus questions</td>
</tr>
<tr>
<td>RW-LES-D-17</td>
<td>RW-F-1</td>
<td>01-12-93</td>
<td>classroom</td>
<td></td>
<td>copy of test for 12E class</td>
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<tr>
<td>RW-LES-N-20</td>
<td>RW-L-1-91-94</td>
<td>01-12-93</td>
<td>staff room</td>
<td>spare 2nd period</td>
<td>RW's explanation to clarify my questions and problems re Julia Sets and Mandelbrot Set, RW animated in his explanations and pointing out interesting phenomena</td>
</tr>
<tr>
<td>RW-DIS-N-21</td>
<td>RW-L-1-94-96</td>
<td>01-12-93</td>
<td>classroom and later math prep room</td>
<td>lunch, 45 minutes of period 3, after class, morning of 02-12-93</td>
<td>RW providing opportunity to use program to plot cycles of 0 for various 'o' inputs to Mandelbrot eq'n., series of experiments, last experiment left running over night</td>
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<tr>
<td>RW-DIS-D-21-e-g</td>
<td>RW-F-1</td>
<td>01-12-93</td>
<td>staff room</td>
<td></td>
<td>photographs of computer screen after various experiments with different 'o' values</td>
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<tr>
<td>RW-LES-N-18</td>
<td>RW-L-1-97</td>
<td>01-12-93</td>
<td>classroom</td>
<td>12A lesson</td>
<td>test triangle applications of trig, modified version of 12E test, whole period employed, bonus questions including take home</td>
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<td>RW-LES-D-18</td>
<td>RW-F-1</td>
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<td>staff room</td>
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<td>copy of test for LES-18</td>
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<td>RW-DIS-N-22</td>
<td>RW-L-1-97-98</td>
<td>01-12-93</td>
<td>math prep room</td>
<td>after school</td>
<td>discussion re direct teaching, views of Dept. Head and support</td>
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<tr>
<td>RW-DIS-N-23</td>
<td>RW-L-1-98</td>
<td>01-12-93</td>
<td>Lake Side Park SS</td>
<td>4-7pm</td>
<td>Euclid and Descartes contest practice session and pizza supper for Grade 12 and OAC students, RW one of 2 Northern HS staff present</td>
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<tr>
<td>RW-LES-N-19</td>
<td>RW-L-1-99-110</td>
<td>02-12-93</td>
<td>classroom</td>
<td>12E lesson</td>
<td>intro to trig identities, using graphing calculators, formal deductive system approach, proof vs demonstration</td>
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<td>RW-LES-T-19</td>
<td>RW-T-6A-115-691, RW-T-6B-000-296</td>
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<td>tape of LES-19</td>
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<td>RW-DIS-D-24</td>
<td>RW-L-1-110</td>
<td>02-12-93</td>
<td>spare &amp; lunch</td>
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<td>attempt to assemble computer equipment, RW demonstrating LOGO programs to generate curves of given dimension</td>
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<td>RW-DIS-D-24</td>
<td>RW-F-1</td>
<td>02-12-93</td>
<td>staff room</td>
<td></td>
<td>photograph of 3 curves of dimension 1,26</td>
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<td>RW-LES-N-20</td>
<td>RW-L-1-111-117</td>
<td>02-12-93</td>
<td>classroom</td>
<td>12A lesson</td>
<td>solving trig eq’ns, algorithmic, interest in calculator technology</td>
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<td>RW-INT-N-06</td>
<td>RW-L-1-117</td>
<td>02-12-93</td>
<td>math prep room</td>
<td>4:20-5:00pm</td>
<td>interview on variety of topics; group work/investigations, student response to Fractal Geometry, outcomes for project from LES-12a, logical development vs excitement</td>
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<tr>
<td>RW-INT-T-08a</td>
<td>RW-T-8A-000-081</td>
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<td>group work/ student investigations to develop mathematics</td>
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<tr>
<td>RW-INT-T-08b</td>
<td>RW-T-8A-081-145</td>
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<td>student responses to Fractal Geometry, motivating, incident of Gery, Grade 10 student</td>
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<td>RW-INT-T-08c</td>
<td>RW-T-8A-145-186</td>
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<td>outcomes of project from LES-12a</td>
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<tr>
<td>RW-INT-T-08d</td>
<td>RW-T-8A-186-241</td>
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<td>logical development of topics vs excitement, RW's changes in view</td>
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<td>RW-DIS-N-25</td>
<td>RW-L-1-118</td>
<td>03-12-93</td>
<td>math prep room</td>
<td>before classes</td>
<td>RW demonstrating output of LOGO programs to generate fractals from various generators, excitement with images</td>
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<tr>
<td>RW-LES-N-21</td>
<td>RW-L-1-118-124</td>
<td>03-12-93</td>
<td>classroom</td>
<td>12E lesson</td>
<td>Intro of topic (exponential f’ns) via problem, trig identities home work, practice problem solving session for Euclid contest</td>
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<tr>
<td>RW-LES-T-21</td>
<td>RW-T-9A-000-042</td>
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<td>tape of first portion of LES-21, sol’n of problem and discussion of exponential f’ns</td>
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<td>RW-LES-D-21</td>
<td>RW-F-1</td>
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<td>Euclid contest problem page, copy of page given out DIS-23</td>
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<td>RW-INT-N-07</td>
<td>RW-L-1-124</td>
<td>03-12-93</td>
<td>math prep room</td>
<td>spare 2nd period</td>
<td>construction of 2nd Subjects RepGrid with meth replaced by fractal geometry, algebra, and calculus</td>
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<tr>
<td>RW-LES-N-22</td>
<td>RW-L-1-125-129</td>
<td>03-12-93</td>
<td>classroom</td>
<td>12A lesson</td>
<td>intro with problem using exponents, solving trig eq’ns home work, review of exponent laws</td>
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<tr>
<td>RW-LES-N-23</td>
<td>RW-L-1-129-132</td>
<td>03-12-93</td>
<td>STEP lab</td>
<td>9 STEP lesson</td>
<td>special lesson, Fractal Geometry, slides of zooms of Mandelbrot Set, Chaos Game explained, predictions, played, data collected, structure vs random, other generators and outcomes</td>
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<td>RW-T-9A-043-895</td>
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<td>tape of LES-23, Fractal Geometry</td>
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<td>RW-L-1-132-133</td>
<td>03-12-93</td>
<td>math prep room</td>
<td>after school</td>
<td>longer look at 3 RepGrids, comparison of two Subject grids, RW's observations</td>
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<tr>
<td>RW-INT-D-08a</td>
<td>RW-F-1</td>
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<td>grids and questions for interview</td>
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<td>15-12-94</td>
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<tr>
<td>RW-DIS-N-28</td>
<td>RW-L-1-133</td>
<td>27-03-94</td>
<td>home, Kingston</td>
<td>phone cell, evening</td>
<td>Phone call to RW to change arrangements for next visit, not April but June, discussion re his teaching</td>
</tr>
<tr>
<td>RW-DOC-D-08</td>
<td>RW-F-1</td>
<td>12-04-94</td>
<td></td>
<td>letter of support</td>
<td>Letter of support for RW's nomination for the Prime Minister's Awards for teaching Excellence in Science, Technology, and Mathematics</td>
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<tr>
<td>RW-DOC-N-08</td>
<td>RW-L-1-133-134</td>
<td>12-04-94</td>
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<td>note re letter of support RW-DOC-D-08</td>
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<tr>
<td>RW-DOC-D-09</td>
<td>RW-F-1</td>
<td>12-05-94</td>
<td>Kingston, Queen's University</td>
<td>May 12, 1994, OAME'94</td>
<td>Page from OAME'94 program booklet announcing RW's 2-hour (double) session Fractals and Chaos for the 1890's Classroom</td>
</tr>
<tr>
<td>RW-DOC-D-10a</td>
<td>RW-F-1</td>
<td>12-05-94</td>
<td>Kingston, Queen's University</td>
<td>10:15am May 12, 1994</td>
<td>Handout #1, Fractals In Your Future, provided by RW for his talk, Fractals and Chaos for the 1890's Classroom, plus post-it notes added by RW to explain the use of a number of pages. RW worried that package does not stand alone as it is a series of worksheets to be used during workshop.</td>
</tr>
<tr>
<td>RW-DOC-D-10b</td>
<td>RW-F-1</td>
<td>12-05-94</td>
<td>Kingston, Queen's University</td>
<td>10:15am May 12, 1994</td>
<td>Second handout, No Chaos In This Iteration?, provided by RW for his talk, Fractals and Chaos for the 1990's Classroom, Especially an up-dated version of his OAME'93 paper.</td>
</tr>
<tr>
<td>RW-DIS-N-27</td>
<td>RW-L-1-134-135</td>
<td>14-05-94</td>
<td>Roulet's home</td>
<td>BBQ after OAME'94</td>
<td>Discussion between JO, FZ, and RW and others re mathematics and teaching</td>
</tr>
<tr>
<td>RW-DIS-N-28</td>
<td>RW-L-1-135-139</td>
<td>23-06-94</td>
<td>school hall</td>
<td>9:15-10:30am, while RW doing hall monitoring for exams</td>
<td>RW's frustrations with career, plans for next year and teaching assignments, TVO award, alternate curricula, use of Geometer's SketchPad</td>
</tr>
<tr>
<td>RW-DIS-N-29</td>
<td>RW-L-1-139</td>
<td>23-08-94</td>
<td>Math Dept Room</td>
<td>mid-morning</td>
<td>example of RW's commitment to precision</td>
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<td>RW-DOC-N-09</td>
<td>RW-L-1-139</td>
<td>23-06-94</td>
<td>Math Dept Room</td>
<td>mid-morning</td>
<td>example of RW's interest in discipline, reading Eves, history of mathematics</td>
</tr>
<tr>
<td>RW-DIS-N-30</td>
<td>RW-L-1-140</td>
<td>23-06-94</td>
<td>Manchu Wok restaurant near school</td>
<td>lunch</td>
<td>Potential future career options, difficulty of making predictions</td>
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<td>RW-INT-N-09</td>
<td>RW-L-1-140-141</td>
<td>23-06-94</td>
<td>Science Lab, quiet &amp; air conditioned</td>
<td>2:00-3:30pm</td>
<td>Interview re Concept Map of discipline of Mathematics, motivation to and rationale for study of mathematics, personal love of subject need to deal with applications</td>
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<td>RW-INT-D-09a</td>
<td>RW-F-1</td>
<td>23-06-94</td>
<td>given to me in am. meeting</td>
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<td>Concept Map of discipline of Mathematics</td>
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<td>RW-INT-D-09b</td>
<td>RW-F-1</td>
<td>23-06-94</td>
<td>created in Math Dept Room</td>
<td>during am, while RW marked exams</td>
<td>Questions re Concept Map to guide INT-09</td>
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<td>RW-INT-T-08</td>
<td>RW-T-10A-000-485</td>
<td>23-06-94</td>
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<td>RW-INT-D-09c</td>
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<td>17-08-94</td>
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<td>RW-DIS-N-31</td>
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<td>23-06-94</td>
<td>Science lab after INT-09</td>
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<td>Discussion re publishing some of RW's work</td>
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<td>RW-DIS-N-32</td>
<td>RW-L-1-142</td>
<td>23-06-94</td>
<td>Math Dept Room</td>
<td>4:00pm, while RW packing up to go home</td>
<td>need for support to maintain interest in subject and to explore alternate curricula and methods</td>
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<tr>
<td>RW-DIS-N-33</td>
<td>RW-L-1-142-143</td>
<td>23-06-94</td>
<td>RW's home</td>
<td>dinner at RW's and talk after</td>
<td>Other aspects of RW's life, family, gardening, scientific approach to gardening, pride in success</td>
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<tr>
<td>RW-DOC-D-11</td>
<td>RW-F-1</td>
<td>23-06-94</td>
<td>Math Dept Room</td>
<td>given to me during am.</td>
<td>RW's Mathematical Life History Time-Line, very detailed and extensive</td>
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<tr>
<td>RW-INT-N-10</td>
<td>RW-L-1-144</td>
<td>24-06-94</td>
<td>School library</td>
<td>morning, before school's promotion meeting</td>
<td>First interview re Mathematical Life History, pre-school, elementary and secondary school, obvious impact of social sphere, need for support</td>
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<td>RW-INT-D-10a</td>
<td>RW-F-1</td>
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<td>Questions used to initiate INT-10, developed evening of 23-06-94 while studying time-line</td>
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<td>Math Dept Room</td>
<td>noon</td>
<td>Copy of newspaper article on RW winning the William N. Roman,</td>
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<td>teacher of the Year Award, NCSB, 1992-93</td>
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<td>RW's nomination package for Prime Minister's Award</td>
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<td>24-06-94</td>
<td>Math Dept Room</td>
<td>noon</td>
<td>Reference to RW's Fractal geometry course in paper</td>
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<td>RW-DOC-N-12,</td>
<td>RW-L-1-145</td>
<td>24-06-94</td>
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<td>Acquisition of DOC-12,13,14</td>
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<td>13, 14</td>
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<td>RW-DOC-N-15</td>
<td>RW-L-1-145-146</td>
<td>24-06-94</td>
<td>Math Dept Room</td>
<td>noon</td>
<td>Books on RW's desk, evidence of philosophical interest in mathematics,</td>
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<td>RW later dismisses this and claims to not have strong interest</td>
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<td>RW-INT-N-11</td>
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<td>24-06-94</td>
<td>Math Dept Room</td>
<td>4:30-5:45pm</td>
<td>Second interview re Mathematical Life History, activities beyond school</td>
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<td>during elementary and secondary school years, university years, focus on</td>
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<td>Questions prepared to guide INT-11</td>
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<td>RW-INT-N-12</td>
<td>RW-L-1-148-147</td>
<td>25-07-94</td>
<td>Office, MoArthur</td>
<td>9:00-11:00am</td>
<td>RW in Kingston for programming work with Sam Lock, meet for</td>
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<td>Hall</td>
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<td>interview on Mathematical Life History, B.Ed. year and first years of</td>
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<td>teaching</td>
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<td>25-07-25</td>
<td></td>
<td></td>
<td>Questions prepared to guide INT-12</td>
</tr>
<tr>
<td>RW-INT-T-12</td>
<td>RW-T-11B-113-689</td>
<td>25-07-25</td>
<td></td>
<td></td>
<td>Audio tape of INT-12</td>
</tr>
<tr>
<td></td>
<td>RW-T-12A-000-287</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RW-INT-D-12b</td>
<td>RW-F-1</td>
<td>19-12-95</td>
<td></td>
<td></td>
<td>Hardcopy of transcription of RW-INT-T-12</td>
</tr>
<tr>
<td>RW-INT-C-12</td>
<td>RW-INT12.LH3</td>
<td>19-12-95</td>
<td></td>
<td></td>
<td>Computer file of transcription of RW-INT-T-12</td>
</tr>
<tr>
<td>RW-DOC-D-16</td>
<td>RW-F-1</td>
<td>25-07-94</td>
<td>Office, MoArthur</td>
<td>11:00am</td>
<td>RW's Concept Map with arrows and some labels added to connecting lines.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hall</td>
<td></td>
<td>RW concerned about the &quot;correctness&quot; of map</td>
</tr>
<tr>
<td>RW-DOC-N-18</td>
<td>RW-L-1-147</td>
<td>25-07-94</td>
<td></td>
<td></td>
<td>Notes re RW's comments on revised map</td>
</tr>
<tr>
<td>Code</td>
<td>Location Code</td>
<td>Date</td>
<td>Place</td>
<td>Event</td>
<td>Content</td>
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<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>RW-DIS-N-34</td>
<td>RW-L-1-147-148</td>
<td>25-07-94</td>
<td>McArthur Hall &amp; 77 Queen Mary Rd</td>
<td>afternoon and evening</td>
<td>Conversations with RW while he scanned in pictures for textbook and visited at home</td>
</tr>
<tr>
<td>RW-INT-N-13</td>
<td>RW-L-1-148-149</td>
<td>25-07-94</td>
<td>77 Queen Mary Rd, office</td>
<td>8:00-10:00pm</td>
<td>Mathematical Life History continued, Introduction to and work with Fractal Geometry and past year at NHS</td>
</tr>
<tr>
<td>RW-INT-D-13</td>
<td>RW-F-1</td>
<td>19-12-95</td>
<td></td>
<td></td>
<td>Hardcopy of transcript of RW-INT-T-13</td>
</tr>
<tr>
<td>RW-DIS-N-35</td>
<td>RW-L-1-152</td>
<td>25-07-94</td>
<td>77 Queen Mary Rd, office</td>
<td>approx. 9:30pm</td>
<td>Pause during INT-13 to discuss possible problem with thesis report in identifying Math Dept of NHS as lacking in mathematical interest, of little concern to RW</td>
</tr>
<tr>
<td>RW-DIS-N-36</td>
<td>RW-L-1-149-150</td>
<td>25-07-94</td>
<td>77 Queen Mary Rd</td>
<td>10:00-11:30pm</td>
<td>RW's interest in publishing work, my help in selecting possible journals, 3-Coin Toss problem, my simulation in LOGO</td>
</tr>
<tr>
<td>RW-DIS-N-37</td>
<td>RW-L-1-150-151</td>
<td>26-07-94</td>
<td>77 Queen Mary Rd and McArthur Hall</td>
<td>8:00am-2:00pm</td>
<td>RW's activities in Kingston, 26-07-94, and discussions</td>
</tr>
<tr>
<td>RW-DIS-N-38</td>
<td>RW-L-1-151</td>
<td>26-07-94</td>
<td>Office, McArthur Hall</td>
<td>11:30am</td>
<td>Observations on RW's mood during visit</td>
</tr>
<tr>
<td>RW-DIS-N-39a</td>
<td>RW-L-1-153</td>
<td>28-08-94</td>
<td>Home</td>
<td>Phone call approx. 11:00am</td>
<td>RW calls to ask me how to construct expression in LOGO to generate exponentials</td>
</tr>
<tr>
<td>RW-DIS-N-39b</td>
<td>RW-L-1-153-154</td>
<td>28-08-94</td>
<td>Home</td>
<td>Phone call approx 2:00pm</td>
<td>RW calls to report that he has found so/.solution to problem</td>
</tr>
<tr>
<td>RW-DOC-D-17</td>
<td>RW-F-1</td>
<td>05-09-94</td>
<td>Faq of Ed</td>
<td>Letter in the MT</td>
<td>Letter from RW in <em>Mathematics Teacher, 87</em>(7), 556 October 1994, discussion of solution of interesting problem provided earlier by Ron Lancaster, alternate sol'n provided by RW, statement at end of letter &quot;... a curriculum that is in a hurry to cover, rather discover mathematics.&quot;</td>
</tr>
<tr>
<td>RW-DOC-D-18</td>
<td>RW-F-1</td>
<td>07-11-94</td>
<td>Faq of Ed</td>
<td>Letter</td>
<td>Letter from RW along with draft of paper on 3-Coin Toss problem, written in narrative style to capture the sense of adventure in the investigation, asking for my comments</td>
</tr>
<tr>
<td>RW-DIS-N-40</td>
<td>RW-L-1-154-158</td>
<td>13-11-94</td>
<td>Kingston</td>
<td>Phone call, Sunday evening</td>
<td>Call to RW to discuss his draft of paper for 3-Coin Toss paper and to report on plans for RW's April visit to Queens, discussion of progress at school, some progress in opportunities to teach Fractal Geometry</td>
</tr>
<tr>
<td>Code</td>
<td>Location Code</td>
<td>Date</td>
<td>Place</td>
<td>Event</td>
<td>Content</td>
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<td>------------------------------------------------------------------------</td>
</tr>
<tr>
<td>RW-DOC-D-19a</td>
<td>RW-F-1</td>
<td>21-11-94</td>
<td>Fac of Ed</td>
<td>Letter</td>
<td>Letter from RW to accompany disk, information re software, formal presentation of directions, asking for comments on work</td>
</tr>
<tr>
<td>RW-DOC-D-19b</td>
<td>RW-F-1</td>
<td>21-11-94</td>
<td></td>
<td></td>
<td>Disk of new versions of Fractal software and procedures for logo to draw structured walk curves of any given dimension</td>
</tr>
<tr>
<td>RW-DIS-N-41</td>
<td>RW-L-1-158-159</td>
<td>12-02-95</td>
<td>phone call from Kingston</td>
<td>evening</td>
<td>Phone call to RW to discuss details of April visit, especially QSLMA Conference and teacher-student conference, talk of RW's Fields Institute experience, enthused about future possibilities, support for ideas of teachers students in the groups at conference, talk of most recent experiments</td>
</tr>
<tr>
<td>RW-DOC-D-20</td>
<td>RW-F-1</td>
<td>24-02-95</td>
<td>Faculty of Education</td>
<td>via Fac of Ed member from press conference</td>
<td>Press release of February 16, 1995 announcing the Prime Minister's Awards for Teaching Excellence in Science, Technology and Mathematics. RW receives a local award with description making reference to points in my supporting statement re. interest outside of class and mentoring of future mathematicians and teachers.</td>
</tr>
<tr>
<td>RW-DIS-N-42</td>
<td>RW-L-1-158-161</td>
<td>17-04-95</td>
<td>Faculty of Education</td>
<td>late afternoon after drive from Sudbury</td>
<td>RW drives from Golden City to spend week at Faculty of Education as Visiting Scholar and to present at QSLMA Conference and Mathematical Explorations Day. Talk of recent explorations in mathematics, RW excited and making links between many ideas</td>
</tr>
<tr>
<td>RW-DOC-D-21a,b</td>
<td>RW-F-1</td>
<td>17-04-95</td>
<td>Faculty of Education</td>
<td></td>
<td>Posters announcing presentations by RW as Visiting Scholar at Queen's</td>
</tr>
<tr>
<td>RW-DIS-N-43</td>
<td>RW-L-1-161</td>
<td>17-04-95</td>
<td>Restaurant</td>
<td>During dinner</td>
<td>Talk concerning RW's sabbatical activities, and his presentations at the Faculty</td>
</tr>
<tr>
<td>RW-INT-N-14</td>
<td>RW-L-1-162</td>
<td>18-04-95</td>
<td>Home, Kingston</td>
<td>After dinner</td>
<td>Interview to explore RW's views on subject interaction and link to his view of mathematics. Much focus on impediments to integration, school structures, lack of leadership, limited opportunities and time for planning</td>
</tr>
<tr>
<td>RW-INT-D-14a</td>
<td>RW-F-1</td>
<td>18-04-95</td>
<td></td>
<td></td>
<td>Summary of RW's writing and interviews concerning nature of mathematics to be used in RW-INT-14, to remind RW of his statements and compare to his concerns with integration</td>
</tr>
<tr>
<td>RW-INT-D-14b</td>
<td>RW-F-1</td>
<td>22-06-95</td>
<td></td>
<td></td>
<td>Hardcopy of transcript of tape of RW-INT-T-14</td>
</tr>
<tr>
<td>RW-INT-C-14</td>
<td>RW-INT14</td>
<td>22-06-95</td>
<td></td>
<td></td>
<td>Computer file of transcript of tape of RW-INT-T-14</td>
</tr>
<tr>
<td>Code</td>
<td>Location Code</td>
<td>Date</td>
<td>Place</td>
<td>Event</td>
<td>Content</td>
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<td>--------------------------------------------</td>
<td>--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>RW-LES-N-24</td>
<td>RW-L-1-162-163</td>
<td>19-04-95</td>
<td>Faculty of Education</td>
<td>1:30-3:30 class, CURR343</td>
<td>RW’s talk to CURR343 class, changing teaching approaches, using iterative procedure to find cube roots as an example, message to new teachers - be positive, take risks and change the system</td>
</tr>
<tr>
<td>RW-LES-D-24a</td>
<td>RW-F-1</td>
<td>19-04-95</td>
<td></td>
<td></td>
<td>Copy of overhead transparency used by RW to illustrate excitement in mathematics, coloured picture of a fractal, dendrite of Mandelbrot set.</td>
</tr>
<tr>
<td>RW-LES-D-24b</td>
<td>RW-F-1</td>
<td>19-04-95</td>
<td></td>
<td></td>
<td>Copy of overhead transparency used by RW to illustrate problems with today’s mathematics teaching. Board full of symbols and professor claiming (joking) &quot;None of this means a thing!&quot;</td>
</tr>
<tr>
<td>RW-LES-D-24c</td>
<td>RW-F-1</td>
<td>19-04-95</td>
<td></td>
<td></td>
<td>Copy of handout, The Cube Root to Chaos, to accompany presentation to CURR343. Ideas not fully developed in class left to students to explore.</td>
</tr>
<tr>
<td>RW-LES-N-25</td>
<td>RW-L-1-164-165</td>
<td>20-04-95</td>
<td>Faculty of Ed</td>
<td>QSLMA Spring Conference</td>
<td>RW gives presentation in second session time-slot of QSLMA Spring Conference, &quot;Chaos in the Candy Store and Other Rare Events&quot;. Well attended and lively presentation and debate</td>
</tr>
<tr>
<td>RW-LES-D-25a</td>
<td>RW-F-1</td>
<td>20-04-95</td>
<td></td>
<td></td>
<td>Announcement for QSLMA Spring Conference with RW’s presentation</td>
</tr>
<tr>
<td>RW-LES-D-25b</td>
<td>RW-F-1</td>
<td>20-04-95</td>
<td></td>
<td></td>
<td>Handout, Chaos in the Candy Store and Other Rare Events, from RW’s talk</td>
</tr>
<tr>
<td>RW-DIS-N-43</td>
<td>RW-L-1-165-166</td>
<td>21-04-95</td>
<td>Home, Kingston</td>
<td>over and after dinner</td>
<td>Dinner at home after work at Mathematical Explorations day. Exciting discussions concerning randomness and probability, sense of intellectual movement.</td>
</tr>
</tbody>
</table>
Appendix B: Information Given to Participants re Writing of Personal Statement on the Nature of Mathematics

The instructions on the following page were given to each participant to request that they write a brief personal statement concerning the nature of mathematics. The response was employed as the starting point for subsequent interview(s) exploring images of mathematics. Each participant was asked to fit the writing task into their schedule and interview times were arranged once the position statement was completed.
Exemplary Mathematics Teachers: 
Subject Conceptions and Instructional Practices

Images of Mathematics

To initiate the investigation of your views of the nature of mathematics I would like to ask you to write some brief personal statements. Please note that this is not meant to be a research essay but is designed to capture your personal position. There are no "correct" answers against which your responses will be measured.

Please respond to the following as if they were posed by some one who has not studied mathematics beyond the Grade 8 level.

1) What is mathematics?

2) Where does mathematical knowledge come from? How is mathematics formed?

3) How do I know that all these things in mathematics are true?

4) Why should I continue to study mathematics? What good is it to me or anyone else?

We (you and I) will use your responses to these questions as starting points in a future interview.
Appendix C: Information Given to Participants re the Construction of a Repertory Grid

The information on the following pages, concerning repertory grids and how they are constructed, was given to each participant at least two weeks prior to developing their grid. Further information concerning the construction and interpretation of repertory grids was provided in person prior to beginning the development of the grid. Participants were requested to consider, prior to our meeting, the school subjects that they would like to use as elements for their grid.

An interview was scheduled after the completion of the grid to permit collaborative analysis of the results. At these interviews the principle components (PrinCom) display of the grid was employed as the focus of discussion.
Repertory grid technique originated in the theory of personal constructs developed during the 1950's by the psychologist, George Kelly. A "personal construct" is an individual's unique image of some aspect of the world around them. Our personal constructs are the way we understand and interpret the world. Repertory grid technique is a way of helping one identify constructs that may be well hidden in the mind.

To begin the process of building a grid we identify a set of items or Elements to think about. For this project, school subjects or academic disciplines will be the Elements. A random triple out of the set of Elements is selected and these are examined for similarities and differences by identifying some way in which two are alike and dissimilar from the third. This process generates a Construct or a scale upon which the Elements may be arranged. For instance a set of TV shows could be arranged along a line with ends labelled "like" and "dislike". The process of selecting triples and identifying Constructs is repeated a number of times. Once this activity is completed the computer program, RepGrid, will be used to analyse the data. This software looks for clustering in the arrangements of Elements along the various Constructs and arranges these in a matrix (FOCUS) with similarly viewed items placed in close proximity. A more graphical display (PrinCom) with Elements arranged as points and Constructs treated as line segments is also provided. Attached are samples of the FOCUS and PrinCom displays of a grid produced by one of my students when thinking about elementary school pupils she had met. Note all names are pseudonyms.

All of the above will be further explained and illustrated when we begin developing the grid. The output from the computer analysis will be used as the focus of a subsequent interview.
To prepare for building the repertory grid I would like you to begin identifying possible Elements. Obviously Mathematics will be an Element and I would also like to include English and Physical Education. The other Elements I would like to select from various groupings. Please pick one or two examples from each group - subjects with which you may have had some experience as a student or teacher or ones about which you have some opinions. I think that a set of about eight to ten subjects should be sufficient.

<table>
<thead>
<tr>
<th>Biology</th>
<th>French</th>
<th>Dance</th>
<th>Economics</th>
<th>any Technology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemistry</td>
<td>German</td>
<td>Drama</td>
<td>Geography</td>
<td>any Commercial</td>
</tr>
<tr>
<td>Physics</td>
<td>Spanish</td>
<td>Music</td>
<td>History</td>
<td>Computer Studies</td>
</tr>
<tr>
<td></td>
<td>any Language</td>
<td>Visual Art</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix D: Information Given to Participants re
the Construction of a Concept Map for Mathematics

The information on the following page, outlining concept
mapping technique, was left with each participant to provide
initial guidance. Participants were requested to fit the
development of the concept map into their schedules. Once the
map was completed, a time was set up for an interview in which we
conducted a further exploration of the map.
Concept map for Mathematics

Concept mapping is a method for attempting to analyse how one has a body of knowledge arranged in one's mind. If an individual draws a concept map of the discipline of mathematics, we can get some idea of his or her mental structure for the subject, how the various topics, concepts, and procedures are linked. Below is an example of a concept map employed by Novak and Gowin (1984) to guide interviews regarding individuals understanding of art.


I would like you to construct a personal concept map for Mathematics. Begin by placing the title, Mathematics, in the middle of the large chart paper and then just let the ideas associated with this title flow. Place the words associated with related ideas in clusters on the page and add lines to show connections. As in the example above, the connecting lines may be labelled to show the nature of the links between concepts. You may wish to do your work in pencil to allow adjustments if later ideas require rearrangements.
Appendix E: Concept Map for Mathematics: Jonathan Ode

(JO-INT-D-12a)
Appendix F: School Subjects Repertory Grid: Jonathan Ode

(JO-INT-D-04)

FOCUS:
Elements: 8, Constructs: 11, Range: 1 to 10, Context: School Subjects/Academic Disciplines

PrinCom:
Elements: 8, Constructs: 11, Range: 1 to 10, Context: School Subjects/Academic Disciplines
Annabelle Arable was a successful farmer and landowner. She started out with only ten acres of land in the year after the Big Drought.

What year was the Big Drought?

Annabelle Arable was so successful that after every fall harvest, she bought all the fields that shared a fence with her own.

Those other farmers left Cakewalk County for the Big City to seek their fortunes.

In Cakewalk County, where Annabelle lived, each field is perfectly square and shares a fence with the four fields that surround it. Each field is also exactly ten acres.

In the summer of 1914, Annabelle had 410 acres of land.

When was the Big Drought in Cakewalk County?

Every year, Annabelle Arable’s total landholdings were in the shape of a square (sort of), though she never held a square number of acres or fields.

By the way, there are 640 acres in a square mile.

In May 1915, at the age of seventy-three, Annabelle retired.

She gave 200 acres to each of her three children and kept ten for herself, where she raised prizewinning asparagus for many years.
Appendix H: Instruction Pages Used by Jonathan Ode for Algebraic Modelling Lesson (JO-LES-D-20b)

TEAM INSTRUCTIONS:

TEAM ROLES:

1. Task Manager/Coach
   a. Read task definition to team members
   b. Make sure team members understand the problem and their role in its solution
   c. Keep everyone on task
   d. Provide help for the problem solver by acting as a coach.

2. Problem Solver
   a. Performs any constructions or tasks outlined by the experiment
   b. Keeps the focus of the discussion on solving the required problem
   c. Together with the team members solves the problem.

3. Recorder
   a. Keeps a record of the progress of the group
   b. Completes any charts, diagrams, graphs, etc. necessary for solving the problem
   c. Works with the presenter in the creation of posters, acetates for the overhead projector etc.

4. Presenter
   a. Presents the results of the team investigation to the rest of the class
   b. Answers any questions the class may have about the solution.

PROBLEM:

In this exercise each team is given a geometric pattern to construct with the blocks. Measurements are taken regarding the surface area and volume of the blocks at each level. This data is recorded in the chart. The team will then investigate the number patterns to see if they can come up with a pattern or rule for predicting what the surface area or volume will be at any level. Your team will then use this pattern to predict the number at level 10.

At the end of the investigation selected teams will have the opportunity to present their results to the class. You must make sure that the class understands how you solved the problem. Your presentation should include a demonstration of what you did, using blocks or the overhead projector. Marks will be awarded for the quality of the demonstration as well as the spoken report.
What was the strongest part of your teamwork?

What was the weakest part of your teamwork?

What improvements could you make in the future and how could you prevent the problems you encountered?

What mark (out of 100%) would you like to assign to yourself for your teamwork?

If you would like to justify your mark, please do so here:
Appendix I: Student Products from Periodic Decimal Art Project

(JO-LES-D-07e,j,m,n)
Appendix J: First School Subjects Repertory Grid: Randy Walker

(RW-INT-D-01a)

FOCUS:

Elements: 8, Constructs: 10, Range: 1 to 10, Context: School Subjects/Academic Disciplines

![Repertory Grid Diagram]

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Appendix K: Second School Subjects Repertory Grid: Randy Walker

(RW-INT-D-07)

FOCUS:
Elements: 7, Constructs: 8, Range: 1 to 10, Context: School Subjects/Academic Disciplines

Descriptive of natural law
Objective
Problem solving
Concise
Symbolic expression
Technical language
Limited by ability
Rigid

1 Subject matter diffuse
5 Subjective
3 Descriptive
8 Verbal
2 Explicit language
4 English language
6 Limited by interest
7 Allows expression and exploration

PrinCom:
Elements: 7, Constructs: 8, Range: 1 to 10, Context: School Subjects/Academic Disciplines

Problem solving
Descriptive of natural law
Rigid
Concise
Objective
Algebra
Calculus
Symbolic expression
Limited by ability
Technical language

English language
Limited by interest
History
Explicit language
English
Subjective
Verbal
Allows expression and exploration
Subject matter diffuse
Descriptive

Music
Appendix L: Concept Map for Mathematics: Randy Walker

(RW–INT–D–09a)
IMAGE EVALUATION
TEST TARGET (QA-3)

APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989

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