Applications of Reflection to Topology

by

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Abstract
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This work looks at reflection properties for topological spaces both in the large cardinal/forcing context and in the elementary submodel context. We present a general technique valid for an interesting class of spaces (e.g. locally compact, Čech-Complete, first-countable) in both contexts and which allows us to reproduce results, present new results and also to obtain interesting upwards and downwards reflection results and examples.
To my loving family.
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Introduction

This thesis consists of three chapters and an Appendix, the Appendix contains definitions used in this work and the reader may choose to start by reading it, even before reading the rest of this introduction, if he is unfamiliar with concepts such as elementary submodels, elementary embeddings, collectionwise normal spaces etc.

The reader that wishes to avoid, in a first moment, elementary embeddings, large cardinals and forcing may wish also to start reading Chapter 3 (and even skip Section 3.2), which uses elementary submodels and has a simpler approach to the technique, and later read the other chapters.

For topological definitions (that are not in the Appendix) and theorems we refer the reader to [9]. For results about elementary embeddings and large cardinals we refer the reader to [16] and [8]. For results and definitions on forcing we refer the reader to [20] and [16].

The effect of set theoretic axioms, combinatorial properties and techniques in general has, for many years, been a source of interesting results for topologists. A topological space has a simple definition, it is a set and a family of subsets of this set that satisfies a few axioms. This structural simplicity allows us to use set theoretic tools to build topological spaces, to obtain properties for known topological spaces and to prove (consistent) results about topological spaces.

A simple example is the real line: if $2^{\aleph_0} = \aleph_1$ then there is a family of $\omega_1$ first category sets whose union is the real line (take the singletons), however, under different consistent axioms, the union of a family of $\omega_1$ first category sets is of first category ([20]).

Things get more complicated when one introduces forcing, large cardinals, elemen-
tary submodels and combinatorial principles, one can start by looking at the many articles in [22] to appreciate the variety of results that can obtained using set theoretic tools.

Reflection of a topological property is a technical term which means:

"If a topological space $X$ does not have the property $\mathcal{P}$, then there is a "small" (e.g. of cardinality less than $|X|$, or less than some fixed cardinal) subspace of $X$ which does not have property $\mathcal{P}$."

Loosely, it has come to mean results proved using supercompact cardinals or elementary submodels.

First, in this work we explore the large cardinal/elementary embeddings technique. In this context, at first the "small" subspace turns out to be $j''(X) \subseteq j(X)$, (where $j$ is the elementary embedding and $X$ is the topological space). By elementarity, this yields a "small" subspace of $X$. We are able, then, to extend the concept of reflection by using "a small nice image of a subspace" instead of "a small subspace" in the previous sentence, so as to be able to get away with weaker hypotheses on $X$ than are needed for full reflection.

We work with properties like (hereditary) normality, collectionwise normality, countable paracompactness, expandability. They all involve subfamilies of the topology with special properties (discreteness, local finiteness...) that we want to expand to families of open sets having the same property. To see how delicate this is, notice that if $X$ is a normal space, one would have some difficulties to prove that $X$ remained normal after forcing as, although the basis for the topological space does not change, new open and thus new closed disjoint sets might appear.

The large cardinal/elementary embedding technique allows us to relate $X$ "nicely" to $j(X)$, (which has, by elementarity, the same properties that $X$ has that can be stated in a formula - normality, for instance). For an interesting class of topological spaces, $X$ turns out to be a "nice" image of a subspace of $j(X)$. This, added to nice technical forcing techniques available to us when adding Cohen or random reals, make it possible to obtain results.
We then explore the elementary submodel technique. We first use elementary submodels as a tool to prove a result on the preservation of normality after Cohen forcing in a non large cardinal context.

After that, we use elementary submodels to obtain new spaces related to the original ones via the elementary submodel. This space is then considered to be a reflection of the original space, although it is not always a subspace. We study how sequential properties like tightness, sequentiality and Fréchet behave and provide some results and examples. We also study the case when this space obtained from a topological space and an elementary submodel is a cardinal. It so happens that, in this context, the topological space and its reflection via the elementary submodel are nicely related, (again, a “nice” image of a subspace) for an interesting class of spaces, to be more precise, the same class that we have in the large cardinal/forcing context.

We finally present a discussion relating both contexts (that have some properties and results in common) and present some examples.

In [8], the authors obtained reflection results for topological spaces using forcing techniques and elementary embeddings of the universe. The results were obtained for spaces of small character (small meaning smaller than the critical point of the elementary embedding). This condition was enough to identify the topological space $X$ with a subspace of $j(X)$, namely $j''(X)$ (which inherits its topology from $j(X)$ making this identification not always valid). Then the authors proceeded with the forcing and reflection results and proved for instance:

**Theorem 0.0.1** [8] In the model obtained after the addition of $\kappa$ supercompact many Cohen or random reals normal spaces of character less than $\kappa$ are collectionwise normal.

The following question arises:

**Question1:** How is $X$ related to $j(X)$ if the character of the space is not “small”? Can they be related in such a way that the same techniques can be applied and similar results proved?
In this work, we show that for a bigger class of spaces which includes locally compact spaces and Čech complete spaces, although $X$ cannot be identified to a subspace of $j(X)$ it can be identified to a perfect image of a subspace of $j(X)$. This technique allows us to produce both new results and, more straightforward proofs of known results.

Using this technique, we prove that in the model obtained after adding supercompact many Cohen or random reals:

1. Normal spaces with $\mathfrak{h} < 2^{\aleph_0}$ (in particular, locally compact spaces) are collectionwise normal [2].

2. Normal $k'$-spaces are collectionwise normal [4] (see Definition 2.4.1 for the definition of $k'$-spaces).

3. Hereditarily normal spaces with $\mathfrak{h} \leq 2^{\aleph_0}$ (in particular, locally compact spaces) remain hereditarily normal after the addition of any number of Cohen reals.

4. Countably paracompact spaces with $\mathfrak{h}(X) = \omega$ are expandable [2].

We then change context and start exploring elementary submodels. First we use elementary submodels to give a partial answer for the following question:

**Question 2:** For which spaces can we (consistently) preserve normality without the use of large cardinals after adding Cohen reals?

Then we start working with spaces obtained via elementary submodels:

**Question 3:** If $X_M$ (see Definition 3.1.1) has a certain topological property $\mathcal{P}$, is it true that $X$ also has $\mathcal{P}$? (This kind of question was explored in [18].)

We present some results and examples concerning properties related to sequentiality.

Using the perfect mapping technique, replicated in the elementary submodel context in [18], we study the following question for cardinal spaces, providing results and examples.
**Question 4:** If $X_M$ is known, can we describe $X$ up to homeomorphism (see Definition 3.1.1)? (This kind of question was explored in [29].)

In the first chapter, two proofs of the main theorem, characterizing $X$ as a perfect image of a subspace of $j(X)$ for $X$ of pointwise countable type are given, and some examples are presented. The second chapter gives applications of this technique to characterizations of normality and related properties. The third chapter deals with elementary submodels, in particular establishing the undecidability of the assertion that if $X_M$ is homeomorphic to $\omega_1$, so is $X$. 
Chapter 1

Reflection with large cardinals

1.1 Introduction

This chapter deals with the proof of the perfect mapping result mentioned in the Introduction of this Thesis. It presents two proofs of the same result. The one in Section 1.2 appeared first, based on [7] which gives a proof of Theorem 2.3.2 in the locally compact case, using the ring of continuous functions. The one in Section 1.3 appeared later but is considered to be simpler. Both are shown to be equivalent and some examples are presented.

Recall that the definitions used in this chapter can be found in the APPENDIX.

Lemma 1.1.1 Let $j : V \rightarrow M$ be an elementary embedding such that $M^\omega \subseteq M$ and let $P$ be a countable chain condition partial order such that $P \in M$ and $P \subseteq M$. Then if $G$ is $P$-generic over $M$, it is also $P$-generic over $V$.

Proof: Let $A \subseteq P$ be a maximal antichain. $A \subseteq M$ and $M^\omega \subseteq M$ so $A \in M$ and $M \models A$ is a maximal antichain (since $M \subseteq V$). So $G \cap A \neq \emptyset$. □

It is worth mentioning that if $P$ is the Cohen partial order then you don’t need $M^\omega \subseteq M$ as $P \cap M$ is completely embedded in $P$.

In addition to the “normal implies collectionwise normal” result presented in the Introduction of this Thesis (Theorem 0.0.1 ) the following Theorem, proved using
techniques presented in [8] is a good motivation for this chapter.

**Lemma 1.1.2** [8] Let $G$ be a generic subset of $Fn(\lambda, 2)$ or of $M_\lambda$ (the partial order for adding $\lambda$ random reals), where $\lambda$ is a cardinal. Suppose that in $\mathcal{V}$, $<X, \mathcal{T}>$ is a topological space and $\mathcal{Y}$ is a discrete collection of subsets of $X$ and such that $|\mathcal{Y}| \leq \lambda$. If $\mathcal{Y}$ is normalized in $\mathcal{V}[G]$, it is separated in $\mathcal{V}$.

**Theorem 1.1.3** In the model obtained by adding $\kappa$ many Cohen reals, where $\kappa$ is supercompact, let $X$ be a hereditarily normal topological space such that for every $x \in X$, $\chi(x, X) < 2^{\aleph_0} = \kappa$. Then $X$ remains hereditarily normal after adding any number of Cohen reals.

**Proof of Theorem 1.1.3.** Suppose the theorem is false. Pick an $\alpha$ such that

$$\mathcal{V} \models "1_\mathbb{P}_\alpha \vDash X \text{ is a normal space and forcing with } \mathbb{P}_\alpha \text{ destroys its normality."}$$

Choose a "convenient" elementary embedding (for the technical details, see the proof of Theorem 2.2.1) $j : \mathcal{V} \rightarrow \mathcal{M}$ with $j(\kappa) > \alpha$. Let $G_{j(\kappa)}$ be $\mathbb{P}_{j(\kappa)}$-generic over $\mathcal{M}$ (and so over $\mathcal{V}$ by Lemma 1.1.1 - because "convenient" implies $\mathcal{M}^\omega \subseteq \mathcal{M}$) and let $G_\kappa = G_{j(\kappa)} \cap \mathcal{V}_\kappa$. Note that $G_\kappa$ is $\mathbb{P}_\kappa$-generic over both $\mathcal{M}$ and $\mathcal{V}$.

Since $\mathbb{P}_\kappa$ has the countable chain condition, we can (see e.g. [8]), extend $j$ to:

$$\bar{j} : \mathcal{V}[G_\kappa] \rightarrow \mathcal{M}[G_{j(\kappa)}].$$

From now on we will abuse notation and refer to $\bar{j}$ as simply $j$.

In $\mathcal{M}[G_{j(\kappa)}]$, $j(X)$ is hereditarily normal, so since $\mathcal{M}[G_{j(\kappa)}] \models j''(X)$ is a subspace of $j(X)$, and $\mathcal{M}[G_{j(\kappa)}] \models j(X)$ is hereditarily normal, $\mathcal{M}[G_{j(\kappa)}] \models j''(X)$ is hereditarily normal. So in $\mathcal{M}[G_{j(\kappa)}]$, $X$ is homeomorphic to a normal space. Now, the choice of $j$, the fact that $j(\kappa) > \alpha$, the fact that forcing with Cohen reals preserves non-normality (see Lemma 1.1.2), and that $\mathcal{M}[G_{j(\kappa)}] \subseteq \mathcal{V}[G_{j(\kappa)}]$ give us the result. □
Lemma 1.1.4 [8] Suppose that \( j \) is an elementary embedding and that the critical point of \( j \) is \( \kappa \). If \( X \) is a topological space such that for every \( x \in X \), \( \chi(x,X) < \kappa \) then \( X \) is homeomorphic to a subspace of \( j(X) \), namely \( j''(X) \).

By looking at the proof of Theorem 1.1.3, the reader will notice that the main result used concerning the topological space \( X \) was that the fact that its having "small character" made it possible to identify \( X \) with a subspace of \( j(X) \). The hereditary normality of \( j(X) \) together with forcing techniques then made it possible to obtain the result.

The theorem that will be proved in this chapter allows us to present a somewhat similar proof of the above result for a more general class of spaces, the ones with pointwise type smaller than the critical point of the elementary embedding \( j \). It should be noted that such space need not be homeomorphically embeddable in \( j(X) \).

1.2 The perfect projection using \( C^*(X) \).

In this section we shall prove the following theorem:

Theorem 1.2.1 Suppose \( j : V \rightarrow M \) is an elementary embedding, \( \kappa \) is the critical point of \( j \), \( P \in V \) is a partial order, \( G \) is \( P \)-generic over \( V \), \( H \) is \( j(P) \)-generic over \( M \) and \( j \) extends to \( j : V[G] \rightarrow M[H] \). Suppose also that \( V[G] \models X \) is a completely regular space and \( \text{Fr}(X) \leq \kappa \). Suppose the elementary embedding \( j : V[G] \rightarrow M[H] \) also satisfies the condition

\[(*) j(\kappa) > \lambda \geq |[0,1]^{C^*(X)}| \text{ and } M[H] \text{ is closed under } \lambda \text{-sequences in } V[H] \text{ whose elements have names in } M.\]

Then \( M[H] \models X \) is homeomorphic to a perfect image of a subspace of \( j(X) \).

In particular, taking \( j(\kappa) \) sufficiently large, this applies to the case of adding supercompact many Cohen or random reals. Also, the perfect mapping has the property that it is precisely \( j^{-1} \) when restricted to \( j''(X) \).
Remark 1.2.2 Let $\mathbb{P}$ denote the partial order for adding supercompact many Cohen or random reals. To see that the mapping $j$ can be extended when you add Cohen or random reals, see Proposition 2.1 and Proposition 2.2 in [8]. To see that you can satisfy (*), just look at the definition of supercompact cardinal (Definition 7 in the APPENDIX). Let $\dot{x}$ be a $\mathbb{P}$-name such that $1_{\mathbb{P}} \Vdash [0,1]^{C^*(X)}$. Let $A \subseteq \mathbb{P}$ be a maximal antichain such that for every $p \in \mathbb{P}$ there is a cardinal $\kappa_p$ such that $p \Vdash |\dot{x}| = \kappa_p$. As $\mathbb{P}$ has the c.c.c (see [20]), $A$ is countable. Pick $\lambda > \sup\{\kappa_p : p \in A\}$. You can choose $j : V \rightarrow M$ such that $j(\kappa)$ is bigger than $\lambda$. When you extend $j$ to $V[G]$, the extension will satisfy (*).

On our way toward finding the copy of $X$, having an elementary embedding with the above conditions, we follow the technique of [7], and in $V[G]$ see $X$ as its image in $[0,1]^{C^*(X)}$, using the canonical embedding

$$e : X \rightarrow [0,1]^{C^*(X)},$$

defined by:

$$e(x) = (f(x))_{f \in C^*(X)}.$$

In $M[H]$, we will see $j(X)$ as its image inside $[0,1]^{C^*(j(X))}$ under the embedding $j(e)$. Note that $[0,1]$ could now be larger!

Noticing that $C^*(j(X)) = j(C^*(X))$ and consequently, $j''(C^*(X)) \subseteq C^*(j(X))$. Define $f : j(X) = \beta(j(X)) \rightarrow [0,1]^{j''(C^*(X))}$ to be the projection described as follows:

$$(x_g)_{g \in C^*(j(X))} \mapsto (x_{j(h)})_{h \in C^*(X)} = (x_g)_{g \in j''(C^*(X))}.$$

Let

1. $j(X)_f = f''(j(X)) \setminus f''(\overline{j(X)} \setminus j(X))$,
2. $Z = f^{-1}(j(X)_f)$,
3. $\pi : Z \rightarrow j(X)_f$ be the restriction of $f$ to $Z$. 

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We use:

Lemma 1.2.3 [9] If \( f : X \rightarrow Y \) is a perfect mapping and \( Z \subseteq Y \), then \( f \upharpoonright f^{-1}(Z) : f^{-1}(Z) \rightarrow Z \) is also perfect.

We then have:

Lemma 1.2.4 \( \pi : Z \rightarrow j(X)_f \) is a perfect mapping.

**Proof:** Observe that as \( \overline{j(X)} \) is compact, \( f \) is perfect and apply Lemma 1.2.3 using that \( j(X)_f \subseteq f^\prime(\overline{j(X)}) \). \( \square \)

In order to prove Theorem 1.2.1, the following alternate characterization of \( \mathcal{H} \) will be useful.

Lemma 1.2.5 [1] For \( X \) completely regular, if \( \mathcal{H}(X) \leq \tau \) then \( \beta X \setminus X \) is the intersection of a family of \( \mathcal{F}_\tau \) sets (i.e. sets which are the union of fewer than \( \tau \) closed sets) in \( \beta X \).

**Proof of Theorem 1.2.1**

We will prove that \( j(X)_f \) includes a subspace that is homeomorphic to \( X \). First of all, note that condition (*) is needed to get that \( \{X, C^*(X), j''(C^*(X))\} \subseteq \mathcal{M}[H] \) and that the homeomorphism that we will describe shortly is also in \( \mathcal{M}[H] \). Also, if \( r \) is a real number, note that \( j(r) = r \). Since \( j \) is an elementary embedding, observe that:

\[
(\forall g \in C^*(X)) (\forall x \in X) ((j(g))(j(x)) = j(g(x)) = g(x)).
\]

Define now,

\[
E_X = \{(j(h))(j(x)) : j(h) \in j''(C^*(X)) : x \in X\}.
\]

Thus:

\[
\mathcal{M}[H] \models E_X \subseteq [0, 1]^j(C^*(X)).
\]
$E_X$ contains the same sequences of real numbers as $X$, but instead of being indexed by elements of $C^*(X)$ they are indexed by elements of $j''(C^*(X))$, and so, using that $[0, 1]^V[G]$ is first countable, we get that

$$[0, 1]^V[G] = j''([0, 1]^V[G]) \text{ is a subspace of } [0, 1]^M[H] = j([0, 1]^V[G]).$$

Consequently

$$([0, 1]^V[G])j''(C^*(X)) \text{ is a subspace of } ([0, 1]^M[H])j''(C^*(X)).$$

(By $[0, 1]^V[G]$, we mean $[0, 1]$ in the sense of $V[G]$, and similarly for $[0, 1]^M[H]$.)

We conclude that $E_X$ is homeomorphic to $X$. Explicitly, $h : X \to E_X$ is described by $h(x) = (j(f))(j(x)))j(t)^{j''(C^*(X))}$.

To simplify our notation, if $h \in C^*(X)$, let's refer to $j(h)(j(x))$ as $x_h$. If $h \in C^*(j(X))$ (or, $C^*(X)$) denote by $\overline{h}$ its extension to $\beta(j(X))$ ($\beta X$, respectively), where for any completely regular topological space $Y$, we consider $\beta Y$ as the closure of $Y$ when embedded in $[0, 1]C^*(Y)$.

We will prove that

$$E_X \subseteq j(X)_f.$$

To see that $E_X \subseteq j(X)_f$, suppose otherwise. Then,

$$\exists z = (j(h)(j(x)))_h \in C^*(X) \in E_X, \exists y \in \beta(j(X)) \setminus j(X) = j(\beta X \setminus X)$$

such that

$$j(h)(y) = j(h)(j(x)), \forall h \in C^*(X).$$
We then have

\[(***) \ M[H] \models y \in j(\beta X \setminus X) \cap \bigcap \{j^{-1}(x_h) : j(h) \in j''(C^*(X))\}.\]

Claim

\[V[G] \models (\beta X \setminus X) \cap \bigcap \{h^{-1}(x_h) : h \in C^*(X)\} \neq \emptyset.\]

Assuming the claim, pick \(z\) in the above intersection. Pick \(g\) a continuous function on \(\beta X\) such that \(g(z) = 0\) but \(g(x) = 1\). Now \(g = g \upharpoonright X \in C^*(X)\). Therefore \(z_g \neq x_g\) contradicting the fact that \(z\) is in the intersection. It remains to prove the claim.

**Proof of Claim:**
Suppose the claim is false. Then

\[V[G] \models \beta X \setminus X \cap \bigcap \{h^{-1}(x_h) : h \in C^*(X)\} = \emptyset.\]

Since \(h(X) \leq \kappa\), by Lemma 1.2.5 we may write

\[\beta X \setminus X = \bigcap \{R^i : i \in I\} \text{ where each } R^i \text{ is a } F_{\aleph_0} \text{ in } \beta X.\]

Then, since \(\bigcap \{h^{-1}(x_h) : h \in C^*(X)\}\) consists of just one point, namely \((x_h)_{h \in C^*(X)}\),
we get

\[V[G] \models \exists i \in I \text{ such that } \bigcap \{h^{-1}(x_h) : h \in C^*(X)\} \cap R^i = \emptyset.\]

Now we can write \(R^i = \bigcup\{K_\sigma : \sigma \in \tau_i\}\), for some \(\tau_i < \kappa\), where each \(K_\sigma\) is closed and thus compact in \(\beta X\).

We have

\[V[G] \models (\forall \sigma \in \tau_i) \ K_\sigma \cap \bigcap \{h^{-1}(x_h) : h \in C^*(X)\} = \emptyset.\]
Using that each $K_\sigma$ is compact, pick for every $\sigma \in \tau_i$, $f^\sigma_1, \ldots, f^\sigma_{k_\sigma} \in C^*(X)$ such that
\[
K_\sigma \cap \overline{f^\sigma_1^{-1}(x_{f^\sigma_1})} \cap \ldots \cap \overline{f^\sigma_{k_\sigma}^{-1}(x_{f^\sigma_{k_\sigma}})} = \emptyset.
\]

We have
\[
\mathcal{V}(G) \models R^i \cap \bigcap \{\overline{f^\sigma_t^{-1}(x_{f^\sigma_t})} : \sigma \in \tau_i, \ t = 1, \ldots, k_\sigma\} = \emptyset.
\]

Apply $j$ to the above formula, and get by the elementarity of $j$ that
\[
\mathcal{M}(H) \models j(R^i) \cap j\left(\bigcap \{\overline{f^\sigma_t^{-1}(x_{f^\sigma_t})} : \sigma \in \tau_i, \ t = 1, \ldots, k_\sigma\}\right) = \emptyset.
\]

But
\[
j\left(\bigcap \{\overline{f^\sigma_t^{-1}(x_{f^\sigma_t})} : \sigma \in \tau_i, \ t = 1, \ldots, k_\sigma\}\right) = \bigcap \{\overline{j(f^\sigma_t)^{-1}(x_{f^\sigma_t})} : \sigma \in \tau_i, \ t = 1, \ldots, k_\sigma\},
\]
as it is a set of size $\tau_i < \kappa$.

Finally, we have
\[
\mathcal{M}(H) \models j(\beta X \setminus X) \subseteq j(R^i).
\]

Thus,
\[
\mathcal{M}(H) \models j(\beta X \setminus X) \cap \bigcap \{\overline{j(f^\sigma_t)^{-1}(x_{f^\sigma_t})} : \sigma \in \tau_i, \ t = 1, \ldots, k_\sigma\} = \emptyset,
\]
contradicting (**). The claim is proved.

Thus we have
\[
\pi : Z \longrightarrow (jX)_f \text{ is a perfect mapping and } E_X \subseteq (jX)_f.
\]
Using Lemma 1.2.3, finally we have:

\[ \pi : \pi^{-1}(E_X) \subseteq j(X) \rightarrow E_X \text{ is perfect}. \]

Remark. In huge cardinal applications, it is useful to note that the condition that \( \lambda \geq |[0,1]|^{C^*(X)} \) can be replaced by \( \lambda \geq \max(2^{|X|}, |C^*(X)|) \) if the application only requires the homeomorphism between \( X \) and \( E_X \) to be in \( V[H] \).

1.3 The perfect projection without functions

Now we obtain the same result as in Section 1.2 with a different technique. By avoiding the trouble of identifying \( X \) with a subspace of \( [0,1]|^{C^*(X)} \), we are able to produce a seemingly simpler projection and the Theorem is valid for regular spaces. Some results of this Section appear in [15]. For the sake of clarity, we restate the Theorem:

**Theorem 1.3.1** Suppose \( j : V \rightarrow M \) is an elementary embedding, \( \kappa \) is the critical point of \( j \), \( P \in V \) is a partial order, \( G \) is \( P \)-generic over \( V \), \( H \) is \( j(P) \)-generic over \( M \) and \( j \) extends to \( j : V[G] \rightarrow M[H] \). Suppose also that \( V[G] \models (X,T) \) is a regular space and \( \mathcal{H}(X) \leq \kappa \). Suppose the elementary embedding \( j : V[G] \rightarrow M[H] \) also satisfies the condition

\[ (*) \quad j(\kappa) > \lambda \geq 2^{|T|} \text{ and } M[H] \text{ is closed under } \lambda \text{-sequences in } V[H] \]

whose elements have names in \( M \).

Then \( M[H] \models X \) is a perfect image of a subspace of \( j(X) \).

Again, in particular, this applies to the case of adding supercompact many Cohen or random reals (see Remark 1.2.2) and the perfect mapping is exactly \( j^{-1} \) when restricted to \( j''(X) \).

From now on \( X \) will denote a topological space (with no extra requirements) with topology \( T \) and we will use the notation of Theorem 1.3.1.
We start by presenting some notation and necessary results to be able to prove the Theorem:

**Definition 1.3.2**

1. Let $\mathcal{K} = \{ K \subseteq X : K \text{ is compact and } \chi(K, X) < \kappa \}$.

2. For $x \in X$, let $\mathcal{K}_x = \{ K \in \mathcal{K} : x \in K \}$.

3. For $x \in X$, let $\mathcal{V}_x = \{ V : V \text{ is open and } x \in V \}$.

4. For $K \in \mathcal{K}$, let $\mathcal{V}_K = \{ V_\alpha : \alpha < \tau \}, \tau < \kappa$, be a fixed outer base for $K$.

Notice that because of $(*)$, $\{ j''(\mathcal{K}), j''(\mathcal{K}_x), j''(\mathcal{V}_x) \} \subseteq M[H]$ and also, supposing without loss of generality that $X$, as a set, is an ordinal, $(X, T) \in M[H]$.

We define in $M[H]$:

**Definition 1.3.3** For $x \in X$, let $K_x = \bigcap j''(\mathcal{V}_x)$.

This idea of taking the intersection of “old” open sets about $x$ also appears in Balogh’s proof [2].

From now on, we assume $X$ is Hausdorff.

**Definition 1.3.4**

$$\pi : Z = \bigcup \{ K_x : x \in X \} \longrightarrow X,$$

by

$$\pi(z) = t \text{ if and only if } z \in K_t.$$

We need $X$ to be a Hausdorff space as in this case, $x \neq y$ implies that $K_x \cap K_y = \emptyset$. This implies that the mapping $\pi$ defined above is indeed well defined, one point in $Z$ belongs to one $K_x$ at most.

**Theorem 1.3.5** If $X$ is $T_3$ and $h(X) \leq \kappa$, then $\pi$ is a perfect mapping.

We need the following results:
Proposition 1.3.6 [1] If $X$ is a $T_2$ space and $K \subseteq F$ are compact subsets of $X$ then $\chi(K,X) \leq \chi(K,F) \cdot \chi(F,X)$.

In [1] a proof of the above result for the Tychonoff case is presented. We will present a proof that holds for regular spaces:

Proof: Suppose $\chi(K,F) \cdot \chi(F,X) = \kappa$, and let $\{W_\alpha\}_{\alpha < \kappa}$ be an outer base for $F$ in $X$ and $\{V_\alpha\}_{\beta < \kappa}$ an outer base for $K$ in $F$. As $F$ and $K$ are compact, we get that for every $\beta < \kappa$, we can find disjoint open sets (in $X$) $G_\beta$ and $H_\beta$, such that $K \subseteq G_\beta$ and $F \setminus V_\beta \subseteq H_\beta$.

We claim that $\{G_\beta \cap W_\alpha : \alpha, \beta < \kappa\}$ is an outer base for $K$ in $X$. Suppose that $K \subseteq V$, where $V$ is an open set in $X$. Then there is a $\beta < \kappa$ such that $K \subseteq V_\beta \subseteq V \cap F$. Also if $G_\beta$ and $H_\beta$ are as above, we have that $F \subseteq H_\beta \cup V$. But then there is an $\alpha < \kappa$ such that $F \subseteq W_\alpha \subseteq H_\beta \cup V$. Clearly, $K \subseteq G_\beta \cap W_\alpha$. It remains to prove that $G_\beta \cap W_\alpha \subseteq V$. Suppose $x \in G_\beta \cap W_\alpha$. Then $x \in G_\beta$ implies that $x \notin H_\beta$, and therefore $x \in W_\alpha$ implies that $x \in V$, and we are done.

$\Box$

Proposition 1.3.7 Suppose $X$ is such that $h(X) \leq \tau$. If $V$ is an open set and $x \in V$ then there exists $K$ compact, $\chi(K,X) < \tau$, $x \in K$, $K \subseteq V$.

Proof: Suppose $V$ is open and $x \in V$. Pick $K$ compact such that $\chi(K,X) < \tau$. Using that $K$ is regular, construct a sequence $\{V_n : n \in \omega\}$ of open sets such that $V_0 = V$, $x \in V_n$ for all $n$, and $\overline{V_{n+1}} \cap K \subseteq V_n \cap K$. Observe that $K' = \cap_{n \in \omega} \overline{V_n} \cap K$ is a compact set, $x \in K'$, and $\chi(K',X) \leq \chi(K',K) \cdot \chi(K,X) = \aleph_0 \cdot \chi(K,X) < \tau$.

$\Box$

Lemma 1.3.8 If $X$ is Hausdorff and $\overline{h}(X) \leq \kappa$ then $K_x = \cap j''(K_x)$.

Proof: Let $z \in \cap j''(K_x)$. We will prove that $z \in j(V)$, for all $V \in \mathcal{V}_x$. Let $V \in \mathcal{V}_x$. By Proposition 1.3.7, there is a $K \in \mathcal{K}_x$ such that $K \subseteq V$. By definition, $z \in j(K) \subseteq j(V)$.

Here is precisely where we use that $\overline{h}(X) \leq \kappa$. Suppose now $z \notin \cap j''(K_x)$. There is some $K \in \mathcal{K}_x$ such that $z \notin j(K)$. Now $\mathcal{V}[G] \models \mathcal{V}_K$ is an outer base for $K$, so by
elementarity, $M[H] \models j(V_K)$ is an outer base for $j(K)$. Since $|V_K| < \kappa$, we have that $j(V_K) = j''(V_K)$ and consequently $M[H] \models j''(V_K)$ is an outer base for $j(K)$. There is, then, a $V \in V_K$ such that $z \notin j(V)$.

We are able now prove the Theorem.

**Proof of Theorem 1.3.5.** Lemma 1.3.8 implies that inverse images of points are compact, so it remains to show $\pi$ is continuous and closed.

Since the topology $\mathcal{T}$ on $X$ in $V[G_\kappa]$ is a basis for the topology $X$ gets in $M[G_{j(\kappa)}]$, to prove continuity it is enough to prove that $\pi^{-1}(F)$ is closed, where $X \setminus F \in \mathcal{T}$. Let $z \in Z \setminus \pi^{-1}(F)$, then $z \in K_x$ for some $x \notin F$. By regularity of $X$, take disjoint open sets $V, W$ containing $x$ and including $F$ respectively. By the definition of $K_x$, $K_x \subseteq j(V)$ and $\pi^{-1}(F) \subseteq j(W)$. Therefore $j(V) \cap \pi^{-1}(F) = \emptyset$ which implies $Z \setminus \pi^{-1}(F)$ is open. Notice that to prove that $\pi$ was continuous the only thing we used about $X$ was its regularity.

To see that $\pi$ is closed, let $A$ be closed in $Z$ and let $x \in X \setminus \pi(A)$. We first show that there is a $K \in K_x$ such that $j(K) \cap A = \emptyset$. Assume otherwise and set $\mathcal{F} = \{j(K) \cap A : K \in K_x\}$. Since $K_x$ is closed under finite intersections, $\mathcal{F}$ is a collection of compact sets with the finite intersection property (if $\mathcal{F}' \subseteq \mathcal{F}$ is finite then $j(\mathcal{F}') = j''(\mathcal{F}')$), and so there is a $z \in \bigcap \mathcal{F}$. But then $z \in K_x \cap A$ and so $x \in \pi(A)$, contradiction.

Since $j(K) \cap A = \emptyset$, there is a $V \in V_K$ such that $j(V) \cap A = \emptyset$ (same argument as in the proof of Lemma 1.3.8- here we use $\mathcal{H}(X) < \kappa$). Since $x \in V$, it remains to show that $V \cap \pi(A) = \emptyset$. If $y \in V \cap \pi(A)$, then for some $z \in A$, $\pi(z) = y$. This would imply $z \in j(V) \cap A$, contradiction.

\qed
1.4 The equivalence of the projections

In this section we will prove that the two mappings that we have defined in Section 1.3 and Section 1.2 are equivalent. During this section, we will refer to the mapping defined on Section 1.2 as $f_1 : H \to X$ and the one defined in Section 1.3 as $f_2 : Z \to X$. Other than that we follow the notation of the previous sections.

By equivalent we mean that we will define $h : Z \to H$ a homeomorphism such that $f_1(x) = f_2(h(x))$.

From now on $X$ will denote a Tychonoff topological space (not necessarily of small pointwise type) with topology $\mathcal{T}$.

As before, we will denote the reals between 0 and 1 in the sense of the ground model by $[0,1]^{V[G]}$ and the ones in the sense of the extension by $[0,1]^{M[H]}$.

Theorem 1.4.1 If $f \in C^*(X)$ (in the sense of $V[G]$) then if $z \in K_x$ we have:

$$j(f)(x) = j(f)(j(x)) = j(f(x)) = f(x) \in [0,1]^{V[G]}.$$ 

Proof:

Let $I_n = ((f(x) - 1/n, f(x) + 1/n))^{M[H]}$ and $I_n' = ((f(x) - 1/n, f(x) + 1/n))^{V[G]}$.

Clearly $j(I_n') = I_n$. Define $O_n = f^{-1}(I_n')$. Using that $z \in K_x \subseteq j(O_n)$ we have that:

$$\forall n \in \omega \ z \in j(f^{-1}(I_n')) = (jf)^{-1}(j(I_n')) \ (\text{by elementarity of } j) = (jf)^{-1}(I_n).$$

So $\forall n \in \omega$ we have that $(jf)(z) \in (f(x) - 1/n, f(x) + 1/n)$ which implies $(jf)(z) = f(x)$.

\[\square\]

Definition 1.4.2 Define $h : j(X) \to [0,1]^{j(C^*(X))}$ by $h(z) = (g(z))_{g \in j(C^*(X)) = C^*(j(X))}$.

It is known that $h$ is a homeomorphic embedding. We will prove that if we restrict $h$ to $Z$, it maps onto $H$. 

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Theorem 1.4.1 gives us that $f_1(z) = f_2(h(z))$ and also

$$f_1(z) = f_2(h(z))$$

If $z \in Z$ then $h(z) \in H$,

as when you project $h(z)$ onto $[0, 1]^{\mathcal{C}(X)}$, via $f_1$ it goes to the point which is identified to $x$, meaning that $h(z)$ is in the preimage of $X$.

It remains to show that $h$ is onto:

**Theorem 1.4.3** $h''(Z) = H$.

**Proof:**

Pick $w \in H \subseteq h''(j(X))$ (by the definition of $H$ and $f_1$). We have that $\exists t \in j(X)$ such that $h(t) = w$. Now, $w = (g(t))_{g \in \mathcal{C}(X)}$. We will prove that $t \in Z$.

As $w \in H$ we have that $\exists x \in X$ such that $f_1(w) = (g(x))_{g \in \mathcal{C}(X)}$.

Claim : $t \in K_2$.

**Proof:** If $t \notin K_2$ then $\exists V$ open set such that $x \in V$ but $t \notin j(V)$. Using that $X$ is Tychonoff, we may assume that $\exists g \in \mathcal{C}(X)$ such that $V = g^{-1}(I)$ where $I \subseteq [0, 1]$ is open. Notice that $j(V) = j(g^{-1}(j(I)))$, so as $t \notin j(V)$ we have that $j(g)(t) \notin j(I)$. Now, $h(t) = w$ which is projected into $(g(x))_{g \in \mathcal{C}(X)}$, so $j(g)(t) = g(x)$, precisely the $g^{th}$ coordinate of the projection (by the definition of $h$ and $f_1$) which implies $j(g)(t) \in I \subseteq j(I)$. Contradiction. □

1.5 Some immediate consequences and examples.

The following examples and results will try to better explain the nature of the mapping defined in Section 1.3 by exploring questions like the ones below. We chose to proceed with the mapping defined without the functions for its apparent simplicity. Unless otherwise stated $j, V, M, V[G], M[H], \pi$ are as in Theorem 1.3.1.

(1) When is $\pi^{-1}(X) = j(X)$?

(2) Is $\pi^{-1}(X)$ closed in $j(X)$?
(3) Is the perfect mapping theorem valid for a broader class of spaces?

We first explore compactness.

**Theorem 1.5.1** If $V[G] \models \chi(X) < \kappa$, then $M[H] \models \pi$ is a homeomorphism between $\pi^{-1}(X)$ and $X$.

**Proof:**

Pick $x \in X$ and fix $V_x = \{V_\alpha : \alpha < \tau\}$ a local basis for $x$ where $\tau < \kappa$. As $\tau < \kappa$ we have that $j''(V_x) = j(V_x)$. So $\bigcap j''(V_x) = \bigcap j(V_x)$. To finish the proof, just notice that $M[H] \models j(V_x)$ is a local base for $j(x)$ at $j(X)$ which implies $\bigcap j''(V_x) = \bigcap j(V_x) = \{j(x)\}$. So $\pi$ is a 1-1 perfect mapping which is a homeomorphism.

\[\square\]

**Theorem 1.5.2** Suppose that $M[H] \models X$ is compact. Then $M[H] \models \pi^{-1}(X) = j(X)$.

**Proof:**

Suppose the conclusion is false and there is some $z \in j(X) \setminus \pi^{-1}(X)$. This means that $M[H] \models \forall x \in X$ there is $V_x \in \mathcal{V}_x$ such that $z \notin j(V_x)$. Notice that $\mathcal{W} = \{j(V_x) : x \in X\}$ is an open cover of $\pi^{-1}(X)$ and that $z \notin \bigcup \mathcal{W}$. Also $M[H] \models \pi^{-1}(X)$ is compact (as $M[H] \models X$ is compact and $\pi$ is perfect).

It has, consequently a finite subcover $\mathcal{T} = \{j(V_1), \ldots, j(V_k)\}$.

Claim: $V[G] \models \mathcal{V} = \{V_1, \ldots, V_k\}$ covers $X(\ast)$.

Otherwise $V[G] \models y \notin \bigcup \mathcal{V}$, so $M[H] \models j(y) \notin \bigcup \mathcal{T}$. But $j(y) \in \pi^{-1}(X)$ and $M[H] \models \bigcup \mathcal{T}$ covers $\pi^{-1}(X)$. Contradiction.

\[\square\]

Now $(\ast)$ implies $M[H] \models j(\mathcal{T})$ covers $j(X)$. This contradicts $z \notin \bigcup \mathcal{W}$. Contradiction.

\[\square\]

**Example 1.5.3** Suppose we are adding supercompact many Cohen or random reals. In the previous Theorem it is not enough to have $V[G] \models X$ is compact. Just pick $X = [0, 1]^{V[G]}$. Being a first countable space, by Theorem 1.5.1, $\pi^{-1}(X) = X$. But $j(X) = [0, 1]^{M[H]}$ which is much bigger.
One might ask the question whether the converse of Theorem 1.5.2 is true, that is if $\pi^{-1}(X) = X$, can we conclude that $X$ is compact?

The following example will show that the conjecture fails.

**Example 1.5.4** Let $X = \kappa + \omega$ with the order topology. Notice that:

- $\forall \alpha < \kappa \rightarrow \pi^{-1}(\{\alpha\}) = \{\alpha\}$.
- $\pi^{-1}(\{\kappa\}) = [\kappa, j(\kappa)]$.
- $\forall n \in \omega, \pi^{-1}(\{\kappa + n\}) = \{j(\kappa) + n\}$

In this case $X$ is locally compact but not compact and $\pi^{-1}(X) = j(X) = j(\kappa) + \omega$.

**Example 1.5.5** Take $X$ to be the rationals. $j(X)$ is also the rationals, so $\pi^{-1}(X) = X$ and $X$ is not even locally compact.

**Example 1.5.6** $X = \kappa$. Clearly $j(X) = j(\kappa)$. By Theorem 1.5.1, $\pi^{-1}(\kappa) = \kappa$ which is not even closed in $j(X)$.

The following example appears in [18], in the elementary submodel context:

**Example 1.5.7** A Tychonoff space $X$, such that $X$ is not a perfect image of a subspace of $j(X)$.

Let $X$ be $\kappa + 1$ where for every $\alpha < \kappa$, $\{\alpha\}$ is open and $\{[\beta, \kappa] : \beta < \kappa\}$ is a base at $\kappa$. $j(X)$ is $j(\kappa + 1) = j(\kappa) + 1$ and for every $\alpha < j(\kappa)$, $\{\alpha\}$ is open and $\{[\beta, j(\kappa)] : \beta < j(\kappa)\}$ is a local base at $j(\kappa)$.

Suppose that $p : B \subseteq j(X) \longrightarrow X$ is perfect. $\pi^{-1}(\{\kappa\})$ cannot be open, for otherwise $F = B \setminus \pi^{-1}(\{\kappa\})$ is closed and then $p''(F) = X \setminus \{\kappa\}$ would be closed. Contradiction.

So $j(\kappa) \in B$ and $p(j(\kappa)) = \kappa$. Notice that

$$j(\kappa) \in cl(\pi^{-1}(\kappa))(\ast),$$

as otherwise there is $\gamma < j(\kappa)$ such that $V = [\gamma, j(\kappa)] \cap B$ satisfies $V \cap \pi^{-1}(\kappa) = \emptyset$. Then $F = B \setminus V$ is a closed set and $p''(F) = \kappa$. So $\kappa$ is closed in $X$ and hence $\{\kappa\}$ is open in $X$. Contradiction.
As $κ < j(κ)$ and $j(κ)$ is regular, there is some $β < κ$ such that $p^{-1}(\{β\})$ is cofinal in $j(κ)$ by $(\ast)$. But then $p^{-1}(]β + 1, κ])$ cannot be open in $j(X)$, contradicting the continuity of $p$.

The simple examples above show that trying to explain the subspace $π^{-1}(X)$ is not an easy task, and in most cases, each space has to be studied separately.

Now we will present some simple extensions of the result:

**Theorem 1.5.8** In $M[H]$, suppose $(X_i)_{i ∈ I}$ is a family of topological spaces such that for every $i ∈ I$ there is a perfect $p_i : Z_i ⊆ j(X_i) → X_i$. Then $M[H] \models$ there is $Z ⊆ j(\prod_{i ∈ I} X_i)$ and a perfect $p : Z → \prod_{i ∈ I} X_i$.

**Proof:** We can define $\tilde{Z} = \prod_{i ∈ I} Z_i ⊆ \prod_{i ∈ I} j(X_i)$ and $p : \tilde{Z} → \prod_{i ∈ I} X_i$ as follows:

$$p((z_i)_{i ∈ I}) = (p_i(z_i))_{i ∈ I}.$$

By Theorem 3.7.9 in [9], since the projections are perfect, this mapping is perfect. Notice now that:

$\mathcal{V}[G] \models X = \prod_{i ∈ I} X_i$ is a product space whose factors are indexed by $I$. So $M[H] \models j(X)$ is a product space whose factors are indexed by $j(I)$.

$\mathcal{V}[G] \models X_i$ is the $i$-th factor in the product space $X$, for every $i ∈ I$. So $M[H] \models j(X_i)$ is the $j(i)$-th factor in the product space $j(X)$.

Notice that $\tilde{Z}$ is not a subset of $j(\prod_{i ∈ I} X_i)$, so from $\tilde{Z}$ we will define such $Z$.

Pick $q ∈ j(X)$ and call $\tilde{q} = (q_m)_{m ∈ j(I) \setminus j(\emptyset)}$. Let $T = \prod_{i ∈ I} j(X_i) × \tilde{q}$. Up to permutation of indices (or up to homeomorphism) $T$ is a subspace of $j(X)$ homeomorphic to $\prod_{i ∈ I} j(X_i)$. $Z = \tilde{Z} × \tilde{q}$ can be seen in the same fashion and clearly the mapping you obtain after composing $p$ and this homeomorphism induced by the permutation on the indices remains perfect.

\[\square\]

Notice that although the mapping is perfect it is not canonically nor naturally defined by $j$ as before.
Theorem 1.5.9 Suppose that $M[H] \models Y$ is a perfect image of a subspace of $j(Y)$ and that $V[G] \models \exists f : X \to Y$ which is perfect and onto. Then $M[H] \models X$ is a perfect image of a subspace of $j(X)$.

Proof: We follow the notation of the diagram below:

First we will define, for every $x \in X$, a compact set $K_x$. For distinct points in $X$ the defined compact sets will be disjoint. Call $K_f(x) = \pi_Y^{-1}(f(x))$, for $x \in X$. Define:

1. $K_x = (j(f))^{-1}(\tilde{K}_{f(x)})$.

2. $Z_X = \bigcup_{x \in X} K_x = (j(f))^{-1}(Z_Y)$.

3. $\pi_X : Z_X \to X$ mapping $z \in K_x$ to $x$.

As the compact sets are clearly disjoint, $\pi_X$ is well defined. Also it is clearly onto. Also notice that $j(f)$ maps $Z_X$ into $Z_Y$. We will prove now that it is a perfect mapping.

The above diagram commutes:

$$f(\pi_X(z)) = \pi_Y(j(f)(z)).(*)$$

Proof:

Just notice that if $z \in K_x$ then $j(f)(z) \in \tilde{K}_{f(z)}$ by the definition. Now $f(\pi_X(z)) = f(x) = \pi_Y(j(f)(z))$. 

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Claim: \( \pi_X \) is closed.

**Proof:** Pick \( F \subseteq Z_X \) closed and \( x \notin \pi''_X (F) \). We have then \( \pi_X^{-1}(x) \cap F = \emptyset \) which implies \( j(f)(\pi_X^{-1}(x)) \cap jf(F) = \emptyset \) (by the elementarity of \( j(f) \)).

Now, \( j(f)(\pi_X^{-1}(x)) = \pi_Y^{-1}(f(x)) \). So \( f(x) \cap \pi_Y(j(f)(F)) = \emptyset \). We know that \( \pi_Y \) is perfect so \( H = \pi''_Y (j(f)(F)) \) is closed in \( Y \). There is, then a \( V \) open in \( Y \) such that \( f(x) \in V \) and \( V \cap H = \emptyset \). So \( f^{-1}(V) \cap f^{-1}(H) = \emptyset \) Now, by \((*)\) we have that \( H = f''(\pi''_X (F)) \) which gives us that \( f^{-1}(V) \cap f''(\pi''_X (F)) = \emptyset \). To finish the proof just notice that \( \pi''_X (F) \subseteq f^{-1}(f''(\pi''_X (F))) \).

Claim: \( \pi_X \) is continuous.

To see that, let \( F \subseteq X \) be closed and \( z \in Z_X \setminus \pi_X^{-1}(F) \). As \( f \) is perfect \( f''(F) \) is closed in \( Y \). So that \( \pi_Y^{-1}(f''(F)) \) is closed in \( Z_Y \).

Notice that by the definition of \( Z_X \) we have that \( j(f)(z) \notin \pi_Y^{-1}(f''(F)) \). So there is an open set \( V \) in \( Z_Y \) about \( j(f)(z) \) such that \( V \cap \pi_Y^{-1}(f''(F)) = \emptyset \). Notice now that \( (j(f))^{-1}(V) \cap \pi_X^{-1}(F) = \emptyset \).

**Definition 1.5.10** We say that \( X \) has property \( \mathcal{P} \) if \( M[H] \models X \) is a perfect image of a subspace of \( j(X) \).

**Remark 1.5.11** In the model \( M[H] \), Theorem 1.5.8 and Theorem 1.5.9 say that \( \mathcal{P} \) is productive and an inverse invariant under surjective perfect mappings. This implies that a variety of spaces that can be constructed from a space \( X \) that has property \( \mathcal{P} \) also have property \( \mathcal{P} \). One such example is the Iliadis Absolute of a space \( X \), \( E(X) \), which is a subspace of the Gleason space of \( X \) (the space of ultrafilters on the family of regular closed subsets of \( X \) - they coincide when \( X \) is compact). In Chapter 6 of [24] it is proven that \( E(X) \) is an extremally disconnected zero dimensional space and if \( X \) is regular, it is a perfect image of \( E(X) \). So \( E(X) \) also has property \( \mathcal{P} \) if \( X \) does.
Chapter 2

Some Applications

2.1 Introduction

In this chapter we present applications of the general method that was proved in Section 1.3 and we follow its notation.

The first section deals with the preservation of hereditary normality. The second one presents a new proof of Balogh’s Theorem regarding the “normal implies collectionwise normal” problem for locally compact spaces.

The third one presents a new proof of Daniels’ result that extends Balogh’s to $k'$-spaces and the last one gives another proof of Balogh’s result that “countably paracompact implies expandable” for spaces with pointwise countable type.

2.2 Preserving hereditary normality for spaces of small pointwise type

We can now state and prove the preservation theorem for spaces that are not of small character but of small pointwise type. We will present a proof dealing with the addition of Cohen reals, but with slightly more complicated details the same machinery will work for random reals.

By examining the proof, the reader will notice that except for the fact that we do not
have \( X \) identified to a subspace of \( j(X) \) the proof is similar to the one of Theorem 1.1.3. The perfect mapping is "enough" to obtain the result. Also, the same machinery works for other topological properties that are invariant under perfect mappings although in particular cases one may also need results depending on the particular problem being investigated; here the particular result is Lemma 1.1.2.

Most of the results in this section and the next one appear in [15], and are joint work with my supervisor Franklin D. Tall.

**Theorem 2.2.1** Let \( \kappa \) be a supercompact cardinal. Add \( \kappa \) many Cohen reals. In the resulting model, hereditary normality is preserved for Hausdorff spaces with \( \mathcal{H} \leq 2^{\aleph_0} = \kappa \) by adding any number of Cohen reals.

**Lemma 2.2.2** (see [9]) Suppose that there is a perfect mapping \( f \) from a topological space \( X \) onto a topological space \( Y \). If \( X \) is hereditarily normal then \( Y \) is hereditarily normal (in fact we just need the mapping to be continuous and closed).

We are ready to prove Theorem 2.2.1.

**Proof:** As before we may assume that the base set for the topological space is an ordinal.

We proceed by contradiction and repeat the same initial procedure as in Theorem 1.1.3. Let \( P_\alpha \) denote the partial order for adding \( \kappa \) Cohen reals. There is a \( p \in P_\alpha \) forcing that there is a topological space \( \langle X, T \rangle \), with \( \mathcal{H}(X) \leq \kappa \), and an \( \alpha \) such that \( X \) has its hereditary normality destroyed after forcing with \( P_\alpha \). As \( X \) is an ordinal, we may pick \( \theta \) (much bigger than the size of a given base for \( X \)), such that \( H_\theta \) contains all the interesting information about open and closed sets (more precisely, nice names for open and closed subsets of \( X \)).

Pick a cardinal \( \lambda \) bigger than \( \alpha, |H_\theta|, 2^{\aleph_1} \) (notice that as \( |H_\theta| = 2^{<\theta} \) which can be bigger than \( \theta \), so \( \lambda \) and \( \theta \) might be different), and pick \( j : V \rightarrow M \) with \( j(\kappa) > \lambda \) and \( M^\lambda \subseteq M \) and \( (M)_\theta = (V)_\theta \) (here we mean the objects of rank \( < \theta \)). Let \( G_{j(\kappa)} \) be \( P_{j(\kappa)} \)-generic over \( M \) (and so over \( V \)) and such that \( p = j(p) \in G_{j(\kappa)} \). Again, let \( G_\kappa = G_{j(\kappa)} \cap V_\kappa \).
Pick in $V[G_\kappa]$, $\dot{Y}$, $\dot{F}$ and $\dot{K}$, nice names for subsets of $X$, and $1 = 1_{P_\alpha} \in P_\alpha$ such that

$$V[G_\kappa] \models \text{"} 1 \Vdash \dot{F} \text{ and } \dot{K} \text{ are closed disjoint unseparated subsets of } \dot{Y} \subseteq \dot{X}.\text{"}$$

Observe that we have that $(M[G_\kappa])_\theta = (V[G_\kappa])_\theta$ (remember that $\theta$ was chosen to be large enough). So, for the properties that concern us, truth in one of them is equivalent to truth in the other one. We mean,

$$M[G_\kappa] \models \text{"} 1 \Vdash \dot{F} \text{ and } \dot{K} \text{ are closed disjoint unseparated subsets of } \dot{Y} \subseteq \dot{X}.\text{"}$$

This implies

$$M[G_\kappa] \models \text{"} 1 \Vdash \dot{X} \text{ is not hereditarily normal. }\text{"}$$

Pick $H$ $P_\alpha$-generic over $M[G_\kappa]$ and $I$ such that (see [8] and Chapter VIII in [20]):

$$M[G_\kappa][H][I] = M[G_{j(\kappa)}].$$

We have that

$$M[G_\kappa][H] \models X \text{ is not hereditarily normal.}$$

Therefore since the addition of Cohen reals preserves non-normality (Lemma 1.1.2),

$$M[G_{j(\kappa)}] \models X \text{ is not hereditarily normal.}$$

Now $j(X)$ is hereditarily normal, so $\pi^{-1}(X)$ is hereditarily normal and as $\pi$ is perfect Lemma 2.2.2 implies that:

$$M[G_{j(\kappa)}] \models X \text{ is hereditarily normal},$$

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Contradiction.

\[ \square \]

**Corollary 2.2.3** Add $\kappa$ supercompact many Cohen reals. In the resulting model, if $X$ is hereditarily normal and $\kappa(X) \leq \kappa$ then $X$ is hereditarily collectionwise normal.

**Proof:** We know that $X$ remains hereditarily normal after adding any number of Cohen reals. Just apply Lemma 1.1.2 for every subspace $Y$ of $X$ and get the result.

\[ \square \]

We have been unable to answer the question of whether Theorem 2.2.1 remains valid if "hereditarily normality" is weakened just to "normality". It is also interesting to note that it is unknown whether Corollary 2.2.3 is true if both instances of "hereditarily" are removed. None of the proofs of Balogh [2], Fleissner [10], or ourselves (see below) work for the case $\kappa = 2^{\aleph_0}$.

### 2.3 Balogh’s Theorem

In this section we present a different proof of Balogh’s result that “in the model obtained by adding supercompact many Cohen reals, normal spaces of pointwise countable type are collectionwise normal” [2]. Dow’s proof [7] of the locally compact case of Balogh’s theorem was our starting point but the one presented here avoids $C^*(X)$ and elementary submodels and is considerably more general as it works for regular spaces of pointwise type $< 2^{\aleph_0}$.

It is known that there are normal spaces whose normality can be destroyed by the addition of one Cohen real [13], but these spaces are not of pointwise countable type.

As in the previous section, our proof will depend on the characterization of $X$ as a perfect image of a subspace of $j(X)$ plus the usual forcing reflection argument for normality (normalized in the extension implies separated in the ground model - see Lemma 1.1.2 ). So this proof depends on the fact that normality is a topological property invariant under perfect mappings (in fact we just need the mapping to be closed), and the fact that normality has the forcing reflection property described above.
A similar procedure could be followed, as we do in Section 2.5, to obtain similar results for other topological properties with similar reflection and invariance properties.

**Lemma 2.3.1** Suppose $X$ is a Hausdorff space, $\bar{h}(X) \leq \kappa$, and $\mathcal{Y}$ is a disjoint collection of subsets of $X$ such that $\forall Z \in [\mathcal{Y}]^{\leq \kappa}$, $Z$ is discrete. Then $\mathcal{Y}$ is discrete.

**Proof:** Since $\mathcal{Y}' = \{Y : Y \in \mathcal{Y}\}$ also has subcollections of size $< \kappa$ discrete, we may suppose that $\mathcal{Y}$ is a collection of closed sets. Also, as $\mathcal{Y}$ is a collection of pairwise disjoint sets, to prove it is discrete it is enough to show that for every $Z \subseteq \mathcal{Y}$ if $x \in \overline{\bigcup Z}$, then $x$ is in some $Z \in Z$. By contradiction assume there is $Z \subseteq \mathcal{Y}$ and $x \in \overline{\bigcup Z} \setminus \bigcup Z$. Let $K_0$ be a compact set containing $x$, $\chi(K_0, X) = \lambda < \kappa$. Then $K_0$ meets only a finite number of elements of $Z$ (otherwise it would include an infinite closed discrete set.) If $K_0$ meets $Z_0, \ldots, Z_n$, notice that $K_0 \cap \bigcup_{i \leq n} Z_i$ is closed and does not contain $x$, so $V = X \setminus (K_0 \cap \bigcup_{i \leq n} Z_i)$ is an open set containing $x$. Working in $K_0$, there is $K$, a closed $G_\delta$ subset of $K_0$, such that $x \in K \subseteq K_0 \cap V$. We have that $K$ is compact in $X$ and, by Lemma 1.3.6, $\chi(K, X) \leq \chi(K, K_0) \cdot \chi(K_0, X) \leq \aleph_0 \cdot \lambda = \lambda < \kappa$. Also $K \cap \bigcup Z = \emptyset$.

Let $\{U_\alpha\}_{\alpha < \lambda}$ be an outer base for $K$ in $X$. Since $x \in U_\alpha$, for each $\alpha$ we may pick $Y_\alpha \in Z$ such that $U_\alpha \cap Y_\alpha \neq \emptyset$. Since subcollections of size $< \kappa$ are discrete, $U = X \setminus \bigcup_{\alpha < \lambda} Y_\alpha$ is an open set including $K$ which does not include any $U_\alpha$ contradicting the fact that $\{U_\alpha\}_{\alpha < \lambda}$ was chosen to be an outer base for $K$.

$\square$

**Theorem 2.3.2** With the notation of Theorem 2.2.1,

$V[G_\kappa] \models \text{ if } X \text{ is a normal space with } \bar{h}(X) < 2^{\aleph_0}, \text{ then } X \text{ is collectionwise normal }.

**Proof:** Suppose $V[G_\kappa] \models "1 \vdash X \text{ is a normal space, } \bar{h}(X) < 2^{\aleph_0}, \text{ and } \mathcal{C} \text{ is a discrete family of closed subsets of } X"$. Choose an elementary embedding as in Theorem 2.2.1. Apply Theorem 1.3.1, and let $\pi : Z \rightarrow X$ be the perfect mapping. Observe that $\langle X, \mathcal{T} \rangle$ and $\mathcal{C}$ are in $M[G_\kappa]$.

We have 29
Claim: $\mathcal{D} = \{\pi^{-1}(C) : C \in \mathcal{C}\}$ is discrete in $j(X)$.

Proof:

Since $M[G_{j(\kappa)}] \models \overline{h}(j(X)) = j(h(X)) < \kappa$, by Lemma 2.3.1 we just need to prove that every subfamily of $\mathcal{D}$ of size $< \kappa$ is discrete. In $M[G_{j(\kappa)}]$, let $\mathcal{E}$ be a subset of $\mathcal{D}$ of size $< \kappa$. Notice that $\mathcal{P}_{j(\kappa)} = \mathcal{P}_\kappa \ast \mathcal{P}_{j(\kappa)} \setminus \kappa$ and that $\mathcal{P}_{j(\kappa)} \setminus \kappa$ has the countable chain condition. Since $\mathcal{C}$ is in $V[G_\kappa]$, there is an $\mathcal{H} \in V[G_\kappa], |\mathcal{H}| < \kappa, \mathcal{H} \subseteq \mathcal{C}$, such that $1 \Vdash_{\mathcal{P}_{j(\kappa)},\kappa} \mathcal{E} \subseteq \{\pi^{-1}(C) : C \in \mathcal{H}\}$ ( $\mathcal{H}$ is also in $M[G_{j(\kappa)}]$), so without loss of generality, we may assume that $\mathcal{E} = \{\pi^{-1}(C) : C \in \mathcal{H}\}, \mathcal{H} \subseteq \mathcal{C}, |\mathcal{H}| < \kappa, \mathcal{H} \in V[G_\kappa]$.

In $V[G_\kappa]$, as $X$ is normal and by the discreteness of $\mathcal{H}$, we can find open sets $\{U_C : C \in \mathcal{H}\}$ with $U_C \supseteq C$, such that $C \neq C'$ implies $\overline{U_C} \cap C' = \emptyset$. Again by normality, we can find a (possibly empty) open $U \supseteq X \setminus \bigcup \{U_C : C \in \mathcal{H}\}$, with $\overline{U} \cap \bigcup \mathcal{H} = \emptyset$. Let $\mathcal{U} = \{U\} \cup \{U_C : C \in \mathcal{H}\}$. Observe that $\mathcal{U}$ covers $X$ so $j(\mathcal{U}) = j''\mathcal{U}$ covers $j(X)$. Observe that $j(\mathcal{U})$ "witnesses the discreteness" of $\mathcal{E}$. First of all, by the definition of $\pi$, $\pi^{-1}(C) \subseteq j(U_C)$. Second, $j(U_C) \cap \pi^{-1}(C') = \emptyset$ if $C \neq C'$. For take open $V_{C'} \supseteq C'$, $V_{C'} \cap U_C = \emptyset$. Then $\pi^{-1}(C') \subseteq j(V_{C'})$ and $j(U_C) \cap j(V_{C'}) = \emptyset$. Finally, suppose $z \in j(\mathcal{U})$. Then $z$ is in no $\pi^{-1}(C)$, for if $z \in K_c, c \in C$, then $c \in X \setminus \overline{U}$, so $z \in j(X \setminus \overline{U}) = j(X) \setminus j(\overline{U})$, contradiction.

Remark 2.3.3 For the case of when the space is of pointwise countable type, the above Claim can be proved just by using that in this case $X$ is a $k$-space [9] and so it is enough to prove discreteness for countable subcollections, which follows since normality implies countable collectionwise normality (see e.g. [28]).
Let $S \subseteq C$. Since $\mathcal{D}$ is normalized, pick $U, V$ disjoint open in $Z$ such that $\bigcup \{ \pi^{-1}(C) : C \in S \} \subseteq U$ and $\bigcup \{ \pi^{-1}(C) : C \in C \setminus S \} \subseteq V$. Notice that $\bigcup \{ C : C \in S \} \subseteq X \setminus \pi(Z \setminus U)$, $\bigcup \{ C : C \in C \setminus S \} \subseteq X \setminus \pi(Z \setminus V)$ and that $X \setminus \pi(Z \setminus U)$ and $X \setminus \pi(Z \setminus V)$ are disjoint open sets in $X$. So $M[G_{\xi(\alpha)}] \models C$ is normalized in $X$. Again, apply Lemma 1.1.2 (collections normalized in a Cohen extension are separated in the ground model), and get that $M[G_\alpha] \models C$ is separated. But then $C$ is separated in $V[G_\alpha]$ and we are done. \(\square\)

2.4 $k'$-spaces

**Definition 2.4.1** We say that a topological $X$ is a $k'$-space if whenever $x \in X$ is such that $x \in \text{cl}(F)$ where $F \subseteq X$, then there is $K \subseteq X$ compact such that $x \in \text{cl}(F \cap K)$.

It is easy to see that $k'$-spaces are $k$-spaces but there are $k$-spaces which are not $k'$-spaces (see [1]). Also there are $k'$-spaces which are not of pointwise countable type (also in [1]) and spaces of pointwise countable type which are not $k'$-spaces as the following example will show.

We need the following proposition to be able to build the example. It is stated in [14]. The proof is simple and we present it here:

**Proposition 2.4.2** Suppose that $N \subseteq X \subseteq \beta N$ (where $N$ represents the natural numbers) and $X$ is a $k'$-space. Then $X \subseteq \beta N$ is open and consequently locally compact.

**Proof:** We think of $\beta N$ as the set of non-principal ultrafilters on $N$ together with $N$ with the topology generated by $N(B) = \{ U \in \beta N : B \in U \}$ for all infinite $B \subseteq N$ and all points in $N$ open.

Let $Y = X \setminus N$. Notice that for every $y \in Y$ we have that $y \in \text{cl}_X(N)$, and as $X$ is a $k'$-space there is a compact set $K_y \subseteq X$ such that $x \in \text{cl}_X(K_y \cap N)$. Call $A = K_y \cap N$.

Now, $N(A)$ is a clopen subset of $\beta N$ and also $y \in N(A) = cl_\beta N(A) \subseteq K_y = cl_\beta N(K_y) \subseteq X$. So $X$ is an open subspace of a locally compact space which implies it is locally compact.
The example is a clear consequence of the previous proposition:

**Example 2.4.3** A space of pointwise countable type which is not a $k'$-space.

Let $A \subseteq \beta N \setminus N$ be infinite and countable and $X = \beta N \setminus A$. The space $X$ is a $G_\delta$ subspace of $\beta N$ which is compact and thus Čech-complete and consequently it is Čech-complete which implies pointwise countable type (see [9]).

$X$, however cannot be a $k'$-space for otherwise it would be open, meaning that $A$ would be closed. This is a contradiction since $A$ is countably infinite and $|\text{cl}_{\beta N}(A)| = 2^\omega$ (see [9]).

This definition and a more detailed study of those spaces appear in [1].

A natural next step after proving the consistency of the “normality implies collectionwise normality” result for locally compact (pointwise countable type) spaces would be to obtain the same result for $k$-spaces. Although this has not been shown, in [4], Peg Daniels shows that in the model obtained after adding supercompact Cohen reals normal $k'$-spaces are collectionwise normal. One would wish to be able to apply Theorem 1.3.1 to obtain Daniels' Theorem. Our proof has some elements in common with Fleissner's measure-theoretic proof of the $k'$-result in the random real model [11].

Theorem 1.3.1 allowed us to use the defined projection in the above model to prove collectionwise normality for those normal spaces for which the projection was perfect. Unfortunately, although the mapping is continuous, it is not necessarily perfect or even closed for $k'$-spaces. Example 7.11 in [18] provides a counterexample in the elementary submodel context which can be translated into the large cardinal context:

**Example 2.4.4** A $k'$-space $X$ such that $X$ is not a closed image of a subspace of $j(X)$. Our ground model is the model obtained after the addition of $\kappa$ supercompact Cohen reals. Let $X = \omega \times \omega \cup \{s\}$, where $\omega \times \omega$ has points open and a neighborhood of $s$ is of the form $V_f = \{< \alpha, \beta > \in \omega \times \omega : \beta > f(\alpha)\}$ where $f : \omega \rightarrow \omega$.

The proof is almost identical to the one in [18]; the result comes mainly from the fact that $\chi(s, X) = \kappa$ but $\chi(s, j(X)) = j(\kappa) > \kappa$.  

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We are able, however to overcome the above difficulty and obtain the result below by using Theorem 1.3.1 together with parts of the proof of the result for spaces of small character (see [8]) and a seemingly technical but really simple idea which will be explained later.

**Theorem 2.4.5** In the model obtained by adding supercompact many Cohen (or random) reals, normal \( k' \)-spaces are collectionwise normal.

To prove the Theorem we need the following Lemma due to Arhangel’skii [1]:

**Definition 2.4.6** We say that a mapping \( f : X \rightarrow Y \) is a pseudo-open mapping if whenever \( V \subseteq X \) is open and \( f^{-1}(C) \subseteq V \), then \( C \subseteq \text{int}(f(V)) \).

It is worth mentioning that pseudo-open mappings are precisely the hereditarily quotient mappings [9].

**Lemma 2.4.7** [1] Every \( k' \) topological space is a pseudo-open image of a locally compact space.

The locally compact space above is the \( k \)-leader of the topological space, meaning the disjoint sum of all compact subsets of the topological space, and the mapping assigns each point in the \( k \)-leader to itself inside the topological space. This mapping is continuous for every \( k \)-space but it is pseudo-open if and only if the topological space is a \( k' \)-space.

**Remark 2.4.8** Observe that the \( k \)-leader of a topological space \( Y \), \( K(Y) \) is a locally compact normal space (as it is the disjoint sum of compact spaces). Its normality does not depend on the normality of \( Y \) as closed disjoint subsets of \( K(Y) \) are disjoint sums of closed disjoint subsets of compact spaces. They can be separated in each compact set and so the disjoint sum of this separation is the desired separation for the original disjoint closed sets.

From now on, \( Y \) is a \( k' \) normal space, \( X \) is a locally compact space and \( f : X \rightarrow Y \) is a pseudo-open continuous onto mapping. We follow the notation of the previous
sections. If $P$ is the partial order for adding $\kappa$ (supercompact) many Cohen or random reals then $G$ will be a $P$-generic filter and $H$ will be a $j(P)$-generic filter such that the mapping $j : V \longrightarrow M$ extends to $j : V[G] \longrightarrow M[H]$.

Recall that the mapping defined in Section 1.3 was continuous provided the topological space was regular (see the second paragraph of the proof of Theorem 1.3.5). So we can define $\pi_Y$ as described in Definition 1.3.3 and Definition 1.3.4, although we must have in mind that $\pi_Y$ is not necessarily perfect, just continuous. We can also define $\pi_X$, which will be perfect as $X$ is locally compact.

With all the notation as described above we are able to draw the following diagram:

```
\[ j(\emptyset) : j(X) \longrightarrow j(Y) \]
\[ \bigcup \quad \bigcup \]
\[ Z_X \quad Z_Y \]
\[ \pi_X \quad \pi_Y \]
\[ \downarrow \quad \downarrow \]
\[ f : X \longrightarrow Y \]
```

The above diagram can help us to describe the proof of Theorem 2.4.5. The details of this explanation are best described in the real proof of the Theorem.

If $C = \{ C_i : i \in I \}$ is a family of closed discrete subsets of $Y$, the first idea would be to apply Theorem 2.3.2 directly to $\{ f^{-1}(C_i) : i \in I \}$ which is a discrete family of subsets of $X$ (see Remark 2.4.8) and obtain a disjoint open expansion of it. By applying first $f$ and later the interior operator to each element of its expansion one would hope to get the desired expansion. It is clearly an open expansion, however it might not be disjoint and this comes mainly from the fact that the open sets in $X$ are not necessarily of the form $f^{-1}(V)$ where $V$ is open in $Y$.

To deal with that we "travel" through the above diagram as follows:
Notice that $M[H] \models \{ \pi_Y^{-1}(C_i) : i \in I \}$ is discrete in $j(Y)$. Now, $j(Y)$ is normal, so every generic partition of the family generates two disjoint closed sets that can be separated by “generic” open sets. Apply $j(f)^{-1}$ to those open sets, use $\pi_X$ which is perfect to obtain “generic” open sets in $X$ as in the proof of Theorem 2.3.2, and then as in [8] (see Lemma 1.1.2), using endowments, obtain a disjoint separation of $\{ f^{-1}(C_i) : i \in I \}$. The difference now, is that these open sets are obtained (after applying a few operations) from $j(f)^{-1}$ of the “generic” open sets. This, together with the elementarity of $j$ allows us to construct an open disjoint expansion of $C = \{ C_i : i \in I \}$ from the open disjoint expansion of $\{ f^{-1}(C_i) : i \in I \}$ obtained through the above described process, use the normality of $Y$ and get an open discrete expansion.

Definition 2.4.9 Let $I$ be a set and $n$ a positive integer. An $n$-dowment for $Fn(I,2)$ is a family $\mathcal{L}_n$ of finite subsets of $Fn(I,2)$ that satisfies the following conditions:

1. For each maximal antichain $A \subseteq Fn(I,2)$ there is $L \in \mathcal{L}_n$ such that $L \subseteq A$.

2. For any element $p \in Fn(I,2)$ with domain of size $n$ and for any collection 
   \{ $L_1, \ldots, L_n$ \} of elements of $\mathcal{L}_n$ there exists $f_1 \in L_1, \ldots, f_n \in L_n$ such that
   \{ $p, f_1, \ldots, f_n$ \} has a common lower bound.

In [8], it is proved that there are $n$-dowments for $Fn(I,2)$ for every positive integer $n$.

The following technical Lemmas are also needed:

Lemma 2.4.10 If $A \subseteq Y$ and $A \in V[G]$ then $M[H] \models (j(f))^{-1}(j(A)) = j(f^{-1}(A))$.

Proof:

$V[G] \models \forall x, x \in f^{-1}(A)$ iff $\exists y \in A$ such that $f(x) = y$.

By the elementarity of $j$, we have then :

$M[H] \models \forall x, x \in j(f^{-1}(A))$ iff $\exists y \in j(A)$ such that $j(f)(x) = y$. 

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Lemma 2.4.11 If $z \in \pi_X^{-1}(x)$ then $j(f)(z) \in \pi_Y^{-1}(f(x))$.

Proof:
In $V[G]$, let $V$ be an open set such that $f(x) \in V$. We need to prove that $M[H] \models j(f)(x) \in j(V)$. Now, $f^{-1}(V)$ is an open set containing $x$, so $z \in j(f^{-1}(V))$. By Lemma 2.4.10 we then obtain that $z \in (j(f))^{-1}(j(V))$ and finally $j(f)(z) \in j(V)$.

Lemma 2.4.12 [11] Let $Z$ be a compact space and $U$ be a family of open subsets of $Z$ such that for every $U, V \in U \exists W \in U$ such that $W \subseteq U \cap V$. Then $U$ is an outer base for $\cap U$.

Recall that outer base is defined in the APPENDIX, together with other concepts mentioned in this chapter.

We can prove the Theorem now.

Proof:
Let $C = \{C_i : i \in I\}$ be a discrete family of closed sets in $Y$. Using the normality of $Y$ in $V[G]$ and that $M[H] \models j(Y)$ is a $k$-space, one uses the usual argument (see Remark 2.3.3) to prove that:

$M[H] \models \{\pi_Y^{-1}(C_i) : i \in I\} \text{ is discrete in } j(Y)$.

Now $M[H] \models j(Y)$ is normal, so given $h : I \to 2$ a generic function ($h = \bigcup H$) there are $V_0, V_1$ open sets in $j(Y)$ such that:

$M[H] \models \bigcup_{h(i)=0} \pi_Y^{-1}(C_i) \subseteq V_0 \text{ and } M[H] \models \bigcup_{h(i)=1} \pi_Y^{-1}(C_i) \subseteq V_1.$
Define, open subsets of $X$ in $M[H]$ as follows:

$$W_0 = X \setminus \pi_X(Z_X \setminus (j(f))^{-1}(V_0))$$
and

$$W_1 = X \setminus \pi_X(Z_X \setminus (j(f))^{-1}(V_1)).$$

As $\pi_X$ is perfect both $W_0$ and $W_1$ are disjoint open subsets of $X$ in $M[H]$.

We also have that:

$$f^{-1}(C_i) \subseteq W_{h(i)} \quad (\ast)$$

Proof: Suppose otherwise, then $\exists x \in f^{-1}(C_i) \cap \pi_X(Z_X \setminus (j(f))^{-1}(V_{h(i)}))$ we have then $\exists y \in Z_X \setminus (j(f))^{-1}(V_{h(i)})$ such that $\pi_X(y) = x$. So $y \in \pi_X^{-1}(x)$ and by Lemma 2.4.11 we have that $j(f)(y) \in \pi_Y^{-1}(f(x))$. But $f(x) \in C_i$ implies $\pi_Y^{-1}(f(x)) \subseteq \pi_Y^{-1}(C_i) \subseteq V_{h(i)}$.

Finally $y \in (j(f))^{-1}(\pi_Y^{-1}(f(x))) \subseteq (j(f))^{-1}(V_{h(i)})$. Contradiction.

We have then:

$$M[H] \models \bigcup_{h(i) = 0} f^{-1}(C_i) \subseteq W_0 \quad \text{and} \quad M[H] \models \bigcup_{h(i) = 1} f^{-1}(C_i) \subseteq W_1.$$ 

Now we will define open sets about each $f^{-1}(C_i)$ in $M[G]$ using the endowment technique [8]:

Fix $\mathcal{L}_2$ a 2-dowment in $\mathbb{P}$.

For $y \in f^{-1}(C_i)$, fix $A_y$ a maximal antichain in $\mathbb{P}$ such that

$$\forall p \in A_y \text{ there is } V_p \text{ open subset of } X, y \in V_p \text{ and } p \Vdash V_p \subseteq W_{h(i)}.$$ 

Pick $B_y \subseteq A_y$ such that $B_y \subseteq \mathcal{L}_2$, and define $V_y = \cap_{p \in B_y} V_p$.

Define $O_i = \cup_{y \in f^{-1}(C_i)} V_y$.

Remember that $f$ is a pseudo-open mapping so we have that $C_i \subseteq \text{int}(f(O_i))$ as $f^{-1}(C_i) \subseteq O_i$.

Claim : $\forall i \in I \quad C_i \cap \bigcup_{j \neq i} f(O_j) = \emptyset$.

Proof: Suppose that $x \in C_i \cap \bigcup_{j \neq i} f(O_j)$. As $X$ is a $k'$-space there is $K$ compact in

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$X$ such that:

$$x \in C_i \cap \bigcup_{j \neq i} f(O_j) \cap \overline{K}(**) .$$

Now $1 \models \exists U_x$ open set such that $x \in U_x$ and $j(U_x \cap K) \subseteq V_{h(i)}$, by Lemma 2.4.12.

To see that, consider $U = \{j(U \cap K) : U$ is open and $x \in U\} = j''(\{U \cap K : x \in U \text{ open }\})$ and $Z = j(K)$ and notice that $\bigcap U = \pi_X^{-1}(x) \cap j(K)$.

Pick $A$ a maximal antichain in $\mathcal{P}$ such that $\forall p \in A \exists U_p \ p \models j(U_p \cap K) \subseteq V_{h(i)}$.

Pick $B \subseteq A$ such that $B \in \mathcal{L}_2$ and define:

$$U = \bigcap_{p \in B} U_p .$$

Now, by (**), $U \cap K \cap \bigcup_{j \neq i} f(O_j) \cap K \neq \emptyset$

So

$$\exists j \neq i \text{ such that } U \cap K \cap f(O_j) \neq \emptyset .$$

This implies

$$\exists y \in f^{-1}(C_j) \text{ such that } U \cap K \cap f(V_y) \neq \emptyset .$$

Let $p = \{< i, 0 >, < j, 1 >\}$, and using that $\mathcal{L}_2$ is a 2-downment, find $p_y \in B_y$ and $q \in B$ such that $p_y, p, q$ have a common lower bound $r$.

Note that:

1. $r \models h(i) = 0$ and $h(j) = 1$.
2. $r \models j(U \cap K) \subseteq V_0$.
3. $r \models V_y \subseteq W_1$.

Pick $z \in U \cap K \cap f(V_y)$. Then

$$\exists t \in V_y \ f(t) = z .$$
Item(2) gives us
\[ r \models j(f(t)) = j(z) \subseteq V_0, \]
which implies
\[ r \models j(f)(j(t)) = j(f(t)) = j(z) \in V_0. \]

We then have that
\[ (a) \ r \models j(t) \in (j(f))^{-1}(V_0). \]

As \( t \in V_y \), item(3) gives us that:
\[ r \models t \in W_1, \text{ so } r \models \pi_X^{-1}(t) \subseteq \pi_X^{-1}(W_1) \subseteq (j(f))^{-1}(V_1). \]

We have then:
\[ j(t) \in \pi_X^{-1}(t) \subseteq (j(f))^{-1}(V_1) \text{ which implies } (b) \ j(t) \in (j(f))^{-1}(V_1). \]

(a) and (b) imply that \((j(f))^{-1}(V_0) \cap (j(f))^{-1}(V_1) \neq \emptyset\). Contradiction!
\[ \square \]

Having the claim, just define \( V_i = \text{int}(f(O_i)) \setminus \bigcup_{j \neq i} f(O_j) \).

The claim above shows that \( C_i \subseteq V_i \) and the \( V'_i \)'s are obviously disjoint.
\[ \square \]

Alan Dow has very recently managed to simplify our machinery, leading to the hope of applying it to more general classes of spaces.

### 2.5 Expandability in countably paracompact spaces

Balogh proves in [2] that:
Theorem 2.5.1 Suppose that in $V$, $\kappa$ is a supercompact cardinal and $P$ is either the poset for adding $\kappa$ Cohen reals or $\kappa$ random reals. Let $G$ be $P$-generic over $V$. Then

$$V[G] \models \text{every countably paracompact space with } E(X) = \omega_1 \text{ is expandable.}$$

We will present a simpler proof of Balogh's result using the perfect mapping theorem. Using Theorem 1.3.1 we are able to do this by a simple modification of the argument in [26] analogous to the modification we did for [8] and normality.

From now on $X$ will denote a topological space (not necessarily regular or of small pointwise type) with topology $\mathcal{T}$ and we will use the notation of the Theorem.

As usual, the proof has technical parts but the idea is very similar to the one we use to prove Theorem 2.3.2. First we note that if a topological space $X$ is countably paracompact after one adds "enough" Cohen reals (the statement of the lemma will clarify the meaning of "enough"), then $X$ is expandable in the ground model (Theorem 2.5.5). Recall that a similar result was used in the proof of Theorem 2.3.2, that is if the topological space is normal after the addition of "enough" Cohen reals, then it is collectionwise normal in the ground model. So if $X$ is countably paracompact and $C$ is a locally finite family of sets then $j(X)$ is countably paracompact and contains a locally finite family of sets $D$ which is related to $C$ via the perfect projection $\pi$. Countable "generic" partitions of $D$ have open locally finite expansions, and $\pi$ being perfect gives us that countable "generic" partitions of $C$ have open locally finite expansions (in Theorem 2.3.2, $\pi$ being closed assured us that the desired collection was normalized). Theorem 2.5.5 finishes the proof.

We need the following results:

Definition 2.5.2 We call a space $X$ $\theta$-expandable if for every locally finite family of subsets of $X$, $\mathcal{L} = \{L_i : i \in I\}$ there is a sequence $E^n = \{E_i^n : i \in I\}$ of open expansions of $\mathcal{L}$ such that for all $x \in X$ there is an open set $V$ containing $x$ and $n \in \omega$ such that $V$ meets at most a finite number of elements of $E^n$. We require that for all $i \in I$, $E_i^{n+1} \subseteq E_i^n$. 

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Lemma 2.5.3 ([25]) A countably paracompact $\theta$-expandable space is expandable.

Lemma 2.5.4 ([25]) A space $X$ is countably paracompact iff every countable locally finite family has a locally finite open expansion.

Proofs for the two previous Lemmas can be found in [10].

The following Theorem is proved using a technique of Tall ([26],[27]).

**Theorem 2.5.5** Let $\mathcal{Y}$ be a locally finite collection of subsets of $X$. Add $\lambda$ Cohen reals where $\text{card}(\lambda) \geq \max(\text{card}(\mathcal{Y}), \text{card}(X))$. If $V[G] \models X$ is countably paracompact, then $V \models \mathcal{Y}$ is expandable.

**Proof:**
Throughout this proof the Cohen forcing will be represented by $Fn(\lambda, \omega, \omega)$. By a result of Dow ([27]), this partial order is $n$-dowed even if we make the extra restriction that $\text{range}(p) \subseteq n$ in the definition of $n$-dowed (Definition 2.4.9, item (2) and $Fn(\lambda, \omega, \omega)$ instead of $Fn(I, 2)$).

Let $\mathcal{Y} = \{Y_\alpha : \alpha < \lambda\}$ be a locally finite collection of subsets of $X$.

In $V[G]$ pick $h = \bigcup G : \lambda \to \omega$ the generic mapping and define

$$Z_n = \{Y_\alpha : h(\alpha) = n\}$$

and

$$Z_n = \bigcup Z_n.$$ 

Clearly $\{Z_n : n < \omega\}$ is a locally finite family, so that in $V[G]$ it has a locally finite expansion $\{W_n : n \in \omega\}$.

For every $n < \omega$ pick an $n$-dowment $L_n$. Also for every $x \in Y_\gamma \forall \gamma < \lambda$ use that $1 \models Y_\gamma \subseteq Z_{h(\gamma)} \subseteq W_{h(\gamma)}$ and define:

$$A^x_n = \{p \in P : \exists V_p \text{ open such that } x \in V_p \text{ and } p \models \forall V_p \subseteq W_{h(\gamma)}\}.$$
Clearly $A_n^x \subseteq \mathbb{P}$ is dense. Pick $B_n^x \in \mathcal{L}_n$ such that $B_n^x \subseteq A_n^x$ and define:

$$h_n(x) = \bigcap_{p \in B_n^x} V_p,$$

$$V_n(\gamma) = \bigcup_{x \in V_n(\gamma)} h_n(x),$$

$$O_n = \{V_n(\gamma) : \gamma < \lambda\}.$$

Notice that all $O_n$ are expansions of $\mathcal{Y}$.

Next we will prove the following claim which implies that the space is expandable as it is countably paracompact.

Claim: $\{O_n : n \in \omega\}$ is a $\theta$-expansion for $\mathcal{Y}$.

Proof:

Suppose that $x \in X$ is such that no $O_n$ is locally finite for $x$. Pick $p \in \mathbb{P}$ and $U$ an open set about $x$ and $m \in \omega$ such that $p \vdash \check{U}$ meets $\leq m$ elements of $\{W_n : n \in \omega\}$.

Let $k = \text{card}(\text{dom}(p)) + \text{sup}(\text{range}(p))$. Since we are assuming that $U$ meets infinitely many elements of $\{V_n \cup \check{\text{dom}}(p) : \gamma < \lambda\}$ and since $\text{dom}(p)$ is finite, we may pick $\gamma_1, \ldots, \gamma_{m+1} > \text{max}(m+k+1, \cup \text{dom}(p))$ such that $U \cap V_{m+k+1}(\gamma_i) \neq \emptyset$ and choose $y_i \in Y_{\gamma_i}$ such that $U \cap h_{m+k+1}(y_i) \neq \emptyset$.

Define:

$$q = p \cup \{< \gamma_i, k + i : 1 \leq i \leq m + 1\}.$$

Clearly $\text{range}(q) \subseteq m + k + 1$, so we can use that for every $1 \leq i \leq m + 1$ the set $B_{m+k+1}^y \in \mathcal{L}_{m+k+1}$ and consequently pick $p_i \in B_{m+k+1}^y$ and $s \in \mathbb{P}$ such that $s \leq q$ and $\forall i 1 \leq i \leq m + 1 s \leq p_i$.

Notice now that $s \vdash h(\gamma_i) = k + i$ and that $p_i \vdash h_{m+k+1}(y_i) \subseteq W_{h(\gamma_i)}$ which gives that $s \vdash U \cap W_{k+i} \neq \emptyset \forall i 1 \leq i \leq m + 1$. But as $s \leq q \leq p$ we have that
s \vdash U \text{ meets } m + 1 \text{ elements of } \{W_n : n \in \omega\}. \text{ Contradiction!}

\square

As \( X \) is a \( \theta \)-expandable countably paracompact space it is expandable and we have the result.

\square

We now need to prove Theorem 2.5.1. We will use the notation of Theorem 1.3.1 and its following paragraph.

\textbf{Proof of Theorem 2.5.1:}

Let \( \mathcal{C} = \{C_\gamma : \gamma < \lambda\} \) be a locally finite family of subsets of \( X \). Again, pick \( h = \bigcup H : \lambda \to \omega \) a generic partition of \( \lambda \) and define \( C_n = \{\pi^{-1}(C_\gamma) : h(\gamma) = n\} \) and \( Y_n = \bigcup C_n \). Notice that by an argument used in the discrete case, as \( X \) has small type \( D = \{\pi^{-1}(C_\gamma) : \gamma < \lambda\} \) is locally finite in \( j(X) \). To see that, as in Remark 2.3.3, one just needs to show that countable subcollections are locally finite. If \( \{C_n : n \in \omega\} \subseteq \mathcal{C} \), as \( X \) is countably paracompact, there is a locally finite collection of open sets \( \mathcal{O} = \{O_n : n \in \omega\} \) such that for every \( n \in \omega \), \( C_n \subseteq O_n \). As in the discrete case, \( \pi^{-1}(C_n) \subseteq j(O_n) \) and \( j(O) = j''(O) \) is locally finite and so we have the result.

Consequently \( \{Y_n : n < \omega\} \) is also locally finite in \( j(X) \) and so it has an open locally finite expansion \( \{O_n : n \in \omega\} \).

Define now \( \mathcal{D}_n = \{C_\gamma : h(\gamma) = n\} \) and \( D_n = \bigcup \mathcal{D}_n \). Clearly \( \mathcal{D}_n = Y_n \).

We can define now:

\[ W_n = X \setminus \pi(j(X) \setminus O_n). \]

We obtain:

1. \( D_n \subseteq W_n \): If \( x \in D_n \) then \( \pi^{-1}(x) \subseteq \pi^{-1}(D_n) = Y_n \subseteq O_n \). So \( \pi^{-1}(x) \cap j(X) \setminus O_n = \emptyset \) which implies \( x \notin \pi(j(X) \setminus O_n) \) and so \( x \in W_n \).

2. \( W_n \) is open as \( \pi \) is closed.

3. \( \{W_n : n \in \omega\} \) is locally finite : Pick \( x \in X \), \( \pi^{-1}(x) \) is compact. So there is an open set \( O \), \( \pi^{-1}(x) \subseteq O \) meeting just a finite number of elements of \( \{O_n : n \in \omega\} \).
Define $V = X \setminus \pi(j(X) \setminus O)$.

We will prove that $V$ meets just a finite number of elements of $\{W_n : n \in \omega\}$:

Claim: If $V \cap W_n \neq \emptyset$ then $O \cap O_n \neq \emptyset$:

Pick $y \in V \cap W_n$.

As $y \in W_n$ then $y \notin \pi(j(X) \setminus O_n)$ which implies $\pi^{-1}(y) \subseteq O_n$.

As $y \in V$ then $y \notin \pi(j(X) \setminus O)$ which implies $\pi^{-1}(y) \subseteq O$.

So $O \cap O_n \neq \emptyset$ and we have the result.

Repeat the proof of Theorem 2.5.5 for $\{W_n : n < \omega\}$ and we will have the result.

\[ \square \]

Notice that Theorem 2.5.1 is the only place in this thesis where we use the fact that the mapping is perfect rather than merely closed. It would be interesting to get some class of topological spaces $X$ for which we could obtain simply a closed mapping between a subspace of $j(X)$ and $X$, but we were unable to answer this question.

Theorem 2.5.1 is true for $\kappa < 2^{\aleph_0}$ but there is a difficulty in proving $\pi^{-1}$ of a locally finite collection is locally finite. We were unable to produce such a proof. Very recently, A. Dow presented a proof that if $\kappa(X) < \kappa = 2^{\aleph_0}$ and $X$ is countably paracompact then $\pi^{-1}$ of a locally finite collection is locally finite. The remaining steps of the proof would be the same, making the result true for any countably paracompact space with $\kappa < 2^{\aleph_0}$. This provides a positive answer to Balogh's question on whether his result could be extended to spaces with pointwise type $< 2^{\aleph_0}$. 

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Chapter 3

Reflection with elementary submodels

3.1 Introduction

The definition of elementary submodel is in the APPENDIX. For a basic reference about the subject see e. g. [19].

In the first section we will simply use elementary submodels as a tool to obtain a forcing result concerning the preservation of normality after forcing.

Then we will consider the general following situation:

\textbf{Definition 3.1.1} Let \( X, T \) be a topological space and \( M \prec H_\theta \) containing \( X \) and \( T \). We look at \( X_M = X \cap M \) with the topology \( T_M \) on \( X_M \) generated by \( T_M = \{ U \cap M : U \in T \cap M \} \).

With this definition we are able to state the analogue of Theorem 1.3.1 in the elementary submodel context:

\textbf{Theorem 3.1.2} [18] Let \( \langle X, T \rangle \) be a regular space with \( h(X) \leq \kappa \) and let \( M \) be an elementary submodel of \( H_\theta \) such that \( \langle X, T \rangle \in M \) and such that \( \kappa \subseteq M \). Then there is \( Y \subseteq X \) and \( \pi : \langle Y, T \rangle \rightarrow X_M \) such that \( \pi \) is perfect.

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The mapping is defined as follows:

Let
\[ \mathcal{K} = \{ K \subseteq X : K \text{ is compact and } \chi(K, X) \leq \kappa \}, \]
\[ \mathcal{K}_x = \{ K \in \mathcal{K} \cap M : x \in K \}, \text{ for } x \in X_M, \]
\[ \mathcal{V}_x = \{ V \in \mathcal{T} \cap M : x \in V \}, \text{ for } x \in X_M. \]

For each \( x \in X_M \) define \( K_x = \bigcap \mathcal{V}_x \).

Note that, since \( X \) is Hausdorff, a simple elementary submodel argument shows that if \( x, y \in M \) and \( x \neq y \), then \( K_x \cap K_y = \emptyset \).

Define
\[ Y = \bigcup \{ K_x : x \in X_M \}, \]
and
\[ \pi : (Y, \mathcal{T}) \to (X_M, \mathcal{T}_M), \]
by
\[ \pi(y) = x \text{ if and only if } y \in K_x. \]

This Theorem is explored in the next section where we deal with the problem of characterizing topological spaces \( X \) when we know that \( X_M \) is a cardinal (ordinal). We present the definition of \( \pi \) and not just the statement of the existence of a perfect mapping since it will be useful later in the proofs.

The following is analogous to Lemma 1.1.4 and will be very useful:

**Lemma 3.1.3** [18] Suppose that \( M \) is an elementary submodel, \( < X, \mathcal{T}> \in M \) is a topological space and \( \kappa \subseteq M \). If for every \( x \in X \) we have that \( \chi(x, X) \leq \kappa \), then \( \mathcal{T}_M \) is the subspace topology \( X \cap M \) gets from \( X \). Moreover, following the notation of the previous Theorem, \( Y = X \cap M \) and consequently \( \pi \) is the identity.

The third section presents the study of upwards reflection of topological properties related to sequentiality. "Upwards reflection" means a property of \( X_M \) transferring to \( X \).

Finally, we discuss the analogy between the large cardinal / forcing approach and the elementary submodel approach.
3.2 Forcing over an elementary submodel

The question of preservation of normality in a non large cardinal context is explored in this section. Here, the role of elementary submodels will be of a tool to prove the result and not as an essential part of the study as in the following sections. Some results in this section appear in [15]. In this section we will prove:

**Theorem 3.2.1** Let $\kappa$ be a regular cardinal. Adjoin $\kappa$ Cohen reals, then for any $\alpha < \kappa$, normality of spaces of size $< \kappa$ in which each point has character $< \kappa$ is preserved by adjoining $\alpha$ Cohen reals.

We could have stated Theorem 3.2.1 for spaces of pointwise type $< \kappa$. The two formulations are equivalent as, for Hausdorff spaces of size $< \kappa$, $\overline{h}(X) \leq \kappa$ implies that every point has character $< \kappa$. To see this, note that for every compact Hausdorff space $K$, $\omega(K) \leq |K|$, so $\psi(x,K) < \kappa$ which implies $\chi(x,K) < \kappa$ as $K$ is compact. Now, apply Proposition 1.3.6 and get the result.

Although the proof has some technical details, the general idea is that we see the addition of Cohen reals as an iteration. We will show that as the space is "small", it appears in an earlier stage (the set and its topology). If the addition of a small number of Cohen reals could kill its normality, the same would be true over this stage where it appears. We finally use that non-normality is preserved by the addition of Cohen reals to see that the space would not be normal in the final extension, which would contradict the hypothesis.

We will follow the notation of and use the following result from [7]:

**Proposition 3.2.2** Suppose $P \in M \prec H_\theta$ is a partial order and $G$ is $P$-generic over both $V$ and $M$. Let $\sigma = \min \{ \tau : \tau$ is a cardinal and $\tau \cap M \in \tau \}$.

Then

$$(M \cap \check{H}_\sigma)[G \cap M] = (M \cap \check{H}_\lambda)[G \cap M] \prec H_\lambda[G],$$

where $\lambda = \min((M \cup \{\theta\}) \cap [\sigma, \theta]).$
Notice that since $M \prec H_\theta$ and $P \in M$, $G \cap M$ is $P \cap M$-generic over $H_\theta$. $M \cap \check{H}_\kappa = \{ \tau : \tau \text{ is a } P\text{-name in } H_\kappa \cap M \}$ and $(M \cap \check{H}_\kappa)[G \cap M] = \{ \tau_{G\cap M} : \tau \in M \cap \check{H}_\kappa \}.$

**Proof of Theorem 3.2.1:**

Suppose that the theorem is false. Pick $G \mathcal{P}_{\kappa}$-generic and, in $V[G]$, pick $(X, B)$ a counterexample (we work with a basis for the topology instead of a topology, as a basis remains a basis after forcing even if it generates a new topology). We also suppose, without loss of generality that $X = \xi < \kappa$ and $B$ is a basis for $\mathcal{T}$ of size $\leq \xi$.

We have a $\mathcal{P}_{\kappa}$-name $\check{B} \in V$ such that $|\check{B}| \leq \xi$ and

$V \models "1_{\mathcal{P}_{\kappa}} \forces \check{B} \text{ is a base for } \check{X} \text{ and } (\check{X}, \check{B}) \text{ is normal but forcing with } \check{P}_{\alpha} \text{ destroys its normality.} "$

As $\mathcal{P}_{\kappa}$ has the countable chain condition, we may suppose that $\check{B}$ is of the form:

$$\{ (\check{\eta} (\check{\pi}_\zeta, 1), 1) : \zeta < \xi \},$$

where each $\check{\pi}_\zeta$ is a nice $\mathcal{P}_{\kappa}$-name for a subset of $\xi$. (We follow the notation of [20].)

Each $\check{\pi}_\zeta$ is of the form:

$$\{ (\check{\eta}) \times A^\check{\eta}_\zeta : \eta < \xi \},$$

where each $A^\check{\eta}_\zeta$ is an antichain of $\mathcal{P}_{\kappa}$, and consequently is countable.

Define

$$\beta_\zeta = \sup \{ \rho : (\exists \eta \in \xi) (\exists p \in A^\check{\eta}_\zeta) \rho \in \text{dom}(p) \},$$

and let

$$\beta = \sup \{ \beta_\zeta : \zeta < \xi \}.$$

Since $\kappa$ is a regular cardinal, $\beta < \kappa$, and so we can pick $\gamma$ such that $\gamma < \kappa$, but $\gamma > \beta, \alpha.$

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Choose $M \prec H_\theta$ (for $\theta$ sufficiently large) such that $|M| < \kappa$, $M \cap \kappa = \mu > \gamma$, and 
$\{\kappa, \dot{B}, \mathbb{P}_\kappa, \gamma\} \cup \gamma \subseteq M$.

As $M \prec H_\theta$ we get that $G$ is also $\mathbb{P}_\kappa$-generic over $M$.

We get:

$$\mathbb{P}_\kappa \cap M = \mathbb{P}_\mu \text{ and } G_\mu = G \cap M.$$ 

Applying Proposition 3.2.2 with $|\mu|^+ \leq \kappa$ playing the role of $\sigma$ and $\kappa$ playing the role of $\lambda$ we obtain:

$$N = (M \cap \dot{H}_\kappa)[G_\mu] \prec H_\kappa[G].$$

By hypothesis we have that $V[G] \models \exists \dot{F}, \dot{K}$ nice $\mathbb{P}_\alpha$-names for subsets of $X$ ($= \bar{x}$) such that

$$1_{\mathbb{P}_\alpha} \Vdash \dot{F}, \dot{K} \text{ are closed disjoint non-separated subsets of } X.$$ 

If $H$ is $\mathbb{P}_\alpha$-generic over $V[G]$, it is also over $H_\kappa[G]$, and, since $\mathbb{P}_\alpha, < X, B >, \dot{F}, \dot{K} \in H_\kappa[G]$, we have that

$$H_\kappa[G] \models "1_{\mathbb{P}_\alpha} \Vdash \dot{F}, \dot{K} \text{ are closed disjoint unseparated sets."}$$

Remember that $\dot{B}$ is a $\mathbb{P}_\tau$-name and $\mu > \gamma$; consequently $(\dot{B})_{G_\mu} = B$, and so $< X, B > \in N$. By elementarity, we may take $\dot{F}, \dot{K} \in M \cap \dot{H}_\kappa[G_\mu]$ and have

$$(M \cap \dot{H}_\kappa)[G_\mu] \models "1_{\mathbb{P}_\alpha} \Vdash \dot{F}, \dot{K} \text{ are closed disjoint unseparated sets."}$$

Consequently there are $\mathbb{P}_\mu$ and hence $\mathbb{P}_\kappa$-names $\dot{Q}, \dot{R} \in M \cap \dot{H}_\kappa$ such that $(\dot{Q})_{G_\mu} = \dot{F}$ and $(\dot{R})_{G_\mu} = \dot{K}$. 

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Consequently

\[ \mathcal{V} \models \exists \sigma < \kappa \text{ such that } \hat{Q}, \hat{R} \text{ are } \mathbb{P}_\sigma\text{-names.} \]

Since \( \hat{Q}, \hat{R} \in M \) and \( M \prec \mathcal{V} \), this \( \sigma \) may be taken in \( M \) and so that \( \sigma < \mu \) and \( \sigma > \alpha \). Notice that if \( H \) is \( \mathbb{P}_\alpha \)-generic over \((M \cap H_\kappa)[G_\mu]\), it is as well over \((M \cap H_\kappa)[G_\sigma]\), where \( G_\sigma = G_\mu \cap \mathbb{P}_\sigma \). Using that \( G_\mu = G \cap M = G \cap \mathbb{P}_\mu \) and that \( G_\sigma = G_\mu \cap \mathbb{P}_\sigma = G \cap \mathbb{P}_\sigma \), we can obtain \( H \) as above such that there is an \( I \) (see Chapter VIII in [20]) such that

\[ \mathcal{V}[G_\mu][H][I] = \mathcal{V}[G](\ast). \]

As \( \hat{F} \) is a nice name for a subset of \( \xi \), we get that

\[ \hat{F} = \{ \{ \hat{\eta} \} \times A_\eta : \eta < \xi \}. \]

This implies

\[ \mathcal{V}[G_\sigma] \models (\exists \iota < \kappa) (\forall \eta < \xi) (\forall p \in A_\eta) (\text{dom}(p) \subseteq \iota \text{ and } \iota < \kappa). \]

Notice that \( \mathbb{P}_\sigma \in M \) which implies \((M \cap H_\kappa)[G_\sigma] = (M \cap H_\kappa)[G_\sigma] \prec \mathcal{V}[G_\sigma] \), so \( \iota \) can be defined in \((M \cap H_\kappa)[G_\sigma] \), and consequently can be taken in \( M \). We can do the same process for \( \hat{K} \) and get a common \( \iota \) for both. Obviously, \( \iota < \mu \). We may then take \( \sigma > \iota, \beta \).

It follows that

\[ \hat{F}, \hat{K} \subseteq (M \cap H_\kappa)[G_\sigma]. \]

This implies:
\[ N_1 = (M \cap H_\kappa)[G_\sigma][H] \models (\hat{F})_H, (\hat{K})_H \text{ are not separated}(**). \]

If they were, the separation would be in \((M \cap \dot{H}_\kappa)[G_\mu][H]\), which contradicts what we had above.

From (**), we obtain:

\[ V[G_\sigma][H] \models (\hat{F})_H \text{ and}(\hat{K})_H \text{ are closed disjoint unseparated}. \]

This is a contradiction to the fact that \(1_{\mathbb{P}_\kappa} \Vdash X \) is normal, since by Lemma 1.1.2 the unseparatedness is preserved after one adds Cohen reals, and by \((*)\), \(V[G] \) is a Cohen real extension of \(V[G_\sigma][H]\).

\[ \square \]

One can actually remove the restriction \(\alpha < \kappa\) in the previous Theorem.

**Corollary 3.2.3** (A. Dow) Let \(\kappa\) be a regular cardinal. Adjoin \(\kappa\) Cohen reals, then for any \(\alpha\), normality of spaces of size \(< \kappa\) in which each point has character \(< \kappa\) is preserved by adjoining \(\alpha\) Cohen reals.

**Proof:** We use the notation of the previous theorem. We work in \(V[G]\). Suppose \(\alpha \geq \kappa\). We will prove that if there is an \(\alpha \geq \kappa\) contradicting the result then there is \(\tilde{\alpha} < \kappa\) that also contradicts the result.

Looking at the proof of the previous Theorem at some point we had:

\[ V[G] \models \exists \dot{F}, \dot{K} \text{ nice } \mathbb{P}_\alpha\text{-names for subsets of } X (= \xi) \text{ such that} \]

\[ 1_{\mathbb{P}_\alpha} \Vdash \dot{F}, \dot{K} \text{ are closed disjoint non-separated subsets of } X. \]

As \(|X| < \kappa\) by Lemma 2.2 in Chapter VIII of [20], there is an \(I \subseteq \kappa\) with \(|I| < \kappa\) such that \(\dot{F}\) and \(\dot{K}\) are \(\mathbb{P} = Fn(I, 2)\) names.

Now \(Fn(\alpha, 2)\) is isomorphic to \(Fn(I, 2) \times Fn(\alpha \setminus I, 2)\). By absoluteness after forcing with \(Fn(I, 2)\) the above sets are disjoint. By the fact that the basis of the topology
remains the same after forcing they are closed. If they were separated after forcing with $Fn(I, 2)$, they would be separated after forcing with $Fn(\alpha, 2)$, so in $V[G]$,

$$1_{Fn(I, 2)} \Vdash \hat{F}, \hat{K} \text{ are closed disjoint non-separated subsets of } X.$$ 

Let $\check{\alpha} = |I|$. $\check{\alpha} < \kappa$ and $Fn(I, 2)$ is isomorphic to $Fn(\check{\alpha}, 2)$. We obtain:

$$1_{Fn(\check{\alpha}, 2)} \Vdash \hat{F}, \hat{K} \text{ are closed disjoint non-separated subsets of } X.$$ 

$\Box$.

### 3.3 Upwards reflection of ordinal spaces

In this section we research and provide answers to the question of what can we say about $X$ if we know that $X_M$ is homeomorphic (equal) to a cardinal (ordinal) in many cases.

This is a particular step in the study of what can we say about different $X_M$'s for a given topological space $X$, and if we know precisely how $X_M$ looks like what can we say about $X$. We mean both questions not only in the topological sense but even with regards to a mathematical structure related to the topology (algebraic, order...).

Previous works have addressed the questions above. In [21] the first question is explored in the case when $X$ is the real line with its usual order and its field structure. In [29] the second question is explored for a variety of topological spaces, e.g. uncountable compact metric spaces, the reals, the rationals.

When appearing in this section, "$\pi$" will be the mapping described after Theorem 3.1.2.

**Theorem 3.3.1** Let $\kappa$ be an ordinal, $<X, T>$ a topological space and let $M < H(\theta)$ be an elementary submodel such that $X, T, \kappa \in M$. If $X_M = \kappa$ (meaning real equality, not homeomorphism) then $X = \kappa$. 

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Proof:

Notice that as $X_M = \kappa$:

1. $M \models \forall x \forall y \in X \ x \in y$ or $y \in x$.
2. $M \models \forall x \in X$, $x$ is an ordinal.
3. $M \models \forall x \in X \ \forall y \in x$ we have that $y \in X$.
4. $M \models \forall A \subseteq X \ A$ has an $\in$-minimal element.

All of the above imply that $M \models X$ is an ordinal, which implies that $H(\theta) \models X$ is an ordinal. Notice that this fact holds in the set sense, i.e. we still have to prove that it is also true in the topological sense. Also notice that we used here that $\in$ is an absolute relation, so the same arguments would not hold for “homeomorphic” instead of “equal”.

Claim 1: For $\gamma_1 < \gamma_2 \in \kappa$ we have that $M \models (\gamma_1, \gamma_2]$ is open in $X$.

Proof: Otherwise we would have $M \models \exists \gamma_1, \gamma_2 < \kappa$ such that $\exists x \in I = ]\gamma_1, \gamma_2] \ \text{int}(]\gamma_1, \gamma_2]).(*)$

Pick $\gamma_1, \gamma_2, x \in M$ satisfying the above sentence. We may say that $x = \gamma < \kappa$. As $I \in M$, $x \in I$ and $X_M = \kappa$, $I$ is open in $X_M$. So there is a $V \in T_M$ such that $x \in V \cap M \subseteq I$. Therefore $M \models V$ is open, $x \in V$ and $V \subseteq I$. This contradicts $(*)$ and we have the result.

Claim 2: $M \models \forall V \in T$ and $\forall \alpha \in X$ with $\alpha \in V$ there is $\beta < \alpha$ such that $(\beta, \alpha] \subseteq V$.

Proof: Pick $V \in T \cap M$. $V \cap M \in T_M$. Using that $X_M = \kappa$ topologically we may find $\beta < \alpha$ such that $(\beta, \alpha] \subseteq V \cap M$. As before $(\beta, \alpha] \in M$ and so we have that $M \models (\beta, \alpha] \subseteq V$.

Claim 1 and Claim 2 combined say that $M \models X$ is homeomorphic to an ordinal and consequently $H(\theta) \models X$ is homeomorphic to an ordinal. All we need to prove now is that this ordinal is actually $\kappa$. 

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Suppose not, then $X = \lambda > \kappa$. As $\kappa \subseteq M$ and $\kappa \in M$ we would have $X \cap M = \lambda \cap M$ which includes $\kappa + 1$. This would contradict $X \cap M = \kappa$.

We will now state and prove a version of the Theorem when we have “homeomorphic” instead of “equal”. As we do not have the topology on the space defined by an absolute order, $\in$, the technique is rather different.

First we need the following Lemmas:

**Lemma 3.3.2** Let $\kappa$ be a cardinal, $\text{cof}(\kappa) \geq \omega_1$, with the topology induced by the order and let $Y \subseteq \kappa$ be such that with the subspace topology, $Y$ is homeomorphic to $\kappa$. Then $Y$ is a closed subset of $\kappa$.

**Proof:** Fix $h : Y \rightarrow \kappa$ a homeomorphism and notice that:

$$|Y| = \kappa$$

as they are homeomorphic (**).

Suppose that $Y$ is not closed. Pick $\alpha = \min\{\theta : \theta \in cl(Y) \setminus Y\}$.

Claim:

$$h''(Y \cap [0, \alpha])$$

is bounded in $\kappa$ (**).

To see that, suppose otherwise. Then $C_1 = h''(Y \cap [0, \alpha])$ is a closed unbounded subset of $\kappa$. In this case $C_2 = h''(Y \setminus [0, \alpha])$ is a closed subset of $\kappa$ which must be bounded as $\text{cof}(\kappa) \geq \omega_1$ and in this case two closed unbounded sets must meet and $C_1$ and $C_2$ are obviously disjoint.

So $C_2 \subseteq [0, \beta]$, which is compact and consequently $Y \setminus [0, \alpha] \subseteq h^{-1}([0, \beta])$ which is also compact. So $Y \setminus [0, \alpha] \subseteq [0, \gamma] \subseteq \kappa$, for some $\gamma < \kappa$.

Notice now that this implies $Y = (Y \cap [0, \alpha]) \cup (Y \setminus [0, \alpha]) \subseteq [0, \max(\alpha, \gamma)]$, which implies $|Y| < \kappa$, contradicting (**).

Now, let $\lambda = \text{cof}(\alpha)$. Fix $f : \lambda \rightarrow \alpha$ a cofinal mapping.
By induction, we will construct a strictly increasing sequence \( \sigma = \{ \alpha_\epsilon : \epsilon < \lambda \} \subseteq Y \) converging to \( \alpha \notin Y \) with the following induction hypothesis:

1. \( \forall \beta < \lambda, \{ \alpha_\epsilon : \epsilon < \beta \} \) is strictly increasing.

2. If \( \beta < \lambda \) is a limit ordinal then \( \alpha_\beta = \sup\{ \alpha_\epsilon : \epsilon < \beta \} \).

Choose \( \alpha_0 \in (f(0), \alpha] \cap Y \). If \( \alpha_\epsilon \) is chosen then:

\[
\alpha_{\epsilon+1} \text{ is chosen in } (\alpha_\epsilon, \alpha] \cap (f(\epsilon + 1), \alpha] \cap Y \ (**).
\]

It remains to choose \( \alpha_\beta \) for \( \beta < \lambda \) a limit ordinal. Notice that \( \sigma_\beta = \{ \alpha_\epsilon : \epsilon < \beta \} \) is not closed in \( \kappa \). If it were closed, as it is bounded, it would have an upper bound in \( \sigma_\beta \), and this would contradict the fact that it is strictly increasing. It has then a limit point, this limit point has to be in \( Y \) by the minimality of \( \alpha \) and as \( \beta < \cof(\alpha) \). We choose \( \alpha_\beta \) to be this limit point. By (**), \( \sigma \) converges to \( \alpha \).

Notice that by the way we constructed it, \( \sigma \cup \{\alpha\} \) is closed in \( \kappa \). So \( \sigma \) is closed in \( Y \).

We have that \( h''(\sigma) \) is closed in \( \kappa \) and is bounded in \( \kappa \) by (**). It is then a compact subset of \( \kappa \). As \( h \) is a homeomorphism, \( \sigma \) is compact in \( Y \). As \( Y \) is a subspace of \( \kappa \), it is compact in \( \kappa \). This is a contradiction since it includes a sequence converging to a point outside of it.

\[\Box\]

The condition that \( \cof(\kappa) \geq \omega_1 \) is a necessary condition for Lemma 3.3.2:

**Example 3.3.3 (A. Dow)** Let \( Y = \omega_1 \setminus \{\omega\} \). Observe that \( Y \) with the subspace topology is homeomorphic to \( \omega \) but is clearly not closed in \( \omega \).

We will define the homeomorphism \( f : \omega_1 \longrightarrow \omega_1 \setminus \{\omega\} \) as follows:

1. \( \forall n \in \omega, f(n) = \omega + n + 1 \).

2. If \( \alpha = k\omega + l \), where \( k, l \in \omega, k > 1 \) and \( l \geq 0 \) then \( f(\alpha) = (k + 1)\omega + l \).
3. If $\alpha = \kappa_n + 1$ where $n \in \omega$, $n \geq 1$ then $f(\alpha) = n - 1$.

4. If $\alpha = \kappa_n + k$ where $n, k \in \omega$, $n \geq 1$, $k \geq 2$ then $f(\alpha) = \kappa_n + k - 1$.

5. Elsewhere, the function is the identity.

\[\]

Lemma 3.3.4 (see [17]) Suppose that $M$ is an elementary submodel and $X$ is a topological space such that $<X, T> \in M$. If $X_M$ is compact then $X$ is compact. Moreover $\text{domain}(\pi) = X$.

The previous lemma is essentially proved earlier in this thesis in the supercompact context (see Theorem 1.5.2).

Lemma 3.3.5 (see [29]) Suppose that $M$ is an elementary submodel and $X$ is a topological space such that $<X, T> \in M$. If $X_M$ is locally compact then $X$ is locally compact.

Lemma 3.3.6 (see [21]) Suppose that $M$ is an elementary submodel, $X, Y \in M$ and $Y \subseteq M$. If $|X| \leq |Y|$ then $X \subseteq M$.

Theorem 3.3.7 Let $M$ be an elementary submodel, $<X, T>$ a topological space and $\kappa$ a cardinal with $\text{cof}(\kappa) \geq \omega_1$ and such that $\kappa, X, T \in M$ and also $\kappa \subseteq M$. If $X_M$ is homeomorphic to $\kappa$, then $X = X_M$ and hence it is homeomorphic to $\kappa$.

Proof: As $X_M$ is homeomorphic to $\kappa$, it is locally compact. So by Lemma 3.3.5 $X$ is locally compact and we can use Theorem 3.1.2. There is , then, $\pi : Z \subseteq X \rightarrow X_M$ perfect. Fix $f : \kappa \rightarrow X_M$ a homeomorphism and call $x_\alpha = f(\alpha)$.

Then:

$\forall \alpha < \kappa \exists V_\alpha \in \mathcal{T} \cap M$ such that $x_\alpha \in V_\alpha \cap M \subseteq f''([x_0, x_\alpha])$.

Notice that:
\[ |V_\alpha \cap M| < \kappa \] \text{(*)}.

Claim 1: \( V \models |V_\alpha| < \kappa \).

Otherwise \( M \models |V_\alpha| \geq \kappa \) which implies \( M \models \exists B \subseteq V_\alpha \text{ such that } |B| = \kappa \).

Pick this \( B \in M \). As \( \kappa \subseteq M \), by Lemma 3.3.6 we get that \( B \subseteq M \). So \( B \subseteq V_\alpha \cap M \) has size \( \kappa \) which contradicts \text{(*)}.

\[ \square \]

Claim 2: \( V \models \forall x \in X \text{ there is a } V \in T \text{ such that } x \in V \text{ and } |V| < \kappa \).

It is enough to prove that \( M \) models the previous sentence.

Suppose otherwise, that:

\[ M \models \exists x \in X \text{ such that } \forall V \in T \text{ such that } x \in V, |V| \geq \kappa \] \text{(**)}

Pick this \( x \in X \cap M \). There is some \( \alpha \) such that \( x = x_\alpha \) and so the previously defined \( V_\alpha \) contradicts \text{(**)}.

\[ \square \]

Now,

\( V \models \text{Every point in } X \text{ has a neighborhood of size } < \kappa \). So \( V \models \forall x \in X \psi(x, X) < \kappa \). Use now that \( X \) is locally compact and get that:

\( V \models \forall x \in X \chi(x, X) < \kappa \).

The previous sentence implies by Lemma 3.1.3 that \( X_M = X \cap M \) has the subspace topology inherited from \( X \).

Claim 3: \( X \cap M \) is open in \( X \). To see that for \( x \in X \cap M \), use Claim 2 and get \( V \) a neighborhood of \( x \) of size \( \lambda < \kappa \). This \( V \) may be taken in \( M \) (as \( x \in M \)) and so by Lemma 3.3.6 we have that \( V \subseteq M \). Now, \( x \in V \subseteq X \cap M \).

\[ \square \]

We will prove now that \( |X| \leq \kappa \). Suppose otherwise that:
\[ V \models |X \setminus X \cap M| \geq \kappa \ (\ast\ast\ast). \]

We obtain:
\[ V \models \exists Y \text{ open subset of } X, \text{ which is homeomorphic to } \kappa \text{ and such that } |X \setminus Y| \geq \kappa. \]
So,
\[ V \models \exists Y \text{ open subset of } X, \text{ which is homeomorphic to } \kappa \text{ and such that } \exists T \subseteq X \setminus Y \text{ such that } |T| = \kappa. \]

By elementarity,
\[ M \models \exists Y \text{ open subset of } X, \text{ which is homeomorphic to } \kappa \text{ and such that } \exists T \subseteq X \setminus Y \text{ such that } |T| = \kappa. \]

Pick those \( Y, T \in M \).

Again, by Lemma 3.3.6, \( T \subseteq X \cap M \) and \( Y \subseteq X \cap M \). In particular \( Y \) is open in \( X \cap M \).

Remember that \( X \cap M = X_M \) which is homeomorphic to \( \kappa \). So fix \( g : \kappa \to X \cap M \) a homeomorphism.

\( C = g^{-1}(Y) \) is a subset of \( \kappa \) homeomorphic to \( \kappa \). So it is closed in \( \kappa \) by Lemma 3.3.2. As \( Y \) is open, \( C \) is also open in \( \kappa \). So \( D = \kappa \setminus C \) is a closed subset of \( \kappa \). Because two clubs in \( \kappa \) always meet (as \( \text{cof}(\kappa) \geq \omega_1 \)), \( D \) is a bounded set in \( \kappa \) having then cardinality less than \( \kappa \). The contradiction comes from the fact that \( g^{-1}(T) \subseteq D \) and has size \( \kappa \). So \( \ast \ast \ast \) is false which implies \( |X| = \kappa \), and by Lemma 3.3.6 \( X \subseteq M \), so \( X = X \cap M \) and hence is homeomorphic to \( \kappa \).

\[ \square \]

The assumption that \( \kappa \subseteq M \) is a strong one; however the following two Theorems will show that the condition that the cardinal be included in the model cannot be eliminated from the hypothesis of Theorem 3.3.7 at least when \( \kappa \) is not weakly inaccessible:

**Theorem 3.3.8** Suppose that \( M \) is an elementary submodel, \( \kappa \) is a cardinal and \( < X, T > \) is a topological space such that \( X, T, \kappa, \kappa^+ \in M \) and \( \kappa \subseteq M \) but \( \kappa^+ \not\subseteq M \). If
**Proof:**

We have two cases. If $\exists x \in X \ (x, X) \geq \kappa^+$ then trivially $X$ is not homeomorphic to $\kappa^+$. Suppose then that $\forall x \in X \ (x, X) \leq \kappa$. Now $\forall x \in X \ (x, X) \leq \kappa$ implies (by Lemma 3.1.3) that $X_M$ is a subspace of $X$ and so $X \cap M$ and $\kappa^+$ are homeomorphic.

If $X$ is homeomorphic to $\kappa^+$, fix $f : \kappa^+ \to X = \{x_\alpha : \alpha < \kappa^+\}$ a homeomorphism. We may pick $f \in M$. Notice that $Y = f^{-1}(X \cap M)$ is a subset of $\kappa^+$ homeomorphic to $\kappa^+$. By Lemma 3.3.2 $Y$ is closed in $\kappa^+$, which implies $X \cap M$ closed in $X$.

As $\kappa^+ \not\subseteq M$, $X$ is not a subset of $M$ (again just use that $f \in M$). Pick now $\gamma = \min\{\alpha : x_\alpha \notin X \cap M\}$. As $f \in M$, we have that $\gamma$ is an infinite limit ordinal. Now $\sigma = \{x_\alpha : \alpha < \gamma\} \subseteq X \cap M$ and $x_\gamma \in \text{cl}(\sigma)$ as $\gamma \in \text{cl}(\{\alpha : \alpha < \gamma\})$ and $f$ is a homeomorphism.

$\Box$

**Theorem 3.3.9** Suppose that $M$ is an elementary submodel, $\kappa$ is a cardinal with $\omega_1 \leq \text{cof}(\kappa) = \tau < \kappa$ and $< X, T >$ is a topological space such that $X, T, \kappa \in M$ and $\tau \subseteq M$ but $\kappa \not\subseteq M$. If $X_M$ is homeomorphic to $\kappa$ then $X$ is not homeomorphic to $\kappa$.

**Proof:** Again, we have two cases. If $w(X) < \kappa$ or $w(X) > \kappa$ then clearly $X$ is not homeomorphic to $\kappa$. We may suppose then that $w(X) = \kappa$. Pick $B = \{B_\beta : \beta < \kappa\} \in M$ a basis for $X$ of size $\kappa$. Fix $f : \tau \to \kappa$ a strictly increasing cofinal map and define for $x \in X \cap M$:

$$\forall \alpha < \tau, \ C_{x, \alpha} = \bigcap\{B_\beta : \beta < f(\alpha) \text{ and } x \in B_\beta\}$$

Notice that as all variables are in $M$, $C_{x, \alpha} \in M$.

Define also:

$$P_x = \{C_{x, \alpha} : \alpha < \tau\}.$$

Notice that:
(1) $P_x \in M$.

(2) $P_x \subseteq M$ (as $\tau \subseteq M$).

(3) $\bigcap P_x = \{x\}$ as if $y \neq x$ then there is $B_\beta$ such that $y \notin B_\beta$ but $x \in B_\beta$. Pick $\alpha \in \tau$ such that $f(\alpha) > \beta$ and notice that $y \notin C_{x,\alpha}$.

To finish the proof observe that:

$$\bigcap \bigcap_{x \in V, \tau \in \tau \cap M} V \subseteq \bigcap P_x = \{x\}.$$  

So, again our usual $\pi$ is a homeomorphism and the proof proceeds as in Theorem 3.3.8. This is the one place in this Chapter where we actually use the particular definition of $\pi$.

\[\square\]

We next need

**Theorem 3.3.10** [21] *If $0^#$ does not exist and $|M| \geq \kappa$ then $\kappa \subseteq M$.***

Theorem 3.3.7 together with the previous theorem leads to the result:

**Corollary 3.3.11** *Suppose $0^#$ does not exist. Let $M$ be an elementary submodel, $<X, \tau>$ a topological space and $\kappa$ cardinal with $\text{cof}(\kappa) \geq \omega_1$ such that $\kappa, X, \tau \in M$. Then if $X_M$ is homeomorphic to $\kappa$, then $X$ is homeomorphic to $\kappa$.***

The reader may get information about $0^#$ in [16]. It is a special subset of $\omega$. Its existence has many consequences, among them existence of large cardinals in inner models. Also $V = L$ implies $0^#$ does not exist, in fact the nonexistence of $0^#$ is equivalent to Jensen's Covering Lemma for $L$ [16], [5].

Any definable $\kappa$ such as $\omega_1$, $\omega_2$, etc.. is automatically in $M$. Also, the condition that $\kappa \in M$ is sometimes a consequence of $\kappa \subseteq M$. This holds for instance for all successor cardinals:
Proposition 3.3.12 Suppose that $\kappa = \aleph_\beta$ and $\beta < \kappa$. If $M$ is an elementary submodel and $\kappa \subseteq M$ then $\kappa \in M$.

Proof: The proof is simple. We have that $\kappa = \aleph_\beta$ and $\beta \in M$ so $\aleph_\beta$ is defined in $M$, so $\kappa \in M$.

We now present some examples:

Lemma 3.3.13 Let $M$ be an elementary submodel and $\kappa \in M$ a cardinal with the order topology $\mathcal{T}$. Then $\kappa_M$ is homeomorphic to $\text{ot}(\kappa \cap M)$, where by $\text{ot}(X)$ we mean the order type of $X$.

Proof: To see that, $B = \{[\alpha, \beta], [\alpha, \beta) : \alpha < \beta < \kappa\}$ is a basis for the topology of $\kappa$ that lies in $M$. Because of that $\mathcal{T}_M$ and $\mathcal{B}_M$ generate the same topology on $\kappa_M$.

Now $[\alpha, \beta) \in M$ if and only if $\alpha, \beta \in M$ (the same for $[\alpha, \beta]$).

So if $f : \text{ot}(\kappa \cap M) \rightarrow \kappa \cap M$ is strictly increasing and onto, it is a homeomorphism between $\text{ot}(\kappa \cap M)$ and $\kappa_M$.

Example 3.3.14 An elementary submodel $M$, and a topological space $< X, \mathcal{T} > \in M$ such that $X$ is not homeomorphic to $\omega$ but $X_M$ is homeomorphic to $\omega$.

Just pick $X$ a discrete space of uncountable cardinality and $M$ a countable elementary submodel such that $X \in M$. In this case $X_M$ is a countable discrete space and thus homeomorphic to $\omega$.

Notice that in the above example $\omega \in M$, $\omega \subseteq M$ but $\text{cof}(\omega) = \omega < \omega_1$.

If we are looking for an $X$ such that $X_M$ is homeomorphic to $\omega_1$ but $X \neq X_M$, by Theorem 3.3.7 we need to find an $M$ with $\omega_1 \not\subseteq M$. Such an $M$ must have $|\omega_1 \cap M|$ countable and yet be uncountable since $X_M$ is uncountable. We thus assume Chang's Conjecture in the following form:

There is an elementary submodel $M$ such that $|M| = \aleph_1$, $|\omega_1 \cap M| = \aleph_0$ and $|\omega_2 \cap M| = \aleph_1$.
Example 3.3.15 Assuming Chang’s Conjecture there is an elementary submodel \( M \), and a topological space \( \langle X, T \rangle \in M \) such that \( X \) is not homeomorphic to \( \omega_1 \) but \( X_M \) is homeomorphic to \( \omega_1 \).

Pick an elementary submodel \( M \) such that \( |M| = \aleph_1 \), \( |\omega_1 \cap M| = \aleph_0 \) and \( |\omega_2 \cap M| = \aleph_1 \).

Observe that:

\[
\omega_2 = \min\{\tau \in M : |\tau \cap M| = \aleph_1\}(*).
\]

To see this, pick \( \tau < \omega_2 \in M \). Then there is \( f \in M \) a bijection between \( \omega_1 \) and \( \tau \). Because \( f \in M \), we have that \( |\tau \cap M| = |\omega_1 \cap M| \).

Claim : The order type of \( \omega_2 \cap M \) is \( \omega_1 \).

To see that suppose otherwise that \( \exists \beta > \omega_1 \) and \( g : \beta \rightarrow \omega_2 \cap M \) strictly increasing onto. Now \( f(\omega_1) < \omega_2 \) would contradict \((*)\).

Pick \( X = \omega_2 \) with the order topology. By Lemma 3.3.13, \( X_M \) is homeomorphic to \( \omega_1 \).

We now deal with ordinals:

Example 3.3.16 An elementary submodel \( M \), an ordinal \( \gamma \) and a topological space \( \langle X, T \rangle \in M \) such that \( X \) is not homeomorphic to \( \gamma \) but \( X_M \) is homeomorphic to \( \gamma \).

Pick \( M \) a countable elementary submodel of \( H(\theta) \). Notice that in this case \( \omega_1 \cap M = \gamma < \omega_1 \) (as \( \omega \subseteq M \) which implies that every countable element of \( M \) is a subset of \( M \)-Lemma 3.3.6). Now by Lemma 3.3.13 \( (\omega_1)_M = \gamma \).

Notice that in the previous example \( \gamma \notin M \).

We have two remaining questions:

1- What happens with Theorem 3.3.7 if \( \kappa \) is an uncountable cardinal with countable cofinality? Because uncountable cofinality is necessary for Lemma 3.3.2 to hold (see Example 3.3.3), we could not extend the result to this case and also we were not able to provide a counterexample.
2- What can we say in the general case when $\gamma$ is an ordinal and not a cardinal? Even if one could get a proof of Lemma 3.3.2 for ordinals, the serious difficulty of producing "small" neighborhoods in the proof of the theorem (see Claim 1 in the proof) would remain.

3.4 Upwards reflection of countable tightness, sequentiality and Fréchet

In [18] the authors provide an example of a topological space $X$, which is a Fréchet space and consequently a sequential space and a countably closed (needed in the proof) elementary submodel $M$ such that $t(X_M) > \aleph_0$. This example shows that countable tightness and sequentiality do not necessarily reflect downwards.

In this section we will explore the other direction, upwards reflection.

**Theorem 3.4.1** Suppose that $M$ is an $\omega$-covering elementary submodel and that $< X, T > \in M$ is a topological space. If $t(X_M) = \aleph_0$ then $t(X) = \aleph_0$.

**Proof:**
Suppose the conclusion is false. Then:

$$\forall V \subseteq X \text{ and } \exists x \in \text{cl}(A) \text{ but } \forall B \subseteq A, |B| \leq \aleph_0, \text{ we have that } x \notin \text{cl}(B).$$

By elementarity,

$$M \models \exists A \subseteq X \text{ and } \exists x \in \text{cl}(A) \text{ but } \forall B \subseteq A, |B| \leq \aleph_0, \text{ we have that } x \notin \text{cl}(B).(*)$$

So we may take such $A \in M$ and $x \in M$.

**Claim:** $x \in \text{cl}_{rM}(A \cap M)$.

To see that, notice that if the claim is false then:

$$\exists V \in T \cap M \text{ such that } x \in V \text{ but } V \cap A \cap M = \emptyset.$$
The above sentence is equivalent to:

\[ M \models \exists V \in \mathcal{T} \text{ such that } x \in V \text{ and } V \cap A = \emptyset, \]

which clearly contradicts (\(*)\).

\[ \square \]

As \( t(X_M) = \aleph_0 \), there is \( B \subseteq A \cap M \) such that \( |B| = \aleph_0 \) and \( x \in \text{cl}_{\text{rm}}(B) \).

Using that \( M \) is \( \omega \)-covering, there is \( C \in M, |C| = \aleph_0 \) and \( B \subseteq C \).

Call \( D = C \cap A \). Notice that \( D \in M, D \subseteq A, |D| = \aleph_0 \) and \( x \in \text{cl}_{\text{rm}}(D \cap M) \).

To finish the proof note that \( V \models x \in \text{cl}(D) \). For otherwise

\[ V \models \exists V \in \mathcal{T} \text{ such that } x \in V \text{ and } V \cap D = \emptyset. \]

This implies,

\[ M \models \exists V \in \mathcal{T} \text{ such that } x \in V \text{ and } V \cap D = \emptyset. \]

Pick this \( V \) in \( \mathcal{T} \cap M \). The previous sentence tells us that \( V \cap D \cap M = \emptyset \). This contradicts the fact that \( x \in \text{cl}_{\text{rm}}(D \cap M) \). So \( t(X) = \aleph_0 \).

\[ \square \]

**Theorem 3.4.2** Suppose that \( M \) is an \( \omega \)-closed elementary submodel and that \( < X, \mathcal{T} > \in M \) is a topological space. If \( X_M \) is sequential then \( X \) is sequential.

**Proof:**

Again, by contradiction, suppose that \( V \models \exists A \) sequentially closed but \( \exists x \in \text{cl}(A) \setminus A \). By elementarity,

\[ M \models \exists A \text{ sequentially closed but } \exists x \in \text{cl}(A) \setminus A(\ast). \]

We again can pick such \( A \) and \( x \) in \( M \).

**Claim:** \( A \cap M \) is sequentially closed in \( < X_M, \mathcal{T}_M > \).

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To see that, let

\[ f : \omega \rightarrow A \cap M \] be a convergent sequence to \( y \in X \cap M \)(**).

(we mean convergence in the \( T_M \) sense). Call \( B = \{ f(n) : n \in \omega \} \).

Notice that \( f \subseteq \omega \times B \subseteq M \) and \( f \) is countable. So, using that \( M \) is \( \omega \)-closed we can conclude that \( f \in M \) and \( B \in M \).

Notice now that

\[ M \models f : \omega \rightarrow B \] converges to \( y \) (in the \( T \) sense).

To see that notice that otherwise \( M \models \exists V \in T \) such that \( y \in V \) and \( \forall n \in \omega \exists m > n \) such that \( f(m) \notin V \). Pick this \( V \in T \cap M \). It contradicts (**).

So we have that

\[ V \models f : \omega \rightarrow B \subseteq A \] converges to \( y \) (in the \( T \) sense).

As \( V \models A \) is sequentially closed then \( V \models y \in A \). So \( y \in A \cap M \), which finishes the proof of the Claim. \( \square \)

As \( X_M \) is sequential, we have that \( A \cap M \) is closed in \( X_M \). Now \( M \models x \in cl(A) \setminus A \), so \( M \models \forall V \in T, x \in V \) implies \( V \cap A \neq \emptyset \). This, by the definition of \( T_M \) means that \( x \in cl(T_M)(A \cap M) = A \cap M \). So \( x \in A \cap M \subseteq A \). This contradicts (*).

\( \square \)

**Theorem 3.4.3** Suppose that \( M \) is an \( \omega \)-closed elementary submodel and that \( < X, T > \in M \) is a topological space. If \( X_M \) is Fréchet then \( X \) is Fréchet.

**Proof:**

Suppose that \( V \models X \) is not a Fréchet space. Then \( M \models X \) is not a Fréchet space.

We get then, \( M \models \exists A \subseteq X, \exists z \in X \) such that \( z \in cl(A) \) but no sequence contained in \( A \) converges to \( z \)(*).
Again, pick \( x \) and \( A \) in \( M \).

Claim 1: \( x \in c_{\tau M}(A \cap M) \).

Otherwise \( \exists V \in \tau \cap M \) such that \( x \in V \) and \( V \cap A \cap M = \emptyset \). This is equivalent to \( M \models x \in V \) and \( V \cap A = \emptyset \), contradicting (\(*\)).

As \( X_M \) is Fréchet we can pick \( f : \omega \to A \cap M \) a sequence converging to \( x \).

Now, \( M \) is \( \omega \)-closed, \( f \subseteq M \) and \( f \) is countable so \( f \in M \).

As in the previous proof \( M \models f \) converges to \( x \) (in the \( \tau \) sense). We have then a contradiction to (\(*\)).

\( \square \)

It is important to notice that some restriction on the elementary submodel is necessary for the above results to hold, one trivial example is:

**Example 3.4.4** A topological space with uncountable tightness \( X \) and an elementary submodel \( M \) such that \( X_M \) is Fréchet.

Just pick any countable elementary submodel \( M \) and any \( \langle X, \tau \rangle \in M \) with uncountable tightness. \( X_M \) has a countable base so it is first countable and hence Fréchet.

The condition that \( M \) is not only \( \omega \)-covering but \( \omega \)-closed required for Theorem 3.4.2 and Theorem 3.4.3 is also (consistently) necessary. We thank Alan Dow for providing the following example.

**Definition 3.4.5** For two sets \( A \) and \( B \), we say that \( A \subseteq^* B \) if \( A \setminus B \) is finite.

**Definition 3.4.6** We say that a set \( A \) is a pseudo intersection of a family \( \mathcal{F} \) if \( A \subseteq^* F \) for every \( F \in \mathcal{F} \). We say that a family of countable sets has the strong finite intersection property - s.f.i.p. if every nonempty finite subfamily has infinite intersection.

**Definition 3.4.7** \( p = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a subfamily of } [\omega]^\omega \text{ with s.f.i.p. which has no infinite pseudo intersection }\} \).

The following is true about \( p \):

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Theorem 3.4.8 [3] For a cardinal $\kappa$ we have that $\kappa < p$ if and only if $MA_\kappa$ for $\sigma$-centered posets holds.

Lemma 3.4.9 (apparently folklore) If $X$ is a countable topological space and $w(X) = \kappa < p$ then $X$ is a Fréchet space.

Proof: Without loss of generality we may suppose that $X = \omega$.
Fix $B = \{B_\alpha : \alpha < \kappa\}$ a basis for $X$ and define for $x \in X$:

$$B_x = \{B_\alpha \in B : x \in B_\alpha\}.$$

Suppose that $A \subseteq X$ and $x \in cl(A)$. Notice that $\forall B \in B_x$, $|B \cap A| = \omega$. Call for $B_\alpha \in B_x$:

$$A_\alpha = B_\alpha \cap A,$$

and

$$\mathcal{F} = \{A_\alpha : B_\alpha \in B_x\}.$$

Using that $x \in cl(A)$ and that $B_x$ is a local basis for $x$, we obtain that $\mathcal{F}$ is a subfamily if $[\omega]^\omega$ with s.f.i.p. and as $|B_x| \leq \kappa$, we have that $|\mathcal{F}| \leq \kappa$.

Use that $\kappa < p$ and obtain $B' \subseteq^* A_\alpha$ for every $A_\alpha \in \mathcal{F}$. In particular $B' \subseteq^* A_0 \subseteq A$, which implies that there is a finite such that $B' \setminus F \subseteq A$. Define $B = B' \setminus F$ and notice that $B \subseteq A$ and $B \subseteq^* A_\alpha$ for every $\alpha \in \mathcal{F}$.

It is easy to notice that $B = \{b_n : n \in \omega\}$ converges to $x$, for just suppose that $B_\alpha \in B_x$ is arbitrary, then $B \subseteq^* B_\alpha \cap A = A_\alpha$, so there is $m \in \omega$ such that for every $n > m$ we have that $b_n \in B_\alpha \cap A \subseteq B_\alpha$.

\[
\square
\]

Example 3.4.10 ($p > \omega_1$) A topological space $X$ and an $\omega$-covering elementary submodel $M$ of size $\aleph_1$ containing $X$ (you can get such a model from an elementary
chain of countable elementary submodels - see [6]) such that $X$ is not sequential but $X_M$ is Fréchet.

Define $X = \omega \times \omega \cup \{ s \}$, where points in $\omega \times \omega$ are open and neighborhoods of $s$ are of the form $V_f^k = \{ < m, n : m > k \text{ and } n > f(m) \} \cup \{ s \}$, where $k \in \omega$ and $f \in \omega^\omega$.

$X$ is a countable space and $|M| = \aleph_1$ so $w(X_M) \leq \omega_1$. By Lemma 3.4.9, $X_M$ is a Fréchet space. $X$ is not a sequential space as it does not have converging sequences and it is not discrete.

To see that $X$ does not have converging sequences, suppose that $\{ x_n : n \in \omega \}$ is a sequence in $X$, where $x_n =< a_n, b_n >$. We will prove that $\{ x_n : n \in \omega \}$ does not converge to $s$.

We have two cases:

Case 1: Infinitely many $a_n$'s are different.

Just pick $f : \omega \rightarrow \omega$ such that for those $a_n$'s, $f(a_n) = b_n + 1$. $V_f^k$ does the job.

Case 2: There is a finite number of $a_n$'s, say $\{ a_{n_1}, \ldots, a_{n_k} \}$

Pick $k = \max\{ a_{n_1}, \ldots, a_{n_k} \} + 1$ and any $f \in \omega^\omega$, again $V_f^k$ does the job.

One cannot get a ZFC example because L. Junqueira [17] has proved that, under $CH$, all $\omega$—covering elementary submodels are $\omega$—closed.

### 3.5 Some remarks on the large cardinal versus the elementary submodel approach and the space $X(M)$

Another way to reflect a topological space through an elementary submodel is with the spaces of the form $X(M)$.

A general definition of $X(M)$ appears in [18]:

**Definition 3.5.1** Let $< X, \mathcal{T} >$ be a topological space and $\mathcal{F}$ be a (not necessarily open) cover of $X$. Let $M$ be an elementary submodel such that $X, \mathcal{T}, \mathcal{F} \in M$. For every $x, y \in X$, define $x \sim y$ if and only if $x \in V \iff y \in V$, for every $V \in \mathcal{F} \cap M$. 

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Clearly $\sim$ is an equivalence relation. We can then define the quotient space

$$X_{\mathcal{F}}(M) = X/\sim.$$ 

Also let

$$\varphi = \varphi_{M}^{\mathcal{F}} : X \longrightarrow X(M)$$

be the quotient map. We will denote by $[x]$ the equivalence class of $x$. We will suppress the "$\mathcal{F}$" in $X(M)$ when it is clear from context.

The spaces $X(M)$ and $X_M$ are related in some particular cases:

**Theorem 3.5.2** [18] Suppose $X$ is a $T_3$ space with pointwise countable type. If $x \in X \cap M$ and $\mathcal{F} = \{ K \subseteq X : K \text{ is compact with countable character} \}$, then $y \sim x$ if and only if $y \in K_{x}$. Thus $X_M$ is homeomorphic to a subspace of $X(M)$.

In this section we will deal with a particular case that relates to the general framework of this thesis in a way which will be explained after the definition.

**Definition 3.5.3** Let $X$ be a topological space and $M$ an elementary submodel such that $X, C^*(X) \in M$. We define the following equivalence relation on $X$:

$$x \sim_{C^*(X)} y \text{ if and only if for every } f \in C^*(X) \cap M \text{ we have that } f(x) = f(y).$$

We will denote by $[x]$ the equivalence class of $x$.

**Definition 3.5.4** We define the topological space $X(M)_{C^*(X)}$ as the quotient space of $X$ under the equivalence relation $\sim_{C^*(X)}$ and $\pi : X \longrightarrow X(M)_{C^*(X)}$ as the projection: $\pi(x) = [x]$.

Notice that if $X$ is a Tychonoff space and $\mathcal{F}$ is a basis for $X$ composed of functionally open sets then both definitions coincide [18].

One can see the proof of the perfect mapping theorem using functions (Theorem 1.2.1) as a $X(M)_{C^*(X)}$ type construction and the proof of the perfect mapping
theorem in section 1.3 (Theorem 1.3.1) as a $X_M$ type construction where for a topological space $Y$, $j(Y)$ plays the role of $X$ and $Y$ plays the role of $X_M$. Notice that in the context of the perfect mapping theorem using functions, first a "kind of $X(M)$" space is constructed and then it is proved that it contains a subspace, ("kind of $X_M$") which is homeomorphic to $X$ (analogy to Theorem 3.5.2).

This analogy described above makes one wonder if everything that can be proved using (or about) the perfect mapping theorem in the large cardinal context can be proved in the elementary submodel context and vice-versa.

Although in some occasions the proofs of certain results in both contexts could not be considered similar, we were not able to produce a statement that could be proved in one context and refuted in the other context.

We give here an example of this difference between proofs. Remember that in Theorem 1.5.8, we proved that in the large cardinal context the property of a topological space $X$ being a perfect image of a subspace of $j(X)$ was productive. The same is true in the elementary submodel context, however as it is not true that $(\prod_{\alpha < \lambda} X_\alpha)_M = \prod_{\alpha < \lambda} (X_\alpha)_M$, an analogous proof would not be enough.

We need the following:

**Lemma 3.5.5** Suppose that $M$ is an elementary submodel, $\lambda \in M$ and $\{X_\alpha : \alpha < \lambda\} \in M$ is a family of topological spaces (the family of the topologies is also in $M$). Then $(\prod_{\alpha < \lambda} X_\alpha)_M$ is homeomorphic to a subspace of $\prod_{\alpha \in \lambda \cap M} (X_\alpha)_M$.

**Corollary 3.5.6** With the above notation if each $(X_\alpha)_M (\alpha \in M)$ is a perfect image of a subspace of $X_\alpha$, then $(\prod_{\alpha < \lambda} X_\alpha)_M$ is a perfect image of a subspace of $\prod_{\alpha < \lambda} X_\alpha$.

**Proof:**

For $\alpha \in \lambda \cap M$, each $(X_\alpha)_M$ is a perfect image of a subspace $Z_\alpha$ of $X_\alpha$. As before we have a perfect mapping (the product mapping) $\pi : \prod_{\alpha \in \lambda \cap M} Z_\alpha \rightarrow \prod_{\alpha \in \lambda \cap M} (X_\alpha)_M$.

Now $(\prod_{\alpha < \lambda} X_\alpha)_M$ is homeomorphic to a subspace $B$ of $\prod_{\alpha \in \lambda \cap M} (X_\alpha)_M$. To finish the proof just notice that $\pi \upharpoonright \pi^{-1}(B)$ is perfect and that $\prod_{\alpha \in \lambda \cap M} Z_\alpha$ is homeomorphic to a subspace of $\prod_{\alpha < \lambda} X_\alpha$.

$\Box$
Now we can prove the above stated lemma:

**Proof of Lemma 3.5.5:**

Define $f : (\prod_{\alpha < \lambda} X_\alpha)_M \rightarrow \prod_{\alpha \in \lambda \cap M}(X_\alpha)_M$ as follows:

$$\bar{x} = (x_\alpha)_{\alpha < \lambda} \mapsto (x_\alpha)_{\alpha \in \lambda \cap M}.$$  

Clearly $\forall \alpha \in M$ we have that $x_\alpha \in M$ as $\bar{x} \in M$ so the mapping is well defined.

To see that it is 1-1, pick $\bar{x} = (x_\alpha)_{\alpha < \lambda}, \bar{y} = (y_\alpha)_{\alpha < \lambda} \in M$. If $f(\bar{x}) = f(\bar{y})$ then $\forall \alpha \in \lambda \cap M$ we have that $x_\alpha = y_\alpha$. So $M \models \bar{x} = \bar{y}$, which implies that they are equal.

To see that $f$ is continuous pick $\bar{x} = (x_\alpha)_{\alpha \in \lambda \cap M}$ and $V = \prod_{\alpha \in \lambda \cap M \setminus F}(X_\alpha)_M \times \prod_{\alpha \in F}(V_\alpha \cap M)$ an open set about $\bar{x}$ where $F$ is a finite subset of $\lambda \cap M$ (so $F \in M$) and $V_\alpha \in T_\alpha \cap M$.

Call $W = \prod_{\alpha \in \lambda \setminus F} X_\alpha \times \prod_{\alpha \in F} V_\alpha$. $W$ is an open set and also $W \in M$. Notice that $f^{-1}(V) = W$, so $f$ is continuous.

To see that it is open, call $T$ the topology of $\prod_{\alpha < \lambda} X_\alpha$ and $Y = f''((\prod_{\alpha < \lambda} X_\alpha)_M)$. Let $V$ be a basic open set of $T_M$.

$V$ is of the form $(\prod_{\alpha < \lambda} H_\alpha) \cap M$ where $(\prod_{\alpha < \lambda} H_\alpha) \in T \cap M$, so:

$$H_\alpha = X_\alpha \text{ for } \alpha \notin I_V,$$

and

$$H_\alpha = V_\alpha \text{ for } \alpha \in I_V \text{ and } V_\alpha \in T_\alpha \cap M,$$

where $I_V \in M$ is a finite subset of $\lambda$.

Pick $\bar{x} \in Y \cap f''(V)$ and define:

$W = \prod_{\alpha \in \lambda \cap M} W_\alpha$ where for $\alpha \in I_V$ we put $W_\alpha = V_\alpha \cap M$ and otherwise $W_\alpha = X_\alpha \cap M$.

Clearly $\bar{x} \in W$ as $f$ is a projection and $I_V$ is a finite subset of $M$ which is open in $\prod_{\alpha \in \lambda \cap M}(X_\alpha)_M$.  

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Claim: $W \cap Y \subseteq f''(V)$.

$\exists \bar{y} \in W \cap Y$ implies $\exists \bar{x} \in \prod_{\alpha < \lambda} X_{\alpha}$ and $\forall \alpha \in \lambda \cap M$, $z_{\alpha} = y_{\alpha}$. Clearly $\bar{x} \in V$ so

$f(\bar{x}) = \bar{y} \in f''(V)$

The claim finishes the proof. \(\Box\)

Again, as in Remark 1.5.11 if we fix an elementary submodel $M$, and work with topological spaces that are in $M$, the property of $X_M$ being a perfect image of a subspace of $X$ is both productive and an inverse invariant under perfect mappings (the proof is analogous to the one in Section 1.5), so, again, a variety of spaces that can be constructed from spaces having this property also have this property (e.g. the Iliadis absolute - see Remark 1.5.11).

To better explain the analogy between the construction in Section 1.2 and the $X(M)_{C^*(X)}$:

**Proposition 3.5.7** If $X$ is compact then $X(M)_{C^*(X)}$ is compact and thus Tychonoff.

*Proof:* The proof is trivial: $X(M)_{C^*(X)}$ is the continuous image of a compact space. \(\Box\)

In Section 1.2, we start with $j(\beta X) = \beta(j(X))$ (here notice that $C^*(j(X))$ is isomorphic to $C^*(\beta j(X))$ ) and get $(j(\beta X))(M)$, where the copy of $X$ lies.

One has to be careful with the space $X(M)$ as we shall see next. First a technical lemma:

**Lemma 3.5.8** If $V \subseteq X$ and $V = f^{-1}(I)$ and $I \subseteq [0,1]$, $f \in M$, then $\pi^{-1}(\pi(V))) = V$.

*Proof:*

Clearly $V \subseteq \pi^{-1}(\pi(V)))$. If $x \in \pi^{-1}(\pi(V)))$ then $\exists y \in V$ such that $x \sim_{C^*(X)} y$. In particular as $f \in M$ we have that $f(x) = f(y) \in I$. So $x \in V$.

\(\Box\)

**Theorem 3.5.9** $X(M)_{C^*(X)}$ is a Hausdorff space.
Proof:

Suppose \([x] \neq [y]\). Then \(\exists f \in C^*(X) \cap M\) such that \(f(x) \neq f(y)\). Without loss of generality, we may assume \(f(x) < f(y)\) and pick \(q, l\) rational numbers (consequently elements of \(M\)) such that \(f(x) < q < l < f(y)\). Define \(V = f^{-1}([0, q])\) and \(W = f^{-1}([l, 1])\). By Lemma 3.5.8, and the definition of the topology on \(X(M)_{C^*(X)}\) both \(\pi(V)\) and \(\pi(W)\) are open disjoint sets about \([x]\) and \([y]\).

\[\Box\]

In fact the following example gives us a regular non-normal topological space \(X\) such that \(X(M)\) is compact.

Example 3.5.10 Let \(X\) be a regular topological space such that \(C^*(X)\) contains only constant functions (see Problem 2.7.17 in [9]). Clearly \(X\) cannot be normal or even Tychonoff but any two points are equivalent, meaning that \(|X(M)_{C^*(X)}| = 1|\).

Notice that the behavior of \(X(M)_{C^*(X)}\) is pathological compared to that of \(X_M\), where all basic separation axioms \((T_1, T_2, T_3)\) [18] reflect both upwards and downwards. This comes from the fact that some of its equivalence classes may not contain a representative that lies in \(M\).
APPENDIX

I- Logical/Set theoretical definitions

Definition 1  $M$ is an elementary submodel of $N$ (denoted by $M < N$) if $M \subseteq N$ and for every $n < \omega$ and formula $\varphi$ with at most $n$ free variables and all $\{a_1, \ldots, a_n\} \subseteq M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(a_1, \ldots, a_n)$$

Definition 2  Let $\theta$ be a cardinal. $H_\theta$ is the collection of all sets that have hereditary cardinality less than $\theta$ (informally, elements of $H_\theta$ have cardinality less than $\theta$, so do elements of elements of $H_\theta$, elements of elements of elements of $H_\theta$, and so forth...).

For $\theta$ big enough, $H_\theta$ satisfies "enough" axioms of Set Theory, so it is a set that can "represent" $V$, the collection of all sets. Using $H_\theta$ rather than $V$ avoids some technical problems - see [19], [21], [29] for amplification of this remark.

Definition 3  We say that an elementary submodel $M$ is $\omega$-closed or countably closed if every countable subset of $M$ is an element of $M$.

Definition 4  We say that an elementary submodel $M$ is $\omega$-covering if for every countable $A \subseteq M$, there is a countable $B \in M$ such that $A \subseteq B$.

Definition 5  We say that $j : V \rightarrow M$ is an elementary embedding iff for every $n < \omega$ and formula $\varphi$ with at most $n$ free variables and all $\{a_1, \ldots, a_n\} \subseteq V$,

$$V \models \varphi(a_1, \ldots, a_n) \iff M \models \varphi(j(a_1), \ldots, j(a_n)).$$

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Definition 6 We say that $\kappa$ is the critical point of an elementary embedding $j : V \rightarrow M$ if $\kappa = \min\{\alpha : \alpha$ is an ordinal and $j(\alpha) \neq \alpha\}$.

If $j$ is not the identity such $\kappa$ exists and is a cardinal.

Definition 7 We say that a cardinal $\kappa$ is supercompact if for every cardinal $\lambda$ there exists an elementary embedding $j : V \rightarrow M$ such that:

1. $\kappa$ is the critical point of $j$.
2. $j(\kappa) > \lambda$.
3. $M^\lambda \subseteq M$ ($M$ is closed under $\lambda$-sequences).

For all basic definitions on forcing we refer the reader to Chapter VII in [20].

Definition 8 If $\mathbb{P}$ is a partial order consisting of functions and $G$ is a $\mathbb{P}$-generic filter over $V$, we say that $h = \bigcup G$ is a generic mapping.

It is not difficult to see that if $\mathbb{P}$ is a partial order consisting of functions and $G$ is $\mathbb{P}$-generic over $V$ then $\bigcup G$ is indeed a function.

II-Topological definitions

A more elaborate discussion on the concepts introduced can be found in [9].

Definition 9 Let $X$ be a topological space and let $x \in X$. $\chi(x, X)$ is the smallest cardinal $\kappa$ such that there is a local base for $x$ in $X$ of size $\kappa$. We define $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. If $\chi(X) = \omega$ we say that $X$ is first countable.

Definition 10 Let $X$ be a topological space and $Y \subseteq X$. A family $\mathcal{B}$ of open sets all including $Y$ is called an outer base for $Y$ if for every open $V \supseteq Y$, there is a $B \in \mathcal{B}$ such that $B \subseteq V$. The character of $Y$ in $X$ is $\chi(Y, X) = \min\{\tau : \text{there is an outer base for } Y \text{ of cardinality } \tau\}$. 
Definition 11 For $X$ a topological space, the pointwise type of $X$, denoted by $\bar{n}(X)$, is the least cardinal $\kappa$ such that $X$ can be covered by compact sets, each of which has character less than $\kappa$. If $\bar{n}(X) = \aleph_1$, we say that $X$ is of pointwise countable type. We define $h(X)$ as the least cardinal $\kappa$ such that $X$ can be covered by compact sets, each of which has character less than or equal to $\kappa$.

Some examples of spaces of pointwise countable type are locally compact spaces, first countable spaces and Čech complete spaces.

Definition 12 We say that a topological space $X$ is a $k$–space if for every subset $A$ of $X$, $A$ is closed in $X$ if and only if $A \cap K$ is closed in $K$, for every $K$ compact subset of $X$.

Definition 13 A topological space $X$ is normal if for every $x \in X$, $\{x\}$ is closed in $X$ and for every pair of disjoint closed subsets of $X$, $F,T$, there are disjoint open sets $V,W$ (in $X$) such that $F \subseteq V$ and $T \subseteq W$. We say that $X$ is hereditarily normal if every subspace of $X$ is normal.

Definition 14 We say that a family $\mathcal{F}$ of subsets of a topological space $X$ is normalized if for every partition of $\mathcal{F}$ into two subcollections $\mathcal{Z}$ and $\mathcal{H}$, there exist two open disjoint subsets of $X$, $V$ and $W$, such that $\bigcup \mathcal{Z} \subseteq V$ and $\bigcup \mathcal{H} \subseteq W$.

Definition 15 If $\mathcal{F} = \{F_\alpha : \alpha < \lambda\}$ is a family of pairwise disjoint subsets of $X$, we say that $\mathcal{F}$ is separated if there is a family of open pairwise disjoint subsets of $X$, $\mathcal{W} = \{W_\alpha : \alpha < \lambda\}$ such that $F_\alpha \subseteq W_\alpha$ for every $\alpha < \lambda$. In this case we say that $\mathcal{W}$ is a separation of $\mathcal{F}$.

Definition 16 We say that a family of subsets of a topological space $X$ is discrete if for every $x \in X$ there is an open set $V$ such that $x \in V$ and $V$ meets at most one element of $\mathcal{F}$.

Definition 17 We say that a family of subsets of a topological space $X$ is locally finite if for every $x \in X$ there is an open set $V$ such that $x \in V$ and $V$ meets at most a finite number of elements of $\mathcal{F}$.
Definition 18 We say that a family of subsets of $X$, \( \{W_\alpha : \alpha < \lambda\} \) is an expansion of another family \( \{F_\alpha : \alpha < \lambda\} \) if \( F_\alpha \subseteq W_\alpha \) for every \( \alpha < \lambda \).

Definition 19 We say that a topological space $X$ is collectionwise normal if every discrete family of subsets of $X$ has a discrete expansion consisting of open sets (open expansion).

If $X$ is normal, separated discrete collections have discrete expansions.

Definition 20 We say that a topological space $X$ is expandable if every locally finite family of subsets of $X$ has a locally finite expansion consisting of open sets (open expansion).

Definition 21 A topological space $X$ is Fréchet if for every $x \in X$ and $A \subseteq X$, $x \in \text{cl}(A)$ iff there is a sequence $\sigma \subseteq A$ such that $\sigma$ converges to $x$.

Definition 22 A subset $S$ of a topological space $X$ is sequentially closed if every converging sequence which is a subset of $S$ converges to a point in $S$. A topological space $X$ is sequential if every sequentially closed subset of $X$ is closed.

Definition 23 If $X$ is a topological space we define the tightness of $X$, denoted by $t(X)$ to be the smallest cardinal $\lambda$ such that if $x \in X$ and $A \subseteq X$ are such that $x \in \text{cl}(A)$, then there is $B \subseteq A$ with $|B| \leq \lambda$ such that $x \in \text{cl}(B)$.

Notice that if $X$ is Fréchet then $X$ is sequential and $t(X) = \aleph_0$.

Definition 24 We say that $p : X \rightarrow Y$ is a perfect mapping if it is continuous, closed, onto and inverse images of points are compact.

Definition 25 We say that a topological space is countably paracompact if every countable open cover of $X$ has an open locally finite refinement.
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