The Calculus of Conformal Metrics and Univalence

Criteria for Holomorphic Functions

by

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Abstract

Let $D$ be the unit disc in the complex plane, and $\lambda(z) = \frac{1}{1-|z|^2}$ the hyperbolic line element. Let $S$ be the class of holomorphic functions

$$\{ f : D \rightarrow \mathbb{C} | f \ 1 - 1, \ f(0) = 0, \ f'(0) = 1 \}$$

Define higher order Schwarzian derivatives by

$$\sigma_3(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2, \quad \sigma_{n+1}(f) = \sigma_n(f)' - (n - 1) \frac{f''}{f'} \sigma_n(f)$$

We show that for any $f \in S$, $|\sigma_n(f)(z)| \leq 6 \cdot 4^{n-3}(n - 2)! \lambda(z)^{n-1} \forall z \in D$. These bounds are sharp. We also give sharp two-point geometric distortion theorems for these in terms of hyperbolic distance.

Furthermore, we develop the calculus of the non-holomorphic derivatives of Minda and Peschl for conformal metrics, in order to facilitate function-theoretic computations. The relation of these derivatives to the Levi-Civita connection is given in terms of the curvature of the conformal metric. Necessary tools for differentiation and integration are also developed.
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Chapter 1

Introduction

A geometric viewpoint is used in univalent function theory only in scattered places. Where it is used, it is often very successful. This thesis is devoted to developing the proper geometric tools, and exploiting them for some particular extremal problems for univalent functions. Further promising applications are indicated.

It is surprising that in spite of the pervasiveness of the use of conformal metrics in function theory, the basic tools for integration and differentiation in these metrics has not been developed. Even defining the right quantities greatly simplifies things. Peschl [19] defined some non-holomorphic hyperbolic derivatives that were later generalized to arbitrary conformal metrics by Minda [14] and extensively exploited (for instance [2], [15]). Part of this thesis gives a full geometric description of these derivatives and lays the foundation for calculation. The relation to the derivatives with respect to the Levi-Civita connection is explained with an interpretation in terms of representation theory. Tools for differentiation and integration are given, as well as some integral formulae for these derivatives generalizing
the Cauchy formulae to complete constant curvature metrics. Even though function-theoretic results are usually in terms of the hyperbolic, Euclidean, or spherical metrics, we work with arbitrary conformal metrics in the belief that this will give a better understanding of the geometry. Evidence supporting this belief is given by Osgood and Stowe's univalence criterion for conformal maps between Riemannian manifolds [18], which gives most existing univalence criteria as specific cases by choosing particular metrics.

Secondly, we define some higher order Schwarzian derivatives $\sigma_n(f)$ which are invariant with respect to composition with Möbius transformations. Using the classical methods of Loewner, sharp necessary conditions for univalence are derived. Specifically, we show that for any $f \in \mathcal{S}$, $|\sigma_n(f)(z)| \leq 6 \cdot 4^{n-3}(n - 2)! \lambda(z)^{n-1} \forall z \in \mathcal{D}$. The proof is interesting for its relative simplicity; such theorems are generally very hard to derive for infinite numbers of derivatives. The powerful methods of deBranges are not necessary in this proof, nor is the density of slit functions in the class of univalent functions.

The calculus developed here makes it possible to make the following suggestive observation about the extremal functions. The upper bound is taken on not just at a single point, but rather along an entire hyperbolic geodesic; along this geodesic, the tensor $\sigma_n(f)dz^{n-1}$ is parallel in the hyperbolic metric. A similar observation holds for the Bieberbach conjecture: bounds on the $n$th coefficient are equivalent to bounds on the $n$th Minda/Peschl derivatives; the corresponding tensor is parallel along a hyperbolic geodesic in the hyperbolic metric. This strongly suggests the possibility of a variational proof of both theorems in terms of hyperbolic geometry. The calculus of conformal metrics is necessary in order to exploit this.

Towards this end, two-point geometric distortion theorems for the Schwarzian derivatives are given in terms of hyperbolic distance (inspired by Minda's recognition that the classical
growth theorem is properly formulated as a two-point theorem [15]). This theorem, which implies the original bounds, should be more amenable to variational techniques. Hopefully this will serve as a model for a new variational proof of deBranges theorem, as well as other extremal problems. Variational techniques have been developed extensively by Schiffer [23].
Chapter 2

Preliminaries

2.1 Spaces of Differentials

In this section we establish some notation and define the space of differentials, identifying some isomorphisms. Everything developed here is local, so we will work in open planar domains \( M, M_1, M_2 \subseteq \mathbb{C} \), with standard complex structure \( J \). We use the global chart on \( M, M_1, M_2 \) inherited from \( \mathbb{C} \) to simplify the discussion, though everything can easily be made intrinsic.

Let \( TM \) and \( T^*M \) denote the tangent and cotangent bundles of \( M \), and \( T^*_sM \) denote the space of tensors over \( M \) of covariant degree \( s \) and contravariant degree \( r \). \( T_{\mathbb{C}}M \) decomposes into

\[
T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z} \right\} \oplus \text{span} \left\{ \frac{\partial}{\partial \bar{z}} \right\}.
\]
Similarly

\[ T_z^* M = T_{1,0} M \oplus T_{0,1} M = \text{span}_C \{dz\} \oplus \text{span}\{d\bar{z}\}, \]

and we also have for all \( \omega \in C \otimes_R T_z^* M \),

\[ \omega = \sum \frac{a_{i_1 \ldots i_r j_1 \ldots j_s}}{(i_1 \ldots i_r j_1 \ldots j_s = 1, 1)} \frac{\partial}{\partial z^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial z^{i_r}} \otimes dz^{j_1} \otimes \cdots \otimes dz^{j_s}, \]

where \( \frac{\partial}{\partial z^1} = \frac{\partial}{\partial z}, \frac{\partial}{\partial z^2} = \frac{\partial}{\partial \bar{z}} \), etc. (This decomposition is closely related to the decomposition of \( T_z^* M \) into irreducible \( O(2) \) submodules, which will be given in Section 2.2.)

Let \( T_{s,0}^* M \) denote the subspace of \( C \otimes_R T_z^* M \) spanned by \( \frac{\partial}{\partial z} \otimes \cdots \otimes \frac{\partial}{\partial \bar{z}} \otimes dz \otimes \cdots \otimes dz \), which is of course isomorphic to \( (\otimes^r T_{1,0}^* M) \otimes_C (\otimes^s T_{0,1}^* M) \). The canonical isomorphisms

\[ \phi^{1,0} : T^{1,0} M \rightarrow TM \]

\[ Z \mapsto 2 \text{Re}(Z) \]

\[ \phi_{1,0} : T_{1,0} M \rightarrow T^* M \]

\[ \alpha \mapsto 2 \text{Re}(\alpha) \]

can be extended to \( T_{s,0}^* M \) via

\[ \phi \left( \alpha \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) = 2 \text{Re} \left( \alpha \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right). \]

Denote the image of \( T_{s,0}^* M \) under this isomorphism by \( D_z^* M \). For example,

\[ D_z^1 M = \text{span}_R \left\{ \frac{\partial}{\partial x} \otimes dx + \frac{\partial}{\partial y} \otimes dy, \frac{\partial}{\partial x} \otimes dy - \frac{\partial}{\partial y} \otimes dx \right\}, \]

and

\[ D_z^0 M = \text{span}_R \{ dx \otimes dx - dy \otimes dy, dx \otimes dy + dy \otimes dx \}. \]
Remark 1. $D^*_x M$ can be identified with the space of $\mathbb{R}$-multilinear maps

$$S : T_x M \times \cdots \times T_x M \rightarrow \bigotimes^r T_x M.$$ 

It is immediately apparent that these maps are symmetric.

$J$ acts on each of the components of elements of $T^*_x M$ separately, e.g.

$$J_1 \left( \frac{\partial}{\partial x} \otimes dy \right) = \left( J \frac{\partial}{\partial x} \right) \otimes dy = \frac{\partial}{\partial y} \otimes dy$$
$$J_2 \left( \frac{\partial}{\partial x} \otimes dy \right) = \frac{\partial}{\partial x} \otimes (Jdy) = -\frac{\partial}{\partial x} \otimes dx.$$

For elements of $D^*_x M$, these actions are the same.

Proposition 1. There is a well defined action of $J$ on $D^*_x M$. Under this action, $\phi(i\alpha) = J\phi(\alpha)$ for all $\alpha \in D^*_x M$.

Proof. First, it is easy to check that for all $X \in T^{1,0} M$, $\alpha \in T_{1,0} M$, $\text{Re}(iX) = J\text{Re}(X)$, $\text{Im}(iX) = J\text{Im}(X)$, and $\text{Re}(i\alpha) = J\text{Re}(\alpha)$, $\text{Im}(i\alpha) = J\text{Im}(\alpha)$. Now assume that the above holds in $D^*_x$ and $D^*_y$. Let $\alpha \in D^*_x$ and $\beta \in D^*_y$. We have

$$\text{Re}(\alpha \otimes \beta) = \text{Re}(\alpha) \otimes \text{Re}(\beta) - \text{Im}(\alpha) \otimes \text{Im}(\beta),$$

and

$$\text{Re}(i(\alpha \otimes \beta)) = \text{Re}((i\alpha) \otimes \beta) = \text{Re}(\alpha \otimes (i\beta)),$$

(similarly for the imaginary part). So,

$$J(\text{Re}(\alpha)) \otimes \text{Re}(\beta) - J(\text{Im}(\alpha)) \otimes \text{Im}(\beta) = \text{Re}(\alpha) \otimes J(\text{Re}(\beta)) - \text{Im}(\alpha) \otimes J(\text{Im}(\beta)).$$

Similarly for $\text{Im}(\alpha \otimes \beta)$. Inductively this shows that the action of $J$ is well-defined and becomes multiplication by $i$ under $\phi$. \qed
Finally, we place an algebraic structure on $\cup_{r,s=1,2,\ldots}D_r^*M$ which is computationally useful. For $a, b \in C$, we have the multiplication

$$T_{s_1,0}^{r_1}M \times T_{s_2,0}^{r_2}M \rightarrow T_{s_1+s_2,0}^{r_1+r_2}M$$

$$a \left( \frac{\partial}{\partial z} \right)^{r_1} \otimes dz^{s_1} \times b \left( \frac{\partial}{\partial z} \right)^{r_2} \otimes dz^{s_2} \rightarrow ab \left( \frac{\partial}{\partial z} \right)^{r_1+r_2} \otimes dz^{s_1+s_2}.$$

Also, if $r_2 \leq r_1$, $s_2 \leq s_1$, we can define a division in the obvious way. This carries over to a multiplication and division on $D_r^*M$ via the isomorphism $\phi$.

### 2.2 Decomposition of $\otimes^n R^2$ into irreducible $SO(2, R)$-submodules

Let $V = R^2$. In this section, we find the irreducible $SO(2, R)$-submodules of $\otimes^n R^2$. This is equivalent to finding the irreducible $CO^+(2, R)$-submodules of $\otimes^n R^2$.

We denote the standard basis of $V^*$ by $dx, dy$. The action of $S^1$ on $V^*$ is given by

$$\rho(\theta) dx = \cos \theta \, dx - \sin \theta \, dy$$

$$\rho(\theta) dy = \sin(\theta) \, dx + \cos \theta \, dy.$$

There is a natural inclusion of $\otimes^n V^*$ into $C \otimes (\otimes^n V^*)$ as a real subspace; we will treat $\otimes^n V^*$ as a subset without explicitly mentioning the inclusion.

Now $C \otimes V^* = \text{span}\{dz\} \oplus \text{span}\{d\bar{z}\}$. The action of $S^1$ on $C \otimes V^*$ is given by

$$\rho(\theta)(adz + b d\bar{z}) = ae^{i\theta} \, dz + be^{-i\theta} \, d\bar{z}.$$

$\rho$ and $\rho$ extend to $\otimes^n V^*$ and $C \otimes (\otimes^n V^*)$ respectively. We have that
Proposition 2. 1) \( \text{Re}(\tilde{\rho}(\theta)\alpha) = \rho(\theta)\text{Re}(\alpha), \) and \( \text{Im}(\tilde{\rho}(\theta)\alpha) = \rho(\theta)\text{Im}(\theta) \) for \( \alpha \in C \otimes (\bigotimes^n V^*). \)

2) \( \bigotimes^n V^* \) is a real \( SO(2,\mathbb{R}) \)-submodule of \( C \otimes (\bigotimes^n V^*) \) under the representation \( \tilde{\rho}. \)

Proof. 1) It is easy to check this for \( n = 1. \) If it is true for \( 1, \ldots, n, \) then for \( \alpha \in V^*, \) \( \beta \in \bigotimes^n V^*, \)

\[
\text{Re}(\tilde{\rho}(\alpha \otimes \beta)) = \text{Re}(\tilde{\rho}(\theta)\alpha \otimes \tilde{\rho}(\theta)\beta)
= \text{Re}(\tilde{\rho}(\theta)\alpha) \otimes \text{Re}(\tilde{\rho}(\theta)\beta) - \text{Im}(\tilde{\rho}(\theta)\alpha) \otimes \text{Im}(\tilde{\rho}(\theta)\beta)
= \rho(\theta)(\text{Re}(\alpha) \otimes \text{Re}(\beta)) - \rho(\theta)(\text{Im}(\alpha) \otimes \text{Im}(\beta))
= \rho(\theta)\text{Re}(\alpha \otimes \beta).
\]

Similarly for the imaginary part.

2) The proof of part 1) immediately implies that \( \bigotimes^n V^* = \text{Re}(C \otimes \bigotimes^n V^*) \) is invariant under \( \tilde{\rho}. \)

So by the above proposition, to decompose \( \bigotimes^n V^* \) into irreducible \( \rho \)-submodules, it suffices to decompose \( C \otimes (\bigotimes^n V^*) \) into irreducible \( \tilde{\rho} \) submodules. The task is now easy.

Let \( \alpha \in \bigotimes^n V^*. \) In \( C \otimes (\bigotimes^n V^*) \) it has the decomposition

\[
\alpha = \sum_{i_1, \ldots, i_n = 1, \bar{1}} \alpha_{i_1 \ldots \cdot i_n} d\zeta_{i_1} \otimes \cdots \otimes d\zeta_{i_n},
\]

where \( \alpha_{i_1 \ldots \cdot i_n} = \bar{\alpha}_{i_1 \ldots \cdot i_n} \) (i.e. \( \alpha_{i1} = \bar{\alpha}_{i1} \)). Each invariant subspace is of the form

\[
W = \{ \text{Re}(ad\zeta_{i_1} \otimes \cdots \otimes d\zeta_{i_n}) : a \in C \},
\]

and \( S^1 \) acts on this by \( \rho(\theta)\text{Re}(ad\zeta_{i_1} \otimes \cdots \otimes d\zeta_{i_n}) = \text{Re}(e^{im\theta}ad\zeta_{i_1} \otimes \cdots \otimes d\zeta_{i_n}) \) where \( m = \# \)
of 1's \( - \# \) of \( \bar{1} \)'s. If \( n \) is even, some of these subspaces have the trivial action, and so divide
into two one-dimensional irreducible subspaces. (If we were looking at irreducible $O(2, \mathbb{R})$ submodules, one of these would be $\rho \equiv 1$, and the other would be $\rho(A) = \det(A) = \pm 1$, but this will not concern us here).

For example,

$$V^* \otimes V^* = \{\text{Re}(adz \otimes dz) : a \in \mathbb{C}\} \oplus \{\text{Re}(bd\bar{z} \otimes dz) : b \in \mathbb{C}\}$$

$$= \text{span}\{dx \otimes dx - dy \otimes dy, dx \otimes dy + dy \otimes dx\}$$

$$\oplus \text{span}\{dx \otimes dx + dy \otimes dy\} \oplus \text{span}\{dx \otimes dy - dy \otimes dx\}.$$

Here $S^1$ acts on the first space by $\rho(\theta) = e^{i\theta}$, and trivially on the second two. Of course, $D^0_n\mathbb{R}^2$ is the two-dimensional subspace with action $\rho(\theta) = e^{in\theta}$, and this is the space which we will be concerned with.

**Remark 2.** An alternate characterization of $D^0_n\mathbb{R}^2$ is that the elements are symmetric (as maps $V \times \cdots \times V \to \mathbb{R}$) and trace-free in every pair of arguments.
Chapter 3

The partial connection $\tilde{\nabla}$ on $D^r_s M$

Here we define a partial connection which acts on $D^r_s M$ or $T^r_{s,0} M$, and depends on a choice of metric conformal to the Euclidean. This partial connection is the 1,0 component of the Hermitian connection acting on sections of $T^r_{s,0} M$, or equivalently the Levi-Civita connection on $D^r_s M$. We give the definitions and basics in Section 3.1. In Section 3.2, some expressions for parallel transport in different conformal metrics are derived, which are necessary for differentiation and integration. The purpose is to make it possible to do calculus with the non-holomorphic derivatives of holomorphic functions of Peschl and Minda. This will be discussed in section 4.

3.1 Definition of $\tilde{\nabla}$

Definition 1. Let $M$ be a 2-dimensional manifold. Two metrics $g_1$, $g_2$ are said to be conformal if $g_2 = e^{2\phi} g_1$ for some smooth function $\phi : M \rightarrow \mathbb{R}$. 
A metric $g$ on $\mathcal{M}$ is said to be \textit{conformal} if it is conformal to the Euclidean metric $g_0$; $g = \rho^2 g_0$. We will often refer to the function $\rho^2$ itself as the metric, and $\rho$ as the line element.

Consider the Levi-Civita connection $\nabla^\rho$ associated to the conformal metric $\rho$. Complex linearly extend it to act on vectors in $T^{1,0} \mathcal{M}$.

**Proposition 3.** If $\rho^2$ and $\nabla^\rho$ are as above, then

\[ \nabla^\rho \frac{\partial}{\partial z} = -\nabla^\rho \frac{\partial}{\partial \bar{z}} = 2 \operatorname{Re}(\nabla^\rho \frac{\partial}{\partial z}) \]

\[ \nabla^\rho \frac{\partial}{\partial \bar{z}} = \nabla^\rho \frac{\partial}{\partial z} = -2 \operatorname{Im}(\nabla^\rho \frac{\partial}{\partial z}). \]

**Proof.** This follows directly from the fact that $\nabla_X JY = J \nabla_X Y$ and the fact that the connection is torsion free. \hfill \square

Also,

**Proposition 4.** If $\rho^2$ and $\nabla$ are as above then

\[ \nabla^\rho \frac{\partial}{\partial z} = 2 \frac{\rho}{\rho} \frac{\partial}{\partial z} = \Gamma^\rho \frac{\partial}{\partial z} \]

\[ \nabla^\rho \frac{\partial}{\partial \bar{z}} = \nabla^\rho \frac{\partial}{\partial \bar{z}} = 0. \]

**Proof.** This can be calculated from $\nabla^\rho \frac{\partial}{\partial z}$ and Proposition 3. (See [11], vol. 2, pp. 155-158.) \hfill \square

Using Propositions 3 and 4, it can be checked that for a function $g : \mathcal{M} \rightarrow \mathbb{C}$,

\[ \nabla^\rho g \frac{\partial}{\partial z} = g \Gamma \frac{\partial}{\partial z} \otimes dz + \frac{\partial g}{\partial z} \frac{\partial}{\partial z} \otimes dz + \frac{\partial g}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}} \otimes d\bar{z}, \]

\[ \nabla^\rho gdz = -g \Gamma dz \otimes dz + \frac{\partial g}{\partial z} dz \otimes dz + \frac{\partial g}{\partial \bar{z}} dz \otimes d\bar{z}. \]
We now define the ‘connection’ \( \tilde{\nabla} \) on \( T_{*0}^{r,0}M \). One metric is used to differentiate the contravariant components and another to differentiate the covariant ones. This is necessary so that \( \tilde{\nabla} \) can be used to differentiate functions between domains with different metrics.

**Definition 2.** For \( gdz \in T^{1,0}M \), \( h \frac{\partial}{\partial z} \in T^{1,0}M \), \( \tau_1, \tau_2 \) two line elements on \( M \), set

\[
\tilde{\nabla}^\tau_1 gdz = \left( \frac{\partial g}{\partial z} - \Gamma^g \right) dz \otimes dz,
\]

\[
\tilde{\nabla}^\tau_2 \frac{\partial}{\partial z} = \left( \frac{\partial h}{\partial z} + \Gamma^h \right) \frac{\partial}{\partial z} \otimes dz.
\]

Now let \( \tilde{\nabla}^\tau_{r,0} : T_{*0}^{r,0}M \rightarrow T_{*1}^{r,0}M \) be given by

\[
\tilde{\nabla}^\tau_{r,0} \left( g \frac{\partial}{\partial z} \otimes \cdots \otimes \frac{\partial}{\partial z} \otimes dz \otimes \cdots \otimes dz \right) + \cdots + = \tilde{\nabla}^\tau_2 \left( g \frac{\partial}{\partial z} \otimes \cdots \otimes \frac{\partial}{\partial z} \otimes dz \otimes \cdots \otimes dz \right.
\]

\[+ g \frac{\partial}{\partial z} \otimes \cdots \otimes \tilde{\nabla}^\tau_2 \left( \frac{\partial}{\partial z} \right) \otimes dz \otimes \cdots \otimes dz
\]

\[+ g \frac{\partial}{\partial z} \otimes \cdots \otimes \frac{\partial}{\partial z} \otimes \tilde{\nabla}^\tau dz \otimes \cdots \otimes dz
\]

\[+ \cdots + g \frac{\partial}{\partial z} \otimes \cdots \otimes \frac{\partial}{\partial z} \otimes dz \otimes \cdots \otimes \tilde{\nabla}^\tau dz
\]

\[= \left( \frac{\partial g}{\partial z} + r \Gamma^g - s \Gamma^g \right) \frac{\partial}{\partial z} \otimes dz \otimes \cdots \otimes dz.
\]

**Remark 3.** Redistributing coefficients would not change the definition above; i.e.

\[
\tilde{\nabla}^\tau_2 \left( g \frac{\partial}{\partial z} \otimes dz \right) + g \frac{\partial}{\partial z} \otimes \tilde{\nabla}^\tau dz = \tilde{\nabla}^\tau_2 \left( g \frac{\partial}{h \partial z} \right) \otimes h dz + g \frac{\partial}{h \partial z} \otimes \tilde{\nabla}^\tau h dz.
\]

The action of \( \tilde{\nabla} \) carries over to \( D^r_*M \) under the isomorphism \( \phi \). We use the same notation \( \tilde{\nabla} \) for the action on both spaces.
Proposition 5. $\tilde{\nabla}$ satisfies a product rule with respect to the multiplication. Explicitly, if $\alpha \in \mathcal{D}_r^s M$ and $\beta \in \mathcal{D}_r^s M$, then

$$\tilde{\nabla}\alpha \beta = \tilde{\nabla}(\alpha)\beta + \alpha\tilde{\nabla}(\beta).$$

Similarly if $r_2 \leq r_1$ and $s_2 \leq s_1$ then

$$\tilde{\nabla}\frac{\alpha}{\beta} = \frac{\tilde{\nabla}(\alpha)\beta - \alpha\tilde{\nabla}(\beta)}{\beta^2}.$$

Now we give the relation between the connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ acting on $\mathcal{D}_r^s M$. $\tilde{\nabla}$ can be thought of as $\nabla$ composed with a projection on to $\mathcal{D}_r^{s+1} M$. For holomorphic sections of $\mathcal{D}_r^s M$, these coincide.

Proposition 6. If $\nabla$ and $\tilde{\nabla}$ are connections corresponding to a metric $\rho^2$, then $\phi \left( \tilde{\nabla} Z_1 Z_2 \right) = \nabla_{\phi(Z_1)} \phi(Z_2)$ for every pair of holomorphic vector fields $Z_1$ and $Z_2$.

Proof. Let $Z_1 = f \frac{\partial}{\partial z}$, $Z_2 = g \frac{\partial}{\partial z}$, with $f = f_1 + if_2$ and $g = g_1 + ig_2$ holomorphic. We have

$$\phi(\tilde{\nabla} f \frac{\partial}{\partial z} g \frac{\partial}{\partial z}) = \phi(fg'\frac{\partial}{\partial z} + fg\tilde{\nabla}^\rho \frac{\partial}{\partial z})$$

$$= \phi(f_1(g_1 x \frac{\partial}{\partial z} + ig_2 y \frac{\partial}{\partial z}) + if_2(g_2 y \frac{\partial}{\partial z} - ig_1 y \frac{\partial}{\partial z}))$$

$$+ \phi(((f_1 g_1 - f_2 g_2) + i(f_1 g_2 + f_2 g_1))\tilde{\nabla}^\rho \frac{\partial}{\partial z})$$

$$= f_1 g_1 x \frac{\partial}{\partial x} + f_1 g_2 y \frac{\partial}{\partial y} + f_2 g_2 y \frac{\partial}{\partial y} + f_2 g_1 y \frac{\partial}{\partial x}$$

$$+ (f_1 g_1 - f_2 g_2)2 \text{Re}(\tilde{\nabla}^\rho \frac{\partial}{\partial z}) - (f_1 g_2 + f_2 g_1)2 \text{Im}(\tilde{\nabla}^\rho \frac{\partial}{\partial z})$$

and

$$\phi(\nabla_{\phi(Z_1)} \phi(Z_2)) = \nabla_{f_1 \frac{\partial}{\partial z} + f_2 \frac{\partial}{\partial y}} g_1 \frac{\partial}{\partial x} + g_2 \frac{\partial}{\partial y}$$

$$= f_1 g_1 x \frac{\partial}{\partial x} + f_1 g_2 y \frac{\partial}{\partial y} + f_2 g_1 y \frac{\partial}{\partial x} + f_2 g_2 y \frac{\partial}{\partial y}$$

$$+ f_1 g_1 \tilde{\nabla}^\rho \frac{\partial}{\partial z} + f_1 g_2 \tilde{\nabla}^\rho \frac{\partial}{\partial y} + f_2 g_1 \tilde{\nabla}^\rho \frac{\partial}{\partial z} + f_2 g_2 \tilde{\nabla}^\rho \frac{\partial}{\partial y}.$$
Now apply Proposition 3.

The Levi-Civita connection $\nabla^{r_1, r_2}$ acts on $T^r_x M$ via a Leibniz rule, e.g.

$$\nabla^{r_1, r_2} a \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

$$= da \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

$$+ a \nabla^{r_2} \left( \frac{\partial}{\partial x^{i_1}} \right) \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

$$+ \cdots + a \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes \nabla^{r_2} \left( dx^{j_s} \right),$$

where it is understood that the order of the factors within the term is rearranged so that the new covariant factor is the first of the covariant factors. (This is just a convenient convention.

With this choice, it works out that the covariant derivative of, say, $\alpha \in T^r_x M$ in the direction of a vector $X$ is just $i_X \nabla(\alpha)$. Alternatively one could symmetrize.)

In general we have

**Proposition 7.** For $g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \in T^r_{s,0} M$,

$$\nabla \phi \left( g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) = \phi \left[ \tilde{\nabla} \left( g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) + \frac{\partial g}{\partial \bar{z}} \left( \frac{\partial}{\partial z} \right)^r \otimes d\bar{z} \otimes dz^s \right]$$

**Proof.** If we extend $\nabla$ complex linearly, then it satisfies a Leibniz rule on $C \otimes T^r_x M$, so using the calculations following Proposition 4,

$$\nabla \phi \left( g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) = \phi \left( \nabla \left( g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) \right)$$

$$= \phi \left( \tilde{\nabla} \left( g \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) + \frac{\partial g}{\partial \bar{z}} \left( \frac{\partial}{\partial z} \right)^r \otimes d\bar{z} \otimes dz^s \right).$$

Thus, we can think of $\nabla$ as a $\nabla$ followed by a projection onto $D^r_{s+1} M$. We also have that if $\alpha \in D^r_x M$ is holomorphic (i.e. the coefficient of $\phi^{-1}(\alpha)$ is holomorphic) then $\nabla \alpha = \tilde{\nabla} \alpha$.  

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3.2 Parallel transport in conformal metrics

We now work out how to parallel transport elements of $T^*_s M$ or $D^*_s M$. Since $\nabla J = 0$ for an arbitrary conformal metric, we have that the parallel transport of an element of $D^*_s M$ remains in $D^*_s M$.

Let $\gamma$ be a smooth curve, and let $P_{\gamma(s)\gamma(t)}$ denote the parallel transport map from $\gamma(s)$ to $\gamma(t)$ along $\gamma$:

$$P_{\gamma(s)\gamma(t)} : T^*_s (\gamma(s))M \to T^*_t (\gamma(t))M$$

(Or, $P_{\gamma(s)\gamma(t)} : D^*_s (\gamma(s))M \to D^*_t (\gamma(t))M$; we will not notationally distinguish between $\phi \circ P_{\gamma(s)\gamma(t)} \circ \phi^{-1}$ and $P_{(\gamma(s)\gamma(t))}$.) Parallel transport of elements of $T^{r,0}_s M$ can be represented by the complex number

$$P_{\gamma(s)\gamma(t)}^c = \frac{P_{\gamma(s)\gamma(t)} \left( \frac{\partial}{\partial z} \right)}{\frac{\partial}{\partial z}}$$

(this is independent of the choice of complex parameter). To see this, note that since

$$P_{\gamma(s)\gamma(t)}(dz)P_{\gamma(s)\gamma(t)}\left( \frac{\partial}{\partial z} \right) = 1,$$

$$P_{\gamma(s)\gamma(t)}^c dz = \frac{1}{P_{\gamma(s)\gamma(t)}^c} dz = P_{\gamma(t)\gamma(s)}^c.$$

So clearly

$$P_{\gamma(s)\gamma(t)} \left( \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s \right) = \left( P_{\gamma(s)\gamma(t)}^c \right)^{r-s} \left( \frac{\partial}{\partial z} \right)^r \otimes dz^s.$$ 

Next we need an expression for the change of geodesic curvature of a smooth curve under conformal change of metric. For this we need to know how the connection changes.
Proposition 8. Let $\phi : M \to \mathbb{R}$ be a smooth function and $\bar{g} = e^{2\phi} g$ be two conformally related metrics with connections $\bar{\nabla}, \nabla$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + (X\phi)Y + (Y\phi)X - g(X, Y) \text{grad}_g \phi.$$ 

Proof. See [12], p. 4.

Proposition 9. Let $\phi, \bar{g}, g$ be as above, and let $\gamma(t)$ be a smooth curve. Then the geodesic curvatures $k_{\bar{g}}$ and $k_g$ of $\gamma$ are related by

$$e^{\phi} k_{\bar{g}} = k_g - \frac{g(\text{grad}_g \phi, J\gamma)}{g(\gamma, \gamma)^{1/2}}.$$

If $ds_{\bar{g}}$ and $ds_g$ are the line elements of $\gamma$, then this can be written

$$k_{\bar{g}} ds_{\bar{g}} = k_g ds_g + 2 \text{Im} \left( \frac{\partial \phi}{\partial z} dz \right).$$

Proof. Using Proposition 8,

$$k_{\bar{g}} = \frac{\bar{g}(\nabla_{\bar{\gamma}} \gamma, J\bar{\gamma})}{g(\gamma, \gamma)^{3/2}} = e^{-\phi} \frac{g(\nabla_{\gamma} \gamma, J\gamma)}{g(\gamma, \gamma)^{3/2}} + 2e^{-\phi} \frac{g((\bar{\gamma}\phi)\gamma, J\gamma)}{g(\gamma, \gamma)^{3/2}} - e^{-\phi} \frac{g(\text{grad}_g \phi, J\gamma)}{g(\gamma, \gamma)^{1/2}}$$

The second form follows easily using $g_0(X, JY) = \text{Re}(\bar{X} JY) = \text{Im}(\bar{X} Y)$ and $(\text{grad}_{g_0} \phi) = 2\phi_{\bar{z}}$, where we identify $X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}$ with $X_1 + iX_2$.

Proposition 10. Let $\gamma$ be a smooth curve parametrized by arclength $s$ in the conformal metric $\rho^2$. Then

$$\frac{d}{ds} \bigg|_{s=0} P^\gamma_{\rho(s)\gamma(t)} = \Gamma \circ \gamma(0) \frac{d\gamma}{ds}(0),$$

where $\frac{d\gamma}{ds}$ is treated as a complex number.
Proof.

\[ ik_2(0) \frac{d\gamma}{ds}(0) = \frac{d}{ds} \bigg|_{s=0} \left( P_{\gamma(0)} \frac{d\gamma}{ds}(0) \right). \]

So

\[ \left( \frac{d}{ds} \bigg|_{s=0} P_{\gamma(0)} \right) \frac{d\gamma}{ds}(0) = ik_2(0) \frac{d\gamma}{ds}(0) - \frac{d^2\gamma}{ds^2}(0). \]

Now if \( t \) is Euclidean arc length, the second term is

\[ -\left( \frac{1}{\rho} \frac{d}{dt} \bigg|_{t=0} \left( \frac{1}{\rho} \frac{d\gamma}{dt} \right) \right) = -\frac{1}{\rho^2} \frac{d^2\gamma}{dt^2}(0) + \frac{1}{\rho^2} 2 \text{Re} \left( \frac{\rho z}{\rho} \frac{d\gamma}{dt}(0) \right) \frac{d\gamma}{dt}(0) \]

\[ = -\frac{i}{\rho} k_{z_0}(0) \frac{d\gamma}{ds}(0) + \text{Re} \left( \Gamma \circ \gamma(0) \frac{d\gamma}{ds}(0) \right) \frac{d\gamma}{ds}(0). \]

So applying Proposition 9,

\[ \left( \frac{d}{ds} \bigg|_{s=0} P_{\gamma(0)} \right) \frac{d\gamma}{ds}(0) = \frac{2i}{\rho} \text{Im} \left( \Gamma \circ \gamma(0) \frac{d\gamma}{dt}(0) \right) \frac{d\gamma}{ds}(0) \]

\[ + \text{Re} \left( \Gamma \circ \gamma(0) \frac{d\gamma}{ds}(0) \right) \frac{d\gamma}{ds}(0) \]

\[ = \Gamma \circ \gamma(0) \left( \frac{d\gamma}{ds}(0) \right)^2. \]

For some specific domains and metrics, there is a unique geodesic between each pair of points. In this case there is a well-defined linear map

\[ P_{z_1 z_2} : T_{z_1} M \rightarrow T_{z_2} M \]

given by parallel transport along the geodesic connecting \( z_1 \) and \( z_2 \). This is the case for the unit disc \( D \) with hyperbolic metric \( \lambda^2_D \), as well as \( \mathbb{C} \) with the Euclidean metric \( g_0 \). In
the case of $\mathcal{C}$ with the spherical metric, there are two geodesics connecting any two points; however parallel transport along these two is the same, so again a linear map can be defined. Finally, if $f : M \rightarrow \mathcal{C}$ is a locally univalent map, it is possible to pull back the Euclidean transport operator on $\mathcal{C}$. This can be done even though $M$ might not be convex in the metric $|f'|^2 g_0 = f^* (g_0)$, just by pushing forward a vector $X \in T_z M$ to $f_* X \in T_{f(z)} \mathcal{C}$, transporting to $T_{f(z)} \mathcal{C}$, and then pulling back via the appropriate branch of $f^{-1}$ to $T_{z''} M$. We calculate this here, and also the linear map for $(D, \lambda_D)$, for use later in examining certain $\lambda$-linear differentials.

**Proposition 11.** The parallel transport map for the metric $|f'|^2 g_0$ on $M$ described above is

$$P^c_{z_1 z_2} = \frac{f'(z_1)}{f'(z_2)}.$$  

**Proof.** It is clear that $P_{f(z_1), f(z_2)} \equiv 1$. Now use the fact that

$$\phi^{-1} (f \cdot z_1, X) = f'(z_1) \phi^{-1} (X).$$

$\square$

**Proposition 12.** For $M = D$, with the hyperbolic metric $\lambda^2_D = \frac{1}{1-|z|^2}$, we have that

a) the parallel transport map is given by

$$P^c_{z_1 z_2} = \frac{(1-|z_2|^2)(1-\bar{z}_1 z_2)}{(1-|z_1|^2)(1-\bar{z}_1 \bar{z}_2)}.$$  

b) An element $g \xi^n \in T^0_{r, 0}$ is parallel along a hyperbolic geodesic $\gamma$ iff $\frac{g(z)}{\lambda^n} e^{in\theta}$ is constant where $\dot{\gamma} = re^{i \theta}$.

**Proof.** a) Let $\gamma$ be the unit speed geodesic through $z_1, z_2$ with $\gamma(0) = z_1, \gamma(l) = z_2$. Let $T(z) = \frac{z - z_1}{1-\bar{z}_1 z}$; $T \circ \gamma$ is a straight line through the origin, of the form $T \circ \gamma(s) = e^{i\theta} \tan \pi s$ since
\[
\frac{\tanh's}{1 - \tanh's} = 1. \text{ Using this same identity,}
\]
\[
\dot{\gamma}(s) = T'^{-1} \left( e^{i\theta \tanh s} \right) e^{i\theta \tanh s} = e^{i\theta} \frac{1 - |e^{i\theta \tanh s}|^2}{T'(\gamma(s))}
\]
\[
= e^{i\theta} \frac{1 - |T \circ \gamma(s)|^2}{T'(\gamma(s))} = e^{i\theta} \frac{|T'(\gamma(s))|}{T'(\gamma(s))} \left(1 - |\gamma(s)|^2\right).
\]
Since \(\nabla_{\dot{\gamma}} \dot{\gamma} = 0\),
\[
P_{z_1 z_2}^\varepsilon = \frac{\dot{\gamma}(l)}{\gamma(0)} = \frac{|T'(z_2)|}{T'(z_2)} \frac{T'(z_1) (1 - |z_2|^2)}{|T'(z_1)| (1 - |z_1|^2)} = \frac{(1 - |z_2|^2) (1 - \bar{z}_1 z_2)}{(1 - |z_1|^2) (1 - z_1 \bar{z}_2)}.
\]

b) Let \(T\) be as in the proof of a). We have \(\frac{T' \circ \gamma'}{|T' \circ \gamma'|} = 1\) so \(\frac{T' \circ \gamma'}{|T' \circ \gamma'|} = \frac{i\|\|}{\gamma}\). Now by the above,
\[
g(z_1) dz^n = P_{z_2 z_1}(g(z_2) dz^n) = (P_{z_1 z_2}^\varepsilon)^n g(z_2) dz^n
\]
\[
= \left[ \frac{|T'(z_2)|}{T'(z_2)} \frac{T'(z_1) (1 - |z_2|^2)}{|T'(z_1)| (1 - |z_1|^2)} \right]^n
\]
\[
= \left[ \frac{\dot{\gamma}(z_2)}{\dot{\gamma}(z_1)} \frac{\dot{\gamma}(z_1)}{\lambda(z_1)} \frac{\dot{\gamma}(z_2)}{\lambda(z_2)} \right]^n g(z_2) dz^n.
\]

\[\square\]

**Remark 4.** If \(T\) is a disc automorphism, and \(T(z_1) = w_1, T(z_2) = w_2\), the following identity holds:
\[
\frac{T'(w_1) (1 - |w_1|^2) (1 - \bar{w}_2 w_1)}{T'(w_2) (1 - |w_2|^2) (1 - \bar{w}_1 w_2)} = \frac{(1 - |z_1|^2) (1 - \bar{z}_2 z_1)}{(1 - |z_2|^2) (1 - z_1 \bar{z}_2)}.
\]
This is equivalent to the fact that parallel transport of a vector along a geodesic commutes with disc automorphisms.
Chapter 4

Covariant derivatives of a holomorphic function

In this section we study the non-holomorphic derivatives of Peschl and Minda. In 4.1 we show that these are elements of $T^r_{s0}M$ given by successive differentiation with $\tilde{\nabla}$. In 4.2 it is shown how $\tilde{\nabla}f$ and $\nabla f$ differ for metrics with non-zero curvature.

4.1 Definitions

Definition 3. Let $f : M_1 \rightarrow M_2$ be a holomorphic function, and $\rho, \sigma$ be conformal line elements on $M_1, M_2$ respectively. Let $f^*(\sigma) = \sigma \circ f | f'|$ be the pullback of $\sigma$ under $f$. Define $\tilde{\nabla}^\rho\sigma f$ inductively by

\[ \tilde{\nabla}^\rho\sigma_1 f = \frac{\partial}{\partial z} \otimes dz \quad (= f_{-1}^{-1}(f'dz)) \]

and

\[ \tilde{\nabla}^\rho\sigma_{n+1} f = \tilde{\nabla}^\rho f^*(\sigma) \left( \tilde{\nabla}^\rho\sigma_n f \right), \]
and denote \( \nabla_n^\rho \sigma f = \frac{\partial}{\partial f} D_n^\rho \sigma f \frac{\partial}{\partial z} \otimes (dz)^n \).

This is equivalent to Minda's definition for general conformal metrics [14] with a slight change of notation. For particular metrics, the notation \( \frac{D_n f}{D f} \) is used in the literature ([15], [19]).

For \( \rho = \frac{1}{1-|z|^2} \) and \( \sigma = 1 \), we have the following expressions and invariances. Let \( T \) be a disc automorphism, and \( S \) an affine map. Since disc automorphisms \( T(z) = e^{i\theta} \frac{z+a}{1+\bar{a}z} \) are isometries of the hyperbolic metric, \( \nabla_n(f \circ T) = T^*(\nabla_n f) \); so, using the identity \( \lambda(z) = \lambda(T(z))|T'(z)| \), we have

\[
(f \circ T)^{-1} \lambda(z)^n D_n(f \circ T)(z) \frac{\partial}{\partial z} \otimes dz^n = \nabla_n(f \circ T)(z) = T^*(\nabla_n f)(z) \\
= f^{-1}(\lambda \circ T(z))^n(D_n f) \circ T(z)T'(z)^{n-1} \frac{\partial}{\partial z} \otimes dz^n \\
= f^{-1} \lambda(z)^n(D_n f) \circ T(z) \frac{T'(z)^{n-1}}{|T'(z)|^n} \frac{\partial}{\partial z} \otimes dz^n.
\]

so

\[
D_n(f \circ T) = (D_n f) \circ T \frac{T'(z)^n}{|T'(z)|^n}
\]

(4.1)

A similar argument shows that

\[
D_n(S \circ f) = D_n f
\]

(4.2)

for all affine maps \( S \). Now for the hyperbolic metric, \( \Gamma(0) = 0 \), so that

\[
\nabla^{\lambda,1} f(0) = \frac{f(n)(0)}{f'(0)} \frac{\partial}{\partial z} \otimes dz^n.
\]

This gives the formula (Peschl's definition)

\[
D^{\lambda} f(a) = \frac{\partial^n}{\partial z^n} \bigg|_{z=0} (f \circ T)(0),
\]

(4.3)
where $T_a(z) = \frac{z + a}{1 + az}$. Calculating the first few,

$$D_1^2 f(z) = (1 - |z|^2)f'(z)$$

$$D_2^2 f(z) = (1 - |z|^2)^2 f''(z) - 2\bar{z}(1 - |z|^2)f'(z)$$

$$D_3^2 f(z) = (1 - |z|^2)^3 f'''(z) - 6\bar{z}(1 - |z|^2)^2 f''(z) + 6\bar{z}^2(1 - |z|^2)f'(z).$$

### 4.2 Relation between $\tilde{\nabla}_n f$ and $\nabla_n f$

The $n$th covariant derivative of $f$ with respect to the Levi-Civita connections on $M_1$ and $M_2$ is a multilinear map

$$\nabla^\rho f : T_z M_1 \times \cdots \times T_z M_1 \rightarrow T_{f(z)} M_2$$

given inductively by

$$\nabla^\rho f (X_0, \ldots, X_n) = \nabla^\rho \nabla_{X_0} f (\nabla^\rho f (X_1, \ldots, X_n))$$

$$-\nabla^\rho f (X_1, \ldots, X_n)$$

$$- \cdots - \nabla^\rho f (X_1, \ldots, \nabla^\rho_{X_0} X_n),$$

with $\nabla^\rho f = f_*$ (see [5]).

**Remark 5.** If $f' = 0$, this can't be defined. So, we either assume $f$ is locally univalent or remove the singular points from the domain.
Remark 6. The multilinear map $\nabla^\rho_\alpha f_z$ can also be seen as an element of $T_z M_1 \otimes (T^*_z M_1)^n$ by pulling back vectors in $T_{f(z)} M_2$ under the appropriate branch of $f^{-1}$. So

$$\nabla^\rho_\alpha f(X_0, \ldots, X_n) = \nabla^\rho_\alpha f_z \left( \nabla^\rho_\alpha f(X_0, \ldots, X_n) \right)$$

- $\nabla^\rho_\alpha f(X_0, \ldots, X_n)$
- $\cdots \cdots \nabla^\rho_\alpha f(X_0, \ldots, \nabla^\rho_\alpha X_n)$

In general $\nabla^\rho_\alpha f$ need not be $J$-linear, so $\nabla^\rho_\alpha f$ and $\tilde{\nabla}^\rho_\alpha f$ will not coincide, even if $f$ is holomorphic and $\rho$, $\sigma$ are conformal. The curvature of $\rho$ and $\sigma$ causes the difference.

Proposition 7 shows that in order to relate $\phi^{-1} \circ \nabla^\rho_\alpha f \circ \phi \circ \tilde{\nabla}^\rho_\alpha f$ to $\tilde{\nabla}^\rho_\alpha f$, it is necessary to know $\frac{\partial}{\partial \bar{z}} \left( \frac{\rho^\sigma}{\sigma o f f} D^\rho_\sigma \right)$. If it is holomorphic, then $\phi^{-1} \circ \nabla^\rho_\alpha f \circ \phi \tilde{\nabla}^\rho_\alpha f = \tilde{\nabla}^\rho_\alpha f$. Now $\frac{\rho^\sigma}{\sigma o f f} D^\rho_\sigma f = 1$ is holomorphic, so $\phi^{-1} \circ \nabla^\rho_\alpha f \circ \phi = \tilde{\nabla}^\rho_\alpha f$. Now let

$$\Psi_1 \equiv \Gamma^\rho_\alpha f = -\frac{\rho^2 K_\rho}{2}, \quad \Psi_\sigma f = \frac{\partial}{\partial z} \left( \Psi_\sigma f \right) = n\Gamma^\rho f \Psi_\sigma f, \quad \Phi_1 \equiv \Gamma^{*}\sigma = -\frac{f^* f K_{f^* f}}{2}, \quad \Phi_\sigma f = \frac{\partial}{\partial z} \left( \Phi_\sigma f \right) = n\Gamma^\rho f \Phi_\sigma f,$$

where $K_\rho$ and $K_{f^* f}$ are the Gaussian curvatures of $\rho$ and $f^* f$ respectively. We have

Proposition 13.

$$\nabla^\rho_\alpha \left( \tilde{\nabla}^\rho_\alpha f \right) = \nabla^\rho_\alpha f + \sum_{k=1}^{n-1} \left[ \frac{(n-1)}{(k-1)} \Phi_{n-k} - \frac{n}{(k-1)} \Psi_{n-k} \right] \frac{\partial}{\partial z} \otimes d\bar{z} \otimes \tilde{\nabla}^\rho_\alpha f.$$

Proof. For simplicity, denote

$$A_n = \frac{\rho^n}{\sigma o f f} D^\rho_\sigma f.$$

By definition,

$$A_{n+1} = A_n + \left( \Gamma^{*}\sigma - n\Gamma^\rho \right) A_n.$$
We have that

\[
A_{n\bar{z}} = \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right].
\]

This is easily checked for \( n = 2 \). Assume it holds for \( n \):

\[
A_{n+1\bar{z}} = A_{n\bar{z}} + \left( \Gamma_{\bar{z}}^f - n\Gamma_{\bar{z}}^g \right) A_n + \left( \Gamma_{\bar{z}}^f - n\Gamma^g \right) A_{n\bar{z}}
\]

\[
= \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] A_k
\]

\[
+ \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] A_{k\bar{z}}
\]

\[
+ (\Phi_1 - n\Psi_1) A_n
\]

\[
+ (\Gamma^\rho_2 - n\Gamma^\rho_1) \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] A_k
\]

\[
= (\Phi_1 - n\Psi_1) A_n
\]

\[
+ \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] \left( A_{k\bar{z}} + (\Gamma^\rho_2 - k\Gamma^\rho_1) A_k \right)
\]

\[
+ \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] \left( \frac{n}{k-1} \right) \Gamma^\rho_1 \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k} - \left( \frac{n}{k-1} \right) \Psi_{n-k} \right] A_k
\]

\[
= (\Phi_1 - n\Psi_1) A_n
\]

\[
+ \sum_{k=2}^{n} \left[ \left( \frac{n-1}{k-2} \right) \Phi_{n-k+1} - \left( \frac{n}{k-2} \right) \Psi_{n-k+1} \right] A_k
\]

\[
+ \sum_{k=1}^{n-1} \left[ \left( \frac{n-1}{k-1} \right) \Phi_{n-k+1} - \left( \frac{n}{k-1} \right) \Psi_{n-k+1} \right] A_k
\]

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Corollary 1. Let $\rho$ be a line element of constant curvature $K$, and $\sigma = 1$. Then,

$$
\nabla^{n+1}(\rho^n D_n f dz^n) = \rho^{n+1} D_{n+1} f dz^{n+1} + \frac{n(n-1)}{4} \rho^n K D_{n-1}^2 f d\bar{z} \otimes dz^n.
$$

Proof. Apply the previous theorem using $\Psi_1 = -\frac{e^2 K}{2}$, $\Psi_n = 0$ for all $n \geq 2$, and $\Phi_1 = 0$. 

4.3 Some observations about extremal functions

We now use some of the machinery of the last few sections to describe some suggestive properties of the extremal function for deBranges' theorem and the bounds on the Schwarzian described later. In this section $D_n f$ is understood to be given by $\tilde{\nabla}_n f = \lambda^n \frac{D_n f}{f' \frac{\partial}{\partial z}} \otimes dz^n$ where $\lambda$ is the hyperbolic metric.

Represent $f(z) \in S$ by their power series $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. It was proven by deBranges [3] that $\sup_{f \in S} |a_n| = n$ and that the only functions in $S$ taking on this upper bound are the Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its rotations $\tilde{k}(z) = e^{-i\theta} k(e^{i\theta} z)$. Given an arbitrary $f \in S$, the function $\tilde{f}(z) = \frac{f \circ T(z) - f \circ T(0)}{f'(0)}$ is also in $S$ for any disc automorphism $T$. Applying the coefficient estimate to $\tilde{f}$ for different $T$, and using equation 4.3, we have that the coefficient estimate is equivalent to

$$
\sup_{f \in S} \left| \frac{D_n f(z)}{D_1 f(z)} \right| = n \cdot n!.
$$

(4.7)
Also, because of the affine invariance (4.2), the normalization \( f(0) = f'(0) - 1 = 0 \) can be removed. Denote the non-normalized family of holomorphic functions by

\[
\tilde{S} = \{ f : D \rightarrow \mathbb{C}, f 1 - 1 \}.
\]

By deBranges' theorem, the set of extremal functions for (4.7) are just the Koebe functions and its transforms

\[
\tilde{k}(z) = \frac{k \circ T(z) - k \circ T(0)}{(k \circ T)'(0)},
\]

and if the normalization is removed, the extremal functions are all of the form \( S \circ k \circ T \) where \( S \) is affine and \( T \) is a disc automorphism.

The Koebe function and its transforms \( \tilde{k} \) have the following property. The upper bound on \( \frac{D_0 f}{D_1 f}(z) \) over the family \( \tilde{S} \) is \( n \cdot n! \), independent of \( z \); what is more interesting, is that if this value is taken on at \( z_0 \), then it is taken on along an entire hyperbolic geodesic through \( z_0 \). Along this geodesic, the tensor \( \tilde{\nabla}_n f \) is parallel in the hyperbolic metric \( \lambda \). This strongly suggests that the problem of maximizing the \( n \)th coefficient is at heart a geometric problem which could give way to variational techniques. Such techniques have been developed extensively by Schiffer [23]. In section [6] we tackle a similar problem, involving higher order Schwarzian derivatives, for which the Koebe functions are also the extremals. We give a proof of the upper bounds using the Loewner method. The hope is that an alternate variational proof can be given, and that this will serve as a prototype for a variational proof of deBranges theorem and other extremal problems in which the geometry is made explicit.

To see that the Koebe functions have this property, it can be calculated first that

\[
\frac{D_2 k(z)}{D_1 k(z)} = 4 \text{ for all } z \in (-1, 1).
\]

Let \( x_1, x_2 \in (-1, 1) \); we get that the parallel transport
map (see Proposition 12) becomes

\[ p_{z_{2}z_{1}}^c = \frac{1 - |z_{2}|^2}{1 - |z_{1}|^2} = \frac{\lambda(z_{1})}{\lambda(z_{2})}. \]

So

\[ p_{z_{2}z_{1}}^\lambda \left( \lambda(z_{2}) \frac{D_{2} k(z_{2})}{D_{1} k(z_{2})} dz \right) = p_{z_{2}z_{1}}^\lambda \left( \lambda(z_{2}) 4 dz \right) = \lambda(z_{1}) 4 dz = \frac{\lambda(z_{1})}{\lambda(z_{2})} \frac{D_{2} k(z_{1})}{D_{1} k(z_{1})} dz. \]

In other words, \( \frac{\tilde{\psi}_{2k}}{\psi_{1k}} \) is parallel along \((-1, 1)\). Using the invariance (4.1) and the identity in Remark 4, we get that for a disc automorphism \( T \), \( k \circ T \) is parallel along the hyperbolic geodesic \( T^{-1} ((-1, 1)) \).

Next, we need to compute the higher hyperbolic derivatives using this information.

**Proposition 14.** Let \( \gamma \) be a smooth curve in \( D \) and \( \dot{\gamma}(t) = re^{i\theta(t)} \). If \( \frac{\tilde{\psi}_{2f}}{\psi_{1f}} = \lambda^{n-1} \frac{D_{n} f}{D_{1} f} dz^{n-1} \) is parallel along the curve \( \gamma \), then

\[
\left( \lambda^{n} \frac{D_{n+1} f}{D_{1} f} - \lambda^{n} \frac{D_{2} f}{D_{1} f} \frac{D_{n} f}{D_{1} f} \right) e^{i\theta} = n(n - 1)\lambda^{n} \frac{D_{n-1} f}{D_{1} f} e^{-i\theta}.
\]

**Proof.** Computing using Corollary 1, we have that

\[
\frac{\partial}{\partial z} \left( \lambda^{n-1} \frac{D_{n} f}{D_{1} f} \right) = -n(n - 1)\lambda^{n} \frac{D_{n-1} f}{D_{1} f}.
\]

Thus by Propositions 7 and 5,

\[
\nabla \left( \lambda^{n-1} \frac{D_{n} f}{D_{1} f} dz^{n-1} \right) = \tilde{\nabla} \left( \lambda^{n-1} \frac{D_{n} f}{D_{1} f} dz^{n-1} \right) - n(n - 1)\lambda^{n} \frac{D_{n-1} f}{D_{1} f} d\bar{z} \otimes dz^{n-1}
\]

\[ = \lambda^{n} \frac{D_{n+1} f}{D_{1} f} dz^{n} - \lambda^{n} \frac{D_{2} f}{D_{1} f} \frac{D_{n} f}{D_{1} f} dz^{n} - n(n - 1)\lambda^{n} \frac{D_{n-1} f}{D_{1} f} d\bar{z} \otimes dz^{n}.
\]

When this is evaluated in the direction \( re^{i\theta} \), it is zero by hypothesis, which gives the result.

\[ \Box \]
Remark 7. Note the similarity to the Marty criterion for extremality [7]. The Marty criterion will be examined in Section 4.4.

Corollary 2. If \( \lambda \frac{D_{z}f}{D_{z}f}dz \) is parallel along a hyperbolic geodesic \( \gamma \), then so are \( \lambda^{n} \frac{D_{z}+1}{D_{z}f}dz^{n} \) for all \( n \geq 2 \).

**Proof.** Let \( \gamma(t) = re^{i\theta(t)} \). Assume that \( \lambda \frac{D_{z}f}{D_{z}f}dz, \cdots, \lambda^{n-1} \frac{D_{z}f}{D_{z}f}dz^{n-1} \) are parallel. Then, by Proposition 12, \( \lambda^{k} \frac{D_{z}+1}{D_{z}f}dz^{k} = \lambda^{k}c_{k+1}e^{ik\theta} \) for \( k = 1, \cdots, n - 1 \). By Proposition 14, \( \lambda^{n} \frac{D_{z}+1}{D_{z}f}dz^{n} = \lambda^{n}c_{n+1}e^{in\theta(t)} \).

Alternatively, one could solve for all functions which are holomorphic on \( D \) and for which \( \lambda \frac{D_{z}f}{D_{z}f}dz \) are parallel, and show that they have the above property. We show here that these functions are just the ‘generalized Koebe functions’ (see [6]) and compositions with affine and disc automorphisms.

By the invariance 4.1 and the identity in remark 4, it suffices to determine the functions such that \( \lambda \frac{D_{z}f}{D_{z}f}dz \) is parallel along the real axis; for other geodesics the functions will be \( S \circ k \circ T \) for some \( S \) affine and \( T \) disc automorphism. Now \( \frac{D_{z}f}{D_{z}f} = (1 - |z|^{2}) \frac{f''}{f'} - 2z \). By Proposition 12 we want to solve, for \( x \in (-1, 1) \),

\[
(1 - x^{2}) \frac{f''}{f'} - 2x = 2c.
\]

Since \( f \) is holomorphic, we must have

\[
(1 - z^{2}) \frac{f''}{f'} - 2z = 2c
\]

for all \( z \in D \). The solutions are

\[
f'(z) = \frac{K_{1}}{(1 - z^{2})} \left( \frac{1 + z}{1 - z} \right)^{c},
\]

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These are also characterized by the invariance (see [6])
\[
g(z) = \frac{g \circ T_z(z) - g \circ T_z(0)}{(g \circ T_z)'(0)},
\]
where \(T_z(z) = \frac{z + \pi}{1 + \pi z}, x \in \mathbb{R}\). By expanding both sides in a Taylor series around 0, (with the coefficient depending on \(x\)), it is easily shown that the above invariance is equivalent to the condition that all the quantities \(\lambda^{n-1} \frac{Dnf}{Dz^n} dz\) are parallel for \(n = 1, 2, \ldots\).

### 4.4 The Marty Criterion

We interpret the Marty criterion in terms of hyperbolic derivatives. The Marty criterion (see [7]) states that if \(f(z) = z + a_2 z^2 + \cdots\) maximizes \(\text{Re} \ a_n\) over the class \(\mathcal{S}\),

\[
(n + 1)a_{n+1} - 2a_2 a_n = (n - 1)a_{n-1}.
\]

(Note that maximizing \(\text{Re} \ a_n\) is the same thing as maximizing \(|a_n|\) by considering rotations \(\tilde{f}(z) = e^{i\theta} f(e^{i\theta} z)\).) Equation 4.8 is easily derived by using the disc automorphisms as a family of variations. Here, rather than fixing the point and varying the function, we fix the function and vary the point. This is really no different from the usual treatment. It is included here only to explicitly compare with the condition of Proposition 14.

Since \(\sup_{f \in \mathcal{S}} \left| \frac{Dnf(z)}{Dz^n} \right|\) is independent of \(z\), if this quantity is maximized at \(z_0\) by a function \(f\), then fixing the function, it attains a maximum at \(z_0\) in \(z\). The Marty criterion is simply that the derivative in every direction must be zero.
Explicitly, let $\gamma$ be a smooth curve, and $g dz^n$ a tensor such that $\lambda^{-n}|g|$ has a local maximum at $\gamma(0)$. Then

$$0 = \frac{d}{dt} \bigg|_{t=0} \frac{|g \circ \gamma|}{\lambda^n \circ \gamma}$$

$$= \frac{|g|}{\lambda^n} \left[ -n \left( \frac{\lambda z}{\lambda} \dot{\gamma} + \frac{\lambda z}{\lambda} \ddot{\gamma} \right) + \frac{1}{2} \left( \frac{g z}{g} \dot{\gamma} + \frac{g z}{g} \ddot{\gamma} + \frac{\ddot{g}}{g} \gamma + \frac{\ddot{g}}{g} \dddot{\gamma} \right) \right]$$

$$= \frac{|g|}{\lambda^n} \text{Re} \left( \frac{\nabla g}{g} \dot{\gamma} + \frac{g z}{g} \dddot{\gamma} \right).$$

This is true for all $\dot{\gamma} = re^{i\theta}$, so at $z_0$, we have

$$\frac{\nabla g}{g} = -\left( \frac{g z}{g} \right).$$

Applying this to $\lambda^{n-1} D_n f dz^{n-1}$, and using the computations of the last section, we get that

$$\left( \frac{D_n f}{D_1 f} \right)^{-1} \frac{D_{n+1} f}{D_1 f} - \frac{D_2 f}{D_1 f} = n(n - 1) \left( \frac{D_n f}{D_1 f} \right)^{-1} \frac{D_{n-1} f}{D_1 f},$$

which, when written in terms of coefficients, gives

$$(n + 1)a_{n+1} - 2a_2a_n = (n - 1)a_{n-1} \frac{a_n}{a_n},$$

which is the Marty criterion. (The inessential factor $\frac{a_n}{a_n}$ appears because we maximized the norm rather than the real part.)

### 4.5 Integral kernels and Taylor series for constant curvature metrics

In this section we introduce reproducing kernels which are analogous to the Cauchy kernel, but for metrics with constant curvature. The usual Cauchy kernel comes from setting the
curvature to zero. Some analogues of the Cauchy formulae for the derivatives of a holomorphic function are given as well.

First we give the complete metrics of constant curvature on discs and their isometries, taken from Minda [14].

**Definition 4.** Let

$$\Delta_k = \begin{cases} 
\{ z : |z| < \frac{1}{\sqrt{|k|}} \} & k < 0 \\
\mathbb{C} & k = 0 \\
\overline{\mathbb{C}} & k > 0 
\end{cases}$$

**Definition 5.**

$$\lambda_k = \frac{1}{1 + k|z|^2}.$$ 

A calculation using the well-known formula $K = -\frac{\Delta \log \lambda_k}{\lambda_k}$ shows that the curvature of $\lambda_k$ is $4k$. Another calculation shows that the maps

$$T(z) = e^{i\theta} \frac{z + a}{1 - k\bar{a}z}$$

satisfy the identity

$$1 + k|T'(z)|^2 = (1 + k|z|^2)|T'(z)|,$$

which shows that these are isometries. These are in fact all of them [14], and they preserve the domain $\Delta_k$.

Using the fact that $\lambda_k$ are radially symmetric, and moving the geodesics to straight lines
through the origin, it’s easy to show that

\[ d_{\lambda_k}(w,z) = \begin{cases} \frac{1}{\sqrt{-k}} \arctanh(\sqrt{-k} \frac{|w-z|}{1+k|wz|}) & k < 0 \\ |w-z| & k = 0 \\ \frac{1}{\sqrt{k}} \arctan(\sqrt{k} \frac{|w-z|}{1+k|wz|}) & k > 0 \end{cases} \]

The Cauchy kernel is closely related to the Euclidean mean value property for harmonic function; e.g. for a circle centred around the point \( a \),

\[ d\theta = \frac{dz}{i(z-a)}. \]

So, for a given holomorphic function \( f(z) = u(z) + iv(z) \), if \( \gamma \) is a circle centred at \( a \), since \( u \) and \( v \) satisfy the mean value property for the Euclidean metric,

\[ f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz, \]

and since the kernel is holomorphic, this is true for any smooth curve \( \gamma \).

For another conformal metric, say \( \rho^2g_0 \), the Laplacian is \( \rho^{-2}\Delta g_0 \) where \( \Delta g_0 \) is the usual Laplacian. Thus the harmonic functions are the same for all conformal metrics; however, the ‘circles’ do change when the metric changes, and this gives rise to different mean value properties. (Not all metrics have a mean value property, but the metrics given above certainly do, as will be shown.) When the mean value property for the metrics \( \lambda_k \) is used, we get analogues of the Cauchy kernel.

When restricted to \( \lambda_k \)-circles, these kernels are just \( d\theta \) where the angle is determined by \( \lambda_k \). We explicitly compute these kernels. Let \( f(z) = u(z) + iv(z) \) be an arbitrary holomorphic function on \( \Delta_k \), and \( \gamma \) a \( \lambda_k \)-circle centred on \( a \). The isometry \( T(z) = \frac{z-a}{1+ka} \) takes \( \gamma \) to a \( \lambda_k \)-circle centred at the origin, which is an ordinary circle by the radial symmetry of \( \lambda_k \). Thus
we have that

\[ f(a) = \frac{1}{2\pi} \int_{T^{-1}(a)} f(z) \frac{dz}{iz} = \frac{1}{2\pi} \int_{\gamma} f(z) \frac{T'(z)}{iT(z)} dz \]

\[ = \frac{1}{2\pi i} \int_{\gamma} f(z) \frac{1 + k|a|^2}{(1 + k\bar{a}z)(z - a)} dz. \]

Since the kernel is holomorphic, this is true for any curve \( \gamma \) in \( \Delta_k \) that winds once about \( a \).

We have also shown that along hyperbolic circles,

\[ d\theta^\lambda_k = \frac{1 + k|a|^2}{i(1 + k\bar{a}z)(z - a)}, \]

and that harmonic functions \( u \) satisfy the \( \lambda_k \) mean value property

\[ u(a) = \frac{1}{2\pi} \int_{\gamma} u(z) d\theta^\lambda_k \]

for \( \lambda_k \) circles \( \gamma \). For \( k = -1 \), we will refer to this kernel as the 'hyperbolic Cauchy kernel'.

**Remark 8.** When restricted to the boundary of the disc, the hyperbolic Cauchy kernel is just the Poisson kernel up to a reparametrization. However, along other curves in the disc, it is not the Poisson kernel for the enclosed domain.

**Remark 9.** One could of course derive many more such kernels by using other constant curvature metrics, by pulling back \( d\theta \) under some holomorphic map onto a domain. In particular, we get hyperbolic Cauchy kernels on every simply connected domain \( G \). They are related to the Green’s function of the domain as follows. If \( f_a \) is a Riemann map from \( G \) to \( D \) such that \( f(a) = 0 \), and \( h \) is a holomorphic function on \( G \), then given a smooth curve \( \gamma \) in \( G \) that winds once about \( a \),

\[ \frac{1}{2\pi i} \int_{\gamma} h(z) \frac{f_a'(z)}{f_a(z)} dz = \frac{1}{2\pi i} \int_{f_{\gamma}(w)} h \circ f^{-1}(w) \frac{dw}{w} \]

\[ = h(a), \]
and so \( \frac{1}{i} \frac{f_a'(z)}{f_a(z)} \) is the hyperbolic Cauchy kernel for \( G \). Along a hyperbolic circle around \( a \) it agrees with the hyperbolic angle \( d\theta \) centred on \( a \) in \( G \). Since \( g_G(z, a) = -\log |f_a(z)| \),

\[
\frac{1}{i} \frac{f_a'(z)}{f_a(z)} = -\frac{2}{i} \frac{\partial g_G}{\partial z}.
\]

**Remark 10.** Along hyperbolic circles, (and only along hyperbolic circles), this agrees with the usual Green’s formula: for \( u \) harmonic,

\[
u(z) = \frac{1}{2\pi} \int_\gamma u(w) \frac{\partial g}{\partial \nu}(z, w) ds,
\]

where \( \nu \) is the Euclidean unit normal to \( \gamma \) and \( ds \) is Euclidean arc length. To see this, just note that the Green’s function is constant in \( z \) along hyperbolic circles centred on \( w \), so

\[
\Re \left( 2 \frac{\partial g}{\partial z}(z, w) dz \right)_{\gamma} = 0.
\]

So

\[
\frac{\partial g}{\partial \nu} ds = -2 \Im \left( \frac{\partial g}{\partial z}(z, w) dz \right) = -\frac{2}{i} \frac{\partial g}{\partial z}.
\]

We now give analogues of the Cauchy formulae

\[
f^{(n)}(w) = \frac{n!}{2\pi i} \int f(z) \frac{dz}{(z - w)^{n+1}}
\]

for the metrics \( \lambda_k \). This can also be done for all constant curvature metrics on simply connected domains.

**Proposition 15.** Let \( \gamma \) be a smooth simple closed curve in \( \Delta_k \) and \( f \) be holomorphic on a domain containing \( \gamma \). Then,

\[
D^{\lambda_k}_n f(w) = \frac{n!}{2\pi i} \int_\gamma f(z) \frac{(1 + k|z|^2)(1 + k\bar{w}z)^{n-1}}{(z - w)^{n+1}} dz.
\]

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Proof. The formula has already been shown for $n = 0$. Now differentiate under the integral sign using

$$D_{n+1}^{\lambda_k} f = \lambda_k^{-1} \left( \frac{\partial}{\partial z} (D_n^{\lambda_k} f) - \frac{n}{2} \Gamma \lambda_k D_n^{\lambda_k} f \right).$$

The inductive step is

$$(1 + k|w|^2) \frac{\partial}{\partial w} \left[ \frac{(1 + k|w|^2)}{(z - w)(1 + k\bar{w}z)^{n-1}} \right] + n k\bar{w} \frac{(1 + k|w|^2)^2 (1 + k\bar{w}z)^{n-1}}{(1 + k|w|^2)(z - w)^{n+1}}$$

$$= (n + 1) \frac{(1 + k|w|^2)(1 + k\bar{w}z)^n}{(z - w)^{n+2}}.$$

One could use these to estimate $D_n f$, and prove convergence of a Taylor series involving them, without reference to the ordinary Taylor series (i.e. one could reconstruct the usual function theory of a complex variable in the metrics $\lambda_k$). The Taylor expansion is motivated as in the Euclidean case by

$$f(w) = \frac{1}{2\pi i} \int f(z) \frac{(1 + k|w|^2)}{(z - w)(1 + k\bar{w}z)} dz$$

$$= \frac{1}{2\pi i} \int f(z) \left[ \frac{(1 + k|w|^2)}{(z - w)(1 + k\bar{w}z)} + k \frac{\bar{w} - \bar{a}}{(1 + k\bar{a}z)(1 + k\bar{w}z)} \right] dz$$

$$= \frac{1}{2\pi i} \int f(z) \frac{(1 + k\bar{w})}{(1 + k\bar{a}z)(z - w)} dz$$

$$= \frac{1}{2\pi i} \int f(z) \frac{1 + k|a|^2}{(1 + k\bar{a}z)(z - a)} \left( \frac{1}{1 - \frac{(w-a)(1+k\bar{a})}{(1+k\bar{w})(z-a)}} \right)$$

$$= \frac{1}{2\pi i} \int f(z) \frac{1 + k|a|^2}{(1 + k\bar{a}z)(z - a)} \left( 1 + \frac{(w-a)(1 + k\bar{a})(z-a)}{(1+k\bar{w})(z-a)} + \frac{(w-a)^2 (1 + k\bar{a})^2}{(1+k\bar{w})^2 (z-a)^2} + \cdots \right) dz.$$
So, \[
f(w) = f(a) + D_1^\lambda f(a) \frac{w-a}{1+k\bar{a}w} + \frac{1}{2} D_2^\lambda f(a) \left( \frac{w-a}{1+k\bar{a}w} \right)^2 + \cdots. \tag{4.9}
\]

This could also be derived simply by looking at the power series at 0 of \( f \circ T \) where \( T(z) = \frac{z+a}{1-k\bar{a}z} \), using \( D_n^\lambda f(a) = \frac{\partial^n}{\partial z^n} |_{z=0} f \circ T \).

We can use these Taylor series to give a non-recursive relation between the covariant derivatives \( \nabla_n f \) and \( \widetilde{\nabla}_n f \) for the metrics \( \lambda_k \). In order to do this, we compare the coefficients of two kinds of Taylor series.

Let \( a, w \in \Delta_k \), and let \( \gamma \) be the unique geodesic connecting these points. The isometry \( T(z) = \frac{z+a}{1-k\bar{a}z} \) takes 0 to \( a \) and takes a radial line through the origin to the curve \( \gamma \). This line has argument \( \theta \), equal to the argument of \( \gamma(a) \). Now let \( X_\theta \) denote the vector field of \( \lambda_k \)-unit tangent vector fields along \( \gamma \). Let \( t_k \) be the \( \lambda_k \) arc length of \( \gamma \) from \( w \) to \( a \). This is given by

\[
t_k = \begin{cases} 
\frac{1}{\sqrt{-k}} \arctanh(\sqrt{-k} | \frac{w-z}{1+k\bar{a}z} |) & k < 0 \\
|w - z| & k = 0 \\
\frac{1}{\sqrt{k}} \arctan(\sqrt{k} | \frac{w-z}{1+k\bar{a}z} |) & k > 0 
\end{cases}
\]

So we get that

\[
\frac{w-a}{1+k\bar{a}w} = \left| \frac{w-a}{1+k\bar{a}w} \right| \exp(i \arg \theta) \left( \frac{w-a}{1+k\bar{a}w} \right) \tag{4.10}
\]

\[
= \tanh t e^{i\theta}, \tag{4.11}
\]

where \( \theta \) is constant for \( w \) along \( \gamma \).

Now let \( f \) be a holomorphic function in a neighborhood of \( a \). Expand it in a Taylor series along the geodesic \( \gamma \):
\[ f(w) = f(a) + X_\theta f(a) t_k + \frac{1}{2} X_\theta^2 f(a) t_k^2 + \cdots. \]

Now since \( \nabla_{X_\theta} X_\theta = 0 \), we have that \( \nabla_n f(a) = X_\theta^n f(a) \), since \( \forall n \),

\[
\nabla_{n+1} f(X_\theta, \ldots, X_\theta) = X_\theta (\nabla_n f(X_\theta, \ldots, X_\theta)) \\
- \nabla_n f(\nabla_{X_\theta} X_\theta, \ldots, X_\theta) - \cdots \\
- \nabla_n f(X_\theta, \ldots, \nabla_{X_\theta} X_\theta).
\]

So

\[
f(w) = f(a) + \nabla_1 f(X_\theta)(a) t_k + \frac{1}{2} \nabla_2 f(X_\theta, X_\theta)(a) t_k^2 + \cdots. \tag{4.12}
\]

Comparing (4.9) and (4.12), using 4.10, we get

\[
\frac{1}{n!} \nabla_n f^k(X_\theta, \ldots, X_\theta)(a) = \sum_{m=1}^{n} \frac{1}{m!} c_{nm} \xi_m \lambda^k D_m \xi_m f(a). \tag{4.13}
\]

where \( c_{nm} \) is the \( nth \) coefficient of the Taylor series of \( \left(\frac{1}{\sqrt{k}} \tan \sqrt{k} t\right)^m \) if \( k < 0 \), and of \( \left(\frac{1}{\sqrt{k}} \tan \sqrt{k} t\right)^m \) if \( k > 0 \). This could also be derived directly from Corollary 1.
Chapter 5

Conformal change of metric

5.1 Introduction

In section 4.2 we saw that in metrics with non-zero curvature the $n$th derivative of a function with respect to the Riemannian connection may have a component which is not complex linear, for $n > 2$, and that the Peschl/Minda derivatives naturally arise as projections onto the complex linear component. In a way, they can be seen as a compromise between conformal and Riemannian geometry. We typically prove distortion theorems for classes of conformal mappings, but these are expressed in terms of how they change the Riemannian geometry of the underlying domain for some particular convenient metrics. Thus many of these theorems are conveniently expressed in terms of the Peschl/Minda derivatives ([2], [15])

In this section we examine the behaviour of certain quantities under conformal change of metric. In (5.2), we construct derivatives which do not depend on choice of conformal metric, and use this point of view to show that a distortion theorem of Pommerenke is essentially
an index theorem. In (5.3), we give higher order versions of Osgood and Stowe's Schwarzian tensor corresponding to a conformal change of metric. This is preliminary; it is hoped that it will be possible to give sufficient conditions for univalence in terms of higher order Schwarzian derivatives in a geometrically unified way, following [18]. Univalence criteria involving higher order Schwarzian derivatives have been related to comparison theorems for ordinary differential equations by Lavie [13].

5.2 The Gauss-Bonnet theorem and the argument principle

From the point of view of conformal geometry, it would be natural to construct derivatives of a holomorphic function which depend only on the conformal structure. So we seek derivatives that can be computed by choosing any particular metric, using only the geometry of that metric, which are unchanged when a different metric is chosen. For example, fix two planar domains \((M_1, \rho_1)\) and \((M_2, \rho_2)\) where \(\rho_1\) and \(\rho_2\) are expressed with respect to the conformal parameter \(\frac{\partial}{\partial z}\). Using the inductive relation from section 4.1, we compute the first three derivatives with respect to these metrics.

\[
\begin{align*}
\tilde{\nabla}_1 f &= f' \\
\tilde{\nabla}_2 f &= f'' + (\Gamma^{\rho_2} \circ f)f'^2 - \Gamma^{\rho_1} f' \\
\tilde{\nabla}_3 f &= f''' - 3\Gamma^{\rho_1} f'' + 3\Gamma^{\rho_2} \circ f f'' f' + (2\Gamma^{\rho_1} - \Gamma^{\rho_1}) f' - 3\Gamma^{\rho_2} \circ f \cdot \Gamma^{\rho_1} \cdot f'^2 + (\Gamma^{\rho_2} \circ f + (\Gamma^{\rho_1} \circ f)^2) f'^3.
\end{align*}
\]
Recursively substituting, and solving for \( f', f'', \) and \( f''', \)

\[
\begin{align*}
f' & = \tilde{\nabla}_1 f \\ f'' & = \tilde{\nabla}_2 f + \Gamma^{\rho_1} \tilde{\nabla}_1 f - \Gamma^{\rho_2} \circ f (\tilde{\nabla}_1 f)^2 \\ f''' & = \tilde{\nabla}_3 f + 3\Gamma^{\rho_2} f - 3\Gamma^{\rho_2} \circ f \tilde{\nabla}_2 f \tilde{\nabla}_1 f + (\Gamma^{\rho_1^2} + \Gamma^{\rho_2^2}) \tilde{\nabla}_1 f \\
& \quad - 3\Gamma^{\rho_2} \circ f \Gamma^{\rho_1} (\tilde{\nabla}_1 f)^2 + (2(\Gamma^{\rho_2} \circ f)^2 - \Gamma^{\rho_2^2} \circ f)(\tilde{\nabla}_1 f)^3.
\end{align*}
\]  

Here the subscript \( w \) denotes differentiation with respect to the natural parameter on \( M_2 \subset \mathbb{C} \). The right hand sides of these expressions are calculated in terms of the metrics \( \rho_1 \) and \( \rho_2 \), but do not depend on the choice of metric.

**Remark 11.** These quantities of course depend on the choice of parameter \( z \) and \( w \); they transform as jets under change of parameter. Change of parameter is analogous to a local change of frame in Riemannian geometry, but here a 'frame' contains not just tangent vectors but jets of all orders.

From these, quantities with invariance under composition with global biholomorphic maps can be constructed. For instance, if \( M_2 = \mathbb{C} \), then the quantities

\[
\frac{f''}{f'} = \frac{\tilde{\nabla}_2 f}{\tilde{\nabla}_1 f} + \Gamma^{\rho_1} - \Gamma^{\rho_2} \circ f \cdot \tilde{\nabla}_1 f, \quad \frac{f^{(n)}}{f'} = \ldots
\]

are invariant under post-composition with affine transformations. The Schwarzian derivative is invariant under post and pre-composition with Möbius transformations.

The local Gauss-Bonnet theorem relates a topological invariant, namely the rotation index of a simple closed curve, to Riemannian geometric quantities on the enclosed domain. The argument principle, when applied to the derivative \( f' \) of a holomorphic map, computes
the change in the rotation index of a closed curve under the map. The change in the rotation index can also be computed by applying the Gauss-Bonnet theorem on the domain and image. The argument principle then appears as the specific case of this using the Euclidean metric on domain and image.

Here we assume the argument principle and apply it to the expression for \( f'' / f' \) from the last section to derive the desired formula, rather than assuming the Gauss-Bonnet theorem. For the specific case of the hyperbolic metric on the disc and the Euclidean metric on the image viewed as a subset of \( \mathbb{C} \), this is applied to get a necessary condition for local univalence due to Pommerenke [20].

Denote the rotation index of a curve \( \alpha \) by \( r(\alpha) \); it is the topological index of the map \( \alpha : [0, \tau] \rightarrow S^1, t \mapsto \alpha(t) / |\alpha(t)| \). The argument principle applied to \( f' \) asserts that

\[
\Delta r(\alpha; f) \equiv r(f \circ \alpha) - r(\alpha) = \# \text{ of zeros of } f' = \frac{1}{2\pi i} \int_{\alpha} f'' d\alpha
\]

(we assume that \( f' \) has no zeros on \( \alpha \)). Expressing \( f'' / f' \) invariantly using equations (5.4),

\[
\Delta r(\alpha; f) = \frac{1}{2\pi i} \int_{\alpha} \frac{\tilde{\nabla} f}{\nabla f} d\alpha + \frac{1}{2\pi i} \int_{\alpha} \Gamma_{\rho_1} d\alpha - \frac{1}{2\pi i} \int_{\alpha} \Gamma_{\rho_2} \circ f \cdot \tilde{\nabla}_1 f d\alpha.
\]

Applying Stoke's theorem to the second term, and using the fact that

\[
d(\Gamma_{\rho_1} d\alpha) = -iK_{\rho_1} dA_{\rho_1},
\]

and

\[
d \left( \Gamma_{\rho_2} \circ f \cdot \tilde{\nabla}_1 f d\alpha \right) = -iK^{f'^*(\rho_2)} dB_{f'^*(\rho_2)}.
\]

where \( K_{\rho_i} \) are the Gaussian curvatures of \( \rho_i \) and \( dA_{\rho_i} \) the areas, we get that

\[
\Delta r(\alpha; f) = \frac{1}{2\pi i} \int_{\alpha} \frac{\tilde{\nabla} f}{\nabla f} d\alpha + \frac{1}{2\pi} \left[ \int G K^{f'^*(\rho_2)} dA_{f'^*(\rho_2)} - \int G K_{\rho_1} dA_{\rho_1} \right].
\]

where \( G \) is the domain bounded by \( \alpha \).
Remark 12. The first double integral doesn’t make sense on the image if $f \circ \alpha$ is not simple; this is why we look at the (possibly singular) metric $f^*(\rho_2)$ on the domain. Secondly, even if $f^*(\rho_2)$ is singular, $K_{f^*(\rho_2)}$ will not be (this follows from the fact that $\Delta \log |f'| = 0$). Except at a discrete set of points where $f' = 0$ and $K f^*(\rho_2) dA_{f^*(\rho_2)}$ is not defined, we have that $d(\Gamma^{\rho_2} \circ f \cdot f' dz) = K_{f^*(\rho_2)}$. So the formula above is valid.

Now by proposition 9, with $\exp \phi = f^*(\rho_2)/\rho_1$, denoting by $k_{\rho_i}$ the geodesic curvatures of $\alpha$, and the arc length element by $ds_{\rho_i}$,

$$
\int_\alpha (k_{f^*(\rho_2)} ds_{f^*(\rho_2)} - k_{\rho_i} ds_{\rho_i}) = \int_\alpha 2 \text{Im} \left( \frac{\partial \phi}{\partial z} \right) dz \tag{5.8}
$$

$$
= \frac{1}{i} \int_\alpha 2 \frac{\partial \phi}{\partial z} dz \tag{5.9}
$$

$$
= \frac{1}{i} \int_\alpha \frac{\nabla f_2}{\nabla f_1} dz. \tag{5.10}
$$

So we recognize in (5.7) the expression for the difference in rotation index that would result from applying the Gauss-Bonnet theorem in the domain and image:

$$
\Delta r(\alpha; f) = r(f \circ \alpha) - r(\alpha)
\begin{align*}
&= \frac{1}{2\pi} \left( \iint_G K f^*(\rho_2) dA_{f^*(\rho_2)} + \int_\alpha k_{f^*(\rho_2)} ds_{f^*(\rho_2)} \right) - \frac{1}{2\pi} \left( \iint_G K \rho_1 dA_{\rho_1} + \int_\alpha k_{\rho_1} ds_{\rho_1} \right).
\end{align*}
$$

Choosing $\rho_1 = \rho_2 = 1$ in (5.7) gives the ordinary argument principle. We now can prove

Theorem 1. (Pommerenke) If $f$ is locally univalent, then, with $\lambda(z) = 1/1 - |z|^2$,

$$
\sup_{z \in D} \left| \frac{1}{\lambda} \nabla f_2 \right| \geq 2,
$$

where $\nabla f$ is associated to the hyperbolic metric on the domain and the Euclidean metric on the image.
Proof. Assume for some $c < 2$ that $\forall z \in D$,
\[
\left| \frac{1}{\lambda} \frac{\tilde{\nabla}_2 f}{\nabla_1 f} \right| \leq c.
\]
Then, choosing $\rho_1 = \lambda$, $\rho_2 = 1$, we have $K^\lambda = -4$, $k'^{(1)} = 0$, and
\[
\Delta r(\alpha; f) = \frac{1}{2\pi i} \int \frac{1}{\lambda} \frac{\tilde{\nabla}_2 f}{\nabla_1 f} \lambda dz + \frac{2}{\pi} \iint dA_\lambda.
\]
So
\[
|\Delta r(\alpha; f)| \geq \frac{1}{2\pi} \left( 4A_\lambda(G) - c l_\lambda(\alpha) \right),
\]
where $A_\lambda(G)$ is the hyperbolic area of the domain $G$ enclosed by $\alpha$, and $l_\lambda(\alpha)$ the hyperbolic length of $\alpha$. Choosing $\alpha$ to be a circle about the origin of radius $r$,
\[
A_\lambda(G) = \frac{\pi r^2}{1 - r^2}, \quad l_\lambda(\alpha) = \frac{2\pi r}{1 - r^2},
\]
so
\[
|\Delta r(\alpha; f)| \geq \frac{1}{2\pi} \left( \frac{4\pi r^2}{1 - r^2} - \frac{2\pi cr}{1 - r^2} \right)
= \frac{r}{1 - r^2} (2r - c) > 0.
\]
if $1 > r > c/2$. This contradicts the fact that $f$ is locally univalent, since in this case $\Delta r(\alpha; f) = 0$. \hfill \square

5.3 Higher order Schwarzian tensors

In [17] Osgood and Stowe defined a Schwarzian tensor arising from a conformal change of metric on a Riemannian manifold. If $(M, g)$ is a Riemannian manifold, $\dim M \geq 2$, then for
a conformal change of metric \( \hat{g} = e^{2\phi} g \),

\[
B_g(\phi) = \text{Hess}(\phi) - d\phi \otimes d\phi - \frac{1}{n} (\delta \phi - \| \text{grad}_g \phi \|_g) g.
\]

The third term makes the tensor trace-free. \( B_g(\phi) \) has a natural interpretation in terms of representation theory as follows. The space of Riemann curvature tensors has a decomposition into three irreducible \( O(2, \mathbb{R}) \)-submodules. These correspond to the scalar curvature, the traceless Ricci tensor, and the Weyl curvature tensor. The tensor \( B_g(\phi) \) is just the difference of the traceless Ricci curvatures under conformal change of metric \( \phi \); for details see [17].

We will be concerned only with the case \( n = 2 \). In this case Osgood and Stowe [18], as well as Chuaqui [4], used the geometric formulation of the Schwarzian to unify most existing sufficient conditions for univalence. A geometric unification of existing sufficient conditions was also achieved earlier by Epstein using a different approach [8], [9].

In order to completely characterize univalence, higher order information is necessary (as will be discussed in Section 6.1). This is not surprising because of the comparative lack of rigidity of conformal geometry in two dimensions. Here we define natural higher-order Schwarzian tensors, which have an interpretation in terms of a component of derivatives of curvature on an \( O(2, \mathbb{R}) \)-irreducible submodule.

First, we make a slight change of notation from Osgood and Stowe. An easy computation shows that if \( g = \rho^2_1 g_0 \), then

\[
B^2_g(\phi) = \text{Re} \left[ (2\phi_{zz} - 2\phi_z^2 - 2\Gamma^p_{\mu} \phi_z) dz \otimes dz \right].
\]
Definition 6. Let \((M, g)\) be a planar domain with metric \(g = \rho^2 g_0\), and let \(\hat{g} = e^{2\phi} g\) be a conformal change of metric. The tensors \(B^\rho_g(\phi)\) are defined by

\[
B^\rho_g(\phi) = (2\phi_{zz} - 2\phi_z^2 - 2\Gamma^\rho\phi_z) dz \otimes dz
\]
\[
B^{\rho+1}_g(\phi) = \tilde{\nabla}^\rho B^\rho_g(\phi) - 2n\phi_z dz \otimes B^n_g(\phi).
\]

Remark 13. It can easily be checked that these tensors are independent of conformal parameter \(z\) (using the fact that for \(\rho(z)|dz| = \hat{\rho}(w)|dw|\) and \(w = f(z)\), \(\Gamma^\rho \circ f \cdot f' = \Gamma^\rho - \frac{L^\rho}{f'}\)).

Now let \(\phi\) be given by the pullback of a metric on another domain under a holomorphic map \(f: (M_1, \rho^2 g_0) \rightarrow (M_2, \sigma^2 g_0)\), e.g. \(\phi = \log \frac{\sigma(f)|f'|}{\rho}\) so that \(f^*(\sigma^2 g_0) = e^{2\phi} \rho^2 g_0\). We will sometimes denote \(f^*(\sigma) = (\sigma \circ f) \cdot |f'|\). In this case,

\[
B^\rho_g(\phi) = \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f} - \frac{3}{2} \left( \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f} \right)^2.
\]

Definition 7. Let

\[
s_3^\rho(\sigma)(f) dz^{n-1} = \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f} - \frac{3}{2} \left( \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f} \right)^2
\]
\[
s_{n+1}^\rho(\sigma)(f) dz^n = \tilde{\nabla}^\rho(\sigma_{n+1}^\rho(\sigma)(f) dz^{n-1}) - (n-1) \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f} \sigma_{n+1}^\rho(\sigma)(f) dz^{n-1}.
\]

Remark 14. Using the fact that \(2\phi_z = \frac{\tilde{\nabla}^\rho f}{\nabla^\rho f}\), it immediately follows that for this choice of \(\phi\),

\[
B^n_g(\phi) = \sigma_{n+1}^\rho(\sigma)(f) dz^n.
\]

We will be concerned in particular with the tensors \(\sigma_{n+1}^\rho(\sigma)(f) dz^{n-1}\) where \(\lambda(z)\) is the hyperbolic line element on the unit disc. Note that for this choice of metric on the domain and image, we have that
$B_g(\phi) = \sigma^\lambda_{n+1}(f)dz^n$. 

These can be written entirely in terms of the hyperbolic metric of the image domain. To see this, note that if $f$ is a univalent map $f : D \rightarrow \mathbb{C}$ onto a domain $\Omega$, then the hyperbolic line element on $\Omega$ is given by 

$$\lambda_\Omega(w) = \frac{1}{|f' \circ f^{-1}|(1 - |f^{-1}(w)|^2)}.$$ 

The quantity 

$$-S^2(\lambda_\Omega) \circ f \cdot f'^2 dz^2 = \sigma^\lambda_3(f)dz^2$$ 

is entirely in terms of the hyperbolic metric. Extending Minda's notation [14] and letting 

$$S^{n+1}(\lambda_\Omega) = \frac{\partial}{\partial w} S^n(\lambda_\Omega),$$ 

we get that 

$$-S^n(\lambda_\Omega) \circ f \cdot f'^m dz^n = \sigma^\lambda_{n+1}(f)dz^n.$$ 

So the higher order Schwarzians $\sigma^\lambda_n(f)$ can be written entirely in terms of the hyperbolic metric of the image domain, and are thus very natural to study. Section 6.2, gives necessary conditions for a function $f$ to be univalent in terms of $\sigma^\lambda_n(f)$. 

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Chapter 6

Estimates for higher order Schwarzian derivatives

6.1 Introduction

Let \( \tilde{S} \) denote the class of holomorphic functions \( \{ f : D \to \mathbb{C} : f \neq 1 \} \). Let

\[
S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2
\]

denote the Schwarzian derivative, and \( \lambda(z) = \frac{1}{(1-|z|^2)} \) be the hyperbolic line element on the disc. It has long been known that univalence is partly characterized by bounds on the Schwarzian derivative. Nehari [16] proved that \( f \in \tilde{S} \Rightarrow |S(f)| \leq 6\lambda(z)^2 \), and also proved that if \( f \) is holomorphic on \( D \), then \( |S(f)| \leq 2\lambda(z)^2 \) implies that \( f \) is one to one. Aharanov and Harmelin [10] studied higher order Schwarzian derivatives \( \psi_n(f) \) with invariance under composition on the left by Möbius transformations \( T \), \( \psi_n(T \circ f) = \psi_n(f) \), and their relation to univalence. Zemyan [26] derived sharp estimates on the coefficients of the Schwarzian,
which amounts to pointwise estimates on these invariants. Lavie [13] improved Nehari's sufficient condition in terms of these kinds of invariants. Tamanoi [25] considered some of the combinatorial and algebraic aspects.

In this section a differential equation for the Loewner flow of the Schwarzian derivative of a univalent map is derived. This is used to give bounds on higher order Schwarzians $\sigma_n(f)$ which are invariant under composition on the right with Möbius transformations; these bounds are sharp for suitable transforms of the Koebe function. The proof is of interest for its simplicity; it is very similar to Loewner's proof of the bounds on coefficients of inverse functions [24]. Bertilsson [1] recently derived sharp bounds on another series of Schwarzians with this invariance property as a corollary of estimates on coefficients of negative powers of the derivative, using the methods of deBranges. (I am grateful to David Minda for bringing this to my attention). We also derive two-point geometric distortion theorems for univalent functions in terms of higher order Schwarzians, making use of some interesting properties of the extremal functions.

**Definition 8.** The higher order Schwarzian derivatives are defined inductively as follows. For $f : D \to \mathbb{C}$, let

$$\sigma_3(f) = \frac{f'''(z)}{f''(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

and

$$\sigma_{n+1}(f) = \sigma_n(f)' - (n-1) \frac{f''(z)}{f'(z)} \sigma_n(f).$$

For example,

$$\sigma_4(f) = \frac{f''''(z)}{f''(z)} - 6 \frac{f'''(z)f''(z)}{f'(z)^2} + 6 \left( \frac{f''(z)}{f'(z)} \right)^3$$
\[
\sigma_5(f) = \frac{f'''''}{f'} - 10 \frac{f'''' f''}{f''^2} - 6 \left(\frac{f'''}{f'}\right) + 48 \frac{f''' f''^2}{f'^3} + 36 \left(\frac{f''}{f'}\right)^4.
\]

These are just the expressions \(\sigma_n^{\lambda, f}\) of Section 5.3. They can also be regarded as \(\nabla f'(z) (\sigma_3(f) dz \otimes dz)\).

It is easy to show by induction that for all Möbius transformations \(R\), the following invariance property holds:

\[
\sigma_n(f \circ R) = (\sigma_n(f) \circ R) \cdot (R')^{n-1}.
\]

In particular for disc automorphisms \(T\) and affine maps \(S\),

\[
\lambda(T'(z))^{n-1} |\sigma_n(S \circ f \circ T)(z)| = \lambda^{n-1}(z)|\sigma_n(f) \circ T(z)|.
\]

This invariance is suited to families of functions whose domain is fixed as the disc \(D\); for instance for the family \(\mathcal{S}\), we have that \(\sup_{f \in \mathcal{S}} \lambda(z)^{n-1}|\sigma_n(f)(z)|\) is independent of \(z\). Derivatives with this invariance property relate formally via a simple identity to derivatives with invariance under post-composition with Möbius transformations. Namely,

\[
\left(\frac{\partial^{n-3}}{\partial z^{n-3}} \sigma_3(f^{-1})\right) \circ f = \frac{\sigma_n(f)}{f'^{n-1}}. \tag{6.1}
\]

In the first section we derive the differential equation for the Schwarzian of the Loewner flow of a univalent function \(f\) and use this to prove the bounds on \(|\sigma_n(f)|\). In the second section, we derive the geometric two-point distortion theorems directly from some elementary observations about the extremal functions.
6.2 A sharp estimate for $\sigma_n(f)$

In this section we derive a differential equation for the Schwarzian derivative as a corollary of the Loewner differential equation. Using this it is possible to prove that

$$\sup_{f \in \tilde{S}} \lambda(z)^{n-1}|\sigma_n(f)(z)| = 6 \cdot 4^{n-3}(n-2)!.$$  

This is sharp for a suitable transform of the Koebe function.

We briefly summarize the Loewner method here (for details and theorems see [21], [22]). Let $S$ denote the class of functions in $\tilde{S}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. Every $f \in S$ can be embedded in a Loewner chain; that is, there is a map $f : D \times \mathbb{R}^+ \rightarrow G_t \subset \mathbb{C}$ with the following properties:

i) $f_t(0) = 0$, $f'_t(0) = e^t$, and $f_0 = f$,

ii) Each $f_t$ is analytic and univalent on $D$,

iii) $\lim_{t \to 0} f_t = f_{t_0}$ locally uniformly,

iv) $G_s \subseteq G_t$ whenever $s \leq t$,

In the following, dot denotes differentiation with respect to time and prime denotes differentiation with respect to $z$. Define the functions $\Phi(t, \zeta) = f_t \circ f^{-1}(\zeta)$, with domain $f(D)$ for each fixed $t$. For fixed $z$, the function $\Phi(z, \cdot)$ is absolutely continuous on compact intervals, so the time derivative exists almost everywhere. Furthermore, for all $t$ at which it exists,

$$\lim_{h \to 0} \frac{\Phi(\zeta, t + h) - \Phi(\zeta, t)}{h} = \Phi_t(\zeta, t)$$

locally uniformly. Also, $\Phi(\zeta, 0) = \zeta$, and $\lim_{t \to \infty} \Phi(e^{-t}\zeta, t) \rightarrow f^{-1}$ locally uniformly, and $\Phi$
satisfies the Loewner differential equation

\[ \dot{\Phi}(\zeta, t) = \zeta p(\zeta, t) \Phi'(\zeta, t) \]  \hspace{1cm} (6.2)
a.e., where \( \Re(p) \geq 0, p(0, t) = 1 \).

**Theorem 2.** Let \( \Phi \) and \( p \) be as above. Then,

i) \( \sigma_3(\Phi)(\zeta, 0) = 0 \), \( \lim_{t \to \infty} \sigma_3(f(e^{-t}\Phi))(\zeta, t) = \sigma_3(f^{-1})(\zeta) \) locally uniformly,

ii) \( \lim_{t \to t_0} \sigma_3(\Phi)(\zeta, t) = \sigma_3(\Phi)(\zeta, t_0) \) locally uniformly,

iii) \( \lim_{h \to 0} \left[ \sigma_3(\Phi)(\zeta, t + h) - \sigma_3(\Phi)(\zeta, t) \right] / h \) exists a.e., and where it exists, the limit converges locally uniformly.

iv) \( \sigma_3(\Phi_t) \) satisfies the differential equation

\[ \sigma_3(\Phi) = \zeta p \sigma_3(\Phi)' + (2p + 2 \zeta p') \sigma_3(\Phi) + (3p'' + \zeta p''') \]

**Proof.** i) follows from the fact that \( \Phi(\zeta, 0) = \zeta \), and \( \lim_{t \to \infty} \Phi(e^{-t}\zeta, t) = f^{-1}(\zeta) \) locally uniformly. Similarly ii) holds since \( \Phi(\zeta, t) \to \Phi(\zeta, t_0) \) locally uniformly, using Cauchy estimates and the fact that \( \Phi'(0) \neq 0 \) so that \( \Phi'(0)^{-1} \) is uniformly bounded away from zero on compact sets in \( \zeta \) and \( t \). To show iii), apply Cauchy estimates to \( \left[ \Phi^{(j)}(\zeta, t + h) - \Phi^{(j)}(\zeta, h) \right] / h \) for \( j = -1, 1, 2, 3 \), using the fact that \( \lim_{h \to 0} \left[ \Phi(\zeta, t + h) - \Phi(\zeta, t) \right] / h \) converges locally uniformly. This also justifies interchanging differentiation with respect to \( t \) and \( \zeta \), and the following calculations can be made using the differential equation (6.2) in order to prove iv).

\[
\begin{align*}
\dot{\Phi}' &= \Phi'(p + \zeta p') + \Phi''\zeta p \\
\dot{\Phi}'' &= \Phi'(2p' + \zeta p'') + \Phi''(2p + 2\zeta p') + \Phi''\zeta p \\
\dot{\Phi}''' &= \Phi'(3p'' + \zeta p''') + \Phi''(6p' + 3\zeta p'') + \Phi'''(3p + 3\zeta p') + \Phi'''''\zeta p.
\end{align*}
\]
Also,

\[ \frac{\partial}{\partial t} \left( \frac{\Phi'''}{\Phi'} - \frac{3}{2} \left( \frac{\Phi''}{\Phi'} \right)^2 \right) = \frac{\Phi'''}{\Phi'} - \frac{\Phi'''}{\Phi'} \frac{\Phi'}{\Phi'} - 3 \frac{\Phi''}{\Phi'} \frac{\Phi'}{\Phi'} + 3 \left( \frac{\Phi''}{\Phi'} \right)^2 \frac{\Phi'}{\Phi'} . \]  

(6.4)

Plugging (6.3) into (6.4),

\[ \sigma_3(\Phi) = \zeta p \frac{\Phi'''}{\Phi'} + (3p + 3\zeta p') \frac{\Phi'''}{\Phi'} + (6p' + 3\zeta p'') \frac{\Phi'''}{\Phi'} + (3p'' + \zeta p''') \]
\[ - \frac{\Phi'''}{\Phi'} \left( (p + \zeta p') + \zeta p \frac{\Phi'''}{\Phi'} \right) \]
\[ - 3 \frac{\Phi'''}{\Phi'} \left( (2p' + \zeta p'') + (2p + 2\zeta p') \frac{\Phi'''}{\Phi'} + \zeta p \frac{\Phi'''}{\Phi'} \right) \]
\[ + 3 \left( \frac{\Phi''}{\Phi'} \right)^2 \left( (p + \zeta p') + \zeta p \frac{\Phi''}{\Phi'} \right) \]
\[ = \zeta p \left( \frac{\Phi'''}{\Phi'} - 4 \frac{\Phi'''}{\Phi'} \frac{\Phi''}{\Phi'} + 3 \left( \frac{\Phi''}{\Phi'} \right)^3 \right) \]
\[ + (2p + 2\zeta p') \left( \frac{\Phi''}{\Phi'} - 3 \frac{\Phi''}{\Phi'} \right)^2 \]
\[ + (3p'' + \zeta p''') . \]

\[ \square \]

In order to prove the bounds on \( \sigma_n(f) \), we will need Caratheodory's lemma, which states that the coefficients \( c_n \) of an analytic function \( p \) on the unit disc with \( p(0) = 1 \) and \( \text{Re}(p(z)) \geq 0 \) satisfy \( |c_n| \leq 2, \ n \geq 2 \). For all \( n \), equality \( |c_1| = \cdots = |c_n| = 2 \) holds iff \( p \) is given by

\[ p(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} = 1 + 2e^{i\theta}z + 2e^{2i\theta}z^2 + \cdots . \]

If we let \( p(z, t) \) be one of the extremal functions given above, then the solution \( \Phi \) of the differential equation (6.2) generates the inverse of a rotation of the Koebe function \( \tilde{k}(z) = \frac{z}{(1 - e^{i\theta}z)^2} \) as \( t \to \infty \) (see [24]).
Theorem 3.

$$\sup_{f \in \mathcal{S}} \lambda(z)^{n-1} |\sigma_n(f)(z)| = 6 \cdot 4^{n-3} (n-2)!.$$ 

This is sharp iff \( f = S \circ k \circ T \) where \( k \) is the Koebe function, \( S \) an affine transformation, and \( T \) a disc automorphism.

Proof. Using the invariances of \( \sigma_n(f) \), it suffices to prove this at the origin for functions with \( f(0) = 0 \) and \( f'(0) = 1 \). Let \( \Phi \) be the transition function for a normalized Loewner flow of \( f \), and \( p \) be the infinitesimal generator, as above. Now

$$\sigma_3(\Phi) = s_0(t) + s_1(t)\zeta + s_2(t)\zeta^2 + \cdots , \quad (6.5)$$

where

$$s_n(t) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} \sigma_3(\Phi)(0).$$

By Theorem 2,

$$\lim_{t \to \infty} e^{-(n+2)t} s_n(t) = \frac{1}{n!} \left( \frac{\partial^n}{\partial z^n} \sigma_3(f^{-1}) \right)(0).$$

By (6.1), and the normalization \( f'(0) = 1 \),

$$\lim_{t \to \infty} e^{-(n+2)t} s_n(t) = \frac{1}{n!} \sigma_{n+3}(f)(0). \quad (6.6)$$

We can expand the differential equation of Theorem 2 in a power series and equate coefficients. Suppressing \( t \),

$$(\sigma_3(\Phi)) = \delta_0 + \delta_1 \zeta + \delta_2 \zeta^2 + \cdots , \quad (6.7)$$
and

\[
\zeta p\sigma_3(\Phi)' = \zeta \left( \sum_{n=0}^{\infty} c_n \zeta^n \right) \left( \sum_{n=0}^{\infty} (n+1) s_{n+1} \zeta^n \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_{n-k} (k+1) s_{k+1} \right) \zeta^{n+1} \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_{n-k} (k+1) s_{k+1} \right) \zeta^{n+1}
\]  \hspace{1cm} (6.8)

Also,

\[
(2p + 2\zeta p') = 2 \left( \sum_{n=0}^{\infty} c_n \zeta^n + \sum_{n=1}^{\infty} n c_n \zeta^n \right) = 2 \left( \sum_{n=0}^{\infty} (n+1) c_n \zeta^n \right).
\]

So

\[
(2p + 2\zeta p')\sigma_3(\Phi) = 2 \left( \sum_{n=0}^{\infty} (n+1) c_n \zeta^n \right) \left( \sum_{n=0}^{\infty} s_n \zeta^n \right) \\
= 2 \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (n+1-k) c_{n-k} s_k \right) \zeta^n,
\]

\[
3p'' + \zeta p''' = 3 \sum_{n=2}^{\infty} n(n-1)c_n \zeta^{n-2} + \zeta \sum_{n=3}^{\infty} n(n-1)(n-2)c_n \zeta^{n-3} \\
= \sum_{n=2}^{\infty} (n+1)n(n-1)c_n \zeta^{n-2} \\
= \sum_{n=0}^{\infty} \frac{(n+3)!}{n!} c_{n+2} \zeta^n.
\]

So applying Theorem 2 to the expansion (6.5) using (6.7), (6.8) (6.9), and (6.9), and equating coefficients, we get the following simple recursive differential equation for \( s_n \).

\[
\dot{s}_n = \sum_{k=0}^{n} (2(n+1) - k) c_{n-k} s_k + \frac{(n+3)!}{n!} c_{n+2}, \hspace{1cm} (6.9)
\]

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or, bringing the \( k = n \) term to the left hand side,

\[
\frac{d}{dt} \left( e^{-(n+2)t} s_n \right) = e^{-(n+2)t} \left( \sum_{k=1}^{n} (2n + 2 - k) c_{n-k} s_k + \frac{(n + 3)!}{n!} c_{n+2} \right).
\] (6.10)

So (compare Schober [24])

\[
e^{-(n+2)t} s_n = \int_0^t e^{-(n+2)\tau} \left( \sum_{k=1}^{n} (2n + 2 - k) c_{n-k}(\tau) s_k(\tau) + \frac{(n + 3)!}{n!} c_{n+2}(\tau) \right) d\tau.
\] (6.11)

By Caratheodory's lemma, each \( c_k \) is bounded in modulus by 2. Inductively, we see that the maximum of the right hand side occurs for each fixed \( t \) iff \(|c_1| = \cdots |c_n| = |c_{n+2}| = 2\). By the discussion preceding the theorem, then, \( p(z, t) = \frac{1+2e^{it}}{1-2e^{it}} \) for all \( t \) (it is not hard to check that the maximum is indeed attained for all of these). Thus,

\[
|\sigma_n(f)(0)| = |\lim_{t \to \infty} e^{-(n+2)t} s_n(t)| \leq |\sigma_n(\tilde{k})(0)|
\] (6.12)

where \( \tilde{k} \) is a rotation of the Koebe function. A calculation shows that

\[
|\sigma_n(\tilde{k})(0)| = 6 \cdot 4^{n-3}(n - 2)!.
\]

\[\square\]

### 6.3 Geometric two-point distortion theorems

**for the class \( \tilde{S} \)**

We now derive a two-point distortion theorem for the higher order Schwarzians. First it is necessary to derive a two-point geometric version of the classical distortion theorem for the class \( S \). Though easy to prove, it is of interest in its own right. It is similar to a two-point
reformulation of the classical growth theorem due to Kim and Minda [15] (which is in fact also sufficient for univalence).

Theorem 4. If \( f \in \tilde{S} \), then \( \forall z_1, z_2 \in D, \)

\[
\exp(-4d_\lambda(z_1, z_2)) \leq \frac{|D_1f(z_2)|}{|D_1f(z_1)|} \leq \exp(4d_\lambda(z_1, z_2)),
\]

where \( d_\lambda(z_1, z_2) \) is the hyperbolic distance between \( z_1 \) and \( z_2 \). This is sharp in the following way: fix \( z_1 \), and let \( z_2 \) vary along a hyperbolic geodesic through \( z_1 \). If \( T \) is any disc automorphism taking this geodesic to the real axis, \( S \) any affine transformation of the plane, and \( k \) the Koebe function, then the function \( \tilde{k} = S \circ k \circ T \) takes on the upper bound if \( T(z_2) \geq T(z_1) \), and the lower bound if \( T(z_1) \geq T(z_2) \). Furthermore for this function, \( \arg \frac{D_1f(z_1)}{D_1f(z_2)} \) is constant while \( z_2 \) remains on one side of \( z_1 \).

Proof. Let \( T_\alpha(z) = \frac{z+\alpha}{1+\alpha z} \). Given \( f \in \tilde{S} \), the function

\[
g(z) = \frac{f \circ T_\alpha(z) - f \circ T_\alpha(0)}{(f \circ T_\alpha)'(0)}
\]

is in \( S \), so the second coefficient of \( g \), namely \( \frac{1}{2} \frac{D_2f(z)}{D_1f(z)} \), is bounded in modulus by 2. Thus \( \forall f \in \tilde{S}, \left| \frac{D_2f(z)}{D_1f(z)} \right| \leq 4 \). Now by an easy computation

\[
d \log |D_1f| = \Re \left( \lambda \frac{D_2f(z)}{D_1f(z)} dz \right).
\]

So for all smooth curves \( \gamma \) joining \( z_1 \) to \( z_2 \),

\[
\log |D_1f(z_2)| - \log |D_1f(z_1)| = \int_\gamma \Re \left( \lambda \frac{D_2f(z)}{D_1f(z)} dz \right).
\]  

(6.13)

Now for the Koebe function \( k, \frac{D_2k(z)}{D_1k(z)} = 4 \forall z \in (-1, 1) \) (i.e. \( \lambda(z) \frac{D_2k(z)}{D_1k(z)} dz \) is parallel along the real axis). Let \( \alpha \) be the geodesic segment joining \( z_1 \) to \( z_2 \) (parametrized with respect to arc
length, increasing from $z_1$ to $z_2$). Let $T$ be a disc automorphism taking $\alpha$ to the real line, with $T(z_2) \geq T(z_1)$. Then $k \circ T$ satisfies (using (4.1))

$$\text{Re} \left( \lambda \circ \alpha \frac{D_2(k \circ T) \circ \alpha}{D_1(k \circ T) \circ \alpha} \right) = \text{Re} \left( \lambda \circ \alpha \frac{D_2(k \circ T) \circ \alpha}{D_1(k \circ T) \circ \alpha} \right)'$$

$$= \text{Re} \left( \lambda \circ (T \circ \alpha) \frac{D_2(k \circ T) \circ \alpha}{D_1(k \circ T) \circ \alpha} (T \circ \alpha)' \right)$$

$$= 4.$$  

Similarly, if we choose $T$ such that $T(z_2) \leq T(z_1)$, we get that

$$\text{Re} \left( \lambda \circ \alpha \frac{D_2(k \circ T) \circ \alpha}{D_1(k \circ T) \circ \alpha} \right) = -4.$$  

So when the lower and upper bound of the right hand side of (6.13) are taken on, it is along the entire geodesic segment. Since the integral of $d \log |D_1 f|$ is independent of path, we can fix the path as a hyperbolic geodesic $\alpha$ connecting $z_1$ to $z_2$. We get that the maximum and minimum of the integral (6.13) must be taken on by these functions, so

$$- \int_\alpha 4|dz| \leq \log \left| \frac{D_1 f(z_2)}{D_1 f(z_1)} \right| \leq \int_\alpha 4|dz|,$$

which implies that

$$-4d_\lambda(z_1, z_2) \leq \log \left| \frac{D_1 f(z_2)}{D_1 f(z_1)} \right| \leq 4d_\lambda(z_1, z_2).$$

That these are all the sharp functions follows directly from the fact that the only functions in $S$ with $|a_2|=2$ are the Koebe function and its rotations.

We now use this to get a two point distortion theorem involving $\sigma_n(f)$, also in terms of hyperbolic distance.
Theorem 5. Let $c_n = 4^{n-3}(n - 2)!6$. Then,

$$
\left| \frac{f'(z_1)^{n-1}}{f'(z_2)^{n-1}} \sigma_n(f)(z_2) - \sigma_n(f)(z_1) \right| \leq c_n \lambda(z_1)^{n-1} \exp (4(n - 1)d_\lambda(z_1, z_2))
$$

for all $f \in \hat{S}$. This bound is sharp only for $f = S \circ k \circ T$.

Proof. We have

$$
\frac{\partial}{\partial z} \left( \frac{\sigma_n(f)}{f^{n-1}} \right) = \frac{\sigma_{n+1}(f)}{f^{n-1}},
$$

so

$$
\left| \frac{f'(z_1)^{n-1}}{f'(z_2)^{n-1}} \sigma_n(f)(z_2) - \sigma_n(f)(z_1) \right| = \left| \int_{z_1}^{z_2} \frac{f'(z_1)^{n-1}}{f'(z)^{n-1}} \sigma_{n+1}(f)(z) \, dz \right|
$$

Assume $|\sigma_{n+1}(f)(z)| \leq c_{n+1} \lambda(z)^n$ for all $f \in \hat{S}$. First we observe that it suffices to estimate the right hand side fixing the path of integration as a hyperbolic geodesic. Denote this again by $\alpha$, travelling from $z_1$ to $z_2$. Next, using

$$
\sigma_n(k)(z) = \frac{(-1)^n c_n}{(1 - z^2)^{n-1}},
$$

an argument similar to that in the last theorem shows that for a disc automorphism $T$ taking $\alpha$ to the real axis, the function $\tilde{k} = k \circ T$ satisfies

$$
|\sigma_{n+1}(\tilde{k})| \circ \alpha = c_{n+1}(\lambda \circ \alpha)^n,
$$

and along $\alpha$,

$$
\arg \left( \frac{\tilde{k}'(z_1)^{n-1}}{\tilde{k}'(z)^{n-1}} \sigma_{n+1}(\tilde{k})(z) \, dz \right)
$$

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is constant. Applying Theorem 4, we get
\[
\left| \int f'(z_1)^{n-1} \sigma_{n+1}(f)(z) \, dz \right| \leq \int \exp \left( 4(n-1)d_{\lambda}(z_1, z) \right) \frac{\lambda(z_1)^{n-1}}{\lambda(z)^{n-1}} c_{n+1} \lambda(z)^n \, dz | \lambda(z) | \, dz |
\]
\[
= \lambda(z_1)^{n-1} \int \exp \left( 4(n-1)d_{\lambda}(z_1, z) \right) c_{n+1} \lambda(z) \, dz | \lambda(z) | \, dz |
\]

To integrate the right hand side, let
\[
x(t) = \left| \frac{\alpha(t) - z_1}{1 - z_1 \alpha(t)} \right| = T \circ \alpha(t),
\]
where \( T \) is the disc automorphism above, and note that
\[
\dot{x}(t) = T'(\alpha(t)) \dot{\alpha}(t) = |T'(\alpha(t))| |\dot{\alpha}(t)|.
\]
Also let \( u(x) = \frac{1}{2} \log \frac{1+x}{1-x} \); then \( u(x(t)) = d_{\lambda}(\alpha(t), z_1) \), and
\[
\frac{du}{dt} = \frac{1}{1-x^2} T'(\alpha(t)) \dot{\alpha}(t) = \frac{|\dot{\alpha}(t)|}{1 - |\alpha(t)|^2}
\]
\[
= \lambda(\alpha(t)) |\dot{\alpha}(t)|.
\]
Thus,
\[
\int \exp 4(n-1)d_{\lambda}(z, z_1) c_{n+1} \lambda(z) \, dz | \lambda(z) | \, dz |
\]
\[
= c_{n+1} \int_0^{\alpha^{-1}(z_2)} \exp 4(n-1)u \circ x(t) d(u \circ x)
\]
\[
= c_n \exp 4(n-1)d_{\lambda}(z_1, z_2).
\]

It is easy to see that this theorem also implies Theorem 3.
Bibliography


