Abstract

A Mean-Value Laplacian for Finsler Spaces

Paul Centore

Ph. D. thesis (1998), University of Toronto

Part I of this thesis defines a Laplacian Δ for a Finsler space; we obtain Δ by requiring that $(\Delta f)(x)$ for a function f measures the infinitesimal average of f around x . This Δ is a linear, elliptic, 2nd-order differential operator. Furthermore, Δf can be written in a divergence form, like the Riemannian Laplacian, but with respect to a canonical osculating Riemannian metric and Busemann's intrinsic volume form. We interpret divergence form as the result of minimizing a certain energy functional on Finsler space, and further use this approach to define harmonic forms, and harmonic mappings between Finsler manifolds. As a byproduct of the Laplacian, in Part II we derive a simple volume-form inequality which characterizes Riemannian manifolds, and define a scalar invariant $\mathcal{V}(x)$ for Finsler spaces. We show that, on a Berwald space, the metric's first derivatives vanish in normal co-ordinates, and use that result to conclude that $\mathcal{V}(x)$ is constant on Berwald spaces.

Table of Contents

INTRODUCTION: MAIN RESULTS	1
PART 1: THE MEAN-VALUE LAPLACIAN	
I. Notation	5
II. Geometric Motivation	6
III. Concomitants of the Finsler Metric	8
IV. Existence of the Laplacian and a Co-ordinate Expression for It	17
V. Our Laplacian Generalizes the Riemannian Laplacian	19
VI. Properties of the Laplacian	25
VII. The Laplacian on Forms, and Harmonic Mappings	35
PART 2: A VOLUME INVARIANT FOR FINSLER SPACES	
VIII. Characterizing Riemann Spaces via Volumes: the Volume Invariant	47
IX. Normal Co-ordinates in Finsler Geometry	53
X. Geometric Objects in Normal Co-ordinates	61
XI. The Derivatives of F^2 Vanish in Berwald Normal Co-ordinates	65
XII. The Derivatives of F^2 Vanish: A Geometric Proof	69
XIII. The Volume Invariant is Constant in a Berwald Space	74
REFERENCES	77

INTRODUCTION: MAIN RESULTS

On a Riemannian space, the Laplace operator (both for forms and functions) is a natural and important operator. It leads to the Hodge Decomposition Theorem, which gives topological information about the space, and is essential to investigating the diffusion of heat. These considerations also make sense on the more general Finsler spaces, but so far it is not clear what we should use as a Laplacian on Finsler spaces. In this paper, we seek to generalize the Laplacian (first for functions and then for forms) on a Riemannian space to a Laplacian on a Finsler space. We do this by generalizing an important property of the Laplacian on Riemannian space, and that is that the Laplacian measures (at least infinitesimally) the average value of a function around a point.

Definition. Let M be a differentiable manifold with a Finsler metric F , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. Define the Laplacian Δ at a point p by

$$\Delta f(x) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f \omega(x) dx}{\int_{\Sigma_\varepsilon} \omega(x) dx} - f(x) \right),$$

where $\Sigma_{(x,\varepsilon)}$ is the ball of radius ε about x , and $\omega(x)dx$ is an intrinsic volume form defined by Busemann. $\square$

Let's examine this expression. We note first of all that Δf is invariant, that is, it doesn't depend on the choice of co-ordinates. We note secondly that Δf is linear. There is an obvious question: Is Δ a differential operator? Somewhat surprisingly, perhaps, the answer is yes. To see this, express $f(x)$ and $\omega(x)$ as Taylor series and use the inverse of the exponential map (to be defined precisely later):

$$x^i = X^i - \frac{1}{2} \Gamma_{jk}^i(X) X^j X^k + O(|X|^3),$$

where $|X| = \sqrt{\Sigma(X^i)^2}$, to express the integrals as integrals on the tangent space, instead of on the manifold. We get:

Theorem IV-1. Δf is a linear, elliptic, 2nd order differential operator given by

$$\Delta f = (n+2) \left(\frac{\int_I X^i X^j dX}{\int_I dX} f_{ij} + \right.$$

$$+ \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2\int_I \Gamma_{jq}^i X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right) f_i \Big),$$

where I is the unit indicatrix in the tangent space, and the integrals are evaluated in the tangent space. $\square$

The mean-value Laplacian ties together some disparate strands. The Riemannian Laplacian Δ satisfies several important properties. For example, Δ can be written in a convenient “divergence form”:

$$\Delta f = \frac{1}{\sqrt{g}(x)} \frac{\partial}{\partial x^j} (f_i g^{ij} \sqrt{g}(x)), \quad (0-1)$$

where g^{ij} are Riemannian metric components, and $\sqrt{g}(x)dx$ is the Riemannian volume form. The mean-value Laplacian for Finsler spaces has the same divergence form, and satisfies an equivalent self-adjointness condition.

Theorem VI-2. Δf is given by

$$\Delta f(x) = \frac{1}{\omega(x)} \frac{\partial}{\partial x^i} (f_j K^{ij} \omega(x)),$$

where $\omega(x)dx$ is the volume form of (M, F) , and

$$K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$$

is a canonically associated Riemannian metric. In addition, Δ is self-adjoint, i.e.

$$\int_M u(x) \Delta v(x) \omega(x) dx = \int_M v(x) \Delta u(x) \omega(x) dx,$$

for any smooth functions u and v on a compact manifold M . $\square$

Note that $\omega(x)dx$ does *not* result from the components K^{ij} . Apart from being an infinitesimal mean value, this gives the Laplacian another compelling geometric interpretation: we can see it as the Laplacian on the the osculating Riemannian space, *with respect to a Finsler volume form*. This elegant viewpoint suggests that we have isolated an important operator.

We'll also see another interpretation of the Laplacian: we can think of it as the Euler-Lagrange equations of an energy functional, given by

$$E(f) = \int_M K^{ij} f_i f_j \omega(x) dx.$$

This is a clear generalization of the Riemannian theory of harmonic functions, and provides a method of defining harmonic mappings between two Finsler spaces, as well as a method of defining a Laplacian for forms on a Finsler space. In addition, this formulation gives a handy basis for comparing the known linear Laplacians—they result from different choices for K^{ij} and ω .

The components g^{ij} in expression (1) must take some new form K^{ij} for Finsler space. The determination of the K^{ij} , which turns out to be distinct from the choice of volume form, again leads to a seemingly remote quantity, the coefficients from the “methods of moments” [BCS]. Furthermore, an invariant volume form arising from the K^{ij} characterizes Riemannian spaces, and an inequality allows us to think of Riemannian manifolds as “extremal cases” of Finsler manifolds.

Theorem VIII-2. *Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij} . Then $\omega(x) \geq k(x)$, and $\omega(x) = k(x)$ if and only if (M, F) actually is Riemannian, with metric K^{ij} . $\square$*

This striking property prompts us to define a scalar invariant function, that resembles Bao & Shen's $\text{Vol}(x)$ [BS]:

Definition. *In the notation of Theorem VIII-2, define*

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}. \quad \square$$

$\mathcal{V}(x)$ varies between 0 and 1, taking a constant value 1 only on Riemannian manifolds. Furthermore, $\mathcal{V}(x)$ takes a constant value (different from 1) only on certain spaces. We arrive at these spaces by considering normal co-ordinates on a Finsler manifold. Normal co-ordinates are generally only C^1 at their point of origin, but on Berwald spaces, they are C^2 , which allows us to prove:

Theorem XI-1. *Let (M, F) be a Berwald manifold with a normal co-ordinate system $\bar{x}^i$ around a point p . In this co-ordinate system, for every $X \in T_p M$, and for any k ,*

$$\frac{\partial F^2}{\partial \bar{x}^k}(p, T) = 0. \quad \square$$

Some corollaries of this property suffice to give the final result:

Theorem XIII-1. *Let (M, F) be a Berwald manifold, with volume invariant $\mathcal{V}(x)$. Then $\mathcal{V}(x)$ is constant.* $\square$

PART 1: THE MEAN-VALUE LAPLACIAN

I. Notation

We will use the following notation. M or M^n is a compact manifold of dimension n with local co-ordinates x^i around a point x . If $X \in T_x M$, then $X = X^i \frac{\partial}{\partial x^i}$, where the X^i are co-ordinates for the tangent bundle canonically induced from the x^i for the base manifold. F is a Finsler metric, i.e. a function $F : TM \rightarrow \mathbb{R}$ where

1. F is positive-definite: $F(x, X) \geq 0$, with equality iff $X = \vec{0}$.
2. F is smooth except on the zero-section: $F|_{TM \setminus \{(x, \vec{0}) | x \in M\}}$ is C^∞ .
3. F is strictly convex: at any (x, X) , $\text{rank} \left[\frac{\partial^2 F}{\partial X^i \partial X^j} \right] = n - 1$.
4. F is homogeneous: $F(x, kX) = |k|F(x, X)$, for all $k \in \mathbb{R}$.

Some authors do not demand the full strength of condition 4., but only require *positive* homogeneity: $F(x, kX) = kF(x, X)$, for all $k > 0$. Condition 4, however, will be essential in what follows, because it guarantees that the unit ball

$$I = \{X \in T_x M | F(X) \leq 1\}$$

is centered at $\vec{0}$, i.e.

$$4.' \quad F \text{ is symmetric: } F(x, -X) = F(x, X).$$

In addition to the unit ball, we can consider a ball (in the tangent space $T_x M$) of any radius $\varepsilon > 0$, and we will define the indicatrix I_ε of radius ε to be this (solid) ball:

$$I_\varepsilon = \{X \in T_x M | F(X) \leq \varepsilon\}.$$

Apart from balls I_ε in the *tangent space*, we will see that we can talk about balls Σ_ε on the *manifold*, and we must be careful to distinguish the two.

Associated with any Finsler metric are several commonly used quantities. All these quantities appear also in Riemannian space, and generally have the same interpretation here, but there is an important difference—most important quantities in a Finsler space are not functions solely of M but in fact of TM . This difference is most obvious when we write the argument as (x, X) but in the sequel, to

preserve space, we will often suppress arguments. Some frequently used quantities and relations are:

$$\begin{aligned}
g_{ij}(x, X) &:= \frac{1}{2} \frac{\partial^2 F^2(x, X)}{\partial X^i \partial X^j} \\
F^2(X^i \frac{\partial}{\partial x^i}) &= g_{ij}(x, X) X^i X^j \\
g^{jk} g_{ij} &= \delta_i^k \\
g(x, X) &:= \det [g_{ij}(x, X)] \\
\Gamma_{jk}^i(x, X) &:= \frac{1}{2} g^{ir}(x, X) \left(\frac{\partial g_{jr}}{\partial x^k}(x, X) - \frac{\partial g_{jk}}{\partial x^r}(x, X) + \frac{\partial g_{rk}}{\partial x^j}(x, X) \right) \\
C_{ijk}(x, X) &:= \frac{1}{4} \frac{\partial^3 F^2}{\partial X^i \partial X^j \partial X^k}(x, X)
\end{aligned}$$

In addition, when restricted to a particular tangent space $T_p M$, F^2 is homogeneous of degree 2, and its derivatives (with respect to tangent vector components X^i) are homogeneous of various degrees. Thus Euler's rule ($X^i \frac{\partial H}{\partial X^i} = kH$ if H is a function of X and homogeneous of degree k) generates many relations, such as

$$\begin{aligned}
C_{ijk}(x, X) X^i &= 0 \\
g_{ij}(x, X) X^j &= \frac{1}{2} \frac{\partial F^2}{\partial X^i}.
\end{aligned}$$

We'll use these freely. $f : M \rightarrow \mathbb{R}$ will be a smooth function with derivatives $f_i = \frac{\partial f}{\partial x^i}$, $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$, etc. We are looking for a Laplacian $\Delta f : M \rightarrow \mathbb{R}$.

II. Geometric Motivation

Our general approach to extending the Laplacian to Finsler spaces will be to choose an operator which measures the mean value of a function at a point.

In the Euclidean space $\mathbb{R}^n$, we say a function f is *harmonic*, or has zero Laplacian, if

$$\begin{aligned}
f(x) &= \text{average value of } f \text{ on a solid sphere (of any radius) around } x \\
&= \frac{\int_{\Sigma_\epsilon} f dV}{\int_{\Sigma_\epsilon} dV}
\end{aligned}$$

where

$$\begin{aligned}\Sigma_\varepsilon &:= \{y \in \mathbb{R}^n | d(x, y) \leq \varepsilon\} \\ dV &:= dx^1 \dots dx^n.\end{aligned}$$

Thus

$$\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV} - f(x) = 0$$

on harmonic functions in Euclidean space. To make sense of the expression $\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV}$ for a general function, we must “infinitesimalize,” i.e. consider the above quantity as $\varepsilon \rightarrow 0$. Furthermore, we know from the more usual expression, $\Delta f = f_{ii}$, that we are only interested in the 2nd-order term of this expression, so we should divide by ε^2 as $\varepsilon \rightarrow 0$. In addition, we need a normalizing factor of $2(n+2)$, and we obtain

$$\Delta f(x) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV} - f(x) \right).$$

We would like to extend this to Riemannian space. The extension should be easy, because our definition is based on spheres, which are defined in Riemannian space, and the correct choice of a volume form

$$dV = \sqrt{g} dx.$$

We define, for a Riemannian space,

$$\Delta f(x) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f \sqrt{g} dx}{\int_{\Sigma_\varepsilon} \sqrt{g} dx} - f(x) \right).$$

(In Riemannian space, it is necessary to infinitesimalize even to define harmonic functions. On a Euclidean space, a harmonic function f is a function whose value at any point x equals the average of f on any ball around x . In a Riemannian space, this is generally not true. It holds only on so-called *harmonic spaces*, which are known to contain symmetric spaces and to be contained in Einstein spaces [RWW]. What is true is that the average value of f over a ball of radius ε tends to the Laplacian of f at x as $\varepsilon \rightarrow 0$.)

Now take the final step and extend to a Finsler space. Again, we can talk about spheres because these are well-defined in Finsler spaces, but now we have a difficulty in that we need a canonical volume form. We will see in the next section that there is a canonical volume form, which for now we will denote as $dV = \omega(x)dx$. With this volume form, we can define Δf on a Finsler space:

Definition. Let M be a differentiable manifold with a Finsler metric F , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. Let $dV = \omega(x)dx$, where $\omega(x) = \frac{\kappa_n}{\int dX}$ is the coefficient of the canonical volume form. Define the Laplacian Δ at a point p by

$$\Delta f(x) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV} - f(x) \right),$$

or, if we write $dV = \omega(x)dx$,

$$\Delta f(x) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f \omega(x) dx}{\int_{\Sigma_\varepsilon} \omega(x) dx} - f(x) \right).$$

III. Concomitants of the Finsler Metric

1. Let M be a differentiable manifold with a Finsler metric F . Then we have a distance function between pairs of points, and this function satisfies certain properties. For $x \in M$, let $U \subset M$ be a co-ordinate chart with $x \in U$. Then define a function $\rho_x : U \rightarrow \mathbb{R}$, where $\rho_x(y)$ is the Finsler distance from y to x . ρ_x satisfies:

1. $\rho_x(x) = 0$,
2. $\rho_x > 0$ on $U \setminus \{x\}$,
3. ρ_x is continuous on U ,
4. ρ_x is smooth on $U \setminus \{x\}$,

Define $\Sigma_\varepsilon = \{y \in M | \rho_x(y) \leq \varepsilon\}$; think of Σ_ε as the ball of radius ε around x . These balls are defined intrinsically on the Finsler manifold, and will be essential to our Laplacian.

2. We can also use F to define a nowhere-zero volume form [Busemann, § 6]. To see how, note that, as a metric space, (M, F) has defined on it a natural

Hausdorff measure. Choose the volume form which generates the n -dimensional measure, i.e. take

$$dV = \left(\frac{\kappa_n}{\int_I dX} \right) dx,$$

where

κ_n = volume of the unit sphere in $\mathbb{R}^n$

$$I = \{X \in T_p M | F(X) \leq 1\}$$

= the solid unit ball in the tangent space

$$dX = dX^1 dX^2 \dots dX^n$$

= Euclidean volume form in the tangent space with the co-ordinates X^i .

This generalizes the Riemannian volume form and is written in co-ordinates as

$$dV(p) := \kappa_n \left(\int_{\{X \in T_p M | F(X) \leq 1\}} dX^1 \dots dX^n \right)^{-1} dx^1 \dots dx^n.$$

(In the Riemannian case, $\kappa_n \left(\int_{\{X \in T_p M | F(X) \leq 1\}} dX^1 \dots dX^n \right)^{-1} = \sqrt{\det g_{ij}}$, where $F(X) = \sqrt{g_{ij} X^i X^j}$; this is the standard Riemannian volume form. The factor κ_n is necessary because the standard Riemannian volume form $\sqrt{\det g_{ij}} dx^1 \dots dx^n$ gives the volume of a *parallelepiped* (in the tangent space) with sides $X_1, X_2, \dots, X_n$, whereas the expression $\int_{\{X \in T_p M | F(X) \leq 1\}} dX^1 \dots dX^n$ gives the volume of a Finsler *sphere* (in the tangent space).) Then

$$dV(X_1, \dots, X_n) = \kappa_n \det [X_i^j] \left(\int_{\{X \in T_p M | F(X) \leq 1\}} dX^1 \dots dX^n \right)^{-1},$$

where

$$X_i = X_i^j \frac{\partial}{\partial x^j}.$$

Proposition III-1. dV is a well-defined n -form.

Proof. dV is an n -form if, when we change co-ordinates to $\tilde{x}^i = \tilde{x}^i(x)$, generating a new expression

$$\tilde{dV} = \kappa_n \left(\int_{\{\tilde{X} \in T_p M | F(\tilde{X}) \leq 1\}} d\tilde{X}^1 \dots d\tilde{X}^n \right)^{-1} d\tilde{x}^1 \dots d\tilde{x}^n,$$

and a new co-ordinate expression $\tilde{X}_i$ for X_i , then

$$dV(X_1, \dots, X_n) = d\tilde{V}(\tilde{X}_1, \dots, \tilde{X}_n).$$

1. Write

$$\tilde{X}_i^k \frac{\partial}{\partial \tilde{x}^k} = X_i^j \frac{\partial}{\partial x^j}.$$

Then

$$\tilde{X}_i^k \frac{\partial x^j}{\partial \tilde{x}^k} \frac{\partial}{\partial x^j} = X_i^j \frac{\partial}{\partial x^j},$$

so

$$\det [\tilde{X}_i^k] \cdot \det \left[\frac{\partial x^j}{\partial \tilde{x}^k} \right] = \det [X_i^j]. \quad (\text{III.2-1})$$

2. Calculate

$$\begin{aligned} & dV(X_1, \dots, X_n) \\ &= \kappa_n \det [X_i^j] \left(\int_{\{X \in T_p M \mid F(X) \leq 1\}} dX^1 \dots dX^n \right)^{-1} \\ &= \kappa_n \det [\tilde{X}_i^k] \cdot \det \left[\frac{\partial x^j}{\partial \tilde{x}^k} \right] \left(\int_{\{\tilde{X} \in T_p M \mid F(\tilde{X}) \leq 1\}} \det \left[\frac{\partial x^l}{\partial \tilde{x}^m} \right] d\tilde{X}^1 d\tilde{X}^2 \dots d\tilde{X}^n \right)^{-1} \\ & \quad (\text{using (1)}) \\ &= \kappa_n \det [\tilde{X}_i^k] \left(\int_{\{\tilde{X} \in T_p M \mid F(\tilde{X}) \leq 1\}} d\tilde{X}^1 \dots d\tilde{X}^n \right)^{-1} \\ &= d\tilde{V}(\tilde{X}_1, \dots, \tilde{X}_n). \quad \blacksquare \end{aligned}$$

3. Like a Riemannian manifold, a Finsler manifold also has an intrinsically defined exponential map, which sends one-dimensional subspaces of a tangent space isometrically onto geodesics. We'll express the exponential map $\exp$ in local coordinates, and compute some expressions to be used in later calculations. A geodesic in a Finsler manifold is a parametrized path $x(t) = (x^1(t), x^2(t), \dots, x^n(t))$ which satisfies the differential equation

$$\ddot{x}^i = -\Gamma_{jk}^i(x, \dot{x}) \dot{x}^j \dot{x}^k,$$

where a dot above the x indicates differentiation with respect to t . We note, in contradistinction to the Riemannian case, that the Christoffel symbols Γ_{jk}^i are functions on the tangent bundle rather than the manifold. This equation allows us to write $\exp$ in local co-ordinates around a point $p \in M$ as

$$\begin{aligned} \exp : T_p M &\longrightarrow M \\ x^i(\exp(X)) &= X^i - \frac{1}{2} \Gamma_{jk}^i(X) X^j X^k - \frac{1}{6} \Gamma_{jk|l}^i X^j X^k X^l + \frac{1}{3} \Gamma_{jk}^i \Gamma_{pq}^k X^j X^p X^q - \\ &\quad - \frac{1}{6} \Gamma_{qr}^l \left(\frac{\partial}{\partial X^l} \Gamma_{jk}^i \right) X^j X^k X^q X^r + O(|X|^4), \end{aligned} \quad (\text{III.3-2})$$

where we are writing $X \in T_p M$ as $X = X^j \frac{\partial}{\partial x^j}$, and $\Gamma_{jk|l}^i = \frac{\partial}{\partial x^l} \Gamma_{jk}^i$. Note that the last term, which looks fourth-order because it has four X 's, is actually only third-order, because a derivative of a Christoffel symbol with respect to a tangent vector is homogeneous of degree -1 .

In a Riemannian space, $\exp$ is C^∞ everywhere on $T_p M$; in a Finsler space, $\exp$ is C^∞ everywhere except the origin $(p, \vec{0})$, at which it is only C^1 [Rund, Chap. 3, § 6]. This deficiency, however, will not cause us any trouble. We will use the exponential map to move integrals from the manifold to the tangent space, where the structure of F is exhibited more clearly. In particular, $\exp^{-1}$ takes the sphere Σ_ε on M to the indicatrix of "radius" ε on $T_p M$, and on $T_p M$ indicatrices of different radii have a natural scaling property.

Indeed, $T_p M$ can be thought of as a manifold on its own, and $\exp$ as a map between the manifolds $T_p M$ and M . Since $\exp$ is invertible, we can pull back forms on M to forms on $T_p M$, and we'll calculate pullbacks (in local co-ordinates) for some special cases. To begin with, let X^i be the co-ordinate functions on $T_p M$, canonically induced by the co-ordinate functions x^i on M . Then forms on the manifold will be written as wedge-products of the 1-forms dx^i , and forms on the tangent space will be written as wedge-products of the 1-forms dX^i . Take the exterior derivative of both sides of (2). (This is allowed because $\exp$ is C^1 at p , although higher derivatives do not generally exist.)

$$dx^i = \frac{\partial}{\partial X^p} \left(X^i - \frac{1}{2} \Gamma_{jk}^i(X) X^j X^k - \frac{1}{6} \Gamma_{jk|l}^i X^j X^k X^l + \frac{1}{3} \Gamma_{jk}^i \Gamma_{pq}^k X^j X^p X^q - \right.$$

$$\begin{aligned}
& -\frac{1}{6}\Gamma_{qr}^l\left(\frac{\partial}{\partial X^l}\Gamma_{jk}^i\right)X^jX^kX^qX^r+O(|X|^4)dX^p \\
= & dX^i+\left[-\frac{1}{2}\Gamma_{jk}^i\delta_p^jX^k-\frac{1}{2}\Gamma_{jk}^iX^j\delta_p^k-\frac{1}{2}\left(\frac{\partial}{\partial X^p}\Gamma_{jk}^i\right)X^jX^k-\frac{1}{3}\Gamma_{pk|l}^iX^kX^l\right. \\
& -\frac{1}{6}\Gamma_{jk|p}^iX^jX^k+\frac{1}{3}\Gamma_{pk}^i\Gamma_{rq}^kX^qX^r+\frac{2}{3}\Gamma_{jk}^i\Gamma_{pq}^kX^jX^q- \\
& -\frac{1}{6}\left(\left(\frac{\partial}{\partial X^p}\Gamma_{qr}^l\right)\left(\frac{\partial}{\partial X^l}\Gamma_{jk}^i\right)+\Gamma_{qr}^l\left(\frac{\partial^2}{\partial X^p\partial X^l}\Gamma_{jk}^i\right)\right)X^jX^kX^qX^r- \\
& \left.-\frac{1}{3}\Gamma_{kr}^i\left(\frac{\partial}{\partial X^l}\Gamma_{pj}^i\right)X^jX^kX^r-\frac{1}{3}\Gamma_{pr}^l\left(\frac{\partial}{\partial X^l}\Gamma_{kj}^i\right)X^jX^kX^r\right]dX^p \\
& +O(|X|^3) \\
= & dX^i-\Gamma_{pk}^iX^k dX^p+c_{pjk}^iX^jX^k dX^p+O(|X|^3), \tag{III.3-3}
\end{aligned}$$

where we are writing

$$\begin{aligned}
c_{pjk}^i = & -\frac{1}{2}\left(\frac{\partial}{\partial X^p}\Gamma_{jk}^i\right)-\frac{1}{3}\Gamma_{pj|k}^i-\frac{1}{6}\Gamma_{jk|p}^i+\frac{1}{3}\Gamma_{pl}^i\Gamma_{jk}^l+\frac{2}{3}\Gamma_{jl}^i\Gamma_{pk}^l- \\
& -\frac{1}{6}\left(\frac{\partial}{\partial X^p}\Gamma_{qr}^l\right)\left(\frac{\partial}{\partial X^l}\Gamma_{jk}^i\right)X^qX^r-\frac{1}{6}\Gamma_{qr}^l\left(\frac{\partial^2}{\partial X^p\partial X^l}\Gamma_{jk}^i\right)X^qX^r- \\
& -\frac{1}{3}\Gamma_{kr}^i\left(\frac{\partial}{\partial X^l}\Gamma_{pj}^i\right)X^r-\frac{1}{3}\Gamma_{pr}^l\left(\frac{\partial}{\partial X^l}\Gamma_{kj}^i\right)X^r \tag{III.3-4}
\end{aligned}$$

To work with n-forms, we'll also need to calculate $dx:=dx^1\dots dx^n$ in terms of $dX:=dX^1dX^2\dots dX^n$.

$$\begin{aligned}
dx = & (dX^1-\Gamma_{pk}^1X^k dX^p+c_{pjk}^1X^jX^k dX^p)(dX^2-\Gamma_{pk}^2X^k dX^p+c_{pjk}^2X^jX^k dX^p) \\
& \dots (dX^n-\Gamma_{pk}^nX^k dX^p+c_{pjk}^nX^jX^k dX^p) \\
= & (1-\Gamma_{jk}^jX^k+(\Gamma_{ak}^a\Gamma_{bj}^b-\Gamma_{bk}^a\Gamma_{aj}^b+c_{rjk}^r)X^jX^k)dX^1dX^2\dots dX^n \\
= & \left(1-\Gamma_{jk}^jX^k+\left(\Gamma_{ak}^a\Gamma_{bj}^b-\frac{1}{3}\Gamma_{bk}^a\Gamma_{aj}^b-\frac{1}{3}\Gamma_{ik|j}^i-\frac{1}{6}\Gamma_{jk|i}^i+\right.\right. \\
& \left.\left.+\frac{1}{3}\Gamma_{il}^i\Gamma_{jk}^l-\frac{1}{2}\left(\frac{\partial\Gamma_{jk}^i}{\partial X^i}\right)\right)X^jX^k\right)dX, \tag{III.3-5}
\end{aligned}$$

where we are neglecting terms of higher order than 2. Because $\exp$ is only C^1 , expression (5) is only continuous and not necessarily differentiable at p .

4. The exponential map allows us to change co-ordinates so that the spheres Σ_ε on M become indicatrices of radius ε on T_pM . Thus any integral $\int_{\Sigma_\varepsilon}\omega$ (of an

n-form ω) becomes an integral $\int_{I_\epsilon} \exp^* \omega$ on the tangent space, and, if $\exp^* \omega(X)$ (where $\exp^* \omega = \exp^* \omega(X) dX$) is homogeneous we can use the scaling property of the I_ϵ 's to write $\int_{I_\epsilon} \exp^* \omega$ as $\epsilon^k \int_I \exp^* \omega$ for some appropriate exponent k . In particular, when we come to integrate $\int_{\Sigma_\epsilon} f \omega$ for a function f , we will replace f by its power series around p to get

$$\int_{\Sigma_\epsilon} f \omega = f(p) \int_{\Sigma_\epsilon} \omega + f_i(p) \int_{\Sigma_\epsilon} x^i \omega + \frac{1}{2} f_{ij}(p) \int_{\Sigma_\epsilon} x^i x^j \omega + \int_{\Sigma_\epsilon} O(|x|^3) \omega, \quad (\text{III.4-1})$$

where $|x| = \sqrt{\Sigma(x^i)^2}$. After moving all these new integrals to the tangent space, we will see that we do in fact have homogeneous functions, so we can write them all as integrals over I , which is the same set as I_1 . We'll now determine the exponent k given above for integrals of the form $\int_{\Sigma_\epsilon} x^{|\alpha|} dV$, where α is a multi-exponent for the x^i 's.

Case A. $\int_{\Sigma_\epsilon} dV$. This amounts to finding the volume of the sphere of radius ϵ in the Finsler manifold. We'll develop a power series in terms of ϵ for this volume.

$$\begin{aligned} \text{Vol}(\Sigma_\epsilon) &= \int_{\Sigma_\epsilon} dV \\ &= \int_{\Sigma_\epsilon} \omega(x) dx, \end{aligned}$$

where $\omega(x) = \frac{\kappa_n}{\int_I dX}$ are the coefficients of the volume form as given in Section III.2. Replace $\omega(x)$ by its Taylor series around p :

$$= \int_{\Sigma_\epsilon} \omega(p) + \omega_i(p) x^i + \frac{1}{2} \omega_{ij}(p) x^i x^j + O(|x|^3) dx.$$

Now use $\exp^{-1}$ to move to the tangent space, so that Σ_ϵ is mapped to I_ϵ , and dx becomes the expression given in Equation (III.3-3). Meanwhile, any x^i in the integrand is replaced by an expression in X^i using Equation (III.3-2).

$$\begin{aligned} &= \int_{I_\epsilon} \left(\omega(p) + \omega_i(p) \left(X^i - \frac{1}{2} \Gamma_{jk}^i(x, X) X^j X^k \right) + \frac{1}{2} \omega_{ij}(p) X^i X^j \right) \\ &\quad \cdot \left(1 - \Gamma_{jk}^j X^k + c_{jk} X^j X^k \right) dX. \end{aligned}$$

(Here we are ignoring terms of order higher than 2, and abbreviating

$$c_{jk} := \Gamma_{ak}^a \Gamma_{bj}^b - \frac{1}{3} \Gamma_{bk}^a \Gamma_{aj}^b - \frac{1}{3} \Gamma_{ik|j}^i - \frac{1}{6} \Gamma_{jk|i}^i + \frac{1}{3} \Gamma_{il}^i \Gamma_{jk}^l - \frac{1}{2} \left(\frac{\partial}{\partial X^i} \Gamma_{jk}^i \right)$$

from Equation (III.3-5).)

$$\begin{aligned}
&= \int_{I_\epsilon} \left(\omega(p) + \left(\omega_k(p) - \omega(p) \Gamma_{jk}^j \right) X^k + \right. \\
&\quad \left. + \left(-\frac{1}{2} \Gamma_{jk}^i \omega_i(p) + \frac{1}{2} \omega_{jk}(p) - \omega_j(p) \Gamma_{ik}^l + \omega(p) c_{jk} \right) X^j X^k \right) dX.
\end{aligned}
\tag{III.4-2}$$

Notice that I_ϵ is symmetric on $T_p M$. This is an advantage of working in the tangent space because symmetry ensures that $\int_{I_\epsilon} X^k dX = 0$, or more generally $\int_{I_\epsilon} h(X) dX = 0$ if h is any odd function. In the line above we see the integral $\int_{I_\epsilon} \Gamma_{jk}^j(X) X^k dX$. Consider

$$\Gamma_{jk}^j(X) := \frac{1}{2} g^{jr}(x, X) \left(\frac{\partial g_{jr}}{\partial x^k}(x, X) - \frac{\partial g_{jk}}{\partial x^r}(x, X) + \frac{\partial g_{rk}}{\partial x^j}(x, X) \right). \tag{III.4-3}$$

Since $F^2(X)$ is an even function, and since $g_{ij}(x, X) = \frac{1}{2} \frac{\partial^2 F^2}{\partial X^i \partial X^j}(x, X)$ is obtained by taking two derivatives of an even function, it must be that $g_{ij}(X)$ is also even. Then $g^{ij}(X)$ is even, and derivatives of the form $\frac{\partial g_{ij}}{\partial x^k}(X)$, with respect to coordinates x^k in the manifold are also even (as functions of the tangent space coordinates X). We see from (III.4-3) that $\Gamma_{jk}^j(X)$ must then be even. The function X^k (on the tangent space) is odd, so the product $\Gamma_{jk}^j(X) X^k$ of an even function and an odd function must again be odd. Thus $\int_{I_\epsilon} \Gamma_{jk}^j(X) X^k dX = 0$, and Equation (2) simplifies to

$$= \int_{I_\epsilon} \omega(p) dX + \int_{I_\epsilon} \left(-\frac{1}{2} \Gamma_{jk}^i(X) \omega_i(p) + \frac{1}{2} \omega_{jk}(p) - \omega_j(p) \Gamma_{ik}^l(X) + \omega(p) c_{jk} \right) X^j X^k dX.
\tag{III.4-4}$$

Note that the first term of (III.4-4) is the integral of a constant; because of the scaling property $I_{r\epsilon} = r I_\epsilon$, we can change our region of integration to I_1 , via

$$\begin{aligned}
\int_{I_\epsilon} \omega(p) dX &= \epsilon^n \int_{I_1} \omega(p) dX \\
&= \epsilon^n \int_I \omega(p) dX,
\end{aligned}$$

where n is the dimension of M , and, of course $I = I_1$. Similarly, the second term of (4) is the integral of a function homogeneous of degree 2, so we can again change

our region of integration to I , but this time with a factor of ε^{n+2} :

$$\begin{aligned}
&= \varepsilon^n \int_I \omega(p) dX + \\
&+ \varepsilon^{n+2} \int_I \left(-\frac{1}{2} \Gamma_{jk}^i(X) \omega_i(p) + \frac{1}{2} \omega_{jk}(p) - \omega_j(p) \Gamma_{ik}^i(X) + \omega(p) c_{jk} \right) X^j X^k dX.
\end{aligned}
\tag{III.4-5}$$

Recall that $\omega(p)$ is given by $\omega(p) = \frac{\kappa_n}{\int_I dX}$, where neither the numerator nor the denominator depends on X . Thus we can take $\omega(p)$ outside the integral:

$$= \varepsilon^n \frac{\kappa_n}{\int_I dX} \int_I dX + 0 \cdot \varepsilon^{n+1} + c_{n+2} \cdot \varepsilon^{n+2} + O(\varepsilon^{n+3}),$$

where c_{n+2} will be written explicitly later. In summary,

$$\text{Vol}(\Sigma_\varepsilon) = \kappa_n \varepsilon^n + c_{n+2} \varepsilon^{n+2} + O(\varepsilon^{n+3}). \tag{III.4-6}$$

Case B. $\int_{\Sigma_\varepsilon} x^i dV$. The function x^i is “odd” about the origin, which is the center of Σ_ε , so we would expect the integral to be near zero; in fact, this integral has order ε^{n+2} instead of the ε^{n+1} which seems natural. Again, we’ll work this out by transferring $\int_{\Sigma_\varepsilon} x^i dV$ to $T_p M$, and using the symmetry of the indicatrix there.

$$\begin{aligned}
\int_{\Sigma_\varepsilon} x^i dV &= \int_{\Sigma_\varepsilon} x^i \omega(x) dx \\
&= \int_{\Sigma_\varepsilon} x^i \omega(p) + \omega_j(p) x^i x^j + \frac{1}{2} \omega_{jk}(p) x^i x^j x^k + O(|x|^4) dx \\
&= \int_{I_\varepsilon} \left[\left(X^i - \frac{1}{2} \Gamma_{jk}^i(X) X^j X^k \right) \omega(p) + \right. \\
&\quad + \omega_j(p) \left(X^i - \frac{1}{2} \Gamma_{ik}^i(X) X^l X^k \right) \left(X^j - \frac{1}{2} \Gamma_{pq}^j(X) X^p X^q \right) \\
&\quad + \frac{1}{2} \omega_{jk}(p) \left(X^i - \frac{1}{2} \Gamma_{pq}^i(X) X^p X^q \right) \left(X^j - \frac{1}{2} \Gamma_{rs}^j(X) X^r X^s \right) \\
&\quad \left. \left(X^k - \frac{1}{2} \Gamma_{tu}^k(X) X^t X^u \right) \right] \left(1 - \Gamma_{jk}^j(X) X^k + c_{jk} X^j X^k \right) dX \\
&= \int_{I_\varepsilon} \omega(p) X^i + d_{ij} X^i X^j + O(|X|^3) dX
\end{aligned}$$

for some coefficient d_{ij} . Again, oddness of X^i ensures that $\int_{I_\varepsilon} \omega(p) X^i dX$ vanishes, and homogeneity allows us to evaluate I_ε solely on I by adding an appropriate factor:

$$= d_{ij} \left(\int_I X^i X^j dX \right) \varepsilon^{n+2} + d_{n+3} \varepsilon^{n+3} + O(\varepsilon^{n+4}),$$

for some constant d_{n+3} . The important result here is that we don't need any terms involving ε^{n+1} , so

$$\left| \int_{\Sigma_\varepsilon} x^i dV \right| \leq d_{n+2} \varepsilon^{n+2} + O(\varepsilon^{n+4}).$$

With Cases A and B worked out, the method of finding expressions for $\int_{\Sigma_\varepsilon} x^\alpha dV$ is clear for any value of $|\alpha|$: move to the tangent space and transfer all integrals to the unit indicatrix. By homogeneity, the coefficients of ε^{n+k} will disappear if k is odd, or if $k < |\alpha|$. The first conclusion assures us that integrals of the form $\int_{\Sigma_\varepsilon} x^i dV$ are actually of the same order of magnitude as the integrals $\int_{\Sigma_\varepsilon} x^i x^j dV$, and the second conclusion allows us to ignore higher-order terms when they appear in integrands.

Furthermore, the expression (6) from Case A is interesting because of its parallels with the Riemannian case. The first term in both the Finsler and Riemannian case is $\kappa_n \varepsilon^n$, which is simply the Euclidean volume of a sphere of radius ε in $\mathbb{R}^n$, so even in a Finsler space, volume measurement doesn't differ dramatically from the Euclidean case. (This is actually a direct consequence of using the Hausdorff measure.) Also, in both cases the coefficient of ε^{n+1} vanishes. The most interesting coefficient is c_{n+2} , which measures how our space differs from Euclidean space. In the Riemannian case, this coefficient (up to a constant) is actually the scalar curvature [GV]. In the 2-dimensional case, this is clear from Puiseux's formula

$$\text{Gaussian Curvature} = \lim_{\varepsilon \rightarrow 0} \frac{12}{\pi} \left(\frac{\pi \varepsilon^2 - \text{Vol}(\Sigma_\varepsilon)}{\varepsilon^4} \right). \quad (\text{III.4-7})$$

Since expression (6) is invariant by its very construction, the foregoing clearly suggests thinking of the second coefficient as a scalar curvature for Finsler spaces.

Definition. Define the *Puiseux curvature* P at x of the Finsler manifold (M, F) to be

$$P = \alpha_n c_{n+2},$$

where $\alpha_n = \frac{6(n+2)\Gamma(\frac{n+2}{2})}{\pi^{\frac{n}{2}}}$ is taken from [Hotelling, § 5].

We can get an expression for the Puiseux curvature by starting with (III.4-5) and replacing $\omega(p)$ with $\omega(p) = \int_I \frac{\kappa_n}{dX}$:

$$\begin{aligned} P &= \alpha_n c_{n+2} \\ &= \alpha_n \int_I \left[-\frac{1}{2} \Gamma_{jk}^i \frac{\partial}{\partial x^i} \left(\int_I dX \right)^{-1} + \frac{1}{2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} \left(\int_I dX \right)^{-1} - \frac{\partial}{\partial x^j} \left(\int_I dX \right)^{-1} \Gamma_{lk}^i \right. \\ &\quad + \left(\int_I dX \right)^{-1} \left(\Gamma_{ak}^a \Gamma_{bj}^b - \frac{1}{3} \Gamma_{bk}^a \Gamma_{aj}^b - \frac{1}{3} \Gamma_{ik|j}^i - \frac{1}{6} \Gamma_{jk|i}^i + \right. \\ &\quad \left. \left. + \frac{1}{3} \Gamma_{ik}^i \Gamma_{jk}^i - \frac{1}{2} \frac{\partial}{\partial X^i} \Gamma_{jk}^i \right) \right] X^j X^k dX, \end{aligned}$$

Lemma III-1. $P : M \rightarrow \mathbb{R}$ given by $x \mapsto$ Puiseux curvature at x is C^∞ .

Proof. The expression for c_{n+2} involves integrals over I , but an integral over I is equal to an integral over $I \setminus \{\vec{0}\}$. If c_{n+2} is seen as an integral over $I \setminus \{\vec{0}\}$, then all limits of integration and integrands are smooth functions of x , so P is smooth. ■

IV. Existence of the Laplacian and a Co-ordinate Expression for It

Though the expression for the Laplacian given in Section II is well-motivated, it is not very usable as stated. Indeed, it is not even clear *a priori* that this Laplacian actually exists for any given function. In this section, we'll use the constructions of Section III to show Δ exists, and to derive a co-ordinate expression for Δ , which will make clear that Δ is a differential operator.

Theorem IV-1. Let M be a differentiable manifold with a Finsler metric F , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. Let p be a point in M . Then $\Delta f(p)$ exists and is a linear, elliptic, 2nd-order differential operator with the co-ordinate expression:

$$\begin{aligned} \Delta f &= (n+2) \left(\frac{\int_I X^i X^j dX}{\int_I dX} f_{ij} + \right. \\ &\quad \left. + \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2 \int_I \Gamma_{jq}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right) f_i \right). \quad (IV-1) \end{aligned}$$

Proof. Write f as a Taylor series around p

$$f(x) = f(p) + f_i(p)x^i + \frac{1}{2}f_{ij}(p)x^ix^j + \frac{1}{6}f_{ijk}(p)x^ix^jx^k + O(|x|^4).$$

Substitute this expression into the definition:

$$\begin{aligned} & 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV} - f(p) \right) \\ &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f(p) + f_i(p)x^i + \frac{1}{2}f_{ij}(p)x^ix^j + O(|x|^3) dV}{\int_{\Sigma_\varepsilon} dV} - \frac{\int_{\Sigma_\varepsilon} f(p) dV}{\int_{\Sigma_\varepsilon} dV} \right) \\ &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{\int_{\Sigma_\varepsilon} f_i(p)x^i + \frac{1}{2}f_{ij}(p)x^ix^j + \frac{1}{6}f_{ijk}(p)x^ix^jx^k + O(|x|^4) dV}{\varepsilon^2 \int_{\Sigma_\varepsilon} dV} \\ &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{f_i(p) \int_{\Sigma_\varepsilon} x^i dV + \frac{1}{2}f_{ij}(p) \int_{\Sigma_\varepsilon} x^ix^j dV + \frac{1}{6}f_{ijk}(p) \int_{\Sigma_\varepsilon} x^ix^jx^k dV + \int_{\Sigma_\varepsilon} O(|x|^4) dV}{\varepsilon^2 \int_{\Sigma_\varepsilon} dV}. \end{aligned}$$

Now use the integral estimates from Section III:

$$= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{c_1\varepsilon^{n+2} + c_2\varepsilon^{n+2} + c_3\varepsilon^{n+3} + O(\varepsilon^{n+4})}{\kappa_n\varepsilon^{n+2} + c_5\varepsilon^{n+5} + O(\varepsilon^{n+4})}$$

for some constants c_i ; we have suppressed higher-order terms because they don't affect the limit. In the denominator, we see that the dominating term is of order ε^{n+2} and it isn't 0 (because κ_n is the volume of the unit sphere in $\mathbb{R}^n$). Since the numerator has no smaller terms, the limit exists.

Leaving aside the last two lines, we'll continue deriving the co-ordinate expression for Δ by writing $dV = \omega(x)dx$, replacing ω by its Taylor series around p , and moving to the tangent space.

$$\begin{aligned} &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \frac{\int_{\Sigma_\varepsilon} (f_i(p)x^i + \frac{1}{2}f_{ij}(p)x^ix^j + O(|x|^3)) \omega(x) dx}{\int_{\Sigma_\varepsilon} \omega(x) dx} \\ &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \frac{\int_{I_\varepsilon} (f_i(p)(X^i - \frac{1}{2}\Gamma_{jk}^i X^j X^k) + \frac{1}{2}f_{ij}(p)(X^i - \frac{1}{2}\Gamma_{pq}^i X^p X^q)(X^j - \frac{1}{2}\Gamma_{rs}^j X^r X^s)) \cdot (\omega(p) + \omega_i(p)(X^i - \frac{1}{2}\Gamma_{pq}^i X^p X^q))(1 - \Gamma_{rs}^s X^s) dX}{\int_{I_\varepsilon} (\omega(p) + \omega_i(p)(X^i - \frac{1}{2}\Gamma_{pq}^i X^p X^q))(1 - \Gamma_{rs}^s X^s) dX} \\ &= 2(n+2)\lim_{\varepsilon \rightarrow 0} \frac{\int_{I_\varepsilon} f_i(p)\omega(p)X^i + \frac{1}{2}f_{ij}(p)\omega(p)X^i X^j + f_i(p)\omega_j(p)X^i X^j - f_i(p)\omega(p)\Gamma_{rs}^s X^i X^s - \frac{1}{2}f_i(p)\omega(p)\Gamma_{jk}^i X^j X^k dX}{\varepsilon^2 \int_{I_\varepsilon} (\omega(p) + \omega_i(p)X^i + O(|X|^2)) dX}. \end{aligned}$$

Again, eliminating integrands homogeneous of degree 1, and ignoring terms of higher order than $n + 2$, and moving to I , we have

$$= 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{n+2} \int_I (\frac{1}{2} f_{ij}(p) \omega(p) + f_i(p) \omega_j(p) - f_i(p) \omega(p) \Gamma_{ij}^k - \frac{1}{2} f_k(p) \omega(p) \Gamma_{ij}^k) X^i X^j dX}{\varepsilon^{n+2} \omega(p) \int_I dX}.$$

Separating into coefficients of f_{ij} and f_i ,

$$\Delta f = (n+2) \left(\frac{\int_I X^i X^j dX}{\int_I dX} f_{ij} + \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2 \int_I \Gamma_{ij}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right) f_i \right) \quad (IV-1)$$

This expression clearly shows that Δ is a linear, 2nd-order differential operator.

The proof of ellipticity will be delayed until the next section. $\blacksquare$

(We could have deduced that Δ is a differential operator solely from the facts that Δ is linear and decreases support, but this explicit expression will be more direct to work with.)

V. Our Laplacian Generalizes the Riemannian Laplacian

As a first application of the co-ordinate expression, we will prove directly that Δ generalizes the Riemannian Laplacian. To begin with, note the leading coefficients $K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$ in the co-ordinate expression for Δ . These coefficients have a considerable amount of structure, and lead to a surprising definition.

Lemma V-1. *Let M be a manifold of dimension n with Finsler metric F . Define*

$$K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}.$$

Then K^{ij} are the components of a symmetric, twice-contravariant, positive-definite tensor.

Proof. Clearly $K^{ij} = K^{ji}$, so these components are symmetric. To see that

$$K^{ij} = \frac{\int_I X^i X^j dX}{\int_I dX} \quad (V-1)$$

are the components of a tensor, make a co-ordinate change

$$\bar{x}^i = \bar{x}^i(x) \quad (V-2)$$

on the manifold. Then we have the following relations on the tangent space:

$$\bar{X}^i = \frac{\partial x^i}{\partial \bar{x}^j} X^j \quad (V-3)$$

$$d\bar{X} = \det \left[\frac{\partial x^i}{\partial \bar{x}^j} \right] dX. \quad (V-4)$$

In the barred co-ordinate system,

$$\begin{aligned} \bar{K}^{ij} &= \frac{\int_I \bar{X}^i \bar{X}^j d\bar{X}}{\int_I d\bar{X}} \\ &= \frac{\int_I \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} X^k X^l \det \left[\frac{\partial x^i}{\partial \bar{x}^j} \right] dX}{\int_I \det \left[\frac{\partial x^i}{\partial \bar{x}^j} \right] dX}. \end{aligned}$$

Since none of the terms $\frac{\partial x^i}{\partial \bar{x}^k}$, $\frac{\partial x^j}{\partial \bar{x}^l}$, and $\det \left[\frac{\partial x^i}{\partial \bar{x}^j} \right]$ depends on X , we can take them all outside the integral signs, and cancel the determinants, to get

$$\begin{aligned} \bar{K}^{ij} &= \frac{\frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} \int_I X^k X^l dX}{\int_I dX} \\ &= \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} K^{kl}, \end{aligned}$$

i.e. K^{ij} are the components of a twice-contravariant tensor.

To see that this tensor is positive-definite, simply use the symmetry to choose a co-ordinate system $\bar{x}$ in which $\bar{K}^{ij}$ is diagonal, that is, one for which $\bar{K}^{ij} = 0$ if $i \neq j$. Then

$$\begin{aligned} \bar{K}^{ii} &= \frac{\int_I \bar{X}^i \bar{X}^i d\bar{X}}{\int_I d\bar{X}} \\ &> 0, \end{aligned}$$

so K^{ij} is positive-definite. ■

Since a symmetric, positive-definite, twice-covariant tensor is a metric tensor, we can interpret the K^{ij} as the (inverse) components of a Riemannian metric. Thus we have canonically associated to our Finsler manifold a Riemannian metric.

Definition. The metric $K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$ on a Finsler space (M, F) is called the (inverse of the) osculating Riemannian metric.

This property of the leading term of the Laplacian allows us to complete the proof of Theorem IV-1. If the leading coefficients of Δ are the components of a Riemannian metric, then Δ must be an *elliptic* differential operator. This ellipticity finds an immediate application, in that we can solve the Dirichlet problem for Finsler manifolds.

Proposition V-1. Let Ω be an open, bounded, simply connected region in the Finsler manifold (M^n, F) , such that $\bar{\Omega}$ is diffeomorphic to the unit ball in $\mathbb{R}^n$. Let Ω have a smooth, rectifiable boundary $\partial\Omega$, and let $h : \partial\Omega \rightarrow \mathbb{R}$ be a smooth function. Then there exists a function $u : \Omega \cup \partial\Omega \rightarrow \mathbb{R}$ such that $u|_{\partial\Omega} = h$ and $\Delta u|_{\Omega} = 0$, where Δ is the mean-value Laplacian.

Proof. Choose a smooth co-ordinate chart $\alpha : \Omega \cup \partial\Omega \rightarrow \mathbb{R}^n$, and move the problem to $\mathbb{R}^n$. We now have a region $\alpha(\Omega)$ with boundary $\partial(\alpha(\Omega)) = \alpha(\partial\Omega)$, and a boundary-value function $\alpha_* h$. Since u and Δu are scalar functions, we have $\alpha_*(\Delta u) = (\alpha_* \Delta)(\alpha_* u)$, where $\alpha_* \Delta$ is again a linear, elliptic, 2nd-order differential operator. Thus we're searching for a function $\alpha_* u$ on a domain $\alpha(\Omega)$ of $\mathbb{R}^n$ with boundary conditions $\alpha_* u = \alpha_* h$ and interior conditions $(\alpha_* \Delta)(\alpha_* u) = 0$. This is just the standard Dirichlet problem, which always has a unique solution $\alpha_* u$ (see, e.g. [RR, Ch. 8]). Pulling $\alpha_* u$ back to M solves the Dirichlet problem on a Finsler manifold. $\blacksquare$

Apart from ellipticity, the existence of such a metric (which has also arisen in a different context [BCS]) leads to very interesting questions. For example, we could simply define the Laplacian on the Finsler space (M, F) to be the Laplacian gotten from thinking of (M, F) as a Riemannian space with coefficients K^{ij} , because these coefficients determine their own Laplacian. Is this Laplacian the same as the one we defined in Section II? The fact that $(n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$ are the leading coefficients in the differential operator Δ suggests the two are the same, but in fact we will see that they are different. Would the Riemannian Laplacian make a good Laplacian,

then? Perhaps, but it seems to say nothing about the Finsler structure and to violate our motivation for the Finsler Laplacian.

Since a Riemannian space is also a Finsler space, the K^{ij} are defined canonically on any Riemannian manifold, and presumably they have some relation to the metric tensor. This relation is very simple (and very similar to a result of [Pinsky, Eq.(4-9)]).

Lemma 2. *Let M be a Riemannian manifold of dimension n with metric tensor g_{ij} . Then*

$$g^{ij} = (n+2) \frac{\int_I X^i X^j dX^1 dX^2 \dots dX^n}{\int_I dX^1 dX^2 \dots dX^n}.$$

Proof. *The above statement is really a statement at a point p , though that argument has been suppressed. To simplify calculations, choose co-ordinates so that*

$$g^{ij}(p) = g_{ij}(p) = \delta_{ij}(p), \quad (V-5)$$

i.e. so that I at p is the unit sphere D in $\mathbb{R}^n$, Then, by symmetry of D ,

$$K^{ij} = 0 \quad \text{if } i \neq j, \quad (V-6)$$

and

$$K^{11} = K^{22} = \dots = K^{nn}, \quad (V-7)$$

so just calculate

$$\begin{aligned} K^{nn} &= (n+2) \frac{\int_D (X^n)^2 dX^1 dX^2 \dots dX^n}{\int_D dX^1 dX^2 \dots dX^n} \\ &= (n+2) \frac{\int_{-1}^1 (X^n)^2 \left(\int_{-\sqrt{1-(X^n)^2}}^{\sqrt{1-(X^n)^2}} \dots \int dX^1 \dots dX^{n-2} dX^{n-1} \right) dX^n}{\int_{-1}^1 \left(\int_{-\sqrt{1-(X^n)^2}}^{\sqrt{1-(X^n)^2}} \dots \int dX^1 \dots dX^{n-2} dX^{n-1} \right) dX^n} \\ &= (n+2) \frac{\int_{-1}^1 (X^n)^2 \cdot \left((n-1)\text{-sphere of radius } \sqrt{1-(X^n)^2} \right) dX^n}{\int_{-1}^1 \left((n-1)\text{-sphere of radius } \sqrt{1-(X^n)^2} \right) dX^n} \\ &= (n+2) \frac{\int_{-1}^1 (X^n)^2 \kappa_{n-1} \left(\sqrt{1-(X^n)^2} \right)^{n-1} dX^n}{\int_{-1}^1 \kappa_{n-1} \left(\sqrt{1-(X^n)^2} \right)^{n-1} dX^n}. \end{aligned}$$

Let $X = \sin \theta$ and integrate:

$$\begin{aligned}
 &= (n+2) \frac{2 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^{n-1} \theta \cos \theta d\theta}{2 \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta} \\
 &= (n+2) \frac{\frac{1}{(n+2)} \cos^{n+1} \theta \sin \theta \Big|_0^{\frac{\pi}{2}} + \frac{1}{(n+2)} \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta}{\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta} \\
 &= 1.
 \end{aligned} \tag{V-8}$$

By (5) to (8),

$$K^{ij} = \delta^{ij} = g^{ij}.$$

We know g^{ij} is a twice-contravariant tensor, and since Lemma 1 proved that K^{ij} is a twice-contravariant tensor, it must be that $g^{ij} = K^{ij}$ in any co-ordinate system, i.e.

$$g^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}. \quad \blacksquare$$

As remarked before, the terms $(n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$ appear as the leading coefficients in Δ . If our space is Riemannian, they will also appear in the first-order coefficients, and this fact allows us to prove the following folk theorem.

Theorem V-1. Let (M, F) be a Riemannian space with its canonical Laplacian $L(f)$. Define

$$\Delta f = 2(n+2) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \left(\frac{\int_{\Sigma_\epsilon} f dV}{\int_{\Sigma_\epsilon} dV} - f \right).$$

Then

$$\Delta f = L(f). \tag{V-9}$$

Proof. In the case of a Riemannian space with metric $F(X) = \sqrt{g_{ij} X^i X^j}$, the associated Laplacian is written in co-ordinates as

$$L(f) = g^{ij} (f_{ij} - \Gamma_{ij}^k f_k), \tag{V-10}$$

(or equivalently

$$L(f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} f_j),$$

where $g = \det[g_{ij}]$.

Now consider the co-ordinate expression IV-(1) for Δf :

$$\Delta f = (n+2) \left(\frac{\int_I X^i X^j dX}{\int_I dX} f_{ij} + \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2\int_I \Gamma_{jq}^i X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right) f_i \right) \quad (V-11)$$

Since the Christoffel symbols Γ_{jk}^i don't depend on tangent vectors in a Riemannian space, we can take them outside the integral signs, which leaves the common factor $(n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$:

$$= (n+2) \frac{\int_I X^i X^j dX}{\int_I dX} \left(f_{ij} + f_i \left(2 \frac{\omega_j}{\omega} - 2\Gamma_{jq}^i \right) - f_k \Gamma_{ij}^k \right). \quad (V-12)$$

In a Riemannian space, the volume form $dV = \omega(x)dx$ is

$$dV = \sqrt{g}dx, \quad (V-13)$$

so

$$\frac{\omega_j}{\omega} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j}. \quad (V-14)$$

There is, however, the Riemannian identity

$$\Gamma_{jq}^i = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j}, \quad (V-15)$$

so two of the terms in expression (12) cancel, leaving

$$\Delta f = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX} (f_{ij} - \Gamma_{ij}^k f_k). \quad (V-16)$$

Now use Lemma 2, which asserts the identity

$$g^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}, \quad (V-17)$$

to convert (16) to

$$\begin{aligned} \Delta f &= g^{ij} (f_{ij} - \Gamma_{ij}^k f_k) \\ &= L(f). \end{aligned} \quad \blacksquare$$

VI. Properties of the Laplacian

Although the Laplacian we have defined does not result from any Riemannian metric, it shares some important properties with Riemannian Laplacians. In particular, Δ is “self-adjoint” in the sense that it satisfies a version of Green’s Theorem for Finsler spaces, and as a result of this, it can be written in a divergence form very similar to the divergence form of the Riemannian Laplacian.

Definition. A linear operator $\mathcal{D}$ on a Finsler space (M, F) is self-adjoint if

$$\int_M (u\mathcal{D}(v))\omega(y)dy = \int_M (v\mathcal{D}(u))\omega(y)dy,$$

for any compactly supported functions u and v .

Lemma VI-1. Let M be a manifold with Finsler manifold F and Δ defined as above. Then Δ is self-adjoint, i.e.

$$\int_M (u\Delta v - v\Delta u)\omega(y)dy = 0.$$

Proof. $\int_M (u\Delta v - v\Delta u)\omega(y)dy$

$$\begin{aligned} &= \int_M \left(u(y) \lim_{\varepsilon \rightarrow 0} \frac{\int_{\Sigma_\varepsilon(y)} (v(x) - v(y))\omega(x)dx}{\varepsilon^2 \int_{\Sigma_\varepsilon(y)} \omega(x)dx} - \right. \\ &\quad \left. - v(y) \lim_{\varepsilon \rightarrow 0} \frac{\int_{\Sigma_\varepsilon(y)} (u(x) - u(y))\omega(x)dx}{\varepsilon^2 \int_{\Sigma_\varepsilon(y)} \omega(x)dx} \right) \omega(y)dy \\ &\quad (\text{where } \Sigma_\varepsilon(y) \text{ (resp. } \Sigma_\varepsilon(x)) \text{ is the ball of radius } \varepsilon \text{ around } y \text{ (resp. } x)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z)dz} \right) \cdot \\ &\quad \cdot [u(y)v(x) - v(y)u(x) - u(y)v(y) + u(y)v(y)] \omega(x)\omega(y)dx dy \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z)dz} \right) u(y)v(x)\omega(x)\omega(y)dx dy - \right. \\ &\quad \left. - \int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z)dz} \right) v(y)u(x)\omega(x)\omega(y)dx dy \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z)dz} \right) u(y)v(x)\omega(x)\omega(y)dx dy - \right. \end{aligned}$$

$$\begin{aligned}
& - \int_M \int_{\Sigma_\varepsilon(x)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z) dz} \right) v(y) u(x) \omega(x) \omega(y) dy dx \\
& \text{(because, along the "diagonal" of } M \times M, \\
& \int_M \int_{\Sigma_\varepsilon(x)} f(x, y) dy dx = \int_M \int_{\Sigma_\varepsilon(y)} f(x, y) dx dy) \\
& = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z) dz} \right) u(y) v(x) \omega(x) \omega(y) dx dy - \right. \\
& \quad \left. - \int_M \int_{\Sigma_\varepsilon(y)} \left(\frac{1}{\int_{\Sigma_\varepsilon(x)} \omega(z) dz} \right) u(y) v(x) \omega(x) \omega(y) dx dy \right) \\
& \text{(renaming the parameters in the second term)} \\
& = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_M \left(\int_{\Sigma_\varepsilon(y)} u(y) v(x) \omega(x) \omega(y) \right) \cdot \\
& \quad \cdot \left(\left(\frac{1}{\int_{\Sigma_\varepsilon(y)} \omega(z) dz} \right) - \left(\frac{1}{\int_{\Sigma_\varepsilon(x)} \omega(z) dz} \right) \right) dx dy \\
& = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_M \left(\int_{\Sigma_\varepsilon(y)} u(y) v(x) \omega(x) \omega(y) \right) \cdot \\
& \quad \cdot \left(\frac{1}{\kappa_n \varepsilon^n + c_{n+2}(y) \varepsilon^{n+2} + \dots} - \frac{1}{\kappa_n \varepsilon^n + c_{n+2}(x) \varepsilon^{n+2} + \dots} \right) dx dy \\
& \text{(using (III.4-6))} \\
& = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_M \left(\frac{1}{\varepsilon^n} \int_{\Sigma_\varepsilon(y)} u(y) v(x) \omega(x) \omega(y) \right) \cdot \\
& \quad \cdot \left(\frac{\kappa_n (c_{n+2}(x) - c_{n+2}(y)) \varepsilon^{n+2} + \dots}{\kappa_n^2 \varepsilon^n} \right) dx dy \\
& = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{n+2}}{\varepsilon^{n+2}} \frac{1}{\kappa_n} \int_M \left(\frac{1}{\varepsilon^n} \int_{\Sigma_1(y)} u(y) v(x) \omega(x) \omega(y) \right) (c_{n+2}(x) - c_{n+2}(y)) dx dy \\
& = 0.
\end{aligned}$$

The last line was obtained by letting $\varepsilon \rightarrow 0$. When this happens, $x \rightarrow y$. c_{n+2} is just the Puiseux curvature we defined in Section III, where we also proved it was smooth (Lemma III-2). Since c_{n+2} is smooth, $\therefore c_{n+2}(x) \rightarrow c_{n+2}(y)$. $\blacksquare$

Self-adjointness induces an equivalent condition on the coefficients of the differential operator $\mathcal{D}$. For a second-order, linear differential operator $\mathcal{D}$ with symmetric leading coefficients, i.e.

$$\mathcal{D}(f) = A^{ij} f_{ij} + B^i f_i, \quad (\text{VI-1})$$

$$A^{ij} = A^{ji},$$

this is indeed the case. For the Riemannian Laplacian $L(f)$, we can write

$$L(f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (f_j g^{ij} \sqrt{g}),$$

and give a name to this convenient form.

Definition. We will say an operator $\mathcal{D}$, of the form $\mathcal{D}(f) = A^{ij} f_{ij} + B^i f_i$, is in divergence form if

$$\mathcal{D}(f) = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega),$$

where $\omega(x)dx$ is the canonical Finsler volume form dV , and where h^{ij} is some symmetric, contravariant 2-tensor on M .

It turns out that the two conditions of self-adjointness and divergence form are equivalent for the special class of operators we're interested in.

Lemma VI-2. If $\mathcal{D}$ is a second-order linear differential operator of the form

$$\mathcal{D}f = A^{ij} f_{ij} + B^i f_i$$

with symmetric leading coefficients, then

$$\mathcal{D} \text{ is self-adjoint} \iff \mathcal{D} \text{ is in divergence form.}$$

Proof. $\Leftarrow$. Assume that $\mathcal{D}$ is in divergence form, i.e. that $\mathcal{D}(f) = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega)$, for some h^{ij} . Let u and v be any compactly supported smooth functions on the manifold M . Then

$$\begin{aligned} \int_M u \mathcal{D}(v) \omega(x) dx &= \int_M u \frac{1}{\omega} \frac{\partial}{\partial x^i} (v_j h^{ij} \omega) \omega(x) dx \\ &= - \int_M u_i v_j h^{ij} \omega(x) dx \\ &\quad (\text{integration by parts, integrating } \frac{\partial}{\partial x^i} (v_j h^{ij} \omega) dx) \\ &= \int_M v \frac{\partial}{\partial x^j} (u_i h^{ij} \omega) dx \\ &\quad (\text{integration by parts, integrating } v_j dx) \\ &= \int_M v \frac{1}{\omega} \frac{\partial}{\partial x^j} (u_i h^{ij} \omega) \omega dx \\ &= \int_M v \mathcal{D}(u) \omega(x) dx. \end{aligned}$$

This is just the definition of self-adjointness.

$\implies$. Let $\mathcal{D}$ be written as in expression (1). Assume self-adjointness and choose two arbitrary compactly supported functions u and v . Then

$$\begin{aligned} 0 &= \int_M (u\mathcal{D}v - v\mathcal{D}u) \omega(x) dx \\ &= \int_M (uA^{ij}v_{ij} + uB^i v_i - vA^{ij}u_{ij} - vB^i u_i) \omega(x) dx \end{aligned} \quad (\text{VI-2})$$

$$= \int_M \left[-v_i \left(\frac{\partial A^{ij}}{\partial x^j} u\omega + A^{ij}u_j\omega + A^{ij}u\omega_j \right) \right] \quad (\text{VI-3})$$

$$\begin{aligned} &+ u_i \left(\frac{\partial A^{ij}}{\partial x^j} v\omega + A^{ij}v_j\omega + A^{ij}v\omega_j \right) \\ &+ uB^i v_i\omega - vB^i u_i \omega(x) dx. \end{aligned} \quad (\text{VI-4})$$

To achieve the three lines above, we used integration by parts on the first term in (2) (integrating $v_{ij}dx$) and on the third term in (2) (integrating $u_{ij}dx$). We can now use the symmetry of the coefficients A^{ij} to cancel the second term in (3) with the second term in (4).

$$\begin{aligned} &= \int_M (u_i v - v_i u) \left(\frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) - B^i \right) \omega(x) dx. \\ \therefore B^i &= \frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \end{aligned} \quad (*)$$

(The last line follows by choosing a compact set K , as small as we like, and then taking u and v such that $u_i v - v_i u$ is always positive on K and vanishes outside K , and $u_k v - v_k u$ is always much smaller than $u_i v - v_i u$ for $k \neq i$.)

Sub-lemma: Recall that $\mathcal{D}f = A^{ij}f_{ij} + B^i f_i$. In this notation,

$$\mathcal{D}f = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega) \iff B^i = \frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \quad \text{and} \quad A^{ij} = h^{ij} .$$

Proof: $\implies$.

$$\begin{aligned} \mathcal{D}f &= A^{ij}f_{ij} + B^i f_i = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega) \\ &= f_{ij} h^{ij} + f_i \left(\frac{\partial h^{ij}}{\partial x^j} + h^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \right) \end{aligned}$$

$$\therefore A^{ij} = h^{ij}, \quad (\text{VI-5})$$

$$\begin{aligned} B^i &= \frac{\partial h^{ij}}{\partial x^j} + h^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \\ &= \frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \end{aligned} \quad (\text{by (5)})$$

$\Leftarrow$. Assume $B^i = \frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega)$.

$$\begin{aligned} \text{Let } h^{ij} &= A^{ij}. \\ \text{Then } \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega) &= \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j A^{ij} \omega) \\ &= A^{ij} f_{ij} + f_i \frac{\partial A^{ij}}{\partial x^j} + f_i A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \\ &= A^{ij} f_{ij} + B^i f_i \\ &= \mathcal{D}f. \end{aligned} \quad \square$$

(*) now implies (by the Sub-lemma) that $\mathcal{D}f = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j h^{ij} \omega)$ for some h^{ij} . In fact, we can see from the calculations that we must have $h^{ij} = A^{ij}$. ■

Recall the co-ordinate expression (IV-1) for Δf :

$$\begin{aligned} \Delta f &= (n+2) \left(\frac{\int_I X^i X^j dX}{\int_I dX} f_{ij} + \right. \\ &\quad \left. + \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2\int_I \Gamma_{jq}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right) f_i \right) \end{aligned}$$

Lemma 1 shows that Δ is self-adjoint, and Lemma 2 gives conditions on the coefficients of Δ . In particular, the A^{ij} of Lemma 2 must just be

$$A^{ij} = K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}, \quad (\text{VI-6})$$

i.e. the coefficients of the osculating Riemannian metric. The B^i , of course, are just

$$B^i = (n+2) \left(\frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2\int_I \Gamma_{jq}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX} \right). \quad (\text{VI-7})$$

The sub-lemma gives the condition

$$\begin{aligned} B^i &= \frac{\partial A^{ij}}{\partial x^j} + A^{ij} \frac{\partial}{\partial x^j} (\ln \omega) \\ &= \frac{\partial}{\partial x^j} \left((n+2) \frac{\int_I X^i X^j dX}{\int_I dX} \right) + (n+2) \frac{\int_I X^i X^j dX}{\int_I dX} \left(\frac{\partial}{\partial x^j} \ln \omega \right). \end{aligned}$$

Using $\omega = \frac{\kappa_n}{\int_I dX}$,

$$\begin{aligned}
& \frac{1}{n+2} B^i \\
&= \frac{\frac{\partial}{\partial x^j} \int_I X^i X^j dX}{\int_I dX} - \frac{\int_I X^i X^j dX \cdot \frac{\partial}{\partial x^j} \int_I dX}{(\int_I dX)^2} + \frac{\int_I X^i X^j dX}{\int_I dX} \frac{\partial}{\partial x^j} \left(\ln \left(\frac{\kappa_n}{\int_I dX} \right) \right) \\
&= \frac{\frac{\partial}{\partial x^j} \int_I X^i X^j dX}{\int_I dX} - \frac{\int_I X^i X^j dX \cdot \frac{\partial}{\partial x^j} \int_I dX}{(\int_I dX)^2} - \frac{\int_I X^i X^j dX}{\int_I dX} \frac{1}{\int_I dX} \cdot \frac{\partial}{\partial x^j} \int_I dX \\
&= -2 \frac{\partial}{\partial x^j} \left(\ln \left(\int_I dX \right) \right) \cdot \frac{\int_I X^i X^j dX}{\int_I dX} + \frac{\frac{\partial}{\partial x^j} \int_I X^i X^j dX}{\int_I dX}. \tag{VI-8}
\end{aligned}$$

Of course from (7), we already know

$$\frac{1}{n+2} B^i = \frac{2\omega_j}{\omega} \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2 \int_I \Gamma_{jq}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX},$$

and if we again use $\omega = \frac{\kappa_n}{\int_I dX}$, then

$$\frac{1}{n+2} B^i = -2 \frac{\partial}{\partial x^j} \left(\ln \int_I dX \right) \cdot \frac{\int_I X^i X^j dX}{\int_I dX} - \frac{2 \int_I \Gamma_{jq}^q X^i X^j dX}{\int_I dX} - \frac{\int_I \Gamma_{jk}^i X^j X^k dX}{\int_I dX}. \tag{VI-9}$$

Equating (8) and (9), cancelling the common term, and multiplying through by $\int_I dX$ gives

$$-\frac{\partial}{\partial x^j} \int_I X^i X^j dX = \int_I \left(\Gamma_{jq}^q(X) X^i X^j + \frac{1}{2} \Gamma_{jk}^i(X) X^j X^k \right) dX. \tag{VI-10}$$

This, then, must be an identity in Finsler space. It would be very interesting to verify this identity directly (a fairly straightforward exercise in the Riemannian case).

If we go back to Lemma 2, we can see a way to write Δf in divergence form directly. The proof of Lemma 2 necessitates $h^{ij} = A^{ij} = K^{ij}$, the coefficients of the osculating Riemannian metric, and this is all that's needed to prove

Theorem VI-2. *Let M be a differentiable manifold with a Finsler metric F , and let $f : M \rightarrow \mathbb{R}$ be a smooth function. Let $dV = \omega(x)dx$, where $\omega(x) = \frac{\kappa_n}{\int_I dX}$*

is the coefficient of the canonical volume form, and $K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$ is the osculating Riemannian metric. Then the Laplacian

$$\Delta f(p) = 2(n+2) \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{\int_{\Sigma_\varepsilon} f dV}{\int_{\Sigma_\varepsilon} dV} - f(p) \right)$$

can also be written

$$\Delta f(p) = \frac{1}{\omega(p)} \frac{\partial}{\partial x^i} (f_j K^{ij} \omega(p)). \quad (\text{VI-11})$$

Thus Δ is a self-adjoint operator (that is, $\int_M (u \Delta v - v \Delta u) \omega(y) dy = 0$), in divergence form. ■

This provides an interesting interpretation of our Laplacian : we can see Δ as the Laplacian of the osculating Riemannian space, *but with respect to a volume form which comes from the Finsler metric*. Section VIII will compare the two forms closely.

Besides its equivalence to self-adjointness, we can view the divergence form property from another angle: an operator $\mathcal{D} = A^{ij} f_{ij} + B^i f_i$ can be put in divergence form if and only if $\mathcal{D}f = 0$ is the necessary and sufficient condition for f to minimize the energy functional $E(f) = \int_M e(f) \omega(x) dx$, where $e(f) = \frac{1}{2} A^{ij} f_i f_j$. To see this, let f be a minimum point of E , and differentiate a small perturbation $f + th$ of f , at $t = 0$; this differentiation results in 0 :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} (E(f + th)) \Big|_{t=0} = \frac{\partial}{\partial t} \int_M \frac{1}{2} A^{ij} (f + th)_i (f + th)_j \omega(x) dx \Big|_{t=0} \quad (\text{VI-12}) \\ &= \frac{1}{2} \int_M A^{ij} \frac{\partial}{\partial t} (f_i f_j + 2t f_i h_j + t^2 h_i h_j) \Big|_{t=0} \omega(x) dx \\ &= \int_M A^{ij} f_i h_j \omega(x) dx \\ &= - \int_M \frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (A^{ij} f_i \omega(x)) h_j(x) \omega(x) dx \\ &= \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j A^{ij} \omega). \end{aligned}$$

The operator $\mathcal{D}(f) = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j A^{ij} \omega)$ is in divergence form, and clearly we can reverse the steps to generate a point-wise energy measure $\frac{1}{2} A^{ij} f_i f_j$. In our case, of

course, $A^{ij} = K^{ij}$, the energy measure is $\frac{1}{2}K^{ij}f_i f_j$, and the mean-value Laplacian is written $\Delta = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j K^{ij} \omega)$.

This new interpretation, that the mean-value Laplacian vanishes on minima of the energy functional $\int_M \frac{1}{2} K^{ij} f_i f_j \omega(x) dx$, has been looked at very recently, in a much more general context, by Jost [Jost, Chap. 4]. We'll show that our constructions and his agree, despite their differing approaches. Jost considers a map $f : M \rightarrow N$, where (M, μ) is a measure space with a non-negative symmetric function $h(x, y)$, and (N, d) is a metric space. In our case, M would be a Finsler manifold, whose measure μ is generated by integrating the Busemann volume form $\omega(x) dx$, and whose non-negative symmetric function $h(x, y)$ is just the distance function $\rho(x, y)$. N would simply be the real numbers with the standard metric $d(x, y) = |x - y|$. Jost defines the total energy for a function f , with respect to $h(x, y)$, to be

$$E_h(f) = \int_M \int_M h(x, y) d^2(f(x), f(y)) \mu(dy) \mu(dx).$$

If we switch to our terminology, we get

$$E_h(f) = \int_M \left(\int_M h(x, y) (f(y) - f(x))^2 \omega(y) dy \right) \omega(x) dx.$$

In order to write the global energy in terms of a point-wise energy,

$$E_h(f) = \int_M e_h(f)(x) \omega(x) dx,$$

Jost defines

$$h_\varepsilon(x, y) := \chi_{\Sigma(x, \varepsilon)}(y),$$

the characteristic function of the ε -ball around x , and takes the limit as $\varepsilon \rightarrow 0$; he adds an unspecified constant $c(\varepsilon, \dim M)$, which we can see must be of the order $\frac{1}{\varepsilon^n}$:

$$(e(f))(x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{\Sigma(x, \varepsilon)} (f(y) - f(x))^2 \omega(y) dy.$$

Though Jost only indicates in general how to achieve an expression for this, we will evaluate it explicitly, with the same method we used to derive the expression

for our Laplacian. First, write f as a Taylor series about x :

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{\Sigma(x, \varepsilon)} \left(f(x) + f_i(x) y^i + \frac{1}{2} f_{ij}(x) y^i y^j + O(y^3) - f(x) \right)^2 \omega(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{\Sigma(x, \varepsilon)} (f_i(x) f_j(x) y^i y^j + O(y^3)) \omega(y) dy. \end{aligned}$$

Then change to exponential co-ordinates, and move to the tangent space $T_x M$, ignoring higher-order terms:

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{I_\varepsilon} f_i(x) f_j(x) \left(Y^i - \frac{1}{2} \Gamma_{ab}^i Y^a Y^b \right) \left(Y^j - \frac{1}{2} \Gamma_{cd}^j Y^c Y^d \right) \\ &\quad \cdot (\omega(y) + \omega_k(y) (Y^k - \frac{1}{2} \Gamma_{pq}^k Y^p Y^q)) (1 - \Gamma_{rs}^r) dY. \end{aligned}$$

Multiplying out, extracting 2nd-order terms, changing Y to X , and switching to the unit indicatrix:

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^n}{\varepsilon^n} \int_I f_i(x) f_j(x) X^i X^j \omega(x) dX \\ &= f_i(x) f_j(x) \left(\int_I X^i X^j dX \right) \omega(x), \end{aligned}$$

since none of the terms depends on ε . Now recall that

$$\omega(x) = \frac{\kappa_n}{\int_I dX},$$

so

$$(e(f))(x) = \kappa_n \frac{\int_I X^i X^j dX}{\int_I dX} f_i(x) f_j(x).$$

Since $K^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}$, we get that, up to a constant,

$$(e(f))(x) = K^{ij}(x) f_i(x) f_j(x)$$

is the point-wise energy for a function on a Finsler manifold. Jost's global energy functional, then, is

$$E(f) = \int_M K^{ij}(x) f_i(x) f_j(x) \omega(x) dx, \quad (\text{VI-13})$$

which, using the procedure starting in (VI-12), gives the mean-value Laplacian

$$(\Delta f)(x) = \frac{1}{\omega} \frac{\partial}{\partial x^i} (f_j K^{ij} \omega).$$

Clearly, then, our construction and Jost's agree, and we can think of our Laplacian as a measure of how harmonic a function is with respect to the energy functional (VI-13).

This energy formulation also leads to an easy comparison of currently known linear Laplacians, in particular that of Bao & Lackey [BL], and that of Antonelli & Zastawniak [AZ]. The energy functional

$$E(f) = \frac{1}{2} \int_M A^{ij} f_i f_j \alpha(x) dx$$

clearly isolates two quantities: a volume form $\alpha(x)dx$, and the principal part A^{ij} of the Laplacian. Once we set these quantities, we can use the procedure (VI-12) to generate an expression in divergence form, and then Lemma VI-2 to show that that expression is self-adjoint; we can also easily calculate an explicit expression. The machinery we have developed is indifferent to the A^{ij} and α we start with, and the various Laplacians arise from various starting points.

Bao & Lackey, for example, could generate their Laplacian by taking

$$A_{BL}^{ij} = \frac{-1}{\int_{S_x M} \sqrt{g(y)} \frac{(-1)^{n-1}}{\text{Vol}(S_x M)} \sqrt{g(y)} dy} \int_{S_x M} \sqrt{g(y)} g^{ij}(y) \frac{(-1)^{n-1}}{\text{Vol}(S_x M)} \sqrt{g(y)} dy$$

$$\alpha_{BL}(x) = \frac{-1}{\int_{S_x M} \sqrt{g(y)} \frac{(-1)^{n-1}}{\text{Vol}(S_x M)} \sqrt{g(y)} dy},$$

where the notation is explained in [BS]. After some manipulation, and use of the Sub-Lemma, we get, in our notation,

$$A_{BL}^{ij} = - \frac{\int_I g(X) g^{ij}(X) dX}{\int_I g(X) dX} \quad (\text{VI-14})$$

$$\alpha_{BL}(x) = \int_I g(X) dX. \quad (\text{VI-15})$$

(Recall that $g(x, X) := \det[g_{ij}(x, X)]$.)

Antonelli & Zastawniak's Laplacian, on the other hand, could come from taking (in their notation)

$$A_{AZ}^{ij} = \int_{I_x M} y^i y^j d\omega_x(y),$$

$$\alpha_{AZ}^{ij}(x) = \int_{I_x M} \sqrt{g(y)} d\omega_x(y).$$

After some manipulation, this becomes, in our notation,

$$A_{AZ}^{ij} = \frac{\int_I X^i X^j \sqrt{g(X)} dX}{\int_I \sqrt{g(X)} dx}, \quad (\text{VI-16})$$

$$\alpha_{AZ}^{ij}(x) = \frac{\int_I dX}{\int_I \sqrt{g(X)} dX}. \quad (\text{VI-17})$$

The mean-value Laplacian, of course, results from taking

$$A_{MV}^{ij} = (n+2) \frac{\int_I X^i X^j dX}{\int_I dX}, \quad (\text{VI-18})$$

$$\alpha_{MV}(x) = \frac{\kappa_n}{\int_I dX}. \quad (\text{VI-19})$$

(In all these expressions, constants in front of either the principal parts (such as Bao & Lackey's negative sign) or the volume forms (such as the mean-value κ_n) are of no account, because they divide out of any expression we manipulate.)

This list highlights an important difference between Riemannian and Finsler spaces. In a Riemannian space, there is *one* canonical volume form. In a Finsler space, there are *many* possible choices. Furthermore, any of the three above generalizes the Riemannian definition—in a Riemannian space they all simplify to $\sqrt{g}$. Likewise, the principal parts are all Riemannian metrics intrinsically associated to the Finsler space, and all reduce to the typical g^{ij} 's in a Riemannian space. (To see that all the A^{ij} 's above are Riemannian metrics, just work through the steps of Lemma V-1. Also, saying A^{ij} is a Riemannian metric is another way of saying the associated Laplacian is an elliptic operator.)

VII. The Laplacian on Forms, and Harmonic Mappings

So far, we have defined a Laplacian only for *functions* on a Finsler manifold. Riemannian geometry features two different generalizations. First, there is Hodge theory, which involves defining harmonicity for forms on M . Second, there is the theory of harmonic mappings, that is, defining harmonicity for a map from a Riemannian space M to another Riemannian space N ; a thorough reference for this is [EL].

The first half of this section will extend our definition to arbitrary p -forms. Although the definitions we give are sensible, they are not as natural as in the case of functions. The reason is that it is not clear what properties of the Riemannian case we should seek to generalize. In the case of functions, we generalized the mean-value property of harmonic functions, but there is no similar property for p -forms. It seems natural to insist that the Laplacian comes from some kind of elliptic complex, so that we can get a Hodge Decomposition Theorem, but this condition is not very stringent, and many possible Laplacians satisfy it. The only other condition which generalizes easily is the Riemannian statement that a harmonic 1-form evaluated on a Killing field should be a constant [Yano, Chap.2, § 3]. The concept of Killing field generalizes almost automatically to Finsler manifolds, so we might expect the same statement to hold in a Finsler space.

On the other hand, Killing fields are rather rare, so we'll be guided by our energy density for functions, and arrive at the Laplacian indirectly by constructing an energy density for p -forms. To define the energy density for a p -form at a point, we will use an averaging process, which is in line with thinking of a Laplacian as a mean value. At a point x , we can imagine all possible p -vectors of length 1 or smaller, that is, sets of p ordered vectors of unit (or smaller) length through the origin of $T_x M$. We can evaluate a p -form α (and square the result) on any of these p -vectors. We'll define the energy of α at x as the "average" of its value over all p -vectors.

As in the Riemannian case, start in a given co-ordinate system with the p -form α . We will evaluate this on the unit p -vectors. There are p unit vectors which can take any values in the unit ball, so we'll take as a region of integration p copies of the unit ball, and integrate over each in turn (scaling by the volumes, with a constant of $n+2$ which will be convenient later), with the integrand being the square of α evaluated on these p vectors (we take the square of α to make sure our energy is positive):

$$|\alpha| = \int_I \int_I \dots \int_I (\alpha(X_1, X_2, \dots, X_p))^2 \left(\frac{n+2}{\int_I dX} dX_1 \right) \left(\frac{n+2}{\int_I dX} dX_2 \right) \dots \left(\frac{n+2}{\int_I dX} dX_p \right). \quad (\text{VII-1})$$

Now think of α as being written

$$\alpha = \alpha_{i_1 i_2 \dots i_p}(x) dx^{i_1} dx^{i_2} \dots dx^{i_p}, \quad (\text{VII-2})$$

with the usual skew-symmetry in the components of α . Then

$$\begin{aligned} (\alpha(X_1, X_2, \dots, X_p))^2 &= (\alpha_{i_1 i_2 \dots i_p}(x) X_1^{i_1} X_2^{i_2} \dots X_p^{i_p}) (\alpha_{j_1 j_2 \dots j_p}(x) X_1^{j_1} X_2^{j_2} \dots X_p^{j_p}) \\ &= \alpha_{i_1 i_2 \dots i_p}(x) \alpha_{j_1 j_2 \dots j_p}(x) X_1^{i_1} X_1^{j_1} X_2^{i_2} X_2^{j_2} \dots X_p^{i_p} X_p^{j_p}. \end{aligned} \quad (\text{VII-3})$$

If we substitute this into (1), then we can perform the integrations over $X_1, X_2, \dots, X_p$ separately, and, since the coefficients $\alpha_{i_1 i_2 \dots i_p}(x)$ don't depend on tangent vectors, we can take them outside the integral to get

$$\begin{aligned} |\alpha|^2 &= \alpha_{i_1 i_2 \dots i_p} \alpha_{j_1 j_2 \dots j_p} (n+2)^p \frac{\int_I X_1^{i_1} X_1^{j_1} dX_1}{\int_I dX_1} \frac{\int_I X_2^{i_2} X_2^{j_2} dX_2}{\int_I dX_2} \dots \frac{\int_I X_p^{i_p} X_p^{j_p} dX_p}{\int_I dX_p} \\ &= \alpha_{i_1 i_2 \dots i_p} \alpha_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p}, \end{aligned} \quad (\text{VII-4})$$

which both bears a strong resemblance to the Riemannian definition

$$\alpha_{i_1 i_2 \dots i_p} \alpha_{j_1 j_2 \dots j_p} g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_p j_p},$$

and recalls our energy density definition for a function:

$$e(\phi) = K^{ij} \phi_i \phi_j.$$

This gives the energy density of α at a point x ; we define the energy of α over the entire compact manifold M in the obvious way as

$$\begin{aligned} |\alpha|^2 &= \int_M |\alpha|^2(x) dV \\ &= \int_M \alpha_{i_1 i_2 \dots i_p} \alpha_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} \omega(x) dx. \end{aligned} \quad (\text{VII-5})$$

We defined this energy in a particular co-ordinate system, so we should check that this definition is independent of co-ordinates. This follows from writing

$$\bar{\alpha}_{k_1 k_2 \dots k_p} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}} \alpha_{i_1 i_2 \dots i_p}, \quad (\text{VII-6})$$

$$\bar{\alpha}_{l_1 l_2 \dots l_p} = \frac{\partial \bar{x}^{j_1}}{\partial x^{l_1}} \frac{\partial \bar{x}^{j_2}}{\partial x^{l_2}} \dots \frac{\partial \bar{x}^{j_p}}{\partial x^{l_p}} \alpha_{j_1 j_2 \dots j_p}, \quad (\text{VII-7})$$

$$\bar{K}^{q_1 l_1} = \frac{\partial x^{q_1}}{\partial \bar{x}^{l_1}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} K^{i_1 j_1}, \quad (\text{VII-8})$$

in a barred co-ordinate system $\bar{x}^i = \bar{x}^1(x)$. Substituting (6), (7), and (8) into the integrand of (5) gives

$$\alpha_{i_1 i_2 \dots i_p} \alpha_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} = \bar{\alpha}_{k_1 k_2 \dots k_p} \bar{\alpha}_{l_1 l_2 \dots l_p} \bar{K}^{q_1 l_1} \bar{K}^{q_2 l_2} \dots \bar{K}^{q_p l_p}, \quad (\text{VII-9})$$

so that the integrand is an invariant function; clearly then, $|\alpha|$ both at a point and over the manifold, is an invariant. (As [BL, 3] remark in their paper, this invariance depends solely on the tensor (density) character of $K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} \omega(x)$; they use a different tensor density to construct an energy density, and indeed, there are many other possible densities, too.)

This energy contains all the information we need to construct a Laplacian on forms, with a method very similar to the Riemannian method. The key is to note that expression (8) lends itself readily to an *inner product* on p -forms. Specifically, if α and β are p -forms, with $\alpha = \alpha_{i_1 i_2 \dots i_p} dx^{i_1} dx^{i_2} \dots dx^{i_p}$, $\beta = \beta_{j_1 j_2 \dots j_p} dx^{j_1} dx^{j_2} \dots dx^{j_p}$,

$$\begin{aligned} \langle \alpha, \beta \rangle|_x &:= \alpha_{i_1 i_2 \dots i_p} \beta_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p}, \\ \langle \alpha, \beta \rangle &:= \int_M \alpha_{i_1 i_2 \dots i_p} \beta_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} \omega(x) dx. \end{aligned} \quad (\text{VII-10})$$

Now define a co-differential $\delta : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)$, where $\Lambda^p(M)$ is the space of p -forms on M by requiring (for any p -form α and $(p-1)$ -form γ)

$$\langle \alpha, d\gamma \rangle = \langle \delta\alpha, \gamma \rangle, \quad (\text{VII-11})$$

i.e. that d and δ are adjoints. As usual, define $\delta f = 0$ for any *function* f . Now simply define, for any p -form α ,

$$-\Delta\alpha = d\delta\alpha + \delta d\alpha. \quad (\text{VII-12})$$

This is our Laplacian on p -forms.

We should check that this new definition is compatible with the old one for the case of 0-forms, or functions.

Lemma VII-1. Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Our two definitions of Laplacians agree, i.e.

$$\delta df = \frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (f_i K^{ij} \omega(x)).$$

Proof. We will calculate explicitly $\delta : \Lambda^1(M) \rightarrow \Lambda^0(M)$. Let α be a 1-form. Condition (VII-11) requires

$$\langle \alpha, d\gamma \rangle = \langle \delta \alpha, \gamma \rangle,$$

where γ is a 0-form, i.e. γ is a function from M to $\mathbb{R}$. Use (VII-10), the definition of the inner product, to write

$$\langle \alpha, d\gamma \rangle = \int_M \alpha_i (d\gamma)_j K^{ij} \omega(x) dx.$$

Since $d\gamma$ is just the exterior derivative of γ , $\therefore (d\gamma)_j = \gamma_j$. so

$$\begin{aligned} \langle \alpha, d\gamma \rangle &= \int_M \alpha_i \gamma_j K^{ij} \omega(x) dx \\ &= - \int_M \gamma \frac{\partial}{\partial x^j} (\alpha_i K^{ij} \omega(x)) dx \\ &\quad (\text{integration by parts, integrating } \gamma_j) \\ &= - \int_M \gamma \frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (\alpha_i K^{ij} \omega(x)) \omega(x) dx \\ &= \langle -\frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (\alpha_i K^{ij} \omega(x)), \gamma \rangle. \end{aligned}$$

Since (VII-11) determines δ uniquely, it must be that

$$\delta \alpha = -\frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (\alpha_i K^{ij} \omega(x)).$$

Now let $\alpha = df$. Then $\alpha_i = f_i$, and so

$$\delta \alpha = \delta df = -\frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (f_i K^{ij} \omega(x)),$$

i.e.

$$\Delta(f) = \frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (f_i K^{ij} \omega(x)),$$

which was to be shown. ■

We also note from (VII-11) that Δ as an operator on p -forms is self-adjoint. To see this, let α, β be two p -forms. Then

$$\begin{aligned}
 \langle \alpha, -\Delta\beta \rangle &= \langle \alpha, (d\delta + \delta d)\beta \rangle \\
 &= \langle \alpha, d\delta\beta \rangle + \langle \alpha, \delta d\beta \rangle \\
 &= \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle \\
 &\quad \text{(because } \delta \text{ and } d \text{ are adjoints)} \\
 &= \langle d\delta\alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle \\
 &\quad \text{(because } \delta \text{ and } d \text{ are adjoints)} \\
 &= \langle (d\delta + \delta d)\alpha, \beta \rangle \\
 &= \langle -\Delta\alpha, \beta \rangle.
 \end{aligned}$$

In terms of integrals, we would write

$$\begin{aligned}
 &\int_M \alpha_{i_1 i_2 \dots i_p} (\Delta\beta)_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} \omega(x) dx \\
 &= \int_M \beta_{i_1 i_2 \dots i_p} (\Delta\alpha)_{j_1 j_2 \dots j_p} K^{i_1 j_1} K^{i_2 j_2} \dots K^{i_p j_p} \omega(x) dx,
 \end{aligned}$$

which is a direct generalization of the condition given for functions.

A point on which the Finsler Laplacian disagrees with the Riemannian Laplacian is the case of n -forms. In the Riemannian case, the canonical volume form is always harmonic. In the Finsler case, we'll find a condition under which an n -form η is harmonic, and see that the volume form $dV = \omega(x)dx$ doesn't generally satisfy this condition. Assume.

$$\Delta\eta = 0.$$

Since $d\eta = 0$, $\therefore$

$$\Delta\eta = -d\delta\eta = 0.$$

Then, if β is any n -form,

$$0 = \langle \beta, -d\delta\eta \rangle$$

$$\begin{aligned}
&= -\langle \delta\beta, \delta\eta \rangle \\
&= -\langle d\delta\beta, \eta \rangle \\
&= \lambda(x) \langle dx, \eta(x) dx \rangle
\end{aligned}$$

for some function $\lambda(x) : M \rightarrow \mathbb{R}$, where $\lambda(x)dx = -d(\delta\beta)$,

$$\begin{aligned}
&= \int_M \lambda(x) \eta(x) \det[K^{ij}] \omega(x) dx \\
&= \int_M \eta(x) \det[K^{ij}] \omega(x) (\lambda(x) dx) \\
&= \int_M \eta(x) \det[K^{ij}] \omega(x) d(\delta\beta).
\end{aligned}$$

If η is the canonical volume form $\omega(x)dx$, then

$$0 = \int_M \omega^2(x) \det[K^{ij}] d(\delta\beta). \quad (\text{VII-13})$$

In the Riemannian case, $\det[K^{ij}] = \frac{1}{g}$, $\omega(x) = \sqrt{g}$, and $\omega^2(x) \det[K^{ij}] = 1$, so this expression is just the integral of a closed form, which must be 0. In the Finsler case, however, $\omega^2(x) \det[K^{ij}]$ is not in general 1—in fact, by Theorem VI-2, the expression $\omega^2(x) \det[K^{ij}]$ is 1 only when the Finsler metric actually is Riemannian. To satisfy equation (13), though, we don't necessarily need $\omega^2(x) \det[K^{ij}] = 1$, but we do need

$$\omega^2(x) \det[K^{ij}] = \text{constant}. \quad (\text{VII-14})$$

This clearly is the case for any Minkowski space, for example, but in general equation (14) is too stringent to be satisfied.

With the apparatus of a Laplacian on forms, a natural question is whether there is a Hodge Decomposition Theorem for Finsler spaces. [BL] have already answered this in the affirmative and their method, which uses a different Laplacian from ours, shows how both Laplacians give a Hodge Decomposition. (Their paper also applies to the Laplacian recently obtained by Antonelli and Zastawniak [AZ].)

To see why the method is so powerful, reflect that the Hodge Decomposition Theorem gives rather coarse, topological information about M , whereas a Laplacian gives more refined, local information. Even in the Riemannian case, it is easy

to picture a manifold whose topology stays constant, while its metric undergoes drastic local changes. All that is needed for a Hodge Theorem is a positive-definite inner product $\langle \alpha, \beta \rangle$ on forms. Since the differential d is defined independently of any metric, we can use this inner product to define a co-differential δ , by requiring that

$$\langle \alpha, d\beta \rangle = \langle \delta\alpha, \beta \rangle \quad (\text{VII-15})$$

and then defining

$$-\Delta := d\delta + \delta d. \quad (\text{VII-16})$$

This construction is really a special case of a more general object. The sequence of p -form bundles on M , with the differential operator d , is an *elliptic complex*, whose (positive-definite) inner product allows us to define a Laplacian, as just described, which then yields a Hodge decomposition for every $\Delta^{(p)}$, i.e. Δ restricted to p -forms [Wells, chap. 4, §5]. Briefly, both d and δ satisfy the null-squared property, that is $dd = \delta\delta = 0$, which guarantees, by calculations with exact sequences, that $\Delta^{(p)}$ is elliptic. The symmetry of (VII-16) guarantees that $\Delta^{(p)}$ is self-adjoint, and so we just apply standard decomposition theorems to the self-adjoint, elliptic operator $\Delta^{(p)}$.

The second major generalization of the Laplacian on functions is the theory of harmonic mappings. Consider a mapping ϕ from a compact, boundaryless Finsler manifold (M, F) to a second compact, boundaryless Finsler manifold (N, G) . If we could find an energy density $e(\phi)(x)$ for a point $x \in M$, then we could follow Eells & Lemaire's method [EL, §2] to derive conditions which a harmonic, or energy-minimizing, ϕ must satisfy. Jost's definition from Section VI fits this situation well because of its generality and its agreement with the mean-value Laplacian for functions. To recall, Jost defined the energy density $e(\phi)$, up to a constant, by

$$e(\phi)(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^n} \int_{\Sigma(x, \epsilon)} {}^N\rho^2(\phi(x), \phi(y)) \omega(y) dy, \quad (\text{VII-17})$$

where $n = \dim M$, $\omega(y)dy$ is Busemann's volume form on M , and ${}^N\rho$ is the distance on N in the Finsler metric G .

To see what we get in the simpler case where (N, G) is Riemannian, use our usual method of replacing the integrand with its Taylor series around x . This time, denote the components of ϕ by ϕ^α , for $\alpha = 1, \dots, \dim N$, and, since x is fixed, write

$${}^N\rho^2(y) = {}^N\rho^2(\phi(x), \phi(y)). \quad (\text{VII-18})$$

Now replace both ϕ and ${}^N\rho^2$ by their Taylor series:

$$e(\phi)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{\Sigma(x, \varepsilon)} ({}^N\rho^2(\phi(x)) + \quad (\text{VII-19})$$

$$+ {}^N\rho_\alpha^2(\phi(x))(\phi_i^\alpha(x)(y^i - x^i) + \frac{1}{2}\phi_{ij}^\alpha(x)(y^i - x^i)(y^j - x^j) + O(y^3)) + \quad (\text{VII-20})$$

$$+ \frac{1}{2} {}^N\rho_{\alpha\beta}^2(\phi(x))(\phi_i^\alpha(x)(y^i - x^i) + O(y^2)) \cdot (\phi_i^\beta(x)(y^i - x^i) + O(y^2)) + \quad (\text{VII-21})$$

$$+ O(\phi^3))\omega(y)dy.$$

The term ${}^N\rho^2(\phi(x))$ is just the distance from x to x , so this term is 0. The term ${}^N\rho_\alpha^2(x)$ also vanishes identically at x , for every α , because the distance-squared function achieves a minimum there. Now consider the remaining term, with terms of higher order neglected:

$$e(\phi)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n} \int_{\Sigma(x, \varepsilon)} \frac{1}{2} {}^N\rho_{\alpha\beta}^2(x) \phi_i^\alpha(x) \phi_j^\beta(x) (y^i - x^i)(y^j - x^j) \omega(y) dy. \quad (\text{VII-22})$$

If we now switch to exponential co-ordinates, notice that $\frac{1}{2} {}^N\rho_{\alpha\beta}^2$, which is defined to be $\frac{1}{2} \frac{\partial^2 ({}^N\rho^2)}{\partial x^\alpha \partial x^\beta} \Big|_x$, becomes, after some manipulation, $\frac{1}{2} \frac{\partial^2 G^2}{\partial X^\alpha \partial X^\beta} \Big|_x$. Set

$$h_{\alpha\beta} := \frac{1}{2} \frac{\partial^2 G^2}{\partial X^\alpha \partial X^\beta}, \quad (\text{VII-23})$$

so $h_{\alpha\beta}$ are the co-ordinates of the metric tensor for N . Take the limit in (VII-22), giving

$$e(\phi)(x) = K^{ij} \phi_i^\alpha \phi_j^\beta h_{\alpha\beta}, \quad (\text{VII-24})$$

where K^{ij} is the osculating Riemannian metric for (M, F) . If (M, F) actually is Riemannian, then, apart from a constant of $\frac{1}{2}$, this is just Eells & Lemaire's definition of the energy density for Riemannian mappings, so Jost's definition generalizes their definition.

The previous paragraph dealt with (N, G) as a Riemannian manifold. If (N, G) were a Finsler manifold, the derivation would break down because the term ${}^N\rho^2_{\alpha\beta}$ in (VII-21) is not generally defined—the distance-squared function is only C^1 . Nevertheless, we can find an expression for (VII-17). This expression will avoid the problem of taking two derivatives of the distance function by taking two derivatives of the metric function with respect to specified directions.

To see how this works, choose exponential co-ordinates on M around x , and on N around $\phi(x)$. Now, as $\epsilon \rightarrow 0$, y approaches x on M , and thus $\phi(y)$ approaches $\phi(x)$ on N . Since we have chosen normal co-ordinates on N , we see that

$${}^N\rho^2(\phi(x), \phi(y)) \rightarrow h_{\alpha\beta}(\phi(x), \phi(y) - \phi(x))(\phi^\alpha(y) - \phi^\alpha(x))(\phi^\beta(y) - \phi^\beta(x)), \quad (\text{VII-25})$$

where

$$G(x, X) = h_{\alpha\beta}(x, X)X^\alpha X^\beta, \quad (\text{VII-26})$$

and $\phi(y) - \phi(x)$ is the tangent vector with co-ordinates $(\phi^1(y) - \phi^1(x), \dots, \phi^{\dim N}(y) - \phi^{\dim N}(x))$. In normal co-ordinates, $\phi(x) = (0, 0, \dots, 0)$, so we can write (VII-25) more simply as

$${}^N\rho^2(\phi(x), \phi(y)) \rightarrow h_{\alpha\beta}(\phi(x), \phi(y))(\phi^\alpha(y))(\phi^\beta(y)), \quad (\text{VII-27})$$

Recall that $y \in M$, and we have also chosen exponential co-ordinates for M ; thus, the point $y = (y^1, y^2, \dots, y^n)$ can be identified with the tangent vector $Y = (y^1, y^2, \dots, y^n)$ with the same co-ordinates. Therefore, as $\epsilon \rightarrow 0$

$$\begin{aligned} {}^N\rho^2(\phi(x), \phi(y)) &\rightarrow h_{\alpha\beta}(\phi(x), \phi_*(Y))\phi^\alpha(Y)\phi^\beta(Y) \\ &= (\phi^*h_{\alpha\beta}(x, Y))Y^\alpha Y^\beta \\ &= ((\phi^*G)(Y), \end{aligned} \quad (\text{VII-28})$$

where $Y \in T_x M$.

Now (VII-17) has become

$$e(\phi)(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^n} \int_{\Sigma(x, \epsilon)} (\phi^* G)(Y) \omega(y) dy, \quad (\text{VII-29})$$

where Y has the same co-ordinates as y . If we write (VII-29) as an expression on $T_x M$, instead of on M , and write X 's instead of Y 's, we get

$$e(\phi)(x) = \frac{\int_{T_x M} (\phi^* G)(X) dX}{\int_{T_x M} dX} \quad (\text{VII-30})$$

If we write this in a more co-ordinatized notation, we get a clear generalization of (VII-24):

$$e(\phi)(x) = \phi_i^\alpha \phi_j^\beta \frac{\int_{T_x M} h_{\alpha\beta}(\phi(x), \phi_* X) X^i X^j dX}{\int_{T_x M} dX}. \quad (\text{VII-31})$$

This expression, then, is the energy density for ϕ at a point $x \in M$. It differs from the Riemannian case in that it mixes the two metrics very intimately. We can think of (VII-30) as pulling back the metric $h_{\alpha\beta}$ on N to the manifold M , applying the pulled-back metric to the unit ball—with respect to the metric on M —at the point in question, and then averaging. For the Riemannian case, Eells & Lemaire found a simple “tension field” that vanishes whenever the integral of the energy is minimized: ϕ minimizes $E(\phi)$ if and only if $\tau(\phi) = -d^* d\phi = 0$, where $d\phi$ is a function from $TM \otimes \phi^{-1}TN \rightarrow \mathbb{R}$. If, as we did in (VI-12), we replace ϕ^α in the expression

$$\int_M \left(\frac{\int_{T_x M} (\phi^* G)(X) dX}{\int_{T_x M} dX} \right) \omega(x) dx$$

by a variation $\phi^\alpha + t\hbar^\alpha$, differentiate with respect to t , and evaluate at $t = 0$, we arrive at a similar Finsler tension field:

$$\tau(\phi) = \int_{T_x M} \left(\frac{\partial G}{\partial y^\alpha}(\phi_* X) - \frac{1}{\omega(x)} \frac{\partial}{\partial x^i} \left(\left(\int_{T_x M} \frac{\partial G}{\partial Y^\alpha} X^i dX \right) \omega(x) \right) \right) \omega(x) dx, \quad (\text{VII-32})$$

where (y, Y) are co-ordinates on TN . Apart from its ungainliness, this expression is not linear in ϕ , so probably will not be of much use.

We can, however, find a fairly compact expression when (M, F) is a Finsler manifold, and (N, G) is a Riemannian manifold. In that case [EL, (2.4)], which derives the Riemannian tension field, follows through nearly line for line if we use the Levi-Civita connection of the osculating Riemannian metric K^{ij} . The only difficulty occurs in the last line

$$\int_M \langle dv, d\phi \rangle v_g = \int_M \langle v, d^* d\phi \rangle v_g, \quad (\text{VII-33})$$

where, in our notation, $v_g = \omega(x)dx$. This last expression only makes sense if we assume that d^* is defined with respect to the volume form v_g . Since $\omega(x)dx$ is not the volume form of the osculating Riemannian metric, we must use (VII-33) to *define* d^* , and thus get a d_ω^* depending explicitly on the Busemann volume form. Thus we can state the result: ϕ minimizes the energy functional $E(\phi) = \int_M e(\phi)(x)\omega(x)dx$ if and only if $d_\omega^* d\phi = 0$, where $d\phi = \phi_i^\alpha$ is a function from $TM \otimes \phi^{-1}T^*N \rightarrow \mathbb{R}$.

Another simple case is that of (N, G) being the Euclidean space $\mathbb{R}^n$. Then $h_{\alpha\beta} = \delta_{\alpha\beta}$, so

$$e(\phi)(x) = K^{ij} \phi_i^\alpha \phi_j^\alpha, \quad (\text{VII-34})$$

$$E(\phi) = \int_M K^{ij} \phi_i^\alpha \phi_j^\alpha \omega(x) dx. \quad (\text{VII-35})$$

If we apply procedure (VI-12) to (VII-19), we get

$$0 = \int_M \frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (K^{ij} \phi_i^\alpha \omega(x)) h^\alpha(x) \omega(x) dx, \quad (\text{VII-36})$$

for any function h^α . This is a linear system of equations, all of which must be 0, so

$$\frac{1}{\omega(x)} \frac{\partial}{\partial x^j} (K^{ij} \phi_i^\alpha \omega(x)) = 0 \quad (\text{VII-37})$$

for any α , i.e. ϕ is harmonic in every component.

PART 2. A VOLUME INVARIANT FOR FINSLER SPACES

VIII. Characterizing Riemann Spaces via Volumes: the Volume Invariant

We have two distinct volume forms: Busemann's canonical form for Finsler spaces and the volume form of the osculating Riemannian metric. We'll see that we can use these forms to characterize Riemannian manifolds as a special case of Finsler manifolds.

A natural starting point is to ask whether the osculating Riemannian metric K^{ij} has a Riemannian volume form which *equals* the volume form of the Finsler space which K^{ij} osculates. An example shows that this is not generally the case. Example: Let $M = \mathbb{R}^2$, and let F be the Minkowski metric whose every unit ball is actually a diamond. Since the terms K^{ij} , $\int_I dX$, and so on that we'll use depend solely on the indicatrix at a point, and not on any derivatives, we'll work just at the origin p and say that $T_p M$ (which is canonically isomorphic to $\mathbb{R}^2$) has the diamond with vertices $(1, 0)$, $(0, -1)$, $(-1, 0)$, and $(0, 1)$ as its indicatrix. Strictly speaking, this is not a Finsler unit ball, so we must bow all the edges very, very slightly, and round off the corners microscopically; since we will only be taking *integrals* over I , these adjustments will not significantly affect our final answers. Let us calculate the osculating Riemannian coefficients K^{ij} , $1 \leq i, j \leq 2$. By the symmetry of the indicatrix,

$$K^{12} = K^{21} = (n+2) \frac{\int_I X^1 X^2 dX}{\int_I dX} = 0, \quad (\text{VIII-1})$$

and

$$K^{11} = K^{22}. \quad (\text{VIII-2})$$

We'll calculate K^{11} :

$$\begin{aligned} K^{11} &= (n+2) \frac{\int_I X^1 X^1 dX}{\int_I dX} \\ &= \frac{4}{2} \cdot \left(4 \int_0^1 \int_0^{1-X^1} (X^1)^2 dX^2 dX^1 \right) \\ &= 2 \cdot 4 \int_0^1 (X^1)^2 - (X^1)^3 dX^1 \end{aligned}$$

$$\begin{aligned}
&= 8 \cdot \left(\frac{1}{3}(X^1)^3 - \frac{1}{4}(X^1)^4 \right) \Big|_0^1 \\
&= \frac{2}{3}.
\end{aligned} \tag{VIII-3}$$

By (1), (2), and (3),

$$K^{ij} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}.$$

K^{ij} are the conjugate components of a Riemannian metric. The coefficient of the volume form of that metric is

$$\begin{aligned}
\sqrt{\det[K_{ij}]} &= \sqrt{\frac{1}{\det[K^{ij}]}} \\
&= \frac{1}{\sqrt{\frac{4}{9}}} \\
&= \frac{3}{2}.
\end{aligned} \tag{VIII-4}$$

The coefficient of the Finsler volume form, however, is given by

$$\frac{\kappa_2}{\int_I dX} = \frac{1}{2} \kappa_2 = \frac{\pi}{2}. \tag{VIII-5}$$

Since (4) and (5) are not equal, the Finsler space and the osculating Riemannian space have different volume forms, and therefore our Laplacian cannot in general be induced by a Riemannian metric, as was asserted earlier without proof.

In fact, we can say considerably more. Not only is it possible for a Finsler volume form to differ from the osculating Riemannian volume form, but the two forms agree *only* when the Finsler metric is actually Riemannian. The proof of this fact, presented below, is a pointwise proof, that is, it uses solely the Finsler function restricted to the tangent space (which is isomorphic to $\mathbb{R}^n$) at one point, and takes no account of neighboring points or even infinitesimal changes in the metric. The only data needed for the proof is the unit Finsler ball $\mathcal{B}$ at a particular point, and we don't need either $\mathcal{B}$'s convexity or its symmetry, two hallmarks of a Finsler metric. The proof shows that if the two volume forms agree, then $\mathcal{B}$ must actually be an ellipsoid at that point, and therefore (because that property needs only pointwise

data) at every point—but a space whose every unit ball is an ellipsoid is actually Riemannian. Furthermore, we get an unexpected inequality which characterizes Riemannian manifolds—the volume of the osculating ellipsoid is never less than the volume of the Finsler ball.

Theorem VIII-1. Let $B \subset \mathbb{R}^n = \{(X^1, X^2, \dots, X^n) | X^i \in \mathbb{R}^n\}$ be a bounded, open, measurable set. Let $K^{ij} := (n+2) \frac{\int_B X^i X^j dX}{\int_B dX}$. We know that the components K^{ij} are the inverse components of that ellipsoid $\mathcal{E}$ which is the unit ball of the Euclidean metric $F^2(X) = K_{jk} X^j X^k$, where $K^{ij} K_{jk} = \delta_k^i$. Then

$$\int_B dX \leq \int_{\mathcal{E}} dX,$$

with equality if and only if

$$B = \mathcal{E}.$$

Proof. Choose co-ordinates so that $K^{ij} = \delta^{ij}$, i.e

$$(n+2) \frac{\int_B X^i X^j dX}{\int_B dX} = \delta^{ij}. \quad (\text{VIII-6})$$

The ellipsoid arising from these co-ordinates is just the unit sphere

$$E = \{(X^1, X^2, \dots, X^n) \in \mathbb{R}^n | \Sigma (X^i)^2 = 1\},$$

and if we can obtain the result in this case, then we can obtain the result for any ellipsoid $\mathcal{E}$ simply by an appropriate linear transformation.

We first prove “equality if and only if,” i.e. that $\int_B dX = \int_E dX$ implies $B = E$. Let $(r, \theta_1, \theta_2, \dots, \theta_{n-1})$ be the usual spherical co-ordinates. Then

$$\begin{aligned} \int_B r^2 dX &= \int_B \Sigma (X^i)^2 dX \\ &= \frac{n}{n+2} \int_B dX && (\text{using (VIII-6)}) \\ &= \frac{n}{n+2} \int_E dX && (\text{by hypothesis}) \\ &= \int_E r^2 dX && (\text{because } E \text{ is the unit sphere}) \\ \therefore \int_B r^2 dX &= \int_E r^2 dX. && (\text{VIII-7}) \end{aligned}$$

We need only this equation and the hypothesis

$$\int_{\mathcal{B}} dX = \int_E dX \quad (\text{VIII-8})$$

for the proof of the equality case.

Now consider that

$$\begin{aligned} \mathcal{B} \cup (E \setminus \mathcal{B}) &= E \cup (\mathcal{B} \setminus E); \\ \therefore \int_{\mathcal{B}} h(X) dX + \int_{E \setminus \mathcal{B}} h(X) dX &= \int_E h(X) dX + \int_{\mathcal{B} \setminus E} h(X) dX \end{aligned}$$

for any function $h : \mathbb{R}^n \rightarrow \mathbb{R}$. In particular, for the two cases $h(X) = r^2$ and $h(X) = 1$, expressions (VIII-7) and (VIII-8) allow us to cancel the first integral on each side:

$$\begin{cases} \int_{E \setminus \mathcal{B}} r^2 dX = \int_{\mathcal{B} \setminus E} r^2 dX & (\text{VIII-9}) \\ \int_{E \setminus \mathcal{B}} dX = \int_{\mathcal{B} \setminus E} dX & (\text{VIII-10}) \end{cases}$$

Note, however, that $r^2 \leq 1$ inside E and thus on $E \setminus \mathcal{B}$, while $r^2 \geq 1$ outside E and thus on $\mathcal{B} \setminus E$, i.e.

$$r^2|_{\mathcal{B} \setminus E} \geq r^2|_{E \setminus \mathcal{B}}. \quad (\text{VIII-11})$$

(VIII-10) says that $E \setminus \mathcal{B}$ and $\mathcal{B} \setminus E$ have the same volume, yet (VIII-9) says that the positive function r^2 , which is bigger on $\mathcal{B} \setminus E$ than on $E \setminus \mathcal{B}$ by (VIII-11), has the same integral over these two regions. This is possible only if $E \setminus \mathcal{B}$ and $\mathcal{B} \setminus E$ are empty, i.e.

$$E = \mathcal{B}.$$

To prove the inequality, again use the function r^2 in the set identity

$$\begin{aligned} \mathcal{B} \cup E \setminus \mathcal{B} &= E \cup \mathcal{B} \setminus E : \\ \int_{\mathcal{B}} r^2 dX + \int_{E \setminus \mathcal{B}} r^2 dX &= \int_E r^2 dX + \int_{\mathcal{B} \setminus E} r^2 dX. \end{aligned} \quad (\text{VIII-12})$$

As before,

$$r^2 \geq 1 \text{ on } \mathcal{B} \setminus E \implies \int_{\mathcal{B} \setminus E} r^2 dX \geq \int_{\mathcal{B} \setminus E} dX, \quad (\text{VIII-13})$$

$$r^2 \leq 1 \text{ on } E \setminus \mathcal{B} \implies \int_{E \setminus \mathcal{B}} r^2 dX \leq \int_{E \setminus \mathcal{B}} dX, \quad (\text{VIII-14})$$

$$\text{and } \int_E r^2 dX = \frac{n}{n+2} \int_E dX. \quad (\text{VIII-15})$$

Use the hypothesis $(n+2) \frac{\int_L X^i X^j dX}{\int_L dX} = \delta^{ij}$ to get

$$\int_B r^2 dX = \frac{n}{n+2} \int_B dX \quad (\text{VIII-16})$$

Substitute (13)-(16) into (12):

$$\frac{n}{(n+2)} \int_B dX + \int_{E \setminus B} dX \geq \frac{n}{(n+2)} \int_E dX + \int_{B \setminus E} dX.$$

Use the set identities

$$B = (E \cup B) \setminus (E \setminus B)$$

$$E = (E \cup B) \setminus (B \setminus E)$$

on the first term on each side:

$$\begin{aligned} & \frac{n}{(n+2)} \left(\int_{E \cup B} dX - \int_{E \setminus B} dX \right) + \int_{E \setminus B} dX \\ & \geq \frac{n}{(n+2)} \left(\int_{E \cup B} dX - \int_{B \setminus E} dX \right) + \int_{B \setminus E} dX. \end{aligned}$$

Cancel the term $\frac{n}{(n+2)} \int_{E \cup B} dX$, and divide by $\frac{2}{(n+2)}$ to get

$$\begin{aligned} & \int_{E \setminus B} dX \geq \int_{B \setminus E} dX \\ & \int_{E \cap B} dX + \int_{E \setminus B} dX \geq \int_{E \cap B} dX + \int_{B \setminus E} dX \\ & \int_E dX \geq \int_B dX. \end{aligned} \quad \blacksquare$$

This theorem characterizes Riemannian spaces as a subset of Finsler spaces solely in terms of volume functions. Typically we characterize Riemannian spaces as Finsler spaces whose Cartan tensor vanishes. The Cartan tensor is a third-order tensor, whose components are functions of a manifold's $2n$ -dimensional tangent bundle. The osculating Riemannian volume form is a zero-order tensor density, whose components are functions of the n -dimensional manifold. Thus the volume form criterion is simpler, and should be easier to apply in many circumstances. In more technical terms, we state

Theorem VIII-2. Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij} . Then $\omega(x) \geq k(x)$, and $\omega(x) = k(x)$ if and only if (M, F) actually is Riemannian, with metric K^{ij} .

Proof. If $\mathcal{B}(x)$ is the Finsler unit ball at a point x , then the coefficient of Busemann's volume form is

$$\omega(x) = \frac{\kappa_n}{\int_{\mathcal{B}} dX}.$$

The coefficient of the volume form from the osculating Riemannian metric, on the other hand, is

$$k(x) = \frac{\kappa_n}{\int_{\mathcal{E}} dX},$$

where $\mathcal{E}$ is the osculating Riemannian unit ball at x . By Theorem VIII-1, we always have

$$\int_{\mathcal{B}} dX \leq \int_{\mathcal{E}} dX.$$

Therefore $k(x) \leq \omega(x)$, with equality exactly when $\mathcal{B} = \mathcal{E}$, i.e. when (M, F) is the Riemannian space (M, K^{ij}) . ■

Recall the values $k(x) = \frac{3}{2}$ and $\omega(x) = \frac{\pi}{2}$ from equations (VIII-4) and (VIII-5), which we calculated in our example.

Theorem VIII-2 suggests the definition of a new quantity, an invariant volume function reminiscent of the one defined by Bao & Shen [BS]. Given two volume forms as above, we can always consider their "ratio," that is, the (sole) component of the first form in some co-ordinate system, divided by the (sole) component of the second form in that co-ordinate system. The result is clearly a scalar invariant.

Definition. Let (M, F) be a Finsler manifold with osculating Riemannian metric K^{ij} and Busemann volume form $\omega(x)dx$. Let $k(x)dx$ be the volume form arising from the metric K^{ij} . Define

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}, \quad (\text{VIII-17})$$

or, substituting in expressions for k and ω :

$$\mathcal{V}(x) = \frac{\kappa_n}{\int_I dX} \det \left[\frac{\int_I X^i X^j dX}{\int_I dX} \right] \quad (\text{VIII-18})$$

$$= \kappa_n(n+2)^n \left(\int_I dX \right)^{n+1} \det \left[\int_I X^i X^j dX \right]. \quad (\text{VIII-19})$$

Because $k(x) \leq \omega(x)$, and both $k(x)$ and $\omega(x)$ are always positive, we have

$$0 < \mathcal{V}(x) \leq 1 \quad (\text{VIII-20})$$

for any $x \in M$. Furthermore, by Theorem VIII-1, $\mathcal{V}(x) \equiv 1$ if and only if the Finsler manifold is actually Riemannian. Already we see a difference between $\mathcal{V}(x)$ and Bao & Shen's invariant $\text{Vol}(x)$: on a Riemannian space $\text{Vol}(x) \equiv \kappa_{n-1}$ [BS, §1], but, as they remark in their second-last paragraph, some non-Riemannian spaces also take on the value κ_{n-1} identically. (The disagreement of the numbers 1 and κ_{n-1} is not important here, because this could be remedied by a scaling factor; the important point is that the volume functions are *constant* over M .)

Apart from working out the derivative of $\text{Vol}(x)$, Bao & Shen also prove the important result that $\text{Vol}(x)$ is constant (with a constant value not generally κ_{n-1}) on any Landsberg space. We will prove the similar result that $\mathcal{V}(x)$ is constant on any Berwald space. Bao & Shen's method involved the Chern connection on points of the unit indicatrix. Our method will be radically different. We will use the fact (proven, for example, in [A, §3]) that normal co-ordinates on a Berwald manifold are C^2 , instead of just being C^1 as on a general Finsler manifold. The next section develops Finsler and Berwald normal co-ordinates from scratch. The two sections after that prove the essential fact that the derivatives of a Berwald metric vanish in normal co-ordinates, and the final section proves $\mathcal{V}(x)$ is constant.

IX. Normal Co-ordinates in Finsler Geometry

Normal co-ordinates have played an important part in the development of Riemannian geometry, so it is natural to seek a version of normal co-ordinates for Finsler geometry. What seems to be the only reasonable approach is outlined in [Rund, Chap.3, §6], where a system is obtained which is C^∞ away from a given point p , but only C^1 at p . This contrasts sharply with the Riemannian case, where the system is C^∞ everywhere. Despite this defect, Finsler normal

co-ordinates can still give us some information about the manifold. We will first outline how to obtain normal co-ordinates on a Finsler space (comparing this with the Riemannian case), then derive the properties of some quantities in normal co-ordinates.

To start with, if (M, F) is a Finsler (or Riemannian) manifold, then isolate a point $p \in M$; our normal co-ordinate system $\bar{x}^i : M \mapsto \mathbb{R}$ will be based at p . Without loss of generality, we will translate whatever arbitrary co-ordinate system x^i is given, so that $x^i(p) = 0$ for every i . To obtain normal co-ordinates, we will insist on certain conditions that these co-ordinates should satisfy.

The first condition is that “straight lines” through p in the barred co-ordinates will in fact be geodesics. Since geodesics have an inherent parametrization, we would like the parameter t to be the arclength parameter, i.e.

$$\ddot{\bar{x}}^i(t) = 0, \quad (\text{IX-1})$$

where a dot denotes differentiation with respect to t . A line through p is a set of the form

$$\bar{x}^i(t) = ta^i, t \in \mathbb{R}, \quad (\text{IX-2})$$

for some constants a^i . Clearly (2) satisfies (1), so we know that $\bar{x}^i(t)$ must have the form (2).

Irrespective of any co-ordinate system, there exists, for a point $p \in M$ and a unit tangent vector V at p , a unique geodesic $\mathcal{G}(t, p, V)$, parametrized by arclength $t \geq 0$, with $\mathcal{G}(0, p, V) = p$, and with tangent vector V at p . Furthermore, this geodesic satisfies, in smooth, unbarred co-ordinates, the differential equation:

$$\ddot{x}^i(\mathcal{G}(t, p, V)) = -\Gamma_{jk}^i(x(t), \dot{x}(t))\dot{x}^j(t)\dot{x}^k(t), \quad (\text{IX-3})$$

with initial conditions

$$\dot{x}^i(0) = a^i, \quad (\text{IX-4})$$

where $\dot{x} = \frac{\partial x}{\partial t}$, p has the co-ordinate expression $p = (0, 0, \dots, 0)$, and $V|_p$ has the co-ordinate expression $V|_p = a^i \frac{\partial}{\partial x^i}|_p$. (Every $\dot{x}(t)$ in the line above is actually

a function $\dot{x}(t, p, V)$ of p and V as well, but we suppress these arguments for conciseness.) We can now write $x^i(t)$ as a Taylor series about p :

$$x^i(\mathcal{G}(t, p, V)) = ta^i - \frac{t^2}{2} \Gamma_{jk}^i(p, V) a^j a^k + h(t), \quad (\text{IX-5})$$

where $h(t) = O(t^3)$. In order to reach an expression of the form (2), we will define

$$\bar{x}^i(\mathcal{G}(t, p, V)) := x^i(\mathcal{G}(t, p, V)) + \frac{t^2}{2} \Gamma_{jk}^i(p, V) a^j a^k - h(t). \quad (\text{IX-6})$$

Substituting (6) into (5) gives

$$\bar{x}^i(\mathcal{G}(t, p, X)) = ta^i \quad (\text{IX-7})$$

as an expression in the desired form (2) for a geodesic.

The appearance of the term t in the expressions above is confusing and awkward, but inevitable; t is the distance from p to $\mathcal{G}(t, p, V)$ and is independent of co-ordinates. We'll eliminate t by rewriting (IX-5) as:

$$ta^i = x^i(\mathcal{G}(t, p, V)) + \frac{t^2}{2} \Gamma_{jk}^i(p, V) a^j a^k - h(t). \quad (\text{IX-8})$$

Now substitute (IX-8) into the second term on the right-hand side of (IX-6):

$$\begin{aligned} \frac{t^2}{2} \Gamma_{jk}^i(p, V) a^j a^k &= \frac{1}{2} \Gamma_{jk}^i(p, V) (ta^j)(ta^k) \\ &= \frac{1}{2} \Gamma_{jk}^i(p, V) \left(x^j + \frac{t^2}{2} \Gamma_{lm}^j a^l a^m - h(t) \right) \\ &\quad \cdot \left(x^k + \frac{t^2}{2} \Gamma_{lm}^k a^l a^m - h(t) \right) \\ &= \frac{1}{2} \Gamma_{jk}^i(p, V) \left(x^j x^k + \frac{t^2}{2} a^l a^m (\Gamma_{lm}^j x^k + \Gamma_{lm}^k x^j) + O(t^3) \right). \end{aligned} \quad (\text{IX-10})$$

Again, terms of the form ta^l and ta^m occur. Substitute (IX-8) once more, this time into (IX-10); make another substitution into the resulting expression, and so on, so that t will be replaced by x :

$$\bar{x}^i(x) = x^i + \frac{1}{2} \Gamma_{jk}^i(p, V(x)) x^j x^k + O(x^3), \quad (\text{IX-11})$$

where it is understood that $V(x)$ is the unit tangent vector at p to the geodesic from p to x . Because of the dependence on V , we can think of normal co-ordinates as a map from the “blown-up” manifold of tangent vectors at p to the manifold M ; this map is smooth except possibly at the zero vector at p .

We inquire about the differentiability of (IX-11). First of all, we see that the term $\Gamma_{jk}^i(p, V(x))$ is a continuous (in fact smooth) function on the compact set of unit tangent vectors V ; this fact and the fact that Γ_{jk}^i is homogeneous of degree 0 insures that $\Gamma_{jk}^i(p, V)$ is bounded, for any $V \in T_p M$. For the same reason, any higher-order coefficients, which are simply combinations of derivatives of Γ_{jk}^i , and thus also smooth functions on the compact set of unit tangent vectors, are also bounded. Thus (IX-11) is a valid Taylor series representation of $\bar{x}^i$, so we can read off the putative derivatives directly from the coefficients. Secondly, (IX-11) is clearly smooth away from p , because $V(x)$ is always a well-defined smooth function of x when $x \neq p$. At p , however, it is unclear what the expression will give.

Clearly, the Jacobian of (6) at p is simply the identity

$$\left. \frac{\partial \bar{x}^i}{\partial x^j} \right|_p = \delta_j^i, \quad (\text{IX-12})$$

so the barred co-ordinate system is at least C^1 . The second derivative, though, would have to be

$$\left. \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} \right|_p = \Gamma_{jk}^i(p, V), \quad (\text{IX-13})$$

where, because the derivative must be defined regardless of the path of approach to p , V could be any unit vector. Since $\Gamma_{jk}^i(p, V)$ is different for different V 's, the second derivative doesn't exist. Thus these co-ordinates are in general not C^2 . In the Riemannian case, of course, Γ_{jk}^i doesn't depend on tangent vectors, and the barred co-ordinates are C^∞ .

What can we say about quantities in the barred co-ordinate system? In the Riemannian case, it is clear that the Christoffel symbols all vanish. An unusual but simple way to see this is to consider their transformation law

$$\bar{\Gamma}_{jk}^i = \Gamma_{qr}^p \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^p} + \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}, \quad (\text{IX-14})$$

at the point p . At p , any term $\frac{\partial x^k}{\partial \bar{x}^l}$ or $\frac{\partial \bar{x}^k}{\partial x^l}$ is an element of the Jacobian of the switch to the barred co-ordinates, so

$$\frac{\partial x^k}{\partial \bar{x}^l} = \frac{\partial \bar{x}^k}{\partial x^l} = \delta_l^k. \quad (\text{IX-15})$$

We can evaluate the term $\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}$ by implicitly differentiating the definition (IX-11) of the barred co-ordinates; we get

$$\frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k} = -\Gamma_{jk}^p. \quad (\text{IX-16})$$

Substitute (16) and (15) into (14) to get

$$\begin{aligned} \bar{\Gamma}_{jk}^i &= \Gamma_{qr}^p \delta_r^q \delta_j^r \delta_k^i + \delta_p^i (-\Gamma_{jk}^p) \\ &= 0. \end{aligned} \quad (\text{IX-17})$$

In the Finsler case, matters are not as simple. First of all, the transformation rule for the Christoffel symbols (see [Laugwitz, §15.6.2])

$$\bar{\Gamma}_{jk}^i = \Gamma_{qr}^p \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^p} + C_{sm}^u \frac{\partial \bar{x}^i}{\partial x^u} \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} X^s \frac{\partial x^s}{\partial \bar{x}^j} \quad (\text{IX-18})$$

involves the Cartan tensor

$$C_{ijk} := \frac{1}{4} \frac{\partial^3 F^2}{\partial X^i \partial X^j \partial X^k}. \quad (\text{IX-19})$$

Secondly, the rule involves the term $\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}$, and, since the barred co-ordinates are only C^1 , second derivatives are not in general defined at p , where the other terms are evaluated. We can circumvent this situation by taking the limits of both sides of the transformation rule as we approach p ; we'll save this involved process for later sections, where we'll see that even though Γ_{jk}^i doesn't vanish, terms of the form $\Gamma_{jk}^i X^j$ do.

These barred co-ordinates are still not normal co-ordinates, and to make them so we insist on a second desirable condition: the metric tensor on $T_p M$ should have a particularly simple form. In a Riemannian space, the metric restricted to $T_p M$ corresponds to a positive-definite form with components g_{ij} , and because the

Jacobian of (IX-11) is δ_j^i , these components have not been altered by the switch to barred co-ordinates:

$$\bar{g}_{ij}(p) = g_{ij}(p). \quad (\text{IX-20})$$

We'll find a set $\hat{x}^i$ of hatted co-ordinates for which

$$\hat{g}_{ij} = \delta_{ij}.$$

We accomplish this simply by using the tensor transformation rule

$$\hat{g}_{pq} = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial \hat{x}^p} \frac{\partial \bar{x}^j}{\partial \hat{x}^q} \quad (\text{IX-21})$$

and solving for

$$\begin{aligned} \delta_{pq} &= \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial \hat{x}^p} \frac{\partial \bar{x}^j}{\partial \hat{x}^q} \\ \delta_{pq} \frac{\partial \hat{x}^p}{\partial \bar{x}^i} \frac{\partial \hat{x}^q}{\partial \bar{x}^j} &= \bar{g}_{ij} \\ \frac{\partial \hat{x}^p}{\partial \bar{x}^i} \frac{\partial \hat{x}^p}{\partial \bar{x}^j} &= \bar{g}_{ij}. \end{aligned}$$

If we write $J_i^p := \frac{\partial \hat{x}^p}{\partial \bar{x}^i}$, then we have, for the matrices J and g_{ij} ,

$$JJ^T = g_{ij}. \quad (\text{IX-22})$$

Since g_{ij} is positive-definite, this always has a solution J_i^p . Now just define

$$\hat{x}^p(x) = J_i^p(p) \bar{x}^i(x) \quad (\text{IX-23})$$

This guarantees that $\hat{g}_{ij}(p) = \delta_{ij}$, and a glance at the transformation rule for Christoffel symbols,

$$\hat{\Gamma}_{jk}^i = \bar{\Gamma}_{qr}^p \frac{\partial \bar{x}^q}{\partial \hat{x}^j} \frac{\partial \bar{x}^r}{\partial \hat{x}^k} \frac{\partial \hat{x}^i}{\partial \bar{x}^p} + \frac{\partial \hat{x}^i}{\partial \bar{x}^p} \frac{\partial^2 \bar{x}^p}{\partial \hat{x}^j \partial \hat{x}^k},$$

shows that $\bar{\Gamma}_{qr}^p = 0$ implies $\hat{\Gamma}_{jk}^i = 0$ ($J_i^p(p)$ being constant implies that $\frac{\partial^2 \bar{x}^p}{\partial \hat{x}^j \partial \hat{x}^k}$ vanishes). We can now define normal co-ordinates in Riemannian space by composing the bar and the hat transformations:

$$\begin{aligned} x_{\text{norm}}^p(x) &= J_i^p(p) \bar{x}^i(x) \\ &= J_i^p(p) \left(x^i + \frac{1}{2} \Gamma_{jk}^i(x, \dot{x}) x^j x^k + O(x^3) \right). \end{aligned} \quad (\text{IX-24})$$

We see from equation (IX-22) that we could have multiplied J by any orthogonal matrix O , and still have reached a solution JO . Our final expression (IX-24), then, is only defined up to an orthogonal transformation, i.e. to a choice of different axes for the positive-definite form $g_{ij}(p)$. Since there is no general way to distinguish directions in a Riemannian space, we must retain this amount of freedom.

For a Finsler space, if we consider the metric restricted to $T_p M$, we see that there is no special form as in Riemannian space. Thus most of the last paragraphs cannot be carried over to the Finsler case. One simplification we can make, however, is to require that the unit ball has volume κ_n . This adjustment is accomplished simply by multiplying the barred co-ordinate expression by the constant $\frac{\kappa_n}{(\int_p dX)^n}$. Since this “simplification” won’t make any of our computations simple, we’ll disregard it and define normal co-ordinates for Finsler space by

$$\bar{x}^i_{\text{norm}}(x) = x^i + \frac{1}{2}\Gamma^i_{jk}(p, V)x^j x^k + O(x^3). \quad (\text{IX-25})$$

Like the Riemannian case, there is inevitably some freedom here: the transformation is only defined up to an invertible linear transformation (if we specify a volume at p , it is defined only up to a linear transformation of determinant ± 1).

These normal co-ordinates only exist on the manifold; most Finsler quantities are functions of the tangent bundle. Since reco-ordinatizing a manifold canonically reco-ordinatizes its tangent bundle, apply normal co-ordinates to all TM by requiring

$$(\bar{x}^i, \bar{X}^q)_{\text{norm}}(x, X) = \left(x^i + \frac{1}{2}\Gamma^i_{jk}(p, V)x^j x^k + O(x^3), X^r \left(\frac{\partial \bar{x}^q}{\partial x^r} \right)_x \right), \quad (\text{IX-26})$$

where $(x, X) \in TM$. These co-ordinates are clearly C^∞ away from $T_p M$. On $T_p M$ the situation is more complicated. Say we’d like to find derivatives at some tangent vector (p, X) . We can take derivatives of $\bar{x}$ (base co-ordinates) or $\bar{X}$ (tangent vectors) with respect to either x (base co-ordinates) or X (tangent vectors). Thus we have a $2n \times 2n$ matrix $\frac{\partial(\bar{x}, \bar{X})}{\partial(x, X)}$ of derivatives, which subdivides into four $n \times n$ submatrices. From the explicit expression (IX-26) we can easily find the entries of this matrix for a given $X \in T_p M$:

$$\frac{\partial(\bar{x}, \bar{X})}{\partial(x, X)} \Big|_{(p, X)} = \begin{bmatrix} \delta^i_j & X^r \Gamma^i_{rj}(V) \\ 0 & \delta^i_j \end{bmatrix} \quad (\text{IX-27})$$

We calculated the upper left submatrix previously; now we see it in a broader context. We note that (in contradistinction to normal co-ordinates restricted to the manifold, where only the second derivative depended on a choice of V) on the tangent bundle we cannot define even first derivatives without specifying V . Thus, on TM , normal co-ordinates are continuous but not generally C^1 .

(We could unambiguously make the formal definition

$$\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} \Big|_{(p,X)} = \Gamma_{jk}^i(p, X), \quad (\text{IX-28})$$

when $X \neq \vec{0}$. This definition corresponds to choosing the geodesic through X as the path of approach to (p, X) , so it's clearly invariant. Though (IX-28) might be useful in some situations, it seems unsuited to a general Finsler manifold.)

Note also that the bottom right-hand submatrix of (IX-27) is the identity. This occurs because, in the second slot of (IX-26), the term $\frac{\partial \bar{x}^q}{\partial x^r}$ is independent of tangent vectors. As a result, if we look at any fixed tangent space $T_x M$, we see that, considered solely as a map restricted to $T_x M$, the change to normal co-ordinates is linear; it must then also be C^∞ , so we can take arbitrarily many “vertical” derivatives. To say it otherwise, the map (IX-26) is a vector bundle map which is smooth on each fibre. We can think of normal co-ordinates as leaving individual tangent spaces unaffected, but modifying their attachments to neighboring tangent spaces.

The discussion so far has dealt with an arbitrary Finsler space; now we'll restrict our attention to Berwald spaces, and see that many of the shortcomings of Finsler normal co-ordinates disappear.

Definition [AIM, §3.1.2]. Let (M, F) be a Finsler manifold. Then (M, F) is a Berwald space if, for any $x \in M$, and $X \in T_x M$,

$$\frac{\partial}{\partial X^i} (\Gamma_{jk}^i) \Big|_{(x,X)} = 0; \quad (\text{IX-29})$$

equivalently, we could say that the Christoffel symbols Γ_{jk}^i depend only on the base-point x and not on the tangent vector X .

We can give a simple but telling construction that always results in a Berwald space. Start with a point $x \in M$, and specify a Minkowski metric on $T_x M$ with a non-ellipsoidal indicatrix $\mathcal{B}$. For any other point $y \in M$, assign the indicatrix in $T_y M$ to be $L_y(\mathcal{B})$, where $L_y : T_x M \mapsto T_y M$ is a linear isometry (varying smoothly with y , of course).

Expression (IX-29) in the definition is the very condition needed to guarantee that Berwald normal co-ordinates are C^2 on M and C^1 on TM . Equation (IX-13), which gives second derivatives for normal co-ordinates on M , and Equation (IX-27), which gives first derivatives for co-ordinates on TM , both suffer from a dependence of Γ_{jk}^i on some tangent vector V . (IX-29) says that, in a Berwald space, Γ_{jk}^i is independent of V , so we can evaluate (IX-13) and (IX-27) unambiguously, giving second derivatives on M , and first derivatives on TM . This additional degree of differentiability is essential in what follows.

IX. Geometric Objects in Normal Co-ordinates

Consider a geometric object such as a metric tensor or a volume form. Any such object, whose components allow a co-ordinate expression, has a corresponding transformation rule, which relates the components written in one co-ordinate system to the components written in a second co-ordinate system, and which involves the derivatives of the map which changes co-ordinates. For tensors, the transformation rule only involves first derivatives, while for other objects, such as Christoffel symbols, the transformation rule involves second (or higher) derivatives. Since the co-ordinate change map for normal co-ordinates in Finsler space is only C^1 , we would expect tensors to follow their usual transformation rules. For the Christoffel symbols, however, transformation rules could break down and, perhaps, the Christoffel symbols might not even be defined.

Finslerian geometric objects are typically C^∞ everywhere on the slit tangent bundle $\tilde{TM} = \{(x, X) \in TM | X \neq (0, 0, \dots, 0)\}$. They all derive ultimately from the C^∞ metric function $F^2 : \tilde{TM} \rightarrow \mathbb{R}$, so this is a good place to start looking at how normal co-ordinates affect expressions. F^2 is a scalar invariant on $\tilde{TM}$, so it is well-defined under any change of co-ordinates, no matter how smooth or

non-smooth. F^2 in a smooth co-ordinate system x^i will have a smooth expression. In normal co-ordinates $\bar{x}^i$ around p , which are C^1 on a Berwald manifold M , F^2 can only be expected to have a C^1 expression, that is, F^2 has a linear approximation dF^2 , written in components as

$$dF^2 \Big|_{(x,W)} = \frac{\partial F^2}{\partial \bar{x}^i} \Big|_{(x,W)} d\bar{x}^i \quad (\text{X-1})$$

where W is a *fixed* tangent vector, so F^2 is a function from M to $\mathbb{R}$. Here we see why we must restrict ourselves to Berwald spaces—the expressions $\frac{\partial F^2}{\partial \bar{x}^i}$ don't even exist on a general Finsler space.

The next important construction on a Finsler, or Berwald, space is the metric tensor

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial X^i \partial X^j}. \quad (\text{X-2})$$

F^2 is smooth on $T_x M$ in smooth co-ordinates, and, because the change to normal co-ordinates (IX-26) is linear when restricted to $T_x M$, F^2 in normal co-ordinates is also smooth—when restricted to $T_x M$. Since the derivatives in (X-2) are both purely vertical, it is clear that

$$\bar{g}_{ij} \Big|_{(\bar{x}, \bar{X})} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \bar{X}^i \partial \bar{X}^j} \Big|_{(\bar{x}, \bar{X})}, \quad (\text{X-3})$$

the metric tensor in normal co-ordinates, is well-defined. Furthermore, relations such as

$$F^2(x, \bar{X}) = \bar{g}_{ij}(x, \bar{X}) \bar{X}^i \bar{X}^j, \quad (\text{X-4})$$

$$\bar{C}_{ijk} \bar{X}^i = 0, \quad (\text{X-5})$$

whose proof depends only on Euler's theorem restricted to $T_x M$, still hold.

Since normal co-ordinates are smooth away from p , the above discussion is necessary only at p . At any other point, the scalar F^2 and the tensor $\bar{g}_{ij}$ enjoy the usual smoothness properties, and we can take derivatives freely. We'll use this as we investigate the functions $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$, which are the building blocks of the Christoffel symbols. As said above, $\bar{g}_{ij}$ is continuous everywhere and smooth away

from p . Thus $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$ exists away from p . At p , the partial derivatives $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$ are well-defined, though the expression $d\bar{g}_{ij}$ is not. By a similar argument, we can also define higher derivatives $\frac{\partial^2 \bar{g}_{ij}}{\partial \bar{x}^k \partial \bar{x}^l}$, though, since these aren't continuous at p , their geometric meaning is unclear. (Unlike the Riemannian case, then, normal co-ordinates can tell us nothing about curvature.)

The approach above provides the key to the problem of transformation rules, even for a Finsler manifold M rather than just a Berwald manifold.. Given a geometric object $\mathcal{O}$ on M , its usual transformation rule

$$\bar{\mathcal{O}}(\bar{x}, \bar{X}) = \tau(\mathcal{O})(x, X), \quad (\text{X-6})$$

relating the components in normal and arbitrary components, still holds away from p , because the co-ordinate change is smooth away from p . As long as the terms appearing in the transformation rule are all continuous in normal co-ordinates, we can take the limit approaching p of both sides of the transformation, and arrive at an expression in normal co-ordinates:

$$\lim_{x \rightarrow p} \bar{\mathcal{O}}|_x = \lim_{x \rightarrow p} \tau(\mathcal{O})|_x, \quad (\text{X-7})$$

or

$$\bar{\mathcal{O}}|_p = \lim_{x \rightarrow p} \tau(\mathcal{O})|_x. \quad (\text{X-8})$$

The right-hand side of (X-8) exists if it contains only geometric quantities and derivatives of the co-ordinate change. If $\tau(\mathcal{O})$ contains terms of the form $\frac{\partial \bar{x}^i}{\partial x^j}$ (or $\frac{\partial x^i}{\partial \bar{x}^j}$), then, because $\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial x^i}{\partial \bar{x}^j} = \delta_j^i$ at p , clearly there will be no trouble taking the limit approaching p . Secondly, and not as obviously, if $\tau(\mathcal{O})$ contains terms of the forms

$$\left. \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \right|_{(x, X)}, \quad (\text{X-9})$$

then we can obtain

$$\lim_{x \rightarrow p} \left. \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \right|_{(x, X)} \quad (\text{X-10})$$

by going back to (IX-11),

$$\bar{x}^i(x) = x^i + \frac{1}{2} \Gamma_{jk}^i(p, V(x)) x^j x^k + o(x^3),$$

the defining expression for normal co-ordinates, and, as we did when calculating normal Christoffel symbols for Riemannian spaces, differentiate implicitly twice, getting

$$\left. \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \right|_{(x,X)} = -\Gamma_{jk}^q|_{(p,V)} + O(x)|_{(x,X)}. \quad (\text{X-11})$$

As we approach p , terms of the form $O(x)$ vanish, and the Christoffel symbols $-\Gamma_{jk}^q$, being continuous, approach some value $-\Gamma_{jk}^q|_{(p,V)}$, where V depends on the direction of approach. (In the Berwald case, of course, there is no dependence on V .)

Strictly speaking, this dependence on V will render many expressions undefined, but in some useful cases, the bounded terms $-\Gamma_{jk}^q(p, V)$ will be extraneous because they occur as multiplicands of defined terms which approach 0. This phenomenon actually occurred already in calculating $\frac{\partial \bar{x}^i}{\partial x}$ in (IX-12)—when evaluated at p , expression (IX-11) contained the undefined term $\Gamma_{jk}^i(p, V(x))$. Since the Christoffel symbols are bounded, however, the product $\Gamma_{jk}^i(p, V(x))x^j$ vanished when $x=0$.

For higher derivatives, or terms of the form $\frac{\partial^3 x^q}{\partial \bar{x}^i \partial \bar{x}^j \partial \bar{x}^k}$, $\frac{\partial^4 x^q}{\partial \bar{x}^i \partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^l}$, etc., a similar process works. Since the geodesic equation is only second order, we can calculate the terms in (IX-6) to any order, and again just read off any needed derivatives, which all have expressions at p depending on V .

An important case of geometric objects, to which we can apply this rule, are tensors. Only first derivatives appear in the tensor transformation rule

$$\bar{T}_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p}|_{(x,X)} = T_{s_1 s_2 \dots s_q}^{r_1 r_2 \dots r_p}|_{(x,X)} \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \dots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}}, \quad (\text{X-12})$$

and each derivative approaches a Kronecker delta as $x \rightarrow p$, giving

$$\bar{T}_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p}|_{(p,X)} = T_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p}|_{(p,X)}. \quad (\text{X-13})$$

Thus tensors don't change their components at p under a change to normal co-ordinates. Furthermore, if one tensor is obtained from another by vertical differentiation, that relation still holds in normal co-ordinates. The Cartan tensor C_{ijk} ,

for example, arises from the metric invariant F^2 , by three vertical differentiations:

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 F^2}{\partial X^i \partial X^j \partial X^k}. \quad (\text{X-14})$$

The comments in the analysis of (IX-26) assure us that vertical derivatives always exist in normal co-ordinates, so

$$\bar{C}_{ijk}|_p = C_{ijk}|_p = \frac{1}{4} \frac{\partial^3 F^2}{\partial X^i \partial X^j \partial X^k} \Big|_p = \frac{1}{4} \frac{\partial^3 F^2}{\partial \bar{X}^i \partial \bar{X}^j \partial \bar{X}^k} \Big|_p. \quad (\text{X-15})$$

XI. The Derivatives of F^2 Vanish in Berwald Normal Co-ordinates

The last section outlined an approach to calculating quantities whose transformation rules involve second derivatives, which are not defined at p . We appeal to continuity to ensure a limit going to p exists, and then evaluate that limit by examining terms away from p , where the transformation rules *are* valid. We'll use this to prove an important result.

As is well-known, in Riemannian normal co-ordinates the metric tensor “vanishes to first order,” that is, the horizontal derivatives of the metric are all zero. The same vanishing occurs in Berwald space. There is however an important, unexpected change when we express this fact on a Berwald space. On a Riemannian space, we formally express the vanishing of the matrix's horizontal derivatives by saying

$$\frac{\partial F^2}{\partial x^k}(x, X) = 0, \quad (\text{XI-1})$$

for every k . Equivalently, we could say

$$\frac{\partial}{\partial x^k}(g_{ij}(x)X^iX^j) = 0 \quad (\text{XI-2})$$

or, since X^i is independent of x^k ,

$$\left(\frac{\partial g_{ij}}{\partial x^k}(x) \right) X^i X^j = 0. \quad (\text{XI-3})$$

Linear algebra tells us that the solution to this quadratic equation is

$$\frac{\partial g_{ij}}{\partial x^k}(x) = 0 \quad (\text{XI-4})$$

for every i, j, k . The conditions (XI-4) are equivalent to the conditions (XI-1), and mathematicians generally prefer (XI-4).

In the Finsler case, the derivation of this equivalence breaks down, because the metric tensor g_{ij} depends on both a point x and a tangent vector X . Thus we get

$$\begin{aligned} \frac{\partial F^2}{\partial x^k}(x, X) &= 0, \\ \frac{\partial}{\partial x^k}(g_{ij}(x, X)X^iX^j) &= 0 \\ \left(\frac{\partial g_{ij}}{\partial x^k}(x, X)\right)X^iX^j &= 0. \end{aligned} \quad (\text{XI-5})$$

Because the terms $\frac{\partial g_{ij}}{\partial x^k}(x, X)$ depend on X as well as x , we cannot use linear algebra to solve equation (XI-5), and we *cannot* conclude that $\frac{\partial g_{ij}}{\partial x^k}(x, X) = 0$.

We'll use the method of the last section to calculate normal expressions for $\frac{\partial g_{ij}}{\partial x^k}(x, X)$, and then to calculate a normal expression for $\frac{\partial F^2}{\partial x^k}(x, X)$; we'll see that (XI-1) still holds in a Berwald space, though (XI-4) does not.

Theorem XI-1. *Let (M, F) be a Berwald manifold with a normal co-ordinate system $\bar{x}^i$ around a point p . In this co-ordinate system, for every $X \in T_pM$, and for any k ,*

$$\frac{\partial F^2}{\partial \bar{x}^k}(p, X) = 0.$$

Proof. *Begin by considering terms of the form $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$. We'll prove*

$$\left. \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} \right|_{(p, X)} = -C_{ijm}(p, X)\Gamma_{kt}^m(p)X^t,$$

where, because we are in a Berwald space, the Christoffel symbols depend only on p and not on any tangent vector. The Finsler transformation rule for these quantities (away from p) is

$$\begin{aligned} \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} &= \frac{\partial g_{sl}}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} + g_{sl} \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^j} + g_{sl} \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} + \\ &+ C_{slm} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial \bar{x}^t}{\partial x^u} X^u, \end{aligned} \quad (\text{XI-6})$$

where g_{ij} depends on a point x and a tangent vector X . Taking the limit approaching p on both sides:

$$\lim_{x \rightarrow p} \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} = \lim_{x \rightarrow p} \left(\frac{\partial g_{sl}}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} + g_{sl} \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^j} + g_{sl} \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} + \right. \\ \left. + C_{slm} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^t} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial \bar{x}^t}{\partial x^u} X^u \right). \quad (XI-7)$$

Recalling that

$$\frac{\partial x^k}{\partial \bar{x}^l} = \frac{\partial \bar{x}^k}{\partial x^l} = \delta_l^k \quad (XI-8)$$

at p , and

$$\left. \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} \right|_{(p,X)} = -\Gamma_{jk}^i \Big|_p \quad (XI-9)$$

everywhere away from p , and using the continuity of $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$, we get

$$\left. \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} \right|_{(p,X)} = \lim_{x \rightarrow p} \left(\frac{\partial g_{ij}}{\partial x^k} \Big|_{(x,X)} - g_{sj} \Gamma_{ik}^s - g_{si} \Gamma_{jk}^s - C_{ijm} \Gamma_{kt}^m X^t \right). \quad (XI-10)$$

In a Berwald space, the symbols Γ_{ik}^s are independent of any tangent vector, so we can take $\Gamma_{ik}^s = \Gamma_{ik}^s(x, X)$, so that $\frac{\partial g_{ij}}{\partial x^k}$, g_{sj} , and Γ_{ik}^s are all evaluated at the same tangent vector. Thus, after some manipulation the first three terms cancel (alternately, we can just recognize these three terms as the "Riemannian part" of (XI-6), so we know already that they vanish), leaving

$$\left. \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} \right|_{(p,X)} = \lim_{x \rightarrow p} -C_{ijm} \Gamma_{kt}^m X^t. \quad (XI-11)$$

Now C_{ijm} , Γ_{kt}^m , and X^t are all smooth functions defined everywhere on M , so we can take this limit just by evaluating them directly at p . We get

$$\left. \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} \right|_{(p,X)} = -C_{ijm}(p, X) \Gamma_{kt}^m(p) X^t.$$

Now that we have found an expression for $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$, we use it to find an expression for $\frac{\partial F^2}{\partial \bar{x}^k}(p, \bar{X})$. Because $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial x^i \partial x^j}$ is a tensor obtained from F^2 by vertical differentiation, we know that, in normal co-ordinates $\bar{x}^i$ we still have

$$F^2(\bar{x}, \bar{X}) = \bar{g}_{ij}(\bar{x}, \bar{X}) \bar{X}^i \bar{X}^j. \quad (XI-12)$$

Differentiate to get

$$\frac{\partial F^2}{\partial \bar{x}^k}(\bar{x}, \bar{X}) = \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}(\bar{x}, \bar{X}) \bar{X}^i \bar{X}^j. \quad (XI-13)$$

Now use the transformation rule of the right-hand side, away from p , and take the limit approaching p . We know that $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}|_p = -C_{ijm}(p, X) \Gamma_{kt}^m(p) X^t$, and that $\bar{X}^i|_p = X^i|_p$, so we get

$$\frac{\partial F^2}{\partial \bar{x}^k}(p, \bar{X}) = -C_{ijm}(p, \bar{X}) \Gamma_{kt}^m(p) X^t X^i X^j. \quad (XI-14)$$

The identity

$$C_{ijm} X^i = 0 \quad (XI-15)$$

ensures that

$$\frac{\partial F^2}{\partial \bar{x}^k}(p, \bar{X}) = 0,$$

as was to be shown. $\blacksquare$

Corollary XI-1. Recall the volume form component $\omega = \frac{\kappa_n}{\int_{I_x} dX}$ on a Berwald manifold (M, F) . In normal co-ordinates $\bar{x}^i$ around p ,

$$\frac{\partial}{\partial \bar{x}^i} \frac{\kappa_n}{\int_{I_p} dX} \Big|_p = 0 \quad (XI-16)$$

for any $i = 1, 2, \dots, n$.

Proof. Examine the expression $\frac{\kappa_n}{\int_{I_p} dX}$, where we recall that

$$I_p = \{X \in T_p M | F^2(p, X) \leq 1\}.$$

This is the only place in the expression that the metric function F appears, so the derivative of $\frac{\kappa_n}{\int_{I_p} dX}$ will involve only integrals with integrands containing $\frac{\partial F^2}{\partial \bar{x}^i}$. Since all these terms vanish, the expression (16) must vanish. $\blacksquare$

Corollary XI-2. Recall that $K^{jk} = (n+2) \frac{\int_{I_x} X^j X^k dX}{\int_{I_x} dX}$ is the expression for the osculating Riemannian metric to a Berwald manifold (M, F) . In normal co-ordinates $\bar{x}^i$ around p ,

$$\frac{\partial}{\partial \bar{x}^i} (K^{jk}) \Big|_p = 0 \quad (XI-17)$$

for any $i = 1, 2, \dots, n$. ■

XI. The Derivatives of F^2 Vanish: A Geometric Proof

In the last section, we computed the first derivatives of F^2 in normal co-ordinates on a Berwald space, and found that they were all zero. In this section we give an alternate proof of the same result. Though this second proof is more roundabout, it provides a more descriptive geometric picture of normal co-ordinates. We work on a general Finsler manifold, where everything follows through, until the last step, where we require a Berwald manifold. We start by proving Finsler versions of the First-Variation Formula and the Gauss Lemma.

First-Variation Formula. Let $\Sigma : (-\epsilon, \epsilon) \times [a, b] \rightarrow (M^n, F)$ be a smooth variation. Let $s \in (-\epsilon, \epsilon), t \in [a, b]$. In co-ordinates x^i , $(s, t) \mapsto \Sigma(s, t) = x^i(s, t)$, for all $i = 1..n$. Define

$$\begin{aligned} T &:= d\Sigma \left(\frac{\partial}{\partial t} \right); T^i := \frac{\partial x^i}{\partial t} \\ V &:= d\Sigma \left(\frac{\partial}{\partial s} \right); V^i := \frac{\partial x^i}{\partial s} \end{aligned}$$

We require that $F(T) = c_s$ (a constant depending on s but not on t). Let

$$\begin{aligned} L(s) &:= \text{length of the } s\text{-curve in the variation } \Sigma \\ &= \int_a^b F(T) dt \\ &= \int_a^b F(\dot{x}^i(s, t)) dt \end{aligned}$$

With the above definitions and conditions, we have the Finsler First-Variation Formula:

$$\left. \frac{\partial L(s)}{\partial s} \right|_{s=0} = \frac{1}{c_0} \left(\langle V, T \rangle_T \Big|_a^b - \int_a^b \left\langle V, \left(\frac{\partial T^j}{\partial t} + \Gamma_{kl}^j(T) T^k T^l \right) \frac{\partial}{\partial x^j} \right\rangle_T dt \right)$$

Proof. Calculate:

$$\begin{aligned} \left. \frac{\partial L(s)}{\partial s} \right|_{s=0} &= \frac{\partial}{\partial s} \int_a^b F(T) dt \Big|_{s=0} \\ &= \frac{1}{2c_0} \int_a^b \frac{\partial}{\partial s} (F^2(T)) \Big|_{s=0} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2c_0} \int_a^b \frac{g_{ij}}{\partial s} (T) T^i T^j + 2g_{ij}(T) T^i \frac{\partial T^j}{\partial s} dt \\
&= \frac{1}{2c_0} \int_a^b \frac{g_{ij}}{\partial s} (T) T^i T^j + 2g_{ij}(T) T^i \frac{\partial^2 x^j}{\partial t \partial s} dt \\
&= \frac{1}{2c_0} \left(2g_{ij}(T) T^i \frac{\partial x^j}{\partial s} \Big|_a^b + \int_a^b \frac{g_{ij}}{\partial s} (T) T^i T^j - 2 \frac{\partial x^j}{\partial s} \frac{\partial}{\partial t} (g_{ij}(T) T^i) dt \right) \\
&\quad (\text{integration by parts}) \\
&= \frac{1}{c_0} \left(\langle V, T \rangle_T \Big|_a^b + \int_a^b \frac{1}{2} V^k \frac{\partial g_{ij}}{\partial x^k} (T) T^i T^j - g_{ij} V^i \frac{\partial T^j}{\partial t} - V^i T^k T^l \frac{\partial g_{ki}}{\partial x^l} dt \right) \\
&= \frac{1}{c_0} \left(\langle V, T \rangle_T \Big|_a^b - \int_a^b \left\langle V, \left(\frac{\partial T^j}{\partial t} + \Gamma_{kl}^j (T) T^k T^l \right) \frac{\partial}{\partial x^j} \right\rangle_T dt \right) \quad \blacksquare
\end{aligned}$$

If our Finsler space were Riemannian, then

$$\left(\frac{\partial T^j}{\partial t} + \Gamma_{kl}^j (T) T^k T^l \right) \frac{\partial}{\partial x^j} = \nabla_T T, \quad (\text{XII-1})$$

where ∇ is the Levi-Civita connection; in the Riemannian case, also, the dependence of the inner product and the Christoffel symbols on the tangent vector T disappears, and our expression simplifies to the usual first-variation formula:

$$\frac{\partial L(s)}{\partial s} \Big|_{s=0} = \frac{1}{c_0} \left(\langle V, T \rangle \Big|_a^b - \int_a^b \langle V, \nabla_T T \rangle dt \right).$$

Indeed, if we have on our Finsler space a non-linear connection ∇ which satisfies (1), that is, a ∇ for which

$$\nabla_T T = 0$$

is the differential equation for geodesics, then, written with that ∇ , our formula will be *formally* identical to the Riemannian formula. This is the case for the Cartan connection [AP, §1.5.2], and for Chern's Connection [BC, Eq. 3.33].

Gauss Lemma. *Let Σ be a variation as above, but now insist in addition that $\sigma_s(t) = \Sigma(s, t)$ is a geodesic for every s , and $L(s) = c$ for any s (i.e. every geodesic in the variation has the same length). Also require that $V(s, a) = 0$ for every s , so that all the geodesics originate from the same point $\Sigma(0, 0)$. Thus Σ sweeps out a curve on the sphere of radius c around $\Sigma(0, 0)$. Then*

$$\langle V(s, b), T(s, b) \rangle_{T(s, b)} = 0$$

for any s .

Proof. Consider the first-variation formula. Since $L(s) = c$ regardless of s , we have

$$0 = \frac{1}{c_0} \left(\langle V, T \rangle_T|_a^b - \int_a^b \left\langle V, \left(\frac{\partial T^j}{\partial t} + \Gamma_{kl}^j(T) T^k T^l \right) \frac{\partial}{\partial x^j} \right\rangle_T dt \right).$$

For a geodesic variation, we have

$$\frac{\partial T^j}{\partial t} + \Gamma_{kl}^j(T) T^k T^l \equiv 0,$$

and we also have $V(s, a) = 0$, so the only remaining term is the one we're interested in:

$$\langle V(s, b), T(s, b) \rangle_{T(s, b)} = 0. \quad \blacksquare$$

Lemma XII-1. Let (M, F) be a Finsler manifold with a normal co-ordinate system $\bar{x}^i$ around a point p . In this co-ordinate system, away from p , for any radial tangent vector $\bar{X} = \bar{X}^k \frac{\partial}{\partial \bar{x}^k}$, where $\bar{X}^k = a^k$, we have

$$\bar{X}(F_r^2(\bar{x}, \bar{X})) = \frac{\partial F^2}{\partial \bar{x}^r}(\bar{x}, \bar{X}),$$

where $F_r^2 = \frac{\partial F^2}{\partial \bar{x}^r}$.

Proof. Because we are working away from p , normal co-ordinates are smooth, so we can take derivatives and define Christoffel symbols without any trouble. The geodesic equations for a path $\bar{x}(t)$ are

$$\ddot{\bar{x}}^i + \Gamma_{jk}^i(\bar{x}) \dot{\bar{x}}^j \dot{\bar{x}}^k = 0,$$

or, if $\bar{X}^i = \dot{\bar{x}}^i$,

$$\ddot{\bar{x}}^i + \Gamma_{jk}^i(\bar{X}) \bar{X}^j \bar{X}^k = 0.$$

In normal co-ordinates, the paths

$$\bar{x}(t) = t(a_0, a_1, \dots, a_n)$$

are geodesics for any set of constants a_i . Along these geodesics, $\ddot{\bar{x}}^i = 0$, so

$$\Gamma_{jk}^i(\bar{X}) \bar{X}^j \bar{X}^k = 0.$$

Expand:

$$\begin{aligned} \frac{1}{2} \bar{g}^{ir} \left(\frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k} + \frac{\partial \bar{g}_{kr}}{\partial \bar{x}^j} - \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r} \right) \bar{X}^j \bar{X}^k &= 0 \\ \frac{1}{2} \bar{g}^{ir} \left(2 \frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k} \bar{X}^j \bar{X}^k - \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r} \bar{X}^j \bar{X}^k \right) &= 0. \end{aligned}$$

Since $\bar{g}^{ir}$ is invertible,

$$2 \frac{\partial \bar{g}_{jr}}{\partial \bar{x}^k} \bar{X}^j \bar{X}^k = \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^r} \bar{X}^j \bar{X}^k.$$

Since $\bar{g}_{jr}(\bar{X}) \bar{X}^j = \frac{1}{2} F_r^2$ by the homogeneity of F^2 on each tangent space, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{x}^k} \left(\frac{\partial F^2}{\partial \bar{X}^r} \right) \bar{X}^k &= \frac{\partial}{\partial \bar{x}^r} (\bar{g}_{jk} \bar{X}^j \bar{X}^k) \\ \bar{X} (F_r^2) &= \frac{\partial}{\partial \bar{x}^r} (F^2(\bar{X})). \quad \blacksquare \end{aligned}$$

Lemma XII-2. Let (M, F) be a Finsler manifold with a normal co-ordinate system $\bar{x}^i$ around a point p . In these co-ordinates, if $\bar{T} \in T_p M$, and $\bar{x}$ is a point on the geodesic generated by $\bar{T}$, then

$$\bar{g}_{ij}(p, \bar{T}) \bar{T}^i = \bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i.$$

Proof. Let $\rho : M \rightarrow \mathbb{R}$ be the distance function from p . In normal co-ordinates, then,

$$F^2(p; y^1, y^2, \dots, y^n) = \rho^2(y^1, y^2, \dots, y^n),$$

for any set y^i . Let $\bar{X} \in T_{\bar{T}} T_p M$ be a vertical vector such that $\bar{X} = \bar{V}^i \frac{\partial}{\partial y^i}$. Then $\bar{X}$ is tangent to the indicatrix at $\bar{T}$ if and only if

$$dF^2(\bar{X}) = 0 \quad (\text{XII-2})$$

(consider F^2 as solely a function of the tangent vectors y^i , i.e. restrict F to $T_p M$). Then, because F^2 and ρ^2 are written with respect to the same co-ordinates,

$$d\rho^2(\bar{V}) = 0. \quad (\text{XII-3})$$

Now $\bar{X}$ is tangent to the indicatrix at T if and only if

$$\bar{g}_{ij}(p, \bar{T}) \bar{T}^i V^j = 0. \quad (\text{XII-4})$$

Furthermore, because normal co-ordinates give a geodesic variation about p , the Gauss Lemma tells us that

$$\bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i \bar{V}^j = 0. \quad (XII-5)$$

Statements (2)—(5) are equivalent, so in particular

$$\begin{aligned} \bar{g}_{ij}(p, \bar{T}) \bar{T}^i \bar{V}^j &= \bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i V^j. \\ (\bar{g}_{ij}(p, \bar{T}) \bar{T}^i) \bar{V}^j &= (\bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i) V^j. \end{aligned}$$

The vectors $\bar{V}^j$ for which both sides are 0 form an $(n-1)$ -dimensional vector space, so the “vectors” $\bar{g}_{ij}(p, \bar{T}) \bar{T}^i$ and $\bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i$ are determined and equal up to a multiplicative constant $k(x)$:

$$\bar{g}_{ij}(p, \bar{T}) \bar{T}^i = k(x) \bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^i.$$

Since, however, $F^2(p, \bar{T}) = F^2(\bar{x}, \bar{T})$ (because $\bar{T}$ is the tangent to a geodesic parametrized by arclength), and since

$$F^2(\bar{T}) = \bar{g}_{ij}(\bar{T}) \bar{T}^i \bar{T}^j,$$

the multiplicative constant $k(x)$ must be identically 1. $\therefore$

$$\bar{g}_{ij}(p, \bar{T}) \bar{T}^j = \bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^j. \quad \blacksquare$$

The discussion so far has involved only a general Finsler manifold, and has always worked away from p . The final step specializes only to Berwald manifolds, and explicitly uses their extra degree of differentiability for normal co-ordinates at p .

Theorem XII-1. *Let (M, F) be a Berwald manifold with a normal co-ordinate system $\bar{x}^i$ around a point p . In this co-ordinate system, for every $\bar{T} \in T_p M$,*

$$\frac{\partial F^2}{\partial \bar{x}^i}(p, \bar{T}) = 0.$$

Proof. Since normal co-ordinates on the tangent bundle of a Berwald manifold are at least C^1 everywhere, it follows that

$$\lim_{\bar{x} \rightarrow p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x}, \bar{T})$$

exists, and

$$\lim_{x \rightarrow p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x}, \bar{T}) = \frac{\partial F^2}{\partial \bar{x}^i}(p, \bar{T}).$$

Furthermore, we can evaluate the limit along any path leading to p . Choose as a path the geodesic in the direction $\bar{T}$ (this will allow us to use Lemma 2). From Lemma 1,

$$\begin{aligned} \lim_{x \rightarrow p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x}, \bar{T}) &= \lim_{x \rightarrow p} \bar{T}(F^2_i(\bar{x}, \bar{T})) \\ &= \lim_{x \rightarrow p} \bar{T} \left(\frac{\partial^2 F^2}{\partial \bar{X}^i \partial \bar{X}^j}(\bar{x}, \bar{T}) \bar{T}^j \right) \\ &= \lim_{x \rightarrow p} \bar{T}(2\bar{g}_{ij}(\bar{x}, \bar{T}) \bar{T}^j). \end{aligned}$$

Lemma 2 says that the argument of $\bar{T}$ in the line above is constant along the geodesic generated by $\bar{T}$, so

$$\lim_{x \rightarrow p} \frac{\partial F^2}{\partial \bar{x}^i}(\bar{x}, \bar{T}) = 0. \quad \blacksquare$$

XIII. The Volume Invariant is Constant in a Berwald Space

With the result that the derivatives of F^2 vanish in normal co-ordinates, we can prove the promised theorem.

Theorem XIII-1. *Let (M, F) be a Berwald manifold, with volume invariant $\mathcal{V}(x)$. Then $\mathcal{V}(x)$ is constant, i.e. $d\mathcal{V}(x) = 0$ everywhere.*

Proof. Recall that

$$\mathcal{V}(x) = \frac{k(x)}{\omega(x)}, \quad (\text{XIII-1})$$

where $k(x)$ is the (coefficient in some co-ordinate system of) the volume form of the osculating Riemannian metric, and $\omega(x)$ is (the coefficient in the same co-ordinate system of) Busemann's volume form.

Choose for a co-ordinate system normal co-ordinates $\bar{x}^i$ around a point $p \in M$. By Corollary XI-1, any first derivative of $\omega(x)$ vanishes at p in these co-ordinates, i.e.

$$\bar{d}(\omega(x))|_p = 0; \quad (\text{XIII-2})$$

where $\bar{d}$ is the exterior derivative in normal co-ordinates of the function $\omega(x)$, where $\omega(x)dx$ is Busemann's volume form in normal co-ordinates.

By Corollary XI-2, any first derivative of $K^{jk}(x)$ vanishes at p in normal co-ordinates, where K^{jk} is the osculating Riemannian metric to (M, F) . The K^{jk} generate a volume form, given in co-ordinates by

$$k(x)dx = \sqrt{\det[K_{kl}(x)]}dx, \quad (\text{XIII-3})$$

where $K_{kl}K^{jk} = \delta_l^j$. Since

$$\bar{d}(K^{jk}(x))|_p = 0, \quad (\text{XIII-4})$$

we have

$$\bar{d}(k(x))|_p = 0, \quad (\text{XIII-5})$$

in normal co-ordinates.

Now consider $\bar{d}\mathcal{V}(x)$. By the quotient rule,

$$\begin{aligned} \bar{d}\mathcal{V}(x)|_p &= \frac{\bar{d}k(x)|_p \omega(p) - \bar{d}\omega(x)|_p k(p)}{\omega^2(p)} \\ &= \frac{0 - 0}{\omega^2(p)} \\ &= 0. \end{aligned} \quad (\text{XIII-6})$$

Unlike $k(x)$ and $\omega(x)$, which are not really functions but rather components of volume forms, $\mathcal{V}(x)$ genuinely is a scalar function, so $\bar{d}\mathcal{V}(x)$ is genuinely its exterior derivative. Since $\bar{d}\mathcal{V}(x)$ vanishes in one co-ordinates system at p , it must vanish in any co-ordinate system at p , so

$$d\mathcal{V}(x)|_p = 0. \quad (\text{XIII-7})$$

Since p was chosen arbitrarily, we could work through the above steps for any $p \in M$, and thus get

$$d\mathcal{V}(x) = 0 \quad (\text{XIII-8})$$

everywhere. ■

REFERENCES

- [A] H. Akbar-Zadeh. "Sur les espaces de Finsler à courbures sectionnelles constantes," Acad. Roy. Sci. Belgique Bulletin, Série 5, 74, 1988, pp.281-322.
- [AIM] P. L. Antonelli, R. S. Ingarden, M. Matsumoto. *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*. Kluwer Academic Publishers, Netherlands, 1993.
- [AP] Marco Abate, Giorgio Patrizio. *Finsler Metrics—A Global Approach: with applications to geometric function theory*. Springer, Berlin, 1994.
- [AZ] P. L. Antonelli, T. J. Zastawniak. "Diffusion, Laplacian and Hodge decomposition on Finsler spaces," to be published in *Proceedings of Finsler Laplacian Conference*, Kluwer Academic Press, 1998.
- [BC] David Bao, Shiing-Shen Chern. "On a Notable Connection in Finsler Geometry," Houston Jour. Math., Vol. 19, No. 1, 1993.
- [BCS] David Bao, Shiing-Shen Chern, Zhongmin Shen. *An Introduction to Riemann-Finsler Geometry*. Springer, in preparation.
- [BL] David Bao, Brad Lackey. "A Hodge Decomposition Theorem for Finsler Spaces," C. R. Acad. Sc. Paris, vol. 223(1996), pp. 51-56.
- [BS] David Bao, Zhongmin Shen. "On the Volume of Unit Tangent Spheres in a Finsler Manifold." Results in Mathematics, Vol. 26, 1994.
- [Busemann] Herbert Busemann. "Intrinsic Area," Annals of Math., vol. 48 (1947) pp. 234-267.
- [EL] James Eells, Luc Lemaire. *Selected Topics in Harmonic Maps* (Conference Board of the Mathematical Sciences: Regional Conference Series in Mathematics #50). American Mathematical Society, Providence, 1983.
- [GV] A. Gray, L. Vanhecke. "Riemannian Geometry as Determined by the Volumes of Small Geodesic Balls," Acta Mathematica 142 (1979) no.3-4, pp. 157-198.
- [Hotelling] Harold Hotelling. "Tubes and Spheres in n -Spaces and a Class of Statistical Problems," American Journal of Mathematics 61 (1939), pp. 440-460.
- [Jost] Jürgen Jost. *Nonpositive Curvature: Geometric and Analytic Aspects*. Birkhäuser

Verlag, Basel, 1997.

- [Laugwitz] Detlef Laugwitz. *Differential and Riemannian Geometry*. Academic Press, New York, 1965.
- [Pinsky] Mark A. Pinsky. "Isotropic Transport Process on a Riemannian Manifold," Trans. AMS, Vol. 218(1976) pp. 353-360.
- [RR] Michael Renardy, Robert Rogers. *An Introduction to Partial Differential Equations*. Springer, New York, 1996.
- [Rund] Hanno Rund. *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, 1959.
- [RWW] H. S. Ruse, A. G. Walker, T. J. Willmore. *Harmonic Spaces*. Edizioni Cremonese, Rome, 1961.
- [Wells] R. O. Wells. *Differential Analysis on Complex Manifolds*. Prentice-Hall, Englewood Cliffs, NJ, 1973.
- [Yano] Kentaro Yano. *Integral Formulas in Riemannian Geometry*. Marcel Dekker, New York, 1970.