A Non-Abelian Analogue
of the Least Character Nonresidue

by

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ABSTRACT

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Let \( E/K \) be a Galois extension of number fields with group \( G \), and let \( \chi \) be a character of \( G \). For a prime \( p \) of \( K \), denote by \( \sigma_p \) the conjugacy class of Frobenius automorphisms at \( p \). This thesis presents upper bounds on the prime \( p \) of smallest norm for which \( \chi(\sigma_p) \neq \chi(1) \) under the assumption of Artin’s conjecture (AC).

To state the main results more precisely, let us set

\[
p(\chi) = \min \{ N_p \mid \chi(\sigma_p) \neq \chi(1) \} .
\]

For \( \chi \) irreducible and non-Abelian we prove (Theorem 4.4) that

\[
p(\chi) \leq A^c_{\chi(1)},
\]

where \( A_{\chi} \) is (essentially) the Artin conductor of \( \chi \), and \( c > 0 \) is an absolute effective constant. As well as AC, here we are assuming that the exceptional zero of \( \zeta_K(s) \) does not exist. We also establish (Theorem 4.3) the explicit relationship between a zero-free region for \( \zeta_K(s)L(s, \chi) \) and the size of \( p(\chi) \).

Another result (Corollary 5.6) states that for \( \chi \) not containing the trivial character, the assumption of the Generalized Riemann Hypothesis implies that

\[
p(\chi) \leq c\chi(1)^{-1}(\log A_{\chi})^2.
\]

Finally, a version of the Deuring-Heilbronn phenomenon for non-Abelian Artin
$L$-function $L(s, \chi, E/K)$ has been proved. This result (Theorem 3.4) states that a zero of $L(s, \chi)$ very close to $s = 1$ has the effect of pushing other zeros further away from the line $\sigma = 1$. 
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Chapter 1

The Basics and the Main Results

In this chapter we briefly recall the construction of Artin $L$-functions, as well as their basic properties including the functional equation. This will provide us the opportunity to introduce our notations, set our conventions (which will be applicable throughout the whole thesis), and finally state the main results.

1.1 Frobenius class

Let $K$ be a number field. We employ the following notations:

\[ \mathcal{O}_K = \text{the ring of integers of } K, \]
\[ n_K = [K : \mathbb{Q}] = [\mathcal{O}_K : \mathbb{Z}] = \text{the degree of } K/\mathbb{Q}, \]
\[ d_K = \text{the absolute value of the discriminant of } K, \]
\[ \Sigma_K = \text{the set of (nonzero) prime ideals of } K. \]

Let $E$ be a finite Galois extension of $K$.

\[ G = \text{Gal}(E/K) = \text{the Galois group of } E/K, \]
1.2 Representations of the Galois group

The inner product of any two (not necessarily class) functions \( \varphi \) and \( \psi \) on \( G \) is defined by 

\[
(\varphi|\psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.
\]

If \( \varphi \) and \( \psi \) are characters of \( G \), with \( \psi \) irreducible, then

\[
|I_q| = n = [E : K] = |G| = n_E/n_K.
\]

Let \( p \in \Sigma_K \). In \( E \), \( p \) decomposes as \( p \mathcal{O}_E = (q_1 \cdots q_e)^e \), \( q_i \in \Sigma_E \). A prime \( q \in \Sigma_E \) divides \( p \) (i.e. belongs to the finite set \( Q = \{q_i\} \)) iff \( q \cap \mathcal{O}_K = p \). If \( q|p \), then the finite field \( \mathcal{O}_K/p \) is naturally imbedded in \( \mathcal{O}_E/q \). \( G \) acts transitively on \( Q \). The decomposition group \( D_q \) and the inertia group \( I_q \) are defined by

\[
D_q = \{ \alpha \in G \mid x \equiv y \pmod{q} \implies \alpha(x) \equiv \alpha(y) \pmod{q}, x, y \in \mathcal{O}_E \},
\]

\[
I_q = \{ \alpha \in G \mid \alpha(x) \equiv x \pmod{q}, x \in \mathcal{O}_E \}.
\]

We have \( e = |I_q| \), and \( e \) is called the ramification index of \( p \) (in \( E \)). If \( e = 1 \), \( p \) is said to be unramified in \( E \). We have \( n = efg \), where \( f = [\mathcal{O}_E/q : \mathcal{O}_K/p] \).

There is an exact sequence of finite groups

\[
1 \longrightarrow I_q \longrightarrow D_q \longrightarrow \text{Gal}(\mathcal{O}_E/q/\mathcal{O}_K/p) \longrightarrow 1.
\]

The Galois group \( \text{Gal}(\mathcal{O}_E/q/\mathcal{O}_K/p) \) is cyclic with a distinguished generator \( \varphi \) defined by \( \varphi(x) = x^{N_p} \), where \( N_p = |\mathcal{O}_K/p| \). Any preimage of \( \varphi \) in \( D_q \) is called a Frobenius element at \( q \) and is denoted by \( \sigma_q \). If \( p \) is unramified in \( E \), \( \sigma_q \) is uniquely defined; otherwise it is only defined mod \( I_q \). If \( q' \in Q \), then there is \( \alpha \in G \) with \( \alpha(q) = q' \), and we have \( D_{q'} = \alpha D_q \alpha^{-1}, \sigma_{q'} = \alpha \sigma_q \alpha^{-1} \). If \( p \) is unramified in \( E \), then the set \( \{\sigma_q \mid q \in Q\} \) is a conjugacy class in \( G \). We denote it by \( \sigma_p \) and call it the Frobenius class at \( p \).

1.2 Representations of the Galois group

The inner product of any two (not necessarily class) functions \( \varphi \) and \( \psi \) on \( G \) is defined by

\[
(\varphi|\psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.
\]

If \( \varphi \) and \( \psi \) are characters of \( G \), with \( \psi \) irreducible, then

\[
|I_q| = n = [E : K] = |G| = n_E/n_K.
\]
The Basics and the Main Results

1.3 Ramification under representations

(\varphi|\psi) equals the multiplicity of occurrence of \psi in \varphi. The trivial character of G is denoted by 1_G.

Let \chi be a character of G. Then (\chi|1_G) equals the number of times that the trivial character occurs in \chi. If \rho : G \to GL(V) is the representation whose character is \chi, then \chi(1) = \dim V. As the character \chi determines the representation \rho uniquely, the attributes of \rho are sometimes associated with \chi. In particular by "ker \chi" we mean ker \rho, a normal subgroup of G.

1.3 Ramification under representations

Let \chi be a character of G, and p \in \Sigma_K. Fix a q \in \Sigma_E above p. If \text{I}_q is included in ker \chi, then p is said to be unramified under \chi, or simply \chi-unramified. (As ker \chi is a normal subgroup, the condition is independent of the choice of q above p.) Otherwise p is said to be ramified under \chi.

In order to distinguish the \chi-ramified primes from others, we have to introduce the Artin conductor f_\chi associated with \chi. This is an ideal of K given by

\[ f_\chi = \prod_{p \in \Sigma_K} p^{\text{ram}(p, \chi)}, \]

where \text{ram}(p, \chi) is a nonnegative integer, zero for all but a finite number of p. The exact description of \text{ram}(p, \chi) is as follows. Let \text{I}_q = \text{I}_0 \supseteq \text{I}_1 \supseteq \text{I}_2 \supseteq \cdots be the higher ramification groups. Let V be the underlying representation space for \chi, and V^{I_j} the subspace of V fixed pointwise by \text{I}_j. Then

\[ \text{ram}(p, \chi) = \sum_{j=0}^{\infty} \frac{|I_j|}{|I_0|} \text{codim} (V^{I_j}). \]

Obviously, p is \chi-ramified iff \text{ram}(p, \chi) > 0 iff p|f_\chi. Also p is ramified in E iff p is
1.4 Artin L–functions and Artin’s conjecture

Let $\rho: G \to GL(V)$ be a linear representation of $G = Gal(E/K)$, with character $\chi$. The Artin L–function $L(s, \chi) = L(s, \chi, E/K)$ is constructed as follows.

Let $p \in \Sigma_K$ be unramified under $\chi$. Then, although the Frobenius class $\sigma_p$ is defined only mod $I_q$, the conjugacy class of $\rho(\sigma_p)$ in $GL(V)$ is well-defined. Therefore the following function of the complex variable $s = \sigma + it$ is well-defined:

$$L_p(s, \chi) = \det (1 - \rho(\sigma_p)(Np)^{-s})^{-1},$$

where $Np = N_K/Q_p$, and $1$ is the identity element in $GL(V)$.

For general $p \in \Sigma_K$, $L_p(s, \chi)$ is defined as follows. Although $\sigma_q$ is only defined mod $I_q$, the restriction $\rho(\sigma_q)|_{V/I_q}$ is well-defined. We set

$$L_p(s, \chi) = \det (1 - \rho(\sigma_q)|_{V/I_q}(Np)^{-s})^{-1}.$$

Having defined $L_p(s, \chi)$ for all $p \in \Sigma_K$, we now introduce the Artin L–function (associated with the Galois extension $E/K$, and the character $\chi$) for $Re s = \sigma > 1$, by

$$L(s, \chi) = L(s, \chi, E/K) = \prod_{p \in \Sigma_K} L_p(s, \chi).$$

This is an analytic function on the half-plane $\sigma > 1$.

Artin L–functions behave nicely under operations on the representation. More
precisely, for all characters $\chi_1, \chi_2$ of $G$

$$L(s, \chi_1 + \chi_2) = L(s, \chi_1)L(s, \chi_2).$$

Also,

$$L(s, \text{Ind}^G_H \chi, E/K) = L(s, \chi, E/E^H),$$

where $H$ is a subgroup of $G$, $\chi$ a character of $H$, and $E^H$ the subfield of $E$ fixed by $H$.

Artin $L$–functions have meromorphic continuation to the whole $\mathbb{C}$. Here is how. First note that if $\chi(1) = 1$, then by Artin’s reciprocity $L(s, \chi)$ is the same as a Hecke $L$–function for a ray class character, and so $L(s, \chi)$ has an analytic continuation to the whole $\mathbb{C}$, except possibly a pole at $s = 1$. Now, for a general character $\chi$ of $G$, Brauer induction theorem says that there exist subgroups $H_i$, characters $\chi_i$ of $H_i$ with $\chi_i(1) = 1$, and integers $m_i$ (possibly negative !), such that

$$\chi = \sum_i m_i \text{Ind}^G_{H_i} \chi_i.$$

Therefore

$$L(s, \chi, E/K) = \prod_i L(s, \chi_i, E/E^{H_i})^{m_i}.$$ 

This gives the meromorphic continuation of the general Artin $L$–function $L(s, \chi)$.

Artin’s conjecture is that actually the Artin $L$–function $L(s, \chi)$ is analytic everywhere, except at $s = 1$ where it has a pole of order $(\chi|1_G)$.

Artin’s conjecture is known to be true in some cases. For example, Artin himself showed [1] that when $G$ is a subgroup of $S_4$, his conjecture holds. Also, Artin’s conjecture has plausible consequences. For instance, it implies Dedekind’s conjecture which says that for any (not necessarily Galois) extension $E/K$ of number fields, $\zeta_E(s)/\zeta_K(s)$ is entire. Dedekind’s conjecture is known to be true in some cases.
example if $E/K$ is Galois, or if $\tilde{E}/K$ is solvable, where $\tilde{E}$ is the normal closure of $E$ over $K$, then $\zeta_E(s)/\zeta_K(s)$ is known to be entire (see [19, Chap. 2]).

For all these reasons Artin's conjecture is a very plausible assumption. We will base our work in this thesis on the assumption of Artin's conjecture. As we just need to know that the particular Artin $L$–function we are dealing with is analytic (at $s \neq 1$), we can get unconditional results by restricting our attention to the extensions for which Artin's conjecture is already known. There seems to be the possibility of reducing the general extensions to those for which Artin's conjecture is known, and so another way of getting unconditional results.

### 1.5 The functional equation

In order to describe the functional equation which the Artin $L$–function $L(s, \chi, E/K)$ satisfies, we have to introduce Euler factors at infinite primes of $K$ too. Let $v$ be such a prime. We set

$$L_v(s, \chi, E/K) = \begin{cases} (2\pi)^{-s} \Gamma(s)^{\chi(1)} & \text{if } v \text{ is complex,} \\ \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right)^a \left(\pi^{-\frac{(s+1)}{2}} \Gamma\left(\frac{s+1}{2}\right)\right)^b & \text{if } v \text{ is real,} \end{cases}$$

where $a$ and $b$ are respectively the dimensions of the $+1$ and $-1$ eigenspaces of $\rho$ (complex conjugation), and so $a + b = \chi(1)$. Now we set

$$\gamma(s, \chi, E/K) = \prod_{\substack{v \text{ infinite prime of } K}} L_v(s, \chi, E/K).$$
The last factor we need to introduce is

\[ A_{x} = d_{K}^{x(1)}N_{K/Q}f_{x} . \]

Finally we put all these factors together to get

\[ \Lambda(s, \chi, E/K) = A_{x}^{s/2} \gamma(s, \chi, E/K)L(s, \chi, E/K) . \] (1.1)

The functional equation is

\[ \Lambda(s, \chi, E/K) = W(\chi)\Lambda(1 - s, \overline{\chi}, E/K) , \] (1.2)

where \( W(\chi) \) is a complex number with \( |W(\chi)| = 1. \)

### 1.6 Conventions

(a) A constant \( c > 0 \) is said to be absolute, if it does not depend on any parameter present in the case. Also \( c > 0 \) is said to be effective, if we can assign an explicit value to it. Any constant appearing in this thesis will be absolute and effective. Also any constant implied by the signs \( \ll, \gg \) and \( O(\cdot) \) will be absolute and effective.

(b) By Minkowski's discriminant bound, \( \log d_{K} \gg n_{K} \) (with implied constant actually > 1), if \( K \neq \mathbb{Q} \). Therefore

\[
\log A_{x} = \chi(1) \log d_{K} + \log Nf_{x} \\
\geq n_{K} \chi(1) .
\]
The expression \( \log A_\chi + cn_K\chi(1) \), which appears in our work frequently, is therefore \( \ll \log A_\chi \).

Also, if \( K \neq \mathbb{Q} \), then \( d_K \geq 3 \), \( \log A_\chi \geq \log 3 \), and so \( \log \log A_\chi > 1/11 \).

We therefore assume \( K \neq \mathbb{Q} \). All the results are valid for (the easier) case of \( K = \mathbb{Q} \) with a slight modification: replace \( \log A_\chi \) by \( \log A_\chi + n_K\chi(1) = \log A_\chi + \chi(1) \).

### 1.7 The main results

Let \( E/K \) be a Galois extension of number fields, with group \( G \). Let \( \chi \) be a character of \( G \). A prime \( p \in \Sigma_K \) is said to be "\( \chi \)-nonresidue" if it is \( \chi \)-unramified and \( \sigma_p \notin \ker \chi \). Note that, although the Frobenius class \( \sigma_p \) is defined only mod \( I_q \) (where \( I_q \) is the inertia group at a prime \( q \in \Sigma_E \) above \( p \)), but as \( p \) is assumed to be \( \chi \)-unramified (i.e. \( I_q \leq \ker \chi \)), the condition \( \sigma_p \notin \ker \chi \) is well-defined. Also note that the condition \( \sigma_p \notin \ker \chi \) is equivalent to \( \chi(\sigma_p) \neq \chi(1) \).

Observe also that there exists a prime \( p \in \Sigma_K \) which is \( \chi \)-nonresidue iff \( \chi \) is not a multiple of the trivial character (Chebotarev density theorem is needed to see this).

Let \( \chi \) be a character which is not a multiple of the trivial one. By "the least \( \chi \)-nonresidue" we mean

\[
p(\chi) = \min \{ Np \mid p \in \Sigma_K \ \chi \text{-nonresidue, and of degree one over } \mathbb{Q} \}.
\]

Note that, as the minimum is taken over primes of degree one, the least \( \chi \)-nonresidue \( p(\chi) \) is actually a rational prime.

Our main objective in this thesis is to find upper bounds on the least character nonresidue. In all of the results Artin's conjecture is assumed. The highlights of the thesis are as follows.
1. For irreducible non-Abelian $\chi$, we prove (Theorem 4.4)

$$p(\chi) \leq A_{\chi}^{(1)},$$

where $c$ is an absolute effective constant. Here we are assuming that $\zeta_K(s)$ does not have the exceptional zero (see Definition 3.3).

2. For $\chi$ not containing the trivial character, we prove (Theorem 4.3)

$$p(\chi) \leq A_{\chi}^{(1)\alpha(1+(\hat{\alpha}-\alpha)\log \chi(1))},$$

where $c$ is an absolute effective constant, $-1 \leq \alpha \leq 1$ is a parameter, $\hat{\alpha}$ is given by

$$\hat{\alpha} = \begin{cases} \frac{1}{2}\alpha, & \text{if } -1 \leq \alpha \leq 0, \\ \alpha, & \text{if } 0 \leq \alpha \leq 1, \end{cases}$$

and we are assuming that $\zeta_K(s)L(s, \chi, E/K)$ does not vanish in the region

$$1 - \frac{c_1}{\chi(1)^{\alpha} \log A_{\chi}} < \sigma < 1, \quad |t| \leq \frac{c_1}{\chi(1)^{\alpha} \log A_{\chi}} \left(1 + \frac{c_1 \log \log A_{\chi}}{\chi(1)^{\alpha+1}}\right),$$

where $c_1$ is absolute and given by Proposition 3.1. This result is a generalization of [11, Theorem 1.2] which treats the Abelian (Hecke) character nonresidues.

3. For $\chi$ not containing the trivial character, it is proved (Corollary 5.6) that the assumption of the Generalized Riemann Hypothesis implies that

$$p(\chi) \leq c\chi(1)^{-1}(\log A_{\chi})^2,$$

with $c > 0$ absolute effective.
4. We also prove a version of the Deuring-Heilbronn phenomenon for non-Abelian Artin $L$-functions. More precisely, let $\chi$ be irreducible, and let $\rho_0$ be any (real or complex, trivial or nontrivial) zero of $L(s, \chi)$. Then any other zero $\rho = \sigma + it$ of $L(s, \chi)$ is proved to satisfy

$$\sigma < 1 - c_3 \frac{\log \frac{c_4}{|1-\rho_0|\Delta}}{\Delta},$$

where $\Delta = \Delta(t, \chi) = \chi(1) (\log A_x + n_K \chi(1) \log(|t| + 2))$, and $c_3, c_4$ are absolute, effective, positive constants.

This means that a zero of $L(s, \chi)$, which is very close to $s = 1$, has the effect of pushing other zeros further away from the line $\sigma = 1$. This phenomenon was first observed by Deuring and Heilbronn in the context of Dirichlet $L$-functions. In [11] a version of the Deuring-Heilbronn phenomenon for $\zeta_K(s)$ has been proved. Theorem 3.4 generalizes that result to non-Abelian Artin $L$-functions.

Although we have not used Theorem 3.4 elsewhere in the thesis, the result is of obvious interest in the field.
Chapter 2

Zeros of Artin $L$–functions

In this chapter we investigate the location and density of the zeros of the Artin $L$–function $L(s, \chi) = L(s, \chi, E/K)$.

Here, as well as in the rest of the thesis, we will use notations strictly as introduced in Chapter 1 (e.g. $E/K$ is a Galois extension of number fields, $\chi$ a character of the Galois group $G$, etc.).

2.1 The trivial zeros

Following a tradition set by Riemann, the real and imaginary parts of our complex variable $s$ will be denoted by $\sigma$ and $t$, i.e. $s = \sigma + it$.

The Euler product of $L(s, \chi)$, which defines it for $\sigma > 1$, shows that $L(s, \chi) \neq 0$ for $\sigma > 1$. Then the functional equation implies that any zero of $L(s, \chi)$ with $\sigma < 0$ is located at a pole of the gamma factor $\gamma(s, \chi)$. These zeros of $L(s, \chi)$, which are said to be trivial, are therefore as follows.

Let $n_1$ and $2n_2$ be respectively the number of real and complex imbeddings of $K$. The trivial zeros of $L(s, \chi)$ are
2.2 The nontrivial zeros and their density

$s = 0$ with multiplicity $n_1a + n_2\chi(1) - (\chi|1_G)$,

$s = -2k$, $k \geq 1$, with multiplicity $n_1a + n_2\chi(1)$,

$s = 1 - 2k$, $k \geq 1$, with multiplicity $n_1b + n_2\chi(1)$.

Notice that the sum of the orders of (possible) zeros at $s = 0$ and $s = -1$ is $n_K\chi(1) - (\chi|1_G)$, and hence if $K \neq \mathbb{Q}$, or $\chi$ is not a multiple of $1_G$, then $L(s, \chi)$ vanishes at $s = 0$ or $s = -1$.

2.2 The nontrivial zeros and their density

From the basic factorization

$$\zeta_E(s) = \prod_{\chi} L(s, \chi, E/K)^{\chi(1)},$$

where $\chi$ runs through all irreducible characters of $G$, we see that the assumption of Artin's conjecture implies that (each) $L(s, \chi)$ divides $\zeta_E(s)$. On the other hand $\zeta_E(s)$ is known to be nonzero on the line $\sigma = 1$, and therefore $L(s, \chi)$ is analytic and nonzero on $\sigma = 1$, except a pole of order $(\chi|1_G)$ at $s = 1$. The functional equation then implies that $L(s, \chi)$ is analytic and nonzero on $\sigma = 0$ as well, except a zero of order $n_1a + n_2\chi(1) - (\chi|1_G)$ at $s = 0$.

The zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$, are said to be nontrivial.

The Generalized Riemann Hypothesis (GRH) for Artin $L$-functions is the statement that any nontrivial zero $\rho = \beta + i\gamma$ of $L(s, \chi)$ has $\beta = 1/2$.

While we will assume Artin's conjecture in all of our results, we will not assume the GRH (except in one secondary result). By a conditional (resp. unconditional) result, we mean one that assumes (resp. does not assume) the GRH. By this terminology, a result assuming Artin's conjecture will be called unconditional as long as the GRH is
2.2 The nontrivial zeros and their density

not assumed.

For real \( t \), and \( 0 < r < 2 \), we set

\[
n_x(t) = \# \{ \rho = \beta + i\gamma \mid L(\rho, \chi) = 0, \ 0 < \beta < 1, \ |\gamma - t| \leq 1 \},
\]

\[
m_x(r, t) = \# \{ \rho \mid L(\rho, \chi) = 0, \ |\rho - (1 + it)| \leq r \}.
\]

We are going to estimate these two density functions. The following lemma is a generalization for Artin \( L \)-functions of similar estimates for Hecke \( L \)-functions (see [12, Lemma 5.4], and [11, Lemma 2.2]).

**Lemma 2.1** Assuming Artin's conjecture for \( L(s, \chi) \), we have the following upper bounds:

(i) \( n_x(t) \ll \log A_\chi + n_K \chi(1) \log(|t| + 2) \),

(ii) \( m_x(r, t) \ll \chi(1) + r (\log A_\chi + n_K \chi(1) \log(|t| + 2)) \), \( (0 < r < 2) \).

**Proof:** (i) [This part has been proved in [20, pp.262-264]. For the sake of completeness here we reproduce the proof.] Artin's conjecture implies that the function

\[
(s(s - 1))^{(\chi | 1_\sigma)} \Lambda(s, \chi)
\]

is entire. It is also known to be of order one, and it does not vanish at \( s = 0 \). It therefore has a Hadamard factorization

\[
(s(s - 1))^{(\chi | 1_\sigma)} \Lambda(s, \chi) = e^{\alpha(\chi) + \beta(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \tag{2.1}
\]

where \( \alpha(\chi), \beta(\chi) \) are constants, and \( \rho \) runs through all zeros of \( (s(s - 1))^{(\chi | 1_\sigma)} \Lambda(s, \chi) \), i.e. precisely the nontrivial zeros of \( L(s, \chi) \).
2.2 The nontrivial zeros and their density

By differentiating (1.1) and (2.1) we get

\[-\frac{L'(s, \chi)}{L(s, \chi)} = (\chi|1_\mathcal{O}) \left( \frac{1}{s} + \frac{1}{s-1} \right) - \beta(\chi) - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) \]  

\[+ \frac{1}{2} \log A_\chi + \frac{\gamma'}{\gamma}(s, \chi), \]  

\[-\frac{\Lambda'(s, \chi)}{\Lambda(s, \chi)} = (\chi|1_\mathcal{O}) \left( \frac{1}{s} + \frac{1}{s-1} \right) - \beta(\chi) - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right), \]  

where \( \rho \) runs through the nontrivial zeros of \( L(s, \chi) \).

From the functional equation we get

\[\frac{\Lambda'}{\Lambda}(s, \chi) = -\frac{\Lambda'}{\Lambda}(1 - s, \overline{\chi}) = -\frac{\Lambda'}{\Lambda}(1 - \overline{s}, \chi). \]  

At \( s = 1/2 \) this gives

\[\text{Re} \frac{\Lambda'}{\Lambda}(1/2, \chi) = 0.\]

Choosing \( s = 1/2 \), we consider the real part of the two sides of (2.3). If \( \rho \) is a nontrivial zero of \( L(s, \chi) \), then so is \( 1 - \bar{\rho} \), and the contributions of these two zeros in the sum \( \text{Re} \sum_{\rho} \frac{1}{1/2 - \rho} \) cancel each other. Hence

\[\text{Re} \beta(\chi) = - \text{Re} \sum_{\rho} \frac{1}{\rho}. \]  

(2.3) therefore implies

\[\text{Re} \frac{\Lambda'}{\Lambda}(s, \chi) = \sum_{\rho} \text{Re} \frac{1}{s - \rho} - (\chi|1_\mathcal{O}) \text{Re} \left( \frac{1}{s} + \frac{1}{s - 1} \right). \]
2.2 The nontrivial zeros and their density

In this equality we set \( s = 2 + it \). For each \( \rho = \beta + i\gamma \) with \( |\gamma - t| \leq 1 \) we have

\[
\text{Re} \frac{1}{(2 + it) - \rho} = \frac{2 - \beta}{(2 - \beta)^2 + (t - \gamma)^2} \geq \frac{1}{3}.
\]

Hence (2.6) gives

\[
n_x(t) \ll \text{Re} \frac{\Lambda'}{\Lambda}(2 + it, \chi) + \chi(1). \tag{2.7}
\]

We are going to complete the proof of (i) by bounding \( \text{Re} \frac{\Lambda'}{\Lambda}(2 + it, \chi) \), and for this we need bounds on \( \left| \frac{L'}{L}(2 + it, \chi) \right| \) and \( \left| \frac{\gamma'}{\gamma}(2 + it, \chi) \right| \). For \( \sigma > 1 \) we have

\[
\left| \frac{L'}{L}(s, \chi) \right| = \left| \sum_{p \in \Sigma_K} \sum_{m=1}^{\infty} \chi(\sigma_p^m) (\log N_p)(N_p)^{-ms} \right|
\leq \chi(1) \sum_{m=1}^{\infty} (\log N_p)(N_p)^{-m\sigma}
= -\chi(1) \zeta_K' \zeta_K(\sigma)
\leq -n_K \chi(1) \zeta'(\sigma)
\ll \frac{n_K \chi(1)}{\sigma - 1}. \tag{2.8}
\]

The first equality is obtained from the Euler product of \( L(s, \chi) \), and by \( \chi(\sigma_p^m) \) we mean the following. For unramified primes there is no ambiguity. If \( p \) is ramified in \( E \) then we choose a prime \( q \in \Sigma_E \) above \( p \), and fix a Frobenius element \( \sigma_q \). By \( \chi(\sigma_p^m) \) we really mean

\[
\chi(\sigma_p^m) = \frac{1}{|I_q|} \sum_{\alpha \in I_q} \chi(\sigma_q^m \alpha). \tag{2.9}
\]
2.2 The nontrivial zeros and their density

From this modification it easily follows that for $\sigma > 1$

$$-\frac{L'}{L}(s, \chi) = \sum_{\mathfrak{p} \in \Sigma_K} \sum_{m=1}^{\infty} \chi(\sigma_{\mathfrak{p}}^m) (\log N_{\mathfrak{p}})(N_{\mathfrak{p}})^{-ms}.$$  \hfill (2.10)

The second inequality in (2.8),

$$\frac{\zeta'_K}{\zeta_K}(\sigma) \leq -n_K \frac{\zeta'}{\zeta}(\sigma),$$  \hfill (2.11)

is [12, Lemma 3.2].

From (2.8) we obtain

$$\left| \frac{L'}{L}(2 + it, \chi) \right| \ll n_K \chi(1).$$  \hfill (2.12)

To bound $\frac{\gamma'}{\gamma}$, first note that if $|s + k| \geq \frac{1}{8}$ for all integers $k \geq 0$, then

$$\left| \frac{\Gamma'}{\Gamma}(s) \right| \ll \log(|s| + 2).$$  \hfill (2.13)

(This is [12, Lemma 6.1].) From the definition of the gamma factor $\gamma(s, \chi)$ and (2.13) it follows that for $s$ with $|s + k| \geq \frac{1}{4}$ (for all integers $k \geq 0$) we have

$$\left| \frac{\gamma'}{\gamma}(s, \chi) \right| \ll n_K \chi(1) \log(|s| + 2).$$  \hfill (2.14)

In particular

$$\left| \frac{\gamma'}{\gamma}(2 + it, \chi) \right| \ll n_K \chi(1) \log(|t| + 2).$$  \hfill (2.15)
Finally (1.1), (2.7), (2.12), (2.15) cooperate to give

\[ n_\chi(t) \ll \log A_\chi + n_K \chi(1) \log(|t| + 2) . \]

The proof of (i) is complete.

(ii) From (2.2) we get

\[
\frac{L'(s, \chi)}{L(s, \chi)} - \frac{L'(3 + it, \chi)}{L(3 + it, \chi)} = -(\chi|1_G) \left( \frac{1}{s} + \frac{1}{s - 1} - \frac{1}{3 + it} - \frac{1}{2 + it} \right) + \sum_{\rho} \left( \frac{1}{s - \rho} - \frac{1}{(3 + it) - \rho} \right) - \gamma'(s, \chi) + \gamma'(3 + it, \chi). \tag{2.16}
\]

(Here \( s = \sigma + it \) and \( 3 + it \) have equal imaginary parts.) From (2.14), (2.16) it follows that for \( \frac{i}{2} \leq \sigma \leq 3 \)

\[
\left| \frac{L'}{L}(s, \chi) + \frac{\chi|1_G}{s - 1} - \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} \right| \ll n_K \chi(1) \log(|t| + 2)
\]

\[
+ \sum_{|\gamma - t| > 1} \left| \frac{1}{s - \rho} - \frac{1}{(3 + it) - \rho} \right| + \sum_{|\gamma - t| \leq 1} \left| \frac{1}{(3 + it) - \rho} \right|. \tag{2.17}
\]

The two sums on the right side can be bounded as follows:

\[
\sum_{|\gamma - t| \leq 1} \left| \frac{1}{(3 + it) - \rho} \right| \leq n_\chi(t)
\]

\[
\ll \log A_\chi + n_K \chi(1) \log(|t| + 2), \tag{2.18}
\]

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2.2 The nontrivial zeros and their density

\[
\sum_{|\gamma-t|>1} \left| \frac{1}{s-\rho} - \frac{1}{(3+it)-\rho} \right| = \sum_{|\gamma-t|>1} \frac{3-\sigma}{|s-\rho||(3+it)-\rho|} \leq \sum_{|\gamma-t|>1} \frac{1}{|\gamma-t|^2} \leq \sum_{j=2}^{\infty} \frac{n_\chi(t+j) + n_\chi(t-j)}{(j-1)^2} \leq \log A_\chi + n_K \chi(1) \log(|t|+2). \tag{2.19}
\]

(2.17), (2.18), (2.19) imply

\[
\left| \frac{L'}{L}(s, \chi) + \frac{(\chi|1_G)}{s-1} - \sum_{|\gamma-t|\leq 1} \frac{1}{s-\rho} \right| \leq \log A_\chi + n_K \chi(1) \log(|t|+2), \tag{2.20}
\]

valid for \( \frac{1}{2} \leq \sigma \leq 3 \).

To finish the proof of (ii), we need to bound \( \left| \frac{L'}{L} \right| \) (the bound given by (2.8) is not sharp enough). From (2.2), (2.5), and (2.20) applied to \( L(s, 1, K/K) = \zeta_K(s) \), we obtain

\[
-\frac{\zeta'_K}{\zeta_K}(\sigma) \leq \frac{1}{\sigma} + \frac{1}{\sigma-1} - \sum_{\rho_K} \Re \frac{1}{s-\rho_K} + \frac{1}{2} \log d_K + O(n_K \log(\sigma+2)), \tag{2.21}
\]

where \( \rho_K \) runs through the nontrivial zeros of \( \zeta_K(s) \). If \( 1 < \sigma < 3 \), then \( \Re \frac{1}{s-\rho_K} > 0 \), and therefore

\[
-\frac{\zeta'_K}{\zeta_K}(\sigma) \leq \frac{1}{\sigma-1} + \frac{1}{2} \log d_K + O(n_K). \tag{2.22}
\]
2.2 The nontrivial zeros and their density

The desired bound on $\left| \frac{L'}{L}(s, \chi) \right|$ is now within reach:

$$
\left| \frac{L'}{L}(s, \chi) \right| = \left| \sum_{p \in \Sigma_K} \sum_{m=1}^{\infty} \chi(p^m) \left( \log N_p \right) (N_p)^{-ms} \right|
$$

$$
\leq \chi(1) \sum_{p \in \Sigma_K} \sum_{m=1}^{\infty} \left( \log N_p \right) (N_p)^{-m\sigma}
$$

$$
= -\chi(1) \frac{1}{\zeta_K}(\sigma)
$$

$$
\leq \chi(1) \left( \frac{1}{\sigma - 1} + \frac{1}{2} \log d_K + O(n_K) \right). \quad (2.23)
$$

Now, (2.20), (2.23) imply

$$
\left| \sum_{|\gamma - \xi| \leq 1} \frac{1}{s - \rho} \right| \ll \frac{\chi(1)}{\sigma - 1} + \log A_{\chi} + n_K \chi(1) \log(|t| + 2). \quad (2.24)
$$

We sum (2.24) over $t + j$, $-3 \leq j \leq 3$, to get

$$
\left| \sum_{|\gamma - \xi| \leq 4} \frac{1}{s - \rho} \right| \ll \frac{\chi(1)}{\sigma - 1} + \log A_{\chi} + n_K \chi(1) \log(|t| + 2). \quad (2.25)
$$

On the other hand

$$
\left| \sum_{|\gamma - \xi| \leq 4} \frac{1}{s - \rho} \right| \geq \sum_{|\gamma - \xi| \leq 4} \text{Re} \left( \frac{1}{s - \rho} \right) = \sum_{|\gamma - \xi| \leq 4} \frac{\sigma - \beta}{|s - \rho|^2} \geq \sum_{|\gamma - \xi| \leq 4} \frac{\sigma - 1}{|s - \rho|^2}. \quad (2.26)
$$

Finally, let $0 < r < 2$. We choose $\sigma = 1 + r$, and set $k = \| \{ \rho | |s - \rho| \leq 2r \} $. Note
that $m_\chi(r, t) \leq k$. From (2.26) we have

$$\left| \sum_{|\gamma - t| \leq 4} \frac{1}{s - \rho} \right| \geq \frac{kr}{4r^2} \geq \frac{m_\chi(r, t)}{4r},$$

which together with (2.25) implies

$$m_\chi(r, t) \ll \chi(1) + r (\log A_\chi + n_K \chi(1) \log(|t| + 2)).$$

Lemma (2.1) is now completely proved. $\square$
Chapter 3

Zero-free regions

We continue our program of studying the zeros of Artin $L$–functions under the assumption of Artin's conjecture.

In order to obtain strong results about density of primes (with certain properties), one needs to know that for the nontrivial zeros $\rho = \beta + i\gamma$ of (various types of) $L$–functions, $1 - \beta$ is not too small, especially for small $\gamma$. In other words, one needs to know that the zeros are away from the line $\sigma = 1$, especially from the point $s = 1$.

In this chapter our objective is to present some “zero-free region” results, which will be used later. In the first section we quote a couple of results from [18]. In the second section we will prove a result which shows that if there is any zero too close to $s = 1$, then the other zeros are fairly away from the line $\sigma = 1$. Any result of this type is known as an instance of the “Deuring-Heilbronn phenomenon.” In [11] a version of the Deuring-Heilbronn phenomenon for $\zeta_K(s)$ has been proved. We generalize it to Artin $L$–functions.
3.1 Zero-free regions and the exceptional zero

The following two propositions are [18, Corollary 3.2] and [18, Proposition 3.7], respectively (also see [11, Lemma 2.3]).

**Proposition 3.1** There is an absolute, effective constant $c_1 > 0$ such that the following holds:

Assume Artin's conjecture for $E/K$. If $\chi$ is irreducible, then $L(s, \chi)$ has at most one zero in the region

$$1 - \frac{c_1}{\chi(1) \log A_x} \leq \sigma \leq 1, \quad |t| \leq \frac{c_1}{\chi(1) \log A_x}.$$ 

**Proposition 3.2** There is an absolute, effective constant $c_2 > 0$ such that the following holds:

Assume Artin's conjecture for $E/K$. If $\chi$ is irreducible, with $\chi(1) > 1$ (or $\chi(1) = 1$ but $\chi^2 \neq 1_G$), then $L(s, \chi)$ does not vanish in the region

$$1 - \frac{c_2}{\chi(1)^3 (\log A_x + n_K \log(|t| + 2))} \leq \sigma \leq 1.$$ 

If $\chi(1) = 1$ and $\chi^2 = 1_G$, then $L(s, \chi)$ has at most one zero in this region. Such a zero is necessarily real.

**Definition 3.3** (The exceptional zero of $L(s, \chi)$)

Proposition 3.1 says that in the rectangular neighborhood of radius $\frac{c_1}{\chi(1) \log A_x}$ of $s = 1$, $L(s, \chi)$, with irreducible $\chi$, has at most one zero. If such a zero exists, we call it the exceptional zero of $L(s, \chi)$. 
3.2 The Deuring-Heilbronn phenomenon for Artin $L$-functions

If the exceptional zero of $L(s, \chi)$ exists, it can be a major obstacle for obtaining strong results about density of (or existence of small) primes (of specific types). There are two ways of dealing with this problem: we can assume it does not exist (after all, even the GRH may be true!), or, if one insists on getting absolutely unconditional results, one can show that if the exceptional zero does exist, then it "pushes" other zeros away from the line $\sigma = 1$. This is the phenomenon first observed by Deuring and Heilbronn in the context of Dirichlet $L$-functions. In [11] a version of the Deuring-Heilbronn phenomenon for $\zeta_K(s)$ has been proved. In the following theorem we generalize that result to Artin $L$-functions.

Theorem 3.4 (The Deuring-Heilbronn phenomenon for Artin $L$-functions)
There are absolute, effective, positive constants $c_3, c_4$ such that the following holds:

Assume Artin's conjecture for $E/K$. If $\chi$ is irreducible, and $L(s, \chi)$ has a zero $\rho_0$ (real or complex), then any other zero $\rho = \sigma + it$ of $L(s, \chi)$ satisfies

$$\sigma < 1 - c_3 \frac{c_4}{|1 - \rho_0| \Delta},$$

where $\Delta = \Delta(t, \chi) = \chi(1) (\log A_\chi + n_K \chi(1) \log(|t| + 2))$.

Before we present the proof of Theorem 3.4, we have to pave the way by means of a few lemmas.

Lemma 3.5 We have

(i) $A_\chi = A_\chi$,
(ii) $A_{\chi \overline{\chi}} \leq A_{\chi}^{2(1)}$.

**Proof:** (i) Let $V$ be the underlying representation space for $\chi$. Let $V'$ be the dual space of $V$. It would suffice to show that for any subgroup $H$ of $G$, $\dim V'^H = \dim V^H$. But the bilinear form

$$\left\{ \begin{array}{c} V^H \times V'^H \to \mathbb{C} \\ (x, f) \mapsto f(x) \end{array} \right.$$ 

is easily seen to be non-degenerate on both sides, and hence $\dim V'^H = \dim V^H$.

(ii) First we show that $f_{\chi \overline{\chi}}$ divides $f_{\chi}^{2(\chi(1) - 1)}$. (For the special case $K = \mathbb{Q}$, this has been proved in [21, Lemma 1]. That proof directly works for the general case as well; for the sake of completeness we reproduce the proof.)

Let $V$ be the underlying representation space for $\chi$. It would suffice to show that for any subgroup $H$ of $G$

$$\text{codim}(V \otimes V')^H \leq 2(\chi(1) - 1) \text{codim } V^H.$$  

(3.1)

First observe that

$$\dim V^H = (\chi|_H \mid 1_H).$$

Therefore (3.1) is equivalent to

$$\chi(1)^2 - (\chi \overline{\chi}|_H \mid 1_H) \leq 2(\chi(1) - 1) (\chi(1) - (\chi|_H \mid 1_H)).$$  

(3.2)

$\chi|_H$ decomposes as

$$\chi|_H = \lambda_0 1_H + \sum_{\varphi \neq 1_H} \lambda_{\varphi} \varphi,$$

$$\lambda_0 1_H + \sum_{\varphi \neq 1_H} \lambda_{\varphi} \varphi,$$
where, in the sum, \( \varphi \) runs through the nontrivial irreducible characters of \( H \), and \( \lambda_0 \) and \( \lambda_\varphi \) are nonnegative integers. Note that

\[
(\chi|\bar{\chi}|_H | 1_H) = (\chi|_H | \chi|_H) = \lambda_0^2 + \sum_{\varphi \neq 1_H} \lambda_\varphi^2,
\]

\[
(\chi|_H | 1_H) = \lambda_0,
\]

\[
\chi(1) = \lambda_0 + \sum_{\varphi \neq 1_H} \lambda_\varphi \varphi(1).
\]

Now, (3.2) can easily be seen to be equivalent to

\[
- \sum_{\varphi \neq 1_H} \lambda_\varphi^2 \leq \left( \sum_{\varphi \neq 1_H} \lambda_\varphi \varphi(1) \right) \left( \sum_{\varphi \neq 1_H} \lambda_\varphi \varphi(1) - 2 \right) . \tag{3.3}
\]

The right side of (3.3) is negative only if for one \( \varphi \), \( \lambda_\varphi = 1 \), \( \varphi(1) = 1 \), and for all other \( \varphi \), \( \lambda_\varphi = 0 \). In this case (3.3) reads \(-1 \leq -1\). In all other cases \( \text{LHS} \leq 0 \leq \text{RHS} \). Hence (3.3), and therefore (3.1), is proved. So our claim

\[
f_{x\bar{x}} | f_x^{2(\varphi(1)-1)},
\]

is now established. In particular

\[
Nf_{x\bar{x}} \leq (Nf_x)^{2\varphi(1)}.
\]

Therefore

\[
A_{x\bar{x}} = d_K^{\chi(1)^2} Nf_{x\bar{x}} \leq \left( d_K^{\chi(1)} Nf_x \right)^{2\chi(1)} = A_x^{2\chi(1)}.
\]

(ii) is now proved.
3.2 The Deuring-Heilbronn phenomenon for Artin $L$–functions

Zero-free regions

Lemma 3.6 Let $u, v \in \mathbb{C}$, with $|u|, |v| \geq 1$. Then for $j \geq 1$

$$|u^{-j} - v^{-j}| \leq j|u - v| .$$

Proof: We have

$$|u^{-j} - v^{-j}| = |u^{-1} - v^{-1}||\sum_{k=1}^{j} u^{-(k-1)}v^{-(j-k)}|$$

$$\leq j |u^{-1} - v^{-1}| = \frac{j|u - v|}{|u| |v|} \leq j|u - v| .$$

\(\Box\)

Finally, we quote [11, Theorem 4.2]:

Lemma 3.7 Let $s_m = \sum_{n=1}^{\infty} z_n^m$, and $L = |z_1|^{-1} \sum_{n=1}^{\infty} |z_n|$. Assume that $|z_n| \leq |z_1|$, for all $n \geq 1$.

Then there exists $j_0$, with $1 \leq j_0 \leq 24L$, such that

$$\text{Re} \ s_{j_0} \geq \frac{1}{8} |z_1|^{j_0} .$$

We can now present

Proof of Theorem 3.4: Let $\phi$ be a character of $G$, not necessarily irreducible.

By our assumption of Artin’s conjecture, $(s - 1)^{(\phi|1_{G})}L(s, \phi)$ is entire. It is also of order one, so we have the Hadamard factorization

$$(s - 1)^{(\phi|1_{G})}L(s, \phi) = s^{r} e^{a_1 + a_2 s} \prod_{\omega} \left(1 - \frac{s}{\omega}\right) e^{s/\omega} ,$$

(3.4)
where $r$ is the multiplicity of $s = 0$ as a zero of $L(s, \varphi)$, and $\omega$ runs through all the zeros $\omega \neq 0$ of $L(s, \varphi)$, including the trivial ones. From (3.4) we obtain

$$\frac{L'(s, \varphi)}{L}(s, \varphi) = \frac{(\varphi|1_G)}{s - 1} - \alpha_2 - \sum_{\omega} \left(\frac{1}{s - \omega} + \frac{1}{\omega}\right) - \frac{r}{s}. \quad (3.5)$$

By (2.10) on the other hand, we have

$$\frac{L'(s, \varphi)}{L}(s, \varphi) = \sum_{p \in \Sigma_K} \sum_{m=1}^{\infty} \varphi(\sigma_p^m) (\log N_p)(N_p)^{-ms}, \quad (3.6)$$

valid for $\sigma > 1$. Differentiating (3.5) and (3.6) $2j - 1$ times, we arrive at

$$\frac{1}{(2j - 1)!} \sum_{p \in \Sigma_K} \sum_{m=1}^{\infty} \varphi(\sigma_p^m)(\log N_p)(\log N_p)^{2j-1}(N_p)^{-ms}$$

$$= \frac{(\varphi|1_G)}{(s - 1)^{2j}} - \sum_{\omega} \frac{1}{(s - \omega)^{2j}}. \quad (3.7)$$

Here $\omega$ runs through all the zeros of $L(\varphi, \chi)$ (including the $r$ trivial zeros $\omega = 0$). (3.7) is valid for $\sigma > 1$, $j \geq 1$.

Now, let $\chi$ be, as in the statement of Theorem 3.4, an irreducible character of $G$. The case $\chi = 1_G$ is equivalent to [11, Theorem 5.1]. Therefore we assume $\chi \neq 1_G$. We are going to use (3.7) with $\varphi = (1_G + \chi)(1_G + \overline{\chi})$. Note that

$$( (1_G + \chi)(1_G + \overline{\chi}) \mid 1_G) = ( (1_G + \chi) \mid (1_G + \chi)) = 2.$$
From (3.7) applied to \((1_G + \chi)(1_G + \overline{\chi})\) we get

\[
\frac{1}{(2j - 1)!} \sum_{\sigma \in \Sigma_K} \sum_{m=1}^{\infty} ((1_G + \chi)(1_G + \overline{\chi})) (\sigma_m^m)(\log N_p)(\log N_p^m)^{2j-1} (N_p)^{-m\sigma} (1 + (N_p^m)^{-i\tau}) \\
= \frac{2}{(\sigma - 1)^{2j}} \frac{2}{(s - 1)^{2j}} - \frac{1}{(\sigma - \rho_0)^{2j}} - \frac{1}{(s - \rho_0)^{2j}} - \sum_{n=1}^{\infty} z_n^j,
\]

(3.8)

where \(s = \sigma + it\), and \(z_n\) are of the form \((\sigma - \omega)^{-2}\) or \((s - \omega)^{-2}\), with \(\omega\) a zero of \(L(s,(1_G + \chi)(1_G + \overline{\chi})) = \zeta_K(s)L(s,\chi)L(s,\overline{\chi})L(s,\chi\overline{\chi})\). (Remember that we are assuming that \(\rho_0\) is a zero of \(L(s,\chi)\), and therefore \(\bar{\rho}_0\) is a zero of \(L(s,\overline{\chi})\).)

The real part of the left side of (3.8) is nonnegative, so if we choose \(\sigma = 2\) (and so \(s = 2 + it\)), we obtain

\[
\Re \sum_{n=1}^{\infty} z_n^j \leq \Re \left[ \left( \frac{1}{(\sigma - 1)^{2j}} - \frac{1}{(\sigma - \rho_0)^{2j}} \right) + \left( \frac{1}{(s - 1)^{2j}} - \frac{1}{(s - \rho_0)^{2j}} \right) \right] \\
\leq \left| \left[ \frac{\sum z_n^j}{(\sigma - 1)^{2j} - (\sigma - \rho_0)^{2j}} + \frac{\sum z_n^j}{(s - 1)^{2j} - (s - \rho_0)^{2j}} \right] \right| \\
\leq 2j|1 - \rho_0| + 2j|1 - \bar{\rho}_0| + 2j|1 - \rho_0| + 2j|1 - \bar{\rho}_0| \\
= 8j|1 - \rho_0|.
\]

(3.9)

(In the last inequality, we have used Lemma 3.6, and the choice \(\sigma = 2\) guarantees the assumption of the lemma.)

We are now ready to complete the proof of Theorem 3.4. Suppose that \(\rho = \beta + i\gamma \neq \rho_0\) is a zero of \(L(s,\chi)\). In (3.8) we set \(t = \gamma\), and apply Lemma 3.7 to the left side of (3.9). In accordance with the notations used in Lemma 3.7, we have

\[
|z_1| \geq \max \left( (2 - \beta)^{-2}, 3^{-2} \right).
\]
(3^{-2} is coming from the fact that \( L(s, \chi) \) has a trivial zero at \( s = 0 \) or \( s = -1 \). We have put \( 3^{-2} \) there to take care of the possibility of \( \rho = \beta \) being a trivial zero with \(-\beta \) large.) And therefore

\[
L \ll \min ( (2 - \beta)^2, 1 ) ( S(1_G) + S(\chi) + S(\overline{\chi}) + S(\chi \overline{\chi}) ),
\]

where

\[
S(\chi) = \sum_{\omega} \left( \frac{1}{|2 - \omega|^2} + \frac{1}{|2 + i\gamma - \omega|^2} \right),
\]

\( \omega \) running through the zeros of \( L(s, \chi) \), and similarly for \( S(1_G) \), \( S(\chi) \), and \( S(\chi \overline{\chi}) \). Since multiplicity of any trivial zero of \( L(s, \chi) \) is \( \leq n_K \chi(1) \), the contribution of the trivial zeros to \( S(\chi) \) is \( \ll n_K \chi(1) \). The contribution of the nontrivial zeros is estimated by using Lemma 2.1(i) giving

\[
S(\chi) \ll n_K \chi(1) + \int_0^\infty \frac{1}{1+u^2} dn_\chi(u) + \int_0^\infty \frac{1}{1+u^2} dn_\chi(u + |\gamma|)
\ll \log A_\chi + n_K \chi(1) \log(|\gamma| + 2).
\]

This, together with Lemma 3.5, implies

\[
S(1_G) + S(\chi) + S(\overline{\chi}) + S(\chi \overline{\chi}) \ll (\log d_K + n_K \log(|\gamma| + 2)) + (\log A_\chi + n_K \chi(1) \log(|\gamma| + 2))
+ (\log A_{\overline{\chi}} + n_K \chi(1) \log(|\gamma| + 2)) + (\log A_{\chi \overline{\chi}} + n_K \chi(1)^2 \log(|\gamma| + 2))
\ll \chi(1) \log A_\chi + n_K \chi(1)^2 \log(|\gamma| + 2) = \Delta(\gamma, \chi).
\]

(3.10) and (3.11) give us

\[
L \ll \Delta(\gamma, \chi).
\]
Lemma 3.7 now says that there exists $j_0$ with $1 \leq j_0 \leq 24L$, such that

$$\text{Re} \sum_{n=1}^{\infty} z_n^{j_0} \geq \frac{1}{8} (2 - \beta)^{-2j_0} \geq \frac{1}{8} \exp (-2j_0(1 - \beta)).$$

This and (3.9) imply that

$$\exp (-2j_0(1 - \beta)) \ll j_0 |1 - \rho_0|.$$  \hspace{1cm} (3.12)

Since $j_0 \ll L \ll \Delta$, for some $c > 0$, $j_0 < c\Delta$. Let also $c'$ be the constant implied by $\ll$ in (3.12). Then (3.12) implies

$$-2c\Delta(1 - \beta) < \log(c'\Delta|1 - \rho_0|),$$

and finally this gives

$$1 - \beta > \frac{1}{2c} \frac{\log \frac{1}{c'\Delta|1 - \rho_0|}}{\Delta}.$$  \hspace{1cm}

We set $c_3 = \frac{1}{2c}$, $c_4 = \frac{1}{c'\Delta}$. Theorem 3.4 is now proved.

Theorem 3.4 can be complemented by the following result, which gives a lower bound for the distance between the exceptional zero and $s = 1$.

**Proposition 3.8** There are absolute, effective, positive constants $c_5, c_6$ such that the following holds:

Assume Artin's conjecture for $E/K$. If $\chi$ is irreducible, and $\rho_0$ is a zero of $L(s, \chi)$, then

(i) if $\chi(1) = 1$, then $|1 - \rho_0| \geq A_\chi^{-c_5}$,

(ii) if $\chi(1) > 1$ (or $\chi(1) = 1$, but $\chi^2 \neq 1_G$), then $|1 - \rho_0| \geq \frac{c_6}{\chi(1)^3 \log A_\chi}$. 
**Zero-free regions**

3.2 The Deuring-Heilbronn phenomenon for Artin $L$-functions

**Proof:** (i) There is always a trivial zero $\geq -2$, and therefore Theorem 3.4 implies that

\[
\log \frac{c_4}{|1 - \rho_0| \Delta} \leq 3 ,
\]

(3.13)

where $\Delta = \Delta(0, \chi) \ll \chi(1) \log A_{\chi} = \log A_{\chi}$.

Assume $|1 - \rho_0| < A_{\chi}^{-c_5}$. We will show that for sufficiently large (but absolute) $c_5$, we can reach a contradiction.

(3.13), with $|1 - \rho_0| < A_{\chi}^{-c_5}$, implies

\[
c_5 \log A_{\chi} + \log c_4 - \log \Delta \leq \frac{3}{c_3} \Delta .
\]

(3.14)

But for $c_5 \gg 1$, (3.14) contradicts the fact that $\Delta \ll \log A_{\chi}$. This proves (i).

(ii) This follows immediately from Proposition 3.2.

\[\square\]

**Remark 3.9** Let us here make it more explicit how any zero of $L(s, \chi)$, with $\chi$ irreducible, which is very close to $s = 1$, pushes other zeros further away from $\sigma = 1$.

Let $\rho_0$ be a zero of $L(s, \chi)$ with

\[
|1 - \rho_0| = \frac{\lambda}{\chi(1) \log A_{\chi}} ,
\]

(3.15)

where $\lambda > 0$ is small. Then Theorem 3.4 implies that any other zero $\rho = \sigma + it$ of $L(s, \chi)$ with $|t| \leq 1$ satisfies

\[
1 - \sigma > c_{31} \frac{\log \frac{c_{41}}{\lambda}}{\chi(1) \log A_{\chi}} ,
\]

(3.16)

where $c_{31}, c_{41}$ are absolute, effective, positive constants.
Now, if $\lambda$ satisfies

$$\lambda < c_{41} e^{-\frac{c_{1}}{c_{31}} M}, \quad (M > 1) \quad (3.17)$$

then (3.16) implies that

$$1 - \sigma > \frac{c_{1}}{\chi(1) \log A_{x}} M. \quad (3.18)$$

Note that $\frac{c_{1}}{\chi(1) \log A_{x}}$ is the radius defining the exceptional zero. Comparison of (3.15), (3.17), and (3.18) shows the effectivity of Theorem 3.4.
Chapter 4

The Least Character Nonresidue

[The notations used in this chapter are in full conformity with those introduced in the first chapter. For convenience we recall the definitions of “character nonresidue”, and “the least character nonresidue” \( p(\chi) \).]

4.1 Introduction

**Definition 4.1** Let \( p \in \Sigma_K \) be \( \chi \)-unramified. We say that \( p \) is \( \chi \)-nonresidue if \( \sigma_p \not\in \ker \chi \).

Note that, although the Frobenius class \( \sigma_p \) is defined only \( \mod I_q \) (where \( I_q \) is the inertia group at a prime \( q \in \Sigma_E \) above \( p \)), but since \( p \) is assumed to be \( \chi \)-unramified (i.e. \( I_q \leq \ker \chi \)), the condition \( \sigma_p \not\in \ker \chi \) is well-defined. Also note that the condition \( \sigma_p \not\in \ker \chi \) is equivalent to \( \chi(\sigma_p) \neq \chi(1) \).
4.2 Bounds on the least character nonresidue

**Definition 4.2** Let $\chi$ be a character which is not a multiple of the trivial one. By "the least $\chi$-nonresidue" we mean

$$p(\chi) = \min \{ N_p | p \in \Sigma_K \chi\text{-nonresidue, and of degree one over } \mathbb{Q} \} .$$

Note that, as the minimum is taken over primes of degree one, "the least $\chi$-nonresidue" $p(\chi)$ is actually a rational prime.

Our goal in this chapter is to find good upper bounds for $p(\chi)$. This investigation is one further step in a classical trend:

(a) If we take $K = \mathbb{Q}$, $E$ a quadratic extension of $K$, we get the problem of finding upper bounds on the least quadratic nonresidue.

(b) If we take a general (Galois) extension $E/K$, but choose one dimensional $\chi$, we get the problem of finding upper bounds on the least character nonresidue in the classical sense.

4.2 Bounds on the least character nonresidue

We are going to present two theorems giving bounds on the least character nonresidue. The first one is a generalization of [14, Theorem 1], and [11, Theorem 1.2], which are dealing with the particular cases of $\{K = \mathbb{Q}, G\text{ Abelian}\}$, and Abelian $\chi$, respectively. In this result, the bound on the least character nonresidue depends on the size of a hypothetical zero-free region around $s = 1$ for $\zeta_K(s)L(s, \chi, E/K)$.

Our second theorem gives a bound on the least character nonresidue unconditionally. Here we generalize the method of [11, Theorem 1.1], which gives a bound on the least prime in a conjugacy class.

**Theorem 4.3** There is an absolute, effective constant $c_7 > 0$ such that the following
holds:

Let \(-1 \leq \alpha \leq 1\) be a parameter, and set

\[
\hat{\alpha} = \begin{cases} 
\frac{1}{2} \alpha & \text{if } -1 \leq \alpha \leq 0, \\
\alpha & \text{if } 0 \leq \alpha \leq 1.
\end{cases}
\]

Let \(\chi\) be a character of \(G\) not containing the trivial one, and assume Artin’s conjecture for \(L(s, \chi)\). If \(\zeta_K(s)L(s, \chi)\) does not vanish in the region \(R(\alpha, \chi)\) defined by

\[
1 - \frac{c_1}{\chi(1)^{\alpha} \log A_{\chi}} < \sigma < 1, \quad 0 \neq |t| \leq \frac{c_1}{\chi(1)^{\alpha} \log A_{\chi}} \left(1 + \frac{c_1 \log \log A_{\chi}}{\chi(1)^{\alpha+1}}\right),
\]

(where \(c_1\) is the absolute constant given by Proposition 3.1), and if \(\zeta_K(\sigma) \neq 0\) for

\[
1 - \frac{c_1}{\chi(1)^{\alpha} \log A_{\chi}} < \sigma < 1,
\]

then

\[
p(\chi) \leq A_{\chi}^{c_{1} \gamma_{\chi}(1) (1 + (\hat{\alpha} - \alpha) \log \chi(1))}.
\]

**Theorem 4.4** There is an absolute, effective constant \(c_8 > 0\) such that the following holds:

Let \(\chi\) be an irreducible character of \(G\) with \(\chi(1) > 1\). Assume Artin’s conjecture for \(E/K\). If the exceptional zero of \(\zeta_K(s)\) does not exist, then

\[
p(\chi) \leq A_{\chi}^{c_{8} \chi(1)}.
\]

We will prove Theorem 4.3 and Theorem 4.4 in the next chapter. The rest of this chapter will be devoted to a few corollaries and remarks.

**Corollary 4.5** Let \(\chi\) be a nontrivial character of \(G\) with \(\chi(1) = 1\). If \(\zeta_K(s)L(s, \chi)\)
4.2 Bounds on the least character nonresidue  The Least Character Nonresidue

does not vanish in the region

\[ 1 - \frac{c_1}{\log A_x} < \sigma < 1 \quad 0 \neq |t| \leq \frac{c_1}{\log A_x} (1 + c_1 \log \log A_x), \]

and if the exceptional zero of \( \zeta_K(s) \) does not exist, then

\[ p(\chi) \leq A_{\chi}^{c_7}, \]

(where \( c_7 \) is the absolute constant given by Theorem 4.3).

PROOF: Under the assumptions of the corollary, all conditions of Theorem 4.3 are met: \( L(s, \chi) \) is entire by Artin's reciprocity, and as \( \log d_K \leq \log A_x \), the real segment \( 1 - \frac{c_1}{\log d_K} < \sigma < 1 \) covers the segment \( 1 - \frac{c_1}{\log A_x} < \sigma < 1 \) (where we need to know \( \zeta_K \) does not vanish). (Note that the parameter \( \alpha \) plays no role in the case \( \chi(1) = 1 \)). Therefore by Theorem 4.3, \( p(\chi) \leq A_{\chi}^{c_7}. \)

Corollary 4.6 Let \( \chi \) be a character of \( G \) not containing the trivial one, and assume Artin's conjecture for \( L(s, \chi) \). Assume also that the exceptional zero of \( \zeta_K(s) \) does not exist.

(i) If \( \zeta_K(s) L(s, \chi) \) does not vanish in the region \( R(1, \chi) \), then

\[ p(\chi) \leq A_{\chi}^{c_7 \chi(1)}. \]

(ii) If \( \zeta_K(s) L(s, \chi) \) does not vanish in the region \( R(0, \chi) \), then

\[ p(\chi) \leq A_{\chi}^{c_7}. \]
(iii) If $\zeta_K(s)L(s, \chi)$ does not vanish in the region $R(-1, \chi)$, then

$$p(\chi) \leq A_{\chi}^{\frac{3}{4\log(1+\frac{1}{2}\log \chi(1))}}.$$ 

(The region $R(\alpha, \chi)$ is defined in Theorem 4.3.)

**Proof:** (i),(ii), (iii) are the special cases $\alpha = 1$, $\alpha = 0$, $\alpha = -1$ of Theorem 4.3 respectively. Note that, as $\log d_K \leq \log A_{\chi} \chi(1)$, the assumption of nonexistence of the exceptional zero of $\zeta_K(s)$ is strong enough to satisfy the needs of all three cases.

**Remarks 4.7** (a) Here we compare Theorem 4.3 with Theorem 4.4. The case $\alpha = 1$ in Theorem 4.3 (i.e. part (i) of Corollary 4.6) is closely related to Theorem 4.4. Yet, neither of the two results implies the other: Theorem 4.4 does not imply the case $\alpha = 1$ in Theorem 4.3, because Theorem 4.4 has the restriction of irreducibility of $\chi$ (as well as $\chi(1) > 1$), while Theorem 4.3 is free in this respect. On the other hand the case $\alpha = 1$ in Theorem 4.3 does not imply Theorem 4.4 either, because the hypothetical zero-free region $R(1, \chi)$ is slightly taller than what we know from Proposition 3.1.

(b) Note that Corollary 4.5 covers the missing case of Abelian $\chi$ in Theorem 4.4.

(c) The condition in Theorem 4.3 that $\chi$ does not contain the trivial character is essential: e.g. for $\chi = 1_G$, there is NO $\chi$-nonresidue at all!
Chapter 5

Proof of the Main Theorems

In this chapter we prove Theorem 4.3 and Theorem 4.4. We will actually prove a slightly stronger (and more complicated!) version of Theorem 4.3. This stronger version, Theorem 5.5, will in particular enable us to prove Corollary 5.6, a result assuming the GRH.

5.1 The general set-up

In this section we set up the generalities needed for proving both Theorem 5.5, and Theorem 4.4.

First we introduce a kernel function (which is the same as the one used in [11], [14], [17]). Let $x, y$ be two parameters (to be chosen in terms of other data of the case under consideration later) with $1 < x < y$. The kernel function is

$$k(s) = k(s, x, y) = \left( \frac{y^{s-1} - x^{s-1}}{s - 1} \right)^2. \quad (5.1)$$
We define $\hat{k}$ by

$$\hat{k}(u) = \hat{k}(u, x, y) = \frac{1}{2\pi i} \int_{(2)} k(s) u^{-s} ds,$$

(5.2)

where $u > 0$, and $\int_{(2)}^{2+i\infty}$ means $\int_{2-i\infty}^{2+i\infty}$. We will show that

$$\hat{k}(u) = \begin{cases} 
0 & \text{if } 0 < u \leq x^2, \\
\frac{1}{u} \log \frac{u}{x^2} & \text{if } x^2 \leq u \leq xy, \\
\frac{1}{u} \log \frac{y^2}{u} & \text{if } xy \leq u \leq y^2, \\
0 & \text{if } y^2 \leq u.
\end{cases}$$

(5.3)

Since all the cases are similar, we just prove the case $x^2 \leq u \leq xy$. We do this by using the well-known discontinuous integral

$$\frac{1}{2\pi i} \int_{(c)} v^s ds = \begin{cases} 
0 & \text{if } 0 < v < 1, \\
\frac{1}{2} & \text{if } v = 1, \\
1 & \text{if } v > 1,
\end{cases}$$

(5.4)

where $c > 0$ (see [5, pp.104–105]). By (5.4) we have

$$\int_{(2)} (s-1)^2 u^{-s} ds = \int_{(1)} \frac{y^{2s}}{s^2} u^{-(s+1)} ds$$

$$= \left. -\left( \frac{y^2}{u} \right)^s \right|_{1-i\infty}^{1+i\infty} + \frac{1}{u} \left( \log \frac{y^2}{u} \right) \int_{(1)} \frac{y^2}{s} ds$$

$$= \frac{2\pi i}{u} \log \frac{y^2}{u}.$$

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Similarly we see that

$$\int_{(2)} \frac{x^{2(s-1)}}{(s-1)^2} u^{-s} ds = 0, \quad \int_{(2)} \frac{(xy)^{s-1}}{(s-1)^2} u^{-s} ds = \frac{2\pi i}{u} \log \frac{xy}{u}.$$ 

Therefore

$$\hat{k}(u) = \frac{1}{2\pi i} \int_{(2)} \left( \frac{y^{s-1} - x^{s-1}}{s-1} \right)^2 u^{-s} ds$$

$$= \frac{1}{u} \log \frac{y^2}{u} - \frac{2}{u} \log \frac{xy}{u}$$

$$= \frac{1}{u} \log \frac{u}{x^2}.$$ 

(5.3) is now established.

Note that for all \( u > 0 \)

$$0 \leq \hat{k}(u) \leq \frac{1}{u} \log \frac{y}{x}. \quad (5.5)$$

Next, let \( E/K \) be a Galois extension of number fields, \( G \) the Galois group, and \( \chi \) a character of \( G \). We set

$$I(\chi) = I(\chi, E/K) = \frac{1}{2\pi i} \int_{(2)} -\frac{L'}{L}(s, \chi) \hat{k}(s) ds. \quad (5.6)$$

From the Euler product (2.10) of \( L(s, \chi) \) we obtain

$$I(\chi) = \sum_{p \in \mathcal{P}_K} \sum_{m=1}^{\infty} \chi(\sigma_p^m) (\log Np)\hat{k}(Np^m). \quad (5.7)$$

Finally, we describe the general outline of the type of arguments used in proof of results giving upper bounds on least primes of specified types.

First observe the fact that the double sum in the right side of (5.7) is actually a finite sum: the only pairs \( (p, m) \) which have nonzero contribution in (5.7) are those
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with

\[ x^2 < Np^m < y^2, \]

and for any rational prime power \( p^r \), there are at most \( n_{\mathcal{K}} \) primes \( p \in \Sigma_{\mathcal{K}} \) for which there exists an \( m \) with \( Np^m = p^r \). On the other hand, the integral (5.6) defining \( I(\chi) \) can be written in terms of the zeros (and the possible pole at \( s = 1 \)) of \( L(s, \chi) \) by Cauchy's theorem. Of course the sharpness of the information we get from this is directly proportional to the depth of our knowledge about the location and density of the zeros. The essence of all arguments is that, by choosing the parameters \( x, y \) properly, the denial of the theorem would contradict (via (5.7)) the information about \( I(\chi) \) gained through (5.6) and zero-analysis. (To reach this contradiction we may need to work with certain \( I(\psi) \) with \( \psi \neq \chi \) as well.)

5.2 The lemmas

The present section is devoted to a few preparatory lemmas. The first two estimate the contribution to the sum in the right side of (5.7) made by primes \( p \in \Sigma_{\mathcal{K}} \) which are \( \chi \)-ramified, and by pairs \((p, m)\) for which \( Np^m \) is not a rational prime, respectively. The last lemma relates \( I(\chi) \) to the zeros of \( L(s, \chi) \).

Lemma 5.1 We have

\[
\left| \sum_{\substack{p \in \Sigma_{\mathcal{K}} \atop \chi \text{-ramified}}} \sum_{m=1}^{\infty} \chi(\sigma_p^m) (\log Np) \hat{\kappa}(Np^m) \right| \ll \chi(1)(\log A_{\chi}) \frac{1}{x^2} \log \frac{y}{x}.
\]
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**Proof:** We recall from the section 1.3, that \( p \in \Sigma_K \) is \( \chi \)-ramified iff \( p \) divides the Artin conductor \( f_\chi \). Using (5.5), we have the left side of the above

\[
\leq \chi(1) \left( \log \frac{y}{x} \right) \sum_{p \mid f_\chi} (\log N_p) \sum_{m \geq 1} N_p^{-m}
\]

\[
\leq 2\chi(1) \frac{1}{x^2} \left( \log \frac{y}{x} \right) \sum_{p \mid f_\chi} (\log N_p)
\]

\[
\leq 2\chi(1) \frac{1}{x^2} \left( \log \frac{y}{x} \right) (\log N f_\chi)
\]

\[
\leq 2\chi(1) \frac{1}{x^2} \left( \log \frac{y}{x} \right) (\log A_\chi)
\].

□

**Lemma 5.2** We have

\[
\sum_{\substack{\sigma_p^m \in \Sigma_K \\ m \geq 1 \\ N_p^m \text{ not a rational prime}}} \chi(\sigma_p^m)(\log N_p) \hat{\zeta}(N_p^m) \leq n_K \chi(1) \frac{1}{x} \frac{\log y}{\log x} \frac{\log y}{x}
\]

**Proof:** There are at most \( n_K \) primes \( p \in \Sigma_K \) dividing any rational prime \( p \). Using this fact, and (5.5), we have the left side of the above

\[
\ll n_K \chi(1) (\log y) \left( \log \frac{y}{x} \right) \sum_{p \in \mathbb{P}} p^{-r}
\]

\[
\ll n_K \chi(1) (\log y) \left( \log \frac{y}{x} \right) \frac{1}{x \log x}
\].

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To justify the last inequality, we are going to show that

\[ \sum_{p \in \mathbb{P}\, \\
\text{s.t. } r \geq 2, \\
p > x^2} p^{-r} \ll \frac{1}{x \log x} \]  \tag{5.8} \]

We have

\[ \sum_{p \in \mathbb{P}\, \\
\text{s.t. } r \geq 2, \\
p > x^2} p^{-r} = \sum_{p < x, r \geq 2} p^{-r} + \sum_{p \geq x, r \geq 2} p^{-r} , \]

and the two sums on the right are estimated as follows:

\[ \sum_{p < x, r \geq 2} p^{-r} \ll \left( \frac{x}{\log x} \right) x^{-2} = \frac{1}{x \log x} , \]

\[ \sum_{p \geq x, r \geq 2} p^{-r} \ll \sum_{p \geq x} p^{-2} = \int_{x}^{\infty} u^{-2} d \pi(u) \ll \frac{1}{x \log x} . \]

This proves (5.8), which completes the proof of Lemma 5.2.

\begin{proof}
This follows immediately from Lemma 5.1 and Lemma 5.2.
\end{proof}

\begin{lemma}
Let \( \sum^* \) denote sum over those \( p \in \Sigma_K \) which are \( \chi \)-unramified and of degree one over \( \mathbb{Q} \). We have

\[ \left| I(\chi) - \sum^*_{p \in \Sigma_K} \chi(\sigma_p)(\log N_p) \hat{k}(N_p) \right| \]

\[ \ll \chi(1)(\log A_\chi)^2 \log \frac{y}{x} + n_K \chi(1) \frac{1}{x \log x} \log \frac{y}{x} . \]

\begin{proof}
This follows immediately from Lemma 5.1 and Lemma 5.2.
\end{proof}

\end{lemma}
Lemma 5.4 Assuming Artin's conjecture for \(L(s, \chi)\), we have

\[ I(\chi) = (\chi|1_G)k(1) - \sum_{\rho} k(\rho), \]

where \(\rho\) runs through all zeros of \(L(s, \chi)\).

Proof: First note that \(k(s)\) is an entire function: the singularity at \(s = 1\) is removable, by defining

\[ k(1) = \lim_{s \to 1} k(s) = \lim_{s \to 1} \left( \frac{y^{s-1} - x^{s-1}}{s-1} \right)^2 = \lim_{s \to 1} x^{2(s-1)} \left( \frac{(y/x)^{s-1} - 1}{s-1} \right)^2. \]

Let \(B(T, N)\) be the (positively oriented) rectangle with vertices at \(2 + iT, -N + iT, -N - iT, 2 - iT\). Here \(T > 2\), not equal to the absolute value of the imaginary part of any zero of \(L(s, \chi)\), and \(N = \frac{1}{2} + a\) positive integer. We set

\[ I(\chi, T, N) = \frac{1}{2\pi i} \int_{B(T, N)} \frac{L'}{L}(s, \chi) k(s) ds. \]  

By Cauchy's theorem

\[ I(\chi, T, N) = (\chi|1_G)k(1) - \sum_{\rho \text{ inside } B(T, N)} k(\rho). \]

Therefore the lemma is proved if we show that

\[ I(\chi, T, N) \xrightarrow{T, N \to \infty} I(\chi). \]

This is equivalent to showing that the contribution made by the two horizontal sides and the left side of \(B(T, N)\) to the integral in (5.10) approaches zero as \(T, N \to \infty\).
First we need a good bound on $\left| \frac{L'}{L} \right|$. We have the bound \eqref{2.8} on $\left| \frac{L'}{L} \right|$ for \( \sigma > 1 \).

The functional equation \eqref{1.2} (or \eqref{2.4}) translates that bound to a bound on $\left| \frac{L'}{L} \right|$ for \( \sigma < 0 \) as follows.

From \eqref{2.2}, \eqref{2.3}, \eqref{2.4} we get

\[ -\frac{L'}{L}(s, \chi) = \frac{L'}{L}(1 - s, \bar{\chi}) + \log A_\chi + \frac{\gamma'}{\gamma}(1 - s, \bar{\chi}) + \frac{\gamma'}{\gamma}(s, \chi). \] (5.12)

From \eqref{2.8}, \eqref{2.14}, \eqref{5.12} we obtain

\[ \left| -\frac{L'}{L}(s, \chi) \right| \ll \log A_\chi + n_K\chi(1)\log(|s| + 2), \] (5.13)

valid for \( \sigma \leq -\frac{1}{4} \) with \( |s + m| \geq \frac{1}{4} \) for all integers \( m \geq 0 \).

From \eqref{5.13} it easily follows that as \( T, N \) approach infinity (with \( T \neq \) the absolute value of the imaginary part of any zero of \( L(s, \chi) \), and \( N = \frac{1}{2} + \) a positive integer), the integrals

\[ \int_{-N+iT}^{-N-iT} -\frac{L'}{L}(s, \chi)k(s)ds, \quad \int_{-N-iT}^{-\frac{1}{4}-iT} -\frac{L'}{L}(s, \chi)k(s)ds, \]

\[ \int_{-\frac{1}{4}+iT}^{-N+iT} -\frac{L'}{L}(s, \chi)k(s)ds \]

all go to zero. It remains to show that the integrals

\[ \int_{2+iT}^{-\frac{1}{4}+iT} -\frac{L'}{L}(s, \chi)k(s)ds, \quad \int_{-\frac{1}{4}-iT}^{2-iT} -\frac{L'}{L}(s, \chi)k(s)ds \]
also go to zero when $T \to \infty$. This easily follows from the fact that, by (2.20),

$$\left| \frac{L'}{L}(\sigma + iT) - \sum_{\rho, |\gamma - \sigma| \leq 1} \frac{1}{(\sigma + iT) - \rho} \right| \ll \log A_\chi + n_K \chi(1) \log(T + 2),$$

an easy estimation of $\int_{-\frac{T}{2}}^{T} \frac{k(\sigma + iT)}{(\sigma + iT) - \rho} d\sigma$, for $\rho$ with $|\gamma - T| \leq 1$, and recalling from Lemma 2.1 that the number of such $\rho$ is

$$n_\chi(T) \ll \log A_\chi + n_K \chi(1) \log(T + 2),$$

(see [12, Lemma 6.3]). This completes the proof of Lemma 5.4.

\[ \square \]

5.3 Proof of Theorem 4.3

In this section, we state and prove Theorem 5.5, which is slightly more general than Theorem 4.3. The parameter $\delta$ in Theorem 5.5 below will enable us to prove Corollary 5.6, which gives a bound on the least character nonresidue $p(\chi)$ assuming the GRH.

Theorem 5.5 generalizes [11, Theorem 1.2] both contextwise (from Abelian characters to general characters) and contentwise (by introducing the parameter $\alpha$).

Here we outline the proof. The essence of the argument is the comparison of $I(\chi)$ with $I(1_G)$. We will show that under the assumptions of the theorem, $\Re I(\chi)$ is small, while $I(1_G)$ is (positive and) large, due to the pole at $s = 1$ of $\zeta_K(s)$. If the conclusion of the theorem were false, then $I(\chi)$ and $\chi(1)I(1_G)$ would be closer to each other than what they actually are. This contradiction will then prove the result.

**Theorem 5.5** There is an absolute, effective constant $c_\delta > 0$ such that the following holds:
Let \(-1 \leq \alpha \leq 1\) be a parameter, and set

\[
\hat{\alpha} = \begin{cases} 
\frac{1}{2} \alpha & \text{if } -1 \leq \alpha \leq 0, \\
\alpha & \text{if } 0 \leq \alpha \leq 1.
\end{cases}
\]

Let \(\chi\) be a character of \(G\) not containing the trivial one, and assume Artin's conjecture for \(L(s, \chi)\). Finally, let \(\delta\) be another parameter related to \(\alpha\) by

\[
\frac{c_1}{\chi(1)^{\alpha} \log A_x} \leq \delta \leq \frac{1}{2},
\]

where \(c_1\) is the absolute constant given by Proposition 3.1.

If \(\zeta_K(s)L(s, \chi)\) does not vanish in the region \(R_1(\delta, \chi)\) defined by

\[
1 - \delta < \sigma < 1, \quad 0 \neq |t| \leq \delta + \delta^2 \chi(1)^{-1}(\log A_x)(\log \log A_x),
\]

and if \(\zeta_K(\sigma) \neq 0\) for \(1 - \delta < \sigma < 1\), then

\[
p(\chi) \leq (c_0 \chi(1)^{\hat{\alpha}} (\log A_x) \delta)^{\frac{1}{2}}.
\]

Before we prove Theorem 5.5, we present the

**Proof of Theorem 4.3:** In Theorem 5.5 we take

\[
\delta = \frac{c_1}{\chi(1)^{\alpha} \log A_x}. \tag{5.14}
\]

It is easily seen that

\[
R_1 \left( \frac{c_1}{\chi(1)^{\alpha} \log A_x}, \chi \right) = R(\alpha, \chi),
\]
and with the choice (5.14) for \( \delta \)

\[
(c_9 \chi(1)^{\delta} (\log A_\chi)^{\delta})^{\frac{1}{\delta}} = A_\chi^{\frac{1}{\delta}} \chi(1)^{\delta (\log (c_1 c_9)+(d-a) \log \chi(1))} \leq A_\chi c_8^{\chi(1)^{\delta (d-a) \log \chi(1))}},
\]

where \( c_8 = \frac{1+|\log (c_1 c_9)|}{c_1} \). This proves Theorem 4.3. \( \square \)

PROOF OF THEOREM 5.5:

Since by assumption \( (\chi|1_G) = 0 \), Lemma 5.4 gives

\[
I(\chi) = - \sum_{\rho} k(\rho), \tag{5.15}
\]

where \( \rho \) runs over all zeros of \( L(s, \chi) \). For any real zero \( \rho \), \(-k(\rho)\) is negative, and therefore

\[
\text{Re} I(\chi) \leq - \sum_{\rho \in \mathbb{R}} \text{Re} k(\rho) \leq \sum_{\rho \in \mathbb{R}} |k(\rho)|. \tag{5.16}
\]

Our first goal is to bound \( \text{Re} I(\chi) \), and by (5.16) it is sufficient to find a bound for

\[
\sum_{\rho \in \mathbb{R}} |k(\rho)|, \text{ where the sum is over nonreal zeros, and in particular trivial zeros as well as the exceptional zero (if it is real) have no contribution to it.}
\]

For any zero \( \rho = \beta + i\gamma \) we have

\[
|k(\rho)| = \left| \frac{y^{\rho-1} - x^{\rho-1}}{\rho - 1} \right|^2 \leq \frac{(y^{\beta-1} + x^{\beta-1})^2}{|\rho - 1|^2} \leq \frac{x^{-(1-\beta)} \left( \left( \frac{y}{x} \right)^{-(1-\beta)} + 1 \right)^2}{|\rho - 1|^2} \leq \frac{4x^{-(1-\beta)}}{|\rho - 1|^2}. \tag{5.17}
\]
We divide the strip $1 - 2\delta \leq \sigma \leq 1 - \delta$ into squares of side length $\delta$ (we owe this idea, which is better than integration, to [14, p.166]). By Lemma 2.1(ii), and (5.17), we have the estimate

$$
\sum_{\rho \in \mathbb{R}, 1 - 2\delta \leq \beta \leq 1 - \delta} |k(\rho)| \ll x^{-2\delta} \sum_{j = -\infty}^{\infty} \frac{\chi(1) + 2\delta (\log A_x + n_K \chi(1) \log(\lfloor j \delta + 2 \rfloor))}{\delta^2 (j^2 + 1)} \\
\ll x^{-2\delta} \delta^{-2} \chi(1) + x^{-2\delta} \delta^{-1} \log A_x . \quad (5.18)
$$

Replacing $\delta$ in (5.18) by $2^m \delta$, $m \geq 0$, we obtain

$$
\sum_{\rho \in \mathbb{R}, 1 - 2^{m+1}\delta \leq \beta \leq 1 - 2m\delta} |k(\rho)| \ll x^{-2^{m+1}\delta} 2^{-2m\delta - 2} \chi(1) + x^{-2^{m+1}\delta} 2^{-m\delta - 1} \log A_x . \quad (5.19)
$$

We sum (5.19) over $m \geq 0$ to get

$$
\sum_{\beta \leq 1 - \delta} |k(\rho)| \ll x^{-2\delta} \delta^{-2} \chi(1) + x^{-2\delta} \delta^{-1} \log A_x . \quad (5.20)
$$

We now divide the strip $1 - \delta < \sigma < 1$ into squares of side length $\delta$. The contribution of the zeros in the $j$th square, where $j \neq 0$, again by Lemma 2.1(ii) and (5.17), is

$$
\sum_{\rho \in \mathbb{R}, 1 - \delta \leq \beta < 1} |k(\rho)| \ll \frac{\chi(1) + \delta (\log A_x + n_K \chi(1) \log(\lfloor j \delta + 2 \rfloor))}{\delta^2 j^2} . \quad (5.21)
$$

Now, by the assumption of the theorem, we have $L(s, \chi) \neq 0$ for

$$
1 - \delta < \sigma < 1 , \quad 0 \neq |\epsilon| \leq J \delta ,
$$

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where

\[ J = 1 + \left[ \delta \chi(1)^{-1}(\log A_x)(\log \log A_x) \right]. \]  

(5.22)

Therefore, (5.21) implies

\[ \sum_{\rho \in \mathbb{R} \setminus 1-\delta < \beta < 1} |k(\rho)| \ll \sum_{|j| \geq J} \frac{\chi(1) + \delta(\log A_x + n_K \chi(1) \log(|j|\delta + 2))}{\delta^2 j^2} \ll J^{-1} \delta^{-2} \chi(1) + J^{-1} \delta^{-1} \log A_x \]

\[ + J^{-1} \delta^{-1} \log(J\delta + 2)n_K \chi(1). \]  

(5.23)

Combining (5.16), (5.20), and (5.23) we finally get

\[ \text{Re } I(\chi) \ll x^{-2\delta} \delta^{-2} \chi(1) + x^{-2\delta} \delta^{-1} \log A_x \]

\[ + J^{-1} \delta^{-2} \chi(1) + J^{-1} \delta^{-1} \log A_x + J^{-1} \delta^{-1} \log(J\delta + 2)n_K \chi(1). \]  

(5.24)

We have therefore shown that \( \text{Re } I(\chi) \) is "small". The next step of the proof is to obtain an estimate for \( I(1_G) \). This time Lemma 5.4 gives

\[ I(1_G) = k(1) - \sum_{\rho_K} k(\rho_K), \]  

(5.25)

where \( \rho_K \) runs over all zeros of \( \zeta_K(s) \). The contribution of the real zeros (both trivial and nontrivial) is estimated as follows. First note that for any real zero \( \rho_K = \beta_K \), by the assumption of the theorem, \( \beta_K \leq 1 - \delta \). Hence (5.17) and (5.20) (whose derivation
did not use the fact that the sum was over nonreal zeros) imply that

\[
\sum_{\rho_K \in \mathbb{R}} |k(\rho_K)| = \sum_{\rho_K \text{ trivial}} |k(\rho_K)| + \sum_{0 < \beta_K \leq 1 - \delta} |k(\rho_K)| \\
\ll n_K x^{-2} + x^{-2\delta - 2} + x^{-2\delta - 1} \log d_K .
\] (5.26)

The contribution of the nonreal zeros is estimated by (5.20) and (5.23), where \( J \) is the same as the previous, and since \( \chi(1) \log d_K \leq \log A_\chi \), from (5.25) and (5.26) we obtain

\[
\chi(1) \left| I(1_G) - \left( \log \frac{y}{x} \right)^2 \right| \ll \text{RHS of (5.24)} .
\] (5.27)

Now, we assume that

\[
p(\chi) > y^2 .
\] (5.28)

This means that \( \chi(\sigma_p) = \chi(1) \) for all \( \chi \)-unramified primes \( p \in \Sigma_K \) of degree one over \( \mathbb{Q} \) with \( N_p \leq y^2 \). Using the notation " \( \sum^* \) " of Lemma 5.3, we therefore have

\[
\sum_{p \in \Sigma_K}^* \chi(\sigma_p) (\log N_p) \hat{k}(N_p) = \chi(1) \sum_{p \in \Sigma_K}^* (\log N_p) \hat{k}(N_p) ,
\]

and hence Lemma 5.3 implies

\[
|I(\chi) - \chi(1) I(1_G)| \ll \chi(1)(\log A_\chi) \frac{1}{x^2} \log \frac{y}{x} + n_K \chi(1) \frac{1}{x} \log \frac{y}{x} ,
\] (5.29)

and since \( I(1_G) \in \mathbb{R} \),

\[
|\text{Re} I(\chi) - \chi(1) I(1_G)| \ll \text{RHS of (5.29)} .
\] (5.30)
Now, we choose $x$ and $y$ to be

$$x^2 = \left( \frac{C}{c_1} \delta \chi(1) \log A_x \right)^{\frac{1}{3}}, \quad y = xC^{\frac{1}{3}}, \quad (5.31)$$

where $C$ is a sufficiently large (yet absolute and effective) constant.

Let $c_0$ be the sum of the constants implied by $\ll$ in (5.24), (5.27), and (5.30). We are going to show that

$$\chi(1) \left( \log \frac{y}{x} \right)^2 > c_0 \sum_{i=1}^{7} T_i, \quad (5.32)$$

where the seven terms $T_i$, coming from the right sides of (5.24) and (5.30), will be given in a moment. But (5.24), (5.27), and (5.30) cooperatively deny (5.32). This contradiction will prove the theorem.

The terms $T_i$ are

$T_1 = x^{-2 \delta - 2} \chi(1),

T_2 = x^{-2 \delta - 1} \log A_x,

T_3 = J^{-1} \delta^{-2} \chi(1),

T_4 = J^{-1} \delta^{-1} \log A_x,

T_5 = J^{-1} \delta^{-1} \log(J \delta + 2)n_K \chi(1),

T_6 = \chi(1)(\log A_x)^{-1} \log \frac{y}{x},

T_7 = n_K \chi(1)^{1} \frac{\log y}{x \log x} \log \frac{y}{x}.$

We recall that

$$\frac{c_1}{\chi(1)^{\alpha} \log A_x} \leq \delta \leq \frac{1}{2}, \quad (5.33)$$
and
\[
\hat{\alpha} = \begin{cases} 
\frac{1}{2}\alpha & \text{if } -1 \leq \alpha \leq 0 \\
\alpha & \text{if } 0 \leq \alpha \leq 1,
\end{cases}
\]
hence
\[-1 \leq \alpha \leq \hat{\alpha} \leq 1.
\]
The function $\delta \mapsto (\lambda \delta)^{\frac{1}{2}}$ attains its maximum at $\delta = \frac{\pi}{X}$. This fact, with $\lambda = \frac{C_1}{c_1} \chi(1)^{\frac{1}{2}} \log A_X$, implies that on the interval given by (5.33), $x$ as given by (5.31) is a decreasing function of $\delta$, because
\[
e^\frac{e}{\lambda} = \frac{e c_1}{C \chi(1)^{\frac{1}{2}} \log A_X} < \frac{c_1}{\chi(1)^{\frac{1}{2}} \log A_X}.
\]
Therefore
\[
x \geq \frac{C}{2c_1} \chi(1)^{\frac{1}{2}} \log A_X \quad \text{(5.34)}
\]
Also, note that on the interval (5.33), $x^2 \geq C^{\frac{1}{2}}$, and so $y \leq x^3$. Hence
\[
\frac{\log y}{\log x} \ll 1 \quad \text{(5.35)}
\]
We are now ready to prove (5.32). First note that the left side of (5.32) is
\[
\chi(1) \left( \log \frac{y}{x} \right)^2 = (\log C)^2 \delta^{-2} \chi(1).
\]
The terms $T_i$, $1 \leq i \leq 7$, on the right side of (5.32) are estimated as follows.

\[ T_1 = x^{-2\delta^{-2}} \chi(1) = \left( \frac{C}{c_1} \delta \chi(1)^{\hat{a}} \log A_\chi \right)^{-1} \delta^{-2} \chi(1) \leq C^{-1} \delta^{-2} \chi(1) , \]

\[ T_2 = x^{-2\delta^{-1}} \log A_\chi = \frac{c_1}{C} \delta^{-2} \chi(1)^{-\hat{a}} \leq \frac{c_1}{C} \delta^{-2} \chi(1) . \]

By (5.22), $J \geq 1$, hence

\[ T_3 = J^{-1} \delta^{-2} \chi(1) \leq \delta^{-2} \chi(1) , \]

\[ T_4 = J^{-1} \delta^{-1} \log A_\chi \leq \delta^{-2} \chi(1) (\log \log A_\chi)^{-1} \ll \delta^{-2} \chi(1) . \]

For $T_5$, note that $J \ll (\log A_\chi)(\log \log A_\chi)$ and so

\[ \log(J \delta + 2) \ll \log J \ll \log \log A_\chi . \]

Therefore

\[ T_5 = J^{-1} \delta^{-1} \log(J \delta + 2)n_K \chi(1) \ll n_K \chi(1)^2 \delta^{-2} (\log A_\chi)^{-1} \ll \delta^{-2} \chi(1) . \]
For $T_6$, we use (5.34), and then (5.33) to get

$$T_6 = \chi(1)(\log A_x) \frac{1}{x^2} \log \frac{y}{x} \leq \frac{4c_1^2}{C^2}(\log C)\delta^{-1} \chi(1)^{1-2\hat{a}}(\log A_x)^{-1}$$

$$\leq \frac{4c_1}{C^2}(\log C)\chi(1)^{1+\hat{a}-2\hat{a}}$$

$$\leq \frac{4c_1}{C^2}(\log C)\chi(1)$$

$$\leq \frac{c_1}{C^2}(\log C)\delta^{-2}\chi(1).$$

Finally for $T_7$, using (5.34) and (5.35) we have

$$T_7 = n_K\chi(1)\frac{1}{x} \log \frac{y}{x} \leq \frac{2c_1}{C}(\log C)\delta^{-1} \chi(1)^{1-\hat{a}}n_K(\log A_x)^{-1}$$

$$\leq \frac{2c_1}{C}(\log C)\delta^{-1} \chi(1)^{-\hat{a}}$$

$$\leq \frac{2c_1}{C}(\log C)\delta^{-1} \chi(1)$$

$$\leq \frac{c_1}{C}(\log C)\delta^{-2}\chi(1).$$

Note that the constants implied by $\ll$ used in the above estimations for $T_i$, are all absolute and in particular independent of $C$. From these estimations for $T_i$, we see that, with sufficiently large $C$, (5.32) is proved. The contradiction mentioned above shows that the assumption (5.28), with the choice (5.31), is false. Therefore

$$p(\chi) \leq y^2 = \left(\frac{C^3}{c_1^3}\chi(1)^{\hat{a}}(\log A_x)\delta\right)^{\frac{1}{\hat{a}}}.\n$$

Setting $c_9 = \frac{C^3}{c_1^3}$, we have completed the proof of Theorem 5.5.

**Corollary 5.6** There is an absolute, effective constant $c_{10} > 0$ such that the following holds:
5.4 Proof of Theorem 4.4

Let $\chi$ be a character of $G$ not containing the trivial one. Assuming both Artin’s conjecture and the GRH for $\zeta_K(s)L(s, \chi)$, we have

$$p(\chi) \leq c_{10} \frac{(\log A_\chi)^2}{\chi(1)}.$$ 

**Proof:** Theorem 5.5 is applicable with $\alpha = -1, \delta = \frac{1}{2}$. Setting $c_{10} = \frac{1}{4} c_9^2$, we get the corollary. \qed

5.4 Proof of Theorem 4.4

In this section we prove Theorem 4.4. The method of the proof will be similar to that of Theorem 5.5. This time we are not assuming a zero-free region (except we assume that the exceptional zero of $\zeta_K(s)$ does not exist), but we have the restriction that $\chi$ is assumed to be irreducible of degree $> 1$.

**Proof of Theorem 4.4:** Set

$$\delta_\chi = \frac{c_1}{\chi(1) \log A_\chi}, \quad \delta_K = \frac{c_1}{\log d_K}, \tag{5.36}$$

where, as usual, $c_1$ is the absolute constant given by Proposition 3.1. Proceeding as in the proof of Theorem 5.5, we get to (5.24), with $\delta = \delta_\chi, J = 1$, except we have the contribution of (the possible) exceptional zero $\rho_0$ of $L(s, \chi)$ (which is not known to be real!) as well:

$$\text{Re } I(\chi) \ll x^{-2\delta_\chi} \delta_\chi^{-2} \chi(1) + x^{-2\delta_\chi} \delta_\chi^{-1} \log A_\chi$$

$$+ \delta_\chi^{-2} \chi(1) + \delta_\chi^{-1} \log A_\chi + n_K \chi(1) \delta_\chi^{-1} + |k(\rho_0)|$$

$$\ll \delta_\chi^{-2} \chi(1) + \delta_\chi^{-1} \log A_\chi + |k(\rho_0)|$$

$$\ll \chi(1)^3 (\log A_\chi)^2 + |k(\rho_0)|. \tag{5.37}$$

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Now we estimate $|k(\rho_0)|$. We have

$$|k(\rho_0)| = \left| \frac{y^{\rho_0-1} - x^{\rho_0-1}}{\rho_0 - 1} \right|^2 = x^{-2(1-\rho_0)} \left| \frac{(\frac{y}{x})^{\rho_0-1} - 1}{\rho_0 - 1} \right|^2 \leq x^{-2(1-\rho_0)} \left( \log \frac{y}{x} \right)^2. \tag{5.38}$$

(The inequality in (5.38) comes from the fact that

$$\left| \frac{(\frac{y}{x})^{\rho_0-1} - 1}{\rho_0 - 1} \right| \leq \max_{\beta = \Re \xi, \xi = \lambda + (1-\lambda) \rho_0, 0 \leq \lambda \leq 1} \left( \frac{y}{x} \right)^{\beta-1} \log \frac{y}{x} \leq \log \frac{y}{x}. \right.$$

From (5.37), (5.38) we obtain

$$\Re I(\chi) \leq c_{21} \chi(1)^3 (\log A_{\chi})^2 + \left( \log \frac{y}{x} \right)^2. \tag{5.39}$$

(Note that the "actual" coefficient of the term $|k(\rho_0)|$ in (5.37) is one, and $c_{21}$ is an absolute effective constant.)

Next, we estimate $I(1_G)$. As we are assuming that the exceptional zero of $\zeta_K(s)$ does not exist, we get (5.27), with $\delta = \delta_K$, $J = 1$:

$$\left| I(1_G) - \left( \log \frac{y}{x} \right)^2 \right| \ll x^{-2\delta \delta_K^2} + x^{-2\delta \delta_K^{-1}} \log d_K$$

$$+ \delta_K^{-2} + \delta_K^{-1} \log d_K + n_K \delta_K^{-1} + n_K x^{-2}, \tag{5.40}$$

where the last term is estimating the contribution of the trivial zeros of $\zeta_K(s)$. From (5.36), (5.40) we get

$$\left| I(1_G) - \left( \log \frac{y}{x} \right)^2 \right| \ll \delta_K^{-2} + \delta_K^{-1} \log d_K$$

$$\ll (\log d_K)^2$$

$$\ll \chi(1)^{-2} (\log A_{\chi})^2,$$
and therefore

$$\left| \chi(1)I(1_\sigma) - \chi(1) \left( \log \frac{y}{x} \right)^2 \right| \leq c_{22} \chi(1)^{-1} (\log A_x)^2, \quad (5.41)$$

with $c_{22}$ absolute effective.

Now, we assume that

$$p(\chi) > y^2. \quad (5.42)$$

Again this means that $\chi(\sigma_p) = \chi(1)$ for all $\chi$-unramified $p \in \Sigma_K$ of degree one over $\mathbb{Q}$ with $N_p \leq y^2$, and therefore

$$\sum_{p \in \Sigma_K} \chi(\sigma_p)(\log N_p)\hat{k}(N_p) = \chi(1) \sum_{p \in \Sigma_K} (\log N_p)\hat{k}(N_p),$$

and Lemma 5.3 gives

$$|I(\chi) - \chi(1)I(1_\sigma)| \ll \chi(1)(\log A_x) \frac{1}{x^2} \log \frac{y}{x} + n_K \chi(1) \frac{1}{x} \log \frac{y}{x}, \quad (5.43)$$

and since $I(1_\sigma) \in \mathbb{R}$,

$$|\text{Re} I(\chi) - \chi(1)I(1_\sigma)| \leq c_{23} \text{ (RHS of (5.43))}, \quad (5.44)$$

with $c_{23}$ absolute effective.

Now, we choose $x$ and $y$ to be given by

$$\log x = C\chi(1) \log A_x, \quad y = x^2, \quad (5.45)$$
where $C$ is a sufficiently large (yet absolute and effective) constant.

We set $c_{24} = c_{21} + c_{22} + c_{23}$. We are going to show that

\[
\chi(1) \left( \log \frac{y}{x} \right)^2 > c_{24} \left( \chi(1)^3(\log A_x)^2 + \chi(1)(\log A_x) \frac{1}{x^2} \log \frac{y}{x} \right.
\]
\[
\left. + nK \chi(1) \frac{1}{x \log x} \log \frac{y}{x} \right) + \left( \log \frac{y}{x} \right)^2. \tag{5.46}
\]

But (5.39), (5.41), and (5.44) deny (5.46). The contradiction proves the theorem.

Since we are assuming $\chi(1) > 1$, we have $\chi(1) - 1 \geq \frac{1}{2} \chi(1)$. Therefore (5.46) is proved if we show

\[
\chi(1) \left( \log \frac{y}{x} \right)^2 > 2c_{24} \left( \chi(1)^3(\log A_x)^2 + \chi(1)(\log A_x) \frac{1}{x^2} \log \frac{y}{x} \right.
\]
\[
\left. + nK \chi(1) \frac{1}{x \log x} \log \frac{y}{x} \right). \tag{5.47}
\]

Note that the left side of (5.47) is

\[
\chi(1) \left( \log \frac{y}{x} \right)^2 = C^2 \chi(1)^3(\log A_x)^2.
\]

The first term on the right side of (5.47) is therefore dominated by the left side. For the other two terms on the right side we have

\[
\chi(1)(\log A_x) \frac{1}{x^2} \log \frac{y}{x} \leq C \chi(1)^2(\log A_x)^2,
\]
\[
nK \chi(1) \frac{1}{x \log x} \log \frac{y}{x} \leq 2C \chi(1)(\log A_x)^2. \tag{5.48}
\]

This proves (5.47), and therefore (5.46). The above mentioned contradiction shows that the assumption (5.42), with the choice (5.45), is false. Therefore

\[
p(\chi) \leq y^2 = x^4 = A_x^{4\chi(1)}.
\]
Setting $c_8 = 4C$, we have established Theorem 4.4.

Remarks 5.7 (a) The restriction $\chi(1) > 1$ in Theorem 4.4 can be removed if we add the assumption that the exceptional zero of $L(s, \chi)$ does not exist either. This is because the only time we used the assumption $\chi(1) > 1$ was when we needed to dominate the term $\left( \log \frac{y}{x} \right)^2$ on the right side of (5.46), and that term is contributed by the (possible) exceptional zero of $L(s, \chi)$.

(b) The assumption of irreducibility of $\chi$ was used in the proof of Theorem 4.4, just to ensure the assumption of Proposition 3.1.
References


REFERENCES


