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Resonantly Forced Inhomogeneous Reaction-Diffusion Systems
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Abstract

The phenomenology of spatiotemporal patterns in oscillatory reaction-diffusion systems subject to periodic external forcing with spatially random amplitude is explored and characterized using numerical simulations. Resonant forcing may cause phase locking of the local reaction kinetics, leading to multiple stable oscillatory states of differing phase. A spatially inhomogeneous form of the forced complex Ginzburg-Landau equation is used to study 2:1 and 3:1 resonantly forced systems with quenched disorder. Front roughening and spontaneous nucleation of target patterns are discussed. In 2:1 resonantly forced systems, local front reversals govern the pattern dynamics. Time-varying disorder is studied in the periodically forced FitzHugh-Nagumo system at the 3:1 resonance; it may lead to the breaking of the symmetry between phase locked states. Consequently, inequivalent fronts with different velocities exist, as do stable structures derived from multiple fronts bound by virtue of their different individual velocities. Spiral wave dynamics in these systems is studied.
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1. Introduction

Patterns in reaction-diffusion systems where the kinetics are statiotemporally modulated can display a variety of phenomena that are not found in homogeneous systems. Many reaction-diffusion processes of practical interest take place in inhomogeneous media or may be coupled to external processes that affect the kinetics in a non-uniform manner. A convenient experimental system for studying the effect of spatiotemporal modulations on pattern dynamics in spatially distributed systems is the ruthenium-catalyzed Belousov-Zhabotinsky (BZ) reaction. This reaction is light-sensitive and thus the kinetics may be modulated by projecting a pattern of illumination of varying intensity onto the reaction medium.

Recent studies have made use of the light-sensitive BZ system to investigate wavefront propagation in systems with spatially disordered excitability. Kádár et al. studied stochastic resonance in a system with a periodically regenerated noise pattern. Sendiña-Nadal et al. studied percolation and roughening of wavefronts in systems with quenched spatially disordered excitability. The dynamics of reaction-diffusion waves in inhomogeneous excitable media is thought to be relevant to cardiac fibrillation since electrical waves may be disrupted by irregularities in the heart muscle medium. Noise is thought to play a role in initiation and propagation of waves in neural tissue. Earlier investigations into spatially in-
homogeneous excitable systems include a study of a BZ medium containing catalyst-coated resin beads which served as nucleation sites for wavefronts,\textsuperscript{8} numerical simulations of a spatially-distributed network of coupled excitable elements which exhibited spontaneous wave initiation, stochastic resonance and fragmentation of wavefronts,\textsuperscript{9,10} and an excitable cellular automaton in which the refractory times of the elements were assigned randomly.\textsuperscript{11} The effect of stochastic spatial inhomogeneities on other types of reaction-diffusion systems has been much less studied.

Periodically forced reaction-diffusion systems have also been investigated.\textsuperscript{12,13} Petrov \textit{et al.} subjected an oscillatory version of the light-sensitive BZ reaction to periodic spatially uniform illumination.\textsuperscript{14,15} As the ratio of the forcing frequency to the natural frequency neared various resonances patterns were observed. In subsequent numerical studies the observed transition between labyrinthine and non-labyrinthine two-phased patterns at the 2:1 resonance was reproduced in the periodically forced Brusselator.\textsuperscript{15,16} Belmonte \textit{et al.} observed a transition from a stable spiral to turbulence in a BZ reaction when resonant forcing was applied.\textsuperscript{17} Resonantly forced oscillatory systems have been investigated by numerical simulation as well as theoretically by Couillet and Emilsson,\textsuperscript{18,19} Couillet \textit{et al.},\textsuperscript{20} Elphick \textit{et al.},\textsuperscript{21,22} and Chaté \textit{et al.}\textsuperscript{23} These systems exhibit a number of interesting pattern-forming phenomena.

Given the wide range of phenomena arising from spatial disorder in excitable systems and resonant forcing of oscillatory systems, one might expect that oscillatory systems with spatially inhomogeneous forcing have the potential to exhibit interesting new features. The research presented here explores qualitatively the phenomenology of such systems and characterizes some of the phenomena quantitatively. We restrict the focus of our study to
systems in two spatial dimensions.
2. Periodically Forced Oscillatory Systems

2.1 Forced Oscillatory Systems and Resonance

Consider an externally forced oscillatory reacting system described by the ordinary differential equation,

\[
\frac{dc(t)}{dt} = R(c(t); a, b(t)),
\]

where \(c(t)\) is a vector containing the concentrations of reagents. The reaction rates are described by the nonlinear vector function \(R\) which depends on a collection of parameters \(a\), such as rate constants and constant concentrations of pool chemicals, as well as parameters \(b(t)\) which comprise the periodic forcing and are of the form \(b(t) = \eta_0 \Phi(\omega t)\) with \(\eta_0\) the constant forcing amplitude and \(\Phi\) a \(2\pi\)-periodic function giving the form of the forcing. As mentioned above, such forcing may be implemented by periodic illumination of the system if the reaction is light sensitive.

If \(b(t) = 0\) we suppose the unforced reacting system has a stable limit cycle \(c_0(t)\) with period \(T_0 = 2\pi/\omega_0\). In such a system there exists an infinite number of limit cycle solutions, \(c'_0(t) = c_0(t + \Delta t)\) which differ from \(c_0(t)\) only by an arbitrary phase shift \(2\pi\Delta t/T_0\). Limit cycle attractors are neutrally stable to phase perturbations corresponding to translations along the orbit. A system following a limit cycle will, in general, have undergone a phase
shift when it returns to the limit cycle after experiencing a small perturbation.

These characteristics of the unforced oscillator may be contrasted with those of the forced oscillator.\textsuperscript{25} If $\omega_f/\omega_0$ is sufficiently close to an irreducible ratio of integers $n/m$, and if the forcing amplitude is sufficiently large, then the oscillations will become entrained to the external forcing and the system possesses $n$ stable limit cycle solutions of period $T = nT_f = 2n\pi/\omega_f \approx mT_0$ which are mapped into each other under phase shifts $t \to t + kT/n$ for $k = 0, 1, 2, \ldots$. A system following one of these limit cycles will return to it with no phase shift after a small perturbation. This discrete, finite collection of limit cycles may be contrasted with the infinite and continuous collection of limit cycles in the unforced case. The entrained resonantly forced oscillator is a system with $n$ stable states, defined by the phase of the oscillations rather than by the system's location in phase space.

Suppose that the intensity of the forcing is proportional to a single parameter $\chi$, that is, the forcing amplitude $\eta_0$ is of the form $\eta_0 = \chi \eta_0$. Then, in the $(\omega_f/\omega_0, \chi)$ parameter plane, there exist regions in which the oscillator is entrained to the forcing, called "Arnol'd tongues". For every irreducible rational $n : m$ there exists an Arnol'd tongue which tapers to a point as it meets the $\omega_f/\omega_0$ axis at $n/m$. These tongues may terminate at a finite $\chi$; although every irreducible $n/m$ has an associated Arnol'd tongue, often only those at simple resonances with low $n, m$ such as 1:1, 2:1, 3:2 will be of a size such that the resonance is easily accessible under experimental conditions.

Some insight into the basis of entrainment may be gained by considering the circle map, a one dimensional recurrence relation that is a simple model for resonance.\textsuperscript{24} Suppose that a trajectory in a toroidal phase space $(\phi, \theta)$ is given by $d\phi/dt = \omega_f$, $d\theta/dt = \omega_0$, that is, we have a system exhibiting two uncoupled oscillations. If the plane $\phi = 0$ is taken as
the Poincaré section for this system, then the recurrence relation on this surface is given by

$$\theta_{j+1} = \theta_j + \alpha \quad (\text{mod } 1),$$  \hspace{1cm} (2.2)

where $\alpha = \omega_0 / \omega_\ell$ is the ratio of the two oscillatory frequencies. Here we have renormalized the angle $\theta$ to the unit interval from the interval $[0, 2\pi)$ in order to conform to the convention for one dimensional recurrence relations. Observe that if $\alpha = m/n$, an irreducible ratio of integers, then this mapping exhibits an $n$-periodic cycle $\theta_{j+n} = \theta_j + m = \theta_j \quad (\text{mod } 1)$. Conversely, if $\alpha$ is irrational then the mapping exhibits no cycle of any order.

Now suppose that the system is modified so that the oscillations in $\theta$ and $\phi$ no longer occur independently of each other, but instead $d\theta/dt$ depends on $\phi$. That is, we are forcing the $\theta$ oscillator at frequency $\omega_\ell$ with the external $\phi$ oscillator. The effect of this on the dynamics in the Poincaré section is to add a nonlinear dependence $g(\theta)$ on $\theta$ such that

$$\theta_{j+1} = \theta_j + \alpha + g(\theta_j) \quad (\text{mod } 1),$$  \hspace{1cm} (2.3)

where $g(0) = g(1)$ to maintain periodicity. Note that the Poincaré section is taken at intervals of one forcing period. In the standard treatment of the circle map, the form considered for $g(\theta)$ is $g(\theta) = (K/2\pi) \sin 2\pi \theta$ where $K$ is a nonlinearity parameter analogous to the forcing amplitude. Hence we are concerned with the map

$$\theta_{j+1} = \theta_j + \alpha + \frac{K}{2\pi} \sin 2\pi \theta_j \quad (\text{mod } 1).$$  \hspace{1cm} (2.4)

We define

$$f_\alpha(\theta) \equiv \theta + \alpha + \frac{K}{2\pi} \sin 2\pi \theta \quad (\text{mod } 1),$$  \hspace{1cm} (2.5)

and we denote the $n$th iterate of $f_\alpha$ as $f_\alpha^n$. The system is said to be entrained at the $m/n$
resonance at a given \((K, \alpha)\) if
\[
\theta_{j+n} = f^n_\alpha(\theta_j) = \theta_j + m ,
\]
that is, if it exhibits an \(n\)-periodic cycle in which \(\theta\) advances by \(m\). For a fixed non-zero \(K\) this will happen for \(\alpha\) in some interval of finite length near, but not necessarily including, \(m/n\). Note that this definition of the "\(m/n\)-resonance" is different from that used everywhere else in this paper; apart from in this discussion of the circle map a resonance at which \(m \omega_f \approx n \omega_0\) will be called an \(n : m\) resonance and the forcing ratio will be written \(\omega_f/\omega_0 \approx n/m\) which is equal to \(1/\alpha\) in the notation used here.

The region in the \((K, \alpha)\) parameter plane for which the system is entrained at a given resonance is the Arnol'd tongue. As in the case of the periodically forced oscillator discussed above, the Arnol'd tongue tapers to a point as \(K \rightarrow 0\) and intersects the line \(K = 0\) at a rational number \(m/n\); also, every rational number has an associated Arnol'd tongue. For a fixed \(K\), the endpoints of a given Arnol'd tongue, which we denote \(\alpha_\ast\), must satisfy the properties that there must be an angle \(\theta_\ast\) for which the system is entrained at the \(m/n\) resonance,
\[
f^n_\alpha(\theta_\ast) = \theta_\ast + m ,
\]
and that this fixed point in the map's \(n\)th iterate is marginally stable,
\[
\frac{df^n_\alpha(\theta_\ast)}{d\theta} = 1 .
\]
Thus, for a given \(K\) and \(m/n\), the boundaries of the Arnol'd tongue at \(K\) can be found by solving Eqs. (2.7) and (2.8) for \(\theta_\ast\) and \(\alpha_\ast\).
2.1.1 Kuramoto’s Phase Description of Entrainment

Kuramoto provides the following description of entrainment in terms of the phase, working in the limit where the deformation of the limit cycle from the unforced case by the periodic perturbation is small. His discussion is specific to the 1:1 resonance although his method is general for any resonance. We consider a periodically forced oscillator of the form

$$\frac{dc(t)}{dt} = R(c) + \epsilon p(c, t)$$

(2.9)

where the unforced oscillator $dc/dt = R(c)$ has a limit cycle $c_0(t)$ of period $T$, and $p(c, t)$ is an additive periodic perturbation of period $T'$ and small amplitude $\epsilon$. Note that $p$ is dependent on $c$. The difference $T - T'$ is assumed to be of order $\epsilon$.

We define a phase variable $\phi(c_0)$ on the limit cycle such that for the unforced oscillator $d\phi/dt = 1$. This definition of $\phi$ is then extended to the entire phase space such that $d\phi(c)/dt = 1$ for any trajectory of the unforced oscillator; note that according to this definition all points in the phase space with the same phase tend asymptotically to a limit cycle with the same phase. The system’s dynamics reduce to the evolution equation for $\phi(t)$. Kuramoto shows that for the forced oscillator, this is given by

$$\frac{d\phi}{dt} = 1 + \epsilon (\nabla c_0) \cdot p(c_0(\phi), t)$$

(2.10)

The phase of the unforced oscillator advances by $T$ in time $T$; if the oscillator were entrained to the forcing then the phase would advance by one cycle, $T$, in one period of forcing, $T'$. Hence, if the change in $d\phi/dt$ resulting from the entrainment were realized uniformly throughout the limit cycle, then we would have $d\phi/dt = T/T'$. We can represent the instantaneous deviation from this rate by $d\psi/dt$; then $\psi$ represents the cumulative
deviation in the phase relative to that which would apply if there were uniform entrainment. \(\psi\) is given by

\[
\phi = \frac{T}{T'} t + \psi. \tag{2.11}
\]

Hereafter we search for the evolution of this cumulative phase difference \(\psi(t)\); if \(\psi\) is constant in time the oscillator has become entrained to the forcing.

For notational brevity we define

\[
\Omega(\phi, t) = (\nabla_{\phi} \phi)_{\phi=\phi_0} \cdot p(\phi_0(t), t). \tag{2.12}
\]

Note that the periodicity of \(\Omega\) is \(\Omega(\phi, t) = \Omega(\phi + T', t) = \Omega(\phi, t+T')\). Substituting Eq. (2.11) into Eq. (2.10) we obtain

\[
\frac{d\psi}{dt} = \varepsilon \left[ \Delta + \Omega \left( \frac{T}{T'} t + \psi, t \right) \right], \tag{2.13}
\]

where \(\Delta = \varepsilon^{-1}(T - T')/T'\) is a measure of the extent to which the forcing is off-resonance.

Since \(\psi\) is a slow variable relative to the time scale of one forcing period, and because \(\Omega\) is \(T'\)-periodic in its total dependence on \(t\) in Eq. (2.13), \(\Omega\) may be averaged over one forcing period. Hence the evolution of \(\psi\) is given by the ordinary differential equation

\[
\frac{d\psi}{dt} = \Delta + \frac{1}{T'} \int_0^{T'} \Omega \left( \frac{T}{T'} t + \psi, t \right) \, dt, \tag{2.14}
\]

where the amplitude of the forcing has been absorbed into \(p\) and \(\varepsilon\) has been set to 1. Entrainment occurs when Eq. (2.14) has a stable fixed point, that is, when \(\psi\) exists such that

\[
\int_0^{T'} \Omega \left( \frac{T}{T'} t + \psi, t \right) \, dt = T - T'. \tag{2.15}
\]
Note that $\Omega$ is $T$-periodic in $\psi$, hence each fixed point exists as a family differing by integral multiples of $T$, and thus it is sufficient to consider $0 \leq \psi \leq T$. The bifurcation to entrainment is a limit point bifurcation, hence there also will exist a family of unstable fixed points corresponding to a limit cycle which is unstable to phase perturbations.

2.2 Spatially Distributed Systems

The general form of an oscillatory reaction-diffusion system with spatially inhomogeneous periodic forcing is

$$\frac{\partial c(r, t)}{\partial t} = R(c(r, t); a, b(r, t)) + D \nabla^2 c(r, t),$$

(2.16)

where $D$ is a diagonal matrix of diffusion coefficients. The parameters responsible for the periodic forcing $b(r, t)$ now depend on space as well as time and are of the form $b(r, t) = \eta(r, t) \Phi(\omega t)$. The random variable $\eta(r, t)$ accounts for the fact that the forcing amplitude may vary stochastically in space and time.

In a spatially distributed system with spatially uniform forcing, $b(r, t) = b(t) = \eta_0 \Phi(\omega t)$, the diffusive coupling and the stability of the $n$ limit cycles to phase perturbations leads to the formation of domains of the different phase locked states. At the domain walls separating the different phase locked states the phase of the oscillations shifts by an amount determined by the character of the phase locking. In two dimensions, $n$-armed spiral waves may form, given suitable initial conditions, if the domain walls have non-zero velocity. The core of the spiral wave is a phase singularity at which all $n$ states meet and around which the phase advances by $2m\pi$. The rotation of these phase locked spirals is a result of the propagation of the phase dislocations comprising the domain walls; thus, they rotate much
more slowly than spiral waves in the unforced system, where the spiral rotation frequency is equal to the frequency of the local oscillations.

2.3 The complex Ginzburg-Landau Equation

Suppose an unforced system of ordinary differential equations of the form of Eq. (2.1), with \( \eta_0 = 0 \), has a steady state \( c_0 \). A Hopf bifurcation is one in which, as some scalar control parameter \( \lambda \) passes through a critical value \( \lambda_c \), the fixed point \( c_0 \) becomes unstable and a new attractor, a stable limit cycle \( c \), grows continuously from \( c_0 \). (More precisely, this is a supercritical Hopf bifurcation, but we will never consider the subcritical case.) When a Hopf bifurcation occurs, a complex conjugate pair of eigenvalues of the linear stability matrix \( (\partial R_i/\partial c_j)_{c_0} \) cross the imaginary axis. We let their values at the Hopf bifurcation be \( \pm i\omega_0 \), and we designate the corresponding eigenvectors to be \( u \) and its complex conjugate \( u^* \). The system evolves on the center manifold, a two-dimensional manifold embedded in the phase space which is tangent at \( c_0 \) to the plane spanned by \( Au + \bar{A}u^* \), where \( A, \bar{A} \in \mathbb{C} \) and \( \bar{A} \) is the complex conjugate of \( A \). Thus, near the Hopf bifurcation, solving for the system's dynamics reduces to the problem of finding the evolution equation for the complex amplitude \( A \). \( A \) evolves on a slow time scale, \( \tau \), while the time scale of oscillations in the original system is on the order of \( 2\pi/\omega_0 \) and is described by a fast time variable \( t \). The evolution of \( A \) is given by the Stuart-Landau equation

\[
\frac{\partial A}{\partial \tau} = A - (1 + i\beta)|A|^2A \tag{2.17}
\]
where $\beta$ is a constant which depends on the original system. The concentrations in the original system can be obtained from $A(\tau)$ according to

$$c(t, \tau) - c_0 = A(\tau)ue^{i\omega t} + \bar{A}(\tau)u^*e^{-i\omega t}$$  \hspace{1cm} (2.18)

which describes oscillations of frequency $\omega_0 + (d/dt)\arg A$ and amplitude $|A|$ on the center manifold in the phase space of the original system. Note that the complex amplitude $A$ contains information about the slow modulations of the envelope of the oscillations: $(d/dt)\arg A$ describes the frequency relative to the reference frequency $\omega_0$, $|A|$ describes the amplitude, and $\arg A$ provides information about the phase of the oscillations.

Another method of reduction of the dynamics that discards short time scale information about detailed dynamics of oscillations and retains slow modulations of the envelope is the Poincaré map. Indeed, Gambaudo obtains the complex amplitude equation analogous to Eq. (2.17) for the resonantly forced oscillator by determining the dynamics of the Poincaré map. This equation is

$$\frac{\partial A}{\partial \tau} = (\mu + i\nu)A - (1 + i\beta)|A|^2A + \gamma A^{n-1},$$  \hspace{1cm} (2.19)

where $\mu, \beta, \gamma$ are constants depending on the original system, and

$$\nu = \omega_0 - \frac{m}{n}\omega_f$$  \hspace{1cm} (2.20)

indicates the extent to which the forcing frequency is off-resonance.

For Eq. (2.19) a critical $\gamma_c$ exists such that for $\gamma < \gamma_c$ the equation exhibits a stable limit cycle solution, while for $\gamma \geq \gamma_c$ there are $n$ stable fixed points. These correspond to the $n$ stable limit cycle solutions of the original system and can be mapped into each other by phase shifts $A \rightarrow Ae^{i2\pi/n}$. More specifically, the fixed points can be found from
the stationary solutions of Eq. (2.19) by expressing the complex amplitude in the form $A = Re^{i\phi}$. The modulus, $R_0$, of the non-zero fixed points of Eq. (2.19) depends on $\gamma$ according to

$$\gamma^2 = \frac{(R_0^2 - \mu)^2 + (\nu - \beta R_0^2)^2}{R_0^{2(n-2)}}. \quad (2.21)$$

For some values of $n$ and $\gamma$ Eq. (2.21) permits multiple $R_0$ values, some of which may correspond to unstable fixed points. Figure 2.1 shows a $\gamma$ vs. $R_0$ curve for $n = 3$; the upper branch corresponds to the stable fixed points while the lower branch describes the nonzero unstable fixed points. The parameter values used to construct this figure are $\mu = 1$, $\nu = 0$, $|\beta| = 0.6$. Note the presence of a critical forcing amplitude $\gamma_c \approx 0.58$ below which phase locking does not occur. For $n = 3$, Eq. (2.21) gives $\gamma_c = [2((1+\beta^2)(\mu^2+\nu^2))^{1/2} - 2(\mu+\beta\nu)]^{1/2}$.

In a spatially distributed system, the corresponding equation for the reduced dynamics is the forced complex Ginzburg-Landau equation,

$$\frac{\partial A(r,t)}{\partial t} = (\mu + i\nu)A - (1 + i\beta)|A|^2 A + \gamma A^{n-1} + (1 + i\alpha)\nabla^2 A. \quad (2.22)$$

This equation has been used as a model for an oscillatory reaction-diffusion system with spatially uniform resonant forcing.$^{18-23}$

### 2.4 Scope of this Investigation

Section 3 consists of an investigation of a specific uniformly forced distributed system, the FitzHugh-Nagumo system of partial differential equations. With this as background, Sections 4 and 5 study systems with quenched disorder, $\eta(r,t) = \eta(r)$, where the periodically applied perturbation has a random distribution in space which does not change in the course of the evolution. A modification of Eq. (2.22) in which $\gamma$ is spatially dependent is used.
Figure 2.1: The modulus, $R_0$, of the stable (solid line) and unstable (dashed line) fixed points of Eq. (2.19) as a function of forcing intensity $\gamma$. The parameter values are $n = 3$, $|\beta| = 0.6$, $\mu = 1$, $\nu = 0$. The three arrows indicate the values $\gamma = 0.0$, 0.6 and 1.0, which were used in simulations described in Section 4. The vertical dotted line is located at $\gamma_c$, the critical value for phase locking.
Section 6 studies a system with dynamic disorder, $\eta(r,t)$, where the spatial distribution of the perturbation may change with time. Again, the FitzHugh-Nagumo system is used as the particular example.

In spatially disordered systems such as those in Sections 4-6, a new length, $\ell_s$, related to the spatial correlation range of the noise distribution enters the problem. The behavior of the system will depend on the magnitude of this length relative to that of other important lengths in the system, such as the diffusion length $\ell_D$ and the typical width of a domain wall or propagating front, $w_d$. If $\ell_s$ is sufficiently small compared to both of these lengths the system will appear effectively homogeneous and behave as if it were subject to periodic forcing with amplitude determined by the spatial average $\overline{\eta}$ of $\eta(r)$. If $\ell_s$ is very large compared to these lengths then the system dynamics may be simply represented in terms of the dynamics of a collection of uniformly forced patches. The interesting regime is when these length scales are comparable and our studies focus on these cases.
3. FitzHugh-Nagumo system with spatially uniform forcing

Consider the FitzHugh-Nagumo system of ordinary differential equations

\[
\begin{align*}
\frac{du(t)}{dt} &= u - u^3 - v \\
\frac{dv(t)}{dt} &= \epsilon(u - av + b(t)),
\end{align*}
\]

in which \( b(t) = \eta_0 \cos \omega_t t \) is an external forcing with constant amplitude \( \eta_0 \) and frequency \( \omega_t \). First, consider the behavior of the unforced system with \( b(t) = b_0 \). If \( 0 < a < 1 \) the system possesses a single fixed point. If, in addition, \( b_0 = 0 \), then the fixed point is unstable and the system exhibits a stable limit cycle whenever \( a\epsilon < 1 \). Our attention is restricted to systems with \( 0 < a < 1 \) and \( a\epsilon < 1 \). The effect of varying \( b_0 \) is to shift the \( u \)-nullcline relative to the \( u \)-nullcline. As \( |b_0| \) increases from zero the limit cycle contracts in the phase plane, eventually collapsing to a stable fixed point. This is a Hopf bifurcation that occurs when \( |b_0| = b_H = (1 - (2a + a^2\epsilon)/3)((1 - a\epsilon)/3)^{1/2} \). For \( |b_0| > b_H \) the system exhibits excitable kinetics. Figure 3.1 shows the nullclines and limit cycle as \( b_0 \) is increased from 0 to 0.46, which lies just inside the excitable regime.

As described in Sec. 2.1 for the case of a general oscillatory system, the ordinary differential equations Eq. (3.1) show phase locking with the external forcing for appropriate \( \eta_0 \) and \( \omega_t/\omega_0 \) sufficiently close to an irreducible ratio of small whole numbers.
Figure 3.1: The nullclines and attractors of the unforced Fitzhugh-Nagumo o.d.e.'s (Eq. (3.1) with \( b(t) = b_0 \) as \( b_0 \) is varied (\( a \) and \( \epsilon \) are held constant at 0.3 and 0.1 respectively). The linear \( v \) nullcline and cubic \( u \) nullcline are shown as solid lines. The last three figures show the \( v \) nullcline for the \( b_0 = 0 \) case as a dotted line. The limit cycle is shown using a dashed line. Top left: \( b_0 = 0 \). The system is oscillatory with a limit cycle. Top right: \( b_0 = 0.4 \). The system is still oscillatory, the limit cycle is asymmetric. Bottom left: \( b_0 = 0.45 \). The limit cycle is smaller. Bottom right: \( b_0 = 0.46 \). The limit cycle has collapsed to a stable fixed point (shown by a dot) and the system is now excitable.
Also consistent with the general case described, the spatially distributed system
\[ \frac{\partial u(x,t)}{\partial t} = u - u^3 - v + D_u \nabla^2 u \]
\[ \frac{\partial v(x,t)}{\partial t} = \epsilon(u - av + \eta_0 \cos \omega t) + D_v \nabla^2 v, \tag{3.2} \]
admits spatially homogeneous oscillatory solutions, which yield a spatially homogenous stationary state when viewed stroboscopically at a period \( nT_f = 2n \pi / \omega r \). In the phase locked regime, Eq. (3.2) exhibits domain walls in one and two dimensions, and spiral waves around phase defects.

The rest of this section will be concerned with patterns found in simulations of Eq. (3.2). Our investigations focused on the 3:1 entrainment band, although brief mention will be made of the 2:1 entrainment region. The simulations used the parameters \( a = 0.3, \epsilon = 0.1 \), for which the period of homogeneous oscillations in the unforced system is 28.80. The forcing amplitude was most typically \( \eta_0 = 0.46 \), and most work was done with \( D_u = D_v = 0.25 \).

In all simulations described below, in this and subsequent sections, numerical integration was performed using explicit forward differencing and a second-order discrete Laplacian, given by
\[ \nabla^2 u_{ij} = \frac{1}{6(\Delta x)^2} \left( \sum_{r,s} A_{r,s} u_{i+r,j+s} - 20u_{i,j} \right), \tag{3.3} \]
where the sum is taken over the eight nearest neighbours of the lattice point and \( A_{r,s} \) is 4 for the four nearest neighbours and 1 for the four next-nearest diagonal neighbours.

Figure 3.2 shows the evolution of the \( u \) field in time at a one-dimensional domain wall in the 3:1 entrainment zone. Observe that in the spatially uniform domains separated by the wall, the dynamics reduces to the limit cycle oscillations of the ordinary differential equations Eq. (3.1), but that the limit cycles attained in the two different regions are out
Figure 3.2: Space-time evolution of the $u$ field at a domain wall in the uniformly forced FHN. Parameters: $\omega_f/\omega_0 = 3.05$, $\eta_0 = 0.46$, $a = 0.3$, $\epsilon = 0.1$, $D_u = D_v = 0.25$.

of phase by one third of a cycle. The interface position advances back-and-forth during a cycle, i.e. it is non-monotonic on a short time scale. However, there is a net motion per cycle that results in a nonzero average velocity over longer time scales. This average velocity may be measured by, for example, measuring the front position stroboscopically at intervals $\pi T_f$ (i.e. once a cycle) in which case the front moves monotonically.

The (average) velocity of planar fronts was measured for the 3:1 case as a function of $\omega_f/\omega_0$ for various $\eta_0$ (Fig. 3.3). Measurements were also made of the dependence on $D_v$ and $\omega_f/\omega_0$ for $\eta_0$ fixed at 0.46 and $D_u$ fixed at 0.25 (Fig. 3.4). The local dynamics at a point as a domain wall passes are shown in Fig. 3.5. There is a contraction in the envelope of oscillations as the front passes, which is matched by a contraction in the limit
cycle in the phase plane from that for the unforced case. On either side of the contraction, the oscillations are those of the uncoupled oscillator because the point then lies far inside a spatially homogeneous domain. The two limit cycle oscillations on either side of the contraction are shifted by one third of a cycle; the shift in phase occurs in the contraction corresponding to the front passing.

In the fronts considered above the phase advances by one-third of a cycle as the front is crossed; one may imagine a front connecting the same two phases in which the phase is delayed by two-thirds of a cycle rather. This front will pass transiently through the third state of the system. Such fronts may be formed by an appropriate choice of starting conditions, however, they are not stable. A domain of the third state having a finite width grows at the interface and eventually a travelling structure is obtained in which a strip of the third phase lies between the other two phases and the two interfaces are of the ordinary kind studied above.

For reaction-diffusion fronts in two dimensions, it is commonly observed that the normal velocity of a front, \( v \) depends on the curvature, \( \kappa = 1/R \), where \( R \) is the radius of curvature, according to

\[
v(\kappa) = v_p - \kappa D,
\]

where \( v_p \) is the velocity of a planar front. This relationship predicts that for a front with \( v_p = 0 \), the radius of a disk will vary in time as \( R^2 = R_0 - 2Dt \). This prediction was confirmed for the forcing parameters \( \omega_1/\omega_0 = 3.098, \eta_0 = 0.46 \), for which \( v_p \approx 0 \) (cf. Fig. 3.3); Fig. 3.6 shows \( R^2 \) against time, a linear decrease with slope \(-0.42 \approx -2D = 0.50\) was found.
Figure 3.3: Velocity of a planar front in the uniformly forced FHN as a function of $\omega_f/\omega_0$ for different $\eta_0$. At the bottom of the graph, from left to right, the lines correspond to $\eta_0 = 0.46, 0.40, 0.35, 0.30, 0.25, 0.20$. Other parameters: $a = 0.3$, $\epsilon = 0.1$, $D_u = D_v = 0.25$.

Figure 3.4: Velocity of a planar front in the uniformly forced FHN as a function of $\omega_f/\omega_0$ for different $D_v$. $D_u = 0.25$ and $\eta_0 = 0.46$ are held constant. At the left hand side of the graph, from top to bottom, the lines correspond to $D_v = 0.25, 0.50, 1.00, 2.00$. Other parameters: $a = 0.3$, $\epsilon = 0.1$. 

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Figure 3.5: Local dynamics at a point as a planar front passes. Top left: $u$ versus $t$. Top right: $\bullet$ (solid line) and $v$ (dashed line) versus $t$ at the front passing. Bottom: dynamics in the phase plane; note the contraction of the limit cycle.
Figure 3.6: $R^2$ vs. $t$ for a disk of initial radius 15. Forcing is at $\omega_f/\omega_0 = 3.098$ and $\gamma = 0.46$, for which a planar front is stationary. The radius is measured stroboscopically every three forcing cycles. The slope of the dashed line (obtained by linear regression after discarding the first point as part of the transient behavior) is -0.42, in reasonable agreement with the predicted value of $2D = -0.50$. 
If the initial conditions contain a phase defect, then three-armed spirals may form. Figures 3.7 and 3.8 compare spirals in the unforced and forced cases. The concentration profile demonstrates that the forced spiral has three arms; plateaus are observed at three different concentration values. We establish that these are indeed phase locked in the 3:1 entrainment zone by viewing stroboscopically at various periods (Fig. 3.9). When viewed in real-time, the concentrations within each arm cycle. When viewed at $\pi T_1$ the concentrations remain constant and the spiral rotates slowly. When viewed at $T_1$ the arms of the spiral exchange concentrations, cycling through three different values.

As the corresponding planar front velocity decreases, spirals become less tightly wound (Fig. 3.10). The planar front velocity is approximately zero at $\omega_f/\omega_0 = 3.098$, for $\gamma = 0.46$, as a result the direction of winding of the spirals reverses at this point.
Figure 3.8: Left: Three-phase spiral generated in the forced FHN with $a = 0.3, \varepsilon = 0.1$, $\eta_0 = 0.46$ and $\omega_f/\omega_0 = 2.98$. Boundary conditions are no-flux. The $u$-concentration field is shown. Starting conditions were a spiral formed using the unforced FHN. Right: $u$ versus position profile for a horizontal cut through the spiral core. The three line types indicate three different times.
Figure 3.9: Stroboscopic viewing demonstrates 3:1 phase locking. Shown are consecutive frames from a spiral strobed at $T_f/3$ (top row), $3T_f$ (middle row) and $T_f$ (bottom row). Forcing parameters: $\omega_f/\omega_0 = 3.00$, $\gamma = 0.46$. 
In the case $\omega_f/\omega_0 = 3.10$ the arms are not wound at all. Edge effects are visible: the interfaces between the arms are meeting the walls at right angles; as a result the three arms are not equivalent. The core moves in this case, however this may be a result of the inequivalence of the arms due to the edge effects. It would be necessary to perform this simulation in a large system, or possibly in a system with circular geometry, in order to determine what the asymptotic behavior of the spiral is. If there is in fact a transition to a moving spiral core, it might be a worthwhile subject for further investigation; the motion of spiral cores in excitable media is extremely well studied and has also been studied in oscillatory media. For all other forcing parameters studied, spiral cores were stationary.

Figure 3.11 shows the local dynamics at a point in a system containing a spiral wave with $\omega_f/\omega_0 = 3.00$, $\gamma_0 = 0.46$. The frequency of local oscillations in the $u$ field is too high for individual oscillations to be seen, but periodic contractions in the envelope of oscillations are present which correspond to the passage of the domain walls between the spiral arms (cf. Fig. 3.5). At each contraction the phase of oscillations advances by one-third of a period, at the third contraction the original phase has been regained, corresponding to one spiral rotation. Thus, Fig. 3.11 shows an interval of time a little longer than one spiral rotation period.

The angular velocity $\omega_r = 2\pi/T_r$ of the spiral rotation was measured as a function of $\omega_f/\omega_0$ for $\gamma = 0.46$ (Fig. 3.12). The frequency of spiral rotation is much less than the frequency of oscillations in spatially uniform domains; this is also demonstrated by Fig. 3.11. At $\omega_f/\omega_0 = 3.00$, the period of homogeneous oscillations is about 28.80 time units, while the period for a complete spiral rotation is 2762 time units, a ratio of two orders of magnitude. With a frequency ratio this large it appears unlikely that there could
Figure 3.10: Long time state of phase locked spiral waves in the uniformly forced FHN system. Forcing parameters: $\eta_0 = 0.46$; $\omega_t/\omega_0$ is (top row, from left to right) 2.90, 3.00, 3.05, (bottom row, left to right) 3.10, 3.15, 3.20. Other parameters: $a = 0.3$, $\epsilon = 0.1$, $D_u = D_v = 0.25$. With the exception of the $\omega_t/\omega_0 = 3.10$ case, these rotate uniformly and appear to be the asymptotic state of the system. In the $\omega_t/\omega_0 = 3.10$ case, edge effects are visible and the core moves; the spiral appears to be still evolving.
Figure 3.11: Local dynamics at a point in a system forced at $\omega_f/\omega_0 = 3.00$, $\eta_0 = 0.46$ containing a spiral wave. The value of $u$ is shown.

be resonance between the frequency of spiral rotation and the frequency of local oscillations, and indeed no evidence has been found to suggest that there is resonance. The different time scales for the two oscillations are a result of the fact that they occur as a result of different mechanisms: the uniform oscillations are driven by the oscillatory local dynamics, while the spiral rotation is a property linked to the spatially distributed nature of the system which arises from the propagation of phase dislocations comprising the domain boundaries.

An entrainment regime was found near the 2:1 resonance. The velocities of planar fronts as a function of $\omega_f/\omega_0$ were measured (Fig. 3.13). Because of the symmetry of the 2:1 system, a non-zero front velocity implies the existence of another front solution propagating in the opposite direction with equal speed. At all $\omega_f/\omega_0$ investigated, a non-zero front velocity was found, thus a symmetric pair of counter-propagating fronts (Bloch fronts) exists. Bloch and Ising fronts are discussed more extensively in Sec. 5.1. It is interesting that the front
Figure 3.12: The angular velocity \( \omega_r = 2\pi/T_r \) for spiral rotation in the 3:1 uniformly forced FHN. \( \gamma \) is fixed at 0.46. The line is dashed for \( 3.05 < \omega_r/\omega_0 < 3.15 \) because the shape of the curve as it passes through \( \omega_r = 0 \) has not been determined. The difficulty of obtaining a rotating spiral of stable shape at \( \omega_r/\omega_0 = 3.10 \) prevented \( \omega_r \) from being measured there.
Figure 3.13: Velocity of a planar fronts in the 2:1 entrainment zone as a function of $\omega_f/\omega_0$ for $\eta_0 = 0.46$ and other parameters $a = 0.3$, $\epsilon = 0.1$, $D_u = D_v = 0.25$. Symmetry requires that for non-zero velocities there exist two fronts propagating in opposite directions with equal speeds. The velocity was measured for one curve, the symmetry property was used to draw the second curve.

velocity curve passes through zero without a transition to the Ising regime.
4. Forced CGL with quenched disorder; 3:1 resonance

Studies into systems with quenched disorder used the inhomogeneously forced CGL equation

\[
\frac{\partial A(r, t)}{\partial t} = (\mu + i\nu)A - (1 + i\beta)|A|^2 A + \gamma(r)\tilde{A}^{n-1} + (1 + i\alpha)\nabla^2 A \tag{4.1}
\]

as a model forced oscillatory system. In this case the forcing amplitude \(\gamma(r)\) is a time-independent, spatially-dependent random variable. This equation may be obtained by performing the reduction to the CGL of an inhomogeneously forced oscillatory system; in this case the forcing amplitude remains a time-independent, spatially-dependent random variable through the reduction and hence appears as the stochastic field \(\gamma(r)\). Recall, (cf. Section 2.3 and Fig. 2.1) that Eq. (2.19, the ordinary differential equation obtained from Eq. (4.1) by dropping the diffusion terms, in which \(\gamma\) is a constant, possesses a critical \(\gamma_c\) for phase locking below which it exhibits oscillatory dynamics.

As an example of quenched disorder the \(\gamma(r)\) fields were taken to be dichotomous random variables. Two values for the forcing amplitude, \(\gamma_1\) and \(\gamma_2\), were chosen. The two-dimensional system was partitioned into square cells and the value of \(\gamma(r)\) in each cell was chosen to be either \(\gamma_1\), with probability \(p\), or \(\gamma_2\) with probability \(q = 1 - p\). More precisely, if the noise cells have dimension \(s \times s\) and the system's dimensions are \(W \times L = sN_W \times sN_L\) then

\[
\gamma(r) = \sum_{i=1}^{N_W} \sum_{j=1}^{N_L} \xi_{ij} \Theta_{ij}(r), \tag{4.2}
\]
where

\[
\xi_{ij} = \begin{cases} 
\gamma_1 & \text{with probability } p \\
\gamma_2 & \text{with probability } q = 1 - p ,
\end{cases}
\]  

(4.3)

and

\[
\Theta_{ij}(r) = \Theta_{ij}(x, y) = \theta(x - (i - 1)s) \theta(is - x) \theta(y - (j - 1)s) \theta(js - y) ,
\]  

(4.4)

where \( \theta \) is the Heaviside function and \((ij)\) are the discrete coordinates of a noise cell. For quenched disorder this distribution is fixed for all time. The cell size, the values of \( \gamma_1 \) and \( \gamma_2 \) and the seeding probabilities \( p \) and \( q \) are the relevant parameters to consider. The probabilities \( p \) and \( q \) were typically, but not necessarily, independent of position; exceptions will be noted as they occur. This \( \gamma(r) \) field has the mean value \( \bar{\gamma} = p\gamma_1 + q\gamma_2 \), and spatial autocorrelation

\[
C(r') = \frac{\langle \delta \gamma(r' + r'') \delta \gamma(r'') \rangle}{\langle \delta \gamma(r'') \delta \gamma(r'') \rangle} = \begin{cases} 
\left(1 - \frac{|x'|}{s}\right) \left(1 - \frac{|y'|}{s}\right) & \text{if } |x'| \leq s \text{ and } |y'| \leq s \\
0 & \text{otherwise ,}
\end{cases}
\]  

(4.5)

where \( \delta \gamma(r) = \gamma(r) - \bar{\gamma}, \) \( r' = (x', y') \) and the average \( \langle \cdot \rangle \) is taken over all \( r''. \)

The studies described in this paper investigate patterns at the 3:1 resonance. For such systems there are two interesting cases for resonant forcing with a dichotomous \( \gamma(r) \) field. In the first case both \( \gamma_1 \) and \( \gamma_2 \) lie above the phase locking threshold, \( \gamma_c \), so that all regions of the medium are entrained to the forcing. In the second case only one of \( \gamma_1 \) or \( \gamma_2 \) lies above the threshold and the medium consists of a mixture of entrained and non-entrained regions.
Figure 4.1: The quasi-homogeneous stationary states of the 3:1 inhomogeneously forced CGL. The densities in the complex phase space are plotted for three separate realizations. In each case, the initial condition was a spatially uniform $A$ field with a value near those obtained in the steady state. This figure is a two-dimensional histogram of the lattice points in the final steady state. The grey scale intensity is proportional to the square root of the density in order to show up structure at low densities.

4.1 All sites phase locked; front roughening

When both $\gamma_1$ and $\gamma_2$ lie above the phase locking threshold $\gamma_c$ then all regions of the medium are tristable. The system possesses three quasi-homogeneous stationary states in which the complex amplitude fluctuates about an average value (Fig. 4.1).

Domain walls separating these phase locked states are in general non-stationary in the 3:1 resonant regime since an interface between two phases of three phases in a tristable system breaks the symmetry between directions of motion; however, the velocity may pass through zero as parameters are tuned. Initially planar fronts in these inhomogeneously
Figure 4.2: Interfaces of the 3:1 inhomogeneously forced CGL from a single realization in a moving frame, at three well separated times. The phase, $\phi$ of the complex amplitude $A = R e^{i\phi}$ is shown, using a gray-scale in which $\phi = -\pi = +\pi$ is white and $\phi = 0$ is black. The system size, $L \times W$ is $800 \times 100$. The noise grain size is $s \times s = W/25 \times W/25$. Other parameters are given in the text. Boundary conditions are periodic along $x$ and no-flux along $y$.

forced systems roughen as they propagate. Figure 4.2 shows an example of a rough interface separating two of the three phases.

Front roughening is also observed if $\gamma_1 < \gamma_c < \gamma_2$, (i.e. when the medium consists of a mixture of entrained and non-entrained regions) but is difficult to study because spontaneously nucleated patterns interfere with propagating fronts. This case will be examined below.

The propagating fronts in this system experience local velocity fluctuations arising from spatial variations in the $\gamma(\mathbf{r})$ field. Diffusion will tend to eliminate front roughness generated by random fluctuations in the $\gamma(\mathbf{r})$ field; consequently, the front dynamics should obey the Kardar-Parisi-Zhang (KPZ) equation,\textsuperscript{28}

$$
\frac{\partial h(x,t)}{\partial t} = \bar{v} + D \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \zeta(x,t), \quad (4.6)
$$

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where $h(x,t)$ is the front position, $\bar{u}$ is the average front velocity, $D$ and $\lambda$ are phenomenological coefficients and $\zeta(x,t)$ is Gaussian white noise with zero mean and correlation
\[
\langle \zeta(x',t') \zeta(x'',t'') \rangle = 2 \Gamma \delta(x' - x'') \delta(t' - t'').
\]
In such a circumstance the interface width $w(t) = (L^{-1} \sum_i (h(x_i, t) - \bar{h}(t))^2 \Delta x)^{1/2}$ increases with time as $w(t) \sim t^{\hat{\beta}}$ for short times; the average width of a saturated front, $w_s$, scales with system size as $w_s \sim L^{\hat{\beta}}$, where $\hat{\alpha} = 1/2$ and $\hat{\beta} = 1/3$.

We have verified these KPZ scaling properties for a FCGL system with parameter values ($\alpha = 1$, $\beta = 0.6$, $\mu + i \nu = 1$) and the forcing field parameters ($\gamma_1 = 0.60$, $\gamma_2 = 1$, $p = 0.50$) for which the critical forcing amplitude is $\gamma_c \approx 0.58$ (cf. Fig. 2.1). The front properties were measured in a frame moving with the front. The system dimensions were $W = 100$ and $L = 100, 200, 400$ and 800. The noise grain size was $s \times s = W/25 \times W/25$. The boundary conditions were periodic along the boundaries perpendicular to the front, and no-flux along the boundaries perpendicular to the direction of front motion.

Figure 4.3 shows the interface width, $w(t)$, against $t$ for single realizations at system sizes $L = 100, 200, 400, 800$. These demonstrate that the interface width saturates, that is, after a transient period of width increase it reaches a state where it fluctuates about a mean value, $w_s$. The mean saturated interface width scales with system size as $L^{\hat{\beta}}$, as shown in Fig. 4.4, which is consistent with KPZ scaling properties; also shown is the width of the $w(t)$ distribution in the saturated regime, which scales with $w_s$.

Plotting $\langle w(t) \rangle / L^{\hat{\alpha}}$ against $t / L^{\hat{\beta}/\hat{\alpha}}$ collapses the $\langle w(t) \rangle$ versus $t$ data for four different $L$ onto a single curve when the KPZ values $\hat{\alpha} = 1/2$, $\hat{\beta} = 1/3$ are used, as shown in Fig. 4.5. Here $\langle \cdot \rangle$ denotes an average over realizations. For comparison, Fig. 4.5 also shows the $\langle w(t) \rangle$ curves rescaled according to the exponents $\hat{\alpha} = 1/2$, $\hat{\beta} = 2$ characterizing growth under the
Figure 4.3: Interface width vs. time for single realizations starting from planar fronts. System sizes are, from top to bottom, $L = 100, 200, 400, 800$. Note the different vertical scales. For these plots the width was sampled at intervals of 10 time units.
Figure 4.4: Mean saturated front width $w_s$ against system size $L$ on logarithmic scales. The error bars indicate the standard deviation of the distribution of interface width in the saturated regime, $\sigma_{\log w_s} = \sigma_{w_s}/w_s$ (cf. Fig. 4.3). The dashed line, obtained by linear regression, has slope $0.499 \pm 0.04$. 

\[ \log w_s \]

\[ \log L \]
Edwards-Wilkinson model, in which the growth equation is

$$\frac{\partial h(x,t)}{\partial t} = \bar{v} + D \frac{\partial^2 h}{\partial x^2} + \zeta(x,t).$$

In contrast to the KPZ case, rescaling with the Edwards-Wilkinson exponents fails to collapse the curves to a single curve. Sendiña-Nadal et al. have shown that in the limit where the interface is thin compared to the size of noise domains, KPZ scaling results; they observe this scaling in experiments for which this limit is valid. Our studies are not conducted in a regime where such an approximation can be made.

By inspection of Figure 4.3 it appears that the \( w(t) \) against \( t \) curves show different degrees of temporal autocorrelation as system size varies. To investigate this, the width temporal autocorrelation function

$$C_w(t) = \frac{\langle \delta w(t + t') \delta w(t') \rangle}{\langle \delta w(t') \delta w(t') \rangle},$$

was calculated for fronts in the saturated regime at four system sizes \( L = 100, 200, 400 \) and 800. Here \( \langle \cdot \rangle \) represents a time average within the saturated regime. Temporal correlations were found to decay to zero, with the rate of decay decreasing as system size increases (Fig. 4.6). This is expected, since the larger system size allows larger fluctuations in the front profile which require longer times to form and decay.

To investigate the presence of spatial correlations in saturated front profiles, the height spatial autocorrelation function

$$C_h(x) = \frac{\langle \delta h(x + x') \delta h(x') \rangle}{\langle \delta h(x') \delta h(x') \rangle},$$

was calculated for the various system sizes. In Eq. (4.9) the inner \( \langle \cdot \rangle \) refer to averaging over \( x' \) in a single front profile and the outer \( \langle \cdot \rangle \) refers to averaging over different saturated front profiles. Results for \( L = 100 \) and \( L = 800 \) are shown (Fig. 4.7).
Figure 4.5: Early time $\langle w(t) \rangle$ vs. $t$ curves for initially planar fronts in the 3 : 1 inhomogeneously forced CGL rescaled according to the KPZ (top) and EW (bottom) scaling exponents. Curves are shown for system sizes $L = 100$ (dotted line), 200 (short dashes), 400 (long dashes) and 800 (solid line). Each curve results from the average over 80 realizations of the stochastic dynamics.
Figure 4.6: The width temporal autocorrelation function for saturated interfaces in the 3:1 inhomogenously forced CGL for system sizes $L = 100$ (dotted line), 200 (short dashes), 400 (long dashes) and 800 (solid line).

In the case of a front free of spatial structure, i.e. a periodic random walk that returns to its initial position in $N$ steps, the height spatial autocorrelation function is of the form $C_h^R(x) = 1 - (2\mathcal{D}/L)x(L - x)$, where $\mathcal{D}$ is a phenomenological diffusion coefficient. The deviations of the fit of the numerically determined $C_h(x)$ from a quadratic function indicate the existence of spatial correlations.

The power spectrum of the saturated front profile

$$E(k) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |\hat{h}(k, t)|^2 dt$$

(4.10)

where $t_2$ and $t_1$ lie within the saturated regime and

$$\hat{h}(k, t) = \frac{1}{N} \sum_{i=1}^{N} e^{ikx_i} h(x_i, t)$$

(4.11)

was calculated for the four single realizations at the system sizes $L = 100$, 200, 400 and 800. A purely random front without spatial correlations ought to have a linear dependence.
Figure 4.7: The height spatial autocorrelation functions for saturated fronts in systems of size $L = 100$ (solid line) and $L = 800$ (long dashes). The best fit quadratic functions are shown for $L = 100$ (dotted line) and $L = 800$ (short dashes).

of $\log E(k)$ on $\log k$, up to a cutoff $k$ above which diffusion smooths out features. The power spectra obtained (Figure 4.8) are consistent with this picture with two exceptions. They show a peak at a wavenumber corresponding to a wavelength of $\lambda = 4$, which is the noise grain size. At small $k$ the observed $\log E(k)$ is greater than that predicted by the linear relation, indicating the presence of some long wavelength structure in the fronts. The structure indicated by the power spectra is in agreement with that seen in the spatial autocorrelation plots.
Figure 4.8: The power spectra of the saturated fronts. The small peak corresponds to a wavelength $\lambda \approx 4$. From top to bottom, the curves correspond to system sizes $L = 100, 200, 400, 800$.

### 4.2 A medium with phase locked and oscillatory sites; spontaneous nucleation of target patterns

When $\gamma_1 < \gamma_c$ and $\gamma_2 > \gamma_c$, the system consists of a mixture of tristable regions and oscillatory regions. If the density of oscillatory sites, $p$, is low the diffusive coupling maintains the concentrations within the oscillatory regions near those in the adjacent tristable regions. The medium behaves essentially like a tristable medium and supports three quasi-homogeneous stable stationary states (for finite system sizes), travelling domain walls and three-armed spiral waves. The oscillatory sites provide a spatial inhomogeneity that leads to roughening of domain walls and to fluctuations of the concentrations within domains.

With increasing $p$, target patterns are observed (Fig. 4.9). They consist of concentric,
Figure 4.9: Target patterns generated in the $3:1$ forced CGL from spatially homogeneous initial conditions. Forcing field parameters are ($\gamma_1 = 0$, $\gamma_2 = 1$, $\rho = 0.30$). Other parameters are ($\mu + i\nu = 1$, $\alpha = 1$, $\beta = 0.6$), for which $\gamma_c \approx 0.58$. The system size is $L \times L = 200 \times 200$, the noise grain size is $s \times s = L/200 \times L/200$. Boundary conditions are periodic. The grey scale indicates $\text{arg} \ A$, as described in Fig. 4.1.

approximately circular domain walls moving outwards from a central region (a "pacemaker") where they are initiated periodically. Within each ring of the target the concentration is quasi-homogeneous. Note that all images in Fig. 4.9 are from realizations with identical parameters, however a range of wavelengths is observed. The probability of a realization possessing a target pattern was measured as a function of $p$ and system size (Fig. 4.10). It was found to approach zero for low $p$, one for high $p$, and to increase rapidly around some critical $p_c$. Figure 4.10 shows results for three sets of parameter values. For larger system sizes, the probability of occurrence of a target pattern is higher, consistent with a fixed probability per unit area of the medium nucleating a target pattern. As a consequence it is not possible for a quasi-homogeneous stationary state to exist in a system of infinite size.

The occurrence of target patterns may be explained by supposing that diffusion provides a mechanism for averaging $\gamma(r)$ over some length scale, and that regions of the medium behave qualitatively like a uniform medium with the same local average $\gamma(r)$. Thus the
Figure 4.10: The fraction of realizations in which one or more pacemakers occurred as a function of $p$, the density of oscillatory ($\gamma = \gamma_1 = 0$) sites. The curves are of the form predicted by the pacemaker model described in the text, the parameters $\gamma_p$ and $N_p$ were found from least-squares fit. For all three sets of parameters $\gamma_1 = 0, \gamma_2 = 1, \mu + i\nu = 1$. Other parameter values are: Top left: $\alpha = 1, \beta = -0.6$, lattice spacing $\Delta x = 0.5$, noise grain size $s \times s = 0.5 \times 0.5$, system sizes $L = 50$ (solid line, squares), 100 (long dashes, circles), 200 (short dashes, downward-pointing triangles), 400 (dotted line, upward-pointing triangles); Top right: $\alpha = 1, \beta = 0.6$, lattice spacing $\Delta x = 1$, noise grain size $s \times s = 1 \times 1$, system sizes $L = 100$ (solid line, squares), 200 (dashed line, triangles); Bottom: $\alpha = 2, \beta = -0.82$, lattice spacing $\Delta x = 0.25$, noise grain size $s \times s = 0.25 \times 0.25$, system sizes $L = 25$ (solid line, squares), 50 (dashed line, triangles).
medium is locally tristable for a high $\bar{\gamma}(r)$, but if $\bar{\gamma}(r)$ is sufficiently low it will be oscillatory. As the density of oscillatory sites increases, it becomes increasingly likely that there will exist regions with a local $\gamma$ sufficiently low to exhibit oscillatory dynamics.

The $\gamma(r)$ field for a particular realization in which a pacemaker was seeded is shown in Fig. 4.11. There are no obvious features in the $\gamma(r)$ field at the location of the pacemaker; the criteria for pacemaker formation are more subtle than can be revealed by inspection at this level. Based on the speculation above that diffusion effectively averages $\gamma(r)$ over some length scale and that pacemakers occur when the local $\bar{\gamma}(r)$ becomes sufficiently low, some processing of this $\gamma(r)$ field was performed. The average over a disk of radius 2.5 was taken; Fig. 4.12 shows this averaged $\bar{\gamma}(r)$ in two different grey scales. In the first the grey scale intensity is linear in $\bar{\gamma}(r)$, in the second a cutoff value has been chosen ($\gamma_c$, the critical value for phase locking in the mean field system, was used) and all points where $\bar{\gamma}(r) < \gamma_c$ are colored black, all others are white. We do indeed see that the pacemaker is occurring at one of the places where the local average $\bar{\gamma}(r)$ is lowest. Although this analysis is somewhat simple, it does suggest that there is validity to the foregoing explanation that pacemakers occur when the diffusion-averaged local $\bar{\gamma}(r)$ becomes low.

To provide a more thorough analysis of the structure of pacemakers, we determine a profile of the average $\gamma$ field around the average pacemaker. We define $\bar{\gamma}_p$ as the average value of $\gamma$ a distance $R$ from the center of a pacemaker. This is given by

$$\bar{\gamma}_p(R) = \frac{1}{2\pi R} \frac{d}{dR} \pi R^2 \bar{\gamma}_d(R), \quad (4.12)$$

where

$$\bar{\gamma}_d(R) = \left\langle \frac{1}{\pi R^2} \int_{|r-r_0| \leq R} \gamma(r) \, dr \right\rangle, \quad (4.13)$$
Figure 4.11: Top left: A target pattern formed in a realization with $p = 0.25$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\mu + i\nu = 1$, $\alpha = 1$, $\beta = -0.6$. Top right: The $\gamma(r)$ field for this particular region. White sites have $\gamma = 1$, black sites have $\gamma = 0$. Bottom left: the $\gamma(r)$ field with a box around the pacemaker region, indicating the region enlarged in the bottom right panel. Bottom right: an enlargement of the boxed region in the bottom left panel.
Figure 4.12: The $\gamma$ field for the realization depicted in Figure 4.11 after coarse-graining. Each point in the left hand image shows the average value of $\gamma$ over a disk of radius 2.5 centered at that point. The gray-scale is chosen such that the lowest averaged $\gamma$ value is black and the greatest is white. The right hand image is the $\gamma$ field subjected to the same coarse-graining. Points for which the average value is less than $\gamma_c$, the threshold phase-locking value, are white, all others are black.

$\gamma$ is the average value of $\gamma$ over a disk of radius $R$, averaged over all pacemakers in all realizations with a given set of parameters. In practice, $\bar{\gamma}_p$ is calculated as the discrete derivative

$$\bar{\gamma}_p(R) = \frac{\pi(R + \Delta R)^2 \bar{\gamma}_d(R + \Delta R) - \pi R^2 \bar{\gamma}_d(R)}{\pi(R + \Delta R)^2 - \pi R^2} .$$

Figure 4.13 shows $\bar{\gamma}_p$ and $\bar{\gamma}_d$ for parameter values equal to those used to measure the first of the nucleation probability curves shown in Fig. 4.10 ($\alpha = 1, \beta = -0.6, \mu + i\nu = 1, \gamma_1 = 0, \gamma_2 = 1$) with the $\gamma_1$ seeding probability $p = 0.30$, system size $L \times L = 100 \times 100$, noise grain size $s \times s = L/200 \times L/200$.

$p = 0.30$, system size $L \times L = 100 \times 100$, noise grain size $s \times s = L/200 \times L/200$, and other parameter values equal to those used to measure the first of the nucleation probability curves shown in Fig. 4.10. Qualitatively similar plots, in which $\bar{\gamma}_p$ and $\bar{\gamma}_d$ increase from a low value at $R = 0$ to the mean-field value, $\bar{\gamma}$, at high $R$ were observed for other values of $p$ (Fig. 4.13,
Also shown in Fig. 4.13 are \( \bar{\gamma}_p \) calculated from the neighborhoods of points chosen randomly from the realizations rather than from points centered on pacemakers, and the mean-field value, \( \bar{\gamma} \).

Consider the following simple model of pacemaker formation. We divide the system into \( N \) "sites" of size \( \Delta x \times \Delta x \) such that the value of \( \gamma \) is constant over a site; i.e., the sites may be noise domains or subdivisions of noise domains. We assume that, on average, pacemakers have a radius of \( R \) and contain \( N_p = \pi (R/\Delta x)^2 \) sites. Thus we divide the system into \( N/N_p \) independent domains which are potential pacemakers. We assume that such a domain is a pacemaker if the average value of \( \gamma \) within it is less than some threshold for oscillatory behavior, \( \gamma^* \). If \( N_1 \) is the number of sites on which \( \gamma = \gamma_1 \) and \( N_2 = N_p - N_1 \) is the number of \( \gamma_2 \) sites, then the potential pacemaker is a pacemaker when

\[
\frac{pN_1 + qN_2}{N_p} = \frac{pN_1 + q(N_p - N_2)}{N_p} \leq \gamma^*,
\]

(4.15)

that is, when

\[
N_1 \geq \frac{\gamma_2 - \gamma^*}{\gamma_2 - \gamma_1} N_p.
\]

Defining

\[
N^* = \left[ \frac{\gamma^* - \gamma_2 N_p}{\gamma_1 - \gamma_2} \right],
\]

(4.17)

the probability that a domain is a pacemaker is

\[
P = \sum_{k=N^*}^{N_p} \binom{N_p}{k} p^k q^{N_p-k},
\]

(4.18)

and it is not a pacemaker with probability \( Q = 1 - P \). The probability that the entire system possesses no pacemakers is \( Q^{N/N_p} \) and it possesses one or more pacemakers with probability

\[
P(N, p) = 1 - Q^{N/N_p}.
\]

(4.19)
The parameters $N_p$ and $\gamma^*$ uniquely determine the $P(N, p)$ surface. The values giving the best fit to the experimental data for the parameters $\alpha = 1, \beta = -0.6$ were $\gamma^* = 0.66$ and $N_p = 180$ which implies $R \approx 3.8$. (For a fit done to the cleanest two curves, for system sizes $L = 50, 100$, the value $\gamma^* = 0.65$ was found.) These values are compared with the numerically determined $\tilde{\gamma}_d(R)$ curve in Fig. 4.13 (left panel). The simple model predicts that the average pacemaker should have $\tilde{\gamma}_d(R) = \gamma^*$; the point $(R, \gamma^*)$ lies close to, but not on, the $\tilde{\gamma}_d$ curve.

Consider the $\tilde{\gamma}_p(R)$ curves in Fig. 4.13 (right panel). At all $p$, the $\tilde{\gamma}_p(R)$ curve shows a distinct transition at $R \approx 4$; for $R$ greater than this the curve flattens out and apart from fluctuations is essentially indistinguishable from the mean field value $\tilde{\gamma}$. This suggests that the range over which the presence of a pacemaker is observable in the properties of the $\gamma(r)$ is about $R \approx 4$ units; consistent with the finding that the effective pacemaker radius is $R \approx 3.8$.

For the parameter values $\alpha = 1, \beta = 0.6, \gamma^*$ and $R$ were found to be 0.60 and 7.5, respectively; for $\alpha = 2, \beta = -0.82, \gamma^* = 0.78$ and $R \approx 1.7$ were found.

In order to provide further insight into the criteria necessary for a region to act as a pacemaker, a series of studies was carried out in which the $\gamma$ field consisted of a disk of radius $R$ sites with a density $p_{in}$ of oscillatory sites. This disk was embedded in a field with density of oscillatory sites $p_{out}$. In all cases $p_{in}$ was greater than $p_{out}$, so that the disk could act as a pacemaker. Multiple realizations of the evolution were simulated at various values of $R$, $p_{in}$ and $p_{out}$, and the fraction of realizations $Pr(br)$ in which the central disk emitted target waves into the surrounding medium (i.e. in which "breakout" occurred) was measured. The parameters $\alpha = 1, \beta = -0.6, \mu + i\nu = 1$, noise grain size $s \times s = 0.5 \times 0.5$, ...
Figure 4.13: Left: The solid curve shows $\gamma_p(R)$; the long-dashed curve is $\gamma_d$. Averages are taken over 20 realizations. The short-dashed curve shows $\tilde{\gamma}_p$ calculated around points chosen randomly from the realizations. A horizontal line indicates $\tilde{\gamma} = 0.70$. The values $\gamma^* = 0.66$, $\mathcal{R} = 3.8$, calculated using the pacemaker model, are also shown. Right: plots of $\tilde{\gamma}_p(R)$ against $R$ for (at right-hand side of plot, from bottom to top) $\tilde{\gamma} = 0.65$, 0.70, 0.71, 0.72, 0.73, 0.74. These plots are for realizations with $\alpha = 1$, $\beta = -0.6$, the first set of parameters in Fig. 4.10.
and lattice spacing $\Delta x = 0.5$, were used for these studies.

Figure 4.14 shows $Pr(br)$ as a function of $R$ and $p_{in}$ for $p_{out} = 0.10$. As one would expect, $Pr(br)$ increases as $R$ increases and decreases as $p_{in}$ decreases, with the exception that at $p_{in} = 1$ the probability of breakout increases with $R$ until $R = 3$ and then decreases for $3 < R < 5$, after which it increases. For $p_{in} = 0.95$, when there is a small fraction of tristable sites in the inner disk, the magnitude of the decrease is substantially reduced. Apart from the anomalous region at low $p_{in}$ the trends that $Pr(br)$ increases with increasing $R$ and decreases with decreasing $p_{in}$ are consistent with the notion that pacemakers form when the local density of oscillatory sites is high. The behavior for $p_{out} = 0.15$ was qualitatively similar (Fig. 4.15).

The anomalous decrease between $R = 3$ and $R = 5$ is possibly related to the fact that the anomalous behavior begins when $R \approx \ell_D$, where $\ell_D$ is the diffusion length of the unforced system ($\gamma(r) \equiv \gamma_1 = 0$). We find the diffusion length $\ell_D = \sqrt{D\tau} \approx 3.24$ by taking the diffusion coefficient $D$ to be unity, and the characteristic time $\tau$ to be equal to $2\pi/\beta\mu$, the period of homogeneous oscillations in the unforced system for the parameters $\beta = -0.6$, $\mu = 1$ used in these studies. Parts of the system separated by a length greater than $\ell_D$ evolve independently over time intervals less than $\tau$. For large $R$ one observes that pacemaker nucleation occurs locally on the boundary of the disk. As the disk perimeter increases with $R$ so does the probability of forming a local pacemaker on the disk boundary.

In addition to the target patterns discussed above, one may also observe spiral waves if the initial conditions contain a phase defect. An example of such a spiral is shown in Fig. 4.16. It was formed in a realization with $\bar{\gamma} = 0.50 < \gamma_c$, hence the medium may be thought of as oscillatory, exhibiting three-fold symmetric relaxational local dynamics, rather
Figure 4.14: The fraction of realizations in which target waves were emitted from the central disk region. Parameter values apart from \( p \) are the same as the first set of parameters in Figs. 4.10; the noise grain size is \( 0.5 \times 0.5 \). Here \( p_{\text{out}} = 0.10 \). The data points in the \( p_{\text{in}} = 1 \) and \( p_{\text{in}} \) curves were each obtained from either 20 or 25 realizations; other data points were obtained from five realizations.
Figure 4.15: The fraction of realizations in which target waves were emitted from the central disk region. Parameter values apart from $p$ are the same as the first set of parameters in Fig. 4.10; the noise grain size is $0.5 \times 0.5$. Here $p_{\text{out}} = 0.15$. Each data point is obtained from five realizations.

than as tristable.
Figure 4.16: Spiral wave in the 3:1 inhomogeneously forced CGL. Parameters: $\alpha = 1$, $\beta = -0.6$, $\mu + iv = 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $p = 0.50$, $L = 100$, $s = L/200$, no-flux boundary conditions. The grey scale indicates $\text{arg } A$, as described in Fig. 4.1.
5. The 2:1 forced CGL with quenched disorder

5.1 The non-equilibrium Ising-Bloch bifurcation

Consider a bistable reaction-diffusion system in which the state may be represented by a state vector in some phase plane. Initially we consider a system with one spatial dimension. At a domain wall the phase of the state vector must undergo a change of ±π. An “Ising front” is one in which this change is accomplished via a passage of the state vector through the origin, with no rotation in the phase plane. In a “Bloch front” the state vector rotates through an angle of ±π; Bloch fronts exist in pairs corresponding to clockwise and counterclockwise rotation of the state vector. A non-equilibrium Ising-Bloch (NIB) bifurcation is one in which the Ising front becomes unstable and a pair of stable Bloch fronts solutions emerge. Consider a bifurcation diagram in which front velocity is plotted against the control parameter. In a system where the two stable states are equivalent the NIB bifurcation is a pitchfork bifurcation; symmetry requires that the Ising front be stationary and that the Bloch fronts move in opposite directions with equal speeds. In the generic case where the states are inequivalent the pitchfork is unfolded to a saddle-node bifurcation, and the Ising front need not be stationary. In two-dimensional systems in the Bloch regime, phase defects may exist at points where fronts of opposite type meet; such points are the cores of two-armed spiral waves.
An NIB bifurcation can give rise to some remarkable pattern dynamics, particularly when other instabilities interact with it. For example, Hagberg and Meron describe a system in which front curvature perturbs the NIB bifurcation; curvature may cause a Bloch solution branch to terminate, triggering a transition to the other Bloch front and hence a reversal of front motion. Combined with a lateral instability of the Ising front, phenomena such as spot splitting and stable “breathing” spots are explained. Hagberg, Meron and co-workers have extensively studied pattern dynamics that involve the NIB bifurcation.

The 2:1 forced CGL contains an NIB bifurcation in which $\gamma$ acts as the control parameter, which was located and studied by Coulet et al. It was expected that the most interesting pattern dynamics to be found at the 2:1 resonance would involve the NIB bifurcation, and hence the studies here are directed at investigating this case. We use the parameters of Coulet et al., $(\mu = 1, \nu = 0.1, \alpha = -1, \beta = -0.15)$ for which $\gamma_c \approx 0.25$. The NIB transition was reported to occur at $\gamma_{NIB} \approx 0.44$, hence the uniformly forced system has three regimes, which are the oscillatory regime for $0 < \gamma < \gamma_c$, the bistable Bloch regime for $\gamma_c < \gamma < \gamma_{NIB}$ and the bistable Ising regime for $\gamma > \gamma_{NIB}$. For dichotomous noise, there are two cases which are expected to be interesting and for which results are presented: a mixture of oscillatory and bistable Ising regions, and a mixture of Bloch and Ising regions.

5.2 A mixture of oscillatory and Ising regime sites

For studies of the case in which the medium consists of a mixture of oscillatory regions and Ising regime regions, the values $\gamma_1 = 0$ and $\gamma_2 = 1$ were used. This places $\gamma_2$ deep into the Ising regime. The mean-field value of $\gamma$ is given by $\bar{\gamma} = p\gamma_1 + q\gamma_2 = 1 - p$. In these studies $\bar{\gamma}$ was varied by varying $p$. 

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We first consider systems in which the initial $A$ field is spatially homogeneous. For $\gamma$ well above the phase locking value $\gamma_c$ we expect that, as in the case of the 3:1 system, the medium will behave locally like a bistable medium and thus that the system will support two stable quasi-homogeneous stationary states. This is indeed observed. As $\gamma$ decreases, regions that exhibit oscillatory dynamics due to a high local density of oscillatory sites occur with increasing frequency. In the 3:1 case these emitted kink-like fronts that propagated outwards forming target patterns. In the 2:1 case formation of a target pattern can only occur if the front formed around the oscillatory patch is Bloch-like and of the same sign all the way around the pacemaker, which does not in general happen. We will return to consider the patterns arising from a pacemaker after discussing the behavior of fronts in the noisy bulk medium.

A series of studies were performed using initial concentration fields in which a sharp planar front existed between homogeneous regions corresponding to the two stable states of the $\gamma \equiv 1$ uniformly forced system. This initial front is achiral; it resembles the Ising front rather than either of the Bloch fronts. The time evolution of these systems after initial transients have passed is shown in Fig. 5.1 for three different values of $\gamma$, which are 0.50, 0.39 and 0.30. (Since the boundary conditions are periodic, there are actually two initially planar fronts in the system.) In these figures, the four quadrants of the complex plane were each assigned one of four shades of grey, with $-\pi < \arg A \leq -\pi/2$ corresponding to the darker medium shade, $-\pi/2 < \arg A \leq 0$ corresponding to the lightest shade, $0 < \arg A \leq \pi/2$ corresponding to the lighter medium shade and $\pi/2 < \arg A \leq \pi$ corresponding to the darkest shade.

Although this color-coding scheme loses information about concentration fluctuations
it has the advantage of identifying the direction of rotation of the state vector at the front. Thus, Ising-like fronts show up as an interface between the lighter and the darker medium shades of grey, while the two types of Bloch-like fronts appear as the darkest and lightest shades.

For $\gamma = 0.50$ (Fig. 5.1, left column), which is in the Ising regime, the front has roughened slightly, and reached what is a nearly static configuration. Although it may be difficult to discern from the images, small regions of the front undergo back-and-forth motions of small range. At much of the front, the two grey phases are in direct contact with each other, indicating that the phase vector is not rotating at the interface; however there are small black and white regions where the front develops some chiral component. We summarize this behaviour by saying the front is essentially Ising-like: it is stationary and the state vector does not rotate.

In the middle column of Fig. 5.1, $\gamma = 0.39$, which lies in the Bloch regime. The extent of oscillatory front motion is much greater, and the interface at the right-hand side of the frame periodically emits “islands” that shrinks and disappear, one of which can be seen in the figure. The interfaces have developed Bloch character nearly everywhere in that dark or light regions are visible. These regions are thicker than in the $\gamma = 0.50$ case. Note that the front motion remains localized and the original domains of the initial conditions are still identifiable; overall, there is still substantial correlation between the initial state at a position and the state at times after the initial transient.

For $\gamma = 0.30$, a time-evolving state of many small irregularly shaped domains is reached. The initial fronts and domains have not persisted; there appears to be no correlation between a position’s initial state and the state at times after the initial transient. The interfaces
have Bloch character and show many spiral defects. In the time evolution of the state there are frequent front reversals and frequent events creating and annihilating pairs of defects.

As was done for the 3:1 case, we can explain these pattern dynamics by proposing that diffusion provides some local averaging of the $\gamma(r)$ field over some length scale and that regions of the medium exhibit local dynamics qualitatively similar to that of a spatially uniform medium with the same local average $\bar{\gamma}(r)$. Thus, at high $\tilde{\gamma}$, where $\tilde{\gamma}$ is deep in the Ising regime, nearly all regions of the front are behaving like Ising regime media and there are comparatively few regions with Bloch-like properties. Hence we find the Ising-like behaviour observed for $\gamma = 0.50$. As $\gamma$ decreases, the fraction of the medium exhibiting Bloch-like local dynamics increases, as does the frequency of occurrence of pacemakers, which are regions of the medium where $\tilde{\gamma}(r)$ is sufficiently low for oscillatory local dynamics to be exhibited.

Another fundamental process that plays a key role in the observed pattern dynamics is the reversal of propagating Bloch-like fronts. In such an event, a portion of the front converts to a Bloch front of the opposite sign, thus reversing direction. Such an event nucleates two spiral defects. We are unable to speculate on the detailed mechanism of front reversals beyond that they arise from the spatial inhomogeneity of the medium; this question may be worth investigating in the future.

With this picture of the underlying dynamics of the medium, we can explain the trends observed in Figure 5.1. Consider an initially planar sharp front. Some portions of the front lie in regions with local dynamics in the Bloch regime and thus develop chirality and begin to move. They move until a combination of the increase in local $\tilde{\gamma}(r)$ and front curvature causes them to stop. Hence a rough front is obtained, with the extent of roughening and
the extent of Bloch character in the front increasing as $\gamma$ decreases (left-hand and middle realizations in Fig. 5.1). For $\gamma$ lower still, a front may propagate sufficiently far in the Bloch-like medium to undergo a front reversal. A portion of the front may exhibit periodic behavior, repeatedly traversing a region with front reversals occurring on either side of the region. The periodic emission of "islands" from the interfaces in the middle realization is a phenomenon arising from behavior of this type. As the Bloch character of the medium increases, the spatial range of this localized behavior increases. As pacemakers become common, the system exhibits behavior like that in Fig. 5.1 (right-hand realization), where there are numerous pacemakers and frequent front reversals, and hence no long-range front-like structure (like those seen in the first two realizations) can persist.

Figure 5.2 shows the evolution from spatially homogeneous initial conditions of a system with $\gamma = 0.36$ containing a single pacemaker. The local dynamics over selected time intervals at the four indicated points are shown in Figs. 5.3 and 5.4. The waves are initiated at the pacemaker at point 1 (top row, left-most frame) and travel outwards, undergoing numerous front reversals and defect creation and annihilation events. The returning waves interact with other waves and with the pacemaker to cause a complex transient state that persists for about 13 500 time units, at which point the system settles into a metastable periodic state. After about 18 500 time units it adopts another apparently periodic state, of period about 67 time units, which persists for the duration of the simulation (until $t = 100\ 000$). The complex transient lasts for approximately 200 periods of the final periodic state. Note that the effects of the pacemaker on the surrounding system remain localized in the vicinity of the pacemaker.

In the final periodic state reached, there appear to be no waves initiated by the pace-
Figure 5.1: Realizations of the 2:1 forced CGL with three different values of $\tilde{\gamma}$. Time evolution is from top to bottom of a column; frames are taken at consecutive intervals of 100 time units. Initial conditions had the system divided into two spatially uniform halves, with sharp interfaces. Left hand column: $p = 0.50$, $\tilde{\gamma} = 0.50$; middle column: $p = 0.61$, $\tilde{\gamma} = 0.39$; right hand column: $p = 0.70$, $\tilde{\gamma} = 0.30$. Other parameters: system size $200 \times 200$, noise grain size $L/200 \times L/200 = 1 \times 1$, periodic boundary conditions. The grey scale is described in the text.
Figure 5.2: Frames from a system with a single pacemaker that exhibits complex transient behavior. The forcing field is a mixture of Ising and oscillatory regions and has $\gamma = 0.36$. The points labelled 1, 2, 3, 4 correspond to the local dynamics plots in Figs. 5.3, 5.4. The top row contains frames taken at four times during the chaotic transient. The bottom row shows frames taken at four equally spaced intervals during the periodic (or nearly periodic) behavior reached at $t > 18500$. After this time period of the system is about 67 time units, the frames are about 17 time units apart. The initial concentration field was spatially homogeneous; the left-most frame in the top row shows the initiation of the waves by a pacemaker.
Figure 5.3: Local dynamics at different points in a realization of the 2:1 forced CGL in a regime with a mixture of oscillatory and Ising regions. The plots show Re $A$ vs. time. From top to bottom, the plots correspond to the points labelled 1,2,3,4 in Fig. 5.2.
Figure 5.4: The local dynamics in the complex phase plane for the four points shown in Fig. 5.2. From top to bottom, the rows correspond to the points marked 1,2,3,4. The left-hand column contains a sample of the dynamics during the complex transient (5000 \( \leq t \leq 8000 \) is shown) and the right-hand column contains the local dynamics after the periodic state has been achieved (20 000 \( \leq t \leq 31 000 \) is shown).
maker. The periodic behavior arises from front reversals and associated defect creation and annihilation events. Presumably if the pacemaker were not affected by incoming wavefronts it would initiate waves, as it did at the start of the simulation. It may be that incoming wavefronts "reset" the pacemaker, or that the emergence of the wave from the pacemaker is not distinct from the passage of an incoming wave. Ideally, one might wish to measure the "intrinsic" period of the pacemaker in the absence of effects from waves returning from the bulk medium, however it is not clear how this could be done, or if one can draw a meaningful demarcation between the pacemaker and its immediate surroundings. That is, we wish to separate the dynamics at the pacemaker into those arising from the pacemaker itself and dynamics due to incoming waves external to the pacemaker that return due to their paths of propagation in the outside system; this may not be possible since it is not clear if a distinct boundary between the pacemaker and the bulk medium exists.

We can compare the phase plane dynamics at some of the points in Fig. 5.4 with the positions of the stable states and Bloch fronts in the mean field system (Fig. 5.5). Such a comparison suggests that the dynamics at point 4 can be interpreted as a point alternating from one quasi-homogeneous state to another as Bloch fronts pass over it.

This example of a realization demonstrates that the coupling of a pacemaker to its surroundings can give rise to an oscillatory system with complex dynamics. When two pacemakers exist such that the regions of the surrounding medium perturbed by their emitted waves overlap, then they are coupled to each other via the interactions of the waves. In the 3:1 inhomogeneously forced CGL, pacemakers are screened from interaction with each other because their outgoing waves annihilate when they collide; there is no way that the annihilation of an emitted wave can have an effect that will propagate back to the
Figure 5.5: The homogeneous stable states of the uniformly forced CGL corresponding to the mean field of Fig. 5.2, $\gamma = 0.36$. The Bloch fronts are shown as curves.

pacemaker. In contrast, this may be possible in the 2:1 case, since emitted waves return to the pacemaker. If they interact with waves from another pacemaker, this may alter the pattern of waves arriving at the pacemaker. Thus, it may be possible for a collection of pacemakers near each other to act as a system of coupled oscillators; in principle this could give rise to a variety of complex behaviors. Additionally, in the 3:1 system the fastest pacemaker (i.e. the one with the shortest period) entrains the entire system since the shock line at which waves annihilate travels in the direction of lower frequency waves. In the 2:1 case there is no reason why a single pacemaker should entrain the entire system. No attempt has been made to pursue these questions; an investigation into the nature of the dynamics arising from the mutual interaction of a small number of pacemakers may be a worthwhile area for future study.
5.3 A mixture of Ising and Bloch regime sites

When the system consists of a mixture of Ising and Bloch regime noise domains, no pacemakers are possible. Thus the only phenomena not seen in spatially uniform systems are front reversals. The two extremes possible in the system are Bloch dynamics and Ising dynamics; front reversals occur in the intermediate regime. Figures 5.6, 5.7 show images from simulations in this intermediate regime. These simulations were conducted with $\gamma_1 = 0.3$, which is in the Bloch regime, and $\gamma_2 = 0.5$, which is in the Ising regime; thus the value of $\gamma$ is controlled by the noise field parameter $p$. In Fig. 5.6, in which $\gamma = 0.38$, front reversals are relatively infrequent in comparison with Fig. 5.7, in which $\gamma = 0.40$, because $\gamma$ is deeper in the Bloch regime. In Fig. 5.6 the fronts are able to travel relatively far without front reversals, hence sufficiently large excursions in the front profile may occur for it to intersect itself and cause a domain splitting event. In contrast, in Fig. 5.7, front reversals are frequent and hence the fluctuations in the front profile are smaller; the front does not develop sufficient curvature to intersect itself and there are no new domains formed. Note also the different time scales on which the systems evolve: the interval between the frames in Fig. 5.6 is one-tenth that in Fig. 5.7, yet there is much less correlation between the system's state in consecutive frames. In Fig. 5.7 the front appears to exhibit slow net motion to the left; we note that we expect fronts in this system to be statistically stationary over long times in a single realizations or over an ensemble of realizations.
Figure 5.6: Four frames taken at intervals of 100 time units from a realization of the 2:1 forced CGL with $\gamma_1 = 0.3$, $\gamma_2 = 0.5$, $p = 0.6$, $\gamma = 0.38$, noise grain size is $1 \times 1$, system size is $200 \times 200$. The first frame is recorded 200 time units after the simulation begins. Initial conditions were a pair of sharp achiral planar fronts, boundary conditions are periodic.

Figure 5.7: Four frames taken at intervals of 1000 time units from a realization of the 2:1 forced CGL with $\gamma_1 = 0.3$, $\gamma_2 = 0.5$, $p = 0.5$, $\gamma = 0.40$, noise grain size is $1 \times 1$, system size is $200 \times 200$. The first frame is recorded 2000 time units after the simulation begins. Initial conditions were a Bloch-like (i.e. chiral) planar front, boundary conditions are periodic on the top and bottom, no-flux on the left and right sides.
6. Systems with time-varying spatial disorder

We now consider situations where the spatial distribution of forcing amplitudes varies in time. For this purpose we examine the behavior of the spatially distributed FitzHugh-Nagumo (FHN) system

\[
\frac{\partial u(r, t)}{\partial t} = u - u^3 - v + D_u \nabla^2 u
\]
\[
\frac{\partial v(r, t)}{\partial t} = \epsilon(u - av + b(r, t)) + D_v \nabla^2 v ,
\]  

(6.1)

subject to such time-varying noise distributions. Here \( u(r, t) \) and \( v(r, t) \) are the “concentrations”, \( a \) and \( \epsilon \) are constant parameters, \( D_u \) and \( D_v \) are the diffusion coefficients and \( b(r, t) \) is a forcing function of the form \( \eta(r, t) \cos \omega t \) with the forcing amplitude \( \eta(r, t) \) a random variable. The spatially uniform case \( \eta(r, t) \equiv \eta_0 \) was considered in Sec. 3.

In these studies the forcing amplitude field \( \eta(r, t) \) was similar to the \( \gamma(r) \) fields used in the quenched disorder studies described earlier in that the system was divided into squares which were randomly assigned one of the two forcing intensities \( \eta_1 \) and \( \eta_2 \). This disordered forcing amplitude field was periodically updated, with the new values of the amplitude in the spatial distribution drawn from the same dichotomous distribution of amplitudes. The updating period was taken to be \( nT \) time units, which is the period of the corresponding \( n : m \) entrained ordinary differential equation. (The investigations described here consider only the case where the updating is on resonance, i.e. where the interval between updates
is \( nT_f \). We shall not consider the case of periodic updating where the updating is off-resonance.

The \( nT_f \)-periodic updating of the forcing amplitude field may have a phase offset relative to the \( T_f \)-periodic forcing \( \cos \omega_f t \). We describe this offset with the parameter \( \sigma \) which ranges from 0 to 1 and specifies the phase offset in units of \( nT_f \), i.e. as a fraction of a period. Thus, updates occur at times

\[
\tau_k = ((k - 1) + \sigma)nT_f \quad \text{for } k = 1, 2, 3 \ldots ,
\]

(6.2)
in addition to the initial specification of the random forcing field at \( \tau_0 = 0 \). To formalize the foregoing: the studies were carried out with \( b(r, t) = \eta(r, t) \cos \omega_f t \), where

\[
\eta(r, t) = \sum_{k=0}^{\infty} \sum_{i=1}^{N_L} \sum_{j=1}^{N_L} \xi_{ij}^k \theta(t - \tau_k) \theta(\tau_{k+1} - t) \Theta_{ij}(r),
\]

(6.3)
where \( \theta \) is the Heaviside function, \( \Theta_{ij}(r) \) is the characteristic function selecting the square with discrete coordinates \((ij)\) as in Eq. (4.4), and

\[
\xi_{ij}^k = \begin{cases} 
\eta_1 \text{ with probability } p \\
\eta_2 \text{ with probability } q = 1 - p.
\end{cases}
\]

(6.4)
The spatial average and time average are equal and given by \( \eta(t) = p\eta_1 + q\eta_2 = \langle \eta(r) \rangle \). The space-time autocorrelation function is

\[
C(r, t) = \frac{\langle \delta \eta(r' + r, t' + t) \delta \eta(r', t') \rangle}{\langle \delta \eta(r', t') \delta \eta(r', t') \rangle} = \begin{cases} 
(1 - \frac{|x|}{s})(1 - \frac{|y|}{s})(1 - \frac{|z|}{nT_f}) & \text{if } |x| \leq s, |y| \leq s \text{ and } |t| \leq nT_f \\
0 & \text{otherwise}
\end{cases}.
\]

(6.5)
The investigations described in this section considered systems at the \( n : m = 3 : 1 \) resonance, using the parameters \( a = 0.3, \epsilon = 0.1, \omega_f/\omega_0 = 3.05, D_u = D_v = 0.25 \) with the
forcing field parameters $\eta_1 = 0$, $\eta_2 = 0.92$, $p = q = 0.50$ and noise grain size $s \times s = 4 \times 4$. The corresponding mean field system, Eq. (6.1) with $b(r, t) = \eta_0 \cos \omega t$, $\eta_0 = \bar{\eta} = 0.46$, lies in the entrained regime and admits three-armed phase locked spiral waves (Fig. 6.1, left panel).

Figure 6.1 (right panel) shows an example of a spiral wave in the FHN system with quenched disorder, analogous to that considered in Secs. 4, 5 for the CGL equation. For this case the $\eta(r, t)$ field may be described within Eqs. (6.3)-(6.4) if we take $\tau_1 = +\infty$. Substantial front roughening is apparent. This system exhibits a phenomenon not seen in the studies in the inhomogenously forced CGL. In the uniformly forced FHN, with $\eta(r, t) \equiv \eta_0$ and other parameters the same, the front velocity passes through zero as $\eta_0$ is varied (cf. Fig. 3.3). Variations in the effective local $\eta$ values, combined with front curvature effects, result in frequent local pinning of the fronts. The fronts may be depinned through coupling to mobile portions of the front, or by the perturbation provided by a following front as it approaches near the pinned front. Thus, the fronts move with an irregular stop-start motion that is controlled by the pinning and depinning events. It seems unlikely that the resulting fronts could obey KPZ scaling; indeed, inspection of Fig. 6.1 suggests the front profile is unlikely to even remain a single-valued function of position. Realizations of spirals in a system with smaller size eventually reached a stationary configuration where fronts were pinned everywhere along their lengths.

For the FHN system with time-varying disorder and the aforementioned parameters, three quasi-homogeneous states were observed, similar to the behavior seen in the CGL with quenched disorder. Unlike in the quenched disorder case, in the presence of time-varying noise the system can not remain in one quasi-homogeneous state for all time; eventually
Figure 6.1: Three-armed spirals in the FHN with time-independent forcing. In each case the spatial average $\bar{\eta} = 0.46$. Left: spatially uniform forcing; $\eta(r) \equiv \eta_0 = 0.46$. Right: the forcing field possesses quenched disorder; $\eta_1 = 0$, $\eta_2 = 2\eta_0 = 0.92$, $p = q = 0.50$, $s \times s = 4 \times 4$. System size is $512 \times 512$; boundary conditions are no-flux. The grey scale indicates $\tan^{-1} v/u$.

The noise must provide a perturbation sufficient to cause the seeding of a droplet of another state which is larger than the critical radius for droplet growth.\(^{30}\)

The existence of noise-update events breaks the symmetry between the different entrained states of the system. Similarly, domain walls are now no longer equivalent and may travel at different velocities. Consider the 3:1 forced system and arbitrarily label the phases 1, 2, and 3. In the following discussion, a [31] front means a domain wall between phases 3 and 1, with phase 3 on the left; its opposite front is [13]. There are three front types: [31], [12], [23] (and their opposites). The velocities of these fronts were measured as function of $\sigma$ (Fig. 6.2). It suffices to measure the velocities for $5/6 < \sigma < 1$, the values for other $\sigma$ follow from the system's symmetry under $t \rightarrow t + T_1$ and $(u, v, t) \rightarrow (-u, -v, t + T_1/2)$. All three fronts move to the left (positive velocity).

Depending on $\sigma$, their velocities rank as $v_{12} > v_{23} > v_{31}$ or as $v_{23} > v_{12} > v_{31}$. We note
that for all \( \sigma, u_{12} > v_{31} \). Thus, if a system has initial conditions consisting of two plane fronts, \([31 \ldots 12]\) (where \ldots represents a region of phase 1), the \([12]\) front will move faster than the \([31]\) front and the distance between the two will decrease. Eventually, the \([12]\) front will closely approach the \([31]\) front and a new stable propagating front consisting of a thin layer of phase 1 connecting phases 3 and 2 will result. We term such a front a compound front and denote it \([312]\). Similarly, for \( \sigma \) values where \( v_{23} > u_{12} > v_{31} \), the compound front \([123]\) exists and can be obtained from the starting configuration \([12 \ldots 23]\).

As one might expect, compound fronts cannot be made from a slower moving front following a faster moving front. For example, \([231]\) is not stable; it splits into \([23 \ldots 31]\) because \( v_{23} > v_{31} \).

The velocities of these compound fronts were measured as a function of \( \sigma \) and, within our numerical accuracy, were found to lie between the velocities of the two simple fronts from which they were derived (i.e. \( u_{12} > v_{312} > v_{31} \) and \( v_{23} > v_{123} > v_{12} \)) (Fig. 6.2). Depending on \( \sigma \) either \( v_{312} > v_{23} \) or \( v_{23} > v_{312} \). For \( v_{312} > v_{23} \) one expects the travelling pulse solution \([2312]\) to be stable, and this is indeed the case, while for \( v_{23} > v_{312} \) the pulse \([2312]\) is unstable (it splits into \([23]\) and \([312]\)) while the pulse \([3123]\) is stable (Fig. 6.3). The pulse velocities were measured, they are essentially the same as the velocity of the faster moving component of the pulse.

The existence of stable pulse solutions joining two domains of the same phase raises the question of whether a "one-armed" spiral whose arms consist of the pulse can exist. Figure 6.4 (left panel) shows a stable spiral for \( \sigma = 11/12 \), a regime where the velocity ordering is \( u_{12} > v_{312} > v_{23} > v_{31} \). It is not a "one-armed" spiral with a \([2312]\) pulse front but could be viewed as a two-armed spiral with arms consisting of phases 3 and 2, with
Figure 6.2: Front velocity versus $\sigma$ in the forced FHN with periodically updated spatial disorder. Lines of the thinnest width indicate simple fronts: [23] (long dashes), [31] (short dashes), [12] (solid line); medium width lines indicate compound fronts: [123] (solid line, exists for $0.99\ldots \leq \sigma \leq 1$), [312] (dashes); thick lines indicate pulses: [3123] (dashes, exists for $0.985\ldots \leq \sigma \leq 1$), [2312] (dotted line, exists for $5/6 \leq \sigma \leq 0.985\ldots$). In addition, for $5/6 \leq \sigma \leq 0.848\ldots$, where $v_{31} > v_{23}$ there should exist a [231] compound front; this front has not been characterized. Velocities were measured in a moving frame in a $200 \times 200$ system; the average front position was measured stroboscopically with period $nT_f$ and the velocity was taken as the slope of the line through these points determined by linear regression.
Figure 6.3: Stability of pulses in various regimes. Realizations were performed with periodic boundary conditions, beginning from either the [3123]-like initial conditions (top left) or [2312-like] initial conditions (top right). For $\sigma = 11/12$ both initial conditions reach the [2312] pulse (bottom left) as the asymptotic state; at $\sigma = 1$ both initial conditions lead to the [3123] pulse (bottom right) as the asymptotic state. The grey scale indicates the $u$ field.
Figure 6.4: Spiral waves in the forced FHN with time-varying disorder with (left) \( \sigma = 11/12 \), (right) \( \sigma = 1 \). The system size is \( 1024 \times 1024 \), the noise grain size is \( s \times s = L/256 \times L/256 \). Boundary conditions are no-flux. The grey scale indicates \( \tan^{-1} v/u \). The phases, as referred to in the text, are 1 (light grey), 2 (dark grey), and 3 (medium grey).

Fronts of type [23] and [312]. Since the [312] front velocity is greater than that of the [23] front, one expects that as the waves travel outward phase 3 will shrink and phase 2 will grow, and far from the core the waves will become a train of [2312] pulses.

Figure 6.4 (right panel) shows a spiral for \( \sigma = 1 \). In this regime the velocity ordering is \( v_{23} > v_{123} > v_{312} > v_{12} > v_{312} > v_{31} \). The stable pulse is [3123] rather than [2312]. Far from the core we expect the waves to become a train of [3123] pulses and in Fig. 6.4 (right panel) this can indeed be seen to happen.

The motion of the spiral core was recorded for a realization of the dynamics with \( \sigma = 11/12 \). The trajectory of the core, \( r_c(t) = (x_c(t), y_c(t)) \), describes a "noisy flower pattern" (Fig. 6.5). Both the periodic looping motion and distortions of the simple flower pattern due to the noise are evident. A plot of \( \langle |r_c|^2 \rangle \) vs. \( t \) shows periodic behavior with period \( \sim 17000 \), which is also the mean period of rotation of the spiral (Fig. 6.6). In the mean field system with \( \eta(r,t) \equiv \bar{\eta} \) the core is stationary. It is also stationary for the
uniformly forced systems with \( \eta(r, t) \equiv \eta_1 \) and \( \eta(r, t) \equiv \eta_2 \), the two extreme values of \( \eta \) in the dichotomous noise process. Consequently, the core motion is a result of the time-varying spatial disorder of the forcing amplitude field.

We have also investigated time-varying noise where updates occurred at Poisson-distributed intervals instead of periodically. The Poisson distribution used was \( \Pr(t \leq \Delta t_{\tilde{r}} \leq t + dt) = (1/\tilde{r})e^{-t/\tilde{r}} dt \) where \( \tilde{r} = \langle \Delta t_{\tilde{r}} \rangle = nT_r \). With this choice of \( \tilde{r} \) the mean time between updates is the same as in the on-resonance periodic updating case discussed previously and corresponds to one period of the entrained system. Thus, if \( \Delta t_{\tilde{r}} \) are chosen from this distribution then updates of the forcing field occur at times

\[
\tau_k = \sum_{\ell=1}^{k} \Delta t_{\tilde{r}}
\]

in addition to the initial specification of the forcing field at \( \tau_0 \). With these definitions of \( \tau_k \) in place of Eq. (6.2), \( \eta(r, t) \) is as given in Eqs. (6.3)-(6.4).

We expect that in this system the three phases will be equivalent on average on time scales longer than the average interval between \( \eta(r) \) field updates. The observed spiral shown in Fig. 6.7 confirms this equivalence. The three arms seen in the figure are approximately equivalent and, when the animation of the dynamics is viewed, this equivalence is preserved in time.
Figure 6.5: Three views of the core trajectory for a realization of the forced FHN dynamics with time-varying spatial disorder in a 512 x 512 system. The updating parameter is $\sigma = 11/12$. Top left: space-time plot; top right: trajectory in space; bottom: $x$ (solid line) and $y$ (dashed line) coordinates against time.
Figure 6.6: $\langle |r_c|^2 \rangle$ versus $t$ for the core trajectories in Fig. 6.5.

Figure 6.7: A spiral wave in the forced FHN system with time-varying spatial disorder in which the interval between updates is chosen from a Poisson distribution. The system size is $1024 \times 1024$. The noise grain size is $s \times s = L/256 \times L/256$. Boundary conditions are no-flux. The grey scale indicates $\tan^{-1} u/u$. 
7. Discussion

We have explored the phenomenology of resonantly forced oscillatory reaction-diffusion systems subject to both quenched and time-varying disorder in the forcing amplitude field. Noteworthy phenomena found when there is quenched disorder near the 3:1 resonance are front roughening and spontaneous nucleation of target patterns. Spontaneous nucleation of target patterns arises because diffusion effectively causes averaging of $\gamma(r)$ locally over some length scale; hence, the medium locally behaves like a uniform system with the same $\gamma$. Alternatively but equivalently, we may describe the dynamics as arising from competition between two regimes selected by the dichotomous forcing values which lie on either side of a bifurcation point. In some spatial regions one regime dominates the dynamics while in other regions the second regime dominates.

In the case of the 2:1 resonance in a system with quenched disorder, the interesting effects observed occurred in the proximity of an Ising-Bloch front transition. The medium adopted Bloch-like or Ising-like character depending on the local average of the forcing amplitude. A noteworthy aspect is that local Bloch regime behavior could be obtained even when the dichotomous forcing was between the oscillatory regime and the Ising regime, because the Bloch regime lies between these. Again, this is consistent with the notion that the diffusion-averaged local forcing amplitude determines the local dynamics of the
medium. The possibility of front transitions between Bloch-like fronts propagating in opposite directions enables more complex front dynamics than was observed in the 3:1 case. Because of these front reversals, the dynamics resulting from a pacemaker are also different from the 3:1 case. A pacemaker in the 2:1 system produces effects that are localized over a finite range, in contrast, in the 3:1 system the waves may propagate to infinity. In the 3:1 system the outgoing waves screen a pacemaker from the external system, while in the 2:1 pacemaker the waves fragment and allow external waves to interact with a pacemaker either indirectly or directly. The coupling of a pacemaker to its immediate surroundings via returning waves allows complicated oscillatory behavior, for example, a complex transient was observed for one case before the system settled into a periodic state.

For time-varying disorder, the symmetry of the system's three states is broken, and thus the velocities of different domain walls differ. As a result, travelling front structures other than simple kink-like fronts may exist. These are the compound fronts and pulses. The velocities of the domain walls depend on the phase of the forcing field updating, therefore the updating parameter \( \sigma \) selects between regimes in which different sets of compound fronts and pulses exist. This asymmetry among states also leads to spiral waves with inequivalent arms.

Meandering core motion with a noisy flower-like trajectory is seen with time-varying disorder. This core motion arises purely from inhomogeneities in \( \eta(r, t) \) since none of the uniformly forced systems with \( \eta(r, t) \equiv \eta_1, \eta_2, \eta_3 \) exhibit core meandering.

There are opportunities for further exploration of spatially disordered resonantly forced systems; for example, the effects of noise on various phenomena or bifurcations known in spatially uniform resonantly forced systems have not been studied. Additionally, one may
investigate oscillatory systems possessing richer limit cycle dynamics than the relatively uncomplicated examples considered in this paper.

Systems of the types described here could be readily realized in an experimental setup. Standard methodology for investigation of spatial disorder in the light-sensitive excitable BZ reaction involves use of a computer-controlled video projector to project a precisely controllable spatiotemporal pattern of illumination intensity onto the reaction medium. Thus, the only necessary changes are the use of the light-sensitive BZ in the oscillatory regime and reprogramming of the projector to provide a periodic illumination signal incorporating appropriate stochastic spatiotemporal modulation of the light intensity.
References


