Hankel Operators and Generalizations

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

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Abstract

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In this thesis, we investigate several properties of bounded Hankel operators in Hilbert space and study some operators which generalize Hankel operators.

We prove that a bounded Toeplitz operator $T$ commutes with a Hankel operator $H_{z^p}$ if and only if $T = T_{z^p}$ and $\varphi$ is an affine function of the characteristic function of an "anti-symmetric" set of the unit circle. We also give a partial classification of algebraic Hankel operators, and some results on the reflexivity and transitivity of ultraweakly closed spaces of Hankel operators.

Given a complex number $\lambda$, we introduce the class of $\lambda$–Hankel operators as those that satisfy the operator equation $S^*X - XS = \lambda X$, where $S$ is the unilateral forward shift. We investigate several properties of $\lambda$–Hankel operators. In particular, we provide a sufficient condition for the compactness of a $\lambda$–Hankel operator and a condition which is necessary for the boundedness of a $\lambda$–Hankel operator. We also show that positivity of a $\lambda$–Hankel operator is equivalent to a moment problem, whose solution gives necessary and sufficient conditions for boundedness of the operator. We also solve some other operator equations that involve the unilateral forward shift.

We prove that certain spaces of non–invertible operators have the property that
every compact subset of the complex plane containing zero is the spectrum of an operator in the space; the space of Hankel operators is of this kind, as is the space of $\lambda$-Hankel operators for $\lambda$ purely imaginary.

To finish, we study a generalization of Hankel operators to the Calkin algebra. We investigate some of the properties of this set, show that it is not an algebra and show that it is not the set consisting of compact perturbations of the set of Hankel operators.
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Contents

1 Preliminaries .......................... 1
   1.1 Hankel operators ................................. 3
   1.2 Toeplitz operators ............................. 6

2 Algebraic properties .................... 9
   2.1 Commutativity ................................... 9
   2.2 Algebraic Hankel operators ........................ 21
   2.3 Reflexivity and transitivity of spaces of Hankel operators .... 26

3 Spectra of Hankel operators ............. 36
   3.1 Questions about spectra of Hankel operators ............... 36
   3.2 Idea of Part I of the proof of Theorem 3.1.1 ............... 39
   3.3 Proof of Theorem 3.1.1, Part I ........................ 41
   3.4 Idea of Part II of the proof of Theorem 3.1.1 ............... 43
   3.5 Some lemmas .................................. 44
   3.6 Proof of Theorem 3.1.1, Part II ..................... 46

4 Generalizations ......................... 51
   4.1 Solutions of some equations involving the shift ............ 52
   4.2 Solutions of some equations involving a parameter ............ 53
   4.3 The operator equation $S^*X - XS = \lambda X$ .................. 56
   4.4 Other properties of $\lambda$-Hankel operators ................ 62
4.5 Positivity of $\lambda$-Hankel operators ........................................... 65
4.6 Boundedness and compactness of $\lambda$-Hankel operators ................. 76

5 Essentially Hankel Operators ............................................................ 85
  5.1 Definitions and Basic Properties .................................................... 85
  5.2 Is ess Hank trivial? ................................................................. 90

Bibliography .................................................................................. 99
Chapter 1

Preliminaries

In this chapter we introduce some of the basic concepts concerning Hankel and Toeplitz operators and some of the notation we will use throughout this thesis.

We will restrict ourselves to infinite dimensional separable Hilbert space, denoted by $\mathcal{H}$. We will usually think of $\mathcal{H}$ as $\ell^2$, the Hilbert space of one-sided square summable complex sequences. We denote the canonical basis of $\ell^2$ by $\{e_n\}_{n=0}^{\infty}$ where, as usual, $e_n$ is the sequence consisting of zeroes, except for the $n$-th place, where its value is one.

We identify $\ell^2$ with the Hardy space $H^2$ of analytic functions in the open unit disk $D$, defined as

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$ 

It is clear that the vector $f = (a_0, a_1, a_2, \cdots) \in \ell^2$ is identified with the analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ and vice versa. We will use this identification freely. For example, we say that $e_n = e_n(z) := z^n$ whenever no confusion is possible.

It is customary to identify the functions in $H^2$ with the space of their boundary functions. It is a non–trivial theorem (see, for example, Hoffman [32]) that if $f \in H^2$, then

$$f(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$$ 

exists almost for all $\theta$, with respect to Lebesgue measure on the unit circle $S^1$. The
boundary functions correspond to those functions in $L^2 = L^2(S^1, dm)$ whose negative Fourier coefficients vanish. Here, $dm$ is the normalized Lebesgue measure on $S^1$. According to this identification, we will then also write $e_n = e_n(e^{i\theta}) := e^{in\theta}$. It is also true that $H^2$ is a closed subspace of $L^2$, and we will denote by $H^2_-$ its orthogonal complement in $L^2$, that is, $H^2_\perp = L^2 \ominus H^2$. The subspace $zH^2$ is the subspace resulting from multiplying every function in $H^2$ by $z$. When a function $f \in L^2$ is also in $H^2$ we say that $f$ is analytic. If $f \in L^2$ is in $L^2 \ominus zH^2$, we say that $f$ is co-analytic.

We will need the space $L^\infty \subset L^2$ of functions on the unit circle which are essentially bounded, and we will denote their norm (the sup norm) by $\| \cdot \|_\infty$. We define $H^\infty := H^2 \cap L^\infty$; this is the space of bounded analytic functions on the unit disk.

We now introduce other notation that we will use frequently: If $f \in L^2$, we define $f^* \in L^2$ as $f^*(z) = \overline{f(z)}$, $\tilde{f} \in L^2$ as $\tilde{f}(z) = f(\overline{z})$ and $\tilde{f} \in L^2$ as $\tilde{f}(z) = \overline{f(z)}$.

One of the most important operators in this thesis is the (unilateral) forward shift $S : \ell^2 \to \ell^2$, defined as

$$S(a_0, a_1, a_2, a_3, \cdots) = (0, a_0, a_1, a_2, \cdots).$$

This operator can be naturally identified with the multiplication operator $M_z : H^2 \to H^2$ defined as $(M_zf)(z) = zf(z)$.

The adjoint of the forward shift is the backward shift, $S^* : \ell^2 \to \ell^2$, defined as

$$S^*(a_0, a_1, a_2, a_3, \cdots) = (a_1, a_2, a_3, a_4, \cdots).$$

A function $u \in H^2$ is called inner if $|u(z)| = 1$ almost everywhere for $z \in S^1$. Inner functions are important because, among other things, they provide a complete characterization of the invariant subspaces of $S$. The classical theorem of Beurling says that $M$ is an invariant subspace for $S$ if and only if $M = uH^2$ for some inner function $u$. For more on inner functions, see Hoffman [32]. For more on the invariant subspaces of $S$, see Radjavi and Rosenthal [53].
1.1 Hankel operators

A Hankel operator $H$ is a linear operator on $\mathcal{H}$ such that

$$(He_m, e_n) = a_{n+m},$$

for some complex sequence $\{a_n\}_{n=0}^\infty$. This means that the matrix, with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$, is constant along each diagonal perpendicular to the main one.

It is very easy to see that an operator $H$ is Hankel if and only if $H$ satisfies the operator equation $S^*H = HS$. Although we will mainly talk about bounded Hankel operators, it is possible for a densely defined operator to satisfy the operator equation, and we will also refer to these operators as Hankel operators. As is customary, we will omit the word "bounded" from "bounded operator" whenever no confusion is possible.

The basic theorem in the theory of Hankel operators is the classical theorem of Nehari [42]. In the following, $P$ is the orthogonal projection from $L^2$ to $H^2$, $J : L^2 \rightarrow L^2$ is the (self-adjoint) flip operator, $Jf = \bar{f}$ and $M_\varphi : L^2 \rightarrow L^2$ is the multiplication operator, defined as $(M_\varphi f)(z) = \varphi(z)f(z)$ for every $f \in L^2$.

**Theorem 1.1.1 (Nehari’s Theorem)** A Hankel operator $H$ on $H^2$ is bounded if and only if there exists a function $\varphi \in L^\infty$ such that

$$H = PJM_\varphi.$$  

In this case, it is possible to choose $\varphi$ in such a way that $\|H\| = \|\varphi\|_\infty$.

For a modern proof, the reader is referred to [50].

In other words, what Nehari’s theorem says is that the Hankel operator $H$ with matrix $(a_{n+m})_{n,m=0}^\infty$ is bounded if and only if there exists a function $\varphi \in L^\infty$ such that $a_n$ is the $(-n)$-th Fourier coefficient of $\varphi$, for $n \geq 0$. 
CHAPTER 1. PRELIMINARIES

We will use the following fact (a straightforward calculation) frequently. If \( \varphi \in L^\infty \) and \( H = PJM_\varphi \), then, for \( g, h \in H^2 \),

\[
(Hg, h) = (PJM_\varphi g, h) = (\varphi g, Jh) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta})g(e^{i\theta})h^*(e^{i\theta}) \, d\theta.
\]

We refer to the function \( \varphi \) given by Nehari's theorem as a symbol of \( H \) and we write \( H = H_\varphi \). It is clear from the expression given by Nehari's Theorem that a Hankel operator is linear with respect to its symbol; i.e., for any \( \varphi \) and \( \psi \in L^\infty \) and complex numbers \( a, b \in \mathbb{C} \)

\[
H_{a\varphi + b\psi} = aH_\varphi + bH_\psi.
\]

Notice that if \( \psi \) is a function in \( zH^2 \), then \( PJM_\psi = 0 \) on \( H^2 \), so the Hankel operator \( H_\psi = 0 \). It is because of this that the symbol of a Hankel operator is not unique. Namely, \( H_\varphi = H_\psi \) if and only if \( \varphi - \psi \in zH^2 \).

In fact, we can see that the Banach space \( L^\infty/(zH^\infty) \) (with the quotient topology) is isometrically isomorphic to the Banach space of Hankel operators (with the norm topology), as stated in the following theorem.

**Theorem 1.1.2 ([49])** For \( \varphi + zH^\infty \in L^\infty/(zH^\infty) \), the symbol map \( \xi : L^\infty/(zH^\infty) \to B(H) \), defined by

\[
\xi(\varphi + zH^\infty) = H_\varphi,
\]

establishes a linear isometry between \( L^\infty/(zH^\infty) \) and the Banach space of all Hankel operators.

We will usually write \( \xi(\varphi) = H_\varphi \) instead of the more cumbersome \( \xi(\varphi + zH^\infty) = H_\varphi \) whenever no confusion is possible.

**Definition 1.1.3** Given a set \( A \) of Hankel operators we define the set of symbols of \( A \) as

\[
\text{Sym } A = \{ \varphi \in L^\infty : H_\varphi \in A \}.
\]
It is clear that this set is a subset of $L^\infty$ which is in fact a subspace whenever $A$ is a vector space of Hankel operators. In this case, it will always contain $zH^\infty$.

There is also a complete characterization of compact Hankel operators due to Hartman [29]. Here $C \subset L^\infty$ is the subalgebra of continuous functions on $S^1$ and $H^\infty + C$ is the sum of functions in $H^\infty$ and in $C$. It is a deep theorem of Sarason (see, for example, [26, p. 376]) that $H^\infty + C$ is a closed subalgebra of $L^\infty$.

**Theorem 1.1.4 (Hartman's Theorem)** A Hankel operator $H$ on $H^2$ is compact if and only if there exists a function $\varphi \in H^\infty + C$ such that $H = H_\varphi$.

For a modern proof, the reader is again referred to [50].

A characterization of those Hankel operators of finite–rank is also known. This remarkable theorem is generally attributed to Kronecker (see Power's book [50] for a reference). A simple proof can be found in [44].

**Theorem 1.1.5 (Kronecker's Theorem)** Let the Hankel operator $H$ have matrix $(a_{n+m})_{n,m=0}^\infty$. Then $H$ is a finite–rank matrix if and only if the function $\frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \cdots$ is rational. Furthermore, $H$ is bounded if the poles of $\frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \cdots$ are contained in the open unit disk $D$.

To end this section, we will state some miscellaneous simple facts about Hankel operators. The proofs can be found in Power's book [50].

**Theorem 1.1.6** Let $H$ be a bounded Hankel operator. Then

- $H$ is never invertible. In fact, $0 \in \sigma_e(H)$.
- $H$ is one to one if and only if $H$ has dense range.
- $\text{Ker} \, H$ is an invariant subspace of $S$. The closure of $\text{Ran} \, H$ is an invariant subspace of $S^*$.
• If \( H \) is not one to one, then \( H = H_{\sigma H} \), where \( u, h \in H^\infty \), \( u \) is an inner function and \( u \) and \( h \) have no common inner factors. In this case, \( \text{Ker} \, H = uH^2 \) and \( \overline{\text{Ran} \, H} = (u^*H^2)^\perp \).

• A Hankel operator \( H \) is a partial isometry if and only if it is of the form \( H = H_{\sigma H} \) for some inner function \( u \). The initial and final spaces are \( (uH^2)^\perp \) and \( (u^*H^2)^\perp \) respectively.

• If \( H = H_\phi \) then \( H^* = H_{\phi^*} \).

1.2 Toeplitz operators

A Toeplitz operator \( T \) is a linear operator on \( \mathcal{H} \) such that

\[
(Te_m, e_n) = b_{n-m}
\]

for some complex sequence \( \{b_n\}_{n=-\infty}^{\infty} \). This means that the matrix, with respect to the orthonormal basis \( \{e_n\}_{n=0}^{\infty} \), is constant along each diagonal parallel to the main one.

It is easy to see that an operator \( T \) is Toeplitz if and only if it satisfies the operator equation \( S^*TS = T \). If an operator satisfies the operator equation \( ST = TS \), \( T \) is called an analytic Toeplitz operator. If \( T \) satisfies the equation \( S^*T = TS^* \), \( T \) is called a co-analytic Toeplitz operator.

Hartman and Wintner [30] proved that the Toeplitz operator \( T \) with matrix \( (b_{n-m})_{n,m=0}^{\infty} \) is bounded if and only if there exists a function \( \psi \in L^\infty \) such that, for all \( n \), \( b_n \) is the \( n \)-th Fourier coefficient of \( \psi \). This can also be expressed in the following form.

**Theorem 1.2.1** A Toeplitz operator \( T \) is bounded if and only if there exists a function \( \psi \in L^\infty \) such that

\[
T = PM\psi.
\]

In this case, \( \|T\| = \|\psi\|_\infty \).
A proof of this fact can be found in [7]. We say that \( \psi \) is the symbol of the operator \( T \) and we write \( T = T_\psi \). Just as in the case of Hankel operators, a Toeplitz operator is linear with respect to its symbol. It is also clear that the symbol of a Toeplitz operator is unique. It is easy to check that \( T_\psi^* = T_\bar{\psi} \). It is not hard to see that a Toeplitz operator is analytic if and only if its symbol is analytic, and that a Toeplitz operator is co–analytic if and only if its symbol is co–analytic.

An important fact is that the forward shift \( S \) is just the analytic Toeplitz operator with symbol \( \psi(z) = z \) and the backward shift is the co–analytic Toeplitz operator with symbol \( \psi(z) = \bar{z} \).

Toeplitz operators have been widely studied. For more on their properties the reader is referred to [7, 19, 28, 55]. We only state here the properties of Toeplitz operators that are of interest for our purposes, namely, their relations with Hankel operators.

There are two very important formulas that we will use throughout this thesis; both can be found in Power [49]. For \( f \) and \( g \in L^\infty \) we have
\[
H_z f H_z g = T_f g - T_f T_g, \tag{1.1}
\]
and
\[
T_f H_g + H_z f T_z g = H_f g. \tag{1.2}
\]

We finish this chapter with some more notation. Given two vectors \( f \) and \( g \in \mathcal{H} \), we define the rank–one operator \( f \otimes g \) as
\[
(f \otimes g) h = (h, g) f, \quad \text{for all } h \in \mathcal{H}.
\]
Throughout this thesis we will use the following easy–to–check properties of this operator:

- \( A(f \otimes g)B = (Af) \otimes (B^*g) \),
- \( (f \otimes g)^* = g \otimes f \),
CHAPTER 1. PRELIMINARIES

\begin{itemize}
  \item $\|f \otimes g\| = \|f\| \|g\|$, and
  \item the trace of $f \otimes g$ is $(f, g)$.
\end{itemize}

If $b \in \mathbb{C}$, and $|b| < 1$, we can define $k_b \in \ell^2$ or $\in \mathbb{H}^2$ as

$$k_b = \sum_{n=0}^{\infty} b^n e_n, \quad \text{or} \quad k_b(z) = \frac{1}{1 - \overline{b}z}.$$  

This is usually referred to as the \textit{reproducing kernel} for $\mathbb{H}^2$ since it is easy to check that, for $f \in \mathbb{H}^2$, $(f, k_b) = f(b)$. It is easy to see that

$$\|k_b\| = \frac{1}{\sqrt{1 - |b|^2}}.$$  

The property of the reproducing kernel which is of interest for us is

$$S^* k_b = \overline{b} k_b.$$  

We denote by $\mathcal{K}$ the ideal of compact operators on $\mathcal{H}$. If $A$ and $B$ are two bounded operators, we write $A = B \pmod{\mathcal{K}}$ whenever $A - B \in \mathcal{K}$, and $\|\cdot\|_e$ for the essential norm; i.e., if $A \in \mathcal{B}(\mathcal{H})$ then

$$\|A\|_e = \text{dist}(A, \mathcal{K}).$$
Chapter 2

Algebraic properties of Hankel operators

In this chapter we consider some properties of Hankel operators, including a full classification of when they commute with Toeplitz operators. We also give a partial description of the Hankel operators which are algebraic. Finally, we include some results on transitivity and reflexivity of spaces of Hankel operators. All operators in this chapter are assumed to be bounded.

2.1 Commutativity

It is known that two Hankel operators commute if and only if one is a multiple of the other [49]. This implies, for example, that the only normal Hankel operators are multiples of self-adjoint ones. (More is known: hyponormal Hankel operators are multiples of self-adjoint ones [49]).

There are some partial results classifying the composition operators that commute with Hankel operators [12]. In particular, it is known that only the trivial Hankel operator $0$ and the Hankel operator $e_0 \otimes e_0$ may commute with composition operators which are not elliptic disk automorphisms of finite periodicity. There are examples of non-trivial Hankel operators that commute with certain composition operators (which are elliptic disk automorphisms of finite periodicity: see [12]).
Besides these facts, not much is known about the commutant of a single Hankel operator. In this section we investigate the question of when a Hankel operator and a Toeplitz operator commute. That is, we will completely answer the question of when the commutant of a Hankel operator contains a Toeplitz operator. It turns out that a Hankel and a Toeplitz operator will commute if and only if their symbols are affine functions of characteristic functions of certain "anti-symmetric" sets of the unit circle.

The "if" part of the following lemma is common knowledge among people who study Hankel operators and the proof is really easy. Unfortunately, we have not been able to find a reference for the "only if" part (although it is also undoubtedly known). We include the proof of both parts here. We would like to point out that, although there exist simpler proofs of the "if" part, the one we present here has the same flavour as the "only if" part.

**Lemma 2.1.1** Let $H$ and $T_g$ be a non-zero Hankel operator and a non-zero Toeplitz operator respectively. Then $T_g^*H = HT_g^*$ if and only if $g$ is analytic.

Notice that when $g(z) = z$, the condition $T_g^*H = HT_g^*$ is equivalent to the alternative definition of a Hankel operator: namely, $H$ is Hankel if and only if $S^*H = HS$.

**Proof:** Let $g$ have Fourier coefficients $a_k = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta})e^{-ik\theta} d\theta$ for $k \in \mathbb{Z}$. This means that $T_g$ has matrix

$$(a_{n-m})_{n,m=0}^\infty.$$

Let $H = H_f$, where $f$ has Fourier coefficients $b_{-k} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})e^{-ik\theta} d\theta$ for $k \in \mathbb{Z}$ (the minus sign on $b_{-k}$ will save us from many more later on); that is, $H$ has matrix

$$(b_{n+m})_{n,m=0}^\infty.$$

A straightforward calculation (just remember that, for an operator $C$, $Ce_m$ is the
CHAPTER 2. ALGEBRAIC PROPERTIES

$m$-th column of the matrix of $C$ with respect to the basis $\{e_n\}_{n=0}^{\infty}$ shows that

$$
(T^*_g H f, e_n, e_m) = (H f, e_n, T^*_g e_m) = \sum_{k=0}^{\infty} b_{k+n} \bar{a}_{k-m}
$$

(2.1)

and that

$$
(H f, T^*_g e_n, e_m) = (T^*_g e_n, H f, e_m) = \sum_{k=0}^{\infty} \bar{a}_{k-n} b_{k+m},
$$

(2.2)

for $m \geq 0$ and $n \geq 0$.

Let us suppose first that $g$ is analytic. Then, since $a_k = 0$ if $k < 0$, equation 2.1 becomes

$$
(T^*_g H f, e_n, e_m) = \sum_{k=0}^{\infty} \bar{a}_{k-m} b_{k+n} = \sum_{s=0}^{\infty} \bar{a}_s b_{m+n+s}
$$

and equation 2.2 becomes

$$
(H f, T^*_g e_n, e_m) = \sum_{k=0}^{\infty} \bar{a}_{k-n} b_{k+m} = \sum_{s=0}^{\infty} \bar{a}_s b_{m+n+s}.
$$

Since the right-hand sides of both equations are equal, $T^*_g H = H T^*_g$.

Conversely, assume $T^*_g H = H T^*_g$ and rewrite equation 2.1 as

$$
(T^*_g H f, e_n, e_m) = \sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} + \sum_{k=m}^{\infty} \bar{a}_{k-m} b_{k+n}
$$

(2.3)

and equation 2.2 as

$$
(H f, T^*_g e_n, e_m) = \sum_{k=0}^{n-1} \bar{a}_{k-n} b_{k+m} + \sum_{k=n}^{\infty} \bar{a}_{k-n} b_{k+m},
$$

(2.4)

where the first term of the right-hand side of each of the equations is just thought of as zero if $m = 0$ or $n = 0$.

A change of variables as before shows that both second summands in the right-hand sides of equations 2.3 and 2.4 are equal, and thus

$$
\sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} = \sum_{k=0}^{n-1} \bar{a}_{k-n} b_{k+m}
$$

(2.5)
CHAPTER 2. ALGEBRAIC PROPERTIES

when both \( m, n > 0 \). If \( n = 0 \) and \( m > 0 \) we obtain

\[
\sum_{k=0}^{m-1} \bar{a}_{k-m} b_k = 0. \tag{2.6}
\]

We will assume for the rest of this proof that \( m > n \). The left-hand side of equation 2.5 can then be written as

\[
\sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} = \sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} + \sum_{k=m-n}^{m-1} \bar{a}_{k-m} b_{k+n}
\]

\[
= \sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} + \sum_{s=0}^{n-1} \bar{a}_{s-n} b_{s+m},
\]

but the last summand is equal to the right-hand side of equation 2.5. It follows that

\[
\sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} = 0, \quad \text{for } m > n > 0.
\]

This last equation is also valid for \( n = 0 \), since it then reduces to equation 2.6. Thus we have

\[
\sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} = 0, \quad \text{for } m > n \geq 0. \tag{2.7}
\]

Now, clearly there must exist a non-negative integer \( n_0 \) such that \( b_{n_0} \neq 0 \) (otherwise \( H \) would be the zero operator).

We use equation 2.7 with \( n = n_0 \); that is,

\[
\sum_{k=0}^{m-n_0-1} \bar{a}_{k-m} b_{k+n_0} = 0, \quad \text{for } m > n_0. \tag{2.8}
\]

We use strong induction to prove that \( a_{-(n_0+s)} = 0 \) for all \( s > 0 \). If \( m = n_0 + 1 \) in equation 2.8 we obtain \( \bar{a}_{-(n_0+1)} b_{n_0} = 0 \), which in turn implies \( a_{-(n_0+1)} = 0 \) (recall again that \( b_{n_0} \neq 0 \)). Now assume that \( a_{-(n_0+1)} = a_{-(n_0+2)} = \ldots = a_{-(n_0+s)} = 0 \). Then equation 2.8 with \( m = n_0 + s + 1 \) becomes \( \bar{a}_{-(n_0+s+1)} b_{n_0} = 0 \), which implies \( a_{-(n_0+s+1)} = 0 \). Thus \( a_{-(n_0+s)} = 0 \) for \( s > 0 \).
Now go back to equation 2.7. If we set $m = n_0 + 1$ we get

$$
\sum_{k=0}^{n_0-1} \tilde{a}_{k-n_0-1} b_{k+n} = \sum_{k=1}^{n_0-n} \tilde{a}_{k-(n_0+1)} b_{k+n} = 0, \quad \text{for } n_0 + 1 > n > 0,
$$

(2.9)

since $a_{-(n_0+1)} = 0$ (as proven in the previous paragraph). If we set $n = n_0 - 1$ in equation 2.9 we get $a_{-n_0} b_{n_0} = 0$ and thus $a_{-n_0} = 0$. Proceeding in this fashion (set $n = n_0 - 2$, $n = n_0 - 3$, $\cdots$, $n = n_0 - (n_0 - 1)$, $n = n_0 - n_0$ in equation 2.9) we get $a_{-n_0} = a_{-(n_0-1)} = \cdots = a_{-2} = a_{-1} = 0$.

Therefore $a_{-s} = 0$ for all $s > 0$; i.e., $g$ is analytic. \qed

Now we can prove a result concerning commutativity. What this theorem says is that if a non-zero Hankel operator commutes with a "symmetric" (equal to its transpose) Toeplitz operator, then the Toeplitz operator is just a multiple of the identity operator.

**Theorem 2.1.2** Let $\varphi \in L^\infty$. Suppose that $\varphi = \tilde{\varphi}$ and that $HT_\varphi = T_\varphi H$ for a non-zero Hankel operator $H$. Then $\varphi$ is a constant function (i.e., $T_\varphi$ is a constant multiple of the identity).

**Proof:** Since $\varphi = \tilde{\varphi}$, it follows that $(\tilde{\varphi})^* = \varphi$. Also, as noted before, $T_\varphi = T_\tilde{\varphi}^*$. Putting these two facts together, we see that $T_\varphi H = HT_\varphi$ is equivalent to $T_\tilde{\varphi}^* H = HT_{(\tilde{\varphi})^*}$. The latter equation implies, by Lemma 2.1.1, that $\tilde{\varphi} \in H^\infty$, and, since $\varphi = \tilde{\varphi}$, it follows that $\varphi$ is constant. \qed

We now prove the following theorem. What it says is that if a Hankel operator commutes with a Toeplitz operator, it must also commute with the "transpose" of the Toeplitz operator, since Hankel operators are "symmetric" (equal to their transpose).

**Theorem 2.1.3** Let $f \in L^\infty$. Then $T_f H = HT_f$ if and only if $T_{f^*} H = HT_f$.

\[1\] I would like to thank an unknown referee for considerably shortening the proof of this theorem and of the next one.
Proof: Define the anti-unitary involution $V$ on $H^2$ by $Vf = f^*$. It is easy to check that $V T_f V = T_{f^*}$, for $f \in L^\infty$, and $VHV = H^*$ for any Hankel operator $H$. Clearly, $V^2 = I$.

Thus $T_f H = HT_f$ implies that $V T_f VVHV = VHVVT_fV$, which in turn implies that $T_{f^*}H^* = H^*T_{f^*}$. Taking adjoints we get $HT_f = T_f H$.

Applying the previous calculation to $\tilde{f}$, it follows that $T_f H = HT_f$ implies $T_f H = HT_f$.

The last two theorems allow us to get the following corollary. This is the first necessary condition for a Hankel and a Toeplitz operator to commute.

**Corollary 2.1.4** If $T_f H = HT_f$ for some function $f \in L^\infty$ and a non-zero Hankel operator $H$, then $f + \tilde{f}$ is a constant function.

**Proof:** If $T_f H = HT_f$ then, by Theorem 2.1.3, $T_f H = HT_f$. Both equations imply that $T_{f + \tilde{f}} H = HT_{f + \tilde{f}}$ (remember that a Toeplitz operator is linear with respect to its symbol). Using Theorem 2.1.2 (since if $\varphi = f + \tilde{f}$ then $\varphi = \tilde{\varphi}$) it follows that $f + \tilde{f}$ is a constant function.

The following theorem will provide some necessary conditions for commutativity between a Toeplitz operator and a Hankel operator. This will lead to a full classification. We would like to point out that the proof of this theorem is inspired by the classical paper of Brown and Halmos [7].

**Theorem 2.1.5** Let $T_\varphi$ and $H$ be a Toeplitz operator which is not a multiple of the identity operator and a non-zero Hankel operator respectively. If $T_\varphi H = H T_\varphi$ then $H = H_{\mu \varphi}$, where $\mu$ is a complex number.

**Proof:** Let $\varphi$ have Fourier coefficients $a_k = \int_0^{2\pi} \varphi(e^{i\theta}) e^{-i k \theta} d\theta$, for $k \in \mathbb{Z}$; that is, $T_\varphi$ has matrix

$$ (a_{n-m})_{n,m=0}^\infty. $$
Let $H = H_\psi$, where $\psi$ has Fourier coefficients $b_k = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta}) e^{-ik\theta} d\theta$, for $k \in \mathbb{Z}$; that is, $H_\psi$ has matrix

$$(b_{-(n+m)})_{n,m=0}^\infty.$$

For $m \geq 0$ and $n \geq 0$ we define

$$C_{n,m} := (H_\psi T_\psi e_m, e_n) = (T_\psi e_m, H_\psi^* e_n)$$

$$= \sum_{k=0}^{\infty} a_{k-m} b_{-(k+n)}$$

and

$$D_{n,m} := (T_\psi H_\psi e_m, e_n) = (H_\psi e_m, T_\psi^* e_n)$$

$$= \sum_{k=0}^{\infty} b_{-(k+m)} a_{n-k}.$$

For $m > 0$ and $n \geq 0$ we obtain

$$C_{n,m} = \sum_{k=0}^{\infty} a_{k-m} b_{-(k+n)}$$

$$= a_{-m} b_{-n} + \sum_{k=1}^{\infty} a_{k-m} b_{-(k+n)}$$

$$= a_{-m} b_{-n} + \sum_{s=0}^{\infty} a_{s-(m-1)} b_{-(s+n+1)}$$

$$= a_{-m} b_{-n} + C_{n+1,m-1}$$

and

$$D_{n+1,m-1} = \sum_{k=0}^{\infty} b_{-(k+m-1)} a_{n+1-k}$$

$$= a_{n+1} b_{-(m-1)} + \sum_{k=1}^{\infty} b_{-(k+m-1)} a_{n+1-k}$$

$$= a_{n+1} b_{-(m-1)} + \sum_{s=0}^{\infty} b_{-(s+m)} a_{n-s}.$$
= a_{n+1} b_{-(m-1)} + D_{n,m}.

Since $H_\psi T_\varphi = T_\varphi H_\psi$, it follows that

\[
C_{n,m} = a_m b_{-n} + C_{n+1,m-1} \\
= a_m b_{-n} + D_{n+1,m-1} \\
= a_m b_{-n} + a_{n+1} b_{-(m-1)} + D_{n,m} \\
= a_m b_{-n} + a_{n+1} b_{-(m-1)} + C_{n,m},
\]

that is,

\[-a_m b_{-n} = a_{n+1} b_{-m+1}, \quad \text{for } m > 0 \text{ and } n \geq 0.\]

By Corollary 2.1.4, $\varphi + \tilde{\varphi}$ is constant, and thus $a_m + a_{-m} = 0$ if $m > 0$. Thus the previous equation becomes

\[a_m b_{-n} = a_{n+1} b_{-m+1}, \quad \text{for } m > 0 \text{ and } n \geq 0. \quad (2.10)\]

First, we prove that, for $n \geq 0$, $a_{n+1} = 0$ if and only if $b_{-n} = 0$. If there exists $n_0 \geq 0$ such that $b_{-n_0} = 0$, then equation 2.10 implies that $a_{n_0+1} b_{-m+1} = 0$ for $m > 0$, so either $a_{n_0+1} = 0$ or $b_{-m+1} = 0$ for all $m > 0$. But the latter would imply that $H = 0$, contradicting our hypothesis, so it follows that $a_{n_0+1} = 0$.

Similarly, if there exists $n_0 \geq 0$ such that $a_{n_0+1} = 0$, equation 2.10 implies that $a_m b_{-n_0} = 0$ for $m > 0$, so either $a_m = 0$ for all $m > 0$ or $b_{-n_0} = 0$. But the former would imply that $\varphi$ is a constant (recall that $\varphi + \tilde{\varphi}$ is a constant); i.e., $T_\varphi$ would be a multiple of the identity, contradicting our hypothesis. It follows that $b_{-n_0} = 0$. Thus we have proven that, for $n \geq 0$, $a_{n+1} = 0$ if and only if $b_{-n} = 0$.

Also, there must exist an integer $n_0 \geq 0$ such that $b_{-n_0} \neq 0$ (otherwise $b_{-n} = 0$ for all $n \geq 0$ which would imply, as before, that $H$ is the zero operator). If we let $\lambda = \frac{b_{-n_0}}{a_{n_0+1}} \neq 0$ (which is well defined, by the previous two paragraphs) equation 2.10 implies that $\lambda a_m = b_{-m+1}$ for $m > 0$, or $\mu a_{-m} = b_{-m+1}$ for $m > 0$ and $\mu = -\lambda$. 
CHAPTER 2. ALGEBRAIC PROPERTIES

But $\mu a_{-m} = b_{-m+1}$ for $m > 0$ means that $\mu z \varphi - \psi \in zH^2$, or equivalently, $H = H_\psi = H_{\mu z \varphi}$. This is what we wanted.

The conclusion of this theorem allows us to suppose, without loss of generality, that $H = H_{z \varphi}$ whenever $HT_\varphi = T_\varphi H$. This will allow us to obtain the following necessary condition.

**Theorem 2.1.6** Let $\varphi \in L^\infty$. If the Hankel operator $H = H_{z \varphi}$ commutes with the Toeplitz operator $T_\varphi$ then $\varphi \bar{\varphi}$ is a constant function.

**Proof:** Since $H_{z \varphi} T_\varphi = T_\varphi H_{z \varphi}$, it follows by Theorem 2.1.3 that $H_{z \varphi} T_\varphi = T_\varphi H_{z \varphi}$.

From equation 1.1 in Chapter 1 we get

$$H_{z \varphi}^2 = T_{\varphi \bar{\varphi}} - T_{\varphi \varphi}.$$

Since clearly, $H_{z \varphi}$ commutes with the left hand side of this expression and also commutes with the second term in the right-hand side (because it commutes with both $T_\varphi$ and $T_{\varphi \bar{\varphi}}$), it must also commute with $T_{\varphi \varphi}$. But since $f = \varphi \bar{\varphi}$ implies that $f = \bar{f}$, it follows from Theorem 2.1.2 that $f = \varphi \bar{\varphi}$ is a constant function.

We can summarize the above results as follows.

**Corollary 2.1.7** If $H$ is a non-zero Hankel operator such that $HT_\varphi = T_\varphi H$ for a Toeplitz operator $T_\varphi$ not a multiple of the identity, then $H = \mu H_{z \varphi}$ for some $\mu \in \mathbb{C}$. In this case, $\varphi + \bar{\varphi}$ and $\varphi \bar{\varphi}$ are constant functions.

Of course, this last corollary does not imply that there are any non-trivial Toeplitz operators which commute with a Hankel operator. But it turns out that it does provide good candidates for sufficient conditions.

**Theorem 2.1.8** Let $\varphi \in L^\infty$ be such that $\varphi + \bar{\varphi}$ and $\varphi \bar{\varphi}$ are constant functions. Then $H_{z \varphi} T_\varphi = T_\varphi H_{z \varphi}$. 
Proof: First of all, suppose that \( \varphi + \bar{\varphi} = c \) and \( \varphi \bar{\varphi} = d \) for some complex numbers \( c \) and \( d \). From equation 1.2 in Chapter 1 we get

\[ T_\varphi H_{z\varphi} + H_{z\bar{\varphi}} T_\varphi = H_{z\varphi \bar{\varphi}}. \]

But \( \varphi \bar{\varphi} = d \), so

\[ T_\varphi H_{z\varphi} + H_{z\bar{\varphi}} T_\varphi = H_{z\varphi \bar{\varphi}} = H_{dz} = 0, \]

since \( dz \in zH^\infty \). Using \( \varphi + \bar{\varphi} = c \) we obtain

\[ T_\varphi H_{z\varphi} + H_{z(c-\varphi)} T_\varphi = 0. \]

But \( H_{dz} = 0 \) as above. This means that

\[ T_\varphi H_{z\varphi} - H_{z\bar{\varphi}} T_\varphi = 0, \]

as desired. \( \square \)

What is so special about functions in \( L^\infty \) with the properties in the hypothesis of the last theorem? In fact, are there any non-trivial ones? The following lemma classifies them, and shows that, in fact, there are plenty of them.

First, we introduce some notation. If \( E \subset S^1 \) is a measurable set, we denote by \( \chi_E \) the characteristic function of \( E \). The conjugate of the set \( E \) is \( E^* = \{ z \in S^1 : \bar{z} \in E \} \) and the complement of \( E \) is \( E^c = \{ z \in S^1 : z \notin E \} \). Also, the symmetric difference of two sets \( A \) and \( B \) is \( A \triangle B = (A \setminus B) \cup (B \setminus A) \). In the following, \( m \) denotes normalized Lebesgue measure on \( S^1 \).

**Lemma 2.1.9** Let \( \varphi \in L^2 \). Then \( \varphi + \bar{\varphi} \) and \( \varphi \bar{\varphi} \) are constant functions if and only if \( \varphi(z) = a \chi_E(z) + b \), where \( a \) and \( b \in \mathbb{C} \) and \( E \) is a measurable subset of \( S^1 \) such that \( m(E^* \triangle E^c) = 0 \).

**Proof:** First the "if" part. If \( \varphi(z) = a \chi_E(z) + b \) as in the statement of the lemma, then \( \bar{\varphi}(z) = a \chi_{E^*}(z) + b \) and

\[ \varphi(z) + \bar{\varphi}(z) = a(\chi_E(z) + \chi_{E^*}(z)) + 2b. \]
because $E^*$ and $E^c$ coincide up to a set of measure 0. Similarly

$$
\phi(z)\bar{\phi}(z) = (a\chi_E(z) + b)(a\chi_{E^*}(z) + b)
= a^2\chi_E(z)\chi_{E^*}(z) + ab(\chi_E(z) + \chi_{E^*}(z)) + b^2
= a^2\chi_{E\cap E^*}(z) + ab + b^2
= a^20 + ab + b^2,
$$

so both $\phi + \bar{\phi}$ and $\phi\bar{\phi}$ are constant functions.

Conversely, suppose that $\phi + \bar{\phi}$ and $\phi\bar{\phi}$ are constant. We have two cases:

Case i) Suppose $\phi + \bar{\phi} = 1$ and $\phi\bar{\phi} = 0$. Then

$$
\phi = \phi(\phi + \bar{\phi}) = \phi^2 + \phi\bar{\phi} = \phi^2,
$$

which implies that $\phi = \chi_E$ for some measurable subset $E \subset S^1$. Since

$$
0 = \phi(z)\bar{\phi}(z) = \chi_E(z)\chi_{E^*}(z) = \chi_{E\cap E^*}(z),
$$

it follows that $m(E \cap E^*) = 0$. Analogously,

$$
1 = \phi(z) + \bar{\phi}(z) = \chi_E(z) + \chi_{E^*}(z) = \chi_{E\cup E^*}(z) - \chi_{E\cap E^*}(z) = \chi_{E\cup E^*}(z)
$$

so $m(E \cup E^*) = 1$. Thus $m(E^* \Delta E^c) = 0$ and the conclusion follows with $a = 1$ and $b = 0$.

Case ii) Suppose $\phi + \bar{\phi} = c$ and $\phi\bar{\phi} = d$. Let $b$ satisfy the equation $b^2 - cb + d = 0$ and let $a = c - 2b$.

If $a \neq 0$, then define $\psi(z) = \frac{\phi(z) - b}{a}$. Then

$$
\psi(z) + \bar{\psi}(z) = \frac{\phi(z) - b}{a} + \frac{\bar{\phi}(z) - b}{a} = \frac{c - 2b}{a} = \frac{a}{a} = 1
$$
CHAPTER 2. ALGEBRAIC PROPERTIES

and

\[ \psi(z)\bar{\psi}(z) = \left( \frac{\varphi(z) - b}{a} \right) \left( \frac{\bar{\varphi}(z) - b}{a} \right) = \frac{d - bc + b^2}{a^2} = \frac{0}{a^2} = 0 \]

so, by the previous case, \( \psi = \chi_E \) where \( E \) is as described above. Thus \( \varphi(z) = a\psi(z) + b = a\chi_E(z) + b \).

If \( a = 0 \) then let \( \psi(z) = \varphi(z) - b \). Then

\[ \psi(z) + \bar{\psi}(z) = (\varphi(z) - b) + (\bar{\varphi}(z) - b) = c - 2b = 0 \]

and

\[ \psi(z)\bar{\psi}(z) = (\varphi(z) - b)(\bar{\varphi}(z) - b) = d - bc + b^2 = 0. \]

Thus \( \bar{\psi} = -\psi \) and \( -\psi^2 = 0 \), which implies that \( \psi = 0 \); i.e., \( \varphi(z) = b \), which is what we wanted.

\[ \square \]

We summarize the above in the following theorem.

**Theorem 2.1.10** Let \( \varphi \in L^\infty \). The non-zero Hankel operator \( H \) commutes with the Toeplitz operator \( T_\varphi \) (not a multiple of the identity operator) if and only if \( H = H_{\mu \varphi} \), where \( \mu \in \mathbb{C} \setminus \{0\} \) and \( \varphi(z) = a\chi_E(z) + b \), with \( a \neq 0 \) and \( b \in \mathbb{C} \), and \( E \subset S^1 \) such that \( m(E^* \Delta E^c) = 0 \).

There are some questions related to this theorem that are left for future research. Does the same reasoning apply to block–Hankel and block–Toeplitz operators? (Definition: let \( S \) be a shift of arbitrary multiplicity. A block–Hankel operator \( H \) is one such that \( S^*H = HS \) and a block–Toeplitz operator \( T \) is one such that \( S^*TS = T \).) What is the situation with respect to other operators with “patterned” matrices? For example, do weighted shifts, generalized Cesàro operators (also called Rhaly matrices — see Chapter 5) or \( \lambda \)-Hankel operators (the definition will be given in Chapter 4) commute with a Hankel operator? Is there a complete classification of composition operators that commute with Hankel operators?
2.2 Algebraic Hankel operators

Algebraic operators have been studied for a long time and some really interesting results are known about them (see, for example, Radjavi and Rosenthal [53, p. 63-64]).

An interesting (seemingly open) question is which Hankel operators are algebraic. First, we recall the definition of an algebraic operator.

**Definition 2.2.1** An operator $A$ is said to be algebraic if there exists a polynomial $p(x)$ (called an annihilating polynomial) such that $p(A) = 0$. If we consider only polynomials which are normalized, in the sense that the coefficient of the term of highest degree is 1, then for any algebraic operator there is a unique annihilating polynomial $p$ of minimum degree, called the minimum polynomial.

The first fact we note about algebraic operators $H$ is that $p(\sigma(H)) = \{0\}$ for an annihilating polynomial $p$. Indeed, by the spectral mapping theorem (see, for example, Conway [13, p. 204]) for any polynomial $p$ we have $p(\sigma(H)) = \sigma(p(H))$; thus $p(\sigma(H)) = \{0\}$ for an annihilating polynomial $p$. In particular, this means that $p(0) = 0$, since $0 \in \sigma(H)$ for any Hankel operator $H$. Therefore if $p$ is an annihilating polynomial for $H$, then $p(x) = xq(x)$ for some polynomial $q$. In particular, this implies that an algebraic Hankel operator $H$ must have non-trivial kernel. For suppose it didn't. Then, if the minimal polynomial is $p(x) = xq(x)$, $Hq(H) = 0$, which would imply that $q(H) = 0$. But the degree of $q$ is less than the degree of $p$. Contradiction.

Any (finite dimensional) matrix is algebraic (they all satisfy their characteristic polynomials). This implies that all Hankel operators of finite rank are algebraic (see Kronecker's Theorem in Chapter 1 for a classification of finite rank Hankel operators).

There are algebraic Hankel operators not of finite rank, as shown in the following example.

**Example 2.2.2** Let $u$ be an inner function such that $u = u^*$. Then the Hankel operator $H_{xu}$ is algebraic with annihilating polynomial $p(x) = x^3 - x$. If $u$ is not a
finite Blashke product, then $H_{zu}$ is not of finite rank.

Proof: As stated in Theorem 1.1.6, $H_{zu}$ is a partial isometry with initial and final space $(uH^2)^\perp$ (since $u = u^*$). This means that $H_{zu}^*H_{zu} = H_{zu}^2$ is the orthogonal projection onto $(uH^2)^\perp$; that is, $I - H_{zu}^2$ is the orthogonal projection onto $uH^2$. It is then clear that $H_{zu} = H_{zu}^3$, which proves the first part. An alternative version of Kronecker's Theorem states that an operator of this form is of finite rank if and only if $u$ is a finite Blashke product (see Power's book [50, p. 6] for a proof of this).

It is actually true that the annihilating polynomial in the previous example is the minimal polynomial as long as $H_{zu}$ is not of rank one. For suppose it wasn't. Then there would be a polynomial $q(x) = x^2 + ax$ (no constant term, by a previous remark) such that $q(H_{zu}) = 0$, or equivalently, $H_{zu}^2 = -aH_{zu}$. But it is well known (see [49] for instance) that the square of a Hankel operator $H$ is a Hankel operator if and only if $H$ is a rank one operator. This yields a contradiction.

A particular case of the characterization of algebraic Hankel operators was solved by Power [51]. Recall that an operator $A$ is nilpotent if $A^n = 0$ for some $n$. Power proved that there are no non-zero nilpotent Hankel operators. (A natural generalization, in a different direction, is the question of existence of a quasinilpotent Hankel operator: this is considered in Chapter 3 below). That is, Power solved the case where $p(x) = x^n$ for some $n$. We use Power's ideas to solve a more general case. First we need a lemma, which is implicitly proven in Power's proof of non-existence of non-zero nilpotent Hankel operators. This lemma implies that the intersection of the kernel and the closure of the range of a Hankel operator is always trivial.

Lemma 2.2.3 Let $u$ be an inner function in $H^2$. Then

$$(uH^2) \cap (u^*H^2)^\perp = \{0\}.$$  

Proof: Let $f \in (uH^2) \cap (u^*H^2)^\perp$. We will show that $f = 0$. Since $f \in uH^2$, we must have $f = uf_1$ for some $f_1 \in H^2$. We will show that $f_1 = 0$. 
We claim that $\overline{w}u_1 \in H^2$. Indeed, $\overline{w}u_1 \in L^2$, so we can write $\overline{w}u_1 = g + h$ where $g \in H^2$ and $h \in H^2$. But this means that

$$f = u_1 = u^*g + u^*h. \quad (2.11)$$

Now, $u^*h \in L^2 \ominus (u^*H^2)$ since, for any function $k \in H^2$,

$$(u^*h, u^*k) = (h, k) = 0$$

(remember that $h \in H^2$). Also, $f \in (u^*H^2)\perp \subset L^2 \ominus (u^*H^2)$ by hypothesis. These two facts, together with equation 2.11, imply that $u^*g \in L^2 \ominus (u^*H^2)$. But, since $g \in H^2$, $u^*g \in u^*H^2$. Thus $u^*g = 0$, which implies that $g = 0$, or, equivalently, $\overline{w}u_1 = h \in H^2$. This proves the claim.

Now consider the Toeplitz operator $T_{\overline{w}u}$. The claim implies that $f_1 \in \text{Ker} T_{\overline{w}u}$. But

$$\text{Ker} T_{\overline{w}u} = \text{Ker} T_{u^*u} = \text{Ker} T_{\overline{w}u^*} = (\text{Ker} T_{\overline{w}u})^*,$$

where $\mathcal{M}^* = \{f^* : f \in \mathcal{M}\}$. Coburn's Alternative (see [19, p. 185]) guarantees that for a non-zero Toeplitz operator either the kernel or the co-kernel (the kernel of the adjoint) must be zero. Thus, in this case, they both must be zero. This implies that $f_1 = 0$. 

We can now prove a generalization of Power's result. In fact, we prove the following theorem.

**Theorem 2.2.4** Let $A$ be a bounded operator such that $\text{Ker} A \cap \text{Ran} A = \{0\}$. Then $A$ cannot be an algebraic operator with minimum polynomial of the form $p(x) = x^2q(x)$ for a polynomial $q$.

**Proof:** We may assume that $A \neq 0$. Suppose there was a non-zero algebraic operator $A$ with minimum polynomial $p(x) = x^2q(x)$ and $\text{Ker} A \cap \text{Ran} A = \{0\}$. Let $f \in \mathcal{H}$. Clearly $Aq(A)f \in \text{Ran} A$. Also, since $A^2q(A) = 0$, it follows that $Aq(A)f \in \text{Ker} A$. Since $\text{Ker} A \cap \text{Ran} A = \{0\}$, it follows that $Aq(A)f = 0$ for all $f \in \mathcal{H}$; i.e., $Aq(A) = 0$. But this means that $xq(x)$ is an annihilating polynomial for $A$. Contradiction. 

We obtain the following corollary.

**Corollary 2.2.5** There are no algebraic Hankel operators with minimum polynomial of the form $x^2q(x)$ for some polynomial $q$.

**Proof:** Suppose there was a Hankel operator $H$ with minimum polynomial $x^2q(x)$ (clearly $H$ cannot be the zero operator). If $H$ has trivial kernel, then $\text{Ker } H \cap \text{Ran } H = \{0\}$ so we can apply the previous theorem. If $H$ has non-trivial kernel, then by Theorem 1.1.6 in Chapter 1, there exists an inner function $u$ such that $\text{Ker } H = u\mathcal{H}^2$ and $\text{Ran } H \subseteq (u^*\mathcal{H}^2)^\perp$. But by Lemma 2.2.3, $u\mathcal{H}^2 \cap (u^*\mathcal{H}^2)^\perp = \{0\}$, so we obtain $\text{Ker } H \cap \text{Ran } H = \{0\}$. We can then apply the previous theorem.  

The previous corollary takes care of the cases where the minimum polynomial of an algebraic Hankel operator is of the form $x^2q(x)$. What happens if the minimum polynomial is of the form $xq(x)$ with $q(0) \neq 0$? Example 2.2.2 shows that there are Hankel operators which have minimum polynomials of this form. Also if $H = k_\alpha \otimes k_\alpha$ for some $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ (that this is a Hankel operator will be verified in the next chapter), then

$$H^2 = (k_\alpha \otimes k_\alpha)(k_\alpha \otimes k_\alpha) = \frac{1}{1 - \alpha^2}(k_\alpha \otimes k_\alpha) = \frac{1}{1 - \alpha^2}H,$$

so $H$ is algebraic with minimum polynomial $x^2 + cx$ for a constant $c$. One common trait of this operator and the one from Example 2.2.2 is that a polynomial function of them is a (not necessarily orthogonal) projection onto their kernels: $I - (1 - \alpha^2)(k_\alpha \otimes k_\alpha)$ is a projection onto $\text{Ker } (k_\alpha \otimes k_\alpha) = u\mathcal{H}^2$, where $u(z) = \frac{\alpha - z}{1 - \alpha z}$ is an inner function, and $I - H^2_{x\alpha}$ is a projection onto $\text{Ker } H_{x\alpha} = u\mathcal{H}^2$.

As it turns out, this is part of a more general fact about algebraic operators with minimum polynomial of the form $xq(x)$, with $q(0) \neq 0$.

We need some definitions. We say that subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ are *quasi-complementary* if $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\overline{\mathcal{M}} + \overline{\mathcal{N}} = \mathcal{H}$ (this concept was introduced in [41]). We say that subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ are *complementary* if $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{H}$ (see Conway [13, p. 93] for more on this concept).
Also, given two quasi–complementary subspaces $M$ and $N$ of $H$, we denote by $P_{M, N}$ the projection of $M$ along $N$ (sometimes also called the projection onto $M$ parallel to $N$), defined on the vector sum $M + N$ as

$$P_{M, N}(m + n) = m,$$

for $m \in M$ and $n \in N$. It is true that two quasi–complementary subspaces $M$ and $N$ are complementary if and only if $P_{M, N}$ is a bounded operator (see [43, p. 197]).

**Theorem 2.2.6** Let $A$ be a bounded algebraic operator with minimum polynomial $xq(x)$ such that $q(0) \neq 0$. Renormalize, if necessary, so that $q(0) = 1$. Then $q(A) = P_{M, N}$, where $M = \ker A$ and $N = \text{Ran} A$. In particular, $A$ has closed range, $\ker A$ and $\text{Ran} A$ are complementary subspaces, and $P_{M, N}$ is an orthogonal projection if and only if $\text{Ran} A = (\ker A)^\perp$.

**Proof:** Since $q(0) = 1$ we can write $q$ as $q(x) = xr(x) + 1$ for some polynomial $r$. Then $q(x)(1 - q(x)) = -xq(x)r(x)$. But this implies that $q(A)(1 - q(A)) = -Aq(A)r(A) = 0$ since $Aq(A) = 0$. This implies that $q(A)^2 = q(A)$; that is, $q(A)$ is a projection.

Since $Aq(A) = 0$, it follows that $\text{Ran} q(A) \subset \ker A$. Now let $f \in \ker A$. Then $q(A)f = r(A)Af + f = f$, so $f \in \text{Ran} q(A)$. This implies that $\text{Ran} q(A) = \ker A$.

Since $q(A)A = 0$, it follows that $\text{Ran} A \subset \ker q(A)$. Now let $f \in \ker q(A)$. Then $0 = q(A)f = Ar(A)f + f$, so $A(-r(A)f) = f$, which implies that $f \in \text{Ran} A$. This implies that $\ker q(A) = \text{Ran} A$. In particular, $\text{Ran} A$ is closed.

This means that $q(A)$ is the projection of $M$ along $N$, and since it is bounded, $M$ and $N$ are complementary. Clearly this projection is orthogonal if and only if $\text{Ran} A = (\ker A)^\perp$. $\square$

By Theorem 2.2.3, for any inner function $u$, we have that $uH^2 \cap (u^*H^2)^\perp = \{0\}$. Applying the same theorem to the inner function $u^*$ we obtain that $(uH^2)^\perp \cap u^*H^2 = \{0\}$, which implies that

$$\overline{(uH^2 + (u^*H^2)^\perp)} = H^2$$
by the Lemma on Closed Subspaces in [43, p. 201]. This means that $uH^2$ and $(u^*H^2)\perp$ are always quasi-complementary. In particular for a Hankel operator $H$, Ker $H$ and Ran $\overline{H}$ are always quasi-complementary.

The previous theorem then shows that for a non-zero algebraic Hankel operator $H$ we always have that Ran $H$ is closed and Ker $H$ and Ran $H$ are complementary. In fact, using the Lemma on Closed Subspaces again [43, p. 201], it follows that in this case the angle between Ker $H$ and Ran $H$ is positive (see [43, p. 197] for the definition of angle between subspaces).

It is well known (and easy to prove) that an algebraic Riesz operator with minimum polynomial of the form $xzq(x)$ with $q(0)$ $\neq$ 0 must be compact (recall that an operator $A$ is Riesz if $\sigma_e(A) = \{0\}$). Thus, if $A$ is an algebraic Riesz operator with minimum polynomial $xzq(x)$ as in the theorem ($q(0) = 1$), then $I - q(A) = -Ar(A)$ is compact. But $I - q(A)$ is a (not necessarily orthogonal) projection with range equal to Ran $A$, by the previous theorem. This implies that Ran $A$ is a finite dimensional subspace (compact idempotents have finite rank), and thus that $A$ is of finite rank. This argument allows us to conclude that an algebraic Hankel operator not of finite rank must have non-zero points in its essential spectrum (the operator in Example 2.2.2 certainly has!).

We leave the following question unanswered: are there any algebraic Hankel operators besides those of finite rank and the partial isometries of Example 2.2.2?

**2.3 Reflexivity and transitivity of spaces of Hankel operators**

In this section we will prove some results about transitivity and reflexivity of spaces of Hankel operators. Azoff and Ptak [4] proved a very interesting result giving a dichotomy for spaces of Toeplitz operators; namely, every ultraweakly closed subspace of Toeplitz operators is either reflexive or transitive. We prove below that no proper (ultraweakly closed) subspace of Hankel operators is transitive, but we have not been
able to prove that they are reflexive. Following the techniques in [4] very closely, we prove some partial results in this direction.

We would like to point out that Corollary 2.3.6, Theorem 2.3.7 and Theorem 2.3.12 below were proven by M. D. Choi for the cases of $n \times n$ Hankel matrices (see Azoff [3, p. 17]).

Most of the notation and the following facts can be found in Azoff [3]. We now state some of the main facts we will need.

We denote by $T$ the space of operators on $\mathcal{H}$ of trace class equipped with the trace norm. As is well known, the dual of $T$ is $B(\mathcal{H})$; that is, $T^* = B(\mathcal{H})$ under the pairing

$$< B, T > = \text{tr} (BT),$$

for $B \in B(\mathcal{H})$ and $T \in T$. The weak* topology on $B(\mathcal{H})$ obtained by this pairing is usually referred to as the ultraweak topology. Throughout this section, a subspace of $B(\mathcal{H})$ will always mean a linear manifold of $B(\mathcal{H})$ closed in the ultraweak topology and a subspace of $T$ will always mean a linear manifold of $T$ closed in the (trace) norm topology.

If $k$ is a positive integer, we denote by $F_k$ the set of operators of rank at most $k$. The set $F_k$ will always be thought of as a subset of $T$. Observe that, for an operator $F = g \otimes h$ of rank one, we have

$$< B, F > = \text{tr} (B(g \otimes h)) = (Bg, h),$$

for any $B \in B(\mathcal{H})$.

**Definition 2.3.1** Given a subset $A$ of $B(\mathcal{H})$ we define $A_\perp \subset T$ to be

$$A_\perp = \{ T \in T : < A, T > = 0 \text{ for all } A \in A \}.$$

Given a subset $F$ of $T$ we define $F_\perp \subset B(\mathcal{H})$ to be

$$F_\perp = \{ B \in B(\mathcal{H}) : < B, F > = 0 \text{ for all } F \in F \}.$$
CHAPTER 2. ALGEBRAIC PROPERTIES

It can be proven (for example, see [38, p. 93, 225]) that $A_\perp$ is a subspace of $T$ and that $T^\perp$ is a subspace of $B(H)$.

**Definition 2.3.2** Let $A$ be a set of operators. The reflexive closure of $A$ is defined to be

$$\text{Ref } A = \{B \in B(H) : B f \in \overline{A f}, \text{ for all } f \in H\}.$$  

The set $A$ is said to be reflexive if $\text{Ref } A = A$ and is said to be transitive if $\text{Ref } A = B(H)$.

Although we will only use the following definition for the cases $k = 1$ or $k = 2$, it seems convenient to state it in full generality.

**Definition 2.3.3** Let $A$ be a subspace of operators and $k$ a positive integer. We say that $A$ is $k$-reflexive if the subspace generated by $A_\perp \cap F_k$ is equal to $A_\perp$.

It is well known [3, p. 10] that $k$-reflexivity of a subspace $A$ of operators in $B(H)$ is equivalent to the reflexivity of the subspace $\bigoplus_{j=1}^k A = \{\bigoplus_{j=1}^k A : A \in A\}$ in $B(\bigoplus_{j=1}^k H)$. In particular, 1-reflexivity is the same as reflexivity.

The following lemma is well known and easy to prove. We include a proof for sake of completeness.

**Lemma 2.3.4** A subspace $A$ of $B(H)$ is $k$-reflexive if any operator not in $A$ is separated from $A$ by an operator of rank at most $k$ in $A_\perp$; i.e., for all $B \in B(H) \setminus A$, there exists $F \in A_\perp \cap F_k$ such that $\langle B, F \rangle \neq 0$.

**Proof:** Suppose the conclusion was false. Then the subspace generated by $A_\perp \cap F_k$ would be a proper subspace of $A_\perp$. This means that $A = (A_\perp)^\perp$ is a proper subspace of $(A_\perp \cap F_k)^\perp$; i.e, there is an operator $B \notin A$ such that $B \in (A_\perp \cap F_k)^\perp$. But this means that $\langle B, F \rangle = 0$ for all $F \in A_\perp \cap F_k$. Contradiction. \[\square\]

It is well known that $L^\infty$ is the dual of $L^1$ under the pairing

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta}) \, d\theta,$$
for \( f \in L^\infty \) and \( g \in L^1 \).

We say that no element of a subset \( \mathcal{N} \) of \( L^1 \) annihilates a subset \( \mathcal{M} \) of \( L^\infty \), if, for \( g \in \mathcal{N} \), \(< f, g >= 0 \) for all \( f \in \mathcal{M} \) implies that \( g = 0 \). In other words, if \( \mathcal{N} \cap \mathcal{M}_L = \{0\} \). For example, no element of \( L^1 \) annihilates \( L^\infty \), since \((L^1)^* = L^\infty \) under this pairing.

We can now prove the transitivity of certain sets of Hankel operators. The proof of the following theorem uses the same techniques as the proofs of Azoff and Ptak [4] for the case of Toeplitz operators.

**Theorem 2.3.5** Let \( \mathcal{A} \) be a set of Hankel operators such that no element of \( H^1 \subset L^1 \) annihilates \( \text{Sym } \mathcal{A} \subset L^\infty \). Then \( \mathcal{A} \) is transitive. In particular, the set \( \mathcal{A} \) has no common invariant subspaces.

**Proof:** We proceed by contradiction. Suppose that \( \mathcal{A} \) was not transitive. Then there would exist \( B \in \mathcal{B}(\mathcal{H}) \) such that \( Bf \notin \mathcal{A}f \) for some non-zero vector \( f \in \mathcal{H} \). Of course this implies that \( \mathcal{A}f \neq \mathcal{H} \) so we can choose a non-zero \( g \in (\mathcal{A}f)^\perp \). It then follows that \( (Hf, g) = 0 \) for all \( H \in \mathcal{A} \).

But this implies that, for all \( \varphi \in \text{Sym } \mathcal{A} \),

\[
0 = (H\varphi f, g) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) (f(e^{i\theta})g^*(e^{i\theta})) \, d\theta = < \varphi, fg^* >,
\]

since \( fg^* \in H^1 \). Since no element of \( H^1 \) annihilates \( \text{Sym } \mathcal{A} \) it follows that \( fg^* = 0 \). But, by the F. and M. Riesz Theorem, either \( f = 0 \) or \( g = 0 \). Contradiction.

It is well known (and easy to prove) that transitive sets do not have common invariant subspaces.

**Corollary 2.3.6** The set of all bounded Hankel operators is transitive and thus has no common invariant subspaces

**Proof:** The set of symbols of the set of all bounded Hankel operators is \( L^\infty \). Clearly no element of \( H^1 \subset L^1 \) annihilates \( L^\infty \).
We also obtain the following theorem, based on the proof for the case of Toeplitz operators, as discussed in [4].

**Theorem 2.3.7** The set of all bounded Hankel operators is 2-reflexive.

*Proof:* We will use Lemma 2.3.4. Clearly the set of Hankel operators $\mathcal{A}$ is an ultra-weakly closed subspace (it is closed in the weak operator topology, which is weaker than the ultraweak topology). Let $B \in B(\mathcal{H})$ not a Hankel operator. Then there exists non-zero vectors $f$ and $g \in \mathcal{H}$ such that $((S^*B - BS)f, g) \neq 0$.

Define $F = f \otimes (Sg) - (Sf) \otimes g$. It then follows that, for any Hankel operator $H$,

$$< H, F > = \text{tr}(H(f \otimes (Sg) - (Sf) \otimes g))$$

$$= \text{tr}((Hf) \otimes (Sg)) - \text{tr}((HSf) \otimes g)$$

$$= (Hf, Sg) - (HSf, g)$$

$$= ((S^*H - HS)f, g) = 0;$$

i.e., $F \in \mathcal{A}_\perp \cap F_2$. But, analogously,

$$< B, F > = ((S^*B - BS)f, g) \neq 0.$$

That is $B$ is separated from the set of Hankel operators by a rank 2 operator in $\mathcal{A}_\perp$. $\square$

As mentioned in Chapter 1, the symbol map $\xi(\varphi) = H_\varphi$ for $\varphi + (z\mathcal{H}^\infty) \in \mathbb{L}^\infty/(z\mathcal{H}^\infty)$ establishes a linear isometry from $\mathbb{L}^\infty/(z\mathcal{H}^\infty)$ onto the set of Hankel operators in $B(\mathcal{H})$.

Recall that $(\mathcal{H}^1)^*$ is naturally isometrically isomorphic to $\mathbb{L}^\infty/(z\mathcal{H}^\infty)$ in the following way. For each $\varphi + z\mathcal{H}^\infty \in \mathbb{L}^\infty/(z\mathcal{H}^\infty)$ define a continuous linear functional $\hat{\varphi}$ on $\mathcal{H}^1$ as

$$\hat{\varphi}(f) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta})\varphi(e^{i\theta}) \, d\theta,$$

for $f \in \mathcal{H}^1$. Conversely, for every linear functional $\lambda \in (\mathcal{H}^1)^*$ there is a function $\varphi + z\mathcal{H}^\infty \in \mathbb{L}^\infty/(z\mathcal{H}^\infty)$ such that $\lambda = \hat{\varphi}$. See Douglas [19, p. 161] for a proof of this fact. We will need the following lemma.
Lemma 2.3.8 Let $A$ be the set of all bounded Hankel operators. Then the symbol map
\[ \xi : L^\infty/(\mathbb{H}^\infty) \to A \] is a weak* to weak* homeomorphism. Here the weak* topology on $L^\infty/(\mathbb{H}^\infty)$ is the one given by $(H^1)^* = L^\infty/(\mathbb{H}^\infty)$ and the weak* topology on $A$ is the one given by $(T/A_\perp)^* = A$ (this topology coincides with the relative ultraweak topology of $A$ inherited as a subset of $B(\mathcal{H})$).

Proof: First, since $A$ is ultraweakly closed, it is clear that $(T/A_\perp)^* = A$. This implies that the weak* topology given by this relation coincides with the relative ultraweak topology of $A$ as a subset of $B(\mathcal{H})$.

We will prove first that $\xi$ is sequentially continuous when $L^\infty/(\mathbb{H}^\infty)$ is equipped when the weak* topology and $A \subset B(\mathcal{H})$ is given the weak operator topology. Let $\varphi_n + (\mathbb{H}^\infty) \in L^\infty/(\mathbb{H}^\infty)$ be a sequence such that $\varphi_n + (\mathbb{H}^\infty) \to \varphi + (\mathbb{H}^\infty)$ weak* as $n \to \infty$. Then, for all $f$ and $g \in \mathbb{H}^2$,
\[ (H_{\varphi_n} f, g) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(e^{i\theta}) f(e^{i\theta}) g^*(e^{i\theta}) d\theta. \]

But since $\varphi_n + (\mathbb{H}^\infty) \to \varphi + (\mathbb{H}^\infty)$ weak* and $fg^* \in H^1$ it follows that, as $n \to \infty$, the previous integral converges to
\[ \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) f(e^{i\theta}) g^*(e^{i\theta}) d\theta = (H_\varphi f, g). \]

Thus $H_{\varphi_n} \to H_\varphi$ in the weak operator topology.

Now we can use Theorem 2.7 in Chapter I of Conway [14, p. 12], which says that a sequentially continuous isometry from the dual of a Banach space equipped with its weak* topology to the space of bounded operators with the weak operator topology is a weak* to weak* homeomorphism onto its image. Thus $\xi$ is a weak* to weak* homeomorphism, as desired. \qed

We will need the following classification of the ultraweakly continuous linear functionals on the set of all Hankel operators.
Theorem 2.3.9 For each \( f \in H^1 \), define the functional \( \hat{f} \) on the set of Hankel operators as

\[
\hat{f}(H_\varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \varphi(e^{i\theta}) \, d\theta.
\]

This is an ultraweakly continuous linear functional. Conversely, every ultraweakly continuous linear functional on the set of Hankel operators has the above form.

**Proof:** That this is a well defined linear functional is easy to check (for example, use the duality between \( H^1 \) and \( L^\infty/(zH^\infty) \)). Since \( f \in H^1 \), \( f \) can be written as \( f = gh \) where \( g \) and \( h \in H^2 \). Take \( F = g \otimes h^* \). Then, for any \( \varphi \in L^\infty \) we have

\[
\langle H_\varphi, F \rangle = \text{tr}(H_\varphi F) = (H_\varphi g, h^*)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\varphi}) g(e^{i\theta}) h(e^{i\vartheta}) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\varphi}) f(e^{i\varphi}) \, d\theta
\]

\[
= \hat{f}(H_\varphi),
\]

which implies ultraweak continuity of the functional. (Recall that the ultraweak topology on \( B(H) \) is, by definition, the weak* topology given by \( T^* = B(H) \) under the pairing \( \langle B, T \rangle = \text{tr}(BT) \) for \( T \) in \( T \) and \( B \in B(H) \).)

Now we need to prove that every ultraweakly continuous linear functional on the set of Hankel operators arises in this way.

Let \( A \) denote the set of all bounded Hankel operators. Since the symbol map \( \xi \) is a weak* to weak* continuous linear operator (by the previous lemma), it follows that there exists a bounded linear operator \( \rho : T/A_\perp \to H^1 \) with \( \rho^* = \xi \) (see Theorem 3.1.11 in [38, p. 287]). Since \( \xi \) is an isometric isomorphism it follows by Theorem 3.1.18 in [38, p. 290] that \( \rho \) is also an isometric isomorphism.

Now, assume that \( \lambda \) is an ultraweakly continuous functional on the set of Hankel operators. Then, since by the previous lemma the weak* topology on this set and the relative ultraweak topology coincide, there exists an element \( \bar{\lambda} + A_\perp \in T/A_\perp \) such that

\[
\lambda(H) = \langle H, \bar{\lambda} \rangle,
\]
for all Hankel operators \( H \). But then, if \( f = \rho(\tilde{\lambda} + A_\perp) \) and since \( \rho \) is an isometric isomorphism, it follows that

\[
\lambda(H_\varphi) = < H_\varphi, \tilde{\lambda} > = < H_\varphi, \rho^{-1}(f) > \\
= < \rho^{-1}H_\varphi, f > = < \xi^{-1}H_\varphi, f > \\
= < \varphi, f > = \hat{f}(H_\varphi),
\]

so that every ultraweak linear functional on \( A \) is of the above form. \( \square \)

**Theorem 2.3.10** If \( A \) is a proper (ultraweakly closed) subspace of Hankel operators, then \( A \) is not transitive.

**Proof:** We will first prove that there is a rank one operator in \( A_\perp \). We will then show that \( A \) is not transitive.

First of all, since \( A \) is ultraweakly closed and it is not the space of all Hankel operators, it follows that \((\text{Sym } A) + (zH^\infty)\) is a proper closed subspace of \( L^\infty/(zH^\infty) \) since \((\text{Sym } A) + (zH^\infty) = \xi^{-1}(A) \) and \( \xi \) is weak* continuous.

Since \( H^1 \) is the predual of \( L^\infty/(zH^\infty) \), it follows that there must exist a non-zero function \( f \in H^1 \) such that \(< \varphi, f > = 0 \) for all \( \varphi + zH^\infty \in (\text{Sym } A) + (zH^\infty) \). Factor \( f \) as \( f = gh \) where \( g \) and \( h \in H^2 \) and define \( F = g \otimes h^* \). Then we have, as before,

\[
< H_\varphi, F > = (H_\varphi g, h^*) = < \varphi, gh > = < \varphi, f > = 0,
\]

for all \( H_\varphi \in A \). That is, \( F \in A_\perp \).

The rest of the argument is standard. We include it for completeness. Select an operator \( B \notin \{F\}^\perp \); i.e., such that \(< B, F > \neq 0 \). As before, this implies that \((Bg, h^*) = < B, F > \neq 0 \), or equivalently \( h^* \) is not orthogonal to \( Bg \). But since \( F \in A_\perp \), it follows that \(< A, F > = 0 \) for all \( A \in A \), or equivalently, \( h^* \) is orthogonal to \( Ag \) for all \( A \in A \). If \( A \) was transitive, then \( Bg \in \overline{Ag} \) for all \( g \). But this is impossible since \( h^* \) is orthogonal to \( Ag \) but not to \( Bg \). Thus \( A \) is not transitive. \( \square \)

We will also need the following definition.
Definition 2.3.11 A linear manifold $A$ of $B(H)$ is elementary if $A_\perp + F_1 = T$.

The utility of this definition will be seen in the corollary to the following theorem, which basically says that all ultraweakly continuous linear functionals on the space of Hankel operators are induced by operators of rank one.

Theorem 2.3.12 The space of all bounded Hankel operators is elementary.

Proof: As usual, let $A$ be the set of all bounded Hankel operators. Let $T$ be a trace-class operator. Then $T$ defines an ultraweakly continuous functional $\lambda$ on $B(H)$ of the form $\lambda(B) = \langle B, T \rangle$ for all $B \in B(H)$. The functional $\lambda$ restricted to the space $A$ of all Hankel operators is then an ultraweakly continuous linear functional on $A$. The functional $\lambda$ restricted to $A$ is given by a function $f \in H^1$ as in Theorem 2.3.9. Again, factor $f$ as $f = gh$ where $g$ and $h \in H^1$. As before,

$$\langle H_\varphi, T \rangle = \lambda(H_\varphi) = \hat{f}(H_\varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi(e^{i\theta}) \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) g(e^{i\theta}) h(e^{i\theta}) \, d\theta$$

$$= (H_\varphi g, h^*)$$

$$= \langle H_\varphi, g \otimes h^* \rangle$$

for all Hankel operators $H_\varphi$. This implies that $\langle H, T - g \otimes h^* \rangle = 0$ for all Hankel operators $H$. It then follows that $T - g \otimes h^* \in A_\perp$, which implies that $T \in A_\perp + F_1$. This finishes the proof. $\Box$

We can now give a partial result on the reflexivity of subspaces of Hankel operators.

Corollary 2.3.13 Every subspace of a reflexive space of Hankel operators is reflexive.

Proof: Theorem 1.7 in [4] implies that any reflexive subspace $F$ of an elementary space of operators has the property that every subspace of $F$ is reflexive. $\Box$

In [4], it is proved that every ultraweakly closed intransitive hyperplane in the set of Toeplitz operators is reflexive. An analogous result for Hankel operators would
be very interesting, and together, with the previous Corollary, would imply that all proper (ultraweakly closed) subspaces of Hankel operators are reflexive. Thus, we leave the following question unanswered: is it true that every ultraweakly closed hyperplane in the set of Hankel operators is reflexive?
Chapter 3

Spectra of Hankel operators

In this chapter we will prove that given any compact subset of the complex plane containing zero, there exists a Hankel operator having this set as its spectrum. The idea of the proof of this theorem is due to Sergei Treil', and, in fact, Treil' applied the techniques to prove a less general result in [58]. The details of the proof can be found in [37].

As a matter of fact, we will prove a more general theorem which uses exactly the same techniques. We will prove that certain classes of non-invertible bounded operators have the property that, given any subset of the complex plane containing zero, there is an operator in the class having this set as its spectrum. We will use this theorem in a later chapter to prove some facts about $\lambda$-Hankel operators (see Chapter 4 for the definition).

3.1 Questions about spectra of Hankel operators

The problem of finding the spectrum of a given Hankel operator in terms of its symbol is intriguing and seems daunting, and apparently remains unsolved. For Toeplitz operators, a lot is known in this direction. For example, it is known (see Douglas [19, p. 183, 186]) that the spectrum of an analytic Toeplitz operator $T_\varphi$ is $\overline{\varphi(D)}$, that the spectrum of a self-adjoint Toeplitz operator $T_\varphi$ is $[\text{ess inf } \varphi, \text{ess sup } \varphi]$ and there is also a beautiful characterization of the spectrum of $T_\varphi$, when $\varphi$ is continuous on
$S^1$, in terms of the winding number of $\varphi$. It would be nice (although we believe it is unlikely) to have similar results for Hankel operators.

In other ways, the spectral structure of Hankel operators is well–understood. In [40], Megretskiï, Peller and Treil' give a complete unitary–invariant description for self–adjoint Hankel operators in terms of their spectral structure. Their description involves some “symmetricity” of the spectrum. A different result is that of Abakumov [1] which states that there exists a finite–rank Hankel operator with any given prescribed finite spectrum and prescribed algebraic multiplicities. See [37] for more on this.

Some restrictions on the spectrum of a general Hankel operator are known. It is proven in [40] that $|\dim \ker (H - \lambda I) - \dim \ker (H + \lambda I)| \leq 1$ for any Hankel operator $H$. In particular, this implies: if $\lambda$ is a multiple eigenvalue of a Hankel operator $H$ ($\dim \ker (H - \lambda I) > 1$), then the point $-\lambda$ has to be an eigenvalue. This and the fact that $0 \in \sigma (H)$ for any Hankel operator $H$ seem to be the only restrictions known on the spectral structure for a general Hankel operator.

A famous question of Power [51] asks whether it is possible to have a non–zero quasinilpotent Hankel operator (Power proved that there are no non–zero nilpotent Hankel operators — see Chapter 2 for more on this); that is, whether there is a non–zero Hankel operator whose spectrum equals $\{0\}$. This question was solved by Megretskiï [39] who found a non–zero compact quasinilpotent Hankel operator. Megretskiï also asks the following three questions:

i) Is there a non–compact quasinilpotent Hankel operator?

ii) Is there a non–compact Hankel operator which is Riesz? (An operator is Riesz if its essential spectrum consists of zero only.)

iii) Is there a non–compact Hankel operator which is power–compact? (That is, some power is a compact operator.)

As far as we know, question i) remains open. We would like to point out that there is indeed a non–compact Hankel operator $H$ such that $H^2$ is compact, thus solving
CHAPTER 3. SPECTRA OF HANKEL OPERATORS

questions ii) and iii) (as remarked in [39], an affirmative solution for iii) implies an affirmative solution for ii)). This is the operator \( H_\xi \) defined by the matrix

\[
H_\xi = \left( \frac{\xi^{n+m}}{n+m+1} \right)_{n,m=0}^\infty,
\]

where \( \xi \) is a complex number of modulus one such that \( \xi \neq 1, -1 \). It is proven in Power's book [50, p. 57], that \( H_\xi H_\xi \) is compact as long as \( \xi \xi \neq 1 \); thus \( H_\xi^2 \) is compact if \( \xi \neq 1, -1 \). It is also known (see [6]) that \( H_\xi \) is never compact. Are any of these operators solutions to question i)? This can be reformulated as: does the compact operator \( H_\xi^2 \) have any eigenvalues for some \( \xi \neq 1, -1 \)? We have not been able to find a satisfactory answer for this problem.

We now prove that a Hankel operator can have arbitrary spectrum, except for the restriction that it must contain the origin. We will first state the following theorem without proof and then see how it applies to Hankel operators. We will prove this theorem in the following sections.

**Theorem 3.1.1** Let \( \mathcal{F} \) be a vector space of non-invertible bounded operators acting on a separable Hilbert space \( \mathcal{H} \). Suppose \( \mathcal{F} \) is closed in the strong operator topology of \( \mathcal{B}(\mathcal{H}) \). Let \( \{ \varphi_n \} \) be a sequence of linearly independent unit vectors in \( \mathcal{H} \) with the property that \( (\varphi_n, \varphi_m) \to 0 \) as \( n \to \infty \) and \( m \) remains fixed. Suppose that \( \varphi_n \otimes \varphi_n \in \mathcal{F} \) for every \( n \). Then, given any compact subset \( \sigma \) of the complex plane containing zero, there exist an operator \( F \in \mathcal{F} \), such that \( \sigma(F) = \sigma \)

As we will show in a moment, the hypotheses of this theorem are satisfied by the set of all bounded Hankel operators. We will see in Chapter 4 that some sets of operators which are generalizations of Hankel operators (the \( \lambda \)-Hankel operators for \( \lambda \) imaginary) also satisfy the hypotheses of the previous theorem. We leave the following question unanswered: are there any other interesting classes of operators satisfying the hypotheses of the previous theorem?

The following theorem now follows easily.
Theorem 3.1.2 Let \( \sigma \) be any compact subset of the complex plane containing zero. Then there exists a Hankel operator \( H \) such that \( \sigma(H) = \sigma \).

**Proof of Theorem 3.1.2.** Assume Theorem 3.1.1. Then we only need to check that the set of all Hankel operators satisfies the conditions of Theorem 3.1.1.

Let \( \mathcal{F} \) be the set of Hankel operators. Clearly this set is a vector subspace of \( B(\mathcal{H}) \) consisting of non-invertible operators. Since the set \( \mathcal{F} \) is the solution set of the linear operator equation \( S^*H = HS \), it follows that this set is closed in the strong operator topology (even in the weak operator topology!).

Let \( |a| < 1 \). The rank-one operator \( k_a \otimes k_a \) is a Hankel operator since

\[
S^*(k_a \otimes k_a) = (S^*k_a) \otimes k_a = (ak_a) \otimes k_a = k_a \otimes (ak_a) = (S^*k_a) \otimes k_a = (k_a \otimes k_a)S.
\]

Now, let \( a_n \) be a strictly increasing sequence of real numbers such that \( a_n \to 1 \) as \( n \to \infty \). Define

\[
\varphi_n = \frac{k_{a_n}}{\|k_{a_n}\|}.
\]

Clearly, \( \|\varphi_n\| = 1 \) and the set \( \{\varphi_n\} \) is linearly independent. Now, since \( \mathcal{F} \) is a vector space and \( k_{a_n} \otimes k_{a_m} \in \mathcal{F} \) (remember that \( a_n, a_m \in \mathbb{R} \)), it follows that \( \varphi_n \otimes \varphi_n \) is also in \( \mathcal{F} \). We need to check now that \( (\varphi_n, \varphi_m) \to 0 \) as \( n \to \infty \). But a calculation shows that

\[
(\varphi_n, \varphi_m) = \frac{\sqrt{1 - a_n^2} \sqrt{1 - a_m^2}}{1 - a_na_m}
\]

and the right hand side goes to 0 as \( n \to \infty \) if \( m \) is fixed. Therefore, all conditions of Theorem 3.1.1 are satisfied. This proves Theorem 3.1.2. \( \square \)

### 3.2 Idea of Part I of the proof of Theorem 3.1.1

As is well known, it is very easy to construct an operator with prescribed spectrum \( \sigma \) (a compact set). The usual procedure is to take a countable dense subset of \( \sigma \), and
just take the diagonal operator (with respect to some orthonormal basis) which has the countable dense subset as its diagonal entries. It is clear that the spectrum of this operator is precisely $\sigma$. If we do not require the operator to be invertible ($0 \in \sigma$), we do not need to have an orthonormal basis, merely an infinite orthonormal set (complete this set to a basis, and put zeroes on the diagonal entries corresponding to the vectors in the basis which were not in our original set).

As it turns out, we do not even require our set to be orthonormal, but merely to be "equivalent" to an orthonormal set (equivalence means that there exists an invertible operator taking one set to the other). That is, we require our set to be a Riesz basis on its span (see Young [60] for properties of Riesz bases).

Definition 3.2.1 A system of vectors $\{f_n\}$ is called a Riesz basis (on its span) if there exists a bounded invertible operator $R$ (called the orthogonalizer of the system) such that $\{Rf_n\}$ is an orthonormal set. The number $\|R\| \|R^{-1}\|$ is called the measure of non-orthogonality of the system.

It is easy to see that an orthogonalizer is unique up to a unitary factor and that the measure of non-orthogonality is 1 if and only if the system is orthogonal.

We would like to mimic the above diagonal construction to obtain an operator in the class $\mathcal{F}$ with prescribed spectrum. To do this, we will construct by induction a sequence of operators of finite rank in $\mathcal{F}$, each having one more point in the dense subset as an eigenvalue. We will then see that the limit in the strong operator topology exists and has precisely the desired spectrum.

To begin, notice that, for a point $b_1 \in \mathbb{C}$, the operator $b_1 \varphi_1 \otimes \varphi_1$ is in $\mathcal{F}$ and has spectrum $\{0, b_1\}$. If $\varphi_2$ was orthogonal to $\varphi_1$, we would have that for a number $b_2 \in \mathbb{C}$, the operator $b_1 \varphi_1 \otimes \varphi_1 + b_2 \varphi_2 \otimes \varphi_2$ is also in $\mathcal{F}$ and has spectrum $\{0, b_1, b_2\}$. Unfortunately, $\varphi_2$ is not necessarily orthogonal to $\varphi_1$. Fortunately, we know that $\varphi_n$ is "asymptotically orthogonal" to $\varphi_1$, in the sense that $(\varphi_n, \varphi_1) \to 0$ as $n \to \infty$. This means that for $n$ large enough, the spectrum of $b_1 \varphi_1 \otimes \varphi_1 + b_2 \varphi_n \otimes \varphi_n$ will "almost" be $\{0, b_1, b_2\}$. We will just perturb the operator a little bit and use the
implicit function theorem to obtain an operator with the desired spectrum in the class $\mathcal{F}$. We give a rigorous definition of "asymptotically orthogonal".

**Definition 3.2.2** We say that a set of unit vectors $\{\varphi_n\}$ is asymptotically orthogonal if $(\varphi_n, \varphi_m) \to 0$ as $n \to \infty$ and $m$ fixed.

### 3.3 Proof of Theorem 3.1.1, Part I

We consider the trivial case first. Suppose $\sigma = \{0\}$. Then $\Gamma = 0$ is in $\mathcal{F}$ and has the desired spectrum.

Assume for a moment that $\sigma$ consists of an infinite number of points. Pick a dense countable subset of $\sigma \setminus \{0\}$ and call it $\{b_n\}_{n=1}^{\infty}$. Suppose we have a sequence of operators $\Gamma_n \in \mathcal{F}$, each of finite rank $n$, with the following properties:

i) $\text{Ran} \Gamma_n = (\text{Ker} \Gamma_n)^\perp$; i.e., $\text{Ran} \Gamma_n$ is a reducing subspace of $\Gamma_n$;

ii) $\text{Ran} \Gamma_n \subset \text{Ran} \Gamma_{n+1}$;

iii) the (non–zero) eigenvalues of $\Gamma_n$ are exactly $b_1, b_2, \ldots, b_n$, with corresponding normalized ($\|f_k^{(n)}\| = 1$) eigenvectors $f_1^{(n)}, f_2^{(n)}, \ldots, f_n^{(n)}$;

iv) the measure of non–orthogonality $\|R\|\|R^{-1}\|$ of each system $\{f_k^{(n)}\}_{k=1}^{n}$, for each $n$, is strictly less than 2 (each system is a finite linearly independent system, so it is a Riesz basis in its linear span and thus, it has an orthogonalizer $R$); and

v) $\|f_k^{(n)} - f_k^{(n+1)}\| \leq 2^{-n}$ for $k = 1, 2, \ldots, n$.

We construct a sequence $\Gamma_n$ with the above properties in the next sections. Now, assuming the existence of such sequence, we show that there exists an operator $\Gamma$ with the desired spectrum. Before doing that, let us point out that if $\sigma$ consisted only of a finite number of points, say $N$, the construction of operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ satisfying properties i)–v) would provide us with the desired operator $\Gamma$ (just choose $\Gamma = \Gamma_N$). Thus we can restrict ourselves to the case when $\sigma$ is an infinite set.
We now show the existence of $\Gamma$. First notice that condition v) implies that $f_k^{(n)} \to f_k$ as $n \to \infty$ for some $f_k \in \mathcal{H}$. Since for every fixed $N \leq n$ the finite set $\{f_k^{(n)}\}_{k=1}^N$ is clearly a Riesz basis with measure of non-orthogonality bounded by 2 (by condition iv)), it is easy to prove that the set $\{f_k\}_{k=1}^N$ is also a Riesz basis (since $\{f_k^{(n)}\} \to f_k$ as $n \to \infty$) and that its measure of non-orthogonality is also bounded by 2. Also, since every finite set $\{f_k\}_{k=1}^N$ is a Riesz basis with measure of non-orthogonality bounded by 2, it follows that the system $\{f_k\}_{k=1}^\infty$ is also a Riesz basis (in its closed linear span), and its measure of non-orthogonality $\|R\|\|R^{-1}\|$ is at most 2.

Define an operator $\Gamma$ on $\mathcal{H}$ by

$$\Gamma f_k = b_k f_k, \quad \text{for each } k \geq 1,$$

and

$$\Gamma f = 0 \quad \text{when } f \text{ is orthogonal to all the } \{f_k\}.$$

Then, if diam $\sigma$ denotes the diameter of the set $\sigma$, we have

$$\|\Gamma f_k - \Gamma_n f_k\| \leq \|\Gamma f_k - \Gamma_n f_k^{(n)}\| + \|\Gamma_n f_k^{(n)} - \Gamma_n f_k\|$$

$$\leq |b_k| \|f_k - f_k^{(n)}\| + \|\Gamma_n\| \|f_k - f_k^{(n)}\|$$

$$\leq (\text{diam } \sigma + 2 \text{ diam } \sigma) \|f_k - f_k^{(n)}\|,$$

where the last inequality follows because the norms of the operators $\Gamma_n$ are all bounded by 2 diam $\sigma$ (this is obtained by observing that when restricted to $\text{Ran} \Gamma_n$, $\Gamma_n = R_n^{-1} \text{diag } \{b_1, b_2, \ldots, b_n\} R_n$ where $R_n$ is the orthogonalizer corresponding to the Riesz basis $\{f_k^{(n)}\}_{k=1}^n$).

Also, notice that $\Gamma_n = 0$ when restricted to $(\text{span } \{f_k : k \geq 1\})^\perp$ since $(\text{Ker } \Gamma_n)^\perp = \text{Ran} \Gamma_n \subset \text{span } \{f_k : k \geq 1\}$ by condition ii).

The two previous paragraphs then show that $\Gamma_n \to \Gamma$ in the strong operator topology. Therefore $\Gamma$ is in $\mathcal{F}$. 

\[\]
The spectrum of $\Gamma$ is $\{b_k : k \geq 1\}$ if the system $\{f_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{H}$, and $\{b_k : k \geq 1\} \cup \{0\}$ if it is not. Since for an operator $\Gamma \in \mathcal{F}$ one always has $0 \in \sigma(\Gamma)$, our operator has spectrum $\sigma = \{b_k : k \geq 1\} \cup \{0\}$ in either case, but notice that the system $\{f_k\}_{k=1}^{\infty}$ may be a basis only if $0 \in \{b_k : k \geq 1\}$.

3.4 Idea of Part II of the proof of Theorem 3.1.1

Let us mention that, since the subspace $E_n := \text{Ran} \Gamma_n$ is a reducing subspace for $\Gamma_n$, one can forget about $\text{Ker} \Gamma_n$, and treat $\Gamma_n$ as an operator acting on the finite dimensional space $E_n$.

We will construct the operators $\Gamma_n$ by induction. Consider the operators $\Gamma_{n,\tau}$ defined as

$$\Gamma_{n,\tau} = \sum_{k=1}^{n} b_k (1 + t_k) \varphi_{s_k} \otimes \varphi_{s_k}, \quad (3.1)$$

for some $\tau = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ and an increasing sequence $\{s_k\}$ of integers to be chosen later.

These operators belong to $\mathcal{F}$. If $t_k \neq -1$ for all $1 \leq k \leq n$, these operators have rank $n$ (since the vectors $\varphi_{s_k}$ are linearly independent) and clearly $\text{Ran} \Gamma_{n,\tau} = \text{span} \{\varphi_{s_k} : 1 \leq k \leq n\}$ and $\ker \Gamma_{n,\tau} = (\text{span} \{\varphi_{s_k} : 1 \leq k \leq n\})^\perp$. This means that conditions i) and ii) will hold, if the operator $\Gamma_n$ satisfying conditions i)--v) is of the form $\Gamma_{n,\tau(n)}$ for some particular $\tau(n) = (t_1^{(n)}, t_2^{(n)}, \ldots, t_n^{(n)})$ and a choice of a sequence $\{s_k\}$. We construct such $\Gamma_{n,\tau(n)}$.

The choice of $t_1^{(1)}$ is trivial. In equation 3.1 above put $s_1 = 1$ and $t_1 = 0$. It is clear that $\Gamma_1 := \Gamma_{1,\tau(1)}$ satisfies conditions i)--v).

Suppose we have constructed operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. If we pick $s_{n+1}$ large enough, the function $\varphi_{s_{n+1}}$ will be almost orthogonal to $\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_n}$. Therefore the operator in $\mathcal{F}$

$$\Gamma_n + b_{n+1} \varphi_{s_{n+1}} \otimes \varphi_{s_{n+1}}$$

will be almost the operator $\Gamma_{n+1}$ we need, since its eigenvalues are "almost" the
desired eigenvalues and the eigenvectors are "almost" the desired eigenvectors. To get the desired operator $\Gamma_{n+1}$, we perturb the above operator using the parameters $t_k^{(n)}$.

To show that such a perturbation is possible, we will use the implicit function theorem. We will need one more assumption on our sequence of operators $\Gamma_n$ in order to do this.

Let $\hat{\lambda}^{(n)}(\tau) = (\hat{\lambda}_1^{(n)}(\tau), \hat{\lambda}_2^{(n)}(\tau), \ldots, \hat{\lambda}_n^{(n)}(\tau))$ be the non-zero eigenvalues of the operator $\Gamma_{n,\tau} \in \mathcal{F}$. Then the condition required is

\begin{align*}
&\text{vi) the Jacobian } \frac{d\hat{\lambda}^{(n)}}{d\tau} = \left( \frac{\partial \hat{\lambda}_j^{(n)}}{\partial t_k} \right)_{j,k=1}^n \text{ is non-singular at } \tau = \tau^{(n)}. \end{align*}

Clearly, the ordering of eigenvalues is not essential for this condition to make sense. It is natural for our purposes to order the eigenvalues in such a way that $\hat{\lambda}_k^{(n)}(\tau^{(n)}) = b_k$.

Since $b_k \neq b_j$ whenever $k \neq j$, it follows that $\hat{\lambda}_k^{(n)}(\tau) \neq \hat{\lambda}_j^{(n)}(\tau)$ for $j \neq k$ in a small neighbourhood of $\tau^{(n)}$; so by Lemma 3.5.1 below, the functions $\tau \mapsto \hat{\lambda}_k^{(n)}(\tau)$ are continuously differentiable. Therefore the Jacobian $d\hat{\lambda}^{(n)}/d\tau$ is well-defined.

### 3.5 Some lemmas

To finish the proof of the theorem we need a few known lemmas. Although they hold for general operators, we will only need them for operators on a finite dimensional space (matrices).

Let us recall that a point $\lambda$ is called a simple isolated eigenvalue of an operator $A$ if it is an isolated point of $\sigma(A)$ and the dimension of the corresponding spectral subspace is 1. For matrices this means that $\lambda$ is a simple root of the characteristic polynomial $p(z) := \det(A - zI)$.

The following lemma is a corollary of the implicit function theorem (for example, see [8, p. 61]). The critical step is that if $\lambda$ is a simple root of a polynomial $p(z)$, then $\frac{dp}{dz}(\lambda) \neq 0$. 


Lemma 3.5.1 Let \( t \mapsto A(t) \) be a continuously differentiable matrix-valued function. Suppose that \( A(t) \) is defined on an open subset \( \Omega \subset \mathbb{R}^m \). Let the point \( \lambda \in \mathbb{C} \) be a simple isolated eigenvalue of \( A(t_0) \) for \( t_0 \in \Omega \). Then there exists a continuously differentiable function \( t \mapsto \lambda(t) \), defined in a small neighbourhood of \( t_0 \), such that \( \lambda(t_0) = \lambda \) and \( \lambda(t) \) is a simple isolated eigenvalue of \( A(t) \) for all \( t \) in the neighbourhood.

The next lemma can also be obtained by an application of the implicit function theorem (see Theorem 3.6.1) and by an application of the previous lemma.

Lemma 3.5.2 Let \( A(t) \) and \( A_n(t) \) be continuously differentiable matrix-valued functions. Suppose they are defined on some open subset \( \Omega \subset \mathbb{R}^m \), and let

\[
\lim_{n \to \infty} \|A_n(\cdot) - A(\cdot)\|_{C^1(\Omega)} = 0.
\]

For \( t_0 \in \Omega \), let \( \lambda \) be a simple isolated eigenvalue of \( A(t_0) \). Then there exist a small neighbourhood \( \mathcal{U} \) of \( t_0 \) and continuously differentiable functions \( \lambda(t) \), \( \lambda^{(n)}(t) \), \( n \geq N \) for some large \( N \), on \( \mathcal{U} \) such that

1. \( \lambda(t) \) and \( \lambda^{(n)}(t) \) are simple isolated eigenvalues of \( A(t) \) and \( A_n(t) \), respectively, for all \( t \in \mathcal{U} \);

2. \( \lambda(t_0) = \lambda \); and

3. \( \lim_{n \to \infty} \|\lambda^{(n)}(\cdot) - \lambda(\cdot)\|_{C^1(\mathcal{U})} = 0 \).

Lemma 3.5.3 Let \( A, A_n \) be bounded operators of finite rank on a Hilbert space, and assume that \( \lim_{n \to \infty} \|A_n - A\| = 0 \). Suppose that \( \mu \) is a simple isolated eigenvalue for \( A \) and \( A_n \) for all \( n \) larger than some fixed \( N \). If \( Af = \mu f \), \( \|f\| = 1 \), then there exists a sequence \( \{f_n\} \) of unit vectors such that \( A_nf_n = \mu f_n \) for \( n > N \) and

\[
\lim_{n \to \infty} \|f_n - f\| = 0.
\]
Proof: If $P(A_n)$ and $P(A)$ are the Riesz spectral projections corresponding to the eigenvalue $\mu$, then

$$\|P(A_n) - P(A)\| \to 0 \quad \text{whenever} \quad \|A_n - A\| \to 0,$$

(for a proof of this see, for example, Baumgärtel [5, p. 80]).

Thus, since $f$ is an eigenvector of $A$,

$$\|f - P(A_n)f\| = \|P(A)f - P(A_n)f\| \to 0$$

as $n \to \infty$. Also, since $\|f\| = 1$,

$$|1 - \|P(A_n)f\|| \leq \|P(A)f - P(A_n)f\| \to 0$$

as $n \to \infty$. Let

$$f_n = \frac{P(A_n)f}{\|P(A_n)f\|}.$$

Clearly $f_n$ is a normalized eigenvector of $A_n$ corresponding to $\mu$, and

$$\|f - f_n\| \leq \|f - P(A_n)f\| + \|P(A_n)f - f_n\|
= \|f - P(A_n)f\| + \left\|P(A_n)f - \frac{P(A_n)f}{\|P(A_n)f\|}\right\|
= \|f - P(A_n)f\| + \|P(A_n)f\| - 1 \to 0,$$

as $n \to \infty$. 

\[\square\]

3.6 Proof of Theorem 3.1.1, Part II

As mentioned above, we will construct the operators $\Gamma_n$ by induction by choosing $\tau^{(n)}$ for each $n$ and an increasing sequence $\{s_k\}$.

The case $n = 1$ is trivial: we just pick $s_1 = 1$ and put $\tau_1^{(1)} = 0$. Then $\Gamma_1 = b_1 \varphi_1 \otimes \varphi_1 = \Gamma_{1,\tau_1^{(1)}}$ satisfies conditions i)--vi).

Let us suppose that we have constructed vectors $\tau^{(k)} \in \mathbb{R}^k$ and an increasing set of integers $s_k$, with $1 \leq k \leq n$, satisfying conditions i)--vi). We must show that there
is a vector $\tau^{(n+1)}$ and a positive number $s_{n+1}$ larger than $s_k$ for $1 \leq k \leq n$, such that conditions i)–vi) are satisfied by the operator $\Gamma_{n+1, \tau^{(n+1)}}$.

Let $\tau = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, and let $\bar{\tau} = (\tau, t_{n+1}) = (t_1, t_2, \ldots, t_n, t_{n+1}) \in \mathbb{R}^{n+1}$.

For $s > s_n$ define the operator–valued function $(s, \bar{\tau}) \mapsto G_{n+1}^{(s)}(\bar{\tau})$ by

$$G_{n+1}^{(s)}(\bar{\tau}) = \Gamma_{n, \tau} + b_{n+1}(1 + t_{n+1}) \varphi_s \otimes \varphi_s.$$ 

Clearly, $G_{n+1}^{(s)}(\bar{\tau}) \in \mathcal{F}$. For $s > s_n$, define also the operator–valued function $(s, \bar{\tau}) \mapsto H_{n+1}^{(s)}(\bar{\tau})$ by

$$H_{n+1}^{(s)}(\bar{\tau}) = \Gamma_{n, \tau} + b_{n+1}(1 + t_{n+1}) \varphi_s \otimes h_s.$$ 

where $h_s$ is the normalized ($\|h_s\| = 1$) projection of the vector $\varphi_s$ onto the orthogonal complement of $E_n := \text{Ran} \Gamma_n = \text{span} \{\varphi_j : 1 \leq j \leq n\}$ ($h_s$ is not zero since $\{\varphi_s, \varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_n}\}$ is a linearly independent set).

Note that $H_{n+1}^{(s)}((\tau^{(n)}, 0))$ has the required eigenvalues, but it might not be in $\mathcal{F}$.

Also notice that, for fixed $\bar{\tau}$, the operators $H_{n+1}^{(s)}(\bar{\tau})$, $s > s_n$, are unitarily equivalent to each other. Thus there exists an operator $H_{n+1}(\bar{\tau})$ (acting, say, on $\mathbb{C}^{n+1}$) and, for each $s$, there exists a unitary operator $U_s : \mathbb{C}^{n+1} \rightarrow \text{span} \{E_n, h_s\} = \text{span} \{h_s, \varphi_s : 1 \leq j \leq n\}$ such that

$$U_s^* H_{n+1}^{(s)}(\bar{\tau}) U_s = H_{n+1}(\bar{\tau}),$$

and such that the restriction of $U_s^*$ to $E_n$ does not depend on $s$ (operators $H_{n+1}^{(s)}(\bar{\tau})$ for fixed $\bar{\tau}$ and different $s$ coincide on $E_n$).

The asymptotic orthogonality (Definition 3.2.2) of the vectors $\varphi_n$ implies

$$\lim_{s \to \infty} \|h_s - \varphi_s\| = 0, \quad (3.2)$$

and therefore the operator–valued functions $\bar{\tau} \mapsto U_s^* G_{n+1}^{(s)}(\bar{\tau}) U_s$ converge in $C^1(G)$ to $\bar{\tau} \mapsto H_{n+1}(\bar{\tau})$ (for every bounded domain $G$ in $\mathbb{R}^n$) as $s \to \infty$.

Again, notice that the eigenvalues of $H_{n+1}(\bar{\tau})$ for $\bar{\tau} = (\tau^{(n)}, 0) \in \mathbb{R}^{n+1}$ are exactly the numbers $b_1, b_2, \ldots, b_n, b_{n+1}$, but $H_{n+1}((\tau^{(n)}, 0))$ might not be in $\mathcal{F}$. 
CHAPTER 3. SPECTRA OF HANKEL OPERATORS

Since the \( \{b_j\} \) are distinct, we can apply Lemma 3.5.2. We get that there exists a neighbourhood \( \mathcal{U} \) of the point \( (\tau^{(n)}, 0) \) such that \( \Lambda^{(s)}(\cdot) \rightarrow \Lambda(\cdot) \) in \( C^1(\mathcal{U}) \) as \( s \rightarrow \infty \), where

\[
\Lambda^{(s)}(\tau) = \left( \lambda_1^{(s)}(\tau), \lambda_2^{(s)}(\tau), \ldots, \lambda_{n+1}^{(s)}(\tau) \right)
\]

are the eigenvalues of \( G_n^{(s)}(\tau) \) restricted to span \( \{E_n, \varphi_s\} \), and

\[
\Lambda(\tau) = (\lambda_1(\tau), \lambda_2(\tau), \ldots, \lambda_{n+1}(\tau))
\]

are the eigenvalues of \( H_{n+1}(\tau) \).

The order of eigenvalues is not important for our purposes, as we pointed out before, but it is convenient to order \( \Lambda(\tau) \) such that \( \lambda_k((\tau^{(n)}, 0)) = b_k, k = 1, 2, \ldots, n+1 \).

The Jacobian \( d\Lambda/d\tau \) is non–singular at the point \( (\tau^{(n)}, 0) \) since it can be easily seen to have the following form:

\[
\begin{pmatrix}
\left( \frac{\partial \lambda_j^{(n)}}{\partial \tau_k} \right)_{j,k=1}^n \\
0 \\
0 \\
b_{n+1}
\end{pmatrix}
\]

The upper–left corner is non-singular by the induction hypothesis vi), and \( b_{n+1} \neq 0 \) by the choice of the set \( \{b_k\}_{k=1}^\infty \).

Let \( \hat{N} = N \cup \{\infty\} \) be the one–point compactification of \( N \) and let \( \Omega \subset \hat{N} \times \mathbb{R}^{n+1} \) be a neighbourhood of the point \( (\infty, (\tau^{(n)}, 0)) \in \hat{N} \times \mathbb{R}^{n+1} \). Define the function \( f : \Omega \rightarrow \mathbb{C}^{n+1} \) by

\[
f(s, \tau) = \begin{cases}
\Lambda^{(s)}(\tau), & \text{if } s \in N; \\
\Lambda(\tau), & \text{if } s = \infty.
\end{cases}
\]

If the neighbourhood \( \Omega \) is small enough, the function \( f \) is well–defined and continuous. Moreover, the partial derivative \( \partial f/\partial \tau \) (a Jacobian) exists and is continuous in \( \Omega \). Notice that at the point \( (\infty, (\tau^{(n)}, 0)) \in \hat{N} \times \mathbb{R}^{n+1} \) the partial derivative \( \partial f/\partial \tau \) is exactly the Jacobian \( \partial \Lambda/\partial \tau \), which is non–singular at this point.

We can then apply the following version of the implicit function theorem.
Theorem 3.6.1 (Implicit function theorem) Let $E$ be a topological space, let $F$ and $G$ be Banach spaces, let $L(F,G)$ be the space of bounded linear operators from $F$ to $G$, let $\Omega$ be an open subset of $E \times F$, and let $(a,b)$ be a point in $\Omega$. Let $f$ be a continuous mapping from $\Omega$ into $G$ such that

1) For any fixed $x$, the function $f$ has a partial derivative $\frac{\partial f}{\partial y}(x,y)$, and the mapping $(x,y) \mapsto \frac{\partial f}{\partial y}(x,y)$ is a continuous mapping of $\Omega$ into $L(F,G)$;

2) $\frac{\partial f}{\partial y}(a,b)$ is an invertible mapping from $F$ to $G$.

Assume, in addition, $c = f(a,b)$.

Then there exist neighbourhoods $A$ and $B$ of the points $a$ and $b$ such that for any $x \in A$, the equation $f(x,y) = c$ has a unique solution $y = g(x)$ belonging to $B$, and the function $g$ defined in this manner is a continuous mapping from $A$ to $B$.

This formulation of the implicit function theorem can be found, for example, in [56, p. 278].

If we apply this theorem to the function $f$ defined above, with $x = s$, $y = \tau$, $a = \infty$, $b = (\tau^{(n)}, 0)$, $c = (b_1, b_2, \ldots, b_{n+1})$, we get that, for $s$ large enough, there exists a vector $\tilde{\tau}(s)$ such that

$$\Lambda^{(s)}(\tilde{\tau}(s)) = \Lambda(\tau^{(n)}, 0) = (b_1, b_2, \ldots, b_{n+1}) \in \mathbb{R}^{n+1}.$$ 

This implies that the operator $G_{n+1}^{(s)}(\tilde{\tau}(s))$ satisfies condition (iii).

If we choose $s$ large enough, the operator $U^* G_{n+1}^{(s)}(\tilde{\tau}(s)) U_z$ can be as close as we want to $H_{n+1}(\tau^n, 0)$, since

$$\lim_{s \to \infty} \tilde{\tau}(s) = (\tau^{(n)}, 0)$$

by continuity. Therefore, by Lemma 3.5.3, the normalized eigenvectors of $G_{n+1}^{(s)}(\tilde{\tau}(s))$ can be chosen to be as close as we want to the corresponding normalized eigenvectors of $H_{n+1}^{(s)}(\tau^{(n)}, 0)$, which are exactly the normalized eigenvectors of $\Gamma_n$ and the normalized eigenvector $h_*$ (which is orthogonal to all the other eigenvectors) corresponding to the eigenvalue $b_{n+1}$. So condition (v) is satisfied for the operator $G_{n+1}^{(s)}(\tilde{\tau}(s))$.

If we add to a Riesz basis of unit vectors (in a subspace) a unit vector orthogonal to it, we get a Riesz basis for the higher-dimensional subspace with the same measure of
non-orthogonality \( \| R \| \| R^{-1} \| \) (because, as can easily be seen, neither \( \| R \| \) nor \( \| R^{-1} \| \) change when adding an orthonormal vector). So, by the induction hypothesis iv), the system of normalized eigenvectors of \( H_{n+1}^{(s)}(\tau^{(n)}, 0) \) is a Riesz basis and has measure of non-orthogonality strictly less than 2.

Therefore, for \( s \) large enough, the measure of non-orthogonality of the system of normalized eigenvectors of \( G_{n+1}^{(s)}(\bar{\tau}(s)) \) is strictly less than 2 as well, since we can make this eigenvectors as close as we want to the system of normalized eigenvectors of \( H_{n+1}^{(s)}(\tau^{(n)}, 0) \), and for finite systems this implies that the orthogonalizers can be made as close as we desire. Hence condition iv) is satisfied for \( G_{n+1}^{(s)}(\bar{\tau}(s)) \).

Finally, since \( \Lambda^{(s)}(\cdot) \to \Lambda(\cdot) \) in \( C^1(\Omega) \) as \( s \to \infty \), for \( s \) large enough the Jacobian \( \partial \Lambda^{(s)}(\bar{\tau}) / \partial \bar{\tau} \) is non-singular at \( \bar{\tau} = \bar{\tau}(s) \), which is just condition (vi).

Therefore if we put \( \tau^{(n+1)} = \bar{\tau}(s) \), where \( s \) is large enough such that all of the above hold, and we define \( s_{n+1} = s \), then we get that \( \Gamma_{n+1} = \Gamma_{n+1, \tau^{(n+1)}} = G_{n+1}^{(s_{n+1})}(\tau^{(n+1)}) \) satisfies all the conditions i)–vi). This finishes the proof.
Chapter 4

Generalizations

As mentioned in Chapter 1, an operator $H$ is Hankel if and only if $S^*H = HS$ and an operator $T$ is Toeplitz if and only if $S^*TS = T$. This suggests the study of generalizations and modifications of these equations.

Generalizations of these equations have been investigated for some time. For example, Douglas [18] has studied the solutions to the equation $S^*XT = X$ for arbitrary contractions $S$ and $T$. Pták [52] studied solutions to the equation $S^*X = XT$ when $S$ and $T$ are contractions. Power [46] studied simultaneous solutions to the equations $S^*X = XS$ for all $S \in S$, where $S$ is a commutative family of shifts.

In a different direction, Barría and Halmos [6] asked the following question: What are the solutions of the equation $S^*XS = \lambda X$ for $\lambda$ an arbitrary complex number? This is a spectral problem, and was completely solved by Sun [57]. If $\lambda = 1$ the solutions of this equation are just the Toeplitz operators.

In this chapter, we study the type of equation proposed by Barría and Halmos, but for the case of Hankel operators. We first show that a lot of equations involving the shift have only trivial solutions. We will describe exactly the solutions of the equation $\lambda S^*X = XS$. Unfortunately, this is not a spectral problem. This consideration leads to the study of the equation $S^*X - XS = \lambda X$, solutions of which we will call $\lambda$–Hankel operators. These operators are a generalization of Hankel operators, and as such, they share some of their properties. We should mention that a different generalization (the
"derived" Hankel matrices) has been studied by Heinig [31].

We study some properties of $\lambda$-Hankel operators, including invertibility, spectra, finite-rank and positivity, among others. It is then shown that positivity of $\lambda$-Hankel operators for some $\lambda$ solves a generalization of the classical Hamburger moment problem. We also give a sufficient condition for compactness of $\lambda$-Hankel operators and a necessary condition for boundedness, and offer a conjecture for a sufficient and necessary condition for boundedness.

4.1 Solutions of some equations involving the shift

As we mentioned above, Toeplitz and Hankel operators are characterized as solutions to the operator equations $S^*TS = T$ and $S^*H = HS$ respectively. In this section we investigate the solutions of a few other arrangements of the shift operator and its adjoint in some linear equations. In particular we show that a lot of arrangements have no solutions other than zero.

Theorem 4.1.1 Let $A$ be a bounded operator with $\text{Ker } A = \{0\}$ and $B$ be any bounded operator. If $X$ is a bounded solution of the operator equation $AX = BXS^*$, then $X = 0$.

Proof: It suffices to prove that $Xe_n = 0$ for all $n$. We proceed by induction. For $n = 0$, $AXe_0 = BXS^*e_0 = 0$, so $Xe_0 = 0$. Assume now that $Xe_k = 0$ for some $k$. Then $AXe_{k+1} = BXS^*e_{k+1} = BXe_k = 0$, so $Xe_{k+1} = 0$. This completes the induction.

As a corollary, we note that many of the modifications of the equations defining Toeplitz and Hankel operators have no non-trivial solutions.

Corollary 4.1.2 If $X$ is a bounded solution of any one of the equations $X = SXS^*$, $X = S^*XS^*$, $SX = XS^*$, $SX = SXS^*$, or $SX = S^*XS^*$, then $X = 0$.

For completeness, we state the following observation.
CHAPTER 4. GENERALIZATIONS

Theorem 4.1.3 Let $A$ be a bounded operator with dense range and $B$ be any bounded operator. If $X$ is a bounded solution of the operator equation $SXB = XA$ then $X = 0$.

Proof: Take the adjoint of the equation and notice that if $\text{Ran} A$ is dense, then $\text{Ker} A^* = \{0\}$.

This leads to a solution of other modifications of the Toeplitz and Hankel equations.

Corollary 4.1.4 If $X$ is a bounded solution of any one of the equations $X = SXS$, $XS^* = SXS$, or $XS^* = SXS^*$, then $X = 0$.

We note that some of the above results (for example: if $SX = XS^*$ then $X = 0$) are well-known among people who study Hankel operators.

4.2 Solutions of some equations involving a parameter

In [6], J. Barría and P. Halmos ask for a description of the solutions of the operator equation $S^*XS = \lambda X$ and how these eigen-operators relate to the case $\lambda = 1$ (Toeplitz operators). The equation was solved by S. Sun [57].

Theorem 4.2.1 (S. Sun, [57]) Let $\lambda \in \mathbb{C}$. The operator equation $S^*XS = \lambda X$ has bounded solutions if and only if $|\lambda| \leq 1$. Moreover,

i) If $|\lambda| = 1$, all solutions are of the form $W_\lambda T$, where $T$ is a Toeplitz operator and $W_\lambda$ is the diagonal unitary operator defined as $W_\lambda e_n = \bar{\lambda}^n e_n$ for all $n$.

ii) If $|\lambda| < 1$, all solutions are compact and of the form

$$\sum_{n=0}^{\infty} \lambda^n ((S^n f) \otimes e_n + e_n \otimes (S^n g))$$

for $f, g \in \mathcal{H}$. 
This suggests the study of the operator equation $\lambda S^*X = XS$, although in this case, this equation does not define a spectral problem. (Note that the modifications of the equations in Corollaries 4.1.2 and 4.1.4 by including multiplication by a non–zero $\lambda$ on one side still have no non–zero solutions.)

To completely solve the operator equation $\lambda S^*X = XS$, we first solve a different equation. The proof of the following theorem uses the techniques found in [57].

**Theorem 4.2.2** Let $A$ be a bounded operator with $\|A\| < 1$ and let $S$ be the unilateral forward shift. Then $X$ is a bounded solution of the operator equation $AX = XS$ if and only if $X$ is compact and $X$ has the following form:

$$X = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n$$

for some vector $\varphi \in \mathcal{H}$.

**Proof:** Assume first that

$$X = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n,$$

for some vector $\varphi \in \mathcal{H}$. Then

$$AX = \sum_{n=0}^{\infty} (A^{n+1} \varphi) \otimes e_n = \sum_{n=0}^{\infty} (A^{n+1} \varphi) \otimes (S^*e_{n+1})$$

$$= \sum_{n=0}^{\infty} ((A^{n+1} \varphi) \otimes e_{n+1}) S = \sum_{n=1}^{\infty} ((A^n \varphi) \otimes e_n) S$$

$$= \sum_{n=1}^{\infty} ((A^n \varphi) \otimes e_n) S + (\varphi \otimes e_0) S = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n S = XS,$$

since $(\varphi \otimes e_0) S = \varphi \otimes (S^*e_0) = 0$.

Now, assume that $X$ satisfies the equation $AX = XS$. As is well known, $SS^* = I - (e_0 \otimes e_0)$ so that $SS^* = I \pmod{\mathcal{K}}$. This implies that $AXS^* = X \pmod{\mathcal{K}}$. Let $\| \cdot \|_e$ denote the essential norm. Then, if $\|X\|_e \neq 0$ (and since $\|A\|_e \leq \|A\| < 1$ and $\|S^*\|_e = 1$) we have

$$\|X\|_e \leq \|A\|_e \|X\|_e \|S^*\|_e < \|X\|_e,$$
which is impossible. Thus \( \|X\| = 0 \); that is, \( X \) is compact.

Define an operator–valued linear transformation \( \hat{\tau} : B(\mathcal{H}) \to B(\mathcal{H}) \) as
\[
\hat{\tau}(Y) = AYS^*.
\]
Since \( \|A\| < 1 \), it follows that \( \|\hat{\tau}\| < 1 \), so \( I - \hat{\tau} \) is invertible and its inverse is given by
\[
(I - \hat{\tau})^{-1} = \sum_{n=0}^{\infty} \hat{\tau}^n.
\]
Now, \( AX = XS \) implies that \( AXS^* = XSS^* = X - X(e_0 \otimes e_0) \), so that \( X - AXS^* = \varphi \otimes e_0 \), where \( \varphi = Xe_0 \). This means that
\[
(I - \hat{\tau})(X) = \varphi \otimes e_0,
\]
so that
\[
X = (I - \hat{\tau})^{-1}(\varphi \otimes e_0) = \sum_{n=0}^{\infty} \hat{\tau}^n(\varphi \otimes e_0) = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n.
\]

Using this theorem, we can solve the operator equation \( \lambda S^* X = XS \).

**Corollary 4.2.3** Let \( \lambda \in \mathbb{C} \) with \( |\lambda| < 1 \). Then \( X \) is a bounded solution of the operator equation \( \lambda S^* X = XS \) if and only if \( X \) is compact and is of the form
\[
X = \sum_{n=0}^{\infty} \lambda^n (S^n \varphi) \otimes e_n
\]
for some vector \( \varphi \in \mathcal{H} \).

**Proof:** Let \( A = \lambda S^* \) and use the previous theorem. \( \square \)

**Corollary 4.2.4** Let \( \lambda \in \mathbb{C} \) with \( |\lambda| > 1 \). Then \( X \) is a bounded solution of the operator equation \( \lambda S^* X = XS \) if and only if \( X \) is compact and is of the form
\[
X = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n e_n \otimes (S^n \varphi)
\]
for some vector \( \varphi \in \mathcal{H} \).
CHAPTER 4. GENERALIZATIONS

Proof: The operator \( X \) is a bounded solution of the operator equation \( \lambda S^*X = XS \), if and only if \( X^* \) satisfies \( S^*X = XS^* \), or equivalently, if and only if \( X^* \) satisfies

\[
\frac{1}{\lambda} S^*X^* = X^*S.
\]

Now, taking \( A = \frac{1}{\lambda} S^* \) in the previous theorem, we obtain that \( X^* \) satisfies the previous equation if and only if \( X^* \) is compact and is of the form

\[
X^* = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n (S^m \varphi) \otimes e_n.
\]

Taking adjoints proves the result. \( \Box \)

The only case that remains is \( |\lambda| = 1 \). In this case, the solutions turn out to be just unitary multiples of old friends.

**Theorem 4.2.5** Let \( |\lambda| = 1 \). Then \( X \) is a bounded solution of the equation \( \lambda S^*X = XS \) if and only if \( X = W_\lambda H \), where \( H \) is a Hankel operator and \( W_\lambda \) is the unitary diagonal operator defined as in Theorem 4.2.1.

**Proof:** First of all, a calculation shows that \( \lambda SW_\lambda = W_\lambda S \). This implies that \( \lambda S = W_\lambda SW_\lambda^* \), so \( \lambda S^* = W_\lambda S^*W_\lambda^* \).

Multiply the previous equation by \( X \) to obtain \( \lambda S^*X = W_\lambda S^*W_\lambda^*X \). Since \( \lambda S^*X = XS \) it follows that \( XS = W_\lambda S^*W_\lambda^*X \), from which \( W_\lambda XS = S^*W_\lambda X \); i.e., \( W_\lambda^*X \) is a Hankel operator.

Conversely, let \( X = W_\lambda H \) for some Hankel operator \( H \). A calculation shows that \( W_\lambda S^* = \lambda S^*W_\lambda \). Then \( XS = W_\lambda HS = W_\lambda S^*H = \lambda S^*W_\lambda H = \lambda S^*X \). \( \Box \)

### 4.3 The operator equation \( S^*X - XS = \lambda X \)

More interesting operators arise from solving the equation \( S^*X - XS = \lambda X \). Let us first point out that reversing the order of \( S \) and \( S^* \) results, again, in only trivial solutions: \( SX - XS^* = \lambda X \) implies that \( (S - \lambda)X = XS^* \) which only has the zero solution by Theorem 4.1.1
It is worth pointing out that there are also no non-trivial solutions to the equations $SX - XS = \lambda X$ or $S^*X - XS^* = \lambda X$ when $\lambda \neq 0$, as we will show presently. We first need a lemma, whose proof was suggested to us by Peter Rosenthal.

**Lemma 4.3.1** Suppose $\lambda \neq 0$. Let $f \in H^2$ be such that $\|(S - \lambda)^n f\| \leq K$ for all $n$ and a fixed number $K > 0$. Then $f = 0$.

**Proof:** As is well known, for any $f \in H^2$ we have

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$  

Let $\epsilon > 0$ and define $A_\epsilon = \{\theta \in [0, 2\pi) : |e^{i\theta} - \lambda| \geq 1 + \epsilon\}$. In fact, choose $\epsilon$ in such a way that the measure of $A_\epsilon$ is non-zero. Then,

$$K^2 \geq \|(S - \lambda)^n f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - \lambda|^{2n} |f(e^{i\theta})|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_{A_\epsilon} |e^{i\theta} - \lambda|^{2n} |f(e^{i\theta})|^2 d\theta$$

$$\geq \frac{1}{2\pi} (1 + \epsilon)^{2n} \int_{A_\epsilon} |f(e^{i\theta})|^2 d\theta.$$  

But this is impossible unless $\int_{A_\epsilon} |f(e^{i\theta})|^2 d\theta = 0$, which implies that $f(e^{i\theta}) = 0$ for $\theta \in A_\epsilon$. But $H^2$ functions cannot vanish in sets of non-zero measure unless they are identically zero. Thus $f = 0$. \[\square\]

We can now prove the following theorem.

**Theorem 4.3.2** Let $\lambda \neq 0$. If $X$ is a bounded solution of the equation $SX - XS = \lambda X$ or of the equation $S^*X - XS^* = \lambda X$, then $X = 0$.

**Proof:** Assume $X$ is a bounded solution of $SX - XS = \lambda X$ for a non-zero $\lambda$. It then follows that $(S - \lambda)^n X = XS^n$ and thus that $(S - \lambda)^n X e_0 = X e_n$ for all $n$. Since $X$ is bounded it follows that $\|(S - \lambda)^n X e_0\| \leq \|X\|$. By the previous lemma, $X e_0 = 0$. But this implies that $X e_n = 0$ for all $n$. That is, $X = 0$.  

57
If $X$ is a solution of $S^*X - XS^* = \lambda X$, taking adjoints we obtain the previous case.

The following theorem is somewhat surprising, in light of the previous results.

**Theorem 4.3.3** Let $|\lambda| < 2$. Then the operator equation $S^*X - XS = \lambda X$ has non-zero bounded solutions.

**Proof:** If $|\lambda| < 2$, then it is always possible to choose a number $a \in \mathbb{C}$, with $|a| < 1$, such that $|a - \lambda| < 1$ (for example, choose $a = \lambda/2$). Then the rank-one operator $X = k_\lambda \otimes k_{a-\lambda}$ is a solution of the equation:

\[
S^*X - XS = (S^*k_\lambda) \otimes k_{a-\lambda} - k_\lambda \otimes (S^*k_{a-\lambda}) = a k_\lambda \otimes k_{a-\lambda} - (a - \lambda) k_\lambda \otimes k_{a-\lambda} = \lambda X
\]

Before going any further, let us realize that the problem of solving the equation $S^*X - XS = \lambda X$ is the problem of finding eigen-operators for the bounded operator-valued linear transformation $\tau : B(H) \rightarrow B(H)$ defined as

$$
\tau(X) = S^*X - XS.
$$

The previous theorem tells us that the disk centred at the origin of radius 2 is contained in the spectrum of $\tau$. A theorem of Rosenblum (see, for example, [53, p. 8]) tells us that $\sigma(\tau) \subset \sigma(S^*) - \sigma(S)$, so that $\sigma(\tau) \subset \{z \in \mathbb{C} : |z| \leq 2\}$ (since $\sigma(S^*) = \sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}$ — see, for example, [53, p. 36]). In conclusion, $\sigma(\tau) = \{z \in \mathbb{C} : |z| \leq 2\}$. This means that there are no non-zero solutions of the equation $S^*X - XS = \lambda X$ when $|\lambda| > 2$.

It turns out, as we will see in the rest of this chapter, that the solutions of $S^*X - XS = \lambda X$ have some properties like those of Hankel operators.

**Definition 4.3.4** We call $X$ a $\lambda$-Hankel operator if $S^*X - XS = \lambda X$. Clearly, a $0$-Hankel operator is just a Hankel operator.
For a fixed $\lambda$, the set of $\lambda$–Hankel operators forms a vector subspace of $B(\mathcal{H})$. As we saw in the proof of Theorem 4.3.3, if $|\lambda| < 2$, for $|a| < 1$ and $|a - \lambda| < 1$, $k_\lambda \otimes k_{a - \lambda}$ is a $\lambda$–Hankel operator. If we choose a sequence of distinct complex numbers $\{a_n\}$ such that $|a_n| < 1$ and $|a_n - \lambda| < 1$, then, for any sequence of complex numbers $\{c_n\}$ such that $\sum_{n=0}^{\infty} |c_n| < \infty$, the operator

$$X = \sum_{n=0}^{\infty} c_n \sqrt{1 - |a_n|^2} \sqrt{1 - |a_n - \lambda|^2} k_{a_n} \otimes k_{a_n - \lambda}$$

is a compact $\lambda$–Hankel operator (since eigenspaces are norm closed). This shows that there are $\lambda$–Hankel operators of arbitrary rank: for example, choose the sequence $\{a_n\}$ in such a way as to make $\{a_n - \lambda\}$ a Blaschke sequence (always possible), and notice that the reproducing kernels are linearly independent.

A natural question arises: are there any non-compact $\lambda$–Hankel operators? To answer this, we will show that some subclasses of $\lambda$–Hankel operators have the property that given any compact set containing zero there is an operator in the subclass having this set as its spectrum (Corollary 4.3.6).

First, we mention some basic properties of $\lambda$–Hankel operators. All of them are known for Hankel operators.

**Theorem 4.3.5** The adjoint of a $\lambda$–Hankel operator is a $(-\lambda)$–Hankel operator. A $\lambda$–Hankel operator is never invertible. Its kernel is an invariant subspace for $S$ and the closure of its range is an invariant subspace for $S^*$. A non-zero $\lambda$–Hankel operator can be self-adjoint only when $\lambda$ is purely imaginary.

**Proof:** We prove only the case $\lambda \neq 0$. For the case $\lambda = 0$ the reader is referred to Power's book [50].

If $X$ is a $\lambda$–Hankel operator, we get $S^*X^* - X^*S = -\lambda X^*$ by taking adjoints, so $X^*$ is a $(-\lambda)$–Hankel operator.

If $X$ is a $\lambda$–Hankel operator, then $(S^* - \lambda)X = XS$. If $X$ was invertible, it would mean that $S$ and $S^* - \lambda$ are similar. But they are not (for example, compare their spectra!).
Let \( f \in \text{Ker} \, X \). Then \( Sf \in \text{Ker} \, X \), since \( XSf = (S^* - \lambda)Xf = 0 \). Thus \( \text{Ker} \, X \) is an invariant subspace for \( S \).

Since \( X^* \) is a \((-\overline{\lambda})\)-Hankel operator, \( \text{Ker} \, X^* \) is an invariant subspace of \( S \). But this means that \((\text{Ker} \, X^*)^\perp \) (which is the closure of the range of \( X \)) is an invariant subspace for \( S^* \).

Suppose \( X \) is a non-zero self-adjoint \( \lambda \)-Hankel operator. Then \( X \) is both a \( \lambda \)-Hankel operator and a \((-\overline{\lambda})\)-Hankel operator. But this means that \( \lambda X = S^*X - XS = (-\overline{\lambda})X \), which implies that \( \lambda + \overline{\lambda} = 0 \); i.e., \( \lambda \) is purely imaginary.

As we saw in Chapter 3, a Hankel operator can have any compact subset of the complex plane that contains the origin as its spectrum. The same theorem holds for \( \lambda \)-Hankel operators when \( \lambda \) is purely imaginary.

**Corollary 4.3.6** Let \( |\lambda| < 2 \) be a purely imaginary number and let \( \sigma \) be any compact subset of the complex plane containing zero. Then there exists a \( \lambda \)-Hankel operator \( X \) such that \( \sigma(X) = \sigma \).

**Proof:** Let \( \mathcal{F} \) be the set of \( \lambda \)-Hankel operators. We only need to check that \( \mathcal{F} \) satisfies the conditions of Theorem 3.1.1 in Chapter 3. As mentioned before, the set of \( \lambda \)-Hankel operators is a vector subspace of \( B(\mathcal{H}) \) and consists of non-invertible operators (see Theorem 4.3.5). Since the set of \( \lambda \)-Hankel operators is the set of solutions of the equation \( S^*X - XS = \lambda X \), it follows that this set is closed in the strong operator topology (even in the weak operator topology!).

Let \( \{a_n\} \) be a sequence in the open unit disk with imaginary part equal to \( \lambda/(2i) \), such that \( |a_n| \to 1 \) as \( n \to \infty \) and with increasing real part (clearly such a sequence always exists). Then \( \overline{a_n} = a_n - \lambda \), which implies that \( k_{a_n} \otimes k_{\overline{a_n}} \) is in \( \mathcal{F} \).

We can then define \( \varphi_n = \sqrt{1 - |a_n|^2} \, k_{\overline{a_n}} \). Notice that \( \varphi_n \otimes \varphi_n \in \mathcal{F} \) and

\[
(\varphi_n, \varphi_m) = \frac{\sqrt{1 - |a_n|^2} \sqrt{1 - |a_m|^2}}{1 - a_n \overline{a_m}} \to 0, \quad \text{as} \quad n \to \infty.
\]

Applying Theorem 3.1.1 in Chapter 3 to the set \( \mathcal{F} \) we obtain the desired result. \( \square \)
In particular this result partially answers the question of existence of non-compact \( \lambda \)-Hankel operators.

**Corollary 4.3.7** Let \( |\lambda| < 2 \) be a purely imaginary number. Then there exist non-compact \( \lambda \)-Hankel operators.

**Proof:** Compact operators can only have discrete spectrum which accumulates, at most, at zero.

It turns out that we can also get non-compact \( \lambda \)-Hankel operators if \( |\lambda| = 1 \). We first need a lemma whose proof is also inspired by [57].

**Lemma 4.3.8** Let \( \lambda \) and \( \mu \) be two complex numbers of modulus one. Suppose that \( X \) is a \( \lambda \)-Hankel operator. If \( Y = W_{\bar{\mu} \lambda} X W_{\bar{\mu} \lambda} \), then \( Y \) is a \( \mu \)-Hankel operator. (Here \( W_{\mu \lambda} \) is the diagonal unitary operator as defined in Theorem 4.2.1.)

**Proof:** If \( X \) satisfies \( S^*X - XS = \lambda X \), then \( (\mu \bar{\lambda})S^*X - (\mu \bar{\lambda})XS = \mu X \). Left and right multiply by \( W_{\bar{\mu} \lambda} \) to get

\[
(\mu \bar{\lambda})W_{\bar{\mu} \lambda}S^*XW_{\bar{\mu} \lambda} - (\mu \bar{\lambda})W_{\bar{\mu} \lambda}XSW_{\bar{\mu} \lambda} = \mu W_{\bar{\mu} \lambda}XW_{\bar{\mu} \lambda}.
\]

It is easy to verify that \( (\mu \bar{\lambda})SW_{\bar{\mu} \lambda} = W_{\bar{\mu} \lambda}S \) and \( (\mu \bar{\lambda})W_{\bar{\mu} \lambda}S^* = S^*W_{\bar{\mu} \lambda} \). Using these equalities, we obtain

\[
S^*W_{\bar{\mu} \lambda}XW_{\bar{\mu} \lambda} - W_{\bar{\mu} \lambda}XW_{\bar{\mu} \lambda}S = \mu W_{\bar{\mu} \lambda}XW_{\bar{\mu} \lambda},
\]

so \( S^*Y - YS = \mu Y \).

This lemma tells us that, for unimodular \( \lambda \), it is sufficient to restrict ourselves to one choice of \( \lambda \) when studying \( \lambda \)-Hankel operators. This leads to the following theorem.

**Theorem 4.3.9** Let \( \mu \) be a complex number of modulus one. Then there exist non-compact \( \mu \)-Hankel operators.
Proof: We know that there exist non-compact $\lambda$–Hankel operators for $\lambda$ purely imaginary (Corollary 4.3.7). In particular for $\lambda = i$ there are non-compact $\lambda$–Hankel operators. Let $X$ be one of them. By the previous lemma $Y = W_{\mu \lambda} X W_{\mu \lambda}$ is a $\mu$–Hankel operator, which cannot be compact (if it were, $X$ would also be compact). \qed

Are there any non-compact $\lambda$–Hankel operators in the case where $|\lambda| < 2$ is not purely imaginary and is not of modulus one? We do not have a definite answer yet, but we conjecture there are many of them.

The careful reader has probably noticed that we have not talked about $\lambda$–Hankel operators for the cases $|\lambda| = 2$. Although clearly there exist approximate solutions to the equation $S^*X - XS = \lambda X$, we have not been able to obtain non-zero $\lambda$–Hankel operators in this case. Are there any?

4.4 Other properties of $\lambda$–Hankel operators

In this section, we will study some other properties of $\lambda$–Hankel operators: in particular, symbols, when they are finite–rank, and their relations to analytic and co-analytic Toeplitz operators.

If $X$ is a $\lambda$–Hankel operator, we call $X_{e_0}$ the symbol of $X$. The reason for this name is that, since $S^*X - XS = \lambda X$, we have that $(S^* - \lambda)X = XS$, so $(S^* - \lambda)^n X = XS^n$, which implies that

$$X_{e_n} = (S^* - \lambda)^n X_{e_0};$$

i.e., $X$, as a densely defined operator on the polynomials, is uniquely determined by $X_{e_0}$. In fact, the following formula will be useful

$$\langle X_{e_m}, e_n \rangle = \langle XS^m e_0, e_n \rangle = \langle (S^* - \lambda)^m X_{e_0}, e_n \rangle$$

$$= \langle X_{e_0}, (S - \lambda)^m e_n \rangle = \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \langle X_{e_0}, S^k e_n \rangle$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \langle X_{e_0}, e_{k+n} \rangle. \tag{4.1}$$
The reader should be warned that our definition of symbol differs slightly from the one used for classical Hankel operators. In the case of a classical Hankel operator $H$, if $He_0 = \psi$, then the function $\phi$, defined as $\phi(z) = \psi(\overline{z})$, is a symbol of $H$ in the classical sense.

As we saw before, a $\lambda$–Hankel operator is never invertible. It turns out that they are not even essentially invertible.

**Theorem 4.4.1** Let $X$ be a bounded $\lambda$–Hankel operator. Then $0 \in \sigma_e(X)$.

**Proof:** The case $\lambda = 0$ was discussed in Chapter 1. If $\lambda \neq 0$, then we know that $(S^* - \lambda)X = XS$. If $X$ was essentially invertible $S^* - \lambda$ and $S$ would be essentially similar. But, since $\sigma_e(S) = \sigma_e(S^*)$ equals the unit circle, $S^* - \lambda$ and $S$ cannot be essentially similar. □

A curious property of the set of $\lambda$–Hankel operators is that it is invariant when multiplied on the right by an analytic Toeplitz operator or on the left by a co–analytic Toeplitz operator.

**Theorem 4.4.2** Let $X$ be a $\lambda$–Hankel operator, $T$ an analytic Toeplitz operator and $T'$ a co–analytic Toeplitz operator. Then $XT$ and $T'X$ are $\lambda$–Hankel operators.

**Proof:** As mentioned in Chapter 1, $T$ is an analytic Toeplitz operator if and only if $ST = TS$; and $T'$ is a co–analytic Toeplitz operator if and only if $S^*T' = T'S^*$.

Then, $(XT)S = XST = (S^* - \lambda)(XT)$, so $XT$ is $\lambda$–Hankel. Also $S^*(T'X) = T'S^*X = (T'X)(S + \lambda)$, so $T'X$ is a $\lambda$–Hankel operator. □

The classical theorem of Kronecker states that a Hankel matrix is of finite rank if and only if its symbol is a rational function. We have a similar theorem for $\lambda$–Hankel operators.

**Theorem 4.4.3** Let $X$ be a $\lambda$–Hankel operator with symbol $Xe_0 = \varphi \in \mathbb{H}^2$. Then $X$ has a finite–rank matrix if and only if $\varphi$ is a rational function.
Proof: The columns of the matrix of $X$ are just the vectors $(S^* - \lambda)^n \varphi$. That means that $X$ is of finite–rank at most $N$ if and only if there exist constant numbers $c_0, c_1, c_2, \cdots, c_N$, not all zero, such that

$$\sum_{n=0}^{N} c_n (S^* - \lambda)^n \varphi = 0.$$ 

Let $d_k = \sum_{n=k}^{N} \binom{n}{k} (-\lambda)^{n-k} c_n$. It is not hard to see that $d_0 = d_1 = d_2 = \cdots = d_N = 0$ if and only if $c_0 = c_1 = c_2 = \cdots = c_N = 0$ (for example, do a calculation similar to the one in Lemma 4.5.2 below). But then the equation

$$\sum_{n=0}^{N} c_n (S^* - \lambda)^n \varphi = \sum_{n=0}^{N} c_n \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} S^* S^k \varphi$$

$$= \sum_{k=0}^{N} \left( \sum_{n=k}^{N} \binom{n}{k} (-\lambda)^{n-k} c_n \right) S^* S^k \varphi$$

$$= \sum_{k=0}^{N} d_k S^* S^k \varphi.$$ 

implies that the vectors $\{S^* S^k \varphi\}_{k=0}^{N}$ are linearly dependent if and only if the vectors $\{(S^* - \lambda)^n \varphi\}_{n=0}^{N}$ are linearly dependent; i.e., the Hankel operator with symbol $\varphi$ is of finite rank at most $N$ if and only if $X$ is of finite rank. But the Hankel operator with symbol $\varphi$ is of finite rank if and only if $\varphi$ is a rational function, as stated by Kronecker's Theorem (see Chapter 1). \hfill \square

Notice that this theorem does not say that $X$ is bounded; it says that its matrix is of finite–rank. This of course brings us to the obvious question: when is $X$ bounded? That is, for what symbols $\varphi$ is the $\lambda$–Hankel operator $X$ (densely defined on polynomials), with $X e_0 = \varphi$, bounded? We will formulate some necessary conditions on the symbol of $X$ implied by the continuity of $X$ at the end of this chapter. Is there a Nehari–type theorem?

The same question arises for compactness: for what symbols $\varphi$ is the $\lambda$–Hankel operator $X$ (densely defined on polynomials), with $X e_0 = \varphi$, compact? Again, some
sufficient conditions on the symbol that guarantee compactness will be studied at the end of this chapter. Is there a Hartman–type theorem?

We answer the question of boundedness for positive $\lambda$–Hankel operators in the next section.

4.5 Positivity of $\lambda$–Hankel operators

Among the properties that $\lambda$–Hankel operators share with Hankel operators is the characterization of positivity. Since $\lambda$–Hankel operators may only be self–adjoint when $\lambda$ is purely imaginary, we restrict ourselves throughout this section to $\lambda$ purely imaginary.

Let $X$ be a (not necessarily bounded, but at least defined on polynomials) $\lambda$–Hankel operator. Assume there exists a non–decreasing function $\mu$ on the real line, thought of as a measure $d\mu$ on the real line throughout the rest of this paper, such that

$$ (Xe_0, e_n) = \int_{\mathbb{R}} \left( t + \frac{\lambda}{2} \right)^n d\mu(t). \quad (4.2) $$

This expression completely characterizes the symbol of the operator $X$, and thus it also characterizes $X$ (as a $\lambda$–Hankel operator densely defined on polynomials).

If $f$ and $g$ are polynomials, $f = \sum_{m=0}^{M} a_m e_m$ and $g = \sum_{n=0}^{N} b_n e_n$, then we have (using equation 4.1)

$$ (Xf, g) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_m b_n (Xe_m, e_n) $$

$$ = \sum_{m=0}^{M} \sum_{n=0}^{N} a_m b_n \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (Xe_0, e_{k+n}) $$

$$ = \sum_{m=0}^{M} \sum_{n=0}^{N} a_m b_n \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \int_{\mathbb{R}} \left( t + \frac{\lambda}{2} \right)^{k+n} d\mu(t) $$

$$ = \sum_{m=0}^{M} \sum_{n=0}^{N} a_m b_n \int_{\mathbb{R}} \left( t + \frac{\lambda}{2} \right)^n \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \left( t + \frac{\lambda}{2} \right)^k d\mu(t) $$
In conclusion, we get that whenever \( f \) and \( g \) are polynomials, therefore, if \( f \) is a polynomial we obtain

\[
(Xf, g) = \int_R f \left( t - \frac{\lambda}{2} \right) g \left( t - \frac{\lambda}{2} \right) d\mu(t),
\]

whenever \( f \) and \( g \) are polynomials. Therefore, if \( f \) is a polynomial we obtain

\[
(Xf, f) = \int_R \left| f \left( t - \frac{\lambda}{2} \right) \right|^2 d\mu(t),
\]

so \( (Xf, f) \geq 0 \) for all polynomials \( f \). Thus \( X \) is positive.

Is the converse true? Namely, if \( X \) is a positive \( \lambda \)-Hankel operator, does there exist a non-decreasing function \( \mu \) such that equation 4.2 holds? The answer is yes. The case for Hankel operators is well-known (see Power [49]) and its solution is intimately related to the Hamburger moment problem. The Hamburger moment problem is a classical problem in the theory of moments that relates the positivity of a Hankel matrix with the solution of a moment problem on the real line. For some very interesting results in the theory of moments, the reader should see Curto and Fialkow [16]. For other operator-theoretic problems in the theory of moments, the reader should see Curto [15].

We will use the solution of the Hamburger moment problem to solve the case of \( \lambda \)-Hankel operators. We need some preliminary lemmas.

**Lemma 4.5.1** \( \sum_{s=k}^{n} \binom{n}{s} (-1)^{n-s} = \delta_{n,k} \), where \( \delta_{n,k} \) is the Kronecker delta.

**Proof:** Apply the binomial theorem twice and change the order of the sums to obtain

\[
x^n = \sum_{s=0}^{n} \binom{n}{s} (x+1)^s (-1)^{n-s}
\]
which implies the desired result.

Given a complex-valued sequence \( \{\mu_n\} \), we define a sequence \( \{m_n\} \) as

\[
m_n = \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{\lambda}{2} \right)^{n-k} \mu_k.
\]

It turns out that knowing \( \{m_n\} \) allows us to recover \( \{\mu_n\} \).

Lemma 4.5.2 \( \sum_{s=0}^{n} \binom{n}{s} \left( \frac{\lambda}{2} \right)^{n-s} m_s = \mu_n \).

Proof: Use the definition of \( \{m_n\} \), interchange the sums, and use the previous lemma to see that

\[
\sum_{s=0}^{n} \binom{n}{s} \left( \frac{\lambda}{2} \right)^{n-s} m_s = \sum_{s=0}^{n} \binom{n}{s} \left( \frac{\lambda}{2} \right)^{n-s} \sum_{k=0}^{s} \binom{s}{k} \left( -\frac{\lambda}{2} \right)^{s-k} \mu_k
\]

\[
= \sum_{k=0}^{n} \sum_{s=k}^{n} \binom{n}{s} \binom{s}{k} \left( \frac{\lambda}{2} \right)^{n-s} \left( -\frac{\lambda}{2} \right)^{s-k} \mu_k
\]

\[
= \sum_{k=0}^{n} \left( \sum_{s=k}^{n} \binom{n}{s} \binom{s}{k} (-1)^{n-s} \right) \left( -\frac{\lambda}{2} \right)^{n-k} \mu_k
\]

\[
= \sum_{k=0}^{n} \delta_{n,k} \left( -\frac{\lambda}{2} \right)^{n-k} \mu_k
\]

\[
= \mu_n
\]

We need some notation that comes from the study of moment problems.

**Definition 4.5.3** Let \( \lambda \) be purely imaginary. We say that the sequence \( \{\nu_n\} \) is \( \lambda \)-positive if for all polynomials \( p(x) = a_n x^n + \cdots + a_1 x + a_0 \) such that \( p(x + \frac{\lambda}{2}) \geq 0 \) for \( x \in \mathbb{R} \) we have \( \sum_{k=0}^{n} a_k \nu_k \geq 0 \).
This agrees with the classical terminology when $\lambda = 0$ (see Widder [61, p. 127]). We also agree to say that a sequence is positive whenever it is 0-positive. The following lemma relates $\lambda$-positivity to positivity.

**Lemma 4.5.4** Let $\{\mu_n\}$ be a complex sequence and $\{m_n\}$ be defined by

$$m_n = \sum_{k=0}^{n} \binom{n}{k} (-\frac{\lambda}{2})^{n-k} \mu_k.$$ 

If $\{\mu_n\}$ is $\lambda$-positive then $\{m_n\}$ is positive.

**Proof:** Suppose $q(x) = b_n x^n + \cdots + b_1 x + b_0$ is a polynomial and $q(x) \geq 0$ for all $x \in \mathbb{R}$. Define $p(x) = q\left(x - \frac{\lambda}{2}\right)$. Then $p\left(x + \frac{\lambda}{2}\right) = q(x) \geq 0$ for all $x \in \mathbb{R}$.

A calculation shows that

$$p(x) = \sum_{s=0}^{n} \left( \sum_{k=s}^{n} \binom{k}{s} \left(-\frac{\lambda}{2}\right)^{k-s} b_k \right) x^s,$$

so, since $p\left(x + \frac{\lambda}{2}\right) \geq 0$ for all $x \in \mathbb{R}$, it follows from the definition of $\lambda$-positivity that

$$\sum_{s=0}^{n} \left( \sum_{k=s}^{n} \binom{k}{s} \left(-\frac{\lambda}{2}\right)^{k-s} b_k \right) \mu_s \geq 0.$$

By changing the order of the sums, and recalling the definition of the sequence $\{m_n\}$, we can see that

$$0 \leq \sum_{s=0}^{n} \left( \sum_{k=s}^{n} \binom{k}{s} \left(-\frac{\lambda}{2}\right)^{k-s} b_k \right) \mu_s$$

$$= \sum_{k=0}^{n} \left( \sum_{s=0}^{k} \binom{k}{s} \left(-\frac{\lambda}{2}\right)^{k-s} \mu_s \right) b_k$$

$$= \sum_{k=0}^{n} m_k b_k,$$

which implies $\{m_n\}$ is positive. \qed

We need the following definition.
Definition 4.5.5 Let \( \lambda \) be a purely imaginary number. We define the \( \lambda \)-moment operator \( M_\lambda \) associated to the \( \lambda \)-Hankel operator \( X \) to be

\[
M_\lambda(p) = (X e_0, p^*),
\]

where \( M_\lambda \) operates on polynomials \( p \).

We need the following result.

Lemma 4.5.6 Suppose \( q(x) \) is a polynomial with real coefficients and define \( q_-(x) = q(x - \frac{\lambda}{2}) \) and \( q_+(x) = q(x + \frac{\lambda}{2}) \). Then \( M_\lambda(q^2) = (X q_+, q_+) \).

Proof: Since \( X \) is a \( \lambda \)-Hankel operator and \( \lambda \) is purely imaginary, it follows that

\[
\left( S + \frac{\lambda}{2} \right)^* X = X \left( S + \frac{\lambda}{2} \right);
\]

thus, since \( (S + \frac{\lambda}{2})^n f = (z + \frac{\lambda}{2})^n f \) for all \( f \), it follows that

\[
\left( X \left( z + \frac{\lambda}{2} \right)^n, \left( z + \frac{\lambda}{2} \right)^m \right) = \left( X e_0, \left( z + \frac{\lambda}{2} \right)^n \left( z + \frac{\lambda}{2} \right)^m \right),
\]

for all \( n \) and \( m \). It follows from this, and the fact that \( q \) has real coefficients, that

\[
\left( X q \left( z + \frac{\lambda}{2} \right), q \left( z + \frac{\lambda}{2} \right) \right) = \left( X e_0, q \left( z + \frac{\lambda}{2} \right) q \left( z + \frac{\lambda}{2} \right) \right).
\]

But \( q^* = q_+ \), so \( (X q_+, q_+) = (X e_0, q_+^2) = (X e_0, q_{z^2}) = M_\lambda(q^2) \). \( \square \)

Clearly, \( M_\lambda \) is linear, and if \( p(x) = x^n \), then \( M_\lambda(p) = (X e_0, e_n) \). If \( \mu_n = (X e_0, e_n) \), then it is clear that \( \{\mu_n\} \) is \( \lambda \)-positive if and only if for all polynomials \( p(x) = a_n x^n + \cdots + a_1 x + a_0 \) such that \( p \left( x + \frac{\lambda}{2} \right) \geq 0 \) for \( x \in \mathbb{R} \) we have that \( M_\lambda(p) \geq 0 \). Using this fact, we obtain the following theorem.

Theorem 4.5.7 Let \( X \) be a \( \lambda \)-Hankel operator and \( \mu_n = (X e_0, e_n) \). If \( X \) is a positive operator, then \( \{\mu_n\} \) is a \( \lambda \)-positive sequence.
Proof: Fix a polynomial \( p \) such that \( p \left( x + \frac{1}{2} \right) \geq 0 \) for all \( x \in \mathbb{R} \). By the remark preceding the statement of the theorem, it suffices to show that \( M_\lambda(p) \geq 0 \).

Define \( f \) as \( f(x) = p \left( x + \frac{1}{2} \right) \). Clearly \( f \) is a polynomial and has real coefficients (polynomials that take only real values on the real numbers have real coefficients). Since \( f \) is positive-valued on the reals, it follows, by a theorem of Pólya and Szegő ([45, p. 77]), that \( f \) can be written as \( f(x) = g^2(x) + h^2(x) \) for some real polynomials \( g \) and \( h \).

If we define \( g_- \) and \( h_- \) as in the statement of the previous lemma, we then have \( p = g_-^2 + h_-^2 \), so \( M_\lambda(p) = M_\lambda(g_-^2) + M_\lambda(h_-^2) = (Xg_+, g_+) + (Xh_+, h_+) \). But this last expression is positive, since \( X \) is positive. It follows that \( M_\lambda(p) \geq 0 \).

We can now answer the question about existence of measures corresponding to positive \( \lambda \)-Hankel operators. Notice that this theorem extends the solution of the Hamburger moment problem to horizontal lines in the complex plane.

**Theorem 4.5.8** A \( \lambda \)-Hankel operator \( X \) is positive if and only if there exists a non-decreasing function \( \mu \) on the real line such that

\[
(Xe_0, e_n) = \int_{\mathbb{R}} \left( t + \frac{\lambda}{2} \right)^n d\mu(t),
\]

for all \( n \).

Proof: As we showed at the beginning of this section, the existence of the measure \( d\mu \) satisfying the condition above implies the positivity of \( X \).

We prove the converse. Suppose \( X \) is positive. By Theorem 4.5.7, the sequence \( \{\mu_n\} \), where \( \mu_n = (Xe_0, e_n) \), is \( \lambda \)-positive. By Lemma 4.5.4 this implies that the sequence \( \{m_n\} \), where \( m_n \) is defined as in equation 4.5, is positive. But the solution of the Hamburger moment problem implies that, for a positive sequence, there exists a non-decreasing function \( \mu \) on the real line such that (see, for example, [61, p. 129])

\[
m_n = \int_{\mathbb{R}} t^n d\mu(t).
\]
By Lemma 4.5.2, it follows that
\[
\mu_n = \sum_{s=0}^{n} \binom{n}{s} \left(\frac{\lambda}{2}\right)^{n-s} \int_{\mathbb{R}} t^s d\mu(t) \\
= \int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^n d\mu(t),
\]
by the binomial theorem.

Consider the following example. If \(d\mu\) is the atomic probability measure at \(a \in \mathbb{R}\), we have
\[
\int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^n d\mu(t) = \left(a + \frac{\lambda}{2}\right)^n.
\]
If this is the measure corresponding to a positive bounded \(\lambda\)-Hankel operator, we must have \(|a + \frac{\lambda}{2}| < 1\). Let
\[
\gamma = \sqrt{1 - \left|\frac{\lambda}{2}\right|^2},
\]
so that \(|\gamma + \frac{\lambda}{2}| = 1\). Then, \(|a + \frac{\lambda}{2}| < 1\) if and only if \(a \in (-\gamma, \gamma)\). In this case, the atomic measure \(d\mu\) corresponds to the rank one positive \(\lambda\)-Hankel operator
\[
\frac{k_{-\frac{\lambda}{2}}}{{a + \frac{\lambda}{2}}} \otimes \frac{k_{\frac{\lambda}{2}}}{{a + \frac{\lambda}{2}}}
\]

This suggests studying those measures that are supported on \((-\gamma, \gamma)\). We have the following theorem, analogous to the classical theorem of Widom [62], which characterizes boundedness of the operator in terms of the speed of decay of the measure near the boundary of its support.

Before stating the theorem, we need the following definition.

**Definition 4.5.9** A positive Borel measure supported on the interval \((-\gamma, \gamma)\) is called a Carleson measure if \(\mu(a, \gamma) = O(\gamma - a)\) and \(\mu(-\gamma, -a) = O(\gamma - a)\) for \(a \to \gamma^-\).

As we will see in the proof of the theorem, this definition agrees with the classical definition of a Carleson measure on the disk, as defined in [21, p. 157], when we view our measure as a measure on the disk.
Theorem 4.5.10 Let $|\lambda| < 2$ be a purely imaginary number. Let $X$ be a positive $\lambda$–Hankel operator and suppose that the measure $d\mu$ corresponding to $X$ is supported on $(-\gamma, \gamma)$. Then $X$ is bounded if and only if $d\mu$ is a Carleson measure.

Proof: Assume first that $X$ is bounded. We need the following generalization of the calculation in equation 4.4.

Claim 1 If $|w| < 1$ then

$$ (Xk_w, k_w) = \int_{-\gamma}^{\gamma} \left| \frac{1}{1 - w \left( t - \frac{\lambda}{2} \right)} \right|^2 d\mu(t). $$

Proof of Claim: By equation 4.4, if we define $k_{w^n}(z) = \sum_{k=0}^n w^k z^k$, we have

$$ (Xk_{w^n}, k_{w^n}) = \int_{-\gamma}^{\gamma} \left| k_{w^n}(t) - \frac{\lambda}{2} \right|^2 d\mu(t). $$

Clearly, $k_{w^n} \to k_w$, and, since $X$ is bounded, we have

$$ (Xk_w, k_w) = \lim_{n \to \infty} \int_{-\gamma}^{\gamma} \left| k_{w^n}(t) - \frac{\lambda}{2} \right|^2 d\mu(t) = \int_{-\gamma}^{\gamma} \left| \frac{1}{1 - w \left( t - \frac{\lambda}{2} \right)} \right|^2 d\mu(t), $$

by Lebesgue’s Dominated Convergence Theorem. This establishes Claim 1. \hfill \Box

Now, for $0 < a < \gamma$, let

$$ f = \frac{k_{a + \frac{\lambda}{2}}}{\| k_{a + \frac{\lambda}{2}} \|.} $$

It then follows from Claim 1 that

$$ (Xf, f) = \int_{-\gamma}^{\gamma} \frac{1 - \left| a + \frac{\lambda}{2} \right|^2}{\left| 1 - (a + \frac{\lambda}{2}) \left( t - \frac{\lambda}{2} \right) \right|^2} d\mu(t) \geq \int_{a}^{\gamma} \frac{1 - \left| a + \frac{\lambda}{2} \right|^2}{\left| 1 - (a + \frac{\lambda}{2}) \left( t - \frac{\lambda}{2} \right) \right|^2} d\mu(t). $$

(4.7)
We need to estimate the denominator of the integrand. We have
\[
\left| 1 - \left( a + \frac{\lambda}{2} \right) \left( t - \frac{\lambda}{2} \right) \right|^2 \leq \max \left\{ \left| 1 - \left( a + \frac{\lambda}{2} \right) \left( a - \frac{\lambda}{2} \right) \right|^2, \left| 1 - \left( a + \frac{\lambda}{2} \right) \left( \gamma - \frac{\lambda}{2} \right) \right|^2 \right\}.
\]
for \( t \in (a, \gamma) \), by noticing that the left-hand side is a real quadratic polynomial in \( t \) and the coefficient of the quadratic term is positive. We then have two cases.

Case i) \[ |1 - (a + \frac{\lambda}{2}) (a - \frac{\lambda}{2})|^2 \geq |1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})|^2. \]

From equation 4.7, it follows that
\[
(Xf, f) \geq \int_a^\gamma \frac{1 - |a + \frac{\lambda}{2}|^2}{|1 - (a + \frac{\lambda}{2}) (a - \frac{\lambda}{2})|^2} d\mu(t) = \frac{\mu(a, \gamma)}{1 - |a + \frac{\lambda}{2}|^2}.
\]
But, since \( X \) is bounded and \( \|f\| = 1 \), we have
\[
\mu(a, \gamma) \leq \|X\| \left( 1 - \left| a + \frac{\lambda}{2} \right|^2 \right).
\]
It is easy to see that \( 1 - |a + \frac{\lambda}{2}| \leq \gamma - a \) and that \( 1 + |a + \frac{\lambda}{2}| \leq 2 \). Then it follows from the previous inequality that
\[
\mu(a, \gamma) \leq 2\|X\| (\gamma - a); \]
i.e., \( \mu(a, \gamma) = O(\gamma - a) \).

Case ii) \[ |1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})|^2 \geq |1 - (a + \frac{\lambda}{2}) (a - \frac{\lambda}{2})|^2. \]

From equation 4.7, it follows that
\[
(Xf, f) \geq \int_a^\gamma \frac{1 - |a + \frac{\lambda}{2}|^2}{|1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})|^2} d\mu(t) = \frac{1 - |a + \frac{\lambda}{2}|^2}{|1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})|^2} \mu(a, \gamma).
\]
But, since \( X \) is bounded and \( \|f\| = 1 \), we have
\[
\mu(a, \gamma) \leq \|X\| \frac{1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})}{1 - |a + \frac{\lambda}{2}|^2} \left( \frac{\gamma - a}{1 - |a + \frac{\lambda}{2}|^2} \right). \tag{4.8}
\]
To finish this case, we need the following claim.
Claim 2 Suppose $|z_0| = 1$ and $\Re z_0 > 0$. Then there exists $h_0 > 0$ and $K > 0$ such that, for all $0 < h < h_0$, if $z = z_0 - h$, then $|1 - zz_0|^2 \leq K(1 - |z|^2)(z_0 - z)$.

Proof of Claim: It is clear that if $z = z_0 - h$ then $|1 - zz_0|^2 = |z_0 - z|^2 = h^2$. Let $z_0 = \cos \theta_0 + i\sin \theta_0$. Then, $1 - |z|^2 = 1 - |z_0 - h|^2 = 2h\cos \theta_0 - h^2$.

Since $\cos \theta_0 > 0$, choose $h_0$ such that $0 < h_0 < 2\cos \theta_0$ and $K = 1/(2\cos \theta_0 - h_0)$. Then, if $0 < h < h_0$ and $z = z_0 - h$, it follows that $1/K \leq 2\cos \theta_0 - h$, which implies that $h^2 \leq K(2h\cos \theta_0 - h^2)h$, which implies that $|1 - zz_0|^2 \leq K(1 - |z|^2)(z_0 - z)$. This establishes Claim 2.

Set $z_0 = \gamma + \frac{\lambda}{2}$. Since $\gamma > 0$ there exists a $h_0 > 0$ and a $K > 0$ for which the conclusion of Claim 2 holds. Choose $a_0 = \gamma - h_0$. Then, if $a_0 < a < \gamma$ and if $z = a + \frac{\lambda}{2}$, it follows that $z_0 - z = \gamma - a < h_0$, so $|1 - zz_0|^2 \leq K(1 - |z|^2)(z_0 - z)$ for some fixed $K$. But this implies that

$$\frac{|1 - (a + \frac{\lambda}{2}) (\gamma - \frac{\lambda}{2})|^2}{1 - |a + \frac{\lambda}{2}|^2} \leq K(\gamma - a), \quad \text{for } a_0 < a < \gamma.$$ 

Combine this with equation 4.8 to get

$$\mu(a, \gamma) \leq K\|X\|((\gamma - a), \quad \text{for } a_0 < a < \gamma;$$ 

i.e., $\mu(a, \gamma) = O(\gamma - a)$.

An analogous calculation shows that $\mu(-\gamma, -a) = O(\gamma - a)$.

Now let us assume that $\mu$ is a Carleson measure. We will show that $X$ is bounded. To achieve this, we will first show that $\mu$ is a Carleson measure in the classical sense (see Duren [21, p. 157] for the definition).

We consider $d\mu$ to be a measure on the unit disk supported on the set $(-\gamma, \gamma) + \frac{\lambda}{2} := \{t + \frac{\lambda}{2} : -\gamma < t < \gamma\}$ (just the translation of the interval $(-\gamma, \gamma)$ up by $\frac{\lambda}{2}$) instead of the interval $(-\gamma, \gamma)$.

Given $0 < h < 1$ and $\theta_0 \in [0, 2\pi]$, let $S_h$ be a Carleson sector; i.e.,

$$S_h = \{z = re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\}.$$
We must prove that $\sup_h \frac{\mu(S_h)}{h} < \infty$. It suffices to consider $h$ small. If $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} = \emptyset$, then $\mu(S_h) = 0$. If $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} \neq \emptyset$, then

$$\mu(S_h) = \mu \left( S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} \right) \leq C \text{ length of } \left( S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} \right),$$

where $C$ is a constant (coming from our definition of Carleson measure) and length of $(S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2})$ means the length of an interval inside the sector $S_h$ which contains $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2}$. But the length of said interval is less than the perimeter of the sector $S_h$, which can be easily seen to be less than or equal to $4h$.\(^1\) Thus

$$\frac{\mu(S_h)}{h} \leq 4C,$$

so

$$\sup_h \frac{\mu(S_h)}{h} < \infty;$$

i.e., $d\mu$ is a Carleson measure in the classical sense.

Now, as in equation 4.3, if $f$ and $g$ are polynomials, we have

$$(Xf, g) = \int_{-\gamma}^{\gamma} f \left( t - \frac{\lambda}{2} \right) g \left( t - \frac{\lambda}{2} \right) d\mu(t).$$

But this implies that

$$(Xf, g) = \int_{-\gamma}^{\gamma} f^* \left( t + \frac{\lambda}{2} \right) g^* \left( t + \frac{\lambda}{2} \right) d\mu(t).$$

Thinking of $d\mu$ as the measure in the disk $D$ as before, this becomes

$$(Xf, g) = \int_D f^*(z) g^*(z) d\mu(z).$$

But since $d\mu$ is a Carleson measure (in the classical sense), a theorem of Carleson (see, for example [21, p. 157]) implies that

$$|(Xf, g)| \leq \int_D |f^*(z) g(z)| d\mu(z) \leq C \|fg\|_1,$$

\(^1\)I thank Kobi Snitz for noticing this fact, which simplified the proof enormously.
CHAPTER 4. GENERALIZATIONS

where \( \| \cdot \|_1 \) is the norm on the Hardy space \( H^1 \).

But, as is well known (by the Cauchy–Schwarz inequality), \( \|fg\|_1 \leq \|f\| \|g\| \), so

\[
| (Xf, g) | \leq C \|f\| \|g\|;
\]

i.e., \( X \) is bounded.

4.6 Boundedness and compactness of \( \lambda \)-Hankel operators

In this section we will obtain some conditions that guarantee that a \( \lambda \)-Hankel operator is Hilbert–Schmidt. We also obtain a necessary condition for boundedness, reminiscent of Nehari’s theorem, and we offer a conjecture for a sufficient and necessary condition for boundedness of a \( \lambda \)-Hankel operator. We need some lemmas.

Lemma 4.6.1 Let \( \lambda \in \mathbb{C} \) such that \( |\lambda| < 1 \) and let \( a \in \mathbb{R} \) such that \( 0 < a < 1 - |\lambda| \). Suppose \( \varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n \) is an analytic function such that \( |\varphi_n| \leq a^n \) for all \( n \). Then

\[
\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2 < \infty.
\]

Proof: We need to estimate \( \|(S^* - \lambda)^n \varphi\| \). We have that

\[
\|(S^* - \lambda)^n \varphi\| = \left\| \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^k S^{*(n-k)} \varphi \right\|
\]

\[
\leq \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k \|S^{*(n-k)} \varphi\|
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k \left( \sum_{s=n-k}^{\infty} |\varphi_s|^2 \right)^{1/2}
\]

\[
\leq \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k \left( \sum_{s=n-k}^{\infty} a^{2s} \right)^{1/2}
\]
 CHAPTER 4. GENERALIZATIONS

\[
\begin{align*}
&= \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k \left( \frac{a^{2(n-k)}}{1-a^2} \right)^{1/2} \\
&= \frac{1}{\sqrt{1-a^2}} \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k a^{n-k} \\
&= \frac{1}{\sqrt{1-a^2}} (a + |\lambda|)^n.
\end{align*}
\]

Since \( a < 1 - |\lambda| \), it follows that

\[
\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2 \leq \frac{1}{1 - a^2} \sum_{n=0}^{\infty} (a + |\lambda|)^{2n} < \infty.
\]

\[\square\]

**Lemma 4.6.2** Let \( \lambda \in \mathbb{C} \) such that \( |\lambda| < 1 \) and let \( \psi \) be a polynomial. Then

\[
\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \psi\|^2 < \infty.
\]

**Proof:** Assume that \( \psi(z) = \sum_{s=0}^{N} \psi_s z^s \) is a polynomial of degree \( N \). If \( n \geq N \) we have

\[
(S^* - \lambda)^n \psi = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} S^k \psi
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \sum_{s=0}^{N} \psi_s S^k z^s
\]

\[
= \sum_{k=0}^{n} \sum_{s=0}^{N} \binom{n}{k} (-\lambda)^{n-k} \psi_s S^k z^s
\]

\[
= \sum_{k=0}^{N} \sum_{s=k}^{N} \binom{n}{k} (-\lambda)^{n-k} \psi_s z^{s-k}
\]

\[
= \sum_{k=0}^{N} \sum_{r=0}^{N-k} \binom{n}{k} (-\lambda)^{n-k} \psi_{r+k} z^r
\]

\[
= \sum_{r=0}^{N} \left( \sum_{k=0}^{N-r} \binom{n}{k} (-\lambda)^{n-k} \psi_{r+k} \right) z^r,
\]
by a change of variables and an interchange of the order of the sums. Thus, if \( n \geq N \),
\[
\| (S^* - \lambda)^n \psi \|^2 = \sum_{r=0}^{N} \sum_{k=0}^{N-r} \binom{n}{k} (-\lambda)^{n-k} \psi_{r+k}^2
\]
\[
\leq \sum_{r=0}^{N} \left( \sum_{k=0}^{N-r} \binom{n}{k} |\lambda|^{2(n-k)} \right) \left( \sum_{k=0}^{N-r} |\psi_{r+k}|^2 \right)
\]
\[
\leq \| \psi \|^2 \sum_{r=0}^{N} \sum_{k=0}^{N-r} \binom{n}{k} |\lambda|^{2(n-k)},
\]
where we used the Cauchy–Schwarz inequality and the fact that \( \sum_{k=0}^{N-r} |\psi_{r+k}|^2 \leq \| \psi \|^2 \)
for all \( r \). It then follows that
\[
\sum_{n=N}^{\infty} \| (S^* - \lambda)^n \psi \|^2 \leq \| \psi \|^2 \sum_{n=N}^{\infty} \sum_{r=0}^{N} \sum_{k=0}^{N-r} \binom{n}{k} |\lambda|^{2(n-k)}
\]
\[
= \| \psi \|^2 \sum_{r=0}^{N} \sum_{k=0}^{N-r} \sum_{n=N}^{\infty} \binom{n}{k} |\lambda|^{2(n-k)}
\]
where the order of the sums can be changed if
\[
\sum_{n=N}^{\infty} \binom{n}{k}^2 |\lambda|^{2(n-k)} < \infty.
\]
But it can be easily checked that this last sum converges (for example, use the ratio test) if \( |\lambda| < 1 \). This means that
\[
\sum_{n=N}^{\infty} \| (S^* - \lambda)^n \psi \|^2 < \infty,
\]
and thus that
\[
\sum_{n=0}^{\infty} \| (S^* - \lambda)^n \psi \|^2 < \infty.
\]

\[\square\]

**Lemma 4.6.3** Let \( \lambda \in \mathbb{C} \) be such that \( |\lambda| < 1 \). Suppose that \( \phi \) is an analytic function whose power series centred at zero has radius of convergence \( R > \frac{1}{1-|\lambda|} \). Then
\[
\sum_{n=0}^{\infty} \| (S^* - \lambda)^n \phi \|^2 < \infty.
\]
Proof: Let $\phi(z) = \sum_{s=0}^{\infty} \phi_s z^s$. Since $R > \frac{1}{1-|\lambda|}$ we can choose a number $a$ such that $1/R < a < 1 - |\lambda|$. As is well known,

$$\frac{1}{R} = \limsup_s |\phi_s|^{1/s}.$$ 

Since $1/R < a$, this means that for large $N$ we must have $|\phi_s|^{1/s} < a$ for all $s > N$; i.e., $|\phi_s| < a^s$ for $s > N$. Now define $\varphi(z) = \sum_{s=N+1}^{\infty} \phi_s z^s$ and $\psi(z) = \sum_{s=0}^{N} \phi_s z^s$. We can then apply Lemma 4.6.1 to $\varphi$ and Lemma 4.6.2 to $\psi$ to obtain

$$\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|(S^* - \lambda)^n \psi\|^2 < \infty.$$ 

Since $\phi = \psi + \varphi$, we have that $\|(S^* - \lambda)^n \phi\| \leq \|(S^* - \lambda)^n \psi\| + \|(S^* - \lambda)^n \varphi\|$, and thus

$$\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2 \leq \sum_{n=0}^{\infty} \|(S^* - \lambda)^n \psi\|^2 + 2 \sum_{n=0}^{\infty} \|(S^* - \lambda)^n \psi\| \|(S^* - \lambda)^n \varphi\| + \sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2.$$ 

But each term converges. We obtain the desired conclusion.

We obtain the following theorem.

**Theorem 4.6.4** Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < 1$. Let $\varphi$ be analytic on an open region containing the closed disk of radius $1/(1-|\lambda|)$. Then the $\lambda$–Hankel operator $X$ with symbol $\varphi$ is a Hilbert–Schmidt operator.

**Proof:** If $\varphi$ is analytic on an open region containing the closed disk of radius $1/(1-|\lambda|)$, then we know by the previous lemma that

$$\sum_{n=0}^{\infty} \|(S^* - \lambda)^n \varphi\|^2 < \infty.$$
But this is precisely the sum of the squares of the matrix entries of the $\lambda$–Hankel operator $X$ with symbol $\varphi$. \hfill \triangle

Notice that when $\lambda = 0$ this theorem reduces to the statement that if the symbol of a Hankel operator is a function analytic on the closed unit disk then the Hankel operator is Hilbert–Schmidt, a result which is known (it is known that a Hankel operator $H$ is Hilbert–Schmidt if and only if the derivative of $He_0 = \varphi$ is square integrable on the open unit disk with respect to the area measure; see for example [55]).

Also, notice that if $\varphi$ is a function such that the hypothesis of the theorem holds, then $\varphi \to 0$ exponentially, and thus (see [20]) $\varphi$ is either a rational function in which case $X$ is a finite–rank operator, or $\varphi$ is a cyclic function for $S^*$ in which case $X$ has dense range (the span of $X$ is $\text{span} \{(S^* - \lambda)^n\varphi : n \in \mathbb{N} \cup \{0\}\}$ which is equal to $\text{span} \{S^n\varphi : n \in \mathbb{N} \cup \{0\}\}$).

We will now give a condition on the symbol of a $\lambda$–Hankel operator implied by its boundedness. We first need a definition.

Let $r = 1 + |\lambda|$. We define the Hardy space corresponding to $r\mathbb{D}$ (the open disk of radius $r$ centred at the origin), following [21, Chapter 10].

**Definition 4.6.5** Let $p = 1$ or $2$. The Hardy space $H^p(r\mathbb{D})$ is the space of analytic functions $f$ on $r\mathbb{D}$ such that there exists a harmonic majorant for $|f|^p$ on $r\mathbb{D}$; i.e., a harmonic function $v$ on $r\mathbb{D}$ such that $|f(z)|^p \leq v(z)$ for all $z \in r\mathbb{D}$. We denote the least harmonic majorant of $|f|^p$ by $v_f$. This is a Banach space with 0 as a norming point; that is, the norm defined as $\|f\|_{r,p} = (v_f(0))^{1/p}$, makes $H^p(r\mathbb{D})$ into a Banach space (see [14, p. 206]).

It can be seen [21, p. 28] that this coincides with the definition of $H^2 = H^2(1\mathbb{D})$ when $r = 1$ (that is, when $\lambda = 0$). In fact there is a natural isometric isomorphism between $H^1$ and $H^1(r\mathbb{D})$ which also sends $H^2$ isometrically onto $H^2(r\mathbb{D})$. Also, it can be seen (follow [21] for example) that the boundary functions of $H^1(r\mathbb{D})$ are in
$L^1(rS^1, dm)$, where $rS^1$ is the circle of radius 1 centred at the origin and $dm$ is arc length on $rS^1$, normalized so that $m(rS^1) = 1$. Thus we think of $H^1(rD)$ as a closed subspace of $L^1(rS^1, dm)$.

We need the following lemma.

**Lemma 4.6.6** Let $h \in H^2(rD)$ where $r = 1 + |\lambda|$ and let $q(z) = h(z + \bar{\lambda})$. Then, $q \in H^2$ and there is a constant $C$, independent of $h$, such that $\|q\| \leq C\|h\|_{r,2}$.

**Proof:** First of all, $q$ is analytic in $D$ since if $z \in D$ then $z + \bar{\lambda} \in rD$. But this means that $|q(z)|^2 = |h(z + \bar{\lambda})|^2 \leq v_h(z + \bar{\lambda})$ for all $z \in D$, where $v_h$ is the least harmonic majorant of $|h|^2$. But if $v_h$ is harmonic on $rD$, then $u$, defined as $u(z) = v_h(z + \bar{\lambda})$, is harmonic in $D$. Thus $|q|^2$ has a harmonic majorant $u$ on $D$. Therefore $q \in H^2$.

Let $v_q$ be the least harmonic majorant of $|q|^2$. It then follows from the previous paragraph that $v_q(z) \leq u(z) = v_h(z + \bar{\lambda})$ for all $z \in D$. In particular, $v_q(0) \leq v_h(\bar{\lambda})$. Recall that the norm of $q$ in $H^2$ is $(v_q(0))^{1/2}$. Also, any two norming points define equivalent norms (see [14, p. 207] or [21, p. 183]), thus $v_h(\bar{\lambda}) \leq C_1 v_h(0)$ for some positive constant $C_1$, which implies that $v_q(0) \leq C_1 v_h(0)$. It follows that $\|q\| \leq C\|h\|_{r,2}$ for some number $C$.

We can now prove the following theorem, which establishes a necessary condition for boundedness of a $\lambda$–Hankel operator.

**Theorem 4.6.7** Let $\lambda \in \mathbb{C}$ with $|\lambda| < 2$ and $r = 1 + |\lambda|$. Let $rS^1$ be the circle of radius $r$ centred at the origin and $dm$ be arc length on $rS^1$, normalized so that $m(rS^1) = 1$. If $X$ is a bounded $\lambda$–Hankel operator with $Xe_0 = \psi$, then there exists a function $\varphi \in L^\infty(rS^1, dm)$ such that, if $a_n$ is the $n$–th Fourier coefficient of $\psi$, then

$$a_n = \int_{rS^1} \bar{z}^n \varphi \ dm.$$ 

Notice that for the case $\lambda = 0$ this is the usual necessary and sufficient condition for boundedness of a Hankel operator. As such, it shouldn't be surprising that the
proof follows the usual analytical proof (see Power [49], for example) of necessity for Hankel operators.

**Proof:** We define the linear functional \( \hat{\psi} \) on a dense subset of \( \mathbf{H}^1(\mathbb{D}) \) (namely, the polynomials) as

\[
\hat{\psi}(f) = (f, Xe_0) = (f, \psi)
\]

where \((\cdot, \cdot)\) is the inner product on \( \mathbf{H}^2 \). Clearly \( \hat{\psi} \) is well-defined and linear.

We will show that \( \hat{\psi} \) is bounded. Recall that any function \( f \in \mathbf{H}^1(\mathbb{D}) \) can be written as \( f(z) = g(z)h(z) \) with \( g, h \in \mathbf{H}^2(\mathbb{D}) \) and \( \|f\|_{r,1} = \|g\|_{r,2}\|h\|_{r,2} \). Indeed, this is the case for the usual Hardy space \( \mathbf{H}^1(\mathbb{D}) \): every function can be factored as the product of two \( \mathbf{H}^2 \) functions satisfying the above condition. The space \( \mathbf{H}^p(\mathbb{D}) \) is naturally isometrically isomorphic to \( \mathbf{H}^p(\mathbb{D}) \) [21, p. 168] (independently of \( p \)), which implies that our factorization holds.

If we define \( q \) as \( q(z) = h(z + \lambda) \), by the previous lemma we get

\[
\|q\| \leq C\|h\|_{r,2}.
\]

Also notice that \( \|g\| \leq \|g\|_{r,2} \) for any \( g \in \mathbf{H}^2(\mathbb{D}) \). Putting these two facts together we get

\[
\|q\| \|g\| \leq C\|h\|_{r,2}\|g\|_{r,2} = C\|f\|_{r,1}.
\]

(4.9)

Let \( h \) have power series about \( \lambda \) of the form \( h(z) = \sum_{n=0}^{\infty} q_n(z - \lambda)^n \). This implies that \( q \) has power series about 0 of the form \( q(z) = \sum_{n=0}^{\infty} q_n z^n \). Keeping in mind that \( \mathbf{H}^2(\mathbb{D}) \subset \mathbf{H}^2 \), we have

\[
\hat{\psi}(f) = (f(z), Xe_0) = (h(z)g(z), Xe_0)
\]

\[
= \sum_{n=0}^{\infty} q_n ((z - \lambda)^n g(z), Xe_0) = \sum_{n=0}^{\infty} q_n ((S - \lambda)^n g(z), Xe_0)
\]

\[
= \sum_{n=0}^{\infty} q_n (g(z), (S^* - \lambda)^n Xe_0) = \sum_{n=0}^{\infty} q_n (g(z), XS^n e_0)
\]

\[
= \sum_{n=0}^{\infty} q_n (g(z), Xe_n) = (g(z), Xq^*(z)),
\]
where the last equality follows from the boundedness of \( X \). Since \( X \) is bounded, the previous equality implies
\[
|\hat{\psi}(f)| \leq ||X|| \|g\| \|q\|.
\]
By equation 4.9, this implies
\[
|\hat{\psi}(f)| \leq C \|X\| \|f\|_{r,1};
\]
i.e., \( \hat{\psi} \) is a bounded linear functional densely defined on \( H^1(\mathbb{D}) \), as desired.

Since \( \hat{\psi} \) is bounded and densely defined, we extend it to all of \( H^1(\mathbb{D}) \). Recall that \( H^1(\mathbb{D}) \) is a subspace of \( L^1(\mathbb{S}^1, dm) \). Therefore, by the Hahn–Banach theorem, we can extend \( \hat{\psi} \) to a functional on \( L^1(\mathbb{S}^1, dm) \). This means that \( \hat{\psi} \) is given by some function \( \phi \in L^\infty(\mathbb{S}^1, dm) \) as
\[
\hat{\psi}(f) = \int_{\mathbb{S}^1} f \phi \, dm,
\]
for all \( f \in L^1(\mathbb{S}^1, dm) \).

We can conclude that, if \( a_n \) is the \( n \)-th Fourier coefficient of \( \psi = Xe_0 \), then
\[
\bar{a}_n = (e_n, \psi) = (e_n, Xe_0) = \hat{\psi}(e_n) = \int_{\mathbb{S}^1} z^n \phi \, dm,
\]
which in turn implies, letting \( \varphi = \frac{\phi}{L^\infty(\mathbb{S}^1, dm)} \), that
\[
a_n = \int_{\mathbb{S}^1} \bar{z}^n \varphi \, dm,
\]
as desired. \( \square \)

Although the condition given by this theorem is necessary and sufficient when \( \lambda = 0 \) (it is then just a restatement of Nehari’s theorem), it can be seen that it is not sufficient when \( \lambda \neq 0 \). For example, let \( \psi \) be defined as \( \psi(z) = k_a(z) \) with \( |a| < 1 \) but \( |a - \lambda| > 1 \). The densely defined (on polynomials) \( \lambda \)-Hankel operator with \( Xe_0 = \psi \) can be seen to be unbounded even though there exists a function \( \varphi(z) = \frac{r}{r - az} \in L^\infty(\mathbb{S}^1, dm) \) and
\[
(\psi, e_n) = a^n = \int_{\mathbb{S}^1} \bar{z}^n \frac{r}{r - az} \, dm.
\]
Is there a necessary and sufficient condition? We conjecture the following.
Conjecture 4.8.8 Let $\lambda \in \mathbb{C}$ with $|\lambda| < 2$. Then, $X$ is a bounded $\lambda$-Hankel operator with $Xe_0 = \psi$ if and only if there exists a function $\varphi \in L^\infty(\Omega, dm)$ such that if $a_n$ is the $n$-th Fourier coefficient of $\psi$ then

$$a_n = \int_{\Omega} \bar{z}^n \varphi \, dm,$$

where $\Omega$ is the boundary of the region $\{z \in \mathbb{D} : |z - \lambda| < 1\}$, and $dm$ is arc length on this boundary, normalized so that $m(\Omega) = 1$. 
Chapter 5

Essentially Hankel Operators

In this chapter we study a different generalization of Hankel operators which we call essentially Hankel operators. We study some of their properties and ask a basic question about their structure, which, surprisingly, has a negative answer, in spite of some theorems that seemed to suggest a positive answer.

5.1 Definitions and Basic Properties

We need to recall the definition of an essentially Toeplitz operator.

Definition 5.1.1 We say that a bounded operator $T$ is essentially Toeplitz if $S^*TS - T$ is compact. We denote the set of all essentially Toeplitz operators by $\text{ess Toep}$.

Notice that since $S$ is unitary in the Calkin algebra $B(\mathcal{H})/\mathcal{K}$, it follows that $T$ is essentially Toeplitz if and only if $ST - TS \in \mathcal{K}$; i.e., if and only if $T$ is in the essential commutant of the shift $S$. The essentially Toeplitz operators have been studied by (among others) Barria and Halmos in [6], where many interesting questions are raised. We now define the essentially Hankel operators.

Definition 5.1.2 We say that a bounded operator $H$ is essentially Hankel if $S^*H - HS$ is compact. We denote the set of all essentially Hankel operators by $\text{ess Hank}$.
CHAPTER 5. ESSENTIALLY HANKEL OPERATORS

It will be useful sometimes to realize that \( H \in \text{ess Hank} \) if and only if \( H - SHS \in \mathcal{K} \) if and only if \( S^*HS^* - H \in \mathcal{K} \), since \( S \) is unitary in the Calkin algebra.

Clearly \( \text{ess Toep} \) is a \( C^* \) algebra. Also, it is obvious that \( \text{ess Hank} \) is a (norm closed) vector subspace of \( \mathcal{B}(\mathcal{H}) \). In fact, it is even a self-adjoint set. Unfortunately, \( \text{ess Hank} \) is not an algebra.

**Example 5.1.3** Let \( \varphi = \varphi^* \in L^\infty \) such that \((\bar{z} - z)\varphi \notin H^\infty + C\). Then \( H^2_\varphi \) is not in \( \text{ess Hank} \).

**Proof:** Let \( H = H_\varphi \). Since \( S^*H^2 - H^2S = HSH - HS^*H = H(S - S^*)H \), it follows that \( H^2 \in \text{ess Hank} \) if and only if \( HT_{z - \bar{z}}H \in \mathcal{K} \). But using equation 1.2 in Chapter 1 we get

\[
T_{z - \bar{z}}H_\varphi = H_{(\bar{z} - z)\varphi} - H_{z(z - \bar{z})}T_{\bar{z}\varphi}
\]

which implies that

\[
HT_{z - \bar{z}}H \in \mathcal{K} \quad \text{if and only if} \quad H_\varphi H_{(\bar{z} - z)\varphi} - H_{z(z - \bar{z})}T_{\bar{z}\varphi} \in \mathcal{K}.
\]

Since \( 1 - z^2 \in H^\infty + C \), it follows by Hartman's Theorem that \( H_{1 - z^2} \in \mathcal{K} \). Thus \( H^2 \in \text{ess Hank} \) if and only if \( H_\varphi H_{(\bar{z} - z)\varphi} \in \mathcal{K} \). By equation 1.1 in Chapter 1 we get

\[
H_\varphi H_{(\bar{z} - z)\varphi} = T_{(z\bar{\psi})}\bar{z}(\bar{z} - z)\varphi - T_{z\bar{\psi}}T_{\bar{z}(\bar{z} - z)\varphi}.
\]

Thus \( H^2 \in \text{ess Hank} \) if and only if \( T_{(z\bar{\psi})}(z^2 - 1)\varphi - T_{z\bar{\psi}}T_{(z^2 - 1)\varphi} \in \mathcal{K} \). But by the Axler-Chang-Sarason-Volberg Theorem [2, 59] this last expression is compact if and only if

\[
H^{\infty}[\bar{z}\varphi^*] \cap H^{\infty}[(z^2 - 1)\varphi] \subset H^{\infty} + C,
\]

where \( H^{\infty}[\psi] \) denotes the uniformly closed subalgebra of \( L^\infty \) generated by \( H^{\infty} \) and \( \psi \).

Clearly \( \bar{z}\varphi^* \in H^{\infty}[\bar{z}\varphi^*] \). Since \( \varphi = \varphi^* \), it follows that \( \bar{z}\varphi \in H^{\infty}[\bar{z}\varphi^*] \), which implies that \( z\varphi \in H^{\infty}[\bar{z}\varphi^*] \) (since \( z^2 \in H^{\infty} \)). Therefore \((\bar{z} - z)\varphi = \bar{z}\varphi - z\varphi \in H^{\infty}[\bar{z}\varphi^*] \).

Also, \((\bar{z}^2 - 1)\varphi \in H^{\infty}[(\bar{z}^2 - 1)\varphi] \). Since \( z \in H^{\infty} \), it follows that \((\bar{z} - z)\varphi \in H^{\infty}[(\bar{z}^2 - 1)\varphi] \).
Combining these two results we obtain

\[(\bar{z} - z)\varphi \in H^\infty [\bar{z}\varphi^*] \cap H^\infty [(\bar{z}^2 - 1)\varphi].\]

Assume that \(H^2 \in \text{ess} \ Hank\). Then

\[(\bar{z} - z)\varphi \in H^\infty [\bar{z}\varphi^*] \cap H^\infty [(\bar{z}^2 - 1)\varphi] \subset H^\infty + C.\]

But this is a contradiction, since \((\bar{z} - z)\varphi \notin H^\infty + C\) by hypothesis. Therefore, \(H^2 \notin \text{ess} \ Hank\). \(\Box\)

It is not hard to find functions that satisfy the hypothesis of the example. As a matter of fact, it is known (and it is simple to see) that the condition \((\bar{z} - z)\varphi \in H^\infty + C\) is equivalent to \(H_\varphi \in \text{ess} \ Toep\) [10]. Since for a self-adjoint Hankel operator \(H\) one can always find a bounded symbol \(\varphi\) such that \(\varphi^* = \varphi\), one only needs to find a self-adjoint Hankel operator \(H_\varphi \notin \text{ess} \ Toep\) (this is clearly possible: if all self-adjoint Hankel operators were in \(\text{ess} \ Toep\) it would follow that all Hankel operators are in \(\text{ess} \ Toep\), which is clearly not true).

While \(\text{ess} \ Hank\) is not an algebra, it does have a curious property, reminiscent of the facts that the product of a Hankel operator and an analytic Toeplitz operator is a Hankel operator and that the product of a co-analytic Toeplitz operator and a Hankel operator is a Hankel operator.

**Lemma 5.1.4** Let \(H \in \text{ess} \ Hank\) and \(T \in \text{ess} \ Toep\). Then \(HT \in \text{ess} \ Hank\) and \(TH \in \text{ess} \ Hank\).

**Proof:** Notice that

\[S^*HT - HTS = HST - HTS \pmod{\mathcal{K}},\]

since \(H \in \text{ess} \ Hank\). But the right hand side is compact since \(ST - TS \in \mathcal{K}\).

Analogously,

\[S^*TH - THS = S^*TH - TS^*H \pmod{\mathcal{K}},\]
since \( H \in \text{essHank} \). But the right hand side is compact since \( S^*T - TS^* \in \mathcal{K} \) (remember that \( \text{essToep} \) is a C* algebra).

In particular, this lemma implies the following corollary.

**Corollary 5.1.5** \( \text{essToep} \cap \text{essHank} \) is a C* algebra with no identity. Even more, \( \text{essHank} \) does not contain any non-zero Toeplitz operators.

**Proof:** Suppose that \( A \) and \( B \in \text{essToep} \cap \text{essHank} \). Then \( AB \in \text{essToep} \) since \( \text{essToep} \) is a C* algebra. Since \( A \in \text{essHank} \) and \( B \in \text{essToep} \), the previous lemma applies to obtain that \( AB \in \text{essHank} \). Therefore \( AB \in \text{essToep} \cap \text{essHank} \). The rest of the properties of a C* algebra are easy to check.

If \( T \) is a non-zero Toeplitz operator, then \( S^*T - TS \) is also a non-zero Toeplitz operator. But non-zero Toeplitz operators are never compact so \( T \notin \text{essHank} \). In particular, the identity is not in \( \text{essHank} \).

It is known that \( \text{essToep} \) contains many Hankel operators (see [9, 10]). This allows us to offer an alternative proof of a weaker version of a theorem of Power [48, Theorem 1.3 (i)].

**Corollary 5.1.6** Let \( A \) be an indexing set. Let \( \{H_a\}_{a \in A} \) be a set of Hankel operators all of which are contained in \( \text{essToep} \). Then the C* algebra (with no identity) generated by \( \{H_a\}_{a \in A} \) does not contain any non-zero Toeplitz operators.

**Proof:** The C* algebra (with no identity) generated by \( \{H_a\}_{a \in A} \) is contained in \( \text{essToep} \cap \text{essHank} \).

Another curious property (this time reminiscent of equation 1.1 in Chapter 1) is the following.

**Lemma 5.1.7** Let \( H \) and \( G \in \text{essHank} \). Then \( HG \in \text{essToep} \).

**Proof:** Notice that

\[
S^*HGS - HG = HSS^*G - HG \quad \text{(mod } \mathcal{K},
\]

\[
S^*HGS - HG = HSS^*G - HG \quad \text{(mod } \mathcal{K},
\]
since \( H \) and \( G \in \text{ess Hank} \). But \( SS^* = I \) (mod \( K \)) so the right hand side is compact. □

This allows us to use a theorem of Power [47] to prove the following.

**Theorem 5.1.8** \( \text{ess Toep} + \text{ess Hank} \) is a \( C^* \) algebra.

**Proof:** We know that \( \text{ess Toep} \) is a \( C^* \) algebra. The vector space \( \text{ess Hank} \) is a norm-closed self-adjoint space which is also an \( (\text{ess Toep}) \)-bimodule by Lemma 5.1.4. Also, \( (\text{ess Hank})^2 \subset \text{ess Toep} \) by the previous lemma. We can then use Theorem 2 of [47] to conclude that \( \text{ess Toep} + \text{ess Hank} \) is a \( C^* \) algebra. □

After we completed the final draft of this thesis, we found that the previous theorem has been proved independently by Guo, Liu and Zhang in [35].

Following Barriá and Halmos [6], we are tempted to define an *asymptotic Toeplitz operator in the Calkin algebra* as an operator \( T \) such that the sequence \( S^n T S^n \) converges in the Calkin algebra. Clearly, all essentially Toeplitz operators are asymptotically Toeplitz in the Calkin algebra. If \( T \) is an asymptotically Toeplitz operator in the Calkin algebra, then \( S^n T S^n \) converges in the Calkin algebra to an operator \( T' \) which is an essentially Toeplitz operator. Unfortunately, this doesn't provide us with anything new since

\[
\| S^n T S^n - T' \|_e = \| S^n (T - T') S^n \|_e = \| T - T' \|_e,
\]

which implies that the only asymptotic Toeplitz operators in the Calkin algebra are the essentially Toeplitz operators.

The same reasoning applies to essentially Hankel operators. In this case, what seems like a natural definition for \( H \) to be an *asymptotic Hankel operator in the Calkin algebra* would be \( S^n H S^n \) converges in the Calkin algebra to an operator \( H' \). Notice that again all essentially Hankel operators are asymptotically Hankel in the Calkin algebra. In this case, as before, \( H' \) would be essentially Hankel. We then obtain

\[
\| S^n H S^n - H' \|_e = \| S^n (H - H') S^n \|_e = \| H - H' \|_e.
\]
Thus the asymptotic Hankel operators in the Calkin algebra are just the essentially Hankel operators.

The same problem occurs if we use $S^n HS^n$ converging in the Calkin algebra as the definition for asymptotic Hankel in the Calkin algebra. We want to point out that there are definitions of (uniformly, strongly and weakly) asymptotic Hankel operators (see Feintuch [22, 23]) which are different from the ones proposed here, but which, although very interesting, do not seem to translate well to the Calkin algebra case. Is there a good definition for asymptotic Hankel in the Calkin algebra?

5.2 Is ess Hank trivial?

As mentioned in [6], it is "the experts' conviction" that ess Toep is a huge set. It is known to contain the C* algebra generated by all Toeplitz operators as well as many other operators not contained in this C* algebra. Since the C* algebra generated by all Toeplitz operators contains $\mathcal{K}$, it then follows that ess Toep is not just the C* algebra generated by the sum of Toeplitz operators and compact operators (but see Feintuch [23] for a class of operators which is of this form!).

What happens for ess Hank? Clearly, all sums of Hankel operators and compact operators are in ess Hank. Are there any other operators? In other words, is it true that if $\Gamma \in \text{ess Hank}$ then $\Gamma = H + K$ for some bounded Hankel operator $H$ and some compact operator $K$? The answer to this questions turns out to be no, as we will show at the end of this section.

We first discuss some results, which, surprisingly, could be interpreted as evidence for an affirmative answer. Let us first formalize the question.

**Question:** Does the space ess Hank consist only of compact perturbations of bounded Hankel operators?

We have the following lemma.

**Lemma 5.2.1** If $A \in \text{ess Hank}$, then $A$ is not a Fredholm operator; i.e., $0 \in \sigma_e(A)$. 
Let $A \in \text{ess Hank}$. This means that $S^*A - AS = K$ for some compact operator $K$. Suppose $A$ was Fredholm. Then, since both $S$ are $S^*$ are Fredholm, we obtain the following relations [13, p. 354] for their Fredholm indices:

$$\text{ind} (S^*A) = \text{ind} (S^*) + \text{ind} (A),$$

and

$$\text{ind} (AS) = \text{ind} (A) + \text{ind} (S).$$

But, since $S^*A = AS + K$, it follows that $\text{ind} (S^*A) = \text{ind} (AS)$; i.e., $\text{ind} (S^*) + \text{ind} (A) = \text{ind} (A) + \text{ind} (S)$. But it is well known (and easy to check) that $\text{ind} (S) = -1$ and $\text{ind} (S^*) = 1$. Contradiction. Thus $A$ cannot be Fredholm.

It is known (see Chapter 1) that Hankel operators are never Fredholm, so any sum of a Hankel operator and a compact operator is also never Fredholm. So this lemma provides some (undoubtedly very weak) evidence that the answer to the question is yes.

We need the following definition from [6].

**Definition 5.2.2** An operator $A$ is said to be (strongly) asymptotically Toeplitz if $S^nAS$ converges in the strong operator topology. If it converges, its limit is a Toeplitz operator $T_\varphi$ and we say that $\varphi$ is the “symbol” of $A$. This “symbol” coincides with the symbol of a Toeplitz operator but it does not coincide with the symbol of a Hankel operator.

It is known that all Hankel operators and all compact operators are (strongly) asymptotically Toeplitz (see [6]) and in fact, their “symbol” is 0. The following lemma then offers some more weak evidence of an affirmative answer to our question.

**Lemma 5.2.3** If $A \in \text{ess Hank}$ is (strongly) asymptotically Toeplitz, then its “symbol” is 0.
CHAPTER 5. ESSENTIALLY HANKEL OPERATORS

Proof: Let $S^*A - AS = K$ and suppose that $S^mAS^n$ converges in the strong operator topology to $T_\varphi$. Then $S^*AS = AS^2 + KS$, which implies that

$$S^{(n+1)}AS^{n+1} = S^mAS^nS^2 + S^mKS^n.$$ 

Taking limits in the strong operator topology, we obtain

$$T_\varphi = T_\varphi S^2 + 0,$$

since $S^mKS^n$ converges in the strong operator topology to $0$ [6]. This equation implies that $\varphi = 0$; i.e., the "symbol" of $A$ is zero. \qed

Another piece of weak evidence is ahead, but first we need the following lemma. This type of calculation is well known and has been used frequently (see, for example, Fialkow [24]).

Lemma 5.2.4 Let $Q$ be a compact operator such that

$$C = -\sum_{n=0}^{\infty} S^Q S^{(n+1)}$$

converges in the uniform operator topology. Then $C$ is compact and $S^*C - CS = Q$.

Proof: That $C$ is compact is obvious. Now,

$$S^*C = -\sum_{n=0}^{\infty} S^{(n+1)}QS^{(n+1)},$$

and

$$CS = -\sum_{n=0}^{\infty} S^mQS^m,$$

so $S^*C - CS = Q$. \qed

We obtain the following theorem.

Theorem 5.2.5 There is a dense subset $Q$ of $K$ such that the following holds: if $S^*A - AS = Q$ with $Q \in Q$, then $A$ is a Hankel operator plus a compact operator.
Proof: First, we will show that the set $Q$ of compact operators $Q$ such that
\[ \sum_{n=0}^{\infty} S^m Q S^{n+1} \]
converges in the uniform operator topology is dense. Let $Q = f \otimes g$ where $f$ and $g \in \mathcal{H}$. Then, for any $N \geq M$,
\[
\sum_{n=M}^{N} \| (S^m f) \otimes (S^{n+1} g) \| = \sum_{n=M}^{N} \| S^m f \| \| S^{n+1} g \| \\
= \| g \| \sum_{n=M}^{N} \| S^m f \| .
\]
This implies that $f \otimes g$ will be in $Q$ if $\sum_{n=0}^{\infty} \| S^m f \|$ converges. But this happens for $f$ in a dense subset of $\mathcal{H}$ (it happens for the polynomials, but it does not happen for all $f \in \mathcal{H}$). This means that a dense subset of the operators of rank one is in $Q$. But this implies that a dense subset of all finite rank operators is in $Q$, ergo, a dense subset of the compact operators is in $Q$.

Now, suppose that $S^* A - A S = Q$ with $Q \in Q$. Then, if we define $C$ to be
\[
C = - \sum_{n=0}^{\infty} S^m Q S^{n+1},
\]
then $S^* C - C S = Q$ and $C$ is compact (by the previous Lemma). But this means that $S^* (A - C) - (A - C) S = 0$; i.e., $A - C$ is a Hankel operator. $\square$

This theorem uses Lemma 5.2.4 very strongly and unfortunately, this lemma cannot be extended to a set which includes all compact operators, for the following reason. Define the operator $\tau : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ as $\tau(X) = S^* X - X S$. A theorem of Fialkow [24] establishes that this mapping is onto if and only if $\tau : \mathcal{K} \rightarrow \mathcal{K}$ is onto. But a theorem of Davis and Rosenthal [17] implies that $\tau : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is onto if and only if $\sigma_\delta(S^*) \cap \sigma_\pi(S) = \emptyset$, where $\sigma_\delta$ denotes the approximate defect spectrum and $\sigma_\pi$ denotes the approximate point spectrum (see [17] for the definitions). But it is easy to check that $\sigma_\delta(S^*) = \sigma_\pi(S) = S^1$. Conclusion: $\tau : \mathcal{K} \rightarrow \mathcal{K}$ is not onto.
In fact, given a rank one operator $f \otimes g$, the equation $\tau(X) = f \otimes g$ does not always have a solution $X \in B(\mathcal{H})$, since if it did, a theorem of Fialkow [25] would imply that $\tau : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is onto.

We would like to point out that $\tau : K \rightarrow K$ has dense range (which provides an alternative, but less explicit, proof of the previous theorem), as can be seen by a theorem of Fialkow [24] which implies that $\tau : K \rightarrow K$ has dense range if $\tilde{\tau} : T \rightarrow T$, defined as $\tilde{\tau}(X) = SX - XS^*$, is injective (recall that $T$ is the space of trace class operators). But remember that (Corollary 4.1.2 in Chapter 4) $SX = XS^*$ implies $X = 0$.

Of course, even though the operator $\tau$ is not surjective, the answer to our question could still be yes. The argument following the proof of the previous theorem just says that $\tau(K) \subseteq K$ but the question has an affirmative answer if $\tau(B(\mathcal{H})) \cap K \subset \tau(K)$ (notice that $\tau(B(\mathcal{H})) \cap K \not\subseteq K$ since $\tau : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is not surjective and the theorem of Fialkow [24] which says that $\tau$ is surjective if and only if $\tau(B(\mathcal{H})) \supseteq K$). We will see that the answer to the question is negative, which implies that $\tau(B(\mathcal{H})) \cap K \not\subset \tau(K)$ (incidentally, it is obvious that $\tau(B(\mathcal{H})) \cap K \supset \tau(K)$, so the answer to our question implies that $\tau(B(\mathcal{H})) \cap K \supseteq \tau(K)$).

Before presenting the example of an operator in ess Hank which cannot be written in the form Hankel plus compact, we prove a theorem about Rhaly (or terraced) matrices.

**Definition 5.2.6** An infinite matrix of the form

$$R = \begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
a_1 & a_1 & 0 & 0 & \cdots \\
a_2 & a_2 & a_2 & 0 & \cdots \\
a_3 & a_3 & a_3 & a_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

for some complex sequence $\{a_n\}_{n=0}^{\infty} \in \ell^2$, is called a Rhaly matrix (or a terraced matrix). See [34, 54] for more on this topic.
CHAPTER 5. ESSENTIALLY HANKEL OPERATORS

It is known that a Rhaly matrix determines a bounded operator if \( na_n \) is bounded \([34]\). It is known that a Rhaly matrix determines a compact operator if \( na_n \to 0 \) \([54]\). (Both conditions are known not to be necessary.) We have the following theorem.

**Theorem 5.2.7** Let \( R \) be a bounded Rhaly matrix as in Definition 5.2.6. Then \( R \in \text{ess Toep} \) if and only if \( R \in \text{ess Hank} \).

**Proof:** Define the infinite matrix \( A \) as

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
-\alpha_1 & 0 & 0 & 0 & \cdots \\
-\alpha_2 & -\alpha_1 & 0 & 0 & \cdots \\
-\alpha_3 & -\alpha_2 & -\alpha_1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

A calculation shows that

\[
S^*RS - R = A + \text{diag}(\alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \cdots).
\]

Since clearly \( \text{diag}(\alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \cdots) \) is compact, this implies that \( R \in \text{ess Toep} \) if and only if \( A \in \mathcal{K} \).

Now, a calculation shows that

\[
S^*R - RS = A + \text{diag}(\alpha_1, \alpha_2, \alpha_3, \cdots) + \text{diag}(\alpha_1, \alpha_2, \alpha_3, \cdots)S^*.
\]

Since clearly \( \text{diag}(\alpha_1, \alpha_2, \alpha_3, \cdots) \) is compact, this implies that \( R \in \text{ess Hank} \) if and only if \( A \in \mathcal{K} \). This concludes the proof. \(\square\)

It is worth pointing out that a calculation also shows that, for a bounded Rhaly matrix \( R \), \( S^*RS - R \) is Hilbert–Schmidt if and only if \( S^*R - RS \) is Hilbert–Schmidt.

We can now present a counterexample to our question.

**Example 5.2.8** Let \( C \) be the Cesàro matrix; i.e., the Rhaly matrix corresponding to the sequence \( \{1/(n+1)\}_{n=0}^\infty \in \ell^2 \). Then \( C \in \text{ess Hank} \) and \( C \) is not a compact perturbation of a bounded Hankel operator.
CHAPTER 5. ESSENTIALLY HANKEL OPERATORS

Proof: It is well known that \( C \) is bounded (see, for example, [6]). A direct calculation (see [6]) shows that \( SC - CS \) is Hilbert-Schmidt, so that \( SC - CS \in \mathcal{K} \); i.e., \( C \in \text{ess Toep} \). By the previous theorem, \( C \in \text{ess Hank} \).

Now, assume there was a bounded Hankel operator \( H \) and a compact operator \( K \) such that \( C = H + K \). Take a point \( \lambda \in \{ z \in \mathbb{C} : |z - 1| < 1 \} \). It is known [27] that \( C - \lambda \) is Fredholm and its Fredholm index

\[
\text{ind}(C - \lambda) = -1.
\]

But then, \( C - \lambda = H - \lambda + K \) which implies that \( H - \lambda \) is also Fredholm and \( \text{ind}(H - \lambda) = -1 \). But it is well known (and simple to prove) [49] that \( \text{ind}(H - \lambda) = 0 \) for any bounded Hankel operator \( H \) whenever \( H - \lambda \) is Fredholm. Contradiction. \( \square \)

We leave the following question unanswered: which bounded Rhaly matrices \( R \) with \( R \in \text{ess Hank} \) are compact perturbations of bounded Hankel operators?

The lower-triangular part of the Hilbert matrix, namely,

\[
H_- = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
1/2 & 1/3 & 0 & 0 & 0 & \cdots \\
1/3 & 1/4 & 1/5 & 0 & 0 & \cdots \\
1/4 & 1/5 & 1/6 & 1/7 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

can be seen to define a bounded operator (it is non-negative entrywise and it is less than or equal to the Hilbert matrix entrywise – see Choi [11] for a proof of the boundedness of the Hilbert matrix). A calculation shows that

\[
S^*H_- - H_-S \in \mathcal{K},
\]

so \( H_- \in \text{ess Hank} \).

Is \( H_- \) a compact perturbation of a Hankel operator? (It is easy to see that \( H_- \) minus the Hilbert matrix cannot be compact, but there could be another Hankel operator \( \Gamma \) such that \( H_- - \Gamma \) is compact!).
In general, it is easy to see that if a lower (or upper) triangular part of a bounded Hankel operator is bounded, then it is in ess Hank. Are any of these examples compact perturbations of bounded Hankel operators?

To finish this chapter, we would like to make the following observations. Let $X$ be a $\lambda$–Hankel operator (see Chapter 4). When can $X$ be in ess Hank? If $\lambda = 0$ we are in the case of Hankel operators so the answer is all the time. If $\lambda \neq 0$, then

$$(1/\lambda)(S^*X - XS) = X,$$

so $X \in \text{ess Hank}$ if and only if $X$ is compact.

When are $\lambda$–Hankel operators in ess Toep?

**Theorem 5.2.9** Let $\lambda$ be a complex number which is not purely imaginary and let $X$ be a bounded $\lambda$–Hankel operator. Then $X \in \text{ess Toep}$ if and only if $X$ is compact.

**Proof:** If $X$ is compact then clearly $X \in \text{ess Toep}$. Conversely, suppose that $X \in \text{ess Toep}$; i.e., $S^*XS - X \in \mathcal{K}$. Since $X$ is $\lambda$–Hankel, we have

$$S^*XS - X = S^*(S^* - \lambda)X - X = (S^*(S^* - \lambda) - I)X = T_{\overline{z}(\overline{z} - \lambda)}^{-1}X.$$  

This implies that if $X \in \text{ess Toep}$ then $T_{\overline{z}(\overline{z} - \lambda)}^{-1}X$ is compact. We just need to prove that the Toeplitz operator $T_{\overline{z}(\overline{z} - \lambda)}^{-1}$ is left semi–Fredholm to conclude that $X$ is compact.

Let $f(z) = \overline{z}(\overline{z} - \lambda) - 1$. Clearly $f \in L^\infty$; in fact, $f$ is continuous. If $f(z_0) = 0$ for some $z_0$ of modulus 1, then $\overline{z}_0(\overline{z}_0 - \lambda) = 1$, which implies that $z_0 - \overline{z}_0 = -\lambda$. This is impossible since $\lambda$ is not purely imaginary. Thus $f$ is continuous and never zero, which means that $1/f \in L^\infty$. But then, using equation 1.1 in Chapter 1, we can check that

$$I - T_{1/f}T_f = H_{z/f}.$$

Since $f$ is continuous, by Hartman's theorem it follows that $H_{z/f}$ is compact and thus that $I - T_{1/f}T_f$ is compact; i.e., $T_f$ is left semi–Fredholm. This finishes the proof. □

Notice that the case of Hankel operators is not covered by this theorem, since 0 is a purely imaginary number. It is known that there are plenty of Hankel operators in
ess Toep, as mentioned in the remark following Corollary 5.1.5. What is the situation for \( \lambda \)-Hankel operators when \( \lambda \neq 0 \) is purely imaginary?
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