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A complete conjugacy invariant is given for homeomorphisms of the unit circle possessing cyclic points. The irreducible \( * \)-representations are given for the C*-algebra that is generated by a universal pair of unitaries \((A, B)\) satisfying \(BAB^* = \varphi(A)\), where \(\varphi\) is a homeomorphism of the unit circle possessing cyclic points. When non-cyclic points are also present, the topological structure of the homeomorphism and its inverse are recovered from the C*-algebra. This is then used to give a complete \( * \)-isomorphism invariant for these C*-algebras. It is shown that for any two homeomorphisms with cyclic and non-cyclic points, the corresponding C*-algebras are \( * \)-isomorphic if and only if one homeomorphism is topologically equivalent to the other or its inverse.
To Adamo and Francesca
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CHAPTER 1

Introduction

With any homeomorphism \( \varphi \) of the unit circle, we can associate the \( C^* \)-algebra \( A_\varphi \) generated by the unitary operators

\[
A = \bigoplus_{\lambda} a_\lambda, \quad B = \bigoplus_{\lambda} b_\lambda ,
\]

(1.1)

where \((a_\lambda, b_\lambda)\) range over all pairs of unitary operators on separable Hilbert space, satisfying \( b_\lambda a_\lambda b^*_\lambda = \varphi(a_\lambda) \).

Our objective is to recover the topological features of the homeomorphism \( \varphi \) from the \( C^* \)-algebra \( A_\varphi \), where \( \varphi \) is assumed to be a homeomorphism of the unit circle which is either order preserving possessing cyclic and non-cyclic points, or \( \varphi \) is order reversing.

There are alternate ways of defining \( A_\varphi \). If \( A', B' \) are unitary operators satisfying the equality \( B' A'(B')^* = \varphi(A') \), then there exists a \( * \)-homomorphism of \( A_\varphi \) onto the \( C^* \)-algebra generated by \( A', B' \), mapping \( A \) to \( A' \) and \( B \) to \( B' \). Hence one could define \( A_\varphi \) as the \( C^* \)-algebra generated by a univeral pair of unitary operators \((A, B)\) satisfying the equality \( BAB^* = \varphi(A) \). \( A_\varphi \) is also \( * \)-isomorphic to the \( C^* \)-crossed product of \( C(\mathbb{T}) \) by the \( * \)-automorphism \( f \mapsto f \circ \varphi : C(\mathbb{T}) \to C(\mathbb{T}) \) induced by \( \varphi \) (cf. G.K. Pederson [7]), where \( \mathbb{T} \) denotes the unit circle. The problem of recovering the topological structure of \( \varphi \) from this \( C^* \)-algebra has been solved for rational rotations \( \varphi(\zeta) = \zeta^{e^{2\pi i \alpha}}, \zeta \in \mathbb{T} \) where \( \alpha \in \mathbb{Q} \cap [0, 1) \) is fixed (Høegh-Krohn, Skjelbred [5] (1980)), irrational rotations \( \varphi(\zeta) = \zeta^{e^{2\pi i \alpha}}, \zeta \in \mathbb{T} \) where \( \alpha \in [0, 1) \setminus \mathbb{Q} \) is fixed (Reiffel [12] (1981), Pimsner-Voiculescu [8] (1980)) and Denjoy homeomorphisms (Putnam, Schmidt, Skau [11] (1986)).

Unlike irrational rotations and Denjoy maps which have only one minimal set (a
closed invariant set with no proper closed invariant subsets), the maps we consider have many.

The basic strategy is to first work out the structure of homeomorphisms \( \varphi \) of the unit circle possessing cyclic points. This enables us to write down a family \( \{ \pi_\zeta \}_{\zeta \in \mathbb{T}^2} \) of irreducible \( * \)-representations of \( A_\varphi \) with a member in each unitary equivalence class. It turns out that for each \( g \in A_\varphi \) the map \( \zeta \mapsto \pi_\zeta(g) \), \( \zeta \in \mathbb{T}^2 \), can be expressed as a combination of block-Toeplitz operators with a compact perturbation, and a continuous matrix valued map (Proposition 3.3.3). Since the \( * \)-representation \( \pi := \sum_\zeta \otimes \pi_\zeta \) is a \( * \)-isomorphism of \( A_\varphi \) onto \( \pi(A_\varphi) \), we can think of an element \( g \in A_\varphi \) more tractably as the map \( \zeta \mapsto \pi_\zeta(g) \), \( g \in \mathbb{T}^2 \). In the special case where \( \varphi \) is a rational rotation (no Toeplitz operators necessary) this characterization of \( A_\varphi \) has already been proven useful by M. DeBrabanter ([2]) who uses it to give a more elementary proof of the classification of the rational rotation algebras. In our case this characterization of \( A_\varphi \) enables us to construct elements \( g \in A_\varphi \) satisfying certain algebraic conditions, and to prove that any \( g \in A_\varphi \) with these properties also contains the information sought about \( \varphi \) (Proposition 4.2.1 and Proposition 4.3.2). In the reverse direction, things are decidedly less complicated. For any homeomorphism \( \theta \) of the unit circle, the correspondence \( (a, b) \mapsto (\theta(a), b) \) between the unitary operator pairs \( (a, b) \) satisfying \( bab^* = \varphi(a) \) and the unitary operator pairs \( (a', b') \) satisfying \( b'a'(b')^* = \theta_\varphi \theta^{-1}(a') \), implies \( A_\varphi \) is \( * \)-isomorphic to \( A_{\theta \varphi \theta^{-1}} \). \( A_\varphi \) is also \( * \)-isomorphic to \( A_{\varphi^{-1}} \) because of the correspondence \( (a, b) \mapsto (a, b^*) \), between operator pairs \( (a, b) \) satisfying \( bab^* = \varphi(a) \), and operator pairs \( (a', b') \) satisfying \( b'a'(b')^* = \varphi^{-1}(a') \).

Chapter 2 starts with an analysis of order preserving homeomorphisms of the unit circle with cyclic points. Salient features, are the cycle number, rotation number, and flow structure. Among these features, the flow structure is the primary departure from the examples above. In Section 2.2, we show that these features
essentially comprise a complete invariant under conjugacy by an order preserving homeomorphism. The analogous result for order reversing homeomorphisms is contained in Section 2.3. In Section 3.1, we find a set of separable irreducible \(*\)-representations \(\{\pi_\zeta\}_{\zeta \in \mathbb{T}^2}\) parameterized by \(\mathbb{T}^2\), which contains at least one member from each unitary equivalence class. In Section 3.2 we collect a bit of machinery with accompanying notation, mainly block-Toeplitz operators, \(C^*\)-algebras defined via certain block-Toeplitz operators, and a technical lemma concerning matrix valued maps. Toeplitz operators are used in Section 3.3 to better express the maps \(\zeta \mapsto \pi_\zeta(g)\), \(\zeta \in \mathbb{T}\) for each \(g \in A_\varphi\). These maps are used frequently there-after, beginning with the spectrum in Section 3.4. We have learned that Williams ([15]) worked out the spectrum in a more abstract setting, but we include our approach mainly for its concreteness and ease of use.

In Chapter 4, a complete \(C^*\)-algebraic invariant is given for \(C^*\)-algebras \(A_\varphi\) with \(\varphi\) order preserving, and possessing both cyclic and non-cyclic points. This invariant is a set of four objects of the kind that served as a conjugacy invariant for homeomorphisms in Chapter 2, so we need to construct 'flow maps' and 'rotation numbers' from \(A_\varphi\). This is essentially accomplished in Proposition 4.2.1 and Proposition 4.3.2. In Section 4.4 we present a candidate for a complete \(C^*\)-algebraic invariant (confirmed by Theorem 4.4.6) and arrive at the main results, Theorem 4.4.5 and Corollary 4.4.7.

For order reversing homeomorphisms, the arguments are basically the same except for some technical differences which we highlight in Chapter 5.

Finally, we collect everything we know in Chapter 6, concluding with a complete \(C^*\)-algebraic invariant for the \(C^*\)-crossed product algebras defined by arbitrary homeomorphisms of the unit circle.

Certain symbols will have a fixed meaning. \(\mathbb{T}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\) denote the unit circle, natural numbers, integers, rational numbers, and real numbers respectively.
For any Hilbert space $H$, we denote the bounded linear operators and the compact operators on $H$ by $B(H)$, $K(H)$ respectively. For each $n \in \mathbb{N}$, the $n \times n$ matrices with complex entries are denoted by $M_n(\mathbb{C})$, and the unitary $n \times n$ complex matrices are denoted by $U_n(\mathbb{C})$. 
CHAPTER 2

Homeomorphisms of the unit circle with cyclic points

In this chapter, we look at homeomorphisms of the unit circle for which there exist cyclic points. Some features of these maps are worked out in Section 2.1. We prove these features collectively comprise a complete topological invariant for order preserving homeomorphisms in Section 2.2, and order reversing homeomorphisms in Section 2.3.

2.1 Structure of Homeomorphisms of the Unit Circle Possessing Cyclic Points

Dealing with continuous maps on the unit circle we encounter arcs (open, closed, semi-open etc...). If \( x, y \) are distinct points in \( \mathbb{T} \) the symbol \([x, y]\) will denote the closed arc of points in \( \mathbb{T} \) going counter clockwise from \( x \) to \( y \), and \([x, x] = \{x\}\). \([x, y], (x, y], (x, y)\) have the obvious meaning. We do not attribute any meaning to the symbols \([x, x), (x, x], (x, x)\). On the rare occasions that intervals in \( \mathbb{R} \) are used, we will specify this.

Definition 2.1.1. Let \( \varphi \) be a homeomorphism of the unit circle and let \( \zeta \in \mathbb{T} \). If the set \( \{\varphi^j(\zeta) | j \in \mathbb{Z}\} \) is finite, we say that \( \zeta \) is a cyclic point for \( \varphi \).

Proposition 2.1.2. Let \( \varphi \) be a homeomorphism of the unit circle with non-empty set \( \mathcal{F} \) of cyclic points, then one of the following is true:

(i) \( \varphi \) is order preserving and there exists \( n \in \mathbb{N} \) such that \( \varphi^n(\zeta) = \zeta \) but \( \varphi^j(\zeta) \neq \zeta \) for each \( \zeta \in \mathcal{F}, j \in \{1, \ldots, n-1\} \). If \( n \geq 2 \), there exists \( r \in \{1, 2, \ldots, n-1\} \) with \( (n, r) = 1 \) (greatest common factor) such that, for any \( \zeta \in \mathbb{T} \), the points \( \zeta, \varphi^*(\zeta), \ldots \varphi^{(n-1)r}(\zeta) \) are distinct, and if \( n \geq 3 \) they are also counter-clockwise ordered.
(ii) \( \varphi \) is order reversing and there are exactly two fixed points \( x, y \). \( \varphi((x, y)) = (y, x), \varphi((y, x)) = (x, y) \), and \( \varphi^2(\zeta) = \zeta \) for all points in \( F \setminus \{x, y\} \).

**Proof.** \( \varphi \) is either order preserving or order reversing since it is a homeomorphism.

Assume \( \varphi \) is order preserving. Let \( n \in \mathbb{N} \) be minimal such that \( \varphi^n \) has fixed points, and let \( \zeta \) be a fixed point of \( \varphi^n \). If \( n \geq 2 \) let \( r \in \{1, 2, \ldots, n - 1\} \) be such that \( \varphi^r(\zeta) \neq \zeta \) and \( (\zeta, \varphi^r(\zeta)) \cap \{\zeta, \varphi(\zeta), \ldots, \varphi^{n-1}(\zeta)\} = \emptyset \). So \( \left( \varphi^j(\zeta), \varphi^r(\varphi^j(\zeta)) \right) \cap \{\zeta, \varphi(\zeta), \ldots, \varphi^{n-1}(\zeta)\} = \emptyset \) for all \( j \in \mathbb{Z} \). Hence \( \zeta, \varphi^r(\zeta), \varphi^{2r}(\zeta), \ldots, \varphi^{(n-1)r}(\zeta) \) are distinct, in particular \( (n, r) = 1 \) and if \( n \geq 3 \) they are counter-clockwise ordered.

If \( z \in T \) then \( z \in [\varphi^\ell(\zeta), \varphi^{\ell+1}(\zeta)] \) for some \( \ell \in \{0, 1, \ldots, n - 1\} \), so \( \varphi^{\ell+j}(z) \in [\varphi^{\ell+j}(\zeta), \varphi^{\ell+j+1}(\zeta)] \) for each \( j \). Hence \( z, \varphi^r(z), \ldots, \varphi^{(n-1)r}(z) \) are distinct, and if \( n \geq 3 \) they are also counter-clockwise ordered for each \( z \in T \). Suppose \( z \) is in the open set of points not fixed by \( \varphi^n \). Let \( (x_1, x_2) \) be the component of this set containing \( z \). Then \( \varphi^n(x_1) = x_1 \) and \( \varphi^n(x_2) = x_2 \). If \( \varphi^n(z) \in (x_1, x_2) \) (the only other case, \( \varphi^n(z) \in (x_1, z) \) can be argued similarly) then for any \( k \in \mathbb{Z} \) we have \( \varphi^{nk}(z) \in (\varphi^{n(k-1)}(z), x_2) \subseteq (x_1, x_2) \). In particular, we've shown that if \( \varphi^n(z) \neq z \) then \( z \notin F \). Hence every element of \( F \) is a fixed point of \( \varphi^n \) but not of \( \varphi^j \) for any \( j \in \{1, \ldots, n - 1\} \).

If \( \varphi \) is order reversing, then we can think of \( \varphi \) as a map from \([0, 2\pi)\) onto \([0, 2\pi)\) (intervals in \( \mathbb{R} \)) by considering \( \arg(\varphi) \). The graph of \( \arg(\varphi) \) crosses the \( x = y \) line twice. On the circle this means \( \varphi \) has exactly two fixed points \( x_1, x_2 \). Clearly \( \varphi((x_1, x_2)) = (x_2, x_1), \varphi((x_2, x_1)) = (x_1, x_2) \) since \( \varphi \) is order reversing. Since \( \varphi^2 \) is order preserving and fixes \( x_1, x_2 \), then every element of \( F \setminus \{x_1, x_2\} \) is fixed by \( \varphi^2 \) but not by \( \varphi \). \( \blacksquare \)
Definition 2.1.3. Let $\varphi$ be an order preserving homeomorphism of the unit circle possessing cyclic points. We refer to the number $n$ of Proposition 2.1.2 as the cycle number of $\varphi$. If $n \geq 2$, define the rotation number of $\varphi$ as the number $r$ of Proposition 2.1.2. If $n = 1$, define the rotation number as 0.

Remark 2.1.4. The notion of rotation number was first introduced by Poincare ([9]), where it was taken to mean $r^{-1}(\text{modulo}(n))/n$ (cf. also Van Kampen [14]). The reason for our slight deviation from the standard meaning is that we want to be able to talk about $n$ and $r$ separately.

Remark 2.1.5. If $n \geq 2$, there exists an order preserving homeomorphism $\theta$ such that $\theta_0 \varphi \theta^{-1}(\zeta) = \zeta e^{2\pi r^{-1}(\text{modulo}(n))/n}$ for each $\zeta \in \mathcal{F}$. To see this, let $z \in \mathcal{F}$ and let $\theta_0 : [z, \varphi^r(z)] \to [1, e^{2\pi i/n}]$ be an order preserving homeomorphism. For any $m \in \{1, 2, \ldots, n - 1\}$ and $\zeta \in \varphi^m([z, \varphi^r(z)])$ define $\theta(\zeta) = e^{2\pi im/n} \theta_0(\varphi^{-m}(\zeta))$. A routine check shows $\theta$ satisfies the requirements.

If $\mathcal{F} = \mathbb{T}$, then $\theta_0 \varphi \theta^{-1}(\zeta) = \zeta e^{2\pi r^{-1}(\text{modulo}(n))/n}$ for all $\zeta \in \mathbb{T}$; i.e. the rational rotation by the angle $2\pi r^{-1}(\text{modulo}(n))/n$.

Definition 2.1.6. Let $\varphi$ be an order preserving homeomorphism of the unit circle with non-empty set of fixed points $\mathcal{F}$. Define the flow map $\varepsilon : \mathbb{T} \to \{-1, 0, 1\}$ associated with $\varphi$ as follows:

$$\varepsilon(\zeta) = \begin{cases} 
1 & \text{if } \zeta \in \mathcal{F}^{\text{c}} \text{ and } (\ldots \varphi^{-1}(\zeta), \zeta, \varphi(\zeta), \ldots) \text{ are a counter-clockwise sequence} \\
-1 & \text{if } \zeta \in \mathcal{F}^{\text{c}} \text{ and } (\ldots \varphi^{-1}(\zeta), \zeta, \varphi(\zeta), \ldots) \text{ are a clockwise sequence} \\
0 & \text{if } \zeta \in \mathcal{F}
\end{cases}$$

The map $\varepsilon$ is well defined. To see this, let $\zeta \in \mathcal{F}^{\text{c}}$ and let $(x_1, x_2)$ be the component of $\mathcal{F}^{\text{c}}$ containing $\zeta$. Then $(\zeta, \varphi(\zeta)) \subseteq (x_1, x_2)$ or $(\varphi(\zeta), \zeta) \subseteq (x_1, x_2)$. Suppose $(\zeta, \varphi(\zeta)) \subseteq (x_1, x_2)$, then $(\varphi^k(\zeta), \varphi^{k+1}(\zeta)) \subseteq (x_1, x_2)$ for all $k \in \mathbb{Z}$. Hence $\varphi^k(\zeta) \in (\varphi^{k-1}(\zeta), \varphi^{k+1}(\zeta)) \subseteq (x_1, x_2)$ for all $k \in \mathbb{Z}$. So $\{\varphi^k(\zeta)\}_{k \in \mathbb{Z}}$ is a counter-clockwise ordered sequence (as $k$ increases) in $(x_1, x_2)$, hence the
limits $\lim_{k \to \pm \infty} \varphi^k(\zeta)$ both exist and are fixed points of $\varphi$ in $(x_1, x_2)$, and so

$$\lim_{k \to \pm \infty} \varphi^k(\zeta) = \begin{cases} x_2 \\ x_1 \end{cases}$$

since there are no fixed points of $\varphi$ in $(x_1, x_2)$. If $z \in (x_1, x_2)$, then $z \in [\varphi^j(\zeta), \varphi^{j+1}(\zeta))$ for some $j \in \mathbb{Z}$; so $\{\varphi^k(z)\}_{k \in \mathbb{Z}}$ is a counter clockwise ordered sequence as well. We also note that $\lim_{k \to \pm \infty} \varphi^k(z) = \begin{cases} x_2 \\ x_1 \end{cases}$. Similarly if $(\varphi(z), \zeta) \subseteq (x_1, x_2)$ then $\{\varphi^k(z)\}_{k \in \mathbb{Z}}$ is a clockwise ordered sequence (as $k$ increases) in $(x_1, x_2)$ and $\lim_{k \to \pm \infty} \varphi^k(z) = \begin{cases} x_1 \\ x_2 \end{cases}$ for any $z$ in $(x_1, x_2)$.

We note that if $w$ is a component of $\mathcal{F}^c$, the argument above shows that $\varepsilon$ is constant on $\bigcup_{k=0}^{n-1} \varphi^k(w)$, so $\varepsilon$ can also be thought of as a map defined on the components of $\mathcal{F}^c$. $\mathcal{F}^c$.

2.2. A COMPLETE CONJUGACY INVARIANT FOR ORDER PRESERVING HOMEOMORPHISMS OF THE UNIT CIRCLE POSSESSING CYCLIC POINTS

**Definition 2.2.1.** Let $J = \{(q, \lambda)\mid q \in \mathbb{Q} \cap [0, 1)\}$, $\lambda : S \to \{-1, 0, 1\}$ is a map with domain $S$ a countable subset of isolated points in $\mathbb{T}$.

Let $(q, \lambda)$, $(q', \lambda')$ be elements of $J$ and let $S$, $S'$ be the domains of $\lambda$, $\lambda'$ respectively. We write $(q, \lambda) \sim (q', \lambda')$ if $q = q'$ and there exists an order preserving bijection $\gamma : S \to S'$ such that $\lambda = \lambda' \circ \gamma$. This is an equivalence relation on $J$ and we denote the set of equivalence classes by $\mathcal{I}$. If $(q, \lambda) \in J$, we let $[(q, \lambda)](\in \mathcal{I})$ denote its equivalence class.

**Definition 2.2.2.** Let $\varphi$ be an order preserving homeomorphism of the unit circle with non-empty set $\mathcal{F}$ of cyclic points, the cycle number $n$ and the rotation number $r$. Let $\varepsilon$ be the flow map of $\varphi^n$. Let $S$ be a set consisting of exactly one element from each component of $\mathcal{F}^c$ and $\mathcal{F}^c$. Let $\varepsilon|_S$ denote the restriction of $\varepsilon$ to $S$. Then $(r/n, \varepsilon|_S) \in J$, and we define $I(\varphi) = [(r/n, \varepsilon|_S)](\in \mathcal{I})$, noting that there is no dependence on the elements chosen for $S$.

We note that the set of cyclic points $\mathcal{F}$ can be determined from $\varepsilon|_S$ (up to
homeomorphism). To do this, select an open arc $\omega_\zeta$ for each $\zeta \in S$, such that the arcs $\{\omega_\zeta\}_{\zeta \in S}$ are pair-wise disjoint. By Zorn’s Lemma we can assume $\{\omega_\zeta\}_{\zeta \in S}$ is maximal in the sense that for any other such selection $\{\omega'_\zeta\}_{\zeta \in S}$, with $\omega_\zeta \subseteq \omega'_\zeta$, $\zeta \in S$ we must have $\omega_\zeta = \omega'_\zeta$ for all $\zeta \in S$. The set $\bigcup_{\zeta \in S, \lambda(\zeta) \neq 0} \omega_\zeta$ is then homeomorphic to $\mathcal{F}^c$, hence the complement of this set is homeomorphic to $\mathcal{F}$.

Remark 2.2.3. Suppose $(q, \lambda) \in J$ with $q = r/n$ in reduced form. It can be shown that the element $[(q, \lambda)] \in I$ is equal to $I(\varphi)$ for some order preserving homeomorphism $\varphi$ with cyclic points if and only if the following are true: (i) for any distinct $x, y \in S$ with $\lambda(x) = \lambda(y) = 0$ there exists $z \in S \cap (x, y)$ with $\lambda(z) \neq 0$, and (ii) there exists an order preserving bijection $\tau : S \to S$ satisfying $\tau^n = 1$, $\tau^j \neq 1$ for $j \in \{1, \ldots, n - 1\}$, and $\lambda \circ \tau = \lambda$. To see this, let $\{\omega_\zeta\}_{\zeta \in S}$ be a set of pairwise disjoint open arcs, such that $\zeta \in \omega_\zeta$ for each $\zeta \in S$. By Zorn’s Lemma we can assume the set $\{\omega_\zeta\}_{\zeta \in S}$ is maximal in the sense that for any set $\{\omega'_\zeta\}_{\zeta \in S}$ of pairwise disjoint open arcs satisfying $\omega_\zeta \subseteq \omega'_\zeta$, $\zeta \in S$, we must have $\omega_\zeta = \omega'_\zeta$ for all $\zeta \in S$. Let $\Omega = \bigcup_{\zeta \in S} \omega_\zeta$, a dense open subset of $T$ which contains $S$. There is no loss of generality if we assume $\zeta, \tau(\zeta), \ldots, \tau^{n-1}(\zeta)$ are counterclockwise ordered for each $\zeta \in S$ (if necessary, replace $\tau$ with $\tau^j$ for appropriate $j$). Let $s = r^{-1}$ (modulo $n$). For each $\zeta \in S$, there exist order preserving homeomorphisms from $\omega_\zeta$ onto $\omega_{\tau^s(\zeta)}$, $\omega_{\tau^s(\zeta)}$ onto $\omega_{\tau^{2s}(\zeta)}$, $\ldots$, $\omega_{\tau^{(n-1)s}(\zeta)}$ onto $\omega_\zeta$, such that their composition is a homeomorphism of $\omega_\zeta$ onto $\omega_\zeta$ with flow $\lambda(\zeta)$. Letting $\zeta$ vary over $S$, the resulting homeomorphisms comprise an order preserving homeomorphism of the dense open subset $\Omega$ of $T$. The argument is completed by the following Lemma, which assures us that this homeomorphism of $\Omega$ uniquely extends to an order preserving homeomorphism $\varphi : T \to T$, furthermore $I(\varphi) = [(q, \lambda)]$. The following Lemma will also help us to show that $I(\varphi)$ is a complete conjugacy invariant for each order preserving homeomorphism $\varphi$ possessing cyclic points.
Lemma 2.2.4. Each order preserving bijection whose domain and range are dense in $T$ can be extended to a homeomorphism of $T$.


The following lemma will be used to simplify the proof of the main theorem of this Chapter.

Lemma 2.2.5. Let $\varphi, \psi$ be order preserving homeomorphisms of $T$ with non-empty sets of cyclic points, and suppose $I(\varphi) = I(\psi)$. Then there exists an order preserving homeomorphism $\theta$ such that the maps $\varphi, \theta \circ \psi \circ \theta^{-1}$ have the same cycle number, rotation number, cyclic points, and flow map.

Proof. Let $r, n, \varepsilon, F$ be the rotation number, cycle number, flow map and cyclic points respectively for $\varphi$. Similarly define $r', n', \varepsilon', F'$ for $\psi$. From each component of $F^o \cup F^c$ ($F'^o \cup F'^c$) choose some element, and let $S$ (resp. $S'$) be the resulting set. Then $[(r/n, \varepsilon|_S)] = I(\varphi) = I(\psi) = [(r'/n', \varepsilon'|_{S'})]$. So $n = n', r = r'$ and there exists an order preserving bijection $\gamma : S \to S'$ such that $\varepsilon|_S = \varepsilon'|_{S' \circ \gamma}$. We can assume $\varepsilon, \varepsilon'$ are non-zero. For each $\zeta \in S$ choose any order preserving homeomorphism of the component of $F^o \cup F^c$ containing $\zeta$ onto the component of $F'^o \cup F'^c$ containing $\gamma(\zeta)$. Collectively, these homeomorphisms define on order preserving homeomorphism $\theta$ between the dense subsets $F^o \cup F^c$, $F'^o \cup F'^c$ of $T$, which can be extended to a unique homeomorphism of $T$ (which we continue to denote by $\theta$) by Lemma 2.2.4. $\theta^{-1} \circ \psi \circ \theta$ and $\varphi$ have the same cyclic points and flow map. ■

We're all set for the main result of this chapter.

Theorem 2.2.6. Let $\varphi, \psi$ be order preserving homeomorphisms of the unit circle with cyclic points. Then $I(\varphi) = I(\psi)$ if and only if there exists an order preserving homeomorphism $\theta : T \to T$ such that $\psi = \theta \circ \varphi \circ \theta^{-1}$. 

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Proof. Let $n$, $r$, $F$, $\varepsilon$ be the cycle number, the rotation number, the cyclic points, and the flow map respectively for $\varphi$. Similarly define $n'$, $r'$ $F'$, $\varepsilon'$ for $\psi$.

Suppose $\psi = \theta \circ \varphi \circ \theta^{-1}$ for some order preserving homeomorphism $\theta$. Then $\theta$ maps $F^o$ onto $F'^o$, $F^c$ onto $F'^c$. From each component of $F^o \cup F^c$ pick some point, the result is a countable set $S \subseteq T$ of isolated points. The set $S' := \theta(S)$ is a set of isolated points in $T$ with exactly one member in each component of $F'^o \cup F'^c$. Let $\gamma : S \rightarrow S'$ denote the order preserving bijection $\zeta \mapsto \theta(\zeta)$, $\zeta \in S$. Since $\theta \circ \varphi \circ \theta^{-1} = \psi^n$ then $\varepsilon(\zeta) = \varepsilon'(\theta(\zeta)) = \varepsilon' \gamma(\zeta)$, $\zeta \in S$, and $r'/n' = r/n$. Hence $(r/n, \varepsilon|_S) \sim (r'/n', \varepsilon'|_{S'})$ via $\gamma$.

Conversely suppose $I(\varphi) = I(\psi)$. By Lemma 2.2.5 we can assume (without loss of generality) that $\varphi$, $\psi$ have the same rotation number, cycle number, cyclic points and flow map.

Suppose $F = T$. If $n = 1$ then $\varphi = \psi = \text{id}_T$. So let $n \geq 2$ and let $\theta_0 : [1, \varphi^r(1)] \rightarrow [1, \rho]$ be any order preserving homeomorphism, where $\rho = e^{2\pi i/n}$. For each $j \in \{1, \ldots, n-1\}$, $\zeta \in \varphi^{r^j}([1, \varphi^r(1)])$ define $\theta(\zeta) := \rho^j \theta_0(\varphi^{-r^j}(\zeta))$. $\theta$ is an order preserving homeomorphism such that the map $\theta \circ \varphi \circ \theta^{-1}$ is the rotation $\rho^{r^{-1}(\text{modulo } n)}$. Similarly for $\psi$, hence $\varphi$ and $\psi$ are conjugate via an order preserving homeomorphism of $T$.

Suppose $F \neq T$. We will construct an order preserving homeomorphism $\theta : T \rightarrow T$ satisfying $\theta \circ \varphi \circ \theta^{-1} = \psi$, by defining its restriction to progressively larger countable subsets of $T$ until $\theta$ is defined on all of $T$. We start by defining $\theta$ on a subset $S$ of isolated points in $T$. When $n = 1$, define $S$ by selecting one point from each component of $F^o \cup F^c$, and define $\theta(\zeta) := \zeta$, $\zeta \in S$. Hence we can assume $n \geq 2$, here too we will define a set $S$. To do this we utilize a point in $F$ at which $\varphi^r - \psi^r$ is zero. Suppose $\varphi^r(x) \neq \psi^r(x)$ for all $x \in T$. Let $\zeta_0 \in F$, then $\psi^r(\zeta_0) \in (\zeta_0, \varphi^r(\zeta_0))$ or $\varphi^r(\zeta_0) \in (\zeta_0, \psi^r(\zeta_0))$; both are similarly
reasoned so assume the former. Then \( \psi^r(\zeta) \in (\zeta, \varphi^r(\zeta)) \) for all \( \zeta \in T \), for if
not then \( \varphi^r(x) = \psi^r(x) \) for some \( x \in T \), which is impossible. If \( n = 2 \), this
means that \( \psi(\zeta_0) \in (\zeta_0, \varphi(\zeta_0)) \) and \( \zeta_0 = \psi^2(\zeta_0) \in (\psi(\zeta_0), \varphi(\psi(\zeta_0))) \), which is
impossible. If \( n \geq 3 \), then by induction we see that \( \psi^k(\zeta_0) \in (\psi^{(k-1)r}(\zeta_0), \varphi^r \circ \psi^{(k-1)r}(\zeta_0)) \subseteq (\psi^r(\zeta_0), \varphi^r(\zeta_0)), k \in \{2, \ldots, n - 1\} \). If \( k = n \) we have
\( \psi^n(\zeta_0) \in (\psi^r(\zeta_0), \varphi^n(\zeta_0)) \), i.e. \( \zeta_0 \in (\psi^r(\zeta_0), \zeta_0) \), which is impossible. Hence
\( \varphi^r(x) = \psi^r(x) \) for some \( x \in T \). Moreover we can assume \( x \in F \), for if not let
\([a, b] \) be the component of \( F \) such that \( a \) is a boundary point of the component
of \( F^c \) containing \( x \). Then \( \varphi^r([a, b]) = \psi^r([a, b]) \), in particular \( \varphi^r(a) = \psi^r(a) \)
\( (a \in F) \). From each component of \( (F^c \cup F^c) \cap [x, \varphi^r(x)] \) pick an element, call
the resulting set \( S(x) \), and let \( S := \bigcup_{j=0}^{n-1} \varphi^j(S(x)) \). Define \( \theta \) on \( S \) by setting
\( \theta(\varphi^j(\zeta)) := \varphi^j(\zeta) \) for all \( \zeta \in S(x), j \in \{0, 1, \ldots, n - 1\} \). Thus by considering the
cases \( n = 1, n > 1 \) separately, we have in both cases defined a set \( S \) of points in \( T \)
and a map \( \theta : S \to T \). For any \( x, y \in T \) let \(-1 \ast [x, y] \) be the arc \([y, x], 1 \ast [x, y] \)
the arc \([x, y] \). Similarly for \([x, y], (x, y), (x, y) \). Suppose \( \zeta \in S \cap F^c \) and \( n = 1 \),
or \( \zeta \in S(x) \cap F^c \) and \( n > 1 \). Extend \( \theta \) to an order preserving homeomorphism of
\( e(\zeta) \ast [\zeta, \varphi^n(\zeta)] \) onto itself by defining \( \theta(z) = z \) for all \( z \in e(\zeta) \ast (\zeta, \varphi^n(\zeta)) \) (\( \theta \) is
already defined at \( \zeta \) and \( \theta(\zeta) = \zeta \) from above). For each \( z \in F^c \cup F^c \) let \( c(\zeta) \) be
the component of \( F^c \) or \( F^c \) containing \( z \). Further extend \( \theta \) to a homeomorphism
of \( c(\zeta) \) onto itself by setting \( \theta(\varphi^j(\zeta)) := \varphi^j(\theta(z)) \) for each \( z \in e(\zeta) \ast [\zeta, \varphi^n(\zeta)] \),
\( j \in Z \). If \( n \geq 2 \) extend \( \theta \) to \( \bigcup_{j=0}^{n-1} \varphi^j(c(\zeta)) \) by defining \( \theta(\varphi^j(\zeta)) := \psi^j(\theta(z)) \) for all
\( z \in c(\zeta), j \in \{1, \ldots, n - 1\} \). \( \theta \) is thus defined on \( \bigcup_{\zeta \in S \cap F^c} c(\zeta) = F^c \).

Next we define \( \theta \) on \( F^c \). If \( n = 1 \) and \( \zeta \in S \cap F^c \), or \( n \geq 2 \) and \( \zeta \in S(x) \cap
F^c \) define \( \theta(z) := z \) for all \( z \in c(\zeta) \). When \( n > 1 \) extend this to \( \bigcup_{j=0}^{n-1} \varphi^j(c(\zeta)) \) by
defining \( \theta(\varphi^j(z)) := \psi^j(\theta(z)), z \in c(\zeta), j \in \{1, \ldots, n - 1\} \). In all cases \( (n \geq 1) \)
the result is an order preserving homeomorphism \( \theta \) of \( F^c \cup F^c = \bigcup_{\zeta \in S} c(\zeta) \) onto
itself, such that \( \theta \circ \varphi \circ \theta^{-1}(\zeta) = \psi(\zeta) \) for all \( \zeta \in F^c \cup F^c \). Since \( F^c \cup F^c \) is dense
in $T$, $\theta$ can be extended to an order preserving homeomorphism of $T$ (Lemma 2.2.4) which we still denote by $\theta$, such that $\theta \circ \varphi \circ \theta^{-1}(\zeta) = \psi(\zeta)$, $\zeta \in T$.  

2.3. ORDER REVERSING HOMEOMORPHISMS

We classify order reversing homeomorphisms of the unit circle upto conjugacy by an order preserving homeomorphism. The invariant used is analogous to that of the order preserving case, differences being essentially technical.

**Definition 2.3.1.** Let $J^* = \{(\{x, y\}, \lambda) \mid x, y \text{ are distinct points in } T, \text{ and } \lambda : S \to \{-1, 0, 1\} \text{ is a map with domain } S \text{ a countable subset of isolated points in } T \setminus \{x, y\}\}$.

If $\{(x, y), \lambda\}, \{(x', y'), \lambda'\}$ are elements of $J^*$, and $S, S'$ the domains of $\lambda, \lambda'$ respectively, we write $\{(x, y), \lambda\} \sim \{(x', y'), \lambda'\}$ if there exists an order preserving bijection $\gamma : S \cup \{x, y\} \to S' \cup \{x', y'\}$ such that $\gamma(S) = S'$, $\gamma(\{x, y\}) = \{x', y'\}$ and $\lambda = \lambda' \circ \gamma$. We denote the set of equivalence classes of $J^*$ by $J^*$, and for each $\{(x, y), \lambda\} \in J^*$ we let $\{(\{x, y\}, \lambda)(\in J^*)$ denote its equivalence class.

**Definition 2.3.2.** Let $\varphi$ be an order reversing homeomorphism of $T$ with fixed points $\{x, y\}$, flow map $\varepsilon$, cyclic points $F$, and let $F_2 = F \setminus \{x, y\}$. From each component $\omega$ of $F_2^0 \cup F^c$ choose an element $\zeta(\omega)$, and let $S = \{\zeta(\omega) \mid \omega \text{ is a component of } F_2^0 \cup F^c\}$. Then $\{(x, y), \varepsilon|\omega\} \in J^*$ and we define $I^*(\varphi) = \{(\{x, y\}, \varepsilon|S\}) \in I^*$, noting there is no dependence on the elements chosen for $S$.

**Theorem 2.3.3.** Let $\varphi, \psi$ be order reversing homeomorphisms of the unit circle, then $I^*(\varphi) = I^*(\psi)$ if and only if there exists an order preserving homeomorphism $\theta$ such that $\varphi = \theta \circ \psi \circ \theta^{-1}$.

**Proof.** The idea of the proof is the same as the order preserving case, but there are technical differences. Let $I^*(\varphi) = \{(\{x, y\}, \varepsilon|S)\}$ where $\{x, y\}, F_2, F, \varepsilon, S$ are as in Definition 2.3.2. Similarly, let $\{x', y'\}, F_2', F', \varepsilon', S'$ be defined...
for $\psi$. For each $\zeta \in S$ let $c(\zeta)$ be the component of $F^c_2 \cup F^c$ containing $\zeta$, and let $c = \bigcup_{\zeta \in S} c(\zeta)$. Similarly define $c'(\zeta')$ for $\zeta' \in S'$, and $c'$ for $\psi$.

Suppose $I^*(\varphi) = I^*(\psi)$, so $\epsilon|_S = \epsilon'|_{S'}$ for some order preserving bijection $\gamma : S \cup \{x, y\} \rightarrow S' \cup \{x', y'\}$, with $\gamma(S) = S'$ and $\gamma(\{x, y\}) = \{x', y'\}$. Define $\theta : T \rightarrow T$ by giving its restriction to progressively larger subsets of $T$. Suppose $\xi \in (x, y) \cap S \cap F^c$. Let $\theta : \epsilon(\xi) \ast [\xi, \varphi^2(\xi)] \rightarrow \epsilon'(\gamma(\xi)) \ast [\gamma(\xi), \psi^2(\gamma(\xi))]$ be any order preserving homeomorphism. For each $\ell \in \mathbb{Z}$, $z \in \epsilon(\xi) \ast [\varphi^{2\ell}(\xi), \varphi^{2(\ell+1)}(\xi)]$ we set $\theta(z) = \psi^{2\ell}(\theta(\varphi^{-2\ell}(z)))$, thus defining $\theta$ from $c(\xi)$ onto $c'(\gamma(\xi))$ for each $\zeta \in (x, y) \cap S \cap F^c$. If $\zeta \in (x, y) \cap S \cap F^c$ define $\theta$ on $c(\xi)$ as any order preserving homeomorphism onto $c'(\gamma(\xi))$. The result is an order preserving bijection $\theta : c \cap (x, y) \rightarrow c' \cap (\gamma(x), \gamma(y))$ satisfying $\theta \varphi^2(z) = \psi^2 \theta(z)$, $z \in c \cap (x, y)$. Extend $\theta$ to $c \cap (y, x)$ by defining $\theta(\varphi(z)) := \psi(\theta(z))$ for $z \in c \cap (x, y)$ ($c \cap (y, x) = \varphi(c \cap (x, y))$).

$\theta$ is then an order preserving bijection between the dense subsets $c$, $c'$ of $T$, so by Lemma 2.2.4 can be extended to an order preserving homeomorphism of $T$ which we continue to denote by $\theta$, satisfying $\theta \circ \varphi = \psi \circ \theta$.

Conversely, assume $\psi = \theta \circ \varphi \circ \theta^{-1}$ for some order preserving homeomorphism $\theta$. Then for each $\zeta \in S$ there exists unique $\zeta' \in S'$ such that $c'(\zeta') = c'(\theta(\zeta))$. Since $\theta(\{x, y\}) = \{x', y'\}$ then the map $x \mapsto \theta(x)$, $y \mapsto \theta(y)$, $\zeta \mapsto \zeta'$ for $\zeta \in S$ is an order preserving bijection of $S \cup \{x, y\}$ onto $S' \cup \{x', y'\}$. Via this bijection the elements $(\{x, y\}, \epsilon|_S)$, $((x', y'), \epsilon'|_{S'})$ of $J^*$ are equivalent, i.e. $I^*(\varphi) = I^*(\psi)$.

Remark 2.3.4. If $\varphi$, $\psi$ are conjugate order reversing homeomorphisms of $T$, the conjugating homeomorphism can always be assumed to be order preserving. Indeed, if $\varphi = \theta \circ \psi \circ \theta^{-1}$ with $\theta$ order reversing, then $\varphi = (\theta \circ \psi) \circ \theta \circ (\theta \circ \psi)^{-1}$.

2.4 EXAMPLES

The following special cases illustrate Theorem 2.2.6 for some order preserving
homeomorphisms.

**Example 2.4.1.** A rational rotation is a map \( \zeta \mapsto \alpha \zeta \) for \( \zeta \in T \) where \( \alpha = e^{2\pi i m/n} \) for some \( n \in \mathbb{N}, \ m \in \{1, \ldots, n - 1\} \) such that \((n, m) = 1\). We denote this map by \( \alpha \). Let \( \alpha' = e^{2\pi i m'/n'} \) be another such scalar. Then as maps \( \alpha, \alpha' \) have the same flow map \( \epsilon = 0 \), and cyclic points \( T \). We can take \( S = S' = \{1\} \) with \( \lambda, \lambda' : \{1\} \to \{0\} \). So \( I(\alpha) = [(m^{-1} (\text{modulo } n))/n, 0] \), \( I(\alpha') = [(m')^{-1} (\text{modulo } n')/n', 0] \). Consequently each conjugacy class (by order preserving homeomorphisms) of order preserving homeomorphisms with cyclic point set \( T \), contains exactly one rational rotation \( \alpha \in \mathbb{Q} \cap [0, 1) \).

**Example 2.4.2.** Let \( \rho = e^{2\pi i/9} \) and let \( \varphi, \psi \) be any order preserving homeomorphisms of \( T \) having \( \{\rho^j\}_{j \in \{0, 1, \ldots, 8\}} \) as their cyclic points, with \( \varphi(\rho^j) = \rho^{j+3} \), \( \psi(\rho^j) = \rho^{j-3} \) for each \( j \in \{0, 1, \ldots, 8\} \). \( \varphi, \psi \) have the cycle number \( 3 \), and the rotation numbers \( 1, 2 \) respectively. Let \( \mu \in (1, \rho) \), and let \( S = \{\mu, \rho\mu, \ldots, \rho^8\mu\} \). Let \( \varphi, \psi \) have the same flow map

\[
\epsilon(\zeta) = \begin{cases} 
1 & \text{if } \zeta \in (1, \rho) \cup (\rho^3, \rho^4) \cup (\rho^6, \rho^7) \\
-1 & \text{if } \zeta \in (\rho, \rho^3) \cup (\rho^4, \rho^6) \cup (\rho^7, 1) \\
0 & \text{if otherwise}
\end{cases}
\]

Then \( I(\varphi) = [(1/3, \epsilon |_S)] \), \( I(\psi) = [(2/3, \epsilon |_S)] \). Also, \( I(\varphi^3) = [(0, \epsilon |_S)] \). Therefore \( I(\varphi^3) \sim I(\psi^3) \). Hence \( \varphi^3 \sim \psi^3 \), yet \( \varphi, \psi \) are not equivalent. In fact \( \varphi, \psi \) are not even conjugate by an order reversing homeomorphism. For if they were then \( \varphi \) would be conjugate to \( \psi' \) by an order preserving homeomorphism, where \( \psi'(\zeta) = \overline{\psi(\zeta)} \), \( \zeta \in T \). But \( I(\psi') = [(1/3, \epsilon' |_S)] \), where

\[
\epsilon'(\zeta) = \begin{cases} 
1 & \text{if } \zeta \in (1, \rho^2) \cup (\rho^3, \rho^5) \cup (\rho^6, \rho^8) \\
-1 & \text{if } \zeta \in (\rho^2, \rho^3) \cup (\rho^5, \rho^6) \cup (\rho^8, 1) \\
0 & \text{if otherwise}
\end{cases}
\]

\( I(\varphi) = I(\psi') \), hence there exists an order preserving bijection \( \gamma \) of \( S \) onto itself, such that \( \epsilon = \epsilon' \circ \gamma \), which is impossible since \( \epsilon'(\zeta) = 0 \) for every element \( \zeta \in \{\rho\mu, \rho^2\mu, \rho^5\mu, \rho^7\mu, \rho^8\mu\} \), while \( \epsilon(\zeta) = 1 \) only if \( \zeta \in \{\mu, \rho\mu, \rho^8\mu\} \).
We note that the maps $\varphi$ and $\psi^{-1}$ have the same cycle number 3, and rotation number 1. $\varphi^3$ and $(\psi^{-1})^3$ are conjugate (via an order reversing homeomorphism); yet $\varphi$ and $\psi^{-1}$ are not conjugate (via an order preserving or reversing homeomorphism).

**Example 2.4.3.** Let $\varphi$ be an order preserving homeomorphism of $\mathbb{T}$ with the cycle number $n$, the rotation number $r$, and the flow map $\varepsilon$. Then the inverse map $\varphi^{-1}$ will have the cycle number $n$, the rotation number $= \begin{cases} n - r & \text{if } r \geq 1 \\ 0 & \text{if } r = 0 \end{cases}$, and the flow map $-\varepsilon$.

**Example 2.4.4.** Let $\varphi$, $\psi$ be order preserving homeomorphisms with fixed point set $\{-1, 1\}$ and the flow maps $\varepsilon$, $\varepsilon'$ respectively, where $\varepsilon(\zeta) = 1$ for all $\zeta \in \mathbb{T}\backslash\{-1, 1\}$, and

$$
\varepsilon'(\zeta) = \begin{cases} 
-1 & \text{if } \zeta \in (1, -1) \\ 1 & \text{if } \zeta \in (-1, 1) \\ 0 & \text{if } \zeta \in \{-1, 1\} 
\end{cases}.
$$

Then any order preserving homeomorphism $\beta$ with exactly two fixed points, is conjugate (via an order preserving or reversing homeomorphism) to an element of $\{\varphi, \varphi^{-1}, \psi\}$. In particular $\beta$ is conjugate to $\beta^{-1}$ via a (possibly order reversing) homeomorphism since $\varphi^{-1}(\zeta) = \overline{\varphi(\zeta)}$ and $\psi^{-1} = M_{-1} \circ \psi \circ M_{-1}$, where $M_{-1} : \mathbb{T} \to \mathbb{T}$ is the map $\zeta \mapsto -\zeta$, $\zeta \in \mathbb{T}$. In fact using Theorem 2.2.6 we can show that any order preserving homeomorphism with five or fewer fixed points is conjugate to its inverse via a (possibly order reversing) homeomorphism. But this fails for six fixed points. For example, consider a homeomorphism with fixed point set $\{\rho^j\}_{j \in \{0, 1, \ldots, 5\}}$ where $\rho = e^{2\pi i/6}$, and flow map

$$
\varepsilon(\zeta) = \begin{cases} 
-1 & \text{if } \zeta \in (1, \rho) \cup (\rho^2, \rho^4) \\ 1 & \text{if } \zeta \in (\rho, \rho^2) \cup (\rho^4, 1) \\ 0 & \text{if } \zeta \in \{1, \rho, \ldots, \rho^5\} 
\end{cases}.
$$
CHAPTER 3

Irreducible \(*\)-representations

We find a set \(\{\pi_\zeta\}_{\zeta \in \mathbb{T}^2}\) of irreducible \(*\)-representations for the \(C^*\)-algebra \(A_\varphi\), where \(\varphi\) is any homeomorphism of the unit circle possessing cyclic points. This set is complete in the sense that any irreducible \(*\)-representation of \(A_\varphi\) is unitarily equivalent to \(\pi_\zeta\) for some \(\zeta \in \mathbb{T}^2\). For each \(g \in A_\varphi\) the map \(\zeta \mapsto \pi_\zeta(g)\), \(\zeta \in \mathbb{T}^2\) will be expressed in terms of block Toeplitz operators and a continuous matrix valued map (Proposition 3.3.3). This expression provides a basic tool for subsequent arguments.

3.1 IRREDUCIBLE \(*\)-REPRESENTATIONS

Let \(\varphi\) be a homeomorphism of the unit circle possessing cyclic points. Let \(A, B\) be the generators of the \(C^*\)-algebra \(A_\varphi\) given in (1.1.1). If \(\pi\) is an irreducible \(*\)-representation of \(A_\varphi\), then \((\pi(A), \pi(B))\) is a pair of unitary operators on a separable Hilbert space, with no common non-trivial reducing subspace, satisfying the equality \(\pi(B)\pi(A)\pi(B)^* = \varphi(\pi(A))\). Conversely, if \((A, B)\) are unitary operators on a separable Hilbert space with no common non-trivial reducing subspace, satisfying the equality \(BAB^* = \varphi(A)\), then there exists a unique irreducible \(*\)-representation \(\pi\) such that \(A = \pi(A), B = \pi(B)\). Hence the problem of finding the irreducible \(*\)-representations of \(A_\varphi\) is equivalent to that of finding all such pairs \((A, B)\) of unitary operators.

Proposition 3.1.1. Let \(\varphi\) be a homeomorphism of the unit circle possessing cyclic points. Let \(A, B\) be unitary operators on a separable Hilbert space, with no common non-trivial reducing subspace, satisfying the equality \(BAB^* = \varphi(A)\). Then \(A\) is a diagonalizable operator with eigenvalue set \(\{\varphi^j(\zeta)\}_{j \in \mathbb{Z}}\) for some \(\zeta \in \mathbb{T}\).
Proof. Let $P$ be the spectral measure satisfying $A = \int \mu dP(\mu)$. If $S$ is a Borel set in $T$ then $B \chi_S(A)B^* = \chi_S \circ \varphi(A)$ ($BAB^* = \varphi(A)$) where $\chi_S$ is the characteristic function of $S$, so $\int \chi_S(\mu)dBP(\mu)B^* = \int \chi_S \circ \varphi(\mu)dP(\mu)$. Hence

$$BP(S)B^* = P(\varphi^{-1}(S))$$

for any Borel set $S$. In particular, if $\varphi(S) = S$ then $P(S) = 0$ or $I$.

Suppose $\text{supp}(P) \cap F^c \neq \emptyset$. Let $\zeta$ be in this set, and let $\delta$ be an open arc neighbourhood of $\zeta$ contained in $F^c$. $\bigcup_{j \in \mathbb{Z}} \varphi^j(\delta)$ is an invariant set for $\varphi$, hence $P\left(\bigcup_{j \in \mathbb{Z}} \varphi^j(\delta)\right) = I$. As $\delta$ decreases to the singleton $\{\zeta\}$, the open set $\bigcup_{j \in \mathbb{Z}} \varphi^j(\delta)$ decreases to the set $\{\varphi^j(\zeta)\}_{j \in \mathbb{Z}}$. Hence $P(\{\varphi^j\}_{j \in \mathbb{Z}}) = I$.

Suppose $\text{supp}(P) \subseteq F$. Let $\zeta \in \text{supp}(P)$ ($\subseteq F$), and let $\delta$ be any open arc neighbourhood of $\zeta$. Suppose $\varphi$ is order preserving with the cycle number $n$. Then $\varphi^n(\delta \cap F) = \delta \cap F$ for all $j \in \mathbb{Z}$, hence $\bigcup_{j=0}^{n-1} \varphi^j(\delta \cap F)$ is an invariant set for $\varphi$. Hence $P\left(\bigcup_{j=0}^{n-1} \varphi^j(\delta \cap F)\right) = I$ (since $P(\delta \cap F) = P(\delta) \neq 0$). Again letting $\delta$ decrease to the singleton $\{\zeta\}$, then the set $\bigcup_{j=0}^{n-1} \varphi^j(\delta \cap F)$ decreases to $\{\varphi^j(\zeta)\}_{j=0,1,\ldots,n-1}$. Hence $P(\{\varphi^j\}_{j=0,1,\ldots,n-1}) = I$ as desired. The same argument applies when $\varphi$ is order reversing, if we put 2 in place of the cycle number.

For each $n \in \mathbb{N}$, let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$. For each $i \in \mathbb{Z}$, let $\delta_i : \mathbb{Z} \rightarrow \{0,1\}$ denote the map $\delta_i(k) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$. Take $\{\delta_i \otimes e_j | i \in \mathbb{Z}, j = 1,2,\ldots,n\}$ as the standard basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n$, with the ordering: $\delta_i \otimes e_j < \delta_i' \otimes e_{j'}$, when $i < i'$, or when $i = i'$ and $j < j'$. We will denote operators by their matrix in the standard basis.

Let $\varphi$ be an order preserving homeomorphism of the unit circle, with cyclic point set $F$ and the cycle number $n$.

For each $(\mu, \nu) \in F \times T$ let $(\pi_{\mu, \nu}, \mathbb{C}^n)$ be the irreducible $*$-representation of
\( A_\varphi \) satisfying
\[
\pi_{\mu, \nu}(A) = \begin{pmatrix}
\varphi^{n-1}(\mu) \\
\vdots \\
\varphi(\mu) \\
\mu
\end{pmatrix},
\]
and
\[
\pi_{\mu, \nu}(B) = \nu \begin{pmatrix}
0 & \cdots & \cdots & 0 & 1 \\
1 & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 1 \\
\end{pmatrix} \in M_n(\mathbb{C}). \tag{3.1}
\]

For each \((\mu, \nu) \in \mathcal{F}^c \times T\) let \((\pi_{\mu, \nu}, \ell^2(\mathbb{Z}) \otimes \mathbb{C}^n)\) be the irreducible \(*\)-representation of \( A_\varphi \) satisfying
\[
\pi_{\mu, \nu}(A) = \begin{pmatrix}
\ddots \\
\varphi(\mu) \\
\mu \\
\varphi^{-1}(\mu) \\
\ddots
\end{pmatrix},
\]
and
\[
\pi_{\mu, \nu}(B) = \nu \begin{pmatrix}
\ddots \\
1 & 0 \\
\ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}. \tag{3.2}
\]

**Remark.** If the \(*\)-representation \( \pi_{\mu, \nu} \) were defined on \( \ell^2(\mathbb{Z}) \) (via the above matrices) for each \((\mu, \nu) \in \mathcal{F}^c \times T\), we would get a unitarily equivalent \(*\)-representation. The Hilbert space \( \ell^2(\mathbb{Z}) \otimes \mathbb{C}^n \) is chosen because it turns out to be more natural for expressing general elements \( \pi_{\mu, \nu}(g) \) for \( g \in A_\varphi \) (cf. Proposition 3.3.3).

**Proposition 3.1.2.** Let \( \varphi \) be an order preserving homeomorphism of the unit circle with non-empty set of cyclic points, and let \((\pi, H)\) be an irreducible \(*\)-representation of \( A_\varphi \). Then \( \pi \) is unitarily equivalent to \( \pi_{\mu, \nu} \) for some \((\mu, \nu) \in T^2\).
Proof. Let $n(\in \mathbb{N})$ be the cycle number of $\varphi$. Let $A = \pi(A)$, and $B = \pi(B)$, these are unitary operators in $B(H)$ with no common non-trivial reducing subspace, satisfying $BAB^* = \varphi(A)$. By Proposition 3.1.1, $A$ is diagonalizable with eigenvalue set $\{\varphi^j(\mu)\}_{j \in \mathbb{Z}}$ for some $\mu \in \mathbb{T}$. Let $v$ be a unimodular eigenvector with eigenvalue $\mu$.

If $\mu$ is a cyclic point of $\varphi$, then $B^n A B^* B = \varphi^n(A) = A$. Hence $B^n = \nu I$ for some $\nu \in \mathbb{T}$. The linear space spanned by the orthonormal vectors

$$\{v, \sqrt{\nu}B(v), \ldots, (\sqrt{\nu}B)^{n-1}(v)\}$$

is invariant for $A$ and $B$, hence equal to $H$. With respect to this basis, $A$ has the matrix

$$
\begin{pmatrix}
\mu & \varphi^{-1}(\mu) \\
& \ddots \\
& & \ddots \\
& & & \varphi^{1-n}(\mu)
\end{pmatrix},
$$

and $B$ has the matrix

$$
\sqrt{\nu}
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & 0 & 1 & 0
\end{pmatrix}.
$$

Hence $\pi_\varphi \cong \pi_\varphi^{1-n(\mu)}$, $\sqrt{\nu} = \pi_\varphi(\mu)$, $\sqrt{\nu}$.

If $\mu$ is not a cyclic point of $\varphi$ then $\{B^\ell(v)\}_{\ell \in \mathbb{Z}}$ is an ordered orthonormal basis of $H$, and $A(B^\ell(v)) = \varphi^{-\ell}(\mu)B^\ell(v)$, $B(B^\ell(v)) = B^{\ell+1}(v)$ for all $\ell \in \mathbb{Z}$. Hence $\pi$ is unitarily equivalent to $\pi_{\mu,1}$. □

3.2 Matrix valued functions and block-Toeplitz operators

This section concerns block-Toeplitz operators (cf. R.G. Douglas [4]), a variant of such, and certain $C^*$-algebras defined via these operators (Proposition 3.2.8).
These $C^*$-algebras are used in the recovery of the rotation number. Block-Toeplitz operators are also used to express the values of infinite dimensional irreducible $*$-representations of the $C^*$-algebras that are the subject of this study.

3.2.1. We start with some notation. Let $L^2(T, \mathbb{C}^n)$ be the norm square integrable functions of $T$ into $\mathbb{C}^n$, and let $H^2(T, \mathbb{C}^n)$ be the associated Hardy space. We denote the continuous $n \times n$-matrix valued maps on $T$ by $C(T, M_n(\mathbb{C}))$, and the bounded Lebesgue measurable $n \times n$-matrix valued maps on $T$ by $L^\infty(T, M_n(\mathbb{C}))$.

For each $\phi, \psi \in L^\infty(T, M_n(\mathbb{C}))$, define

$$M_\phi : f \mapsto \phi f : L^2(T, \mathbb{C}^n) \to L^2(T, \mathbb{C}^n),$$

and define the $n \times n$-block Toeplitz operator associated with $\phi$ as:

$$T_\phi := PM_\phi|_{H^2(T, \mathbb{C}^n)} : H^2(T, \mathbb{C}^n) \to H^2(T, \mathbb{C}^n) \text{ (restriction to } H^2(T, \mathbb{C}^n)),$$

where $P : L^2(T, \mathbb{C}^n) \to H^2(T, \mathbb{C}^n)$ is the orthogonal projection. Also define:

$$\tilde{T}_\phi := (1 - P)M_\phi|_{H^2(T, \mathbb{C}^n)} : H^2(T, \mathbb{C}^n)^\perp \to H^2(T, \mathbb{C}^n)^\perp,$$

and

$$T(\phi, \psi) := (1 - P)M_\phi(1 - P) + PM_\psi P = \tilde{T}_\phi(1 - P) + T_\psi P : L^2(T, \mathbb{C}^n) \to L^2(T, \mathbb{C}^n).$$

Recall the maps $\delta_i : \mathbb{Z} \to \{0, 1\}$ for each $i \in \mathbb{Z}$, defined by $\delta_i(k) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$. Also recall the standard basis $\{\delta_i \otimes e_j | i \in \mathbb{Z}, \ j = 1, \ldots, n\}$ of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n$, with the ordering: $\delta_i \otimes e_j < \delta_i' \otimes e_{j'}$, when $i < i'$, or when $i = i'$ and $j < j'$. Let $\ell^2(\mathbb{Z})_+$ be the Hilbert space consisting of the elements of $\ell^2(\mathbb{Z})$ with value zero for the entries $\{-1, -2, \ldots\}$, and let $\ell^2(\mathbb{Z})_- = (\ell^2(\mathbb{Z})_+)^\perp$. Take $\{\delta_i \otimes e_j | i \in \mathbb{Z}_{\geq 0}, \ j = 1, 2, \ldots, n\}$, $\{\delta_i \otimes e_j | i \in \mathbb{Z}_{> 0}, \ j = 1, 2, \ldots, n\}$ as the standard bases for $\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n$ and $\ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n$ respectively. For each $\phi \in L^\infty(T, M_n(\mathbb{C}))$ and $i$, 21
j \in \{1, \ldots, n\}$ let $\phi_{ij}: \mathbb{T} \to \mathbb{C}$ be the map $\phi_{ij}(\zeta) = (\phi(\zeta))_{ij}$, $\zeta \in \mathbb{T}$. Then $\phi_{ij} \in L^\infty(\mathbb{T})$ for each $i$, $j$, and there is a Fourier expansion $\phi_{ij}(\zeta) = \sum_{k \in \mathbb{Z}} \hat{\phi}_{ij}(k)\zeta^k$, $\zeta \in \mathbb{T}$, for some scalars $\{\hat{\phi}_{ij}(k)\}_{k \in \mathbb{Z}}$. Let $\hat{\phi}_k$ be the $n \times n$ matrix whose $(i,j)^{th}$ entry is $\hat{\phi}_{ij}(k)$. Then $\phi(\zeta) = \sum_{k \in \mathbb{Z}} \hat{\phi}_k \zeta^k$ for all $\zeta \in \mathbb{T}$, with convergence in $L^2(\mathbb{T}, \mu_n)$. Define the block-Toeplitz matrix $T_{\phi}: \ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n \to \ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n$ as

$$T_{\phi} = \begin{pmatrix} \hat{\phi}_0 & \hat{\phi}_1 & \cdots \\ \hat{\phi}_1 & \hat{\phi}_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

in the standard basis. Define $\tilde{T}_{\phi}: \ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n \to \ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n$ as

$$\tilde{T}_{\phi} = \begin{pmatrix} \hat{\phi}_0 & \hat{\phi}_1 & \cdots \\ \vdots & \cdots & \cdots \\ \hat{\phi}_1 & \hat{\phi}_0 & \cdots \end{pmatrix}$$

in the standard basis. We remark that the operators $\tilde{T}_{\phi}$, $\tilde{T}_{\phi}$ are unitarily equivalent, where $\tilde{\phi}(\zeta) = \phi(\bar{\zeta})$ for all $\zeta \in \mathbb{T}$. Finally, if $\psi \in L^\infty(\mathbb{T}, \mu_n(\mathbb{C}))$ as well, define $T(\psi, \phi): \ell^2(\mathbb{Z}) \otimes \mathbb{C}^n \to \ell^2(\mathbb{Z}) \otimes \mathbb{C}^n$ in the standard basis as

$$T(\psi, \phi) = \begin{pmatrix} \hat{\psi}_0 & \hat{\psi}_1 & \cdots \\ \hat{\psi}_1 & \hat{\psi}_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $\psi(\zeta) = \sum_{k \in \mathbb{Z}} \hat{\psi}_k \zeta^k$, $\zeta \in \mathbb{T}$, is the Fourier expansion of $\psi$.

The operators $T_{\phi}$ and $T_{\tilde{\phi}}$ are unitarily equivalent, as are $\tilde{T}_{\phi}$ and $\tilde{T}_{\phi}$, $T(\psi, \phi)$ and $T(\psi, \phi)$. Henceforth we consider only the operators $T_{\phi}$, $\tilde{T}_{\phi}$, $T(\psi, \phi)$.

3.2.2. The following basic properties are just a restatement of the basic properties of block-Toeplitz operators (cf. R.G. Douglas [4]).

(i) Let $\phi \in C(\mathbb{T}, \mu_n(\mathbb{C}))$, then $T_{\phi}$ is compact if and only if $\phi = 0$. $T_{\phi}$ is Fredholm if and only if $\det \phi(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$.
(ii) Let $\phi \in C(T, M_n(C))$ and $\det(\phi(z)) \neq 0$ for all $z \in T$ (so $T_\phi$ is Fredholm), then $\text{ind}(T_\phi) = -\text{wn} \det(\phi)$ (ind = index, wn = winding number at the origin).

(iii) Let $\phi \in C(T, M_n(C))$ and $\psi \in L^\infty(T, M_n(C))$, then $T_\psi T_\phi - T_\psi$ and $T_\phi T_\psi - T_\psi$ are compact.

(iv) Let $\phi \in L^\infty(T, M_n(C))$, then $\|M_\phi\| = \|T_\phi\| = \|\phi\|_\infty$.

(v) The $C^*$-algebra generated by $\{T_\phi|\phi \in C(T, M_n(C))\}$ is equal to

$$\{T_\phi|\phi \in C(T, M_n(C))\} + K(\ell^2(Z)_+ \otimes C^n).$$

Definition 3.2.3. Let $n \in \mathbb{N}$, $p \in \mathbb{Z}_{\geq 0}$ and $H = C^p \oplus (\ell^2(\mathbb{N}) \otimes C^n)$. We let $A_{p,n}$ denote the $C^*$-algebra $\{O_{p \times p} \oplus (\ell^2(\mathbb{N}) \otimes Q) + k|Q \in M_n(C), k \in K(H)\}$ (cf. Behnke and Leptin [1]).

Theorem 3.2.4. (Behnke and Leptin [1]). Let $n \in \mathbb{N}$ and let $H$ be a separable Hilbert space. Suppose $A$ is a $C^*$-algebra of operators in $B(H)$ containing $K(H)$ such that $A/K(H)$ is $\ast$-isomorphic to $M_n(C)$. Then there exists unique $p \in \{0,1,\ldots,n-1\}$ such that $A$ is $\ast$-isomorphic to $A_{p,n}$.

If $n \in \mathbb{N}$, let $X_n \in C(T, M_n(C))$ be the map:

$$X_n(z) = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & z \\
1 & \ddots & & & \vdots \\
0 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 1 & 0
\end{pmatrix}_{n \times n}, \text{ for all } z \in T. \quad (3.3)$$

Lemma 3.2.5. Let $n \in \mathbb{N}$, $p \in \{0,1,\ldots,n-1\}$. Then the $C^*$-algebra

$$\{T_{X_n^* Q X_n}|Q \in M_n(C)\} + K(\ell^2(Z)_+ \otimes C^n)$$

is $\ast$-isomorphic to $A_{p,n}$, and the $C^*$-algebra

$$\{\tilde{T}_{X_n^* Q X_n}|Q \in M_n(C)\} + K(\ell^2(Z)_- \otimes C^n)$$
is \ast-isomorphic to \( A_{j,n} \) where \( j = [-p]_n \) ([\cdot]_n means modulo \( n \)).

**Proof.** Let \( Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \) be an element of \( M_n(\mathbb{C}) \) expressed in block matrix form with \( Q_{11} \in M_{n-p}(\mathbb{C}) \), \( Q_{22} \in M_p(\mathbb{C}) \), \( Q_{12} \in M_{n-p,p}(\mathbb{C}) \), \( Q_{21} \in M_{p,n-p}(\mathbb{C}) \). Then \( X_n^p Q X_n^{p*} \) has the Fourier expansion:

\[
X_n^p(\zeta)Q X_n^{p*}(\zeta)^* = \begin{pmatrix} Q_{22} & \zeta Q_{21} \\ \zeta Q_{12} & Q_{11} \end{pmatrix}
= \begin{pmatrix} Q_{22} & 0 \\ 0 & Q_{11} \end{pmatrix} + \begin{pmatrix} 0 & Q_{21} \\ 0 & 0 \end{pmatrix} \zeta + \begin{pmatrix} 0 & 0 \\ Q_{12} & 0 \end{pmatrix} \overline{\zeta}
\]

for all \( \zeta \in T \). Hence

\[
T_{X_n^p Q X_n^{p*}} = \begin{pmatrix} Q_{22} & 0 & 0 & 0 & 0 & \cdots \\ 0 & Q_{11} & Q_{12} & 0 & 0 & \cdots \\ 0 & Q_{21} & Q_{22} & 0 & 0 & \cdots \\ 0 & 0 & 0 & Q_{11} & Q_{12} & \cdots \\ 0 & 0 & 0 & Q_{21} & Q_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\]

for each \( Q \in M_n(\mathbb{C}) \), (3.4)

and

\[
\tilde{T}_{X_n^p Q X_n^{p*}} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{12} & 0 \\ 0 & 0 & Q_{21} & Q_{22} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\]

for each \( Q \in M_n(\mathbb{C}) \). (3.5)

The conclusion follows from (3.4), (3.5). \( \blacksquare \)

**Lemma 3.2.6.** Let \( n \in \mathbb{N} \), and let \( W \in B(\ell^2(\mathbb{Z}^+)) \) be a bounded operator such that \( WT_B - T_B W \) is compact for each \( B \in M_n \). Then \( W = \lambda I_n + k \) for some bounded matrix \( \lambda \) and compact matrix \( k \).

**Proof.** For each \( i, j \in \{1, \ldots, n\} \), let \( E_{ij} \) be the \( n \times n \) matrix unit satisfying

\[
(E_{ij})_{pq} = \begin{cases} 1 & \text{if } (i, j) = (p, q) \\ 0 & \text{if } (i, j) \neq (p, q) \end{cases}
\]

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For each $Q \in B(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n)$ and $r, s \in \mathbb{Z}_{\geq 0}$, let $Q^{rs}$ be the $n \times n$-matrix satisfying:

$$(Q^{rs})_{ij} = (Q(\delta_r \otimes e_i), \delta_s \otimes e_j) \text{ (inner product) for all } i, j \in \{1, \ldots, n\} .$$

If $i, j, k \in \{1, \ldots, n\}$, then

$$T_{E_{ii}} W T_{E_{jk}} T_{E_{kj}} - T_{E_{ii}} T_{E_{jk}} W T_{E_{kj}} \in K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) ,$$

hence

$$T_{E_{ii}} W T_{E_{jj}} - T_{E_{ii}} T_{E_{jj}} W T_{E_{jj}} \in K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) . \quad (3.6)$$

This implies

$$\sum_{i \neq j} T_{E_{ii}} W T_{E_{jj}} \in K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) \quad (3.7)$$

as well. For $i \in \{1, 2, \ldots, n\}$ and $r, s \in \mathbb{Z}_{\geq 0}$ we have

$$(T_{E_{ii}} W T_{E_{ii}})^{rs} = (W^{rs})_{11} E_{ii} ,$$

so

$$\left(\sum_i T_{E_{ii}} W T_{E_{ii}}\right)^{rs} = (W^{rs})_{11} I_n ,$$

hence

$$\sum_i T_{E_{ii}} W T_{E_{ii}} = \lambda \otimes I_n \quad (3.8)$$

where $\lambda$ is the bounded matrix $\lambda_{rs} = (W^{rs})_{11}$ for $r, s \in \mathbb{Z}_{\geq 0}$. Letting $i = j$, $k = 1$ in (3.6) and summing shows $\sum_i T_{E_{ii}} W T_{E_{ii}} - \sum_i T_{E_{ii}} W T_{E_{ii}}$ is compact. So

$$W = \left(\sum_i T_{E_{ii}}\right) W \left(\sum_j T_{E_{jj}}\right) = \sum_{i \neq j} T_{E_{ii}} W T_{E_{jj}} + \sum_i T_{E_{ii}} W T_{E_{ii}}$$

is $\lambda \otimes I_n +$ some compact matrix by (3.7), (3.8). \[\blacksquare\]

**Lemma 3.2.7.** Suppose $X, Y \in C(\mathbb{T}, M_n(\mathbb{C}))$ are unitary valued maps such that the two $C^*$-algebras $\{T_{XX}^* Q | Q \in M_n(\mathbb{C})\} + K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n)$ and $\{T_{YQ}^* Q | Q \in$
Then \( \text{wn det}(X) \equiv \text{wn det}(Y) \mod (n) \).

**Proof.** Both algebras contain \( K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) \). Hence there exists a unitary \( W \in B(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) \), such that for each \( Q \in M_n(\mathbb{C}) \) there corresponds some \( Q' \in M_n(\mathbb{C}) \) for which the operator \( WT_XQW^* - T_YQ'Y^* \) is compact. The correspondence \( Q \mapsto Q' \) is a \(*\)-automorphism of \( M_n(\mathbb{C}) \), hence there exists a unitary \( n \times n \) matrix \( M \) such that the operator \( WT_XT_Q^*WT_X - T_YT_MT_QT_M^*T_Y^* \) is compact for each \( Q \in M_n(\mathbb{C}) \). By Lemma 3.2.6 \( T_M^*T_Y^*WT_X \) is equal to \( \lambda I_n + k \) for some compact matrix \( k \) and bounded matrix \( \lambda \). So \( 0 \equiv \text{ind}(T_M^*)T_Y^*WT_X = \text{ind}(T_M^*) + \text{ind}(T_Y^*) + \text{ind}(W) + \text{ind}(T_X) \) (modulo \( n \)). But \( \text{ind}(T_M^*) = \text{ind}(W) = 0 \) and \( \text{ind}(T_Y^*) = -\text{ind}(T_Y) \), so \( \text{ind}(T_X) \equiv \text{ind}(T_Y) \) (mod \( n \)). Hence \( \text{wn det}(X) \equiv \text{wn det}(Y) \) (modulo \( n \)) by (ii) of subsection 3.2.2. \( \blacksquare \)

**Proposition 3.2.8.** Let \( n \in \mathbb{N} \), and let \( X, Y \in C(T, U_n(\mathbb{C})) \) be unitary valued functions. Then the \( C^* \)-algebra \( \{T_{XQX^*}, Y_{YQY^*} | Q \in M_n(\mathbb{C})\} + K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) \) is \(*\)-isomorphic to \( A_{j,n} \), where \( j = [\text{wn det} Y - \text{wn det} X]_n \).

**Proof.** \( \{T_{XQX^*} | Q \in M_n(\mathbb{C})\} + K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) / K(\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^n) \cong M_n(\mathbb{C}) \). Hence \( \{T_{XQX^*} | Q \in M_n(\mathbb{C})\} + K(\ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n) \cong A_{p,n} \) for some \( p \in \{0, \ldots, n-1\} \) (Theorem 3.2.4), and hence \(*\)-isomorphic to

\[
\{T_{X_i^iQX_i^*} | Q \in M_n(\mathbb{C})\} + K(\ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n)
\]

for some \( i \in \{0,1,\ldots,n-1\} \) (by Lemma 3.2.5), where in fact \( i = [\text{wn det}(X)]_n \) (Lemma 3.2.7). Both algebras contain the compact operators, hence there exists a unitary operator \( N_1 \in B(\ell^2(\mathbb{Z})_- \otimes \mathbb{C}^n) \) ([6] Proposition 10.4.6) such that for each \( Q \in M_n(\mathbb{C}) \) there corresponds some \( Q' \in M_n(\mathbb{C}) \) such that \( N_1 \tilde{T}_{XQX^*}N_1^* - \tilde{T}_{X_i^iQX_i^*} \) is compact. The correspondence \( Q \mapsto Q' \) is a \(*\)-isomorphism of \( M_n(\mathbb{C}) \),
hence there exists some unitary $W_1 \in M_n(\mathbb{C})$ such that for each $Q \in M_n(\mathbb{C})$ we have

$$N_1 \tilde{T}_X Q X^* N_1^* - \tilde{T}_X w_1 Q w_1^* x_i^* \in K(\ell^2(Z_-) \otimes \mathbb{C}^n)$$

where $i = [\text{wn det}(X)]_n$. Similarly, we can show the existence of unitaries $W_2 \in M_n(\mathbb{C})$, $N_2 \in B(\ell^2(Z_+) \otimes \mathbb{C}^n)$ such that

$$N_2 T_Y Q Y^* N_2^* - T_X^* w_2 Q w_2^* x_i^* \in K(\ell^2(Z_+) \otimes \mathbb{C}^n)$$

for all $Q \in M_n(\mathbb{C})$.

where $j = [\text{wn det}(Y)]_n$. Hence via the unitary operator $\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$ we have

$$\{T(XQX^*, YQY^*)|Q \in M_n(\mathbb{C})\} + K(\ell^2(Z_+) \otimes \mathbb{C}^n) \cong \{T(X_n^* w_1 Q w_1^* x_i^*, X_n^* w_2 Q w_2^* x_i^*)|Q \in M_n(\mathbb{C})\} + K(\ell^2(Z) \otimes \mathbb{C}^n),$$

which is $\ast$-isomorphic to $A_{[j-i],n,n}$ (cf. (3.4), (3.5)) where $i = [\text{wn det}(X)]_n$, $j = [\text{wn det}(Y)]_n$.

Since we are on the subject of matrix valued functions, we include two basic lemmas which will be useful at a later time.

**Lemma 3.2.9.** Let $F \subseteq [0,1]$ be a closed set in $\mathbb{R}$. Let $n \in \mathbb{N}$, and suppose $\beta : F \times [0,1] (\subseteq \mathbb{R}^2) \rightarrow U_n(\mathbb{C})$ is a function such that $\zeta \rightarrow \beta(\zeta)Q\beta^*(\zeta)$ is a continuous function from $F \times [0,1]$ to $M_n(\mathbb{C})$ for each $Q \in M_n(\mathbb{C})$, and $\beta(\mu,0) = \beta(\mu,1)$ for all $\mu \in F$. Then there exists a map $\alpha : F \times [0,1] \rightarrow \mathbb{T}$ such that $\alpha \beta \in C(F \times [0,1], U_n(\mathbb{C}))$ and $\alpha(\mu,0) = \alpha(\mu,1)$ for all $\mu \in F$.

**Proof.** Let $\Omega = F \times [0,1]$. $\beta(\zeta) = \begin{pmatrix} \beta_{11}(\zeta) & \cdots & \beta_{1n}(\zeta) \\ \vdots & \ddots & \vdots \\ \beta_{n1}(\zeta) & \cdots & \beta_{nn}(\zeta) \end{pmatrix}$ for all $\zeta \in \Omega$, where $\beta_{ij} : \Omega \rightarrow \mathbb{C}$. Let $\{E_{ij}\}_{i,j=1,\ldots,n}$ be the usual system of matrix units for $M_n(\mathbb{C})$. Since $(\beta E_{ij} \beta^*(\zeta))_{kl} = \beta_{ki}(\zeta)\overline{\beta}_{lj}(\zeta)$ for all $\zeta \in \Omega$, $i,j,k,l \in \{1\ldots n\}$ then $\beta_{ki} \overline{\beta}_{lj} \in C(\Omega)$ for all $i,j,k,l$, hence $|\beta_{ij}| \in C(\Omega)$. Let $\zeta \in \Omega$, then $\beta_{ij}(\zeta) \neq 0$ for some $i$, $j$. Hence, by continuity $|\beta_{ij}|$ is bounded away from zero on some
closed rectangle $R\zeta$ containing $\zeta$ in its interior. Define $\lambda_{\zeta}(z) := |\beta_{ij}(z)|/\beta_{ij}(z)$ for all $z \in \Omega \cap R\zeta$. For any $p$, $q$, we have, $|\beta_{ij}|\lambda_{\zeta}\beta_{pq} = (\lambda_{\zeta}\beta_{ij})(\lambda_{\zeta}\beta_{pq}) \in C(\Omega \cap R\zeta)$ since $\beta_{ij}\beta_{pq} \in C(\Omega)$. So $\lambda_{\zeta}\beta_{pq} \in C(\Omega \cap R\zeta)$, hence $\lambda_{\zeta}\beta \in C(\Omega \cap R\zeta, U_n(C))$ for each $\zeta \in \Omega$. By compactness of $\Omega$ there are finitely many $\zeta_1, \ldots, \zeta_m \in \Omega(m \in \mathbb{N})$ with $\Omega \subseteq R\zeta_1 \cup \ldots \cup R\zeta_m$. Extend all the edges of each $R\zeta_j$ to the boundary of $[0,1]^2$ to get a grid on $[0,1]^2$. For simplicity, say we get six regions $R_1 = [0,x] \times [0,y]$, $R_2 = [0,x] \times [y,y']$, $R_3 = [0,x] \times [y',1]$, $R_4 = [x,1] \times [0,y]$, $R_5 = [x,1] \times [y,y']$, $R_6 = [x,1] \times [y',1]$, for some $x, y \in (0,1)$, $y' \in (y,1)$, with maps $\lambda_j : R_j \to \mathbb{T}$ such that $\lambda_j\beta$ is continuous for each $j = 1, 2, \ldots, 6$. Let $f_1 = \lambda_1$, so $f_1\beta \in C(R_1 \cap \Omega, U_n(C))$. There are two cases:

(i) $R_1 \cap R_2 \cap \Omega = \emptyset$. In this case we let

$$f_2(\zeta) =\begin{cases} f_1(\zeta) & \text{if } \zeta \in R_1 \cap \Omega \\ \lambda_2(\zeta) & \text{if } \zeta \in R_2 \cap \Omega \end{cases}$$

(ii) $R_1 \cap R_2 \cap \Omega \neq \emptyset$. Hence $(\lambda_1\beta)(\lambda_2\beta)^* = \lambda_1\lambda_2 I_n$ is continuous on this closed subset of $[0,x] \times \{y\}$, so $\lambda_1\lambda_2$ can be extended to some $h \in C(R_2 \cap \Omega, \mathbb{T})$ (e.g. $h(\mu, \nu) = \lambda_1\lambda_2(\mu, \nu)$ for $(\mu, \nu) \in R_2 \cap \Omega$), and we let

$$f_2(\zeta) =\begin{cases} f_1(\zeta) & \text{if } \zeta \in R_1 \cap \Omega \\ h\lambda_2(\zeta) & \text{if } \zeta \in R_2 \cap \Omega \end{cases}$$

In either case the product function $f_2\beta$ is in $C((R_1 \cup R_2) \cap \Omega, U_n(C))$. Similarly extend $f_2$ to $R_3$ using $\lambda_3$, so as to get some $f_3 : (R_1 \cup R_2 \cup R_3) \cap \Omega \to \mathbb{T}$ such that $f_3\beta$ is continuous. Likewise, starting with $f_4 := \lambda_4$ on $R_4 \cap \Omega$, extend to $f_5$, and then to $f_6$, with $f_6\beta \in C((R_4 \cup R_5 \cup R_6) \cap \Omega, U_n(C))$.

If $(\{x\} \times [0,1]) \cap \Omega = \emptyset$, define

$$f(\zeta) =\begin{cases} f_3(\zeta) & \text{if } \zeta \in (R_1 \cup R_2 \cup R_3) \cap \Omega \\ f_6(\zeta) & \text{if } \zeta \in (R_4 \cup R_5 \cup R_6) \cap \Omega \end{cases}$$

so $f\beta \in C([0,1]^2, U_n(C))$. If $(\{x\} \times [0,1]) \cap \Omega \neq \emptyset$, we note the function $f_3\overline{f_6}I_n = (f_3\beta)(f_6\beta)^*$ is continuous on $(\{x\} \times [0,1]) \cap \Omega$. As before, it has an extension to
some \( h \in C((R_4 \cup R_5 \cup R_6) \cap \Omega, \mathbb{T}) \), and we define
\[
f(\zeta) = \begin{cases} 
f_3(\zeta) & \text{if } \zeta \in (R_1 \cup R_2 \cup R_3) \cap \Omega \\
h(\zeta)f_5(\zeta) & \text{if } \zeta \in (R_4 \cup R_5 \cup R_6) \cap \Omega \
\end{cases}
\]
So in all cases, there exists a function \( f : \Omega \to \mathbb{T} \) such that \( f \beta \in C(\Omega, U_n(\mathbb{C})) \).

Finally, let \( g : F \to \mathbb{T} \) be the map \( g(\mu) = f(\mu, 0)\bar{f}(\mu, 1) \) for \( \mu \in F \).
Then \( g(\mu)I_n = f(\mu, 0)\bar{f}(\mu, 1)\beta(\mu, 0)\beta^*(\mu, 1) = (f \beta)(\mu, 0)(f \beta)^*(\mu, 1), \mu \in F \), so \( g \in C(F, \mathbb{T}) \). Extend \( g \) to an element of \( C([0, 1], \mathbb{T}) \) and let \( g \) now denote this extension. Finally, let \( G \in C([0, 1]^2, \mathbb{T}) \) satisfy \( G(\mu, 0) = 1, G(\mu, 1) = g(\mu), \mu \in [0, 1] \) (\( g \) is homotopic to the constant map), and take \( \alpha = Gf \).

We will use the following slightly different form of this Lemma

**Corollary 3.2.10.** Let \( F \) be a closed proper subset of \( \mathbb{T} \), and \( \beta : F \times \mathbb{T} \to U_n(\mathbb{C}) \) a function such that \( \zeta \mapsto \beta(\zeta)Q\beta(\zeta)^* \) is a continuous function from \( F \times \mathbb{T} \) into \( M_n(\mathbb{C}) \) for each \( Q \in M_n(\mathbb{C}) \). Then there exists a map \( \alpha : F \times \mathbb{T} \to \mathbb{T} \) such that \( \alpha \beta \in C(F \times \mathbb{T}, U_n(\mathbb{C})) \).

### 3.3 Linear Operators and \( C^* \)-Algebras Determined by Irreducible \( * \)-Representations of \( A_\varphi \)

We find the \( C^* \)-algebras \( \pi(A_\varphi) \), where \( \varphi \) is a homeomorphism of the unit circle with non-empty set \( \mathcal{F} \) of cyclic points, and \( \pi \) is an irreducible \( * \)-representation of \( A_\varphi \). Through-out this section, we let \( n \) be the cycle number of \( \varphi \) and \( \varepsilon : \mathbb{T} \to \{-1, 0, 1\} \) the flow map. Recall from (3.1), (3.2), the irreducible \( * \)-representations \( \pi_{\mu, \nu} \) where \( (\mu, \nu) \in \mathbb{T}^2 \). For any \( g \in A_\varphi \), the map
\[
(\mu, \nu) \mapsto \pi_{\mu, \nu}(g) : \mathcal{F} \times \mathbb{T} \to M_n(\mathbb{C})
\]
will play an important role in expressing the operators \( \pi(g) \) for any irreducible infinite dimensional \( * \)-representation \( \pi \). The following Lemma allows us to describe all such maps more concretely.
Lemma 3.3.1. Let \( \varphi \) be an order preserving homeomorphism of the unit circle with cyclic point set \( \mathcal{F} \neq \emptyset \) and the cycle number \( n \). Let \( \mathcal{C} = C^*-\text{algebra of continuous maps} \ G : \mathcal{F} \times \mathbb{T} \to M_n(\mathbb{C}) \) satisfying

\[
G(\varphi(\mu), \nu) = U_n G(\mu, \nu) U_n^*, \ (\mu, \nu) \in \mathcal{F} \times \mathbb{T}
\]

where

\[
(U_n)_{ij} = \begin{cases} 1 & \text{if } i - j \equiv 1 \text{ modulo}(n) \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
G(\mu, \rho \nu) = V(\rho) G(\mu, \nu) V^*(\rho), \ (\mu, \nu) \in \mathcal{F} \times \mathbb{T},
\]

where \( \rho = e^{2\pi i/n} \) and \( V(\rho) = \text{diag}(1, \rho, \ldots, \rho^{n-1}) \). Then \( \mathcal{C} \) is generated by the maps \( a : (\mu, \nu) \mapsto \text{diag}(\varphi^{n-1}(\mu), \ldots, \varphi(\mu), \mu), \ b : (\mu, \nu) \mapsto \nu U_n, \ (\mu, \nu) \in \mathcal{F} \times \mathbb{T} \).

Proof. Suppose \( G \in \mathcal{C} \). Let \( \{G_{ij}\}_{ij \in \{1, \ldots, n\}} \) be the coordinate maps of \( G \), and for each \( \ell \in \{0, 1, \ldots, n-1\} \) let \( G_\ell = U_n^\ell \text{diag}(G_{\ell+1,1}, G_{\ell+2,2}, \ldots, G_{n,n-\ell}, G_{1,n-\ell+1}, G_{2,n-\ell+2}, \ldots, G_{\ell,n}) \). Since \( G = \sum_{\ell=0}^{n-1} G_\ell \) it is enough to show that each \( G_\ell \) is contained in the \( C^* \)-algebra generated by \( a, b \). \( G_\ell \) satisfies (3.9), hence \( G_\ell(\mu, \nu) = U_n^\ell \text{diag}(G_{\ell,n}(\varphi^{n-1}(\mu), \nu), \ldots, G_{\ell,n}(\varphi(\mu, \nu)), G_{\ell,n}(\mu, \nu)) \), \( (\mu, \nu) \in \mathcal{F} \times \mathbb{T} \). \( G_\ell \) satisfies (3.10), hence \( G_{\ell,n}(\mu, \nu) = f(\mu, \nu^n) \nu^\ell \) for some \( f \in C(\mathcal{F} \times \mathbb{T}) \). Hence \( G_\ell = b^\ell f(a, b^n) \).

Corollary 3.3.2. Let \( \varphi \) be an order preserving homeomorphism of the unit circle with cyclic point set \( \mathcal{F} \neq \emptyset \), and the cycle number \( n \). For each \( g \in A_\varphi \), let \( G(\mu, \nu) = \pi_{\mu, \nu}(g), \ (\mu, \nu) \in \mathcal{F} \times \mathbb{T} \). The correpondence \( g \mapsto G \) is a \( * \)-homomorphism onto the \( C^*-\text{algebra of continuous} \ M_n(\mathbb{C}) \)-valued maps on \( \mathcal{F} \times \mathbb{T} \) satisfying (3.9), (3.10).

Proof. \( a(\mu, \nu) = \pi_{\mu, \nu}(A) \) and \( b(\mu, \nu) = \pi_{\mu, \nu}(B) \) for each \( (\mu, \nu) \in \mathcal{F} \times \mathbb{T} \), where \( a, b \) are as in Lemma 3.3.1.

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Remark. Hence if \( \varphi \) is conjugate to a rational rotation \((\text{i.e. } \mathcal{F} = \mathcal{T})\), then \( A_\varphi \) is \( * \)-isomorphic to the \( \mathbb{C}^* \)-algebra of continuous maps: \( \mathcal{F} \times \mathcal{T} \to M_n(\mathbb{C}) \) satisfying (3.9), (3.10). This alternate characterization of the rational rotation algebras was used by DeBrabanter ([2]) to give a more elementary proof of their classification.

Next, we look at the case where \( \varphi \) has non-cyclic points \((\mathcal{F} \neq \mathcal{T})\) as well as cyclic points. An explicit expression will be given for arbitrary elements in the range of each infinite dimensional irreducible \( * \)-representation. Recall the parameterization \( \{\pi_{\mu,\nu}\}_{(\mu,\nu) \in \mathcal{F}^c \times \mathcal{T}} \) (Proposition 3.1.2) of the infinite dimensional irreducible \( * \)-representations of \( A_\varphi \). For \( n \in \mathbb{N} \) and \( \nu \in \mathcal{T} \), we let \( \sqrt[n]{\nu} \) be the \( n \)-th root of \( \nu \) which is contained in \([1, e^{2\pi i/n}]\). For each ordered pair \((f, g) \in C(\mathcal{T}) \oplus C(\mathcal{T})\), and \( \varepsilon \in \{-1, 1\} \), let

\[
\varepsilon \ast (f, g) = \begin{cases} (f, g) & \text{if } \varepsilon = 1 \\ (g, f) & \text{if } \varepsilon = -1 \end{cases}.
\]

Proposition 3.3.3. Let \( \varphi \) be an order preserving homeomorphism of \( \mathcal{T} \) with cyclic point set \( \mathcal{F} \neq \phi \), \( \mathcal{T} \), the cycle number \( n \), and the flow map \( \varepsilon \). For each \((\mu, \nu) \in \mathcal{F}^c \times \mathcal{T} \) and \( g \in A_\varphi \), there exist unique functions \( f, f' \in C(\mathcal{T}, M_n(\mathbb{C})) \) such that

\[
\pi_{\mu,\nu}(g) - T_{\varepsilon(\mu) \ast (f', f)} \in K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) .
\]

Specifically,

\[
f(\zeta) = V^*(\sqrt[n]{\zeta})\pi_{\alpha,\nu} \sqrt[n]{\zeta}(g)V(\sqrt[n]{\zeta})
\]

and

\[
f'(\zeta) = V^*(\sqrt[n]{\zeta})\pi_{\alpha',\nu} \sqrt[n]{\zeta}(g)V(\sqrt[n]{\zeta})
\]

for \( \zeta \in \mathcal{T} \), where \((\alpha, \alpha')\) is the component of \( \mathcal{F}^c \) containing \( \mu \).

Proof. Let \((\mu, \nu) \in \mathcal{F}^c \times \mathcal{T} \), and let \((\alpha, \alpha')\) be the component of \( \mathcal{F}^c \) containing \( \mu \). Recall the irreducible \( * \)-representation \( \pi_{\mu,\nu} \) (as in (3.1), (3.2)).
\[ \varphi^j(\mu) \in \varphi^j((\alpha, \alpha')) = \varphi^{[j]}((\alpha, \alpha')) \text{ for all } j \in \mathbb{Z}, \text{ since } \alpha, \alpha' \text{ are cyclic points } \] ([j]_n \text{ means modulo } n). \text{ Also, for any } k \in \{0, 1, \ldots, n-1\} \text{ we have }
\[
\lim_{j \to \pm \infty} \varphi^{n_j+k}(\mu) = \begin{cases} \varphi^k(\alpha') & \text{if } \varepsilon(\mu) = 1 \\ \varphi^k(\alpha) & \text{if } \varepsilon(\mu) = -1. \end{cases}
\]

Hence

\[
\pi_{\mu, \nu}(A) - T_{\epsilon(\mu) \ast (a', a)} \in K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n)
\]
where \( a(\zeta) = \text{diag}(\varphi^{n-1}(\alpha), \ldots, \varphi(\alpha), \alpha) = V^* (\sqrt{\zeta}) \pi_{\alpha, \nu} \sqrt{\zeta}(A) V(\sqrt{\zeta}), \) and \( a'(\zeta) = \text{diag}(\varphi^{n-1}(\alpha'), \ldots, \varphi(\alpha'), \alpha') = V^* (\sqrt{\zeta}) \pi_{\alpha', \nu} \sqrt{\zeta}(A) V(\sqrt{\zeta}), \zeta \in T. \)

Also

\[
\pi_{\mu, \nu}(B) - T_{\epsilon(\mu) \ast (b', b)} \in K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n)
\]
where \( b(\zeta) = \nu X_n(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha, \nu} \sqrt{\zeta}(B) V(\sqrt{\zeta}), \) with \( X_n \) as in (3.3), and \( b'(\zeta) = \nu X_n(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha', \nu} \sqrt{\zeta}(B) V(\sqrt{\zeta}), \zeta \in T. \) So we conclude that for any \( g \) a polynomial expression in \( A, A^*, B, B^* \), then

\[
\pi_{\mu, \nu}(g) - T_{\epsilon(\mu) \ast (f', f)} \in K(\ell^2(\mathbb{Z}) \times \mathbb{C}^n)
\]
where \( f(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha, \nu} \sqrt{\zeta}(g) V(\sqrt{\zeta}) \) and \( f'(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha', \nu} \sqrt{\zeta}(g) V(\sqrt{\zeta}), \zeta \in T. \) Now let \( g \in A_{\varphi} \) be arbitrary, then there exists a sequence \( \{g_j\}_{j \in \mathbb{N}} \) of polynomials in \( A, A^*, B, B^* \), approaching \( g \). For each \( j \in \mathbb{N} \) let \( f_j(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha, \nu} \sqrt{\zeta}(g_j) V(\sqrt{\zeta}) \) and \( f_j'(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha', \nu} \sqrt{\zeta}(g_j) V(\sqrt{\zeta}) \) for all \( \zeta \in T. \)

Then for each \( j \in \mathbb{N} \)

\[
\pi_{\mu, \nu}(g_j) - T_{\epsilon(\mu) \ast (f_j, f_j)} = k_j
\]
for some \( k_j \in K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) \). So

\[
\|T_{\epsilon(\mu) \ast (f', f)} + k_j - \pi_{\mu, \nu}(g)\| \leq \|T_{\epsilon(\mu) \ast (f', f)} - T_{\epsilon(\mu) \ast (f_j', f_j)}\|
\]

\[
+ \|\pi_{\mu, \nu}(g_j) - \pi_{\mu, \nu}(g)\| + \|T_{\epsilon(\mu) \ast (f_j', f_j)} - \pi_{\mu, \nu}(g_j) + k_j\| \leq 2\|g - g_j\|
\]

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by (iv) of subsection 3.2.2. Hence \( \{k_j\}_{j} \) converges to some \( k \in K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) \) and \( \pi_{\mu,\nu}(g) = T_{\varepsilon(\mu)}*(f', f) + k \).

Remark. The Fourier coefficients \( \{f_j\}_{j \in \mathbb{Z}} \) and \( \{f'_j\}_{j \in \mathbb{Z}} \) of \( f \) and \( f' \) respectively, can be readily derived from \( \pi_{\mu,\nu}(g) \). In fact if \( \varepsilon(\mu) = 1 \) then for each \( k \in \mathbb{Z} \), \( f_k \) is just the limit of the \( (i + k, i)^{th} \) \( n \times n \)-matrix block of \( \pi_{\mu,\nu}(g) \) as \( i \to \infty \), and \( f'_k \) is the limit as \( i \to -\infty \). If \( \varepsilon(\mu) = -1 \), the \( (i + k, i)^{th} \) block of \( \pi_{\mu,\nu}(g) \) converges to \( f_k \) as \( i \to -\infty \) and to \( f'_k \) as \( i \to \infty \).

An interesting consequence (which is not used later) is:

**Corollary 3.3.4.** Let \( \varphi \) be a homeomorphism of the unit circle with cyclic point set \( \mathcal{F} \neq \emptyset \), \( T \), and the cycle number \( n \). If \( (\mu, \nu) \in \mathcal{F}^c \times T \), then the \( C^* \)-algebra \( \pi_{\mu,\nu}(A_{\varphi}) \) is \(*\)-isomorphic to \( \{T_{(f,g)}| f, g \in C(T, M_n(\mathbb{C}))\} + K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n) \).

**Proof.** Let \( (\mu, \nu) \in \mathcal{F}^c \times T \), then by Proposition 3.3.3 we have

\[
\pi_{\mu,\nu}(A_{\varphi}) \subseteq \{T_{(f,g)}| f, g \in C(T, M_n(\mathbb{C}))\} + K(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^n).
\]

Each element of the set on the right is a compact perturbation of some element of the set on the left (Corollary 3.3.2 and Proposition 3.3.3), which itself contains the compact operators since it is a \( C^* \)-algebra acting irreducibly on \( \ell^2(\mathbb{Z}) \otimes \mathbb{C}^n \), and it contains non-zero compact operators ([6] Theorem 10.4.10).

### 3.4 The Spectrum

In this section, we find the spectrum of \( A_{\varphi} \), the set of unitary equivalence classes of irreducible \(*\)-representations with the topology induced by the Jacobson topology on \( \text{Prim}(A_{\varphi}) \), where \( \varphi \) is a homeomorphism of the unit circle possessing cyclic points.

**3.4.1 First, some notation.** Let \( \varphi \) be an order preserving homeomorphism of \( T \) with non-empty set \( \mathcal{F} \) of the cyclic points, and the cycle number \( n \).

each irreducible $*$-representation $\pi$ of $A$, let $[\pi]$ denote the set of irreducible $*$-representations that are unitarily equivalent to $\pi$. For any $(\mu, \nu), (\mu', \nu') \in T^2$, we will write $(\mu, \nu) \sim (\mu', \nu')$ if $[\pi_{\mu, \nu}] = [\pi_{\mu', \nu'}]$. This defines an equivalence relation on $T^2$, and we let $q : T^2 \to T^2 / \sim$ be the canonical map, and $Q$ the quotient topology that it defines on $T^2 / \sim$. Explicitly, for $(\mu, \nu), (\mu', \nu') \in F \times T$, then $(\mu, \nu) \sim (\mu', \nu')$ if and only if there exist $i, j \in \{0, 1, \ldots, n - 1\}$ such that

\[
\mu' = \phi^i(\mu) \quad \text{and} \quad \nu' = \rho^j \nu \quad (\rho = e^{2\pi i/n});
\]

and for $(\mu, \nu), (\mu', \nu') \in F^c \times T$ then $(\mu, \nu) \sim (\mu', \nu')$ if and only if $\mu' = \phi^i(\mu)$ for some $i \in \mathbb{Z}$.

Suppose $z \in T^2 / \sim$ and $\varepsilon > 0$.

If $z \in q(F \times T)$, let $\delta_\varepsilon(z)$ denote the open neighbourhood $\delta_\varepsilon(z) = \{z' | z' \in q(F \times T) \text{ and } \|((\mu, \nu) - (\mu', \nu'))\| < \varepsilon \text{ for some } (\mu, \nu) \in q^{-1}(z), (\mu', \nu') \in q^{-1}(z')\} \cup \{z' | z' \in q(F^c \times T) \text{ and } |\mu - \mu'| < \varepsilon \text{ for some } (\mu, \nu) \in q^{-1}(z), (\mu', \nu') \in q^{-1}(z')\}$.

(3.12)

If $z \in q(F^c \times T)$, let $\delta_\varepsilon(z)$ denote the open neighbourhood $\delta_\varepsilon(z) = \{z' | z' \in q(F^c \times T) \text{ and } |\mu - \mu'| < \varepsilon \text{ for some } (\mu, \nu) \in q^{-1}(z), (\mu', \nu') \in q^{-1}(z')\}$.

(3.13)

In either case, the sets $\{\delta_\varepsilon(z)\}_{\varepsilon > 0}$ are an open neighbourhood basis at $z$.

Let $\hat{A}_\varphi$ be the set of unitary equivalence classes of irreducible $*$-representations of $A$, with the topology induced by the hull-kernel topology of $\text{Prim}(A)$ (3]). Then a net $\{\theta_j\}_{j \in D}$ in $\hat{A}_\varphi$ ($D$ a directed set) converges to some $\theta (\in \hat{A}_\varphi)$ if for any $a \in A$ with $\pi(a) \neq 0$ for $\pi \in \theta$, there exists $N \in D$ such that if $j \geq N$ and $\pi_j \in \theta_j$ then $\pi_j(a) \neq 0$.

**Proposition 3.4.2.** Let $\varphi$ be an order preserving homeomorphism of $T$ possessing cyclic points. Then $\hat{A}_\varphi$ is homeomorphic to $T^2 / \sim$ via the correspondence $q(\mu, \nu) \mapsto [\pi_{\mu, \nu}], (\mu, \nu) \in T^2$.

**Proof.** Denote the map $q(\mu, \nu) \mapsto [\pi_{\mu, \nu}], (\mu, \nu) \in T^2$ by $\mathfrak{h}$. Let $\mu$ denote the normalized Lebesque measure on $T$, and for each $\zeta \in C^2$ let $(\zeta)_1, (\zeta)_2$ denote
its first and second entries respectively.

Suppose \( \{h(z_j)\}_j \) converges to \( h(z) \) for some \( z, z_j \in T^2/\sim, j \in D \). We want to show that \( \{z_j\}_j \) converges to \( z \). Let \( \eta \in q^{-1}(z) \) and \( \eta_j \in q^{-1}(z_j) \) for each \( j \).

Assume \( z \in q(F^c \times T) \). Let \( (\alpha, \alpha') \) be the component of \( F^c \) containing \((\eta)_1\). If \( \epsilon > 0 \) there exists an open arc neighbourhood \( N_\epsilon \) of \((\eta)_1\) of length \( < \epsilon \) contained in \((\alpha, \alpha')\), satisfying \( \varphi^i(N_\epsilon) \cap \varphi^j(N_\epsilon) = \emptyset \) for \( i \neq j \). Let \( f \in C(T) \) with \( f((\eta)_1) \neq 0 \) and support in \( N_\epsilon \). So \( \pi_\eta(f(A)) = \text{diag}(f(\varphi^{-i}((\eta)_1)))_{i \in \mathbb{Z}} \neq 0 \), and there exists \( N \in D \) such that \( \pi_{\eta_j}(f(A)) \neq 0 \) if \( j > N \). For \( \eta_j \in F^c \times T \) then \( \text{diag}(f(\varphi^{-i}((\eta)_1)))_{i \in \mathbb{Z}} \neq 0 \); so \( \varphi^{-i}((\eta)_1) \in N_\epsilon \) for some \( i \in \mathbb{Z} \), hence \( z_j \in \delta_\epsilon(z) \).

Assume \( z \in q(F \times T) \). Let \( \epsilon > 0 \) and let \( N_\epsilon \) be an open arc neighbourhood of \((\eta)_1\) of length \( < \epsilon \), \( M_\epsilon \) an open arc neighbourhood of \((\eta)_2\) of length \( < \epsilon \). Let \( f, g \in C(T) \) with \( f((\eta)_1) \neq 0 \), \( g((\eta)_2) \neq 0 \) and support in \( N_\epsilon, M_\epsilon \) respectively; so \( \pi_\eta(f(A)g(B)) = (\text{diag}(f(\varphi^{-i}((\eta)_1)))_{j=1, \ldots, n}) \cdot g((\eta)_2 U_n) \neq 0 \). So there exists \( N \in D \) such that \( \pi_{\eta_j}(f(A)g(B)) \neq 0 \) if \( j > N \). For \( \eta_j \in F^c \times T \), then \( \text{diag}(f(\varphi^{-i}((\eta)_1)))_{i \in \mathbb{Z}} \cdot g((\eta)_2 U_n) \neq 0 \); so \( \varphi^{-i}((\eta)_1) \in N_\epsilon \) for some \( i \in \mathbb{Z} \), hence \( z_j \in \delta_\epsilon(z) \). If \( \eta_j \in F \times T \) then \( (\text{diag}(f(\varphi^{-i}((\eta)_1)))_{i=1,2, \ldots, n}) \cdot g((\eta)_2 U_n) \neq 0 \); so \( \varphi^{-i}((\eta)_1) \in N_\epsilon \) and \( \rho^l \cdot (\eta)_2 \in M_\epsilon \) for some \( i, \ell \in \{1, \ldots, n\} \), hence \( z_j \in \delta_\epsilon(z) \).

Conversely, suppose \( \{z_j\}_j \) converges to \( z \) in \( T^2/\sim \). We want to show that \( \{h(z_j)\}_j \) converges to \( h(z) \). So let \( g \in A_\varphi \) with \( \pi_\eta(g) \neq 0 \) for some \( \eta \in q^{-1}(z) \).

Suppose \( z \in q(F^c \times T) \). Then let \( g' = \sum g'_{p,q} A^p B^q \) be a finite polynomial in \( A, A^*, B, B^* \) such that \( \|g - g'\| < \frac{1}{3}\|\pi_\eta(g)\| \). If \( \eta' \in F^c \times T \) then

\[
\|\pi_{\eta'}(g') - \pi_\eta(g')\| = \| \sum g'_{p,q} \cdot (\pi_{\eta'}(A^p B^q) - \pi_\eta(A^p B^q)) \|
\]

\[
\leq \sum |g'_{p,q}| (\sup_{k \in \mathbb{Z}} \|\varphi^k((\eta')_1)\|_p \cdot (\eta')_2^q - (\varphi^k((\eta)_1))_p \cdot (\eta)^q_2) \|
\]

\[
\leq \sum |g'_{p,q}| (p \sup_{k \in \mathbb{Z}} \|\varphi^k((\eta')_1) - \varphi^k((\eta)_1)\| + q|\eta')_2 - (\eta)_2|) .
\]
Hence \( \|\pi_{\eta'}(g') - \pi_\eta(g')\| \) approaches zero as \( \eta'(\in \mathcal{F}^c \times \mathbb{T}) \) approaches \( \eta \). Let \( \varepsilon > 0 \) be small enough that \( \|\pi_{\eta'}(g') - \pi_\eta(g')\| < \frac{1}{3}\|\pi_\eta(g)\| \) if \( \eta' \in \mathcal{F}^c \times \mathbb{T} \) satisfies \( \|(\eta')_1 - (\eta)_1\| < \varepsilon \), and \( (\eta')_2 = (\eta)_2 \). There exists \( N \in D \) such that \( z_j \in \delta_\varepsilon(x) \) if \( j > N \), i.e. \( |(\eta_j)_1 - (\eta)_1| < \varepsilon \) and \( (\eta_j)_2 = (\eta)_2 \) for some \( \eta_j \in q^{-1}(z_j) \). Hence if \( j > N \) then \( \|\pi_{\eta_j}(g') - \pi_\eta(g')\| < \frac{1}{3}\|\pi_\eta(g)\| \) for some \( \eta_j \in q^{-1}(z_j) \) satisfying \( (\eta_j)_2 = (\eta)_2 \). This implies

\[
\|\pi_{\eta_j}(g) - \pi_\eta(g)\| < \|\pi_{\eta_j}(g - g')\| + \|\pi_{\eta_j}(g') - \pi_\eta(g')\| + \|\pi_\eta(g - g')\| < \|\pi_\eta(g)\| ,
\]

hence \( \pi_{\eta_j}(g) \neq 0 \) if \( j > N \). Therefore \( \{h(z_j)\}_j \) converges to \( h(z) \).

If \( z \in q(\mathcal{F} \times \mathbb{T}) \), there exists \( \varepsilon > 0 \) such that if \( z' \in q(\mathcal{F} \times \mathbb{T}) \) with \( \|\eta' - \eta\| < \varepsilon \) for some \( \eta' \in q^{-1}(z') \), then \( \|\pi_{\eta'}(g) - \pi_\eta(g)\| < \|\pi_\eta(g)\| \) (the map \( \mu \mapsto \pi_\mu(g) \) is continuous for \( \mu \in \mathcal{F} \times \mathbb{T} \), by Corollary 3.3.2). Let \( N \in D \) be such that if \( j > N \) then \( z_j \in \delta_\varepsilon(x) \). Hence, if \( j > N \) there exists \( \eta_j \in q^{-1}(z_j) \) such that (i) \( \eta_j \in \mathcal{F} \times \mathbb{T} \) and \( \|\eta_j - \eta\| < \varepsilon \) or (ii) \( \eta_j \in \mathcal{F}^c \times \mathbb{T} \) and \( \|(\eta_j)_1 - (\eta)_1\| < \varepsilon \). If (i) is true then \( \pi_{\eta_j}(g) \neq 0 \). If (ii) holds, let \( (\alpha, \alpha') \) be the component of \( \mathcal{F}^c \) containing \( (\eta_j)_1 \).

Then \( |\alpha - (\eta)_1| < \varepsilon \) or \( |\alpha' - (\eta)_1| < \varepsilon \), hence at least one of the continuous maps

\[
f(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha,(\eta)_2} \sqrt{\zeta}(g) V(\sqrt{\zeta}), \quad f'(\zeta) = V^* (\sqrt{\zeta}) \pi_{\alpha',(\eta)_2} \sqrt{\zeta}(g) V(\sqrt{\zeta}), \quad \zeta \in \mathbb{T},
\]

is non-zero. By Proposition 3.3.3 \( \pi_{\eta_j}(g) - T_{\varepsilon((\eta_j)_1)} \ast (f', f) \) is a compact operator, hence \( \pi_{\eta_j}(g) \neq 0 \) (by (i) of subsection 3.2.2.).

We will denote the points of \( \hat{\mathcal{A}}_\varphi \) corresponding to the infinite dimensional, and the finite dimensional irreducible \( * \)-representations by \( \hat{\mathcal{A}}_{\varphi,\infty}, \hat{\mathcal{A}}_{\varphi,f} \) respectively.

**Corollary 3.4.3.** Let \( \varphi \) be an order preserving homeomorphism of \( \mathbb{T} \) possessing cyclic points. Then \( \hat{\mathcal{A}}_{\varphi,\infty} = \{ \text{the elements of } \hat{\mathcal{A}}_\varphi \text{ having an open neighbourhood homeomorphic to } (0,1) \} \). \( \hat{\mathcal{A}}_{\varphi,f}^o = \{ \text{the elements of } \hat{\mathcal{A}}_\varphi \text{ having an open neighbourhood homeomorphic to a disc} \} \). Finally, each component of \( \hat{\mathcal{A}}_{\varphi,\infty} \) is homeomorphic to \( \mathbb{T} \), and each component of \( \hat{\mathcal{A}}_{\varphi,f}^o \) is homeomorphic to \( (0,1) \times \mathbb{T} \).
Proof. We'll prove this for $(T^2 / \sim, \mathcal{Q})$. The homeomorphism $\iota : T^2 / \sim \to \hat{A}_\varphi$ of Proposition 3.4.2 maps $q(F^c \times T)$, $q(F \times T)$, $q(F^o \times T)$ onto $\hat{A}_\varphi, \hat{A}_\varphi, (\hat{A}_\varphi, \iota)^o$ respectively. Let $z \in T^2 / \sim$, $\zeta \in q^{-1}(z)$, and for each $\varepsilon > 0$ let $B_\varepsilon(\zeta) = \{ \zeta' \in T^2 \mid |(\zeta')_1 - (\zeta)_1| < \varepsilon, |(\zeta')_2 - (\zeta)_2| < \varepsilon \}$. If $z \in q(F^c \times T)$, then $\zeta \in F^c \times T$, and if $\varepsilon$ is small then $q|_{\{ \zeta' \in T^2 \mid |(\zeta')_1 - (\zeta)_1| < \varepsilon, (\zeta')_2 = (\zeta)_2 \}}$ is injective, hence a homeomorphism onto $q(\{ \zeta' \in T^2 \mid |(\zeta')_1 - (\zeta)_1| < \varepsilon, (\zeta')_2 = (\zeta)_2 \}) = q(B_\varepsilon(\zeta))$. Hence $z$ has an open neighbourhood $(q(B_\varepsilon(\zeta)))$ homeomorphic to $(0, 1)$. If $z \in q(F^o \times T)$, then $\zeta \in F^o \times T$. If $\varepsilon > 0$ is small then $B_\varepsilon(\zeta) \subseteq F^o \times T$ and $q|_{B_\varepsilon(\zeta)}$ is injective, hence a homeomorphism onto $q(B_\varepsilon(\zeta))$. So $z$ has an open neighbourhood $(q(B_\varepsilon(\zeta)))$ homeomorphic to a disk $(B_\varepsilon(\zeta))$. If $z \in q(\partial F \times T)$, then $\zeta \in (\partial F) \times T$, and for each $\varepsilon > 0$ the open set $B_\varepsilon(\zeta)$ contains $\varepsilon(\mu) \times \varphi(\mu) \times \{(\zeta)_2 \}$ for some $\mu \in F^c$ with $|(\zeta)_1 - \mu| < \varepsilon$. So any open neighbourhood of $z$ contains $q(B_\varepsilon(\zeta))$ for small $\varepsilon > 0$, hence contains an open set homeomorphic to $T$.

Each component of $q(F^c \times T)$ is equal to $q(\{ \zeta, \varphi(\zeta) \times \{ 1 \})$ for some $\zeta \in F^c$, which is homeomorphic to $T$ since $q$ identifies the points $(\zeta, 1)$ and $(\varphi(\zeta), 1)$.

Finally, each component of $q(F \times T)$ (or $F^o \times T$) is equal to $q(c \times \{ 1, e^{2\pi i/n} \})$ for some component $c$ of $F$ (resp. $F^o$), and this is homeomorphic to $c \times T$ since $q$ identifies the points $(\zeta, 1)$ and $(\zeta, e^{2\pi i/n})$ for each $\zeta \in c$. ■
CHAPTER 4

A complete $C^*$-algebraic invariant for the order preserving case

Consider the $C^*$-algebra $A_\varphi$ where $\varphi$ is an order preserving homeomorphism of $T$ with cyclic point set $\mathcal{F} \neq \emptyset$, $T$. From $A_\varphi$ we will construct a four element subset of $\mathcal{I}$ (Section 4.4), which turns out to be $\{I(\varphi), I(\varphi^{-1}), I(p_{-1} \circ \varphi \circ p_{-1}), I(p_{-1} \circ \varphi^{-1} \circ p_{-1})\}$ where $p_{-1}(\zeta) = \bar{\zeta}$, $\zeta \in T$, thus recovering the set $\{\varphi, \varphi^{-1}\}$ up to conjugacy. An arbitrary element of $\mathcal{I}$ can be written $[(q, \lambda)]$ where $q \in \mathbb{Q} \cap [0, 1)$, and $\lambda : S \to \{-1, 0, 1\}$ is a map whose domain $S$ is a countable set of isolated points of $T$, so our aim is to derive such $q$, $\lambda$, which we later use to define our invariant. Essentially $\lambda$ first appears as the map $\Lambda : \mathcal{G} \to \{-1, 0, 1\}$ of Section 4.2, where $\mathcal{G} = \{\text{components of } \hat{A}_{\varphi,\infty}, (\hat{A}_{\varphi,f})^o\}$. The only thing wrong with $\Lambda$ is that its domain $\mathcal{G}$ is not a countable set of isolated points in $T$. Using the subset $\mathcal{O} \subseteq \mathcal{G}^3$ described in Section 4.1 we are able to define in Section 4.4 a bijection from a countable set $S$ of isolated points in $T$ onto $\mathcal{G}$, which when composed with $\Lambda$ yields a map $\lambda : S \to \{-1, 0, 1\}$ of the desired kind. Section 4.3 is devoted to constructing a number $q \in \mathbb{Q} \cap [0, 1)$ from the algebra. Finally, the invariant is expressed in terms of $q$, $\lambda$ in Section 4.4.

4.1 PLACING THE COMPONENTS OF $\hat{A}_{\varphi,\infty}$ AND $(\hat{A}_{\varphi,f})^o$ ON THE CIRCLE

Let $\varphi$ be an order preserving homeomorphism of $T$ with cyclic point set $\mathcal{F} \neq \emptyset$, $T$. We can characterize $\hat{A}_{\varphi,\infty}$ (Corollary 3.4.3) as the points of $\hat{A}_{\varphi}$ with an open neighbourhood homeomorphic to $(0, 1)$. The purpose of this section is to define a total ordering on the set $\mathcal{G} = \{\text{components of } \hat{A}_{\varphi,\infty}, (\hat{A}_{\varphi,f})^o\}$ (via the topological structure of $\hat{A}_{\varphi}$), which gives rise to a 'circular ordering' $\mathcal{O} \subseteq \mathcal{G}^3$. This is used at a later time to construct an order preserving bijection of $\mathcal{G}$ onto a countable set of isolated points in the unit circle.
Fix any component $\omega_0$ of $\hat{A}_\varphi,\infty$. Then there exists one or two components of $\hat{A}_\varphi,f$ for which every open neighbourhood contains $\omega_0$. Choose one and call it $\omega_1$. For any two components $\omega$, $\omega'$ of $\hat{A}_\varphi,\infty \cup (\hat{A}_\varphi,f)^\circ$, we write $\omega \leq \omega'$ if $\omega = \omega'$ or there exists a connected set $X \subseteq \hat{A}_\varphi$ satisfying $\omega_0 \cap X \neq \emptyset$, $\omega \cap X \neq \emptyset$, and $\omega' \cap X = \omega_1 \cap X = \emptyset$.

Lemma 4.1.1. $\leq$ is a total ordering on the components of $\hat{A}_\varphi,\infty \cup (\hat{A}_\varphi,f)^\circ$, where $\varphi$ is an order preserving homeomorphism of $T$ possessing cyclic and non-cyclic points.

Proof. Let $r$ be the rotation number of $\varphi$ and assume $r \neq 0$. Let $F$ be the cyclic point set of $\varphi$. Let $(\alpha_0, \alpha'_0)$ be a component of $F^c$ such that $\omega_0 = \eta \circ \varphi((\alpha_0, \alpha'_0) \times T)$, where $q : T^2 \to T^2/\sim$ is the quotient map of subsection 3.4.1 and $\eta : T^2/\sim \to \hat{A}_\varphi$ is the homeomorphism of Proposition 3.4.2. Let $\delta : \{\text{components of } \hat{A}_\varphi,\infty \cup (\hat{A}_\varphi,f)^\circ\} \to (\alpha_0, \varphi^r(\alpha_0))$ be a map satisfying $\delta(\eta \circ \varphi((\alpha, \alpha') \times T)) \in (\alpha, \alpha')$ for each component $(\alpha, \alpha')$ of $(F^c \cup F^o) \cap (\alpha_0, \varphi^r(\alpha_0))$, (the map $c \mapsto \eta \circ \varphi(c \times T)$ associates a component of $\hat{A}_\varphi,\infty \cup (\hat{A}_\varphi,f)^\circ$ with each component $c$ of $(F^c \cup F^o) \cap (\alpha_0, \varphi^r(\alpha_0))$, and this association is one to one and onto). Let $[\alpha_1, \alpha'_1]$ be the component of $F \cap (\alpha_0, \varphi^r(\alpha_0))$ satisfying $\omega_1 = \eta \circ \varphi([\alpha_1, \alpha'_1] \times T)$. Either $\alpha_1 = \alpha'_0$ or $\alpha'_1 = \varphi^r(\alpha_0)$. Suppose $\alpha'_1 = \varphi^r(\alpha_0)$.

For any two components $\omega$, $\omega'$ of $\eta \circ \varphi((F^c \cup F^o) \times T)$, we will show that $\omega \leq \omega'$ if and only if $\omega = \omega_0$ or $\omega = \omega'$ or the points $(\alpha_0, \delta(\omega), \delta(\omega'))$ are distinct and counter-clockwise ordered. Let $(x, y)$, $(x', y')$ be the components of $F^c \cup F^o$ containing $\delta(\omega)$, $\delta(\omega')$ respectively (i.e. the components of $(F^c \cup F^o) \cap (\alpha_0, \varphi^r(\alpha_0))$ such that $\omega = \eta \circ \varphi((x, y) \times T)$, $\omega' = \eta \circ \varphi((x', y') \times T)$).

If $\omega = \omega'$ or $\omega = \omega_0$ then $\omega \leq \omega'$. Suppose the points $(\alpha_0, \delta(\omega), \delta(\omega'))$ are distinct and counter-clockwise ordered. Then the connected set $X = \eta \circ \varphi((\alpha_0, y) \times T)$ satisfies $\omega_0 \cap X \neq \emptyset$, $\omega \cap X \neq \emptyset$, $\omega' \cap X = \omega_1 \cap X = \emptyset$. Hence $\omega \leq \omega'$. 

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For the converse, suppose the points \((a_0, \delta(w'), \delta(w))\) are distinct and counter-clockwise ordered. Let \(X\) be a subset of \(\hat{A}_\varphi\) satisfying \(\omega_0 \cap X \neq \emptyset\), \(\omega \cap X \neq \emptyset\), \(\omega' \cap X = \omega_1 \cap X = \emptyset\). Then \(X\) is not connected since it is contained in 
\(h \circ q(((a_0, x') \cup (y', \alpha_1)) \times T), \) with points in each of the separated sets 
\(h \circ q((a_0, x') \times T), h \circ q((y', \alpha_1) \times T)\).

For the case \(\alpha_1 = \alpha_0',\) similar reasoning shows \(\omega \leq \omega'\) if and only if \(\omega = \omega_0\) or \(\omega = \omega'\) or the points \((\alpha_0, \delta(w'), \delta(w))\) are distinct and counter-clockwise. This completes the proof for non-zero \(r\). If \(r = 0\) (i.e. \(\varphi\) has fixed points) the same reasoning applies, with \(T\) in place of the set \((\alpha_0, \varphi^r(\alpha_0))\).

Let \(\mathcal{O} = \{(\omega, \omega', \omega'')| \omega, \omega', \omega''\) are distinct components of \(\hat{A}_\varphi, \infty \cup (\hat{A}_\varphi, f)^{\circ}\) such that \(\omega \leq \omega' \leq \omega''\) or \(\omega' \leq \omega'' \leq \omega\) or \(\omega'' \leq \omega \leq \omega'\}\).

**Lemma 4.1.2.** Let \(\varphi\) be an order preserving homeomorphism of the unit circle with cyclic point set \(F \neq \emptyset, T\). Let \(\omega_0(\omega'_0)\) be a component of \(\hat{A}_\varphi, \infty\), and \(\omega_1\) (resp. \(\omega'_1\)) a component of \(\hat{A}_\varphi, f\), for which every open neighbourhood contains \(\omega_0\) (resp. \(\omega'_0\)). Let \(\mathcal{O}(\mathcal{O}')\) be the set of \((4.1)\), determined by \(\omega_0, \omega_1\) (resp. \(\omega'_0, \omega'_1\)). Then \(\mathcal{O}' = \mathcal{O}\) or \(\mathcal{O}' = \mathcal{O}^{-1} := \{(\omega, \omega', \omega'')| (\omega'', \omega', \omega) \in \mathcal{O}\}\).

**Proof.** Let \(\delta\) be a map corresponding to the component \(\omega_0\) (as in the proof of Lemma 4.1.1), and similarly let \(\delta'\) be a map corresponding to \(\omega'_0\). From the proof of Lemma 4.1.1, it follows that

\[\mathcal{O} = \{(\omega, \omega', \omega'')| \omega, \omega', \omega''\) are components of \(\hat{A}_\varphi, \infty \cup (\hat{A}_\varphi, f)^{\circ}\) such that 
\((\delta(\omega), \delta(\omega'), \delta(\omega''))\) are counter-clockwise ordered\} \tag{4.2}\]

or

\[\mathcal{O} = \{(\omega, \omega', \omega'')| \omega, \omega', \omega''\) are components of \(\hat{A}_\varphi, \infty \cup (\hat{A}_\varphi, f)^{\circ}\) such that 
\((\delta(\omega), \delta(\omega'), \delta(\omega''))\) are clockwise ordered\} \tag{4.3}\]
Likewise $O'$ can be expressed in terms of $\delta'$. For any components $\omega, \omega', \omega''$, the points $({\delta}(\omega), {\delta}(\omega'), {\delta}(\omega''))$ are distinct and counter-clockwise ordered if and only if the points $({\delta'}(\omega), {\delta'}(\omega'), {\delta'}(\omega''))$ are distinct and counter-clockwise ordered. \[\square\]

4.2 Flow maps from algebras

In this section, we describe a procedure that associates with each $C^*$-algebra $A_\varphi$ (where $\varphi$ is an order preserving homeomorphism of $T$ with cyclic point set $F \neq \emptyset$, $T$), a map $\Lambda : \{\text{components of } \hat{A}_{\varphi, \infty} \cup (\hat{A}_{\varphi, f})^\circ\} \to \{-1, 0, 1\}$. It turns out that $\Lambda$ is essentially $\pm$ the flow map of either $\varphi$ or $\varphi^{-1}$.

**Proposition 4.2.1.** Let $\varphi$ be an order preserving homeomorphism of $T$ with the rotation number $r$, the cycle number $n$, the flow map $\varepsilon$, and the cyclic point set $F \neq \emptyset$, $T$. Suppose $(\alpha_0, \alpha'_0), (\alpha_1, \alpha'_1)$ are components of $F^c$ such that $(\alpha_1, \alpha'_1) \subseteq (\alpha_0, \varphi^r(\alpha_0))$. Then there exists $g \in A_\varphi$ such that $\pi(g)$ is a Fredholm operator for each irreducible $*$-representation $\pi$ of $A_\varphi$, with

$$\text{ind}_{\pi_{\mu, 1}}(g) = 0 \quad \text{when } \mu \in F^c \setminus \bigcup_{j=0}^{n-1} \varphi^j((\alpha_0, \alpha'_0) \cup (\alpha_1, \alpha'_1)), $$

$$\text{ind}_{\pi_{\mu, 1}}(g) = 1 \quad \text{when } \mu \in \bigcup_{j=0}^{n-1} \varphi^j((\alpha_0, \alpha'_0)), $$

and

$$\text{ind}_{\pi_{\mu, 1}}(g) = 1 \quad \text{for all } \mu, \text{ or } -1 \quad \text{for all } \mu \in \bigcup_{j=1}^{n-1} \varphi^j((\alpha_1, \alpha'_1)).$$

For each such $g$ and $\mu_0 \in \bigcup_{j=0}^{n-1} \varphi^j((\alpha_0, \alpha'_0))$, $\mu_1 \in \bigcup_{j=0}^{n-1} \varphi^j((\alpha_1, \alpha'_1))$, then

$$\text{ind}_{\pi_{\mu, 1}}(g) = -\varepsilon(\mu_0)\varepsilon(\mu_1).$$

(4.4)

**Proof.** For existence, assume $\varepsilon((\alpha_0, \alpha'_0)) = -1$ and let $g = e + (B - e)\theta(A)$ where $e$ is the identity element of $A_\varphi$ and $\theta$ is any element of $C(T)$ with

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value 0 on $\bigcup_{j=0}^{n-1} \varphi^j([a_0', \alpha_1])$, and value 1 on $\bigcup_{j=0}^{n-1} \varphi^j([a_1', \varphi^r(\alpha_0)])$. For each $(\mu, \nu) \in (\bigcup_{j=0}^{n-1} \varphi^j([a_0', \alpha_1]) \cap \mathcal{F}) \times \mathbb{T}$, we have $\pi_{\mu, \nu}(g) = I_\mu$. For each $(\mu, \nu) \in (\bigcup_{j=0}^{n-1} \varphi^j([a_1', \varphi^r(\alpha_0)]) \cap \mathcal{F}) \times \mathbb{T}$, we have $\pi_{\mu, \nu}(g) = \nu U_\mu$. Hence $g$ has the required properties by (ii) of subsection 3.2.2, and Proposition 3.3.3. If $\varepsilon((\alpha_0, \alpha_0')) = 1$, take $g^*$ instead.

Suppose now that $g$ is any element of $A_\varphi$ with the above properties, and $\mu_0 \in \bigcup_{j=0}^{n-1} \varphi^j((\alpha_0, \alpha_0'))$, $\mu_1 \in \bigcup_{j=0}^{n-1} \varphi^j((\alpha_1, \alpha_1'))$. Let $G(\zeta) = \pi_\zeta(g)$, $\zeta \in \mathcal{F} \times \mathbb{T}$. Let $\mu \in \mathcal{F} \cap \bigcup_{j=0}^{n-1} \varphi^j((\alpha_0, \alpha_0') \cup (\alpha_1, \alpha_1'))$ and $(\alpha, \alpha')$ the component of $\mathcal{F} \cap$ containing $\mu$. By (ii) of subsection 3.2.2, Proposition 3.3.3 and the properties of $g$, we have $0 = \text{ind} \pi_{\mu, 1}(g) = \varepsilon(\mu)(\text{wn} \det G(\alpha', \sqrt{\gamma}) - \text{wn} \det G(\alpha, \sqrt{\gamma}))$. This holds for all $\mu$, hence $\text{wn} \det G(\zeta, \sqrt{\gamma})$ is constant as $\zeta$ varies in $\bigcup_{j=0}^{n-1} \varphi^j([a_0', \alpha_1])$, and as $\zeta$ varies in $\bigcup_{j=0}^{n-1} \varphi^j([a_1', \varphi^r(\alpha_0)])$. In particular $\text{wn} \det G(\alpha_1, \sqrt{\gamma}) = \text{wn} \det G(\alpha_0', \sqrt{\gamma})$, and $\text{wn} \det G(\alpha_1', \sqrt{\gamma}) = \text{wn} \det G(\alpha_0, \sqrt{\gamma})$. So

$$\text{ind} \pi_{\mu, 1}(g) = \varepsilon(\mu_1)(\text{wn} \det G(\alpha_1', \sqrt{\gamma}) - \text{wn} \det G(\alpha_1, \sqrt{\gamma}))$$

$$= \varepsilon(\mu_1)(\text{wn} \det G(\alpha_0, \sqrt{\gamma}) - \text{wn} \det G(\alpha_0', \sqrt{\gamma}))$$

$$= -\varepsilon(\mu_1)\varepsilon(\mu_0)\text{ind} \pi_{\mu_0, 1}(g) \quad (\text{by Proposition 3.3.3})$$

$$= -\varepsilon(\mu_1)\varepsilon(\mu_0).$$

4.2.2. Let $\varphi$ be an order preserving homeomorphism of $\mathbb{T}$, with cyclic point set $\mathcal{F} \neq \emptyset$, $\mathbb{T}$. Let $\omega_0$ be a component of $A_{\varphi, \infty}$. Define a map $\Lambda : \{\text{components of } A_{\varphi, \infty} \cup (A_{\varphi, f})^\circ\} \to \{-1, 0, 1\}$, by setting $\Lambda(\omega_0) = 1$, $\Lambda(\omega_1) = 0$ if $\omega_1$ is a component of $(A_{\varphi, f})^\circ$, and if $\omega_1$ is a component of $A_{\varphi, \infty}$ distinct from $\omega_0$ we define $\Lambda(\omega_1) = -\text{ind} \pi_1(g)$ where $\pi_1$ is an irreducible $*$-representation of $A_{\varphi}$ with $[\pi_1] \in \omega_1$, and $g$ is an element of $A_{\varphi}$ such that $\pi(g)$ is a Fredholm operator for each irreducible $*$-representation $\pi$ of $A_{\varphi}$, with $\text{ind} \pi(g) = 0$ when $[\pi] \notin \omega_0 \cup \omega_1$, $\text{ind} \pi(g) = 1$ when $[\pi] \in \omega_0$, and $\text{ind} \pi(g) = 1$ or $-1$ when $[\pi] \in \omega_1$. Such $g$ always exists, since we can apply Proposition 4.2.1 to any two components.
$(\alpha_0, \alpha'_0), (\alpha_1, \alpha'_1)$ of $\mathcal{F}^c$ satisfying $\omega_j = \{[\pi_{\mu,1}]|\mu \in (\alpha_j, \alpha'_j)\}$, $j = 0, 1$, and $(\alpha_1, \alpha'_1) \subseteq (\alpha'_0, \varphi^r(\alpha_0))$, where $r$ is the rotation number of $\varphi$. We also note that, for each component $\omega_1$ of $\hat{A}_{\varphi, \infty}$ distinct from $\omega_0$, the value of $\Lambda(\omega_1)$ does not depend on the choice of $\pi_1, g$. This is because there always exists some $\mu_1 \in (\alpha_1, \alpha'_1)$ such that $\pi_1 \cong \pi_{\mu_1,1}$. So $\Lambda(\omega_1) = -\text{ind}\pi_{\mu_1,1}(g) = \varepsilon((\alpha_0, \alpha'_0))\varepsilon(\mu_1) = \varepsilon((\alpha_0, \alpha'_0))\varepsilon((\alpha_1, \alpha'_1))$ by Proposition 4.2.1, where $\varepsilon$ is the flow map of $\varphi$. Hence if $\varepsilon((\alpha_0, \alpha'_0)) = 1$ then

$$\Lambda(\omega_1) = \varepsilon((\alpha_1, \alpha'_1)) \text{ for each component } \omega_1 \text{ of } \hat{A}_{\varphi, \infty},$$

and if $\varepsilon((\alpha_0, \alpha'_0)) = -1$ then

$$\Lambda(\omega_1) = -\varepsilon((\alpha_1, \alpha'_1)) \text{ for each component } \omega_1 \text{ of } \hat{A}_{\varphi, \infty},$$

where $(\alpha_1, \alpha'_1)$ is any component of $\mathcal{F}^c$ such that $\omega_1 = \{[\pi_{\mu,1}]|\mu \in (\alpha_1, \alpha'_1)\}$. In particular we see that the set $\{\Lambda, -\Lambda\}$ does not depend on the component $\omega_0$ chosen. So with each algebra $A_{\varphi}$ as above we can associate in a natural way, a pair of maps

$$\Lambda, -\Lambda : \{\text{components of } \hat{A}_{\varphi, \infty} \cup (\hat{A}_{\varphi, f})^0\} \to \{-1, 0, 1\}. \quad (4.7)$$

### 4.3 Rotation Numbers from Algebras

Suppose $\varphi$ is an order preserving homeomorphism of $\mathbb{T}$ with the cycle number $n \geq 2$, the rotation number $r$, and cyclic point set distinct from $\emptyset, \mathbb{T}$. In this section, we recover as much information about $r$ as the $C^*$-algebra $A_{\varphi}$ allows. We start by giving an example of an element $g \in A_{\varphi}$ satisfying certain algebraic properties. Later we show that any element of the algebra satisfying these properties also contains enough information to recover the set $\{r, n - r\}$. We note the
impossibility of recovering \( r \) or \( n-r \) specifically, since the rotation numbers of \( \varphi \), \( \varphi^{-1} \) are \( r \), \( n-r \) respectively, and the \( C^* \)-algebras \( A_\varphi \), \( A_{\varphi^{-1}} \) are \( * \)-isomorphic.

For each \( m \in \mathbb{N} \), let \( \{ e_j \}_{j \in \{1, \ldots, m\}} \) be the standard basis of \( \mathbb{C}^m \), and let \( N_m \in M_m(\mathbb{C}) \), be the shift matrix

\[
(N_m)_{ij} = \begin{cases} 1 & \text{if } i-j = 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.8)
\]

**Lemma 4.3.1.** Let \( \varphi \) be an order preserving homeomorphism of \( T \) with the cycle number \( n \geq 2 \), and the cyclic point set \( \mathcal{F} \neq \emptyset \), \( T \). Let \( \pi_0 \) be an infinite dimensional irreducible \( * \)-representation of \( A_\varphi \). There exists \( g \in A_\varphi \) with the following properties:

(i) \( g \) is nilpotent of order \( n \), \( \| g \| = \| \pi(g^{n-1}) \| = 1 \) for each non-zero \( * \)-representation \( \pi \).

(ii) For each irreducible infinite dimensional \( * \)-representation \( \pi \) with \( [\pi], [\pi_0] \) in different components of \( \hat{A}_{\varphi, \infty} \), \( \pi(g) \) is unitarily equivalent to \( N_n \otimes I_{\ell^2(\mathbb{Z})} \in B(\mathbb{C}^n \otimes \ell^2(\mathbb{Z})) \).

**Proof.** Suppose \( \varepsilon((\alpha_0, \alpha_0')) = 1 \). Let \( \mu \in (\alpha_0, \alpha_0') \) and let \( \theta \) be an element of \( C(T) \) with value 0 on \( [\mu, \varphi^r(\mu)] \), value 1 on \( [\varphi^{-n}(\mu), \varphi^{-n}(\mu)] \), and values in \( (0, 1) \) \((\subseteq \mathbb{R})\) everywhere else. Let \( g = B\theta(A) \in A_\varphi \). Then \( \| g \| \leq 1 \) since \( \| \theta \|_\infty = 1 \).

If \( \pi \) is an infinite dimensional irreducible \( * \)-representation with \( [\pi], [\pi_0] \) in different components of \( \hat{A}_{\varphi, \infty} \), then \( \pi \) is unitarily equivalent to \( \pi_{\zeta, 1} \) for some \( \zeta \in (\alpha_0', \varphi^r(\alpha_0)) \cap \mathcal{F}^c \). But

\[
\theta(\varphi^j(\zeta)) = \begin{cases} 0 & \text{if } j \in n\mathbb{Z} \\ 1 & \text{if } j \notin n\mathbb{Z} \end{cases}
\]

so \( \pi(g) \cong \pi_{\zeta, 1}(g) = U\theta(\pi_{\zeta, 1}(A)) \cong N_n \otimes I_{\ell^2(\mathbb{Z})} \in B(\mathbb{C}^n \otimes \ell^2(\mathbb{Z})) \), where \( U \) is the infinite bilateral shift matrix. In particular, \( \pi(g) \) is also nilpotent of order \( n \) and \( \| \pi(g^{n-1}) \| = 1 \).
If \( \pi \) is a finite dimensional irreducible \( * \)-representation then \( \pi \cong \pi_{\zeta, \nu} \) for some \( \zeta \in [\alpha_0, \varphi^{-n}(\alpha_0)] \cap F, \nu \in \mathbb{T} \). Hence \( \pi(g) \cong \pi_{\zeta, \nu}(g) = \nu U_n \theta(\pi_{\zeta, \nu}(A)) = \nu N_n \), where \( U_n \in M_n(\mathbb{C}) \) is the matrix
\[
(U_n)_{ij} = \begin{cases} 1 & \text{if } i - j \equiv 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}
\]

Thus \( g \) satisfies the required properties. The same proof works if \( \varepsilon((\alpha_0, \alpha'_0)) = -1 \), except now we define \( \theta \) as having the value 0 on \([\mu, \varphi^{-n}(\mu)]\), value 1 on \([\varphi^{-2n}(\mu), \varphi^n(\mu)]\), and values in \((0, 1) \subseteq \mathbb{R} \) everywhere else. \( \square \)

Suppose \( g \in A_\varphi \) satisfies condition (i) of Lemma 4.3.1, and \( \pi \) is an infinite dimensional irreducible \( * \)-representation of \( A_\varphi \). It can be shown that \( k + \pi(g) \cong (N_n \otimes I_{\ell^2(\mathbb{Z})}) \oplus O_{\ell} \) (\( O_{\ell} \) is the \( \ell \times \ell \) zero matrix) for some compact operator \( k \), and unique \( \ell \in \{0,1,\ldots,n-1\} \) which depends on \( g \) and \( \pi \) (cf. proof of next Proposition). Furthermore, if \( \pi' \) is an irreducible \( * \)-representation of \( A_\varphi \) such that \([\pi']\) and \([\pi]\) are in the same component of \( \hat{A}_{\varphi, \infty} \), and \( \ell' \) is the unique element of \( \{0,1,\ldots,n-1\} \) corresponding to \( \pi' \), then \( \ell = \ell' \). If \( g \) also satisfies condition (ii) of Lemma 4.3.1, then we can say much more about \( \ell \).

**Proposition 4.3.2.** Let \( \varphi \) be an order preserving homeomorphism of the unit circle with the cycle number \( n \geq 2 \), the rotation number \( r \), the flow map \( \varepsilon \), and the cyclic point set \( F \neq \emptyset, \mathbb{T} \). Let \( \pi_0 \) be an infinite dimensional irreducible \( * \)-representation of \( A_\varphi \), and let \( g \in A_\varphi \) satisfy conditions (i), (ii) of Lemma 4.3.1. There exists unique \( \ell \in \{0,1,\ldots,n-1\} \) such that for each irreducible \( * \)-representation \( \pi \) with \([\pi]\) and \([\pi_0]\) in the same component of \( \hat{A}_{\varphi, \infty} \) we have
\[
k + \pi(g) \cong (N_n \otimes I_{\ell^2(\mathbb{Z})}) \oplus O_{\ell},
\]
for some compact operator \( k \). Furthermore, if \( \mu \in F^c \) with \( \pi_0 \cong \pi_{\mu, 1} \), then
\[
\ell = \begin{cases} n - r & \text{if } \varepsilon(\mu) = 1 \\ r & \text{if } \varepsilon(\mu) = -1 \end{cases}
\]

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Proof. Let \((\alpha_0, \alpha'_0)\) be the component of \(F^c\) which contains \(\mu\). Let \(\pi\) be a finite dimensional irreducible \(*\)-representation of \(A_\varphi\). \(\pi(g)^n = 0\) so there exists an orthonormal basis in which the matrix of \(\pi(g)\) satisfies \((\pi(g))_{i,j} = 0\) if \(1 \leq i < j \leq n\). So
\[
(\pi(g)^{n-1})_{i,j} = \begin{cases} 
0 & \text{if } (i,j) \neq (n,1) \\
n \prod_{k=2}^{n} (\pi(g))_{k,k-1} & \text{if } (i,j) = (n,1)
\end{cases}
\]
Hence \(\prod_{k=2}^{n} (\pi(g))_{k,k-1} = 1\) (since \(\|\pi(g)^{n-1}\| = 1\)), so \(\|\pi(g))_{k,k-1} = 1\) for each \(k \in \{2, \ldots, n\}\) \((\|\pi(g))_{k,k-1} \leq 1\) for each \(k\) since \(\|\pi(g)\| = 1\). Hence \(\pi(g)\) is unitarily equivalent to \(DN_n\) where \(D\) is a unitary diagonal \(n \times n\) matrix. Thus for each finite dimensional irreducible \(*\)-representation \(\pi\), the operator \(\pi(g)\) is unitarily equivalent to \(N_n\). Let \(G : F \times T \to M_n(C)\) be the continuous map \(\zeta \mapsto \pi_c(g)\), \(\zeta \in F \times T\). Then there exists a unitary valued map \(Y : F \times T \to M_n(C)\) such that \(YN_nY^*\) is equal to the continuous map \((\mu, \nu) \mapsto V^*(\nu)G(\mu, \nu)V(\nu)\), \((\mu, \nu) \in F \times T\) where \(V(\nu) = \text{diag}(1, \nu, \ldots, \nu^{n-1})\), \(\nu \in T\). \(N_n\) generates \(M_n(C)\), hence \(Y\) can be assumed continuous, by Corollary 3.2.10. \(G\) satisfies (3.9) and (3.10), hence \(YN_nY^*(\varphi(\mu), \nu) = V^*(\nu)U_nV(\nu)V(\mu, \nu)N_nY^*(\mu, \nu)V^*(\nu)U_n^*V(\nu)\) and \(YN_nY^*(\mu, \rho \nu) = YN_nY^*(\mu, \nu)\) for all \((\mu, \nu) \in F \times T\) \((\rho = e^{2\pi i/n})\). Hence \(Y(\varphi(\mu), \nu)Y^*(\mu, \nu)\) is a scalar multiple of \(V^*(\nu)U_nV(\nu)\), and \(Y(\mu, \rho \nu)Y^*(\mu, \nu)\) is a scalar matrix for all \((\mu, \nu) \in F \times T\). Let \(\theta \in C(F \times \{1, \rho, \ldots, \rho^{n-1}\}, T)\) satisfy the equality \(\theta(\mu, \rho^j)I_n = Y^*(\mu, \rho^j)Y(\mu, 1)\), for \(\mu \in F\), \(j \in \{0, 1, \ldots, n-1\}\). Since \(F \neq \emptyset\), \(T\), then \(\theta\) extends to an element of \(C(F \times T, T)\), which we also denote by \(\theta\). Let \(W = \theta Y\). Then \(W\) is a unitary element \(C(F \times T, M_n(C))\) such that
\[
W(\varphi(\mu), \nu)W^*(\mu, \nu)V^*(\nu)U_n^*V(\nu) \in CI_n \text{ for each } (\mu, \nu) \in F \times T, \quad (4.9)
\]
\[
W(\mu, \rho^j) = W(\mu, 1) \text{ for all } \mu \in F, \quad j \in \{0, 1, \ldots, n-1\}, \quad (4.10)
\]
and
\[
G(\mu, \nu) = V(\nu)W(\mu, \nu)N_nW^*(\mu, \nu)V^*(\nu), \text{ for all } (\mu, \nu) \in F \times T.
\]
Let \( \omega \) be a component of \( \hat{A}_{\varphi,\infty} \) and \( (\alpha, \alpha') \) any component of \( \mathcal{F}^\circ \) such that \( \omega = \{[\pi_{\mu,1}] | \mu \in (\alpha, \alpha')\} \) (we could have chosen \( \varphi^j((\alpha, \alpha')) \) for any \( j \in \{0, 1, \ldots, n-1\} \), instead of \( (\alpha, \alpha') \)). Let \( (\pi, H_\pi) \) be a \(*\)-representation of \( A_\varphi \) with \([\pi] \in \omega \). Then \( \pi \) is unitarily equivalent to \( \pi_{\mu,1} \) for some \( \mu \in (\alpha, \alpha') \), hence \( C^*(\pi(g)) \) (the \( C^* \)-algebra generated by \( \pi(g) \)) + \( K(H_\pi) \) is \(*\)-isomorphic (unitarily implemented) to \( C^*(\pi_{\zeta,1}(g)) + K(\ell^2(\mathbb{Z})) \) which in turn is equal to \( C^*(T_{\pi((\alpha,\alpha'))(e',f)}) + K(\ell^2(\mathbb{Z})) \) (Proposition 3.3.3), where

\[ f(\zeta) = V^*(\sqrt{\zeta})G(\alpha, \sqrt{\zeta})V(\sqrt{\zeta}) = W(\alpha, \sqrt{\zeta})N_nW^*(\alpha, \sqrt{\zeta}), \quad \zeta \in T \]

and

\[ f'(\zeta) = V^*(\sqrt{\zeta})G(\alpha', \sqrt{\zeta})V(\sqrt{\zeta}) = W(\alpha', \sqrt{\zeta})N_nW^*(\alpha', \sqrt{\zeta}), \quad \zeta \in T . \]

Hence by Proposition 3.2.8 the \( C^* \)-algebra \( C^*(\pi(g)) + K(H_\pi) \) is \(*\)-isomorphic to \( A_{\ell,n} \) where

\[ \ell = [\varepsilon((\alpha, \alpha'))(\operatorname{wn} \det W(\alpha, \sqrt{\gamma}) - \operatorname{wn} \det W(\alpha', \sqrt{\gamma}))]_n , \quad (4.11) \]

and \([\cdot]_n \) means modulo \((n)\). There are two cases:

(i) \( \omega \neq \omega_0 \). By hypothesis, \( C^*(\pi(g)) + K(H_\pi) \) is \(*\)-isomorphic to

\[ C^*(N_n \otimes I_{\ell^2(\mathbb{Z})}) + K(C^n \otimes \ell^2(\mathbb{Z})) \cong B(C^n) \otimes I_{\ell^2(\mathbb{N})} + K(C^n \otimes \ell^2(\mathbb{N})) = A_{0,n} . \]

Hence \( \ell = [\operatorname{wn} \det W(\alpha, \sqrt{\gamma}) - \operatorname{wn} \det W(\alpha', \sqrt{\gamma})]_n = 0 \) by (4.11). Hence \([\operatorname{wn} \det W(\zeta, \sqrt{\gamma})]_n \) is unchanged as \( \zeta \) varies in \( \mathcal{F} \cap (\alpha'_0, \varphi^r(\alpha_0)) \).

(ii) \( \omega = \omega_0 \). Then by (4.11)

\[ \ell = [\varepsilon((\alpha_0, \alpha'_0))(\operatorname{wn} \det W(\alpha_0, \sqrt{\gamma}) - \operatorname{wn} \det W(\alpha'_0, \sqrt{\gamma}))] \\
= [\varepsilon((\alpha_0, \alpha'_0))(\operatorname{wn} \det W(\alpha_0, \sqrt{\gamma}) - \operatorname{wn} \det W(\varphi^r(\alpha_0), \sqrt{\gamma}))]_n . \]
By (4.9) \( W(\varphi^*(\alpha_0), \nu)W^*(\alpha_0, \nu) \) is a scalar multiple of \( V^*(\nu)U_n^\nu V(\nu) \), hence of
\[
X^r_n(\nu^n) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \nu^n \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}
\]
for each \( \nu \in T \) (\( X_n \) as in (3.3)).

So \( W(\varphi^*(\alpha_0), \nu)W^*(\alpha_0, \nu) = \beta(\nu)X^r_n(\nu^n) \), \( \nu \in T \), for some continuous map \( \beta : T \to T \). \( W(\varphi^*(\alpha_0), \rho)W^*(\alpha_0, \rho) = W(\varphi^*(\alpha_0), 1)W^*(\alpha_0, 1) \) by (4.10), so \( \beta(\rho) = \beta(1) \). Hence
\[
\ell = [\varepsilon((\alpha_0, \alpha'_0))(\text{wn det}W(\alpha_0, \sqrt{\cdot}) - \text{wn det}\beta(\sqrt{\cdot})X^r_nW(\alpha_0, \sqrt{\cdot}))]_n
\]
\[
= [\varepsilon((\alpha_0, \alpha'_0))(\text{wn det}W(\alpha_0, \sqrt{\cdot}) - \text{wn det}\beta^n(\sqrt{\cdot}) - \text{wn det}X^r_n - \text{wn det}W(\alpha_0, \sqrt{\cdot}))]_n
\]
\[
= [-\varepsilon((\alpha_0, \alpha'_0))r]_n \text{ since } \det X_n(\zeta) = (-1)^{n+1}\zeta, \quad \zeta \in T.
\]

The proof is completed by noting that any \( * \)-isomorphism between \( C^*(\pi(A)) + K(H_\pi) \) and \( A_{\ell,n} \) is implemented by a unitary. \( \square \)

It follows that, given a \( c^* \)-algebra \( A_\varphi \) where \( \varphi \) is an order preserving homeomorphism of \( T \) with the cyclic point set \( \mathcal{F} \neq \emptyset, \mathcal{T} \), we can associate three integers. First, we have the dimension of some (hence every) finite dimensional irreducible \( * \)-representation (this is the cycle number of \( \varphi \)). If this number is greater than 1 we can pick any infinite dimensional irreducible \( * \)-representation \( \pi_0 \) of \( A_\varphi \), and \( g \in A_\varphi \) satisfying conditions (i), (ii) of Lemma 4.3.1. The \( C^* \)-algebra generated by \( \pi(g) \) and the compact operators, is \( * \)-isomorphic to \( A_{\ell,n} \) for some unique \( \ell \in \{r, n-r\} \) (Proposition 4.3.2). In this way the set \( \{r, n-r\} \) is determined, though we can't determine which element is the rotation number \( r \) and which is \( n-r \).
4.4 A COMPLETE $C^*$-ALGEBRAIC INVARIANT

In this section we take the information yielded by the constructs of the last three sections for an arbitrary $C^*$-algebra $\mathfrak{A} \in \{ A_\varphi | \varphi \text{ is an order preserving homeomorphism of } T \text{ with the cyclic point set } \mathcal{F} \neq 0, T \}$, and use it to define a four element subset of $\mathcal{I}$. This subset of $\mathcal{I}$ will be shown to be a complete $*$-isomorphism invariant, which in the case of the $C^*$-algebra $A_\varphi$ determines the topological structure of $\varphi$ or $\varphi^{-1}$.

4.4.1 We begin the construction of this subset of $\mathcal{I}$ by letting $n$ denote the dimension of some (hence every) finite dimensional irreducible $*$-representation of $\mathfrak{A}$. Let $\mathcal{G} = \{ \omega | \omega \text{ is a component of } \hat{\mathfrak{A}}_\infty \cup (\hat{\mathfrak{A}}_f)^o \}$, and let $\mathcal{O}$ be a subset of $\mathcal{G}^3$ defined via the spectrum as in (4.1). Let $\Lambda : \mathcal{G} \to \{-1, 0, 1\}$ be any one of the two maps of subsection 4.2.2 (the only other is $-\Lambda$).

4.4.2 From $\mathcal{G}, \mathcal{O}$ construct an injection $\beta : \mathcal{G} \to T$ as follows. Let $\{ \Omega_j \}_{j \in \mathbb{N}}$ be a denumeration of $\mathcal{G}$. Let $c_1, c_2$ be any disjoint open arcs in $T$. By (4.2), (4.3), $(\Omega_1, \Omega_3, \Omega_2) \in \mathcal{O}$ or $(\Omega_2, \Omega_3, \Omega_1) \in \mathcal{O}$, but not both.

If $(\Omega_1, \Omega_3, \Omega_2) \in \mathcal{O}$, let $c_3$ be any open arc in $T \setminus (c_1 \cup c_2)$ such that $(c_1, c_3, c_2)$ are counter-clockwise ordered.

If $(\Omega_2, \Omega_3, \Omega_1) \in \mathcal{O}$, let $c_3$ be any open arc in $T \setminus (c_1 \cup c_2)$ such that $(c_2, c_3, c_1)$ are counter-clockwise ordered.

Inductively, suppose $c_1, \ldots, c_k$ are defined for some $k \in \{2, 3, \ldots, \}$. There exist unique $i, j \in \{1, \ldots, k\}$ such that $(\Omega_i, \Omega_l, \Omega_j) \notin \mathcal{O}$ for each $l \in \{1, \ldots, k\}$, and $(\Omega_i, \Omega_{k+1}, \Omega_j) \in \mathcal{O}$. Let $c_{k+1}$ be an open arc contained in $T \setminus (c_1 \cup \ldots, \cup c_k)$, such that the arcs $(c_i, c_{k+1}, c_j)$ are counter-clockwise. For each $j \in \mathbb{N}$, pick some $\zeta_j \in c_j$, and define

$$\beta : \mathcal{G} \to T : \Omega_j \mapsto \zeta_j , \quad j \in \mathbb{N}.$$
4.4.3 We are now ready to assemble the invariant. Choose an infinite dimensional irreducible \(*\)-representation \(\pi_0\) of \(\mathcal{A}\) and let \(\omega_0\) be the component of \(\mathcal{A}_\infty\) containing \(\pi_0\). For each \(m \in \mathbb{Z}\), let \(p_m : T \to T\) be the map \(\zeta \mapsto \zeta^m, \zeta \in T\). Consider the set

\[
\{[(q,-\lambda)], [(1-q)_1, \lambda)], [(q,-\lambda_0 p - 1)], [(1-q)_1, \lambda_0 p - 1]\} \quad (\subseteq I) \quad (4.12)
\]

where

\[
\lambda(z) = \Lambda(\omega_0)A_0 \beta_0^{-1} p_n(z), \quad z \in \{\zeta | \zeta^m \in \beta(\mathfrak{S})\}, \quad (4.13)
\]

so

\[
\lambda_0 p - 1(z) = \Lambda(\omega_0)A_0 \beta_0^{-1} p_n p_0 - 1(z) = \Lambda(\omega_0)A_0 \beta_0^{-1} p_n(z), \quad z \in \{\zeta | \zeta^m \in \beta(\mathfrak{S})\}, \quad (4.14)
\]

where \([::1]\) means modulo 1 (\([\zeta]_1 \in [0,1) \subseteq \mathbb{R}\) for all \(\zeta \in \mathbb{R}\), \(q = 0\) if \(n = 1\), and \(q = \ell/n\) if \(n \geq 2\), where \(\ell\) is as in Proposition 4.3.2.

The set (4.12) will be shown to be a complete \(C^*\)-algebraic invariant for \(\mathfrak{A}\), in particular we must verify independence with respect to the choice of \(\Lambda, \mathcal{O}, \beta, \pi_0\). For this we assume \(\mathfrak{A} = A_\varphi\) where \(\varphi\) is an order preserving homeomorphism of \(T\) with the cycle number \(n\) (\(=\) the dimension of any finite dimensional irreducible representation of \(A_\varphi\)), the rotation number \(r\), cyclic point set \(\mathcal{F} \neq \emptyset, T\), and the flow map \(\epsilon\). Let \((\alpha_0, \alpha_0')\) be a component of \(\mathcal{F}^c\) such that \([\pi_0] = [\pi_{\mu,1}]\) for some \(\mu \in (\alpha_0, \alpha_0')\). Let \(\delta : \{\text{components of } \hat{A}_{\varphi, c}, (\hat{A}_{\varphi}, r)^c\} \to (\alpha_0, \varphi_r(\alpha_0))\) be a map satisfying \([\pi_{\delta(\omega), 1}] \in \omega\) for each component \(\omega\), i.e. as in the proof of Lemma 4.1.1.

To see that (4.12) is independent of the choice of \(\Lambda\), we need only note that \(\epsilon \circ \delta\), and \(-\epsilon \circ \delta\) are the only possibilities for \(\Lambda\) ((4.5), (4.6)), and each of these is independent of the component \((\alpha_0, \alpha_0')\) chosen, and of the map \(\delta\) chosen.

For independence with respect to \(\pi_0\), assume \(\Lambda = \epsilon \circ \delta\), and compute. If \(n = 1\)
then (4.12) is

\[
\{[(0, \lambda), [(0, -\lambda)], [(0, \lambda \circ p_{-1})], [(0, -\lambda \circ p_{-1})]\}
\]

If \( n \geq 2 \), by Proposition 4.3.2 \( \ell = -\varepsilon(\delta(\omega_0))r |_{n} = -\Lambda(\omega_0)r |_{n} \), so (4.12) is

\[
\{[(r/n, \lambda)], [(1 - r/n, -\lambda)], [(r/n, \lambda \circ p_{-1})], [(1 - r/n, -\lambda \circ p_{-1})]\}
\]

\[
= \{[(r/n, \lambda \circ p_{-1}^{-1} \circ p_{n})], [(1 - r/n, -\lambda \circ p_{-1}^{-1} \circ p_{n})], [(r/n, \lambda \circ p_{-1}^{-1} \circ p_{n})],
\]

\[
[(1 - r/n, -\lambda \circ p_{-1}^{-1} \circ p_{n})]\}
\]

(compute for the cases \( \Lambda(\omega_0) = 1, \Lambda(\omega_0) = -1 \)), proving independence with respect to \( \pi_0 \).

Next we show that if \( \beta_1 : \mathcal{G} \rightarrow T \) is any other injection constructed from \( \mathcal{G}, \mathcal{O} \) as was \( \beta \) in subsection 4.4.2, then the resulting set (4.12) is the same. To see this, consider the order preserving bijection \( \theta : \{\zeta | \zeta^n \in \beta_1(\mathcal{G})\} \rightarrow \{\zeta | \zeta^n \in \beta(\mathcal{G})\} \) given by \( \theta(\zeta) = \rho^k \sqrt[2n]{\beta_0 \beta_1^{-1} \circ p_{n}(\zeta)} \) for \( \zeta \in [\rho^k, \rho^{k+1}] \cap \{\zeta | \zeta^n \in \beta_1(\mathcal{G})\} \), \( k \in \{0, 1, \ldots, n - 1\} \), where \( \rho = e^{2\pi i/n} \) and \( \sqrt[n]{\cdot} \) is the \( n \)-th root map on \( T \) with range \( [1, \rho) \). Let \( \lambda_1 = \Lambda(\omega_0) \Lambda_0 \beta_1^{-1} \circ p_{n} : \{\zeta | \zeta^n \in \beta_1(\mathcal{G})\} \rightarrow \{-1, 0, 1\} \), then \( \lambda_1 = \lambda \circ \theta \) since \( \beta_1^{-1} \circ p_{n} = \beta^{-1} \circ p_{n} \circ \theta \). Hence we also have: \( \lambda_1 \circ p_{-1} = \lambda \circ p_{-1} \circ \hat{\theta} : \{\bar{\zeta} | \zeta^n \in \beta_1(\mathcal{G})\} \rightarrow \{-1, 0, 1\} \), where \( \hat{\theta} = p_{-1} \circ \theta \circ p_{-1} \) an order preserving bijection of \( \{\bar{\zeta} | \zeta^n \in \beta_1(\mathcal{G})\} \) onto \( \{\bar{\zeta} | \zeta^n \in \beta(\mathcal{G})\} \). Hence replacing \( \beta \) with \( \beta_1 \) the set (4.12) becomes

\[
\{[(q, -\lambda_1)], [(1 - q, \lambda_1)], [(q, -\lambda_1 \circ p_{-1})], [(1 - q, \lambda_1 \circ p_{-1})]\}
\]

\[
= \{[(q, -\lambda \circ \theta)], [(1 - q, \lambda \circ \theta)], [(q, -\lambda \circ p_{-1} \circ \hat{\theta})], [(1 - q, \lambda \circ p_{-1} \circ \hat{\theta})]\}
\]

\[
= \{[(q, -\lambda)], [(1 - q, \lambda)], [(q, -\lambda \circ p_{-1})], [(1 - q, \lambda \circ p_{-1})]\}
\]

since \( \theta, \hat{\theta} \) are order preserving maps (giving an equivalence between elements of \( T \) as per Definition 2.2.1), and this is just the set (4.12) evaluated for \( \beta \) instead of \( \beta_1 \). So the set (4.12) does not depend on our choice of the injection \( \beta \) of subsection 4.4.2.
Finally, the injection $\overline{\beta} : \mathcal{G} \to T$ has the same defining properties with respect to $\mathcal{O}^{-1} = \{(\omega, \omega', \omega'')|(\omega'', \omega', \omega) \in \mathcal{O}\}$ as the injection $\beta : \mathcal{G} \to T$ has with respect to $\mathcal{O}$. So if we chose $\mathcal{O}^{-1}$ instead of $\mathcal{O}$, then the set (4.12) would be unchanged since substituting $\overline{\beta}$ for $\beta$ in (4.12) does not change this set.

As the set (4.12) does not depend on our choice for $\mathcal{O}$, $\beta$, $\Lambda$, $\pi_0$, the following definition is possible.

**Definition 4.4.4.** Let $\mathcal{A}$ be a $C^*$-algebra in $\{A_\varphi | \varphi \text{ is an order preserving homeomorphism of } T \text{ with cyclic point set } \neq \emptyset, T\}$. Define $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{I}$ as the set (4.12).

At this point, if desired we could find an order preserving homeomorphism $\varphi$ of $T$ with non-trivial set of cyclic points satisfying $\mathcal{I}(\mathcal{A}) = \{\mathcal{I}(\varphi), \mathcal{I}(\varphi^{-1}), \mathcal{I}(p_{\varphi}^{-1}p^{-1}), \mathcal{I}(p_{\varphi}^{-1}p^{-1})\}$. To see this pick any element of $\mathcal{I}(\mathcal{A})$, say $[(q, -\lambda)]$ (the result is the same for any other element of $\mathcal{I}(\mathcal{A})$). By Remark 2.2.3, there exists $\varphi$ such that $\mathcal{I}(\varphi) = [(q, -\lambda)]$. So $\mathcal{I}(\varphi^{-1}) = [[[1 - q_1], \lambda]]$, $\mathcal{I}(p_{\varphi}^{-1}p^{-1}) = [[[1 - q_1], -\Lambda p^{-1}]]$, $\mathcal{I}(p_{\varphi}^{-1}p^{-1}) = [[[q, \Lambda p^{-1}]]]$, but we can say much more.

**Theorem 4.4.5.** Let $\varphi$ be an order preserving homeomorphism of $T$ with the cyclic point set $\mathcal{F} \neq \emptyset, T$. Then

$$\mathcal{I}(A_\varphi) = \{\mathcal{I}(\varphi), \mathcal{I}(\varphi^{-1}), \mathcal{I}(p_{\varphi}^{-1}p^{-1}), \mathcal{I}(p_{\varphi}^{-1}p^{-1})\}.$$

**Proof.** Let $n$, $r$, $\varepsilon$ be the cycle number, the rotation number, and the flow map of $\varphi$ respectively. Let $\mathcal{S}$, $\mathcal{O}$, $\Lambda$ be as in subsection 4.4.1, $\beta$ as in subsection 4.4.2. Let $\pi_0$ be an infinite dimensional irreducible $*$-representation, $\omega_0$ the component of $\hat{A}_{\varphi, \infty}$ containing $\pi_0$, and let $\ell$ be as in Proposition 4.3.2. Let $(\alpha_0, \alpha'_0)$ be a component of $\mathcal{F}^c$ such that $[\pi_{\mu, 1}] \in \omega_0$ for all $\mu \in (\alpha_0, \alpha'_0)$, and let $\delta : \{\text{components of } \hat{A}_{\varphi, \infty} \cup (\hat{A}_{\varphi, \ell})^0\} \to (\alpha_0, \varphi^r(\alpha_0))$ be a map satisfying
where \( q = \ell/n, \) \( \lambda = \Lambda(\omega)\Lambda_0\beta^{-1}p_\omega : \{\zeta|\zeta^n \in \beta(\mathcal{S})\} \to \{-1,0,1\}. \) \( \mathcal{I}(A_\varphi) \) is invariant with respect to the choice of \( \mathcal{S}, \Lambda, \) so we can assume \( \Lambda(\omega) = \varepsilon_0\delta(\omega), \) \( \omega \in \mathcal{S}, \) and by Lemma 4.1.2 we can assume that \( \mathcal{O} \) is given as

\[
\mathcal{O} = \{(x,y,z) \in \mathcal{S}^3|\,(\delta(x),\delta(y),\delta(z)) \text{ are counter-clockwise in } \mathbb{T}\}. \tag{4.15}
\]

Assume \( \varepsilon((\alpha_0,\alpha_0')) = 1 \) (the case \( \varepsilon((\alpha_0,\alpha_0')) = -1 \) is similar). Then \( q = 1 - r/n, \) and \( \lambda = \varepsilon((\alpha_0,\alpha_0'))\varepsilon_0\delta_0\beta^{-1}p_\omega = \varepsilon_0\delta_0\beta^{-1}p_\omega = \varepsilon_0\theta, \) where \( \theta : \{\zeta|\zeta^n \in \beta(\mathcal{S})\} \to \mathbb{T} \) is the injection defined by \( \theta(\zeta) = \varphi^k(\delta(\beta^{-1}(\zeta^n))) \) for \( \zeta \in \{\zeta|\zeta^n \in \beta(\mathcal{S})\} \cap [\rho^k,\rho^{k+1}], \) \( k = 0,1,\ldots,n-1. \) \( \theta \) is an order preserving bijection between its domain and image, by (4.15). We also have \( \lambda_\omega p_{-1} = \varepsilon_\omega \theta p_{-1} = \varepsilon_\omega p_{-1}^\circ \theta p_{-1} = \varepsilon_\omega p_{-1}^\circ \theta \) where

\[
\hat{\theta} = p_{-1}^\circ \theta p_{-1} : \{\overline{\zeta}|\zeta^n \in \beta(\mathcal{S})\} \to \{\overline{\theta(\zeta)}|\zeta^n \in \beta(\mathcal{S})\},
\]

an order preserving bijection. So

\[
\mathcal{I}(A_\varphi) = \{[(1 - r/n,-\varepsilon\theta)], [(r/n,\varepsilon\theta)], [(1 - r/n,-\varepsilon_0 p_{-1}^\circ \theta)], [(r/n,\varepsilon_0 p_{-1}^\circ \theta)]\}
\]

\[
= \{[(1 - r/n,-\varepsilon|S)], [(r/n,\varepsilon|S)], [(1 - r/n,-\varepsilon_0 p_{-1}|\mathcal{S})], [(r/n,\varepsilon_0 p_{-1}|\mathcal{S})]\},
\]

where \( S \) is the countable set \( \{\theta(\zeta)|\zeta^n \in \beta(\mathcal{S})\} \) of isolated points in \( \mathbb{T}. \) Hence

\[
\mathcal{I}(A_\varphi) = \{\mathcal{I}(\varphi^{-1}), \mathcal{I}(\varphi), \mathcal{I}(p_{-1}\varphi p_{-1}), \mathcal{I}(p_{-1}\varphi^{-1} p_{-1})\}. \tag*{\blacksquare}
\]

**Theorem 4.4.6.** Let \( \mathfrak{A}, \mathfrak{A}' \) be \( C^* \)-algebras in the set \( \{A_\varphi | \varphi \text{ is an order preserving homeomorphism of the unit circle with cyclic point set } \mathcal{F} \neq \emptyset, \mathbb{T}\}. \) Then \( \mathfrak{A}, \mathfrak{A}' \) are \(*\)-isomorphic if and only if \( \mathcal{I}(\mathfrak{A}) = \mathcal{I}(\mathfrak{A}^*) \).

**Proof.** Suppose \( \gamma : \mathfrak{A} \to \mathfrak{A}' \) is a \(*\)-isomorphism. Let \( \pi_0 \) be an infinite dimensional irreducible \(*\)-representation of \( \mathfrak{A} \) and let \( \omega_0 \) be the component of
\( \mathfrak{A}_\infty \) containing \( \tau_0 \). \( \mathcal{G} = \{ \omega | \omega \) a component of \( \mathfrak{A}_\infty \cup (\mathfrak{A}_f)^0 \} \), \( \mathcal{O} \), \( \Lambda \) be as in subsection 4.4.1, \( \beta \) as in subsection 4.4.2, \( q \), \( \lambda \) as in subsection 4.4.3, and let \( n \) be the dimension of some (hence every) finite dimensional irreducible \(*\)-representation of \( \mathfrak{A} \). Let \( \gamma : \mathfrak{A}' \to \mathfrak{A} \) be the homeomorphism \([\pi'] \mapsto [\pi' \circ \gamma] \), \( \pi' \) an irreducible \(*\)-representation of \( \mathfrak{A}' \). \( \gamma \) defines a bijection \( \omega \mapsto \gamma(\omega) \) of \( \mathcal{G}' := \{ \omega | \omega \) a component of \( \mathfrak{A}' \cup (\mathfrak{A}_f)^0 \} \) onto \( \mathcal{G} \), and we let \( \gamma \) denote this map as well. Consider the set \( \{ (x, y, z) | x, y, z \in \mathcal{G}' \text{ and } (\gamma(x), \gamma(y), \gamma(z)) \in \mathcal{O} \} \). Since \( \gamma : \mathfrak{A}' \to \mathfrak{A} \) is a homeomorphism, this is one of the two subsets of \((\mathcal{G}')^3\) that can be constructed from \( \mathfrak{A}' \) by the method of Section 4.1, and we denote it by \( \mathcal{O}' \).

If \( g \in \mathfrak{A} \) satisfies the conditions (i), (ii) of Lemma 4.3.1 with respect to \( \tau_0 \), then \( \gamma(g) \in \mathfrak{A}' \) satisfies the analogous conditions with respect to \( \tau_0 \circ \gamma^{-1} \) for the algebra \( \mathfrak{A}' \). Hence \( (n, \mathcal{G}', \mathcal{O}', \beta \circ \gamma, \Lambda \circ \gamma, \pi_0 \circ \gamma^{-1}, q) \) are for \( \mathfrak{A}' \) as \( (n, \mathcal{G}, \mathcal{O}, \beta, \Lambda, \pi_0, q) \) are for \( \mathfrak{A} \). Hence \( \mathcal{I}(\mathfrak{A}') = \{ ([q, -\lambda'], [([1 - q]_1, \lambda'], [[q, -\lambda' \circ p_{-1}]], [[[1 - q]_1, \lambda' \circ p_{-1}])] \} \) (Definition 4.4.4), where \( \lambda' = \Lambda \circ \gamma(\gamma^{-1}(\omega_0)) \Lambda \circ \gamma(\beta \circ \gamma)^{-1} \circ p_n = \Lambda(\omega_0) \Lambda \circ \beta^{-1} \circ p_n = \lambda \), so \( \mathcal{I}(\mathfrak{A}') = \mathcal{I}(\mathfrak{A}) \).

Conversely, suppose \( \mathcal{I}(\mathfrak{A}) = \mathcal{I}(\mathfrak{A}') \). There exists \( \varphi \), \( \psi \) such that \( \mathfrak{A} = A_\varphi \), \( \mathfrak{A}' = A_\psi \). By Theorem 4.4.5 \( \{ \mathcal{I}(\varphi), \mathcal{I}(\varphi^{-1}), \mathcal{I}(p_{-1} \circ \varphi \circ p_{-1}), \mathcal{I}(p_{-1} \circ \varphi^{-1} \circ p_{-1}) \} = \mathcal{I}(A_\varphi) = \mathcal{I}(A_\psi) = \{ \mathcal{I}(\psi), \mathcal{I}(\psi^{-1}), \mathcal{I}(p_{-1} \circ \psi \circ p_{-1}), \mathcal{I}(p_{-1} \circ \psi^{-1} \circ p_{-1}) \} \), hence by Theorem 2.2.6 \( \varphi \) is conjugate to \( \psi \) or \( \psi^{-1} \) or \( p_{-1} \circ \psi \circ p_{-1} \) or \( p_{-1} \circ \psi^{-1} \circ p_{-1} \) via an order preserving homeomorphism, i.e. \( \varphi \) is conjugate to \( \psi \) or \( \psi^{-1} \). Hence \( A_\varphi \) and \( A_\psi \) are \(*\)-isomorphic. \( \square \)

**Corollary 4.4.7.** Let \( \varphi \), \( \psi \) be order preserving homeomorphisms of the unit circle, each possessing cyclic points and non-cyclic points. Then the \( C^* \)-algebras \( A_\varphi \) and \( A_\psi \) are \(*\)-isomorphic if and only if \( \varphi = \theta \circ \psi \circ \theta^{-1} \) or \( \varphi = \theta \circ \psi^{-1} \circ \theta^{-1} \) for some homeomorphism \( \theta \).
Proof. Suppose $A_{\varphi}$ and $A_{\psi}$ are $*$-isomorphic. Then

$$\{\mathcal{I}(\varphi), \mathcal{I}(\varphi^{-1}), \mathcal{I}(p_{-1}\circ\varphi\circ p_{-1}), \mathcal{I}(p_{-1}\circ\varphi^{-1}\circ p_{-1})\}$$

$$= \{\mathcal{I}(\psi), \mathcal{I}(\psi^{-1}), \mathcal{I}(p_{-1}\circ\psi\circ p_{-1}), \mathcal{I}(p_{-1}\circ\psi^{-1}\circ p_{-1})\}$$

by Theorem 4.4.5 and Theorem 4.4.6. Hence $\varphi = \theta_{\circ}\psi_{\circ}\theta^{-1}$ or $\varphi = \theta_{\circ}\psi^{-1}_{\circ}\theta^{-1}$ for some homeomorphism $\theta$, by Theorem 2.2.6. The converse was shown in the introduction. ■
CHAPTER 5

The order reversing case

In this chapter we give a classification of the $C^*$-crossed product algebras associated with order reversing homeomorphisms of the unit circle. Fortunately, this presents no new problems, in fact since order reversing maps have no rotation things are actually easier. Statements and proofs from the order preserving case carry over mechanically to their order reversing analogues, any difference being exclusively of a technical or foreseeable nature. Because of this, our treatment of the order reversing case is confined to giving the statement of the order reversing version of earlier results when differences occur.

It will be assumed in this chapter that $\varphi$ in an order reversing homomorphism of the unit circle with cyclic point set $\mathcal{F}$, and fixed points $\{x, y\}$ arranged so that $1 \in [x, y]$.

For each $(\mu, \nu) \in \mathcal{F} \times T$ define the irreducible $*$-representation $(\pi_{\mu, \nu}, \mathbb{C}^n)$ as in (3.3.1), except that now we put $n = 1$ if $\mu$ is a fixed point of $\varphi$, and $n = 2$ if $\mu$ is not fixed by $\varphi$.

If $(\mu, \nu) \in \mathcal{F}^c \times T$ we distinguish between four possibilities:

(i) $\lim_{j \to \infty} \varphi^j(\mu), \lim_{j \to -\infty} \varphi^j(\mu)$ are fixed points of $\varphi$.

(ii) $\lim_{j \to \infty} \varphi^j(\mu)$ is a fixed point of $\varphi$, $\lim_{j \to -\infty} \varphi^j(\mu)$ is not a fixed point of $\varphi$.

(iii) $\lim_{j \to \infty} \varphi^j(\mu)$ is not a fixed point of $\varphi$, $\lim_{j \to -\infty} \varphi^j(\mu)$ is a fixed point of $\varphi$.

(iv) $\lim_{j \to \infty} \varphi^j(\mu), \lim_{j \to -\infty} \varphi^j(\mu)$ are not fixed points of $\varphi$.

When either (i) or (iv) holds, define $\pi_{\mu, \nu}$ as in (3.3.2), substituting $n = 1$ if (i) holds, and $n = 2$ if (iv) holds. When either (ii) or (iii) holds, define $\pi_{\mu, \nu}$ by giving the matrices $\pi_{\mu, \nu}(A), \pi_{\mu, \nu}(B)$ as in (3.3.2), except a different Hilbert space and basis is chosen in each case. For example in case (ii) the Hilbert space is $\ell^2(\mathbb{Z})_- \oplus (\ell^2(\mathbb{Z})_+ \otimes \mathbb{C}^2)$, with standard basis $\{\delta_i \oplus 0 \mid i < 0\} \cup \{0 \oplus (\delta_j \otimes \ldots \}$
\( e_k \mid j \geq 0, k = 1, 2 \}, \) where \( \delta_i(j) = \begin{cases} 1 & \text{if } i = j, i, j \in \mathbb{Z}. \end{cases} \) The ordering of the basis is \( e_k \otimes e_{\ell} < e_{k'} \otimes e_{\ell'} \) when \( k < k' \), or when \( k = k' \) and \( \ell < \ell' \) \((\ell = 1, \ell' = 2)\); and also \( \delta_i \otimes 0 < \delta_j \otimes 0 < \delta_k \otimes e_\ell \) if \( i < j < 0 \) and \( k = 0 \), \( \ell = 1, 2 \). With \( \pi_{\mu, \nu} \) thus defined for each \( (\mu, \nu) \in \mathbb{T}^2 \), Proposition 3.1.2 also holds when \( \varphi \) is an order reversing homeomorphism. Lemma 3.3.1 is still valid if we substitute \( n = 2 \) (instead of the ‘cycle number’ which is not uniquely defined for order reversing homeomorphisms). Corollary 3.3.2 holds if we substitute \( n = 2 \), and for each \( g \in A_\varphi \), the corresponding continuous function \( G : \mathcal{F} \times \mathbb{T} \to M_2(\mathbb{C}) \) is defined as follows,

\[
G(\mu, \nu) = \begin{cases} \pi_{\mu, \nu}(g), & \text{if } (\mu, \nu) \in (\mathcal{F} \setminus \{x, y\}) \times \mathbb{T} \\ \frac{1}{2} \left( \pi_{\mu, \nu}(g) + \pi_{\mu,-\nu}(g) + (\pi_{\mu,\nu}(g) - \pi_{\mu,-\nu}(g))U_2 \right), & \text{if } (\mu, \nu) \in \{x, y\} \times \mathbb{T}, \end{cases}
\]

where \( U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The analogue of Proposition 3.3.3 is,

**Theorem 5.1.** For each \( (\mu, \nu) \in \mathcal{F}^c \times \mathbb{T} \) and \( g \in A_\varphi \), the operator

\[
\pi_{\mu, \nu}(g) - T_{e(\mu) \circ (f',f)}
\]

is compact, where

\[
f'(\zeta) = \begin{cases} V^*(\sqrt{\zeta})\pi_{\alpha', \nu \sqrt{\zeta}}(g)V(\sqrt{\zeta}), & \zeta \in \mathbb{T}, \text{ if } \varphi(\alpha') \neq \alpha' \\ \pi_{\alpha', \nu \sqrt{\zeta}}(g), & \zeta \in \mathbb{T}, \text{ if } \varphi(\alpha') = \alpha', \end{cases}
\]

\[
f(\zeta) = \begin{cases} V^*(\sqrt{\zeta})\pi_{\alpha, \nu \sqrt{\zeta}}(g)V(\sqrt{\zeta}), & \zeta \in \mathbb{T}, \text{ if } \varphi(\alpha) \neq \alpha \\ \pi_{\alpha, \nu \sqrt{\zeta}}(g), & \zeta \in \mathbb{T}, \text{ if } \varphi(\alpha) = \alpha, \end{cases}
\]

and \((\alpha, \alpha')\) is the component of \( \mathcal{F}^c \) containing \( \mu \).

We remark that here we are stretching the definition of the term \( T_{e(\mu) \circ (f',f)} \) to include mixed block-Toeplitz matrices with block sizes 1 and 2 (these are defined analogously to the block-Toeplitz matrices with unique block size, as defined previously).
For any \((\mu, \nu), (\mu', \nu') \in \mathbb{T}^2\), the irreducible \(*\)-representations \(\pi_{\mu, \nu}, \pi_{\mu', \nu'}\) are unitarily equivalent if and only if one of the following is true:

(i) \(\mu, \mu' \in \{x, y\}, \mu = \mu', \nu = \nu'\) or

(ii) \(\mu, \mu' \in \mathcal{F} \setminus \{x, y\}, \mu \in \{\mu, \varphi(\mu)\}, \nu' \in \{\nu, -\nu\}\) or

(iii) \(\mu, \mu' \in \mathcal{F}^c, \mu' = \varphi^j(\mu)\) for some \(j \in \mathbb{Z}\).

Let \(q : \mathbb{T}^2 \to \mathbb{T}^2 / \sim\) be the canonical map for this equivalence. The analogue of Proposition 3.4.2 has the same statement, except now \(\varphi\) is order reversing. As before, the subset \(\hat{A}_{\varphi, \infty}\) can be characterized as the elements of \(\hat{A}_{\varphi}\) having an open neighbourhood homeomorphic to \((0, 1)\).

As in Section 4.1, we can define a total ordering on the components of \(\hat{A}_{\varphi, \infty} \cup (\hat{A}_{\varphi, f})^\circ\). It can be assumed that \(\hat{A}_{\varphi, \infty} \neq \emptyset\) (if not, then \(\mathcal{F} = \mathbb{T}\), and \(\varphi\) is conjugate to the map \(\zeta \mapsto \bar{\zeta}, \zeta \in \mathbb{T}\)). Hence there are two components of \(\hat{A}_{\varphi, f}\) corresponding to the one-dimensional \(*\)-representations (one contains \([\pi_{x, \nu}]\), the other contains \([\pi_{y, \nu}]\) for all \(\nu \in \mathbb{T}\)). Let \(\omega_0\) denote one of these components. For any components \(\omega, \omega'\) of \(\hat{A}_{\varphi, \infty} \cup (\hat{A}_{\varphi, f})^\circ\) we write \(\omega \leq \omega'\) if \(\omega = \omega'\) or there exists a connected set \(X \subseteq \hat{A}_{\varphi}\) satisfying \(\omega_0 \cap X \neq \emptyset, \omega \cap X \neq \emptyset\), and \(\omega' \cap X = \emptyset\).

The analogue of the map \('\delta'\) mentioned in the proof of Lemma 4.1.1 is defined as a map \(\delta : \{\text{components of } (\hat{A}_{\varphi, f})^\circ \cup \hat{A}_{\varphi, \infty}\} \to (x, y)\) satisfying \([\pi_{\delta(\omega), 1}] \in \omega\) for each component \(\omega\). As before (Lemma 4.1.2), there are two possibilities for this ordering on components: (i) For any distinct components \(\omega, \omega'\) then \(\omega \leq \omega'\) if and only if \((x, \delta(\omega), \delta(\omega'), y)\) are counter-clockwise ordered, or, (ii) For any distinct components \(\omega, \omega'\) then \(\omega \leq \omega'\) if and only if \((x, \delta(\omega'), \delta(\omega), y)\) are counter-clockwise ordered.

The analogue of Proposition 4.2.1 is virtually the same, we have:

**Proposition 5.2.** Let \(\varphi\) be an order reversing homeomorphism of \(\mathbb{T}\) with cyclic point set \(\mathcal{F}\), and fixed points \(\{x, y\}\) arranged so that \(1 \in [x, y]\). Let \((\alpha_0, \alpha'_0\), \((\alpha_1, \alpha'_1)\) be components of \(\mathcal{F}^c\) contained in \((x, y)\), such that \((\alpha_1, \alpha'_1) \subseteq [\alpha_0, y]\).
Then there exists \( g \in A_\varphi \), such that \( \pi(g) \) is a Fredholm operator for each irreducible \( * \)-representation \( \pi \) of \( A_\varphi \), with

\[
\text{ind}(\pi_{\mu_1}(g)) = 0 \quad \text{when} \quad \mu \in \mathcal{F} \setminus \bigcup_{j=0}^{1} \varphi^j((\alpha_0, \alpha_0') \cup (\alpha_1, \alpha_1')),
\]

\[
\text{ind}(\pi_{\mu_1}(g)) = 1 \quad \text{when} \quad \mu \in (\alpha_0, \alpha_0') \cup \varphi(\alpha_0, \alpha_0'),
\]

\[
\text{ind}(\pi_{\mu_1}(g)) = 1 \quad \text{for all} \quad \mu \in (\alpha_1, \alpha_1'), \quad \text{or} \quad \text{ind}(\pi_{\mu_1}(g)) = -1 \quad \text{for all} \quad \mu \in (\alpha_1, \alpha_1');
\]

and \( \pi_{\mu,\nu}(g) = I \) for all \( (\mu, \nu) \in (\{\alpha_1', \varphi(\alpha_1')\} \cup [\varphi(\alpha_0), \alpha_0]) \cap \mathcal{F} \times \mathbb{T} \).

For each such \( g \), \( \mu_0 \in (\alpha_0, \alpha_0') \), \( \mu_1 \in (\alpha_1, \alpha_1') \) then

\[
\text{ind}(\pi_{\mu_1}(g)) = -\varepsilon(\mu_0)\varepsilon(\mu_1). \tag{5.1}
\]

We remark that for existence, we can take \( g = c + (B - c)\theta(A) \) if \( \varepsilon(\mu_0) = 1 \), where \( \theta \in C(\mathbb{T}) \) is such that \( \theta(\zeta) = 0 \) if \( \zeta \in [\alpha_1', \varphi(\alpha_1')] \cup [\varphi(\alpha_0), \alpha_0] \), and \( \theta(\zeta) = 1 \) if \( \zeta \in [\alpha_0', \alpha_1] \cup [\varphi(\alpha_1), \varphi(\alpha_0')] \). If \( \varepsilon(\mu_0) = -1 \), then the element \( g^* \) has the required properties.

Let \( \omega_0 \) be a component of \( \hat{A}_{\varphi,\infty} \). The above example shows that for each component \( \omega_1 \) of \( \hat{A}_{\varphi,\infty} \) distinct from \( \omega_0 \), there exists \( g \in A_\varphi \) as in subsection 4.2.2. The above example enables us to further insist that \( \pi(g) = I \) for each finite dimensional irreducible \( * \)-representation \( \pi \in A_\varphi \) with \([\pi]\) in a component of \( \hat{A}_{\varphi}(\omega_0 \cup \omega_1) \) containing \( 1 \)-dimensional \( * \)-representations. Via such elements \( g \in A_\varphi \), we can associate a map from \( \mathcal{G} = \{ \text{components of } \hat{A}_{\varphi,\infty} \cup (\hat{A}_{\varphi})^2 \} \) into \( \{-1, 0, 1\} \), as in subsection 4.2.2. Again this association gives rise to exactly two maps; if we let \( \Lambda : \mathcal{G} \to \{-1, 0, 1\} \) denote one of them the other will be \(-\Lambda\).

Define a bijection \( \beta : \mathcal{G} \to (-i, i) \) as follows. Take an enumeration \( \{\omega_j\}_{j \in \mathbb{N}} \) of \( \mathcal{G} \) and let \( c_1 \) be an open arc in \( (-i, i) \). If \( \omega_1 \leq \omega_2 \), let \( c_2 \) be an open arc in \( (-i, i) \setminus c_1 \) such that \( (-i, c_1, c_2, i) \) are counter-clockwise ordered. If \( \omega_2 \leq \omega_1 \), pick \( c_2 \) so that \( (-i, c_2, c_1, i) \) are counter-clockwise ordered. Inductively suppose \( c_1, \ldots, c_k \) are defined. Then one of the following holds:
(i) \( \omega_{k+1} \leq \omega_l \) for each \( l \in \{1, \ldots, k\} \) or
(ii) \( \omega_l \leq \omega_{k+1} \) for each \( l \in \{1, \ldots, k\} \), or
(iii) There exists \( s, t \in \{1, \ldots, k\} \) such that \( \omega_s \leq \omega_{k+1} \), \( \omega_{k+1} \leq \omega_t \), and for each \( l \in \{1, \ldots, k\} \) either \( \omega_l \leq \omega_s \) or \( \omega_t \leq \omega_l \).

If (i) holds, let \( c_{k+1} \) be an open arc in \(( -i, i ) \setminus (c_1 \cup \ldots \cup c_k) \) such that \((-i, c_{k+1}, c_l)\) are counter-clockwise for each \( l \in \{1, \ldots, k\} \).

If (ii) holds, let \( c_{k+1} \) be an open arc in \(( -i, i ) \setminus (c_1 \cup \ldots \cup c_k) \) such that \((c_l, c_{k+1}, i)\) are counter-clockwise for each \( l \in \{1, \ldots, k\} \).

If (iii) holds, let \( c_{k+1} \) be an open arc in \(( -i, i ) \setminus (c_1 \cup \ldots \cup c_k) \) such that \((c_s, c_{k+1}, c_t)\) are counter-clockwise.

For each \( j \in \mathbb{N} \), pick some \( \zeta_j \in c_j \) and define
\[
\beta : \mathcal{S} \rightarrow (-i, i) : \omega_j \mapsto \zeta_j , \ j \in \mathbb{N} .
\]

Having selected a map \( \Lambda : \mathcal{S} \rightarrow \{-1, 0, 1\} \), ordering \( \leq \), and a map \( \beta : \mathcal{S} \rightarrow (-i, i) \), we consider the subset of \( T^* \)
\[
\{ \{[-i, i], \lambda\}, \{[-i, i], -\lambda\}, \{[-i, i], p_\lambda p_{-1}\}, \{[-i, i], -\lambda p_{-1}\} \}
\]
where \( \lambda : \beta(\mathcal{S}) \cup (-\beta(\mathcal{S})) \rightarrow \{-1, 0, 1\} \) is the map
\[
\lambda(\zeta) = \begin{cases} \Lambda \circ \beta^{-1}(\zeta) & \text{if } \zeta \in \beta(\mathcal{S}) \\ -\Lambda \circ \beta^{-1}(\zeta) & \text{if } \zeta \in -\beta(\mathcal{S}) \end{cases} .
\]

This set does not depend on our selection of \( \Lambda, \leq, \beta \), so we denote it by \( T^*(A_\varphi) \).

Finally, we get the order reversing analogue of Theorem 4.4.5, Theorem 4.4.6, and Corollary 4.4.7:

**Theorem 5.3.** Let \( \varphi \) be an order reversing homeomorphism the unit circle. Then \( T^*(A_\varphi) = \{ T^*(\varphi), T^*(\varphi^{-1}), T^*(p_{-1} \circ \varphi \circ p_{-1}), T^*(p_{-1} \circ \varphi^{-1} \circ p_{-1}) \} \), where \( p_{-1}(\zeta) = \overline{\zeta} \) for all \( \zeta \in T \).
Theorem 5.4. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be C*-algebras in the set $\{A_\varphi|\varphi$ is an order reversing homeomorphism of the unit circle}. Then $\mathfrak{A}$ and $\mathfrak{A}'$ are $*$-isomorphic if and only if $T^*(\mathfrak{A}) = T^*(\mathfrak{A}')$.

Corollary 5.5. Let $\varphi$, $\psi$ be order reversing homeomorphisms of the unit circle. The C*-algebras $A_\varphi$ and $A_\psi$ are $*$-isomorphic if and only if $\varphi = \theta \circ \psi \circ \theta^{-1}$ or $\varphi = \theta \circ \psi^{-1} \circ \theta^{-1}$ for some homeomorphism $\theta$. 
CHAPTER 6

The general case

In this chapter, we conclude with the classification of $C^*$-crossed product algebras associated with an arbitrary homeomorphism of the unit circle.

Let $\varphi$ be a homeomorphism of the unit circle, and let $\mathcal{F}$ denote the cyclic points of $\varphi$. There are two possibilities: (i) $\varphi$ is order preserving, (ii) $\varphi$ is order reversing. If $\varphi$ is order preserving, exactly one of the following is true:

(i) (a) $\mathcal{F} = \mathbb{T}$. In this case $\varphi$ is conjugate to a rational rotation (Remark 2.1.5).

(i) (b) $\mathcal{F} = \emptyset$ and every $\varphi$-orbit is dense in $\mathbb{T}$. In this case $\varphi$ is conjugate to an irrational rotation (Poincaré [9]).

(i) (c) $\mathcal{F} = \emptyset$ and there are non-dense $\varphi$-orbits. Such a map $\varphi$ is called a Denjoy homeomorphism.

(i) (d) $\mathcal{F} \neq \emptyset, \mathbb{T}$.

At the $C^*$-algebraic level, (ii) is true if and only if $A_\varphi$ has both one and two-dimensional irreducible $*$-representations, or $\hat{A}_{\varphi,\infty}$ is connected while $\hat{A}_{\varphi,1}$ (classes of one-dimensional $*$-representations) is not. (i) (a) is equivalent to the $C^*$-algebra $A_\varphi$ having all irreducible $*$-representations of the same finite dimension, and no infinite dimensional irreducible $*$-representations (Proposition 3.1.2). (i) (b) is true if and only if $A_\varphi$ is simple (Power [10]). (i) (c) holds if and only if $A_\varphi$ is not simple and it has no finite dimensional irreducible $*$-representations (Putnam, Schmidt, Skau [11]). Finally, (i) (d) is true if and only if none of the above; i.e. all finite dimensional irreducible $*$-representations of $A_\varphi$ have the same dimension, $\hat{A}_{\varphi,\infty}$ and $\hat{A}_{\varphi,f}$ are both non-empty, and if $\hat{A}_{\varphi,f}$ is disconnected so is $\hat{A}_{\varphi,\infty}$. Consequently, we have:
Theorem 6.1. Let \( \varphi, \psi \) be homeomorphisms of the unit circle. The \( C^* \)-algebras \( A_\varphi \) and \( A_\psi \) are \(*\)-isomorphic if and only if there exists a homeomorphism \( \theta \) such that \( \varphi = \theta \circ \psi \circ \theta^{-1} \) or \( \varphi = \theta \circ \psi^{-1} \circ \theta^{-1} \).

Proof. When \( \varphi \) is conjugate to \( \psi \) or \( \psi^{-1} \), it was shown in the introduction that the \( C^* \)-algebras \( A_\varphi \) and \( A_\psi \) must be \(*\)-isomorphic. So assume \( A_\varphi \) and \( A_\psi \) are \(*\)-isomorphic. By the preceding argument, \( \varphi, \psi \) simultaneously satisfy exactly one of the conditions (i) (a), (i) (b), (i) (c), (i) (d), (ii). The case (i) (a), rational rotations, is covered in Høegh-Krohn, Skjelbred [5]. (i) (b), the irrational rotations, is found in Rieffel [12], Pimsner-Voiculescu [8]. (i) (c), the Denjoy homeomorphisms, were analysed in Putnam, Schmidt, Skau [11]. Finally, (i) (d) is just Corollary 4.4.7, and (ii) is Corollary 5.5. ■
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