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DEVELOPING CHILDREN'S RATIONAL NUMBER SENSE: A NEW APPROACH AND AN EXPERIMENTAL PROGRAM

by

Joan Moss

A thesis submitted in conformity with the requirements for the degree of Master of Arts
Department of Human Development and Applied Psychology
Ontario Institute for Studies in Education of the University of Toronto

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ABSTRACT

Rational number, decimals, fractions and percent is the most difficult topic in mathematics that students encounter in their elementary school years. In this study a new model for the development of rational number understanding was proposed and an experimental instructional program was devised and tested. The curriculum approached the teaching of rational number through percents rather than fractions and stressed the use of benchmark percents (50%, 25%, 75% etc.) which were derived from "halving and doubling". The focus of the program was on continuous quantity, ratio and measurement. A comparison class of 17 students of similar demographic characteristics and matching mathematical abilities received a traditional curriculum. Students from the experimental group were more successful in all aspects of rational number performance. Flexibility of movement among representations, understanding of magnitude, the ability to perform complex computational items using invented strategies were among the general characteristics that emerged.
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## CONTENTS

### Chapter 1
1. Introduction
   1.1 Difficulties in rational number learning and understanding 1
   1.2 Epistemological research 2
   1.3 Teaching experiments 3
   1.4 Results and recommendations of the teaching research 6
   1.5 Number sense 7
   1.6 Summary 8

### Chapter 2
2. A New Approach To The Teaching Of Rational Number 10
   2.1 Traditional sequence for rational number teaching 10
   2.2 Problems with the part-whole model: Rationale for the change of sequence of rational number teaching 11
      2.2.1 Part-whole teaching of fractions and problems of representation 11
      2.2.2 Part-whole teaching of fractions and discrete vs continuous quantity 12
   2.3 The benefits of teaching percents as a global overarching environment for rational number learning 14
   2.4 The components and rationale of the experimental rational number curriculum 20
      2.4.1 Core representation 20
      2.4.2 Rationale for the teaching methods 21
   2.5 A day in the life of the percent curriculum 22
   2.6 Purposes and hypothesis 24

### Chapter 3
3. Method
   3.1 Subjects 26
   3.2 Design 27
   3.3 Procedure 27
3.4 The experimental rational number curriculum

3.4.1 Instruction schedule

3.4.2 Sequence of program

3.4.3 The lessons; an overview

3.4.3.1 Estimating percents (Lessons 1 through 4)

3.4.3.2 Computing percents (Lessons 5-7)

3.4.3.3 Introduction to decimals using stopwatches (Lessons 8 - 11)

3.4.3.4 Learning about decimals on number lines (Lessons 12 - 16)

3.4.3.5 Playing and inventing decimal board games (Lessons 16 to 19)

3.4.3.6 Fractions (Lessons 19 to 21)

3.4.3.7 Review (Lessons 22 to 25)

3.5 Rational number program for the control group

3.6 Assessment measures

3.6.1 Rational number test

3.6.2 Rationale for five "rational number sense" Subcategories

3.7 Scoring procedures

Chapter 4 Results

4.1 Overall results of rational number test

4.2 Interchangeability

4.3 Compare and order

4.4 Visual perceptual distractors

4.5 Standard algorithms

4.6 Nonstandard computation

4.7 Word involving real world content

4.8 Canadian test of basic skills

4.9 An example of expertise in rational number sense thinking
Chapter 5

5.1 Summary of purposes and hypotheses

5.1.1 Summary of the results

5.2 Other gains and their relationship to the curriculum

5.2.1 Overcoming whole number biases

5.2.2 Affective engagement

5.3 Core structure and its accessibility to intuitive reasoning

5.3.1 Core representation

5.4 The core representation as a context or tool for fostering students' intuitions of proportion, ratio and percent.

5.5 The embeddedness of subconstructs of rational number in the experimental curriculum.

5.5.1 Rational number as quantity

5.5.2 Quotient/ratio subconstructs

5.5.3 Quotient subconstruct

5.5.4 Ratio subconstruct

5.5.5 Operator subconstruct

5.5.6 Measure subconstruct

5.6 Limitations of the experimental curriculum.

5.6.1 Limitations of the halving schema

5.7 Curricular implications

5.7.1 Place of percent teaching in the curriculum

5.7.2 A global number sense approach to teaching

5.7.3 Focus on decimals over fractions

5.8 Further research questions arising from this study

5.9 Methodological limitations

5.10 Conclusion

Bibliography

Appendices
LIST OF TABLES

Table 4-1: Comparison of Mean Pretest Scores on the Rational Number Test
Table 4-2: Comparison of Mean Pretest Scores on the Individual Subtests on
the Rational Number Test
Table 4-3: Comparison of Mean Posttest Scores between Experimental and
Control Groups on the Rational Number Test
Table 4-4: Comparison of Mean of Expanded Posttest Scores between
Experimental and Control Groups on the Rational Number Test
Table 4-5: Responses (with Percents) to Test Items at Pre and Posttest which
Assess Ability to use Rational Number Representations
Interchangeably
Table 4-6: Responses (with Percents) to Test Items at Pre and Posttest on Items
Demonstrating the Ability to Compare and Order Rational
Numbers
Table 4-7: Responses (with Percents) to Test Items at Pre and Posttest which
Include Visual Perceptual Distractors
Table 4-8: Responses (with Percents) to Test Items at Posttest of Standard
Computational Problems
Table 4-9: Responses (with Percents) to Test Items at Pre and Posttest of
Nonstandard Computational Problems
Table 4-10: Percentage of Students Scoring on Real World Questions at Pre
and Posttest

vii
LIST OF FIGURES

Figure 4-1. Pretest and Posttest Scores for Rational Number Test, Experimental and Control Groups

Figure 4-2. Pretest and posttest scores for math computation; Canadian Test of Basic Skills

Figure 4-3. Pretest and posttest scores for math concepts; Canadian Test of Basic Skills
CHAPTER 1
INTRODUCTION

1.1 Difficulties in Rational Number Learning and Understanding

The field of rational numbers, which includes fractions, decimals, and percents, is the most complex domain in mathematics that students encounter in their elementary school years. Scores from assessments of student achievement consistently show that students face significant problems in learning the rational number system. The errors that students make on such assessments are well documented, and illustrate the misconceptions that they develop. For example, on an item that appeared on a national standardized test $1/2 + 1/3 = \, ?$, 30% of 13-year-old students chose 2/5 as the correct answer (Post, 1981). This answer not only reveals faulty procedural knowledge but also a clear lack of conceptual knowledge. In the second National Assessment of Educational Progress, one half of 13-year-olds were unable to tell which is greatest of .19, .036, 195, .2 (Carpenter, Corbit, Kepner, Linquist, & Reys, 1980). A third example comes from the same national assessment measure. When asked to estimate the sum of $9/10 + 11/12 = \, ?$, 67% of grade 11 and 12 students chose 19 or 20 as the answer instead of 2 in a multiple choice format (Carpenter et. al., 1980). Indeed, there is a developing body of research that describes the failure of adults to understand and perform successfully in the domain of rational number (Grossman, 1983; Silver, 1986; Sowder, 1995). These findings are a serious cause for concern as rational number forms a basis for algebra and higher mathematics. Students who encounter difficulties fail to go on in their
mathematics studies; the door to many future professional possibilities is thus closed to them.

1.2 Epistemological Research

A great deal of research has focussed on the reasons for the foregoing difficulties and has found that these problems are in part attributable to deficits in traditional educational practice. Criticism is directed at the style and coverage of textbook rational number units, as well as at the order of instruction and representative models traditionally offered by these texts. In teaching the whole number system, for example, textbooks provide exercises and manipulative experiences that allow students to gain conceptual understanding before learning procedures and algorithms. The exercises typically offered in rational number curricula are significantly different: In these lessons the material is covered very quickly with an emphasis on symbols rather than on meaning. Operations are taught in isolation, divorced from meaning and natural activity (Post, Cramer, Behr, Lesh, & Harel, 1993). As well, textbooks give little attention to students' informal knowledge in this area and the style of presentation of the lessons denies students the opportunity to construct their own understandings (Hiebert & Wearne 1986; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Armstrong & Bezuk 1993; Sowder 1995).

Although these problems with instruction have been widely acknowledged, there is growing evidence that the rational number system has complexities which the whole number system does not. As a result, an impressive body of research has focused on explicating these complexities and methodically
defines its elements, (i.e. part/whole comparison, decimal, ratio, measure of continuous or discrete quantity, quotient, operator etc.) (Kieren, 1976, 1988, 1993; Behr, Harel, Post, & Lesh, 1992).

One outcome of this work has been a summary of the knowledge structures students must acquire, in order to perform with competence and understanding in the above subdomains. Although there are some variations in how these subdomains and subconstructs are named and defined (Kieren 1976; Freudenthal, 1983; Ohlsson 1987), there is consensus among researchers that these subconstructs underlie rational number understanding and that they must all be incorporated by the students in order for this number system to be understood.

1.3 Teaching Experiments

The foregoing research has led to a number of studies which have experimented with more appropriate instructional approaches. The educational analyses and instructional methods that have ensued from this research are broadly based around the knowledge structures that have been identified.

In the forefront of this research is the Rational Number Project (RNP) which has been researching children's learning of rational number concepts since 1979. This impressive project has provided a very solid background for understanding the epistemological constructs of the number system and children's acquisition of this knowledge. The researchers have conducted experimental studies (Cramer, Post, Behr, 1989), surveys (Heller, Post, Behr, &
Lesh, 1990) and teaching experiments, (Behr, Wachsmuth, Post & Lesh, 1984; Behr, Wachsmuth & Post 1985). The teaching experiments have been conducted with students in grades 4, 5, and 7 and have focused on the processes of rational number concept development. These experiments included controlled, detailed lesson plans between 12 and 30 weeks in duration. Data were collected on the learning process as it occurred. The researchers documented the "depth and direction of students understanding resulting from interaction with carefully constructed theory-based instructional materials" p. 326 (Behr, et. al., 1992). The teaching was carefully constructed to interweave a wide variety of embodiments or contexts, for example, cuisenaire rods, paper folding, colour coded circular pieces, etc. with the rational number subconstructs. Spoken symbols and real-world situations were introduced as additional modes of representation and children were taught to use these various representations to learn rational number subconstructs. Translation among these modes was also encouraged. These teaching experiments proved to be beneficial to the participating students; however, many problems still persisted.

For example, in one of these teaching experiments, part/whole activities were used to base a curriculum on the subconstructs of "measure" "quotient" "ratio and "operator" as sources of representations for children's thinking. This study focused on teaching fractions to grade four children. The researchers found that gains in fraction understanding were slow. After instruction students were able to understand, for example, the compensatory relationship between the size of $\frac{1}{n}$ and the number $n$, however, they still experienced difficulties answering fraction questions with re configured units.
Another clinical teaching experiment from the Rational Number Project focused on the understanding of order and equivalence of rational numbers. This experiment, conducted with grade four students, relied on the use of manipulative aids to highlight concepts involving order and equivalence (Behr, Wachsmuth, Post, & Lesh 1984). After extensive instruction most children were successful but some continued to demonstrate inadequate understanding. A significant gap was found between students' abilities, on the one hand to use manipulative aids, and on the other to operate with symbols. It was suggested that specific attention needed to be directed toward helping students to implement the various translations as the "the mental bridge to cross the gap is complex." (Post, 1988, p. 15-16).

Hiebert and Wearne are another team of researchers who have done extensive work investigating students' understanding and performance in the domain of rational numbers, with particular attention to decimals. They have analyzed the knowledge structures that students must acquire and devised many teaching experiments which monitor the students' understandings and provide teaching support for deep understanding. In a study of middle school students, they instructed small groups in the semantic processes for solving decimal addition and subtraction problems (Wearne & Hiebert, 1988). The focus of this study was to help students use semantic processing instead of syntactic processing for solving these problems. The students were taught to associate decimals with Dienes base-10 blocks as an alternate representation to written symbols. A series of nine instructional lessons was designed to help students create meaning to solve problems that were posed symbolically. This experimental instruction was successful for most of the students who had not received previous instruction in decimals,
but of the 15 students who had received prior instruction, only 33% changed to using semantic analyses.

1.4 Results and Recommendations of the Teaching Research

Although these studies have generated student gains in controlled circumstances, a caution is in order. To date, the evidence suggests that addressing the knowledge structures which comprise the rational number system individually does not provide the kind of overall competence and facility with rational number skills and concepts that is needed. In their recent recommendations for directions for curriculum development in rational number, the RNP team (Post, et. al., 1993), observed that more instructional attention needs to be placed on students' integrated use of and access to the totality of the rational number domain. They conclude that the "curriculum developers' attention should be directed away from the attainment of individual tasks toward the development of more global "cognitive processes" in order for students to gain "flexibility in coordinating translations and the emergence of embodiment-independent thought." (p. 343).

This focus on flexibility of thinking is also echoed in the work of Judith Sowder et al. (Sowder, Bezuk, & Sowder 1993) in their research on rational number sense. Sowder asserts that flexibility is essential in operating in the domain of rational number and that rational number sense is the linchpin for this flexibility. She notes that several research reports have emphasized the importance of the development of a quantitative notion or "an awareness of bigness for fractions and decimals." (Behr, Wachsmuth, & Post, 1985;
Sowder & Markovits, 1989). It is the understanding of quantity of rational numbers that enables students for example, in the realm of fractions, to "perceive relative sizes of fractions, to compare and order fractions using size concepts and benchmarks, to find or recognize equivalent fractions, and to estimate the location of a fraction on a number line." Sowder, (1992), asserts that the "ownership of this (quantitative) notion is the essence of rational number sense" (p. 246).

1.5 **Number Sense**

Number sense is a notion that has gained a great deal of prominence in recent years. In fact, the authors of "Everybody Counts: A Report to the Nation on the Future of Mathematics Education" (National Research Council, 1989) state that the "major objective of elementary school mathematics should be to promote number sense." (p 46). Although a definition of number sense is elusive, the characteristics of people in whom it is well developed have been noted. Some of its manifestations include: using numbers flexibly when mentally computing, inventing procedures for calculations (Lemke & Reys 1994), estimating and judging number magnitude, showing good quantitative judgment and being able to judge reasonableness of answers. Finally, students with good number sense have been shown to be able to move easily between number representations as well as to represent numbers in multiple ways (NCTM, 1989; Case & Sowder, 1990; Markovits & Sowder 1991; Kieren 1995).

Greeno (1991) makes an important conceptual shift in characterizing number sense. Rather than analyzing subconstructs and cognitive structures as is
traditionally done, he recommends using the spatial metaphor of a neighbourhood. Greeno suggests that conceptual domains can be likened to environments "in which people can know how to live." (Greeno, 1991). In finding their way around their own neighbourhood, for example, children demonstrate a good understanding of relative distances of the location of houses and streets. They can reason about the relative efficiency and speed of different routes from one place to another. They can discern subtle patterns and solve routine and novel problems and they have knowledge of the salient landmarks that exist. Greeno asserts, "This metaphor highlights the multilinear and multiconnected nature of knowing in a domain," (Greeno 1991, p. 46).

The implication of Greeno's analysis is that a crucial aspect of mathematics competence is not captured by the micro analysis of cognitive subconstructs. Thus a need exists for an overarching context in which a rich and varied set of connections is established among the constructs. This notion fits well with suggestions made by Sowder, (1995), on number sense and with the suggestion by Post, et al. (1993) on teaching, that goes beyond the teaching of rational number tasks or subtasks in isolation.

1.6 Summary

Whereas the current rational number research provides a very useful model of the knowledge domain which students must grasp, and many partially successful teaching attempts have resulted, there is evidence that this research model has not yet provided a sufficient basis for creating
instructional programs to promote a broad based and quantitatively oriented understanding of this number system.
CHAPTER TWO
A NEW APPROACH TO THE TEACHING OF
RATIONAL NUMBER

The purpose of the present study was to create a curriculum that would fit the general specification outlined in the previous chapter: that is, a curriculum that would provide an overarching cognitive context for rational number "sense," and enable students to move flexibly among the various "modes" or "elements" of the rational number system. In this chapter, I will first describe the traditional sequence for teaching rational number. Next, I will provide a rationale for the change in this sequence that was implemented in the present study. Following that, I will offer an analysis of the benefits to young students of one aspect of this sequence: the introduction to rational numbers through teaching percent. Finally, I will explicate the core components of the experimental curriculum and delineate the social constructivist approach that influenced its implementation and delivery. I will conclude with an excerpt from a lesson.

2.1 Traditional sequence for rational number teaching

In traditional programs, common fractions are first taught to students in Grade 2 or 3. At these levels, fractions are presented as parts of wholes. The most common representations students first encounter are shaded regions of geometric figures — e.g., students are presented with one-part shaded of four evenly partitioned "pie" segments, and told that the shaded piece is one-part of four, or one-fourth (1/4). In some programs, the fraction bar is said to symbolically indicate the words "parts of". In the usual sequence of fraction
teaching, the benchmarks "one-fourth" and "one-half" are taught first, tenths are then introduced as a way of creating a bridge to the learning of one-place decimals. Decimal lessons begin with the notion of their equivalency to fractional tenths. Soon after that, students learn to add and subtract decimals by using the rule of lining up the decimal points. Percents are introduced between Grades 6 and 8 in a rule-dominated fashion, as "missing value" problems. Students are told to convert the percent to a decimal and then use an appropriate operation to compute an answer. In current practice, therefore, the common fraction representation of rational number serves as the core building block for subsequent learning and the part-whole model dominates the introduction to fractions.

2.2 Problems with the part-whole model: Rationale for the change of sequence of rational number teaching.

2.2.1 Part-whole teaching of fractions and problems of representation
The part-whole model for teaching fractions has intuitive appeal for young students, as it taps into their existing whole number counting schemas. Students often interpret these fractions in a subtractive fashion, as in the case of 1/4, where they see the quarter as being removed or subtracted from the whole (Case, personal communication). However, it has also been reported that the visual representations offered to students become the ones they rely on, (Sowder, et. al., 1993), and are often carried over into adulthood where they support misconceptions that have great resilience (Kerslake, 1986; Silver, 1986).
Kieren has said that this element of part-whole thinking is "completely visual, static, and only nominally related to the mathematical notion that a unit can be divided into any number of equal parts, and that each part is in itself is an independent entity or amount," (Kieren, 1995, p. 37). Analyses of student errors consistently point to the problems that this limited representation supports. Kerslake (1986), reports that the 13 and 14 year old students that were surveyed in one study on fraction understanding were familiar only with the part-whole model, and that they were unable to attach any meaning to equivalent fractions. In fact, these students had trouble seeing fractions as numbers at all and often asserted that they were "two numbers put on top of one and other," (pg. 246). In Silver's (1986) study of first year college students he reports that these young adults almost exclusively relied on a single mental and physical model for fraction concept: a part-whole model for fractions expressed as sectors of a circle. This study also reported the resilience of the "Freshman Error" which describes students' misconceptions about fraction addition: they added the numerators and the denominators (e.g. 1/2 + 1/3 = 2/5), even after remedial conceptual instruction was given in the course of the study. He found that their reliance on the pie-chart visual representation not only reinforced the erroneous answer but also the faulty strategy used to obtain the answer. (Silver, 1986)

2.2.2 Part-whole teaching of fractions and discrete vs continuous quantity
A second, related problem is that the initial part-whole teaching of fractions highlights the discrete rather than continuous interpretation of rational

1As with most incorrect strategies, this strategy would be appropriate in certain limited circumstances. E.G. in a ratio: if I get 1 hit in my first 3 times at bat, and 2 hits in my next 5 times at bat, then in this situation my overall batting average is 1/3 + 2/5 = 3/8.
number. Quantities referred to as discrete are countable and are measured with whole units. Returning to the example of the circular region that is partitioned into four, students understand the quarter-shaded piece as one of four units that make up the whole, and can interpret these fractions in countable terms. Continuous quantity on the other hand is quantity that can be measured to any degree of accuracy, with parts of units as well as whole units. Continuous quantity includes the property of "denseness" (Hiebert, 1992) – the notion that a third number can always be inserted between two numbers – a concept easily represented on a common meter stick.

To understand the decimal system requires a conceptualization of continuous quantity. As described in Chapter One, many researchers have noted the difficulty that students have with quantitative understanding of decimals. Hiebert (1992) asserts that students' previous instructional experience in the whole number domain and their part-whole fraction learning interfere with their conceptual and quantitative understanding of decimals. Although the shift to decimals flags an entry into the world of continuous rather than discrete quantity, students numerical understandings have been hitherto deeply rooted in discrete interpretations of quantity. As well, they are presented with a notation system highly similar to that of whole number. Symbolically and linguistically, then, there is no real support for the shift in quantitative meanings from discrete to continuous. The net result is that current approaches to teaching decimals, by focussing on fractions and the part/whole representation, actually make it more difficult, rather than easier, for children to develop a good intuitive feel for the rational nature of decimal numbers.
2.3 The Benefits of teaching percents as a global overarching environment for rational number learning

To correct the foregoing problems, Case, (personal communication) has suggested that understanding of decimals should be more directly rooted in the system of percents, which should in turn be rooted in students' intuitions about continuous quantity and ratios. The proposal to introduce the rational number system by teaching percents, a topic that has always been considered one of the most difficult for students to grasp, (Parker, & Leinhardt, 1995) may at first seem counterintuitive. However, it is consistent with suggestions that have been made in the mathematics education literature.

Researchers and educators have generally agreed that percent is a ratio comparison. "Mathematically, the appearance of percent language flags the entry into an intensive world of comparison between quantities," (Parker, & Leinhard 1995, p. 438). Working with percents allows children access to their intuitive understanding of ratio. When children say that a job is 99% done, they naturally understand that this statement can be made for big or little jobs, because it refers to the ratio of what has been undertaken, to what has been completed. The teaching of percent presents students with a description of a proportional relationship between two quantities: when the students are comparing one beaker 50% full, with another, different-sized beaker also 50% full, they are naturally making ratio and proportional assessments.

Parker and Leinhardt (1995), in their review, describe percent as a "privileged proportion," in that it privileges a particular base (100). They assert that the availability of this base offers some of the following advantages.
"Students can: a) locate on a scale of 0 - 100 the size of a part as it relates to a whole; b) locate on an unbounded scale the multiplicative relationship between two referent quantities, compare the magnitude of these relationships quickly based on the natural order of the decimal numeration system." (P. 438)

In my pilot study for the present project I discovered several other ways in which young (Grade 4) students may benefit from the introduction of percents. When students were initially presented with percent "challenges," they spontaneously used a variety of invented strategies to solve them. For example, they initially used a halving strategy in order to estimate and find solutions. To solve the problem, "What is 25% of an 80 cm piece of string?" Students first asserted that 50% would be 40 cm and therefore 25% would be half of that. i.e. 20 cm.

This halving strategy has been noted by many researchers (Kieren 1976; Behr Lesh, Post, & Silver, 1983; Vernaud 1988; Confrey, 1994; Steffe, 1994; Streefland, 1994; Confrey, & Smith, 1995). This primitive intuitive ability of splitting, based on actions such as sharing, dividing symmetrically, growing, magnifying and folding, has been shown to be present in very young children's thinking. (Confrey, 1994). Confrey distinguishes between multiplicative and additive schema in the following way, "Whereas in additive operations (schemas) counting starts at 0 and the successor action is adding one with the unit being one, in a multiplicative splitting world counting starts at 1 and the successor action is splitting by n." (Confrey, 1994, p. 292). Kieren's extensive research and teaching studies have also
demonstrated the strength of the splitting scheme in children's development of multiplicative understanding.

Another strategy that students in my pilot class invented for working with percent challenges appeared to involve spontaneous and appropriate movement between additive and multiplicative thinking. For example, children seemed to easily grasp the notion that to find 75% of a quantity it was first necessary to decompose the 75% into 50% and 25% and then add the two resulting quantities to find a solution. These tendencies were also noted by Lemke & Reys, (1994) in their developmental assessments of percent learning. The ability to formulate and reformulate the unit, a conceptualization that is reportedly difficult for students to acquire, appears to be naturally present in their efforts, in this domain.

In my pilot work on introducing children to percents, I also found they were naturally able to use their substantial existing proportional reasoning skills. For example, when students worked with cut-out paper dolls of different heights they were capable of making astute visual judgments and could thus correctly pair sets of dolls that were in the same proportion to each other. Resnick and Singer refer to this faculty of making non-numerical proportional assessments as "protoquantitative covariation/ratio schema". In their developmental account of ratio and proportional reasoning, they point out that when computing ratio tasks where numbers are involved students initially abandon their protoquantitative ratio intuitions and rely on

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2The term protoquantitative refers to the pre numerical or a qualitative appreciation of a mathematical relationship and the covariation schema refers to children's sensitivity to patterns of direct covariation between two series so that they know that, for example, things grow and shrink proportionally.
additive strategies to try to solve these multiplicatively based operations. When seeking quantified solutions to relational problems children rely on repeated addition and additive compositions.

Developmentally this fits well with Case's neo-Piagetian theories of mathematical development. (Case, 1985, 1992, 1996) According to Case, children's understanding of whole and rational numbers develop in a manner that is very similar. In each instance, children's numerical and global quantitative schemas develop separately at the outset. Then, as they make the transition to a higher level of cognitive development, they gradually coordinate these two schemas, to yield a core understanding of the way in which the "easy" (e.g. low) numbers in the field in question are structured. This core understanding is then extended to harder (e.g. "higher") numbers, until the structure of the entire field is understood. Case has done extensive work in the field of children's acquisition of whole numbers. He has found that the understanding of this number system develops in a series of stages. In the first stage, which occurs as children enter the stage of concrete operations, children begin to coordinate their separate "primitive" schemas for counting with their non-numerically based schemas of global quantity comparison. The coordination of these two structures yields a "mental number line" at about the age of 6. Coordination of different mental number lines with each other yields an understanding of the part-whole structure of addition which is solidly in place by the age of about 8. Finally, this understanding leads to an understanding of the part-whole grouping that underlies the base-ten system, and this understanding gradually extends to include the entire field of whole numbers by the age of about 10 years (Resnick, 1983; Griffin, Case, & Seigler, 1994; Case & Okamoto, 1996; Griffin
The development of proportional reasoning has been hypothesized to occur in a similar way. Children must begin to coordinate their well developed schemas for multiplication and division of the additive-based whole number system, with their protoquantitative, ratio schemas, which provides the conceptual underpinnings for the synthesis of proportional reasoning and ratio operations and form the foundation of rational number. Coordination of these two structures at the age of 11-12 yields the first semi-abstract understanding of relative proportion, which is then coordinated, very gradually, over the next several years to yield an understanding of the way in which proportional and additive thought can be merged (as in the decimal system) and ultimately to a full understanding of the way in which ratios, decimals, fractions and percents are related. (Case & Sandieson, 1988; Resnick & Singer, 1993).

However, although whole number seems to be universally acquired, Case suggests that rational number is not (Case, 1985), and for unschooled adults does not necessarily develop at all. The literature widely reports the difficulties that students have with the ability to use proportional reasoning or multiplicative strategies to solve ratio problems. Resnick concurs and makes the claim that there is no smooth transition from protoquantitative to numerical operations in ratio. This phenomenon, she asserts, is in contrast to the development of addition and subtraction operations where students intuitive non-numerical constructs are readily applied.
In the pilot study, however, I observed that percent problems offered an opportunity to engage students in concrete examples which allow them to begin to assign numerical contexts to their existing (protoquantitative) ratio schemas. In my experience, when the numbers in the tasks readily lent themselves to splitting (e.g. even numbers), the children were able to successfully accomplish numerical ratio operations. Thus this context appeared to allow children to begin to integrate their protoquantitative ratio thinking with numerical ratio operations.

Finally, I noticed in the pilot study, that children already possessed knowledge about percents and expressed a genuine interest in working with this topic. Percent teaching thus allowed children to take advantage of their everyday language and the knowledge they derive from popular culture.

In summation, then, 1) Percent offers a ratio comparison and is a useful vehicle for students learning comparison between quantities; 2) Its privileged base of 100 provide the opportunity for magnitude comparisons and a solid understanding of quantitative notions; 3) Percent allows spontaneous use of invented strategies, particularly the halfing strategy; 4) It supports children's protoquantitative ratio and proportional understandings and provides a conceptual bridge to the field of rational number; 5) As just mentioned above, this topic generates genuine interest in the students because of their substantial knowledge base of percents from real world examples.
2.4 The Components and Rationale of the Experimental Rational Number Curriculum

The core idea which ran throughout the entire curriculum, and which suggested the initial focus on percents, was that children should be presented with a core representation of rational number that would be simple, and consistent with their intuitions about ratio and continuous quantities as well as whole numbers and discrete quantities. This core representation would then be sustained throughout the curriculum, and used to integrate its various parts. In effect, it would serve as the core image or intuition, around which children's further conceptual understanding could be built in the same way that the image of a "number line" serves as the core image around which children's knowledge of a whole number system is typically built (Griffin, Case, & Seigler, 1994; Case & Griffin, 1996).

2.4.1 Core representation

The core representation that was developed for this purpose was a two-dimensional, vertical, narrow rectangle, with some part of the bottom shaded in. One edge was marked in percent, with only the 0 and 100 indicated, the other edge was marked numerically from 0 to n where n could represent the numerical total length or volume of some three-dimensional object (e.g. a beaker, a rug, etc.).

This representation was derived from the initial lessons. The children had to estimate the degree of "fullness" of different containers. The initial props used were cylindrical beakers containing various amounts of water, and large
drainage pipes of assorted lengths, each partially covered by a moveable venting tube. Both these props provided a "side view" that made it easy to draw the proportion of the total object that was covered, using this "number ribbon" style diagram. These amounts were described as a percent of the whole. The students were initially engaged in exercises where they assessed the quantity of liquid in different beakers, or the length of tube that was covered, then used a "halving" strategy to compute the "percent" of the total volume that a particular quantity occupied. Thus, the experimental curriculum began with exercises that were designed to ground students' introduction to rational number in a conceptually simple and perceptually accessible way of representing ratios.

This initial introduction of percents was carried over into their learning of decimals and fractions, in the manner that is described in Chapter 3.

2.4.2 Rationale for the Teaching Methods

Many different sources informed the choice of methods used in delivering the curriculum. The general approach of the teaching was social constructivist in nature and was one that I had found successful in previous teaching experiences. A number of the methods drew on work of such teacher-researchers as Lampert, (1990), Ball, (1993), Streefland, (1993) and Mack, (1993, 1995). The core principle of constructivist teaching is the notion that mathematics understanding is actively constructed rather than passively received, and involves students individually and collectively in finding patterns and building conceptual knowledge in the mathematical domain. Among the social constructivist tenets that I employed were the inclusion of
mathematical discourse, where students explain their own thinking and theories as well as listen and comment on the conjectures of others; a focus on the importance of process, and the establishment of real world connections to mathematics.

I also included other participation structures that I had been using in my classroom. Students were involved in a variety of different types of projects where they took on the role of experts or "consultants" to others, thus allowing them to consolidate their knowledge skills and take a metaposition towards them (Meichenbaum, & Biemiller, in press). Some of these activities involved the students in designing lesson plans, peer coaching, the inventing of teaching games and diagnostic tools as well as project design and presentations.

2.5 A day in the life of the percent curriculum

The following is an excerpt from one lesson that illustrates the type of discussion and problem solving that the students engaged in.

The lesson involved students' inventions of "percent measurement devices". Pairs of children invented devices to show or teach percents and were invited to demonstrate their inventions and describe how they would be used. Stephen was the first student to volunteer (his partner was absent that day). For their percent device, he and his partner had selected a glass jar that could hold a total of 600 ml of sand. They had marked (calibrated) one side of the jar on the outside with horizontal strips of masking tape placed to (exactly) indicate how to fill the jar to 25%, 50% and 75% full. They had also written
the corresponding volumes (quantities) on the tape - e.g. 25% = 150 ml, 50% = 300 ml., etc.

Stephen stands in front of the class holding his jar. The sand is filled to the 25% mark.

Stephen (S): This is our device. It holds a total of 600 ml, and we have marked 25%, 50% and 75%. It is not finished yet, because we are also going to put tape on the other side showing 10%, 20% and 30%, up to one hundred.

Teacher (T): I see, and how many ml's will of sand will there be if you filled the jar 10% full.

(S points to the place on the jar where he estimates the 10% point to be.)

S: I think it will come to about here.

T: Do you know how many ml's of sand there would be if you filled it 10% full?

S: I'm not sure yet.

T (to the class): Does anybody have an idea?

Alexander (A): Well... I think it's going to be about between 55 ml and 70 ml.

T: What makes you say that?

Z: Well, 12 1/2% would be 75, so it has to be less than that. 6 1/4% would be 37 1/2, and its a lot closer to 12 1/2% so I think that maybe it's about 65.

Signy: Wait, I have an idea! 10 is 10% of a hundred, so 20 must be 10% of two hundred and 30 is ten percent of three hundred, so 60 is ten percent of 600.
Through this kind of discourse I hoped to steer the children's thinking towards a flexibility and ease in operating in this number system and to awaken in them a deep understanding and engagement in a rational number world.

Further examples of these lessons are available on video. This includes 8 different segments from the following lessons: making of percent measurement devices; an introduction to decimals using stopwatches; two different lessons showing students' inventions of mixed-representation equations; two student-made videos to teach percent, one that uses the representation of a building blowing up, a second that uses an invented "math-a-long terror quiz;" a paper-and-pencil lesson that highlights a student's discovery of a rational number principle that shows her comprehension of the multiplicative nature of rational number; children playing their invented board game, the "Dragon Game, " that allows them to move through a series of number lines to practice addition and subtraction of decimals as well as to learn to make judgments of decimal magnitude.

2.6 Purposes and hypothesis

This present study had three purposes: 1. To find an initial cognitive context and "image" that would support deep rational number understanding; 2. To create and implement an experimental curriculum based on this context, with the teaching of percent as a basis for translating between representations; 3. To test the impact of this curriculum by assessing changes in children's performance and understanding, and comparing their rational number mastery to that of a more traditionally taught group.
It was hypothesized that an experimental rational number curriculum that promoted the integration of proportional reasoning with additive schemas, and introduced this number system through teaching percent, would enable students to learn rational number with more conceptual understanding than traditionally taught children. Evidence for this deeper understanding would be shown in the students' ability to: 1. Demonstrate a greater understanding of the relationship among percents, decimals and fractions and be able to use these representations interchangability. 2. Show an increased understanding of number magnitude that would enable them to compare and order decimal fractions and other fractional numbers more successfully than traditionally instructed students. 3. Be better able to overcome the influence of visual perceptual distracters and to discriminate between relevant and irrelevant variables. (For the importance of this as a criterion, see Behr, Lesh, Post and Silver, 1989) 4. At the same time as demonstrating superiority in these conceptual areas, it was also anticipated that the experimental group would perform equally well to traditionally trained students in basic rational number computational problems.
CHAPTER 3
METHOD

3.1 Subjects

Thirty-four grade four students participated in this study. The experimental group was composed of sixteen grade four students from a laboratory school located at the University of Toronto, which has a population of predominantly white middle and upper-middle class students. These sixteen students were part of a class of twenty children. Four of the students in the grade four class received remedial (withdrawal) lessons that took place at the same time as the experimental classes. The grade four teacher felt that they should not miss their special program and decided that these four students should not participate in the study. The mean age of the subjects was 9.2 at pretest and the range was 8.10 to 9.7. Eight boys and eight girls comprised the sample. None of these children had received any previous instruction in either decimals or fractions, although they had been given a brief introduction to percents during their grade three year.

A comparison class of seventeen grade four students was drawn from a private school near the University and served as the control group. This school has a comparable population and serves students of average and above average abilities. Eight boys and nine girls participated. The mean age in this sample was 9.4 with the children ranging in age from 8.11 to 9.5. These students had received 15 lessons in decimals and fractions as part of their grade three program. This school was selected because of its similarities to the
Lab school: both schools have strong academic programs, small classes, and an approach to teaching that reflects their focus on the individual child.

3.2 Design

Pre-tests, which consisted of the three sections of the Rational Number Test (see Appendix A) were administered to all the students from both the experimental and control classes in the fall of 1994. The subjects from the two grade four classes received their pre-tests on an individual interview basis. An experimental rational number curriculum was taught to the experimental students by the researcher, once a week, for forty minutes from October until May. These students also received a standard Grade Four mathematics program that was taught by their classroom teacher. (These lessons did not include any rational number content.)

The grade four control class received standard text book instruction in rational number over the course of their year. These units were covered over a four month period and took approximately 20 hours to teach. An extended version of the Rational Number Test was administered to all of the grade four subjects as a post-test measure.

3.3 Procedure

In the Fall, prior to the start of the experimental instruction, the Rational Number Test was administered as a pretest interview to each of the grade four subjects on an in individual basis. The researcher administered one half of the tests and a graduate student who was trained to perform the interview
administered the rest. The interviews were standardized and all the students' answers were recorded verbatim by the interviewers. Six of the tasks; items 9, 11, 16, 20, 40 and 41, probed for the children's explanations of their solutions. The students' responses to these probes were also recorded in the individual interview protocol for each student.

The children were withdrawn from their regular class and brought to a quiet room. Administration time for the pretest varied from 25 minutes to 45 minutes according to the knowledge level of the student. The test in its entirety was read aloud to the students and the students were given the opportunity to respond to all of the items on all three tests. For specific tasks the interviewer used blocks and other small props to visually represent numbers or concepts.

Four weeks after the experimental program had been completed, the post test interviews were administered. These post instructional interviews were administered in an identical fashion to the pretests. They were presented with the same tasks that they had been given in the pre instruction interview with the inclusion of seven new tasks. The posttest is shown in Appendix B. (The new tasks that were added for the posttest are highlighted in bold print on the test pages.) These interviews ranged from 25 minutes to 70 minutes according to the needs of the individual subjects from both groups.

Canadian Test of Basic Skills CTBS, (Nelson, 1988) reading comprehension, math concepts and math computation subtests were administered in a group test situation to all the students in the experimental group in June of their
grade three year and then again in June of the grade four year. These score were also used as a pre-and posttest measures for this study.

3.4 The Experimental Rational Number Curriculum

3.4.1 Instruction schedule:

The experimental sessions were approximately forty minutes in length. All of the lessons were documented and selected classes were video-taped. Each lesson was reviewed and subsequent lessons planned on the basis of the class's understanding and interest level.

3.4.2 Sequence of program:

This approach significantly altered the traditional sequence of instruction for rational number by grounding children's introduction to this number system through the learning of percents, rather than through the traditional introduction of fractions. Following the teaching of percents, decimals were taught and were introduced as the percent of the distance between two numbers that a particular point occupied. In this way the program was able to give children a first intuitive understanding of the decimal number system. Fractions were not taught independently but were considered as alternate symbolic representations of percents and decimals.
3.4.3 The lessons: an overview

3.4.3.1 Estimating Percents (lessons 1 through 4)

The lessons started with an introduction to percents. To begin the unit, the students were challenged to think about all of the instances where percents occurred in their daily lives. A definition of four key benchmark points (100%, 50%, 25% 0%) was discussed. Next, large drainage pipes of varying lengths that were covered with specially fitted sleeves were presented. These sleeves were pieces of flexible venting tube that fit around the pipes and could be pulled up and down and set to various levels. The children were challenged to estimate the percentage of the pipe that was covered. The objective of the first few lessons was to encourage children to think of strategies for making reliable estimates. The perceptual halving strategy was encouraged. "Percent full " estimations were made using beakers and vials filled with sand or water. These estimation exercises were designed to allow the students to integrate their natural halving strategies with percent terminology. The children were then introduced to a standard numerical form of notation for labeling percents. Standard notation for the writing of fractions for benchmarks was also introduced so that the students would be comfortable moving between representations.

3.4.3.2 Computing percents (lessons 5-7):

The visual estimation exercises using the vials and beakers were continued with a new focus on measurement. Children were instructed to compare visual estimates with estimates based on measurement and computation.
For example, if a vial is 20 mm tall 50% of that should be 10 mm. The children then began to estimate and mentally compute percentage of volume, for example, this vial holds 60 ml of water, 50% full should be 30 ml, 25% full should be 15 ml. Other challenges included measuring objects in the classroom and then estimating and calculating different benchmark points such as 50%, 25%, 12 1/2% and 75%. The children were not given any standard rules to perform these calculations. An example of a method that was commonly used is exemplified; 75% of 80 cm (the desk) should be 60 cm because 50% of 80 cm is 40 cm and half of that (25%) is 20 cm and together they equal 75%. Other exercises included comparing heights of children and the teacher and estimating their percentages. For example, "Peter's height is what percent of Joan's"? A series of specially made laminated cut-out dolls ranging in height from 5 cm to 25 cm provided additional practice at comparing heights. Percent lessons were concluded with the students planning and teaching a percent lesson for a child from a lower grade (A sample lesson showing the details of how percents were taught is included in Appendix C).

3.4.3.3 Introduction to decimals using stopwatches (lessons 8-11)

In this unit children were introduced to decimals as an extension of their work on percents. They were told that decimal numbers permit more precise measurement than whole numbers. Two-place decimals were introduced as a way of indicating what "percent" of the distance between two whole numbers a particular quantity occupies. LCD stopwatches with screens that displayed seconds and hundredths of seconds were used as the introduction to decimals so that the continuous nature of these numbers could be
concretely observed by the children. Many activities and games were devised for the purpose of helping the students to actively manipulate the decimal numbers in order to illuminate the conceptually difficult concept of number magnitude and order. The first challenge that was presented to the students was "The Stop/Start Challenge." In this exercise, students attempted to start and stop the watch as quickly as possible, several times in succession. They then compared their personal quickest reaction time with those of their classmates. In this exercise, they had the opportunity to experience the ordering of decimal numbers as well as to have an informal look at computing differences in decimal numbers (scores). Another difficult initial aspect of using decimal symbols is the ordering of decimals when the numbers move from e.g. .09 to .10. This activity also addressed this problem as some students were able to respond quickly enough to the challenge to achieve a score of .09 seconds. Therefore, such traditionally difficult rational number tasks such as, for example, what is bigger .09 or .42 could be naturally introduced. Another stopwatch game that offered active participation in the understanding of magnitude was a game entitled "Stop The Watch Between." The object of this game was for the student to decide which decimal numbers come in between two given decimal numbers and then to stop the watch somewhere in that span of decimal numbers. In the game "Crack the Code," the students had to move between representations, as they were challenged to stop the watch at the decimal equivalent of for example, 1/2 (.50). The students were encouraged to invent variations on these games to use as challenges for their classmates. The first stopwatch lesson is presented in Appendix.
3.4.3.4 **Learning about Decimals on number lines. Lessons (12 - 16)**

The next series of lessons used large laminated number lines which were placed on the classroom floor for children to move (walk) on so that the number line representation for decimal numbers could be reinforced. Initial activities included "percent walks" where the students walked a given percentage of a number line. These distances were then reinterpreted as decimal numbers. Other number line activities included games involving ordering of numbers using symbols for lesser and greater.

3.4.3.5 **Playing and inventing decimal board games (Lessons 16 to 19)**

A board game was devised with the intention of giving the students the opportunity to learn about the magnitude of decimal numbers as well as to add and subtract decimal numbers moving along a series of number lines. (For a detailed description of the “Dragon Game”, see Appendix E). Three lessons followed where the students invented and planned their own rational number board game and then played each other's games.

3.4.3.6 **Fractions (Lessons 19 to 21)**

In keeping with the curriculum focus of translating among representations, fraction lessons were taught in relationship to decimals and percents. In these lessons, the children were challenged to, for example, represent the fraction $1/4$ in as many ways as they could using a variety of fraction representations as well as visual, decimal, and percent representations. They
also worked on problems and invented their own challenges for solving mixed-representation equations involving decimals percents and fractions.

3.4.3.7 Review (Lessons 22 to 25)
Games were played where the students had to add and subtract decimals, fractions and percents by creating their own hands-on concrete materials. For example students invented card games with mixed representations and challenged their classmates to solve a variety of problems that were posed. As a final culminating project, students were invited to either invent their own rational number teaching strategies and lessons that could be taught to another group, or to design a game or video that incorporated specific rational number teaching objectives.

3.5 Rational number program for the control group

The control group followed the program for rational number from the widely used math text series," Math Quest 4" (Kelly, 1988). This Canadian series is considered among the best of all current series that are used in elementary schools. Although the sequence of instruction for Rational numbers in "Math Quest" is the same as it is for the two other well known Canadian math series, "Journeys in Math" and "Starting Points in Mathematics," "Math Quest" has been particularly distinguished for its attempt to balance drill and practice worksheets with exercises that contribute to conceptual understanding. It is important to note that in the Math Quest series the rational number system is briefly introduced in the last month of grade three. A short outline of the material covered in the grade three rational number curriculum is listed below.
a) Paper folding into equal parts, b) Writing of fractions (fraction symbols) 1/2 1/3 1/4 1/5, (in writing the fractional symbols the children are taught that the numerator be thought of as "number of parts coloured" and the denominator as "number of parts in all"), c) Comparing fractions, d) Tenths, e) Ones and tenths, f) Decimals with tenths, g) Adding and subtracting decimal numbers, and h) Dollars and cents.

The grade four program that was followed by the control group carried on from the introduction to rational numbers in grade three. The first unit of the rational number lessons for the control group was titled "Fractions and Decimals". This unit, comprising ten lessons, started with a reintroduction of fractions. In this program, a fractional number was initially defined as "a number that describes parts of a whole." The initial visual representation for this introduction was a circle divided into four equal sections, with one of the sections shaded in. A page of exercises followed that asked the students to write a fraction that indicated what part of the illustrated geometrical shape was shaded. Fractions of a set, equivalent fractions, and comparing fractions followed. Then, in keeping with the sequence of the grade three unit on rational number, tenths were introduced and their relation to decimals was shown. Decimal tenths were briefly taught using pie graphs and number line representations. Hundredths were taught next and their situation on a place value chart was demonstrated. (Lessons on place value charts are commonly taught to reinforce the multiplicative relationship of tenths in adjacent numbers in the natural number system). Finally, equivalent decimals were taught by showing that numbers such as .3 and .30 were equivalent because 3/10 is the same as 30/100.
Fifteen lessons on operations with decimals followed. First rounding from decimal tenths to whole numbers was taught. Linear measurement was used to illustrate the principle of rounding. Addition and subtraction of decimals was taught through the rules for both "lining up the decimal points and then either adding tenths and trading tenths for a one, or trading a one for ten tenths and then subtracting tenths." Addition and subtraction with money was then taught followed by computing the sums and differences of two place decimal numbers. The rules for these operations were also taught: "Line up the decimal points, subtract the hundredths, subtract the tenths and then subtract the whole numbers." Multiplication of one and two place decimals followed. Rules were taught for knowing where to place the decimal in the product, for example, "A whole number times a number with one decimal place equals a product with one decimal place."

The classes in rational number ended with computation with fractions and division of decimals.

3.6 Assessment Measures

3.6.1 Rational Number Test

The Rational Number Test was designed especially for the present study; its purpose was to assess the children's conceptual understanding and procedural knowledge of rational number. The test comprised 41 items and was divided into three sections: Percents, (12 items), Decimals, (14 items) and Fractions (15 items). Each section of the test included several different types of tasks including for example, estimating distances on number lines, shading
specified fractional quantities in standard geometric shapes, and performing traditional rational number computation problems. Some items were designed specifically for this study by the researcher. Other tasks were either taken from Case's Developmental Fraction Measure (Case, Krohn and Bushey, 1992) or from standard grade 4, 5 and 6 mathematics text books. The items for each separate test were arranged in order of difficulty.

The test was constructed so that a percentage (approximately 20%) of the tasks were direct measure questions that were related to the experimental curriculum while the remaining questions were transfer items. The direct measure questions assessed the students' ability to perform rational number tasks in a familiar context. Some examples of these included asking the subjects to identify fractional numbers on a number line, or to estimate 25% of the height of a container. The transfer items were tasks that required the students to work in novel contexts. Some of these tasks were chosen because they reflected a broadened conceptualization e.g. Shade in 0.3 of a circle that is partitioned into 5 sections, or Can these be the same amount, .06 of a tenth and .6 of a hundredth? and some were selected because they were standard types of school tasks. Since standard algorithms for computation were not taught to the students in the experimental program, items such as How much is 3.64 - .8 and How much is .5 + .38? were considered to be transfer items.

This test was administered to the students in both groups both as a pre- and post-measure. For the post test 7 new items were added to provide extended opportunities for analysis. As well as adding the new items at post test, The Rational Number Test was also sub-categorized into six separate but not mutually exclusive data strands. Five of these sub-categories provided data
about the students' abilities in areas that have been shown to indicate rational "number sense" and conceptual understanding. These areas were the ability to 1) Use rational number representations interchangeably and to work with flexibility among symbolic representations; 2) Understand magnitude of rational numbers and to be able to compare and order these numbers in their different symbolic representations; 3) Overcome visual perceptual distractors and not be misled when the solution to a task required that the student ignore inconsistent or irrelevant visual representations; 4) Perform non standard algorithms by inventing strategies and rational number to solve tasks; 5) Perform real-world type problems and be able to interpret mathematical challenges that are presented as objects in the three dimensional world. The sixth (final) category looked at students ability to perform standard algorithms.

3.6.2 Rationale for Five “Rational Number Sense” Subcategories

Interchangeability
The five rational number sense subcategories reflect research findings in current mathematics education literature. Flexibility in moving among symbolic representations in the domain of rational number is considered to be a good indication of rational number understanding as well as an important factor in rational number sense. (Markovits, & Sowder, 1991, 1994; Sowder, 1994). Using multiple representations for quantities allows students to work conceptually and to "transform problems on the basis of useful equivalencies." In fact Lesh, Post, & Behr (1987, see p. 320) make the claim that recognizing or constructing correspondences between different
representational systems is at the heart of knowing or understanding mathematics.

*Compare and Order Numbers*

Closely associated with the seamless movement among representations and the ability to use different rational number symbolic representations interchangeably, is the ability to compare and order rational numbers. Tests that require students to order a series of fractional numbers reveal that students have difficulty attaching a quantitative referent to decimal symbols, (Carpenter et al, 1981; Bright, et., al., 1988; Hiebert, Wearne and Taber, 1991). A source of difficulty in assigning a quantity for decimals stems from the fact that although the decimal symbol notation is familiar to students, the quantities are much less familiar. The shift from discrete quantity of the whole number system to the abstract notion of continuous quantity of rational number is a difficult one.

*Visual Perceptual Distractors*

Knowledge of quantity includes knowledge of what happens when quantities are moved or partitioned or combined or acted upon in other ways. (Hiebert, 1992). Students have a natural expectation that the visual representation (or physical conditions) in which mathematics problems are presented will be relevant and consistent with the task. When students are presented with a problem that does not conform to this expectation, they are required to ignore the distractor and deal with the task with a deeper conceptual understanding in order to perform the task successfully. The ability to overcome visual perceptual distractors is considered to be a good sign of rational number sense. (Behr, Lesh, Post & Silver, 1989).
Non-standard Algorithms

A characteristic of number sense that is generally agreed upon is the ability to use invented strategies to solve standard and non-standard computation problems. The types of errors that are consistently shown in the rational number literature demonstrate that students are overly dependent on the use of procedures. Even when uncertain of the rules, they will misuse a procedure, preferring to accept an improbable answer rather than to invent an alternate strategy. Hatano distinguishes between two types of expertise, routine and adaptive. (Sowder, 1995) People who demonstrate routine expertise are able to perform standard problems with speed and accuracy. It is the adaptive expert that is able to use idiosyncratic and modified procedures to adapt to the constraints of a problem. It is this kind of adaptive ability that allows the problem solver to invent personal strategies to solve mathematical problems.

Real World Problems

The ability to perform real world problems is evident in student's ability to move back and forth between representations i.e. pictures and numbers, as well as the ability to interpret mathematical challenges that are presented as objects in the three dimensional world. Students who are successful in this area show a facility of movement among numerical systems and also a seamlessness of movement from numerical to non-numerical representations.

3.7. Scoring Procedures...
The items were scored dichotomously with one point allocated for each correct answer. For those questions which required that subjects explain their thinking process, one point was allocated for giving both the correct answer and the corresponding explanation. Two items were scored twice. Question 15, asks students to place the fractions 1/2 and 13 on a number line. Students who were able to show that they knew that 1/2 was greater than 1/3, were given 1 point. Those students who also correctly placed the number 1/3 between 0 and 1, were given a second point. Most students were misled by the numberline representation and placed the number 1/3 on a point that they estimated to be 1/3 of the entire line. A second question that was scored twice, asked the students to shade in .3 of the circle. In scoring this question, 1 point was given to students who were accurate in their answer by .10 so that they either shaded two full sections of the circle, (.40 of the circle) or one section (.20 of the circle). An extra point was given to students who accurately shaded in 1.5 portions of the circle.

Percent, Fraction and Decimal subtest scores were added together resulting in three respective subtest totals, and a grand total was also obtained for the rational number test as a whole. Individual scores were also obtained for the six "number sense" subcategories mentioned above.

After the tests had been scored and prior to the analysis of the data, four subjects from the control group were eliminated from the study. These four students were selected based on the fact that they had the lowest overall scores on the rational number test. This procedure was done in order to provide a more equal match with the experimental group as four students from this class had not participated in the study as mentioned previously.
**Canadian Test of Basic Skills**

Canadian Test of Basic Skills is a normed achievement test that measures student achievement in all major skill area. Raw scores are converted to grade equivalents and grade equivalents are also represented as percentile ranks.
CHAPTER 4
RESULTS

This chapter is divided into ten sections. In the first section, the overall results of pre and post scores for both the Experimental and Control groups on the Rational Number Test are presented. In the six sections that follow, the six subtests are analyzed (Interchangeability, Compare and Order, Visual Perceptual Distractors, Standard Algorithms, Nonstandard Algorithms, and Real World Problems). In each of these sections, a quantitative analysis is followed by a qualitative analysis, which presents examples of the students' explanations from the protocols. Selected transcripts are intended to be representative of the thinking of students from both groups and to illustrate the different types of strategies that were used.

In Section Eight, standardized scores from the Canadian Test of Basic Skills are analyzed for the experimental group. The final two sections contain qualitative analyses of the overall performance for the two groups. In section Nine, a verbatim protocol from a discussion that was initiated by a student from the experimental group is presented, followed by an analysis that illustrates how this one student's synthesized knowledge shows his number sense. The final section contains an analysis of the overall performance of the control group and focuses on the common problems and misconceptions that are indicated in the literature.
4.1 Overall Results of Rational Number Test

Table 1 shows the means and standard deviations for the two groups on the combined Rational Number Test at pretest. Although the Experimental group scored slightly higher than the Control group at pretest, there was no statistically significant difference between the two groups.

Table 4-1.

Comparison of Mean Pretest Scores on the Rational Number Test (Standard Deviations in Brackets)

<table>
<thead>
<tr>
<th></th>
<th>Experimental Mean (SD)</th>
<th>Control Mean (SD)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(max = 41)</td>
<td>(max = 41)</td>
</tr>
<tr>
<td>Total raw score</td>
<td>12.25 (6.48)</td>
<td>6.54 (5.71) ns</td>
</tr>
</tbody>
</table>

T-tests were also performed to compare the performance of the two groups on each of the pretest subtests: Percents, Decimals, and Fractions. Table 2 shows the results of these analyses. Again, no significant differences were evident in these tests.
Table 4-2.
Comparison of Mean Pretest Scores on the Individual Subtests on the Rational Number Test

<table>
<thead>
<tr>
<th>Subtest</th>
<th>Experimental Mean (SD)</th>
<th>Control Mean (SD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent Subtest</td>
<td>4.75 (1.91)</td>
<td>4.15 (2.03)</td>
</tr>
<tr>
<td>Fractions Subtest</td>
<td>4.62 (2.46)</td>
<td>4.23 (2.09)</td>
</tr>
<tr>
<td>Decimals Subtest</td>
<td>2.88 (2.92)</td>
<td>3.15 (2.48)</td>
</tr>
</tbody>
</table>

** p <.01; *** p<.001; ****p <.0001; ns = not significant

Table 3 shows the mean posttest scores of the total raw score for the two groups on the Rational Number Test. The experimental group outperformed the control group as expected and the results were significant (t = 4.78; p < .0001). The average raw score gain for the students in the experimental group was 16 points whereas the average gain for subjects in the control group was 5 points (t = 5.8; p = .0001).

Table 4-3.
Comparison of Mean Posttest Scores between Experimental and Control Groups on the Rational Number Test

<table>
<thead>
<tr>
<th></th>
<th>Experimental Mean (SD) (max = 41)</th>
<th>Control Mean (SD) (max = 41)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total raw score</td>
<td>28.62 (16.97)</td>
<td>16.62 (7.35) ***</td>
</tr>
</tbody>
</table>

*** p < .001
A further analysis was conducted to compare the performance of the two groups on the expanded posttest as noted above. The expanded posttest combined all of the original tasks with the eight additional items that were included at post test. The Experimental group show even greater superiority compared to the control group in their performance with these combined scores \(t = 5.12; p = .0001\). Table 4 illustrates the results of the analysis.

A two way analysis of variance with repeated measures (2 x 2 Group: [experimental/control] x Time [pre/post]) was conducted to compare the overall performance of the two groups. As can be seen in Figure 1, the performance by the experimental group improved much more than that of the control group. A strong effect of group was evident: \(F(1,32) = 7.72; p < .009\). The effect of time was also highly significant: \(F(1,32) = 117.7; p < .001\). As well, there was a very strong interaction effect for group and time where \(F(1,32) = 29.06; p < .001\) B, thus showing that the intervention had a significant effect on the Experimental group.
FIGURE 4-1

PRE AND POST TEST SCORES
(RATIONAL NUMBER TEST)

Mean score

Experimental
Control

(Pre) (Post)
Table 4-4.

Comparison of Mean of Expanded Posttest Scores between Experimental and Control Groups on the Rational Number Test

<table>
<thead>
<tr>
<th></th>
<th>Experimental Mean (SD) (max = 47)</th>
<th>Control Mean (SD) (max = 47)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total raw score</td>
<td>32 (18.29)</td>
<td>16.44 (9.31) ***</td>
</tr>
<tr>
<td>*** p &lt; .001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

t = 5.12; p = .0001

* This expanded score includes 6 items that were administered at posttest only.

4.2 *Interchangeability*

Six original test items and three items that were added in the expanded posttest measure of the Rational Number Test assess the ability of the students to use rational number representations interchangeably. Table 5 shows that the children in the experimental group had a significantly higher success rate than the subjects in the control group at posttest on these items. The mean score for the experimental group at posttest was 5.88 (out of 9) compared to 2.72 for the control group. A t-test was conducted and the results were highly significant (t = 5.20; p < .0001).
The advantage that the experimental group demonstrated was particularly evident in the responses of the students at posttest to the following item: *What is 1/8 as a decimal?*  (C = 17%, E = 75%). The strategies that were commonly used by students in the Experimental group are well captured in an explanation that was given by a student from that group:

E (Experimenter): *What is 1/8 as a decimal?*

S (Student): 0.125

E: *How did you get that answer?*

S: Well, 1/4 is 25% ... and 1/8 is half of that so it is 12 1/2/\%

... so 12 1/2% is .12 1/2 or .12.5 ... no I think it is just .125.

In contrast, the explanation below of a student from the control group provides an example of how twelve (67%) students in that group thought about the question.

E: *What is 1/8 as a decimal?*

S: I think it is .8 ... because 1/8 is probably the same as .8.

Another question that highlights the ability of the students in the experimental group to translate from one mode of representation to another is *What is 6% as a decimal?*  (C = 17%; E = 93%). A student in the experimental group answered as follows:

E: *What is 6% as a decimal?*

S: Well it can't be .6 because that is the same as 60% so it must be .06.
By using an elimination strategy that was based on a consideration of the reasonableness of the answer, the student was able to overcome (think beyond) the salient distractor.

Table 4-5.

Responses (with Percents) to Test Items at Pre and Posttest which Assess Ability to use Rational Number Representations Interchangeably

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n = 16)</th>
<th>Control (n = 13)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>How much is 50% of $ 8.00?</td>
<td>88</td>
<td>100</td>
</tr>
<tr>
<td>How many is .5 of all the blocks?</td>
<td>56</td>
<td>100</td>
</tr>
<tr>
<td>What is 1/8 as a decimal?</td>
<td>6</td>
<td>75</td>
</tr>
<tr>
<td>What is 1/3 as a percent?</td>
<td>19</td>
<td>69</td>
</tr>
<tr>
<td>What is seventy-five thousandths as a decimal?</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>How would you write 6% as a fraction?</td>
<td>0</td>
<td>13</td>
</tr>
<tr>
<td>*What is 6% as a decimal?</td>
<td>93</td>
<td></td>
</tr>
<tr>
<td>*What is 1/8 as a percent?</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>*What is thirty-five hundredths as a decimal?</td>
<td></td>
<td>25</td>
</tr>
<tr>
<td>t = 4.42; p &lt; .0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>* New items that were administered at posttest only.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Students in the control group however, did not demonstrate the same understanding. Only one student from that group was able to successfully answer the question. Ten students (56%) from the control group asserted that 0.6 was the correct response and four students responded that the answer was 6.0.

Although the students in the Experimental group were much stronger in moving between forms of representation, there were two questions where the Control group were more successful: i.e. *What is 35 hundredths as a decimal?* (C = 46%; E = 25%) and *How would you write 6% as a fraction?* (C = 15%; E = 13%). This difference in performance was probably due to the fact that the students from the experimental group had not received formal lessons in fractions and were less able to use symbolic fraction representations.

4.3 *Compare and Order*

When children in the experimental group were required to compare and order decimal fractions and other fractional numbers, their performance was superior to that of the control group. On the eight items that were grouped together to assess this ability, the Control group had a mean raw score of 2.94 at posttest, with the mean score for the Experimental group of 4.75 (t = 2.43; p < .03). Table 6 shows these results. A test item that was particularly revealing in this group was, *Can you think of a number that lies between 0.3 and 0.4?* (E = 100%; C = 15%). All of the students in the Experimental group were able to make a correct (plausible) response. Sixty percent of students in that group answered as follows:
E: **Can you think of a number that lies between 0.3 and 0.4?**

(E= 100%; C= 17%)

S: Well .35 is between .3 and .4.

Other responses to that question from students in the experimental group included .3099, 0.31, 0.32, and 0.34.

**Table 4-6.**

**Responses (with Percents) to Test Items at Pre and Posttest on Items Demonstrating the Ability to Compare and Order Rational Numbers**

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n=16)</th>
<th>Control (n=13)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>Which is less, 1/3 or 1/2 of 6 blocks?</td>
<td>44</td>
<td>100</td>
</tr>
<tr>
<td>What number lies between .3 and .4?</td>
<td>44</td>
<td>100</td>
</tr>
<tr>
<td>Place the fractions 1/2 and 1/3 on a number line.</td>
<td>25</td>
<td>75</td>
</tr>
<tr>
<td>Which is bigger, .20 or .89?</td>
<td>38</td>
<td>81</td>
</tr>
<tr>
<td>Draw a picture to show which is greater, 2/3 or 3/4?</td>
<td>31</td>
<td>81</td>
</tr>
<tr>
<td>Which is bigger? tenths, hundredths or thousandths?</td>
<td>6</td>
<td>56</td>
</tr>
<tr>
<td>Which is more, .06 of 1/10 or .6 of 1/100?</td>
<td>0</td>
<td>44</td>
</tr>
</tbody>
</table>

*Name a number that comes between 1/8 and 1/4.

* New test item administered at posttest only.

\[ t = 2.34; p < .027 \]
By contrast, ten students (56%) in the control group asserted that there was no number that could fall between, as one student claimed:

E: *Can you think of a number that lies between 0.3 and 0.4?*
S: There is no number between .3 and .4.

Four students (20%) claimed that the answer was 0.20. while other students reasoned that the answer was .03 or .25.

4.4 *Visual Perceptual Distractors*

Eight items on the test were grouped together to reveal the students' ability to overcome visual distractors. Both irrelevant and inconsistent distractors were included in this group. The results show that the experimental group was much better able to ignore these distractors at posttest. Of 8, the mean score for the control group was 2.46 and the mean score for the experimental group was 5.81 (*t* = 5.67; *p* < .0001). Table 7 illustrates the results of this subtest.

One item that illustrates the difference between the two groups is one that required the students to shade in 3/4 of a pizza that was partitioned in 8 sections (*E* = 100%; *C* = 44%). The experimental students responded as follows:

E: *Shade in 3/4 of this pizza (pizza illustrated with eight sections)*

(*E* = 100%; *C* = 54%)

53
S1: I don't know ... well let me see ... This is a half (shaded in four sections) ... so you would need two more to make 3/4 (shades in two more sections)

S2: There are two slices in a quarter so you need six (slices) to make three quarters.

S3: (Shades in six sections) I just keep the quarters and forget about the eighths.

By contrast, six students (46%) in the control group shaded in three sections of the pizza. A typical explanation from the control group was as follows:

E: Shade in 3/4 of this pizza (pizza illustrated with eight sections)
S: It says 3/4 so you need to shade in three parts.

Another item required the students to use base-ten manipulative blocks to construct a decimal number. A typical experimental answer was as follows:

E: Can you construct the number 23.5 with base 10 blocks. using the ten sticks as ones? (E = 82%; C = 46%)
S: I get it, If this is one (points to a ten stick) then this (points to a hundreds board) has to be ten so these (points to the centicubes) become tenths.

Students in the Control group had more difficulty with this challenge. Two different examples of students' reasoning from that group represent some of the difficulties that they experienced.
E: Can you construct the number 23.5 with base 10 blocks using the ten sticks as ones?

S1: When we were learning fractions we learned that this (ten stick) is tenths so would it be like this?

S2: I think this is how you make it.

Table 4-7.

Responses (with Percents) to Test Items at Pre and Posttest which Include Visual Perceptual Distractors

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n=16)</th>
<th>Control (n=13)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>Find 3/4 of a pizza that has been divided in 8.</td>
<td>75</td>
<td>100</td>
</tr>
<tr>
<td>How far did Mary travel from home to school? Fraction?</td>
<td>25</td>
<td>93</td>
</tr>
<tr>
<td>How far did Mary travel from home to school? Percent?</td>
<td>25</td>
<td>82</td>
</tr>
<tr>
<td>Construct the number 23.5 with base-10 blocks.</td>
<td>56</td>
<td>82</td>
</tr>
<tr>
<td>Place letter on number line B.</td>
<td>6</td>
<td>44</td>
</tr>
<tr>
<td>Place letter on number line A.</td>
<td>6</td>
<td>44</td>
</tr>
<tr>
<td>Shade in .3 of a circle that is divided in 5 sections.</td>
<td>6</td>
<td>25</td>
</tr>
</tbody>
</table>

** Place the fractions 1/2 and 1/3 on a number line.

** New test item administered at posttest only.

\[ t = 6.56; p < .0001 \]
4. 5 *Standard Algorithms*

Five items were grouped together because they included a standard form of computation. As predicted, the performance of the two groups was similar overall and the results of an unpaired t-test revealed no statistically significant differences when the posttest means were computed. The mean score at posttest for the Control group was 1.54 (SD 1.11) compared to the mean score for the Experimental group of 1.93 (SD 1.12) \( t = 1.86; p < .35 \). Table 8 shows the items and the scores for that subtest.

When the students were presented with the standard textbook style fraction subtraction problem for example, *What is 3 1/4 - 2 1/2?*, the students in both groups had trouble responding (E = 38%; C = 0%). Although the students in the Experimental group had not been taught any formal way to work on this type of question, and had never encountered a problem of this sort in their curriculum, six students were able to answer the question successfully. One student in the group reasoned as follows:

E: *What is 3 1/4 - 2 1/2?* (E = 38%; C = 0%)

S: I have to carry it over, but I don't know how to carry it over but since I'm doing a whole, since it is still a whole, shouldn't we use a quarter and a whole and then subtract a half. So the answer is 3/4.

The students in the Control group relied on algorithmic rules to answer this questions and although they indicated that they had learned the rule they
were not able to sensibly answer the item. An example of an attempt at an answer from a student in the Control group is as follows:

E: *What is 3 1/4 - 2 1/2?*
S: First I must find common denominator which is 4, therefore it would become 3 1/4 - 2 2/4 which equals 1 0/4.

A student from the Experimental Group responded in the following manner to another standard computation item:

E: *What is 1/2 of 1/8? (E = 63%; C = 10%)*
S: 1/2 of 1/8 is 1/4. No ... wait, it can't be ... because 1/2 of 1/8 is the same as 1/2 of .12 1/2 % so it is 61/4 % or .0625 So I think that it (the answer) is 1/16.

This preceding reasoning and explanation indicate the ability of this student to move flexibly among representations served to prevent her from making the error that was most common in the control group. Of the 16 students in the control group, 14 responded by asserting that the answer is 1/4.
Table 4-8.
Responses (with Percents) to Test Items at Posttest of Standard Computational Problems

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n=16)</th>
<th>Control (n=18)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>What is 1/3 of 15?</td>
<td>63</td>
<td>88</td>
</tr>
<tr>
<td>How much is 3.64 - .8?</td>
<td>0</td>
<td>44</td>
</tr>
<tr>
<td>How much is 2/3 of 6/7?</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>What is 7 1/6 - 6 1/3?</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*What is 3 1/4 - 2 1/2</td>
<td>0</td>
<td>38</td>
</tr>
</tbody>
</table>

p < .07

*New item administered at posttest only.

4.6 Nonstandard Computation

Seven items from the original subtest and one new item were combined to look at students' ability to perform nonstandard computation. The students in the experimental group were far more successful at these types of questions, achieving a mean score at post test of 6.34 compared to 2.46 (max = 8) for the control group (t = 5.97; p < .0001).

Two test items that are representative of the differences between the groups are as follows. Students in the experimental group answered:
E:  *Another student told me that 7, is 3/4 of 10, is it?*  
(E = 75% C= 8%)

S:  No, because 1/2 of 10 is 5. One half of five is two and a half. So if you add 2 1/2 to 5, that would be 7 1/2 so 7 1/2 is 3/4 of ten.

S2:  No because 1/4 is 2 1/2 so 2 1/2 X 3 isn't 7, it is 7 1/2.

Two students from the control group answered in the following way:

E:  *Another student told me that 7 is 3/4 of 10, is it?*

S1:  No ... seven is not right because it is an odd number so 6 would be right.

S2:  Yes, 7 is 3/4 of 10 because 3 + 4 = 7.

E:  *What is 65% of 160? (E = 69% C= 0%)*

S1:  50% (of 160) is 80. I figure out 10% which would be 16. Then I divided 16 by 2 which is 8 (5%), then 16 plus 8 um ... 24. Then I do 80 plus 24 which would be 104.

S2:  50% of 160 is 80 ... 25% is 40 so 75% (of 160) is 120 so it would be a little less than that (120) it would be 10% less so it would be about 108.

S3:  Ten percent of 160 is 16. 16 x 6 = 96. Then I did 5% and that was 8 so 96 + 8 = 104.

Some students in the Control group reasoned in the following ways:

E:  *What is 65% of 160?*

S1:  The answer is 95 because 160 - 65 = 95
S2: 160 divided by 65 = 2 R 30 — is the answer 2?

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n=16)</th>
<th>Control (n=13)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>How much is 50% of $8.00?</td>
<td>89</td>
<td>100</td>
<td>62</td>
<td>92</td>
</tr>
<tr>
<td>25% of 80?</td>
<td>34</td>
<td>93</td>
<td>62</td>
<td>92</td>
</tr>
<tr>
<td>15 is 75% of what?</td>
<td>50</td>
<td>88</td>
<td>46</td>
<td>46</td>
</tr>
<tr>
<td>How much is 10% of $.90?</td>
<td>19</td>
<td>88</td>
<td>31</td>
<td>23</td>
</tr>
<tr>
<td>Is 7 3/4 of 10?</td>
<td>6</td>
<td>75</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>1% of $4.00?</td>
<td>19</td>
<td>69</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>65% of 160?</td>
<td>0</td>
<td>69</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>.05 of 20?</td>
<td>6</td>
<td>33</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>* 1/2 of 1/8</td>
<td></td>
<td></td>
<td>63</td>
<td>0</td>
</tr>
</tbody>
</table>

* New item administered at posttest only.

4.7 Word Problems Involving Real World Content

Three test items were grouped to form this subtest. As can be seen in Table 12, the students in the Experimental group were more successful with these questions. The mean score for the Experimental group was 2.56 (SD .512) and
the control group 1.5 (SD .707) (t= 4.96; p < .0001). The flexibility that students in the experimental group demonstrated is illustrated below.

E: *What is the discount on the CDs as a percent?*
S: I knew it was 80 cents. I did the quick math in my head. I figured out 80 cents was 10%.

By contrast the most common answer students in the control group was 80%.

Table 4-10.
**Percentage of Students Scoring on Real World Questions at Pre and Posttest**

<table>
<thead>
<tr>
<th>Items</th>
<th>Experimental (n=16)</th>
<th>Control (n=13)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>90% finished?</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Jessica’s height is what % of Joan’s?</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>What is the discount on the CDs as a percent?</td>
<td>6</td>
<td>56</td>
</tr>
</tbody>
</table>

4.8 *Canadian Test of Basic Skills*

In order to put the foregoing results in somewhat broader context, two subtests from the Mathematics Achievement section of the Canadian Test of Basic Skills were also administered. These are shown in Figures 3 and 4. As can be seen from these figures, there was a significant increase in percentile scores on both computation and concepts. It was expected that the students
would increase in their ability to perform questions involving math concepts and their performance in the grade 3 (pre) and grade 4 (post) concepts subtest did reflect this improvement. The students achieved a mean difference score of 11% from pre to post (\(t = 3.06; p < .008\)). The surprising finding was the dramatic increase in the scores in the mathematics computation subtest where the mean difference percentile score from grade 3 to grade 4 is 29.2% (\(t = 3.81; p < .001\)). The students in the experimental group made a full, two academic-year gain in this area (computation) in only one year. Their grade equivalent mean score moved (increased) from 3-9 (ninth month of grade three) to 5-9 (ninth month of grade five).

Figure 2 shows the gain in scores for the experimental group in Math Computation on the Canadian Test of Basic Skills, and Figure 4-3 illustrates the change from pre- to posttest in Math Concepts from the same test.
FIGURE 4-2

PRE AND POSTTEST SCORES FOR MATH COMPUTATION CANADIAN TEST OF BASIC SKILLS

Percentile Score

60  65  70  75  80  85  90

1  2

(Pre)  (Post)
FIGURE 4-3:

PRE AND POSTTEST SCORES FOR MATH CONCEPTS;
CANADIAN TEST OF BASIC SKILLS

Percentile Score

(Pre)  (Post)
An Example of Expertise in Rational Number Sense Thinking.

The following protocol presents a model of synthesized number sense thinking. This is a verbatim account of a conversation that was initiated by a student that shows both rational number sense and illustrates the types of conceptualizations that students in the experimental group were able to make. The conversation began during the posttest interview. The interviewer (I) was in the process of administering the 6-page posttest. She had just completed the second page of the test when the student (S) asked:

S: How many pages long is this?
I: The whole interview is six pages long
S: So then I have just done 1/3 of the test ... hm that is 33.3%.

When he finished the third page
S: OK so now I've finished 1/2 or 50% of the test.

On completing the fourth page he remarked
S: "Now I have done 2/3 of the test which is the same as 66.6%.

When he had completed the penultimate page he wondered out loud
S: So now I have finished 5/6 of the test. What is that?
I: What do you think it is?
S: Well it is over 66.6%. It is also more than 75%.... I'd say that it is about 80%.... No wait, it can't be 80% because that is 4/5 and this (5/6) is more than 4/5.... It is 1/2 + 1/3... so it is 50% + 33.3% which is 83.3%. So I am 83.3% finished."
This child's commentary on the diminishing portion of the test that he was taking provides various concrete examples of the types of number sense strategies that became typical of the students in the experimental program. First and most evident, in every line of this dialogue the student displays his flexibility in representing these amounts as either fractions or percents. He also displays a sophistication in his quantitative understanding and ability to compare and order these numbers. This ability to judge quantity is particularly evident when he rejects his own initial assertion that 5/6 of the test might be equivalent to 80%. In attempting to find a precise answer to the challenge he has posed for himself, he also reveals his knowledge that 5/6 is greater than 4/5. Related to his well-developed understanding of quantity, this student also reveals the inferences he has made concerning the principle of "denseness" of the rational number system.

The use of invented procedures in solving math problems is also evident in this protocol. S has not learned the algorithm to solve "What is N, if N/100 = 5/6?" The strategy that he employs is to first decompose 5/6 into 1/2 plus 1/3, then to translate those quantities into percents and finally to add those percents to get a solution. With this operation, two significant strengths are displayed: On one hand a conceptual understanding of the principle of unit composition and decomposition, on the other hand, an ability to employ useful equivalencies to transform a math problem in order to find a solution. Both these abilities are highly indicative of number sense.

In initiating this discussion he also demonstrates his developing understanding of the application of mathematical problem solving to the real world. The abilities he displays of flexibility, quantitative judgment,
invention and estimation of solutions illustrate what Greeno (1991) calls "knowing in a conceptual domain." They clearly demonstrate the degree of competence that the student has acquired, and, perhaps of greater significance, the sense of commitment and ownership that this student feels towards this topic: He both initiated the conversation and persisted until he was satisfied with his own solution. This illustration echoes the example of many other students, who, in the posttest and lesson protocol, show similar engagement and curiosity.

4.10 Control Group

As can be seen in the quantitative analyses of the performance of the control group, this group made meager gains from pre- to posttest. Their average raw score gain at posttest on all their subtests combined was only 5 points despite the fact that they spent a substantial number of hours in learning rational number. The type of reasoning that the control group demonstrated in their posttest protocol is highly representative of the types of errors and misconceptions that are reported in the mathematics education literature and discussed in Chapters One and Two. Their lack of fundamental understanding of quantity or magnitude is repeatedly illustrated in the protocol from their post test interviews. For example, only 17% of these students could find a correct answer to the item that required them to find a number that falls between .3 and .4. A very substantial number of students in that group (56%) asserted that there was no number at all that could fall between the two. Not only does this error indicate a lack of quantitative understanding but also perhaps demonstrates a type of misconception that stems from interference of the whole number system. Thus, to insert a
number between two numerals that symbolically appear like whole numbers and which are adjacent to each other cardinal, seems counterintuitive.

Another item that showed these students' lack of quantitative understanding was *What is 1/8 as a decimal?* 67% of the students in that group asserted that \( 1/8 = 0.8 \). A knowledge of magnitude would at least alert the students to the fact that this is not possible as, for example, \( 1/8 \) is much less than a half and \( .8 \) is much more than a half and in fact close to one.

The preference that students typically show for using algorithmic solution strategies—(even when they were unsure of the rules)—over attempting to estimate reasonable answers is also consistently shown in the posttest interviews of these students. In answer to the question \( 3 \frac{1}{4} - 2 \frac{1}{2} \) one student replied that you must first get a common denominator which is 4 so the problem is then stated as \( 3 \frac{1}{4} - 2 \frac{2}{4} \) and the senseless answer \( 1 \frac{0}{4} \) is given.

Finally the problems with visual distractors that students with weak conceptual understanding demonstrate are illustrated by this group's performance on all of the questions from that subsection. The test item that required that the students shade in \( 3/4 \) of a pizza that was partitioned into 8. Only 33% of the students in the control were able to successfully answer that question. Their inability to answer this question also points to problems with reconfiguring the unit as well as difficulties with quantity. A surprising finding at posttest was the fact that their performance dropped from \( \text{pre} = 44\% \) post = \( 33\% \). This drop in score suggests that their part whole instruction actually hindered the students in their understanding. 56% of the
control students shaded in 3 segments of the circle and asserted that "3/4 means 3 parts of something." This is a salient example of the students' choice of rules over sense making. The fact that these students were misled by the "irrelevant" visual distracter displays the lack of rational number understanding that Behr et. al (1983) report. The students demonstrated a lack of ability to make conceptually based choices.
CHAPTER 5
DISCUSSION

In this chapter I review and discusses the findings, consider limitations of the study, and explore implications for research and for practice. In the first section I review the purposes of the study and summarize the findings that are related to the hypotheses. In the second section, I discuss and interpret other unanticipated gains that were experienced by the students in the experimental group. In the third section, I present a discussion of the core conceptual features of the experimental curriculum, analyzing how students' significant intuitive knowledge was supported by the core representations. In the fourth section I then go on to speculate about how the subconstructs of rational number that have been identified in the math education research literature may have been addressed through the experimental curriculum. In the fifth section I describe the limitations of both the experimental curriculum and implications for further research and educational practice. In the last section, the methodological limitations are presented.

5.1 Summary of purposes and hypotheses

The purposes of this study were: 1. To develop a model of children's' rational number understanding and to use this model to design and implement a rational number curriculum, and 2. To assess this curriculum by contrasting the performance of students who participated in this curriculum with an equally matched group that had more traditional training. The curriculum that was developed was based on the introduction of the rational number system through the teaching of percents. Percent teaching afforded the
opportunity for students to coordinate their well-developed knowledge of the whole number system with their knowledge of proportion.

The hypothesis was that a curriculum that promotes the integration of proportional reasoning with whole number schemas by introducing the rational number system through percents, would enable students to master this domain with more conceptual understanding than traditionally taught children. Students in the experimental condition would demonstrate this deeper understanding by 1. Demonstrating flexibility in moving among the various representations of rational number, 2. Demonstrating a quantitative awareness of number in this domain, and 3. Demonstrating the ability to overcome the influence of visual perceptual distractors. It was also predicted that the experimental group would perform as well as traditionally trained students in basic rational number computation problems.

5.1.2 Summary of the results

The students in the experimental group demonstrated a superior ability to perform both conceptual and standard types of rational number tasks. This was confirmed at posttest in the three subtests: Percents, Decimals, and Fractions. These results are best interpreted in the analysis of the six separate subcategories: Interchangeability, Compare and Order, Overcome Visual Distractors, Non Standard Computation, Standard Computation, Real World.

The results of the analysis of interchangeability showed that the students were able to both translate among the symbolic representations of rational number and flexibly convert and operate with these numbers from percents to decimals and fractions. For example 93% of the students in the
experimental group could show the decimal notation for 6% and 75% of the experimental group were able to correctly respond to the question *What is 1/8 as a decimal?* Both of these questions elicited a 17% correct response for the control group. Another type of task that probed for the ability to use the representations interchangeably was the item that asked the students to show .5 of 20 blocks. All of the students in the experimental group were able to answer this question compared to only 61% of the control group.

An analysis of the students' ability to compare and order rational numbers, the second subcategory, demonstrates the facility that students gained in their ability to judge magnitude differences in this domain. Two items on the test particularly demonstrate these abilities. All of the students in the experimental group were able to choose the greater number between .20 and .89. Only 38% of the students in the control were able to successfully compare these two numbers. As well, 81% of the students in the experimental group were able to successfully draw a diagram to demonstrate which of the numbers 2/3 or 3/4 was the greater. Only 38% of the control group were able to do so.

The third subcategory analyzed the students' ability to overcome visual perceptual distractors and not be misled by irrelevant or inconsistent visual data. A question that was difficult for the control group asked the students to shade in 3/4 of a circle (pizza) that was divided into eight sections. Whereas 50% of the control group shaded in 3 sections (3/8) of the circle, all of the students in the experimental group answered this question correctly. Another question in this category asked students to use a fraction to describe the distance that Mary had traveled from home to school.
This question posed a significant difficulty for the control group and only 31% of these students were able to respond correctly. All of the other students in that group asserted that Mary had travelled either $7/9$ or $8/10$ of the way to school. Thus illustrating the difficulty that students encounter in interpreting number lines. The response of the students in the experimental group was much different and 82% of that group were able to offer the correct answer.

The fourth and fifth subcategories analyzed the students' ability to compute with rational numbers. As indicated in the result section, 69% of the experimental subjects were able to calculate the answer to 65% of 160, and 93% could tell what is $3/4$ of 10. In this same sub category, the results indicate that 88% of the experimental group could answer the questions 15 is 75% of what number? and What is 10% of $.90$? The control group scored 33% correct for the former question and 6% for the latter. A final category looked at students' ability to solve real world tasks that were presented as word problems with accompanying diagrams.

Not only was the experimental group more successful at performing on the posttest, but protocols from both tests and classroom lessons revealed that they also manifested the types of problem-solving behavior that characterize
number sense for both whole numbers and rational numbers. The students in this group regularly showed flexibility in moving among representations, the ability to invent their own algorithms and systems for problem solving, and an understanding of magnitude.

5.2 Other gains and their relationship to the curriculum

5.2.1 Overcoming whole number biases

In Chapter one, I reported on the difficulties that students typically encounter in learning rational number. Many researchers have suggested that the similarities of the notation system of decimals and fractions to that of whole numbers can often become a stumbling block in students' understanding and performance on rational number tasks. Hiebert (1992) asserts that conceptualizations that become rote for students from their previous whole number learning interfere with their introduction to this number system. Although rational numbers share a language with whole numbers and uses concepts from whole number, it is a distinctive field and not a simple extension of whole number knowledge. Many instances of these kinds of interference from whole number knowledge are demonstrated in the responses and assertions offered by the control group at posttest. For example, students in the control group were unable to correctly find the sum of .38 + .5. Their response of .43 demonstrates that their addition schema was derived from whole number knowledge and as they were unable to see these numbers as representing different quantities. As well, when asked to find 1/2 of 1/8, 61% of students in the control group chose 1/4. This response perhaps shows their conceptualization that "multiplication makes bigger," or
perhaps, they used a rule-based strategy that made dividing 2 into 8 in the denominator seem appropriate. Both of these strategies come from whole number schemas.

By comparison the students in the experimental group seemed to be able to overcome these blocks and were not subject to these typical problems. This could be seen in students' ability, for example, to understand properties of denseness and continuous quantity and to correctly order decimal numbers. These strengths are repeatedly demonstrated in posttest and lesson protocols.

Their success at these tasks can possibly be attributed to the experiences that they had in the experimental program using stopwatches, numberlines, board games etc. These ways of representing the decimal system appeared to support students' understanding about the density of the decimal rationals and their understanding of the diminishing relationship between adjacent digits moving to the left.

5.2.2 Affective engagement

Another important feature of the experimental program that is suggested by the data is the affective engagement displayed by the students in the experimental group. Although not statistically quantified, it was clear that, in the experimental condition, students were persistent in their efforts to find solutions. This positive disposition towards rational number problem solving was evident in the experimental classroom lessons where students seemed confident and willing to invent their own strategies to further their learning. It was particularly evident in the students' efforts in taking the posttest that, even when confronted with novel problems in unfamiliar
contexts, they attempted to find solutions and persisted until they felt they had found reasonable answers.

5. 3 *Core structure and its accessibility to intuitive reasoning*

5.3.1. **Core representation**
Traditionally, the research in curriculum development in rational number is geared towards students understanding and mastery of the individual subconstructs of this number system (Behr et al., 1983; Behr et al., 1992; Hiebert, 1992; Harel, Post, & Lesh, 1993). These subconstructs are treated as specific entities, and it is the goal of these experimental programs to help students gain mastery in each of these areas. The general assumption is that if children can be helped to become competent in the specific subconstructs, they will become competent in the number system in general. Rational number understanding is considered to be the accumulation (and assimilation) of these various knowledge structures.

The experimental curriculum that was constructed for this study had a different orientation. This curriculum was constructed around a core conceptual structure rather than around precisely identified concepts. As has been described previously in this thesis, the organizing principle was a linear measurement context for percent. In this program percent understanding was constructed by the students through their use of benchmarks and invented strategies. No conventional algorithms were taught to the students. The halfing and doubling schema served as a means for getting around the number system and as a vehicle for operations.
The measurement context that was established is represented visually by the concept of the percent ribbon; a vertical numberline representation calibrated from 0 to 100 along one edge and from 0 to \( n \) along the other edge. This vertical percent bar represents a percentage full, and, at the same time, a portion of a quantity. The represented space can gradually be filled or depleted and quantities can be understood in a continuous context. Quantities are seen as measured to a degree of accuracy with the understanding that a third number can always be inserted between any two other numbers.

It is the combination of all of these elements; percent introduction, the use of the halving and doubling schema, and the linear measurement context, that constitutes what I call the core representation of this program. All of the lessons in the experimental curriculum were organized around this structure. All the props that were presented to the students or built by them were consistent with this central structure. And following the introduction of percent, the teaching of both decimals and fractions also were grounded in this structure.

In designing this curriculum the intent was to create a global context that would serve as a conceptually based introduction to rational number learning, and promote the initial mastery and flexibility of thinking in this domain that could be characterized as rational number sense. As the results indicate, the students in the experimental group developed many skills and competencies. How did this context operate for the students? What were the points of entry for the students that supported their construction of rational number knowledge. Freudenthal (1983) maintains that the context in which
the learning of mathematics takes place is very significant. The chosen context should enable students to reconstruct important math ideas because, "the phenomena under exploration beg to be organized by the mathematics of interest," (Freudenthal quoted in Lamon, 1995 pg. 168). How did the chosen context or the core representation promote the students' construction of the mathematics of interest-rational number sense?

It is my contention that it was the instructional focus on this core representation that promoted the conceptual understandings in rational number. The questions that arise are: How did this core structure support the development and mastery of rational number concepts that the students demonstrated? How did the students come to gain a competency in a wide range of tasks or ideas when the instruction did not specifically focus on the knowledge structures to be acquired? The following discussion is based on the researcher's interpretive analysis of the strengths of the intervention.

5.4 The core representation as a context or tool for fostering students' intuitions of proportion, ratio and percent.

In the following sections I will consider some possibilities of how this core structure may have provided the entry point for students to construct their rational number knowledge. I will first demonstrate the significant types of informal knowledge that students possessed prior to the instruction and then I will consider how the core structure acted as a setting in which students could expand on and develop further insights and understandings in this domain.
Children have been shown to possess a significant amount of proportional intuitive knowledge; knowledge that does not depend on formal instruction (Case, 1985; Resnick, 1986; Kieren, 1988; Mack, 1992; Resnick & Singer, 1993; Lamon, 1993, 1994). In both the pilot study and the present experiment, it was evident from the very first lesson that students possessed and were able to express a significant amount of intuitive knowledge about percent, ratio and proportion. What was also in evidence was the usefulness of the percent props for generating the expression, reflection and elaboration of that intuitive knowledge.

The following selections are paraphrased from the protocol that was recorded on the very first day of the experimental unit. They illustrate the types of insights and observations that the students were able to make using the materials provided.

On the first day of the experimental unit the students were asked to display their understanding of percent using as props a series of differently sized drainage pipes covered in expandable (and contractible) venting tubes. (This lesson is described in detail in chapter 3.) Some of the observations that the students made on that first day are as follows:

The children all knew that one whole pipe represented 100%. They also showed that by covering a portion of a tube it was possible to show a percentage of the 100% whole. The students also demonstrated a knowledge of benchmarks percents e.g. 50% 25% and 75%. They demonstrated an awareness of ratio by commenting that 50% of one tube is a different quantity than 50% of a different sized tube. They also illustrated that a given percent
(e.g. 25%) covered on one tube can be (e.g. 75%) of a smaller tube. When probed to consider the possibility that there could exists a percentage that is greater than 100%, they reasoned that if you had an initial 100% tube (e.g. 80 cm) and a second (e.g. 20 cm) this second tube represents 25% of the first. If you then joined them together the new tube would represent 125% of the original 80 cm tube. Another insight that was offered by a student concerning the combing of these two tubes was that in the event that the joined tubes became merged so that they together represented a new unit of 100%, then the original 20 cm (25%) tube, no longer represented 25% of the whole but 20% of the (newly formed) whole.

These ideas that were put forward by the students on the first day of the experimental unit, provide evidence of the substantial sense of proportional reasoning that many of the students possessed. It also seemed clear that the linear measurement aspect of the props afforded students an accessible context for describing and extending their understanding. As well the language of percents and its familiar referent base of 100 helped students to naturally make ratio statements and comparisons. The accessibility of the base referent of 100 seemed to suggest the types of quantitative comparisons both within and between the unit that the students could easily make. The props also afforded students the opportunity to test constructs of order and equivalence of a rational number entity. Because the varying tubes allowed students to see equivalence from the perspective of proportionality, they were able to perceive that a multiplicative transformation is inherent in equivalence. This is a construct that is considered to be very important and again very difficult for students to understand (Behr et al., 1989). As well, the students ability to conceptualize the percent increase and the covarying
adjustments that must be made in situations of proportional increase, illustrates an understanding of concepts in percentages that are considered to be very difficult even for high school students (Parker & Leinhardt, 1995). In summary, the students gained a significant feel for this number system, and although mathematical principles were never explicitly stated by the children, consistent behaviour and response patterns suggest that these understandings were abstracted from principles. The core conceptual structure served as a facilitator for these understandings.

5.5 The embeddedness of subconstructs of rational number in the experimental curriculum.

5.5.1 Rational number as quantity

As mentioned at the start of Section 5.2, standard approaches seek to teach the rational number subconstructs in a separate fashion. Because the experimental curriculum of this study does not take this approach, it is important to consider whether and how it serves as an entree to such conceptual structures.

In chapter 1, I referred to a study by Kerslake (1987) that demonstrates the difficulty that students have in perceiving a rational number as a single number. She reports that students see a fraction as two numbers and not as a single quantity. This situation is of particular concern as she found the same conceptions to be held by teachers. Other researchers concur with Kerslake For example Kieren (1992) and Behr et al. *(1993).
The evidence from the present study shows that in learning percents, students do perceive rational number as a single quantity. They see percents as quantities that can be readily compared to other percents, fractions, decimals and whole number quantities. Perhaps this conceptualization is evident to them because of the natural ordering that is available to them through the linear measurement percent context.

The question that remains is what rational number properties or subconstructs are embedded in the students' understanding of rational number as a number or quantity? The research on rational number teaching and understanding has been driven by analyses of the various subconstructs that underpin this number field. How does the students' quantitative understanding reflect the concerns of this literature? In the following speculative analysis, I will attempt to describe the subconstructs of quotient ratio, operator and measure using Kieren's (1992) analysis and Marshall's schema theory (Marshall, 1993) as referents and consider how these constructs might be represented in this experimental rational number curriculum. (Particular attention will be given to the ratio and quotient subconstructs as their relationship to this curriculum is possibly more obscure).

5.5.2 Quotient/Ratio subconstructs

Kieren (1993), has suggested that rational/ fractional number is simultaneously quotient and ratio. He maintains that as quotients, rational numbers are additive and answer the question, "how much?" (springing from the social act of sharing) e.g. 75% or $\frac{3}{4}$ is a "much" amount. As ratios, rational numbers are intensive quantities and answer questions of relational
properties of quantities (Kieren, 1993 p. 54). Is this complimentarity of ratio and quotient understanding demonstrated by the students in the experimental group? What actions of the students make these understandings visible?

5.5.3 Quotient subconstruct

The evidence of this study suggests that students had a natural understanding of the quantitative or "how much" aspect of rational number. All of the activities that were presented to the students involved them in considering and estimating percent amounts. For example, students estimated or measured percent quantities and filled beakers with sand to specified amounts. These same understandings were naturally transferred to their descriptions of these quantities as fractions. As I have reported, the students were (typically) not misled by the two integers in the fraction representation as they saw these fractional representations as single amounts that could be quantified and ordered. This understanding of quantity was also demonstrated in their schemas for decimal representation. I also speculate that because these rational number representations of percent, decimal and fraction were taught simultaneously, the students were able to experience notions of equivalence in rational number quantity. For example, in the final lessons of the experimental instructional sequence, the students composed their own computational challenges using mixed representations. These exercises required students to compute sums and differences with unlike denominators. An example that was composed by a student in the experimental group typifies the types of skills that the students developed by participating in the experimental curriculum:
Which grouping of numbers is the larger?

\[ \frac{1}{8} + \frac{1}{16} + \frac{1}{2} + 0.0625 \] or \[ \frac{1}{4} + 25\% + 0.125 + \frac{1}{8} \]

This type of challenge naturally flowed from the curriculum as the students were able to conceptualize the notion of equivalence.

5.5.4  **Ratio subconstruct**

At the same time as allowing them to understand rational number as a quotient, the core representation of this curriculum also appeared to afford the students a natural setting to experience the ratio subconstruct of rational number. Percent is a ratio comparison. I have demonstrated that students in this program seemed to naturally perceive percent quantities in relationship to a referent whole (100%). The protocol that has been presented in this chapter repeatedly demonstrates these understandings.

Vergnaud (1983) proposes the notion of two measure spaces in his analysis of multiplicative ratio thinking. Kieren characterizes Vergnaud’s thinking as follows: ratios, and hence rational numbers, can be represented either by pairs of elements in the same measure space or elements in two distinct measure spaces. “Thus equivalence means either the same scalar relationship held between elements in the same measure space, or the same functional relationship held between elements of two measure spaces.” (Kieren, 1992 p. 345). He gives the following example to explain this notion of ratios: 2 pizzas for 5 people is the same as 6 pizzas for 15 people. 2/5 and 6/15 are members of the same function, or class, in a single measure space. Between two spaces, there exists the same scalar relationship which, in the case of the above example is \((X3)\) between persons and pizzas.
In the experimental program, the representation of ratio in the same measure space is constructed when students perform the following type of quantification: 25% of 100% on one dimension of the percent ribbon is the same as 4 cm of 16 cm on the parallel side. In other words, $\frac{25}{100} = \frac{4}{16}$. The scalar in two distinct measure spaces is represented in our program when in the course of the lessons, a 40 cm pipe (Pipe A) is covered 25% of the way up, which means i.e. 10 cm of the tube is covered, and simultaneously, on a second tube that is 80 cm in height, (Pipe B) is also covered 25% of its length. In the case of the second tube 25% covered represents 20 cm. Therefore the relationship is 25% of tube B is $2 \times 25\%$ of tube A.

5.5.5 Operator subconstruct

The operator interpretation of rational number emphasize multiplicative operations of this number system. "The operator concept of rational number suggests that the rational number $\frac{3}{4}$ is thought of as a function that is applied to some number, object or set." (Behr, et al, 1993). It seems evident to me that this program clearly presented rational number as a multiplicative operator. From the first day of class the students were using percents as operators. For example, to answer the question what is 25% of 20 the students could use 25% as an amount and also as an operator. The examples in the result and other sections that illustrated the students strategies in computing percents provide clear evidence of this idea. As well the students came to know the differences about the multiplicative aspect and particularly the operator construct of rational numbers.
5.5.6 Measure subconstruct

The measure subconstruct refers to a fractional quantity e.g. 1/b that is used repeatedly to determine a distance. Marshall (1994) "It is most frequently accompanied by a number line or a picture of a measuring device, and students are expected to measure the distance from one point to another in terms of 1/b units. Usually the line starts at zero and extends to 1, with the distance in between them broken into b segments." (p. 275). In order to gain competency with the measure subconstruct, Marshall asserts that "individuals must possess knowledge about ratio scales, the highest level of measurement." (p. 276). The measurement subconstruct, she suggests, is generally very difficult for students to acquire. She claims that the number line representation is essential for understanding the measure subconstruct and students generally have not sufficient grounding in number lines for this understanding to develop.

This schema appears to be central to the experimental rational number curriculum. Marshall points out that the essential feature knowledge of the measure subconstruct includes that understanding fractional numbers as units of measure. Using a percentage e.g. 25% as a single unit to repeatedly measure a distance is an operation that is regularly performed in this program.
5.6 Limitations of the experimental curriculum.

5.6.1 Limitations of the halving schema

The data analysis repeatedly demonstrates that the students' use of benchmarks and the halving schema support their intuitive understanding of percent problems and help them to generate invented strategies. This benchmark/halving approach, however, had limitations for the students. Although they could readily compute numbers that could be broken easily into halves, they were not as able to use this schema with numbers that were more complex. As well, although students came to know various representations for halves, quarters, eighths and sixteenths, and could use these quantities with ease, they had much less exposure to tenths, and other single unit fractions such as 1/3, 1/5 etc. The excerpt from a lesson that is provided in chapter 2 illustrates a phenomenon that was observed often in situations where the children were asked to find .10 or 10% of a quantity. All of the students in the experimental group could successfully determine 12 1/2% of a quantity. Many could calculate 6 1/4%. However this program appeared to accomplish less for the reinforcement of base ten notions. For example, at posttest only 56% of the experimental students (compared to 66% in the control group) were able to successfully answer the item, "Which is greater, tenths, hundredths, or thousandths?" In future, therefore, it would be useful to allow more time for students to develop their intuitions for tenths, and hundredths.
5.7 Curricular implications

Although the data that were analyzed derive from a very particular situation, and no assumptions can be made about the general application of this curriculum to other populations, it is believed that the ideas discussed here raise productive questions for rational number teaching in other settings.

5.7.1 Place of percent teaching in the curriculum

The limitations of the present curriculum have been described. However, the analysis of the data does support the idea that reordering and altering the traditional teaching sequence to initially introduce percent can be an effective conceptual introduction to rational number learning. The findings suggest that the students in the experimental group were able to successfully learn percent, a topic that is considered very difficult (Parker & Leinhardt, 1995), and is not covered in traditional instruction until grades 7 and 8. It is reported that many of the significant problems that students encounter in learning percent stem from the negative transfer that they experience from their previous rational number learning in fractions and decimals. In viewing percents as a decimal, students reportedly lose the conceptualization of a comparison between referents that is needed to create a percent (Parker & Leinhardt, 1995). It is proposed therefore, that these problems would be minimized by changing the order of instruction to teach percent in advance of these other rational number representations. The data also suggest that the teaching of percents through the use of invented strategies and the halving and doubling schema, is a successful way to introduce this topic. Lemke and Reys' (1994) research shows that after traditional percent instruction students
lose their halving, doubling and build-up strategies that they spontaneously use prior to formal instruction.

5.7.2 **A global number sense approach to teaching**

The analysis of the data also provided support for the notion of a global approach to teaching that focuses on an integrated rather than separate look at the number system. The students gained a flexibility in working in this number system and developed a quantitative focus and understanding that has been characterized by Sowder (1995) as rational number sense. An approach to teaching rational number that encourages the use of mixed representations is one that should be considered. The de-emphasis on rules for computation and problem solving that was an important feature of the experimental curriculum also appeared to strengthen concept development and increase performance. The sequence for instruction in the realm of whole number is commonly organized so that students get a great deal of practice in experimentation and manipulation with numbers before rules are presented. A rational number curriculum that allows for this kind of introduction of concept building before rule learning, and introduces this domain through percent teaching might go a long way in improving current achievement standards.

5.7.3 **Focus on decimals over fractions**

In considering the reordering of the teaching of rational number another proposal for curriculum implementation is a reevaluation of the focus on fraction teaching in the rational number system. Case (personal
communication) has pointed out that early training in rational number should have as its ultimate goal the teaching of decimals. Case noted that for most ratio and rate problems, except those involving very simple numbers, experts appear to rely on the system of decimal numbers rather than fractions in dealing with problems. Of the three major rational number systems, this seems to be the one that plays the most central role in their conceptual thought. Indeed, the decimal representation appears to play the central role that the whole number system plays earlier.

5.8 Further research questions arising from this study

This study proposed to look at children's learning of rational number. It was global in its implementation and drew from many different domains: developmental (cognitive) psychology, mathematics education research and pedagogical theory. The questions that are raised for further study emanate from these various domains. The first class of research questions focuses on the possibilities of a re-implementation of the experimental curriculum. In this chapter, I have suggested that it is the instructional focus on the core conceptual structure that underpins this curriculum that promotes the competencies that students have displayed in this study. There are a number of confounding issues that obscure and challenge this assumption as enumerated in the above limitations. Future research must evaluate this curriculum keeping the core conceptual context intact, but using different instructional methods. As well, it will be important to look at the potential effectiveness of this curriculum in other circumstances; for example larger groups of children in conventional settings and students from less advantaged situations.
This study also brings up another area of questions related to students' intuitions about ratio and proportion. An unsuspected finding that is revealed in the lesson protocols was the significant knowledge that students possessed of ratio, proportion and percents prior to instruction. An analysis of these understandings prior to and following instruction would be useful to further guide curriculum development. What is the impact of this intuitive knowledge on students overall performance in rational number tasks? Also, how can we support these intuitions and provide a bridge for students to move to more formal operations in rational number?

5.9 Methodological limitations

The present study was very useful in yielding general insights into the possibilities of developing an introductory curriculum to support students' rational number sense. It also presented an opportunity for an analysis of the informal knowledge and intuitions that students bring to the learning of rational number, the types of strategies that the students developed and the levels of mastery that they could achieve. The results however are limited due to various constraints of this study.

The study was implemented for a very small and particular population of middle-to-high SES students who attended a (lab) school where creative approaches to teaching are featured, learning is highly valued as is achievement of each individual student. It is necessary to consider then what the applicability to other groups of students might be. Questions arise about the success of this program for low achieving students (or low ses) students.
The short duration of this study (the students were taught once a week for twenty weeks) is another factor that limits the possibilities of a more complete analysis. Although the results of this study indicate that the short term gains for the students were very impressive, the long term effects remain to be seen. It is possible that when these students continue their rational number learning using a more traditional approach, they will abandon their highly conceptual approach to problem solving and come to rely on a more rote and algorithmic method of working. As well, it is not known if their strong affective gains and positive disposition would be sustained over the long term and inspire them to continue to employ conceptually based strategies as math problems become more complex.

Developmental considerations also arise due to the short term nature of this study. The data suggests that the students in this group far outperformed a well matched group in concept acquisition and mastery. It is not known however, how the performance of these students will compare to that of their matched peers by the end of high school.

Because this study was taught by the researcher another series of methodological limitations must be noted. The social constructivist approach that was adopted for the experiment requires that teachers promote knowledge building or theory building through the use of structured discourse. Less experienced teachers might have difficulty eliciting the types of discussion that support the development of conceptual understanding and rational number sense. As well the students in the experiment group may have benefited from the researcher's knowledge and the use that she made of on-line decisions regarding upcoming lesson plans. By contrast, the control
group did not have the benefit of a research teacher and some of the differences in their performance might be attributed to that situation. It is possible that the results of the study would differ if the control group was learning the traditional sequence, but in an experimental mode.

5.10 Conclusion

Although the limitations just cited are real and imply that caution must be experienced in generalizing from the present study, it should be clear that the overall conclusion of the study is positive. The results obtained were extremely promising and may well offer a solution to the difficult problems that arise when rational numbers are taught in a conventional manner.
BIBLIOGRAPHY


APPENDIX A

RATIONAL NUMBER TEST

PRETEST VERSION
Rational Number Test (Pretest)
Percent Test

1. If I said that we were 90% finished, would you think that we had a long way to go?

2. How much is 50% of eight dollars?

3. Draw a line on this beaker to show what it would look like if the beaker was approximately 25% full.

4. If this beaker holds a total of 80 ml. of water, how many ml's. of water would there be if you had filled it 25% full?

5. Fifteen blocks spilled out of a bag. This was 75% of the total number of blocks. How many blocks were in the bag to begin with?

6. How much is 10% of ninety cents?

7B How would you write 6% as a fraction?
N.B. A fraction looks like this: 1/2

8. Suppose that you got 1/3 of the answers correct on a test, what would that be as a percent? How do you say 1/3 as a percent?

9. There is a sale at Sam's. This $8.00 CD (point) is on sale and the new price was $7.20. Sometimes when things go on sale they are say, 25% off the regular price. What do you think the percent discount is for the C.D.?
10. How much is 1% of four dollars?

11. What is 65% of 160? Explain how you got your answer.

12. Joan is 100% taller than Jessica. Jessica’s height is ____% of Joan’s. Don’t forget to look at the picture.
Fraction Test

13. (Divide 10 blocks into 3 groups of 3, 5 and 2. Shift group of 5 blocks ahead.) Is this half of the blocks?

14. (Display 8 blocks). Show me 3/8 of these blocks.

15. Here is another group of blocks. (Show 6 blocks). Please show me which would be less, 1/3 of these blocks or 1/2 of these blocks. First show me 1/2 of the blocks. Now show me 1/3. Which is less?

16. How much is 1/3 of 15. Suppose you wanted to explain what it means to a younger student. How would you show her what it means to take 1/3 of 15?

17. I've got some blocks here in this bag and I'm going to show you 1/5 of them. If this is 1/5 of the blocks (Display 3), how many blocks did I start out with?

18. This is a number line. (Point to the whole line) Can you place the fractions 1/3 and 1/2 on this number line?

19. This is a pizza (Pizza cut in 8 slices). How much is 3/4 of the pizza? Use your pencil to shade in 3/4.

20. Another student told me that 7 is 3/4 of 10. Is it? Explain your answer. (Wait) What is 3/4 of 10?
21. Draw a picture to show which is greater, $\frac{3}{4}$ or $\frac{2}{3}$. You can draw a circle, a rectangle or anything you like.

22. What fraction of the distance has Mary travelled from home to school?

23. How would you express that as a percent?

25. What is $7 \frac{1}{6} - 6 \frac{1}{3}$?

26. How much is $\frac{2}{3}$ of $\frac{6}{7}$?

27. Can you draw a picture to explain how you got the answer?
Decimals Test

28. (Show 10 yellow and 10 blue blocks. Point to all yellow's and ask): Is this .5 of all the blocks? Are the yellow blocks .5 of all blocks?

29. Which is bigger, .20 or .089?

29. Which is bigger, tenths, hundredths, or thousandths?

30. How should you write seventy-five thousandths as a decimal?

31. How much is .5 + .38?

32. (Give 5 hundred-squares, 5 ten-sticks, 5 centicubes). Can you make the number 23.5 with these blocks?

33. Look at this number line. What number is marked by the letter A?
   What number is marked by the letter B?

   ![Number Line]

   0  B  0.1  A  0.2  0.3

34. How much is 3.64 - .8? Show vertically.

35A. Can you tell me a number that comes between .3 and .4?

36A. What is 1/8 as a decimal, do you know?

37. If you had 20 candies and you were told to give away .05 of all the candies, how many candies would you have to give away?
38. Shade in .3 of the circle. Do you know approximately what .3 of something is?

39. Show on a number line $3 \times .4$. What I mean is, show me 3 groups of .4

40. What does 20 divided by .5? Can you draw a picture on the back of the page to show what it means?

41. Could these be the same amount, .06 of a tenth and .6 of a hundredth?

Yes _____ or No _____ (Explain)
APPENDIX B
RATIONAL NUMBER TEST
POSTTEST VERSION
Rational Number Test (Posttest)

Percent Test

1. If I said that we were 90% finished, would you think that we had a long way to go?
2. How much is 50% of eight dollars?
3. Draw a line on this beaker to show what it would look like if the beaker was approximately 25% full.
4. If this beaker holds a total of 80 ml. of water, how many mls. of water would there be if you had filled it 25% full?
5. Fifteen blocks spilled out of a bag. This was 75% of the total number of blocks. How many blocks were in the bag to begin with?
6. How much is 10% of ninety cents?
7A. How would you write 6% as a decimal?
7B How would you write 6% as a fraction?
N.B. A fraction looks like this: 1/2
8. Suppose that you got 1/3 of the answers correct on a test, what would that be as a percent? How do you say 1/3 as a percent?
9. There is a sale at Sam's. This $8.00 CD (point) is on sale and the new price was $7.20. Sometimes when things go on sale they are say, 25% off the regular price. What do you think the percent discount is for the C.D.?
10. How much is 1% of four dollars?

11. What is 65% of 160? Explain how you got your answer.

12. Joan is 100% taller than Jessica. Jessica's height is ____% of Joan's. Don't forget to look at the picture.

Joan

Jessica
13. (Divide 10 blocks into 3 groups of 3, 5 and 2. Shift group of 5 blocks ahead.) Is this half of the blocks?

14. (Display 8 blocks). Show me 3/8 of these blocks.

15. Here is another group of blocks. (Show 6 blocks). Please show me which would be less, 1/3 of these blocks or 1/2 of these blocks. First show me 1/2 of the blocks. Now show me 1/3. Which is less?

16. How much is 1/3 of 15. Suppose you wanted to explain what it means to a younger student. How would you show her what it means to take 1/3 of 15?

17. I've got some blocks here in this bag and I'm going to show you 1/5 of them. If this is 1/5 of the blocks (Display 3), how many blocks did I start out with?

18. This is a number line. (Point to the whole line) Can you place the fractions 1/3 and 1/2 on this number line?

19. This is a pizza (Pizza cut in 8 slices). How much is 3/4 of the pizza? Use your pencil to shade in 3/4.

20. Another student told me that 7 is 3/4 of 10. Is it? Explain your answer. (Wait) What is 3/4 of 10?

21. Draw a picture to show which is greater, 3/4 or 2/3 You can draw a circle, a rectangle or anything you like.
22. What fraction of the distance has Mary travelled from home to school?

23. How would you express that as a percent?

24. What is 3 1/4 - 2 1/2?

25. What is 7 1/6 - 6 1/3?

26. How much is 1/2 of 1/8?

26. How much is 2/3 of 6/7?

27. Can you draw a picture to explain how you got the answer?
Decimals Test

28. (Show 10 yellow and 10 blue blocks. Point to all yellow's and ask): Is this .5 of all the blocks? Are the yellow blocks .5 of all blocks?

29. Which is bigger, .20 or .089?

29. Which is bigger, tenths, hundredths, or thousandths?

30. **How should you write thirty-five hundredths as a decimal?** How should you write seventy-five thousandths as a decimal?

31. How much is .5 + .38?

32. (Give 5 hundred-squares, 5 ten-sticks, 5 centicubes). Can you make the number 23.5 with these blocks?

33. Look at this number line. What number is marked by the letter A?

What number is marked by the letter B?

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0   B   0.1   A   0.2   0.3
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34. How much is 3.64 - .8? Show vertically.

35A. Can you tell me a number that comes between .3 and .4?

35B. Can you tell me a number that comes between 1/3 and 1/4?

36A. What is 1/8 as a decimal, do you know?

36B. **What is 1/3 as a percent?**

37. If you had 20 candies and you were told to give away .05 of all the candies, how many candies would you have to give away?
38. Shade in .3 of the circle. Do you know approximately what .3 of something is?

39. Show on a number line $3 \times .4$. What I mean is, show me 3 groups of .4.

40. What does 20 divided by .5? Can you draw a picture on the back of the page to show what it means?

41. Could these be the same amount, .06 of a tenth and .6 of a hundredth?

   Yes _____ or No _____  (Explain)
BENCHMARK PERCENT MEASURES - TUBES

OBJECTIVES:
1. To review understanding of percents as part/whole relationship between quantities scaled in terms of 100;
2. To review understanding that percents are relative numbers always relating to something;
3. To continue to use visual strategies for estimating percents;
4. To provide support for the use of a halfing "splitting" strategy to calculate percents of various lengths.

In this lesson each pair of children was given a cardboard tube, varying in height from 20 cm to 80 cm. The teacher initiated a discussion about the relative differences in lengths of these tubes. Next, the children were asked to estimate and mark the 50% point on their tube. Rulers were then distributed and the students checked their estimates using standard measures. The students marked these precise spots on their tubes with the accompanying cm measurement and comparisons of the 50% points were made with those marked on other students’ tubes. In the next challenge the students were required to visually estimate and mark 25% of the length of their tube. In order to get an exact calculation for 25% of the length, the halfing strategy was discussed as a method. Students mentally computed half of their 50% quantity to calculate the amount in cm of 25% of the tube’s length. The precise
location of 25% of the length was then marked on the tube. The final challenge was for the students to first estimate and then precisely calculate 75% of their tube. This was done by combining the 50% measure with the 25% measure, or by subtracting 25% from 100%. An extension of the lesson for children who had completed the initial challenges was to compute 12.5% and 62.5% (50% + 12.5%) of the length of their tubes.

These activities in which the students were engaged, demonstrate how the students incorporated the splitting schema to compute a percent value. As well, the integration of the additive and multiplicative schema was evident in the students’ strategy to compute the solution for 75%.
INTRODUCTION TO DECIMALS WITH STOPWATCHES

OBJECTIVES:
1. To learn how the hundredths of a seconds (centi-seconds) are related to percents;
2. To give the students an concrete opportunity to observe the continuous nature of decimal numbers;
3. To enable students to learn to perceive the actual speed of the hundredths of seconds;
4. To learn an understanding of magnitude and compare and order decimal numbers as well as to gain experience in understanding how decimal numbers can be inserted between two given decimal numbers;
5. To learn to order decimals when the numbers move from e.g. .09 to .10;
6. To practice computing differences in decimal numbers (scores) and to invent strategies for subtraction of decimals;

Introduction to decimals

LCD stopwatches with screens that displayed seconds and hundredths of seconds were shown to the students and a discussion was initiated to prompt the students to consider what the small numbers on the screen (hundredths of seconds) represented and measured.

Students offered that these small numbers must be milliseconds. (Milliseconds was a term that they had heard of but not known the meaning of.) Distinctions between milli
and centi-seconds were discussed. The students suggested that these numbers must be centi-seconds as "cent" in French means 100 and cents in English was one part of $1.00. They noticed that the small numbers on the screen moved from 1 to 99.

In the first activity the children were asked to find their best starting-stopping time in 7 practices. The children were then asked to consider any patterns that they could see in their set of numbers (times) that they produced by stopping and starting the watch. The patterns that were most often noticed by the students were the highest and the lowest numbers and which number repeated itself the most often.

The students then wrote their personal times on the chalk board and a further discussion ensued. One child had written the following numbers 0.12, 0.09, 0.11 0.14, 0.12, 0.19, 0.13. The teacher then challenged the children to consider which of these numbers fit least well or which of these numbers was furthest away from the others and why. The majority of the children agreed with the assertion of one student who suggested that 0.09 was the number that was most outside the pattern because the others all had a 1 following the decimal. Two children then suggested that the number 0.19 fit least well because it was much more than the other numbers. The class ended with a further discussion of magnitude and the ordering of decimal numbers.
APPENDIX E

DECIMAL LESSON

DRAGON GAME
THE DRAGON GAME

Adding and Subtracting on an Invented Board Game

OBJECTIVES:

1. To review translating numberline representation to decimals;
2. To use numberlines to visualize addition and subtraction of decimals;
3. To enable the students to invent strategies for addition and subtraction of decimals;
4. To help students to consider magnitude of decimal numbers and to understand the need for more exact quantitative discriminations;

A new board game was made for the children to give them hands-on experience with adding and subtracting using number lines. The game board was approximately 60 cm x 90 cm and was composed of 20 individual laminated 10 cm number lines that were arranged as a winding path. Each number line was marked as a ruler; Ten black thick lines indicated cm measures, Ten slightly shorter blue lines highlighted the .5 cm measures and 100 red lines provided the mm measures. The object of the game was to get from the beginning (the first sidewalk) to the end (20 th sidewalk) before the other players. At each turn, a child picked two cards; an “Add” or “Subtract” card and a “Number” card. The number cards each had two digits written on them. The rule was that each number card had to be expanded by the player with the addition of a decimal and a zero. For example if a child picked a card with the numbers 1 2, they had the options of calling that card .120, 1.20, 12.0, or 120. The game also had other appealing features, for example, good
luck bonuses and hazard areas. Three players could play at once. The rest of the group worked with the teacher to practice adding and subtracting on their own number lines.

This was both a very useful and enjoyable game. All the children were successful at placing the decimal and 0 on their number cards to their advantage. This indicated that they had become very good as certain types of ordering tasks. Because there had not been formal lessons in adding and subtracting, this game and the related exercises provided the children with a reason to discover their own system.

There were several strategies that the children used when adding decimal numbers or advancing on the game board. One approach that was used by some of the students was to first, work only with the numbers to the right of the decimal, count ahead with the decimal number, isolating it from the whole number and then add the whole number after that. For example, a player started on 6.5 and had to advance 7.3. She first added .5 to .3 and moved her marker ahead to 6.8. Next she counted and moved 7 complete side walks touching each number line on the .8 spot. She stopped on the 13th side walk at the .8 spot, and correctly achieved the challenge. Another method was to hold on to the original whole and decimal number as a unit and then advance by the whole number amount and add the decimal number after that.

For the next few lessons the children made their own games to teach addition, subtraction and ordering of decimals numbers.