ESTIMATION AND TESTING OF STRUCTURAL PARAMETRIC SEALED-BID AUCTIONS

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Economics
University of Toronto

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0-612-49879-4
Abstract

Estimation and Testing of Structural Parametric
Sealed-Bid Auctions
Doctor of Philosophy
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In this thesis I examine various aspects of structural parametric sealed-bid auctions. Structural models are attractive to estimate because they allow the econometrician to recover the parameters of the private signals of the bidders. These parameters can be used to simulate different auction forms enabling a seller to compare her expected revenue from these and determine which one is optimal.

Estimation and testing of structural parametric auction models is difficult since the support of the data depends on the parameters of the distribution of the private signals of the bidders; standard asymptotic theory breaks down in this scenario. This thesis takes a Bayesian approach to estimation and testing of structural auction models since the Bayesian approach is unaffected by the dependence of the support on the parameters.

In the first chapter, I develop a posterior odds ratio method to decide whether the symmetric parametric structural common-value or the independent private-values model is more probable once data on winning bids is observed. This method is applied to the low-price, sealed-bid
auctions conducted by the Indian Oil Corporation to purchase crude-oil from the international market.

In the first chapter I had data on winning bids across auctions; I was unable to obtain data on bids. In the second chapter I examine whether an empirical researcher is always losing expected information about the parameters of the distribution of the private signals of the bidders when she has data only on winning bids across auctions. I then provide a link between the loss of expected information in having data on just the winning bids instead of all the bids, and specification of reference priors under the two scenarios.

In the third chapter I provide a statistical analysis of the bidding decision of the sawmills participating in the sales conducted by the county of Simcoe in Southern Ontario for standing timber in the woodlots that it owns. These auctions violate key assumptions made in Chapters 1 and 2: a single, indivisible object is auctioned; bidders are symmetric and they bid competitively.
Acknowledgements

The people I want to acknowledge fall into two mutually exclusive categories: those that made this thesis possible and those that made it impossible. Since there is no identification problem here, the job of acknowledgement is done and my thesis can rest in peace - or in the dustbin since I have been told that the work I am doing in this thesis is not mainstream. I am certain this acknowledgement does not meet the standards of a Ph.D. thesis; hence a proper acknowledgement follows.

A century back when I came to Toronto in pursuit of knowledge, I had naïve views about a random variable. Dale J. Poirier changed all that by bombarding me with his EC 2400 course. I thank him for his time and patience in arousing and nurturing my curiosity in Bayesian econometrics. Some of my best (or worst since the two are kindred emotions) times in the Ph.D. program have been in his office taking stock of my new found wisdom. Despite my Bayesian choice, he refused to mentor me saying that supervision was enough. He then refused to supervise me saying that advising was enough. By the time the last curtain call of this thesis was made his formal role in my thesis was amorphous.

Mike Peters entered my thesis after the first act. One of his many generous gestures was to fund a trip to Jerusalem for a summer school in Auctions and Mechanism Design. Post that the only way Mike could get me to leave his office was by kicking me out! I became interested in the theory of auctions quite apart from the econometrics of auction - and there could not have been a better person than Mike to get me going on this. Chapter 4 in my thesis is testimony of this. Mike even put up with my discourses on Bayesian econometrics and after each dialog I was better-off because he clarified to me what I was trying to communicate.

I thank Gordon Anderson for ensuring that this thesis comes to an end.
Tony Lancaster and Nancy Reid have made many insightful suggestions on which I am likely to work in the future. I thank them for reading my work and making these suggestions.

I am indebted to my teachers at the Delhi School of Economics, especially K.L. Krishna and A.L. Nagar, who first aroused my interest in econometrics with their dedicated teaching. I thank Arthur Hosios for giving me the opportunity to come and study at Toronto by funding my program and travels to conferences. Several people in the academic world have encouraged me on in their own way; amongst them I would like to thank Larry Epstein, Charles Manski, James Pesando, John Rust and Aman Ullah.

I doubt I would have finished this thesis with all the academic support that I have listed had it not been for the emotional support provided by my family and friends. I owe Sophia Knapik a whole wardrobe since most of her clothes are soaked up with my tears. Nimmi, Bala, Sailor and Kuldeep Chatwal provided me with a home-away-from-home; saying thanks to them seems too inadequate. Mummy, Papa, Amita and Chottu were by my side at every moment urging me to go on even though physically they were thousands of miles away from me.

The critical factor that saw me through this thesis was my determination. I dedicate this thesis to my Grand Mother for giving me determination in legacy; and to Mummy, Papa, Amita and Chottu for pointing out a legacy I had forgotten.
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Chapter 1

Introduction

In this thesis I examine various aspects of structural parametric sealed-bid auctions. Structural models are attractive to estimate because they allow the econometrician to recover the parameters of the private signals of the bidders. These parameters can be used to simulate different auction forms enabling a seller to compare her expected revenue from these and determine which one is optimal for her in the sense of maximizing expected revenue. Besides mechanism design, the seller can also study the impact of any policy change on the bidding behavior. For example, the seller can determine the impact of a change in the environment under which the auction is held on the bids and the private signals of the bidders.

Estimation and testing of structural parametric auction models is difficult since the support of the data depends on the parameters of the distribution of the private signals of the bidders; standard asymptotic theory breaks down in this scenario. Christensen and Kiefer (1991), Donald and Paarsch (1996) and Hong (1998) have obtained the asymptotic theory of the maximum likelihood estimator when the support of the data depends on the parameters. The key point of these papers is that the sample minimum or maximum is a superconsistent estimator of the support of the data; that is, the sample minimum or maximum converges to the support evaluated at the "true" parameter values at the same rate as the sample size. The properties of the maximum likelihood estimator are, as a result, based on the properties of the sample minimum or maximum.

While there has been progress on estimation of structural auction model, to the best knowledge of the author, there is no work on testing of structural auction models. Testing of structural
auction models involves non-nested hypothesis testing with the support of the data being different under the null and the alternative structural auction model. Defining a pivotal quantity in this scenario is difficult.

This thesis takes a Bayesian approach to estimation and testing of structural auction models since the Bayesian approach is unaffected by the dependence of the support on the parameters. Estimation proceeds by examining the posterior distribution of the parameters. Hypothesis testing proceeds through the posterior odds ratio.

In the second chapter of my thesis, I develop a posterior odds ratio method to decide whether the symmetric parametric structural common-value or the independent private-values model is more probable once data on bids or winning bids is observed. The method is developed to deal with the difficulties that arise from the support of the data depending on the parameters of the distribution of the private signals of the bidders in structural auction models; I have outlined these above. This method is applied to decide whether the symmetric common-value or the independent private-values paradigm is a more probable explanation of the low-price, sealed-bid auctions conducted by the Indian Oil Corporation in India to purchase crude-oil from the international market. These are auctions for a single object, the object being a barrel of crude-oil. The bidders are assumed to be symmetric and risk neutral with private signals that are independent. I also justify that the auctions are independent so that there is no strategic bidding across auctions.

I conclude that the symmetric independent private-values paradigm is more probable. The private-value component in these auctions is explained by the accessibility of the bidders to the informal world market for crude-oil which is the private information of the bidders, and which is being targeted in these auctions by Indian Oil. This conclusion implies that Indian Oil can use these auctions as a screening device for the most efficient bidder.

In the second chapter of my thesis I had data on winning bids across auctions; I was unable to obtain data on bids within and across auctions. In the third chapter of my thesis, I examine whether an empirical researcher is always losing expected information about the parameters of the distribution of the private signals of the bidders when she has data only on winning bids across auctions. I then provide a link between the loss of expected information in having data on just the winning bids instead of all the bids, and specification of reference priors, as in
Ghosal (1998), under the two scenarios. Note that since the support of the data depends on the parameters, Jeffreys' prior and the reference prior are different, in general. The difference in inference about the parameters when data on all bids instead of just winning bids is available, is also noted.

For a scalar boundary parameter, when the minimum winning bid is sufficient for the parameter, I conclude that the reference prior obtained from all the bids within and across auctions and the minimum winning bid coincide. Inference about this parameter will be unaffected whether data on all the bids or just the winning bids across auctions is available. When the minimum winning bid is not sufficient, the reference prior for the parameters is more dispersed. This follows from the Fisher information about the parameters from all the bids within and across auctions compared with the winning bids across auctions being a psd but not a null matrix. Inference about these parameters will be different due to both the reference prior and the likelihood function being different.

In the fourth chapter of my thesis I provide a statistical analysis of the bidding decision of the sawmills participating in the sales conducted by the county of Simcoe in Southern Ontario for standing timber in the woodlots that it owns. Each sale involves auctioning off multiple woodlots; a first-price, sealed-bid auction is conducted separately for each woodlot. These auctions violate three keys assumptions made in Chapters 1 and 2: the auctions are for a single, indivisible object; the bidders do not bid strategically across auctions, and their private signals are independently distributed.

I show that there is evidence of aggressive/nonaggressive bidding that characterizes the bidding behavior of bidders with multi-unit demand facing budgetary or capacity constraints, as discussed in Engelbrecht-Wiggans and Weber (1979). Bidders bid strategically across auctions within a sale. They bid aggressively in some auctions to ensure that they win at least some woodlots; they bid less aggressively on other woodlots in a sale, expecting to win these woodlots even if other sawmills bid less aggressively, and as a result win them at a bargain price. I also find that three bidders who are bidding consistently through this period are coordinating their bidding strategies violating the assumption, in Chapters 1 and 2, of independent private signals of the bidders.

There is no received theory for the kind of multi-unit auctions conducted by the county of
Simcoe that I have described above. As a result, structural estimation of these auctions was not possible. This is left as an agenda for future research.
Chapter 2

Posterior Odds Comparison of a Symmetric Low-Price, Sealed-Bid Auction Within the Common Value and the Independent Private-Values Paradigms

2.1 Introduction

The Indian Oil Corporation (IOC henceforth) is a leading public-sector oil company in India that conducts low-price, sealed-bid auctions to purchase crude oil on the international market. In its conduct of these auctions, IOC invites bidders to submit sealed bids. On a specified day, all the bids are opened and the bidder with the lowest bid wins the auction. The winning bid is made public to the bidders; individual bids are not, however, public knowledge.

Assuming the bidders are risk neutral, a natural question to ask is why do bidders submit different bids. The auction-theory literature points to several answers, with the symmetric common-value paradigm and the symmetric independent-private-values paradigm being at the two extremes; I will refer to these two paradigms as the common-value paradigm and the
private-values paradigm, respectively. According to the common-value paradigm, the cost of performing the task for IOC is the same but unknown to all bidders *ex ante* (i.e., until the task is actually performed). Therefore, each bidder forms an unbiased estimator of this cost based on his private knowledge and uses this when formulating his bid. The winner of the auction is the bidder who has the lowest-cost estimate and hence the lowest bid. In the private-values paradigm, on the other hand, each bidder has a different cost of performing the task for the IOC. Only he knows this cost. He submits a bid based on this cost. The winner of the auction is the bidder with the lowest private cost, and hence the lowest bid. Other paradigms, such as the affiliated-values model (Laffont and Vuong, 1996, pp. 415), lie between these two extremes and make different assumptions about the risk neutrality, symmetry, and/or independence of the private values of the bidders.

The aim of this paper is to ascertain which paradigm, the common-value paradigm or the private-values paradigm, is more probable once the data on winning bids across auctions are available.¹ Ascertaining which paradigm applies can aid IOC in the design of these auctions. For example, if the private-values paradigm holds, then the low-price, sealed-bid auction serves as a screening device for the most efficient (lowest cost) bidder. On the other hand, if the common-value paradigm holds, then the low-price, sealed-bid auctions yield the lowest revenue for IOC compared to other auction forms like the English auction or the second-price auction; IOC could then procure crude oil at lower prices through an auction form other than the sealed-bid auction (Milgrom and Weber, 1982, p. 1095).

This paper takes a structural parametric approach to the estimation of the auction models; i.e., the private signal of the bidder is assumed to be drawn from a specified family of distributions. The distribution of the private signals is specified such that a natural conjugate prior can be assumed for the parameters of the distribution of the winning bid. A natural conjugate prior is attractive in this instance as it allows interpretation of the parameters of the winning bid. This is a problem in the nonconjugate case, especially when the winning bid does not belong

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¹Within a nonparametric framework, Laffont and Vuong (1996, p. 417) have proved that for first-price, sealed-bid auctions with a reserve price, the common-value model is indistinguishable from the affiliated private-values model from data on bids. Parametric assumptions about the joint distribution of the private signals of the bidders and the common, unknown component identifies the common-value model since it leads to a different mapping from unobservables to the observed bids under the common-value and the affiliated private-values paradigm. This is explained in Section 4.
to any known family of distributions (Laffont, Ossard, Vuong, 1995). Priors are then specified for the parameters of the distribution of the private signals of the bidders. This implies a prior on the support of the winning bid since the support of the winning bid is a function of the parameters of interest in structural auction models; this implied prior may be in conflict with the data on winning bids.

The hyperparameters of the natural conjugate prior are specified by making three assumptions. First, I visualize a hypothetical low-price, sealed-bid auction “prior” to observing the data. For this hypothetical auction, I assume that both paradigms predict the same value for the yet to be observed winning bid. Second, the uncertainty faced by a bidder regarding the private signal of other bidders is made comparable under the two paradigms. Third, the size of the “fictitious” prior sample is assumed to be the same under both paradigms, implying that the same amount of prior information is being used in both paradigms. These assumptions make the common-value paradigm and the private-values paradigm comparable a priori.

The posterior odds ratio is then used to compare the common-value paradigm and the private-values paradigm; i.e., the posterior odds ratio indicates how the prior beliefs about the two paradigms, stated above, are revised on observing the data. In parametric structural auction models, since the support of the data depends on the parameters of interest, standard asymptotic theory breaks down presenting severe theoretical problems for non-nested hypotheses testing. Using the posterior odds ratio to compare models circumvents this problem.

The paper is organized as follows. In Section 2, I describe how IOC procurement auctions work and the data set that I have used in this study. Section 3 discusses the theoretical models of low-price, sealed-bid auction, described in Section 2, within the common-value and the private-values paradigms. Empirical studies that have attempted to validate theoretical auction models are presented in Section 4. The estimation method used in this paper is outlined in Section 5. In Section 6, I present the results using this method. The symmetric affiliated-value model is discussed in Section 7. I conclude in Section 8.
2.2 The IOC Auction Framework

IOC is a public-sector company in India that conducts low-price, sealed-bid auctions to procure crude-oil on the international market. IOC has a three-fold aim in conducting these auctions. Most of India’s crude-oil needs are met through long-term contracts between the government of India and other oil-producing countries; the balance is met through these auctions. Complete dependence on the former source of procuring crude-oil is avoided for strategic reasons. Further, the exact proportion in which oil is secured through these two means is not disclosed. A second aim is to meet seasonal and unplanned demand-supply imbalances. Anecdotal evidence suggests that coexisting with the formal world market for crude-oil is an informal or black market for crude-oil. For example, members of the OPEC (oil producing and exporting countries) cartel have been known to produce crude-oil beyond their quota to the extent of around 1%; this crude-oil beyond the stipulated quota finds its way into the informal world market for crude-oil. A third aim of IOC in conducting these auctions is to capture the gains from the short-term fluctuations in this informal market.2

The bidders are international oil companies. These include national oil companies of oil-producing countries, private oil majors as well as trading companies.3

The auctions conducted are low-price sealed-bid auctions. The auction begins with IOC inviting bidders to submit sealed-bids before a specified date. The bidding variable is the price in U.S. dollar per metric tonne of crude oil. On the specified date, the sealed-bids are opened, and the bidder with the lowest bid wins the auction. The quantity of crude oil that IOC wants to purchase is communicated to the winner of the auction. The winning bid and the identity of the winner is made public to all the bidders by IOC. The individual bids and the number of bidders that participated in the auction is not made public.

IOC would like to procure crude-oil at the cheapest price through these auctions. For this it has to know why bidders are submitting different bids since the design of an optimal auction depends on the answer to this question. If bidders were submitting different bids because they had different costs of procuring crude-oil, that were private knowledge of the bidders, then

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2 This is my personal view; it does not represent an IOC policy.
3 Private oil majors are upward as well as downward integrated companies; i.e., they undertake exploration, production, refining and retailing of oil. Examples are Shell, Mobil, etc.
the bidder with the lowest cost would win the auction. In the symmetric independent-private-values environment, the low-price, sealed-bid auctions as well as the English auction, second-price auction or the Dutch auction, with an appropriately chosen reserve price will maximize the expected revenue of IOC (Myerson, 1981, McAfee and McMillan, 1987, p. 714). On the other hand, if the bidders are submitting different bids because they do not know the "true" cost of procuring crude-oil, but have private signals about this "true" cost, then the low-price, sealed-bid auction yields the lowest expected revenue for IOC compared to other auction forms. Milgrom and Weber (1982, p. 1095) show that under the common-values paradigm, while none of the auction forms is optimal, a ranking of the auctions in terms of the expected revenue of the seller is possible; the English auction yields the highest expected revenue, followed by the second-price auction, with the first-price, sealed-bid/Dutch auction yielding the lowest expected revenue to the seller. The reason for this follows from the private signals of other bidders conveying information about the "true" unknown cost for a bidder leading to more aggressive bidding and therefore higher expected revenue for the seller. Since the low-price, sealed-bid auction conveys least information about the private signals of other bidders compared to the English auction or the second-price auction (Milgrom and Weber, 1982, p. 1110), the ranking of auctions under the common-value paradigm follows.

I have data on winning bids and the number of bidders for 37 auctions that IOC conducted in the year April, 1993 to April, 1994. Summary statistics are reported in table 1. The auctions are not conducted on any periodic basis by IOC. Further for the period under study, there was no instance when IOC called tenders for an auction but did not award the contract to any bidder.

I will now make some assumptions about the auctions conducted by IOC and justify these assumptions; the theoretical auction model presented in Section 3 is based on these assumptions.

The auctions conducted by IOC have been assumed to be independent. The bidders are submitting bids based on the cost of procuring crude-oil. I am modeling the cost of procuring crude-oil as the sum of a nonstochastic and a stochastic component. The nonstochastic component is a function of past world crude-oil prices; the stochastic part is the sum of a common

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4I thank John Rust for pointing out the literature on optimal auctions to me.
5Due to the politically sensitive nature of the oil industry in India, I found it difficult to obtain data across the years.
shock and an idiosyncratic shock, both of which are independently and identically distributed across auctions. Indicating by $C^i_j$ the cost of procuring crude-oil by bidder $i$ in auction $j$,

$$C^i_j = \varphi(P_{-j}) + \epsilon + \eta^i_j.$$ 

$P_{-j}$ indicates past world crude-oil prices, excluding the world price of crude-oil in auction $j$, and $\varphi(P_{-j})$ is a deterministic function of the past crude-oil prices. $\epsilon$ is an independently and identically distributed shock across the auctions; $\eta^i_j$ is an idiosyncratic shock that is independently and identically distributed across auctions and bidders. But as I report in table 3, rows 1-2, the world crude-oil price has remained stable over the period for which these auctions are being studied (fiscal year, 1993-94). Thus to an approximation, world crude-oil price is known to the bidders. Hence my bidding model simplifies to the bids being determined by the cost of procuring crude-oil, which is simply a constant known to the bidders plus the sum of the two shocks, $C^i_j = constant + \epsilon + \eta^i_j$. I will give further justification for the assumption of identical auctions below. Within the common-value model, the bidding model given above is $C = constant + \epsilon$. The “true” cost of procuring crude-oil, $C$, is unknown to the bidders; they do however have private signals about $C$ that are distributed independently across the bidders in an auction conditional on $C$. The assumption of independent auctions implies that in each auction independent draws are made from the joint distribution of the common and unknown cost, $C$, and the private signals of the bidders. Within the independent-private-values model, the bidding model is $C^i_j = constant + \eta^i_j$; the assumption of independent auctions implies that the cost of procuring crude-oil, that is private information of the bidders, is distributed independently across auctions.6

The bidders in the auctions are assumed to be identical. I have data on the identity of some of the winners in these auctions. For example, some private oil majors that won the auctions are Caltex, Mobil and Shell. Examples of trading companies that won the auctions are Marubeni and Yukong. From the identity of some of the winners as well as from conversations with IOC

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6It is possible that the disturbance term is correlated across auctions due to demand or supply side factors leading to learning on part of the firms and hence correlated bids across auctions. Some of the problems with the econometrics of dynamic auctions are the existence of multiple equilibria, irregularities in the likelihood function for the bids/winning bid due to the nonexistence of closed form solutions for the equilibrium bids, etc.. At present, I am unaware of any work that addresses the issue of correlated shocks across auctions.
officials, I have concluded that while these bidders have different costs with respect to other aspects of the oil trade, like refining, no bidder has a distinct cost advantage with respect to the trading of crude-oil. And the key aspect of the auctions that I am studying is the trading of crude-oil. Hence symmetry of the bidders seems a plausible assumption.

The number of potential bidders is the same as the number of participants in the auction, \( N_j \). This is a reasonable assumption to make for the auctions conducted by IOC for two reasons. First, there is no announced reserve price in these auctions. Further, for the auctions that I have studied, there was no single instance when bids were called and the contract not awarded, leading me to conclude that there was no secret reserve price set by IOC. Second, the bidders have no costs to preparing and submitting bids. The quantity that IOC would procure from the winner, which is not announced at the time the bids are invited, could be a possible bidding cost if the bidders had storage constraints. I can rule out this possibility since the bidders, being familiar with the oil scenario in India, are fairly certain about the quantity that IOC is going to demand. Further, the quantities being procured through these auctions are not on a scale that would hinder the bidders to submit sealed-bids on account of capacity constraints.

As I mentioned in the beginning of this section, only a small fraction of India’s crude-oil needs is being met through these auctions. For the data that I have, there was no instance when the winner could not supply the quantity of crude-oil that IOC had demanded.\(^7\)

Further, I also assume that each potential bidder knows \( N_j \) when forming her equilibrium bid. I have mentioned above that IOC is targeting the informal market for crude-oil that coexists with the formal market through these auctions. There are few players operating in this informal market; the maximum number of bidders in the auctions that I studied was 10 (see table 1). While information on the quantity of crude-oil that a bidder will be able to obtain in this market and the price at which she will be able to obtain this is private to the bidder, each bidder knows which oil company has been able to secure crude-oil in the market since the number of bidders in the auction is small.

The bidding variable, as mentioned above, is the price of a metric tonne of crude oil. This ensures that the product for which the auction is being held (a metric tonne of crude oil) is

\(^7\)This is an ex post argument and the bidding costs could be a constraint ex ante. However, the fact that this kind of ex post behavior is not being observed strengthens the argument that the capacity constraints are not a bidding cost.
homogenous. However, the actual amount that IOC demands from the bidder varies across auctions. I am assuming that the quantity demanded by IOC is exogenous. The reason for this is that the amount that IOC is going to procure is decided, in consultation with the Ministry of Petroleum, prior to the sealed-bids being invited. Thus IOC officials cannot use their discretion to increase the quantity to be procured through an auction when they see a low-bid. This is done to prevent collusion between the bidder and the IOC official.

I have also assumed that the distributions of the common, unknown component and the private signals of the bidders, conditional on the common-unknown component, under the common-value paradigm, and the distribution of the private cost of procuring crude-oil under the private-values paradigm, is unchanged across auctions. Since the auctions are held over a financial year, and no instability is noted in any of the economic indicators used by the government of India, it is reasonable to assume that the environment under which the auctions are held is unchanged. In table 3, rows 3-7, I have reported some economic indicators for the Indian economy over the years 1991-95 to support this assumption.

From the discussion above, a priori it would seem that the data should favor the common-value paradigm over the private-values paradigm since the commodity in question is oil and the world market price for crude-oil seems to be a natural candidate for the underlying common and unknown cost in the common-value paradigm. But as I have pointed in my discussion on the assumption of independence of auctions, the world market price for crude-oil was relatively stable over the period when these auctions were held; the cost of procuring crude-oil was changing across auctions on account of an independently and identically distributed shock across auctions. Hence I can rule out the world market price for crude-oil as the common, unknown component of the cost of procuring crude-oil in these auctions.

Another interpretation of the fact that IOC is conducting low-price, sealed-bid auctions is that IOC presumably believes these auctions are not common-value auctions. If it were the case that these were common-value auctions, then IOC would be using an English auction or a second-price auction. This follows from the result of Milogram and Weber, that I have mentioned above, that under the common-value model, the low-price, sealed-bid auction obtains the smallest expected revenue for IOC. Hence, it will be interesting to find out, using the posterior odds ratio, whether the common-values paradigm or the independent private-values
paradigm is supported by the data.

2.3 Theoretical Auction Models

Surveys on theoretical models of auction can be found in Milgrom and Weber (1982) and McAfee and McMillan (1987). In these models, behavior at auctions is modeled as a non-cooperative game. In the discussion that follows, individual bids/bidders are indexed by \( i \) with \( i = 1, \ldots, N_j \), and auctions by \( j = 1, \ldots, n \). Thus \( a_j^i \) refers to the \( i \)-th 'a' in the \( j \)-th auction (for example bid/bidder \( i \) in the \( j \)-th auction). \( a_j \) refers to some 'a' in the \( j \)-th auction (for example all bids/bidders in the \( j \)-th auction, or the number of bidders and the winning bid in the \( j \)-th auction). \( a_i \) refers to the \( i \)-th 'a' in all auctions \( j \). All random variables will be indicated by capital letters, and the realization of a random variable by small letters. As an example, \( a_j^i \) is the realization of the random variable \( A_j^i \). In the simplest set up, following Paarsch (1992), a known number of bidders \( N_j \) compete to purchase an object from one seller; that is, the potential bidders, and the participants are the same in an auction. Each bidder is assumed to be risk neutral.

Within the common-value paradigm, all bidders have the same unknown \emph{a priori} cost \( c \) of performing the task. Each bidder \( i \) forms an independent and unbiased estimator of this cost, \( X_j^i \). The cost estimate, \( x_j^i \), is known only to bidder \( i \) and to no other bidder; they do, however, know the probability distribution from which it has been drawn - \( h(x_j^i|c, \beta) \) - the probability density function of the estimator \( X_j^i \), conditional on the task's unknown cost \( c \) and other parameters \( \beta \) with \( E(X_j^i|c, \beta) = c \). This probability distribution is assumed to be the same across all bidders in an auction making the game symmetric. \( d(c) \), the density of the task's unknown cost is also common knowledge. \( H(x_j^i|c, \beta) \) is the cumulative distribution of the estimator of cost \( X_j^i \) conditional on the unknown cost \( c \). Thus, each bidder \( i \), knowing \( x_j^i, N_j \), the number of potential bidders in auction \( j \), \( h(x_j^i|c, \beta) \), and \( d(c) \), and that all this is common knowledge, formulates his bid, \( b_j^i = e_c(x_j^i) \), to maximize expected profits. The bids, \( b_j^i \), are strictly increasing functions of the estimate of cost, \( x_j^i \).\textsuperscript{8}

The winner of an auction \( j \) is the bidder with the lowest cost estimate, \( x(1 : N_j) \), and

\textsuperscript{8}Note that the assumptions made under the common-value paradigm, imply that only the estimate, \( x_j^i \), of the common unknown cost, \( c \), and the bid, \( b_j^i \) are private information of a bidder \( i \) in auction \( j \).
hence the lowest bid; \( X(1 : N_j) \) is the smallest order statistic for a sample of size \( N_j \) from the distribution of \( X_j^* | c, \beta \). The winning bid in auction \( j \) is (Paarsch, p. 193)

\[
w_j = e_c(x(1 : N_j)) \quad \forall j = 1, ..., n. \tag{1}
\]

In contrast, within the independent-private-values paradigm, each bidder \( i \) has a private cost \( c_j^i \) of performing the task for IOC that is known only to him. This cost is known to him \emph{ex ante} (\textit{i.e.}, before he submits the bid). He does not know the private cost of other bidders. However, each bidder knows that all private costs including his own are drawn independently from the same probability distribution (assumption of symmetry). The probability density function of \( C_j^i \) is given by \( g(c_j^i | \theta_1) \) and cumulative distribution by \( G(c_j^i | \theta_1) \) for all \( i, j \). Knowing \( c_j^i, G(c_j^i | \theta_1), N_j \), and that all this is common knowledge, bidder \( i \) maximizes expected profits to form the equilibrium bid, \( b_j^i \equiv e_p(c_j^i) \), where the equilibrium bid is a strictly increasing function of the private cost, \( c_j^i \).

The winner of an auction \( j \) is the bidder with the lowest cost \( c(1 : N_j) \); \( C(1 : N_j) \) is the smallest order statistic for a sample of size \( N_j \) from the distribution of \( C_j^i \). The winning bid is then given by (Paarsch, p. 195):

\[
w_j = e_p(c(1 : N_j)) = c(1 : N_j) + \frac{\int_{c(1 : N_j)}^{\infty} (1 - G(x_j^* | \theta_1))^{N_j - 1} d_x}{1 - G(c(1 : N_j) | \theta_1)} \quad \forall j = 1, ..., n. \tag{2}
\]

Under the common-value paradigm, once the density of \( X_j^i \), \( h(x_j^i | c, \beta) \), is specified, the density of the smallest order statistic, \( \tilde{h}(\cdot) \), and hence the density of the winning bid can be obtained from equation (1) (Paarsch, p. 196)

\[
f_c(w_j | c, \beta) = \frac{\tilde{h}(e_c^{-1}(w_j), N_j | c, \beta)}{e'_c(e_c^{-1}(w_j))}, \tag{3}
\]

where \( X(1 : N_j) = e_c^{-1}(W_j) \) from equation (1), \( e'_c(\cdot) \) is the Jacobian for the transformation of \( x(1 : N_j) \) to \( w_j \).

\footnote{The private cost, \( c_j^i \), of performing the task for IOC, and the bid, \( b_j^i \), is private information of bidder \( i \) in auction \( j \).}
Similarly, under the private-values paradigm, once \( g(c_j|\theta_1) \) is specified, the density of the smallest order statistic, \( C(1 : N_j) \), \( \tilde{g}(\cdot) \), is known. From equation (2), the density of the winning bid is (Paarsch, p. 196)

\[
f_p(w_j|\theta_1) = \frac{\tilde{g}(e_p^{-1}(w_j), N_j|\theta_1)}{e_p'(e_p^{-1}(w_j))},
\]

where \( C(1 : N_j) = e_p^{-1}(W_j) \) from equation (2); \( e_p'(\cdot) \) is the Jacobian for the transformation of \( c(1 : N_j) \) to \( w_j \), and

\[
w_j > \int_{c(1:N_j)}^\infty (1 - G(\xi|\theta_1))^{N_j-1}d\xi.
\]

The above inequality follows from equation (2). It is the equilibrium bid of a bidder who has a zero cost of procuring oil for IOC. This bid provides a lower bound to any equilibrium bid submitted to IOC.

### 2.4 Empirical Auction Models

This paper takes a structural approach to the estimation of auction models. The structure is represented by the density function of the winning bid, given by equation (3) and (4), which evolve from auction theory as explained in Section 3. Since the distribution of private values is unknown to the researcher, I shall make assumptions about the joint distribution of \( X_j, C \) in the common-value paradigm and the distribution of and \( C_j|\theta \) in the independent-private-values paradigm, as in Paarsch. The density function of the winning bid can then be obtained from (3) under the common-value paradigm and from (4) under the private-values paradigm.

In addition to the assumptions made in Section 3, I also assume that the auctions are sufficiently independent so that the information revealed at one auction is irrelevant to the next one. Within the common-value paradigm, independent draws are made in each auction from the joint distribution of \( X_j, C \); in the independent-private-values auction, independent draws across auctions are made from the distribution of \( C_j \). I have justified the independence assumption for the auctions conducted by IOC in Section 2. This assumption implies that individual bids and the winning bids are also independent across auctions.

Further, as justified in Section 2, the distribution of the private values, stays unchanged across auctions. The distribution of the winning bid (given by (3) and (4)), on the other
hand, varies from auction to auction depending on the competition amongst the bidders. This competition is embodied in the number of bidders in each auction minus one \((N_j - 1)\), which may differ across auctions.

Under the common-value paradigm, using linear equilibrium strategies and making assumptions which will admit these linear equilibrium strategies, Paarsch (p. 202) derives the density of the winning bid. Specifically, the estimator of cost follows a two-parameter Weibull distribution, \(X_j|Q, c \sim WE(Q, \frac{\Gamma^Q}{cQ(Q - 1)}), c, Q > 0\). The parameters \([c, \beta']\) that characterize the distribution of the private signal, \(X_j\), are given by \([c, Q]\)', when the estimator of cost follows a Weibull distribution. \(c\) is the cost of procuring oil for IOC that is unknown to the bidder a priori. \(Q\) is inversely related to the standard deviation of the estimator of cost, \(X_j\) (Smiley, 1979, p. 38); it is interpreted as the scale parameter of the distribution \(X_j|Q, c\). Here, \(\Gamma(\alpha)\) is a gamma function. The distribution of the winning bid, \(W_j|c, Q\), is a two-parameter Weibull distribution; \(W_j \sim WE(Q, \frac{c^Q}{\Gamma})\) \(\forall j\). The probability density function of the winning bid is

\[
f_c(w_j|c, Q) = \gamma \frac{c^Q}{\gamma} Qw_j^{Q-1}e^{-w_j^{\frac{c^Q}{\Gamma}}} \quad \forall j \quad w_j > 0. \tag{6}
\]

\(\gamma \equiv \gamma(N_j, Q)\) depends on the data (number of bidders in the \(j\)-th auction, \(N_j\) and \(Q\); it is given by

\[
\gamma \equiv \gamma(N_j, Q) = \left[\Gamma\left(1 + \frac{1}{Q}\right) \frac{Q(N_j - 1) - 1}{Q(N_j - 1)}\right]^Q.
\]

The parametric assumptions made about \(X_j|Q, c\) and \(C\) identifies the common-value model even though the common-value model and the affiliated private-values model are observationally equivalent in a nonparametric framework from data on bids for a first-price, sealed-bid auction with a reserve price (Laffont and Vuong, 1996). As Laffont and Vuong (p. 416) point out, the

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10 Paarsch (p. 202), following Smiley has assumed that \(d(c) \propto \frac{1}{c}\), where \(d(c)\) is the density function of \(C\). This is one of the sufficient conditions needed to obtain bids that are proportional to the estimate of cost in equation (1).

11 The standard deviation of the estimator of cost, \(\sigma_{X_j}\) (Smiley, p. 38) is

\[
\sigma_{X_j} = \sigma\left(\frac{1}{Q}\right) c = \frac{\Gamma(1 + 2/Q) - \Gamma^2(1 + 1/Q)}{\Gamma(1 + 1/Q)}^{1/2}.
\]

12 \(\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt\).
equilibrium bid is a function of quantities unobserved by the econometrician, the private signals and the common, unknown component, as well as the distribution of these unobservables. By making parametric assumptions about the distribution of the unobservables, $X_j$ and $C$, the common-value paradigm leads to a different mapping from unobservables to the winning bid in equation (1) than the affiliated private-values model.

Within the private-values paradigm, if bids are additive in cost, then as Paarsch has shown $C_j$ follows an exponential distribution with parameter $\theta_1$. The winning bid follows an exponential distribution that is truncated from below; $W_j \sim EXP((\theta_1 N_j)^{-1}, \frac{1}{(N_j-1)\theta_1}),$ with density function

$$f_p(w_j|\theta_1) = \theta_1 N_j e^{-\theta_1 N_j(w_j-(N_j-1)\theta_1)}, \quad w_j > \frac{1}{\theta_1 (N_j - 1)}. \quad (7)$$

2.5 Methodology

The question addressed in this paper is which paradigm, the common-value or the private-values, is a more probable explanation of the low-price, sealed-bid auctions conducted by IOC, once data on these auctions are available. Thus, the aim is to compare the common-value paradigm and the private-values paradigm via the posterior odds ratio.

The basic relation underlying the Bayesian estimation of auction models is

$$f(\theta, w) = f(\theta) f(w|\theta). \quad (8)$$

$w = (w_1, ..., w_n)$ are the realized winning bids. The unknown parameters that govern the distribution of $W_j$ for all $j$ are specified generically by $\theta$; $\theta \equiv [c Q]'$ for the common-value paradigm, and $\theta \equiv \theta_1$ in the private-values paradigm. Beliefs about these unknown parameters, prior to the data on the winning bids and the number of bidders in the $n$ auctions being observed, are given by $f(\theta)$, the prior probability density function of $\theta$. $f(w|\theta)$ when viewed as a function of $\theta$ given the data is the likelihood function. I have data on the winning bids, $w$, and the number

$^{13}$The parameters of the distribution of the latent variable, $X_j$ in the common-value paradigm and $C_j$ in the private-values paradigm, are given by $\theta$. The parameters of the distribution of the winning bid are given by $\theta$ and the number of potential bidders, which is unknown. Since, I have assumed that the number of potential bidders, $N_j$, is identical to the bidders participating in the auction, the parameters of the distribution of the winning bid are given by $\theta$. 

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of bidders, \( N = (N_1, ..., N_n) \), in the \( n \) auctions. Since the bidders in the \( n \) auctions are assumed to be fixed, \( f(w|\theta) \equiv \ell(\theta; w, N) \) is the likelihood function.

To compare models, the method proposed in this paper requires the posterior odds ratio of the two models being compared. The posterior odds ratio for model \( m \) versus model \( m' \), \( P_{m,m'} \), where \( m \neq m' \), is given by

\[
P_{m,m'} = \frac{P(m|w)}{P(m'|w)} = \frac{P(m)P(w|m)}{P(m')P(w|m')} = \frac{P(w|m)}{P(w|m')},
\]

assuming equal prior probability for both models \( m \) and \( m' \). Here \( m, m' \) refer to the common-value and the private-values paradigms. \( P(w|m) \) is referred to as the marginal likelihood under model \( m \). \( P(w|m)/P(w|m') \) is referred to as the Bayes factor for model \( m \) versus model \( m' \). The numerator and the denominator of (9) can be evaluated as follows

\[
P(w|m) = \int_\Theta f(\theta|m)\ell(\theta; w, N, m)d\theta.
\]

\( \ell(\theta; w, N, m) \) is the likelihood function under model \( m \); and \( f(\theta|m) \) represents prior beliefs under model \( m \).

If a sufficient statistic \( t_m \) exists for the likelihood function under model \( m \), the likelihood function under model \( m \) can be written as follows, using the Factorization Theorem (Poirier, 1994, p. 223)

\[
\ell(\theta; w, N, m) = \delta_m f(t_m|\theta),
\]

where \( \delta_m \) is a non-negative constant that is determined by data (winning bids and number of bidders in the 37 auctions), and that differs across the \( m \) models. \( f(t_m|\theta) \) is the probability density function of the sampling distribution of the sufficient statistic \( t_m \) under model \( m \). Using equation (11), the marginal likelihood under model \( m \) given by equation (10), is

\[
P(w|m) = \delta_m \int_\Theta f(t_m|\theta)f(\theta|m)d\theta = \delta_m f(t_m),
\]

where \( f(t_m) \) is the conjugate prior predictive density of the sufficient statistic \( t_m \) under model \( m \). That is, \( f(t_m) \) embodies beliefs about the sufficient statistic \( t_m \) prior to data being observed.

To compute the posterior odds ratio of model \( m \) versus \( m' \), the marginal likelihood for
models $m$ and $m'$ has to be obtained. Computing the marginal likelihood of a model requires the specification of the likelihood and prior elicitation. Prior elicitation requires specification of the distribution of $\theta$ and the specification of the hyperparameters (that is, the parameters of the prior distribution of the parameter $\theta$) under each model. A natural conjugate prior is specified with respect to the likelihood under the two models.\footnote{A natural conjugate prior is a conjugate prior that has the same functional form as the likelihood function.} The likelihood function and the prior elicitation under the two models are now described.

\subsection*{2.5.1 The Common-Value Paradigm}

I have to specify the prior density functions for $\theta = Q, c$ and the likelihood. The parameter $Q$ measures the accuracy with which cost is being estimated.\footnote{$Q$ is measured in U.S. dollars per metric tonne.} Thus given $c$, $Q$ is inversely related to the standard deviation of the estimator of cost, $\sigma_{X_j}$; or $Q$ is directly related to the precision of the estimator of cost (see footnote 10). The precision of the estimator of cost, $\frac{1}{\sigma_{X_j}}$, reflects the uncertainty that a bidder faces regarding the private signal about the unknown cost, $c$, of other bidders.

Since a natural conjugate prior does not exist with respect to a Weibull likelihood, a dogmatic prior is specified for $Q$; that is $Q = 1$. The Weibull probability density function for the winning bid given by (6), then reduces to an exponential probability density function; specifically, $W_j \sim EXP(\gamma/c)$, where $\gamma = \Gamma(2)\frac{N_j - 2}{N_j - 1}$. The density function of the winning bid is given by

$$f_c(w_j|c) = \frac{\Gamma(2) N_j - 2}{c} \frac{e^{\Gamma(2)\frac{N_j - 2}{N_j - 1} \frac{1}{c} w_j}}{N_j - 1}, \quad w_j > 0. \quad (13)$$

The likelihood is now constructed using the density function of the winning bid in equation (13). Using equation (11), the likelihood is

$$\ell(c; w, N) = \delta_c GA(t_c|n, \frac{1}{c}), \quad (14)$$

where $\delta_c$ is determined by data $(w, N, n = 37)$, and $t_c$ is the sufficient statistic for the likelihood given by equation (14). $\delta_c$ and $t_c$ are given in appendix A. $GA(t_c|n, \frac{1}{c})$ in equation (14) is the density function of a Gamma distribution (see appendix A); it is the sampling distribution of
the sufficient statistic \( t_c \) which has been specified generically as \( f(t_m|\theta) \) in equations (11) and (12).

The parameter \( c \) is the common and unknown cost of performing the task for IOC by the bidders. A conjugate prior density with respect to the likelihood given in equation (14), is specified for \( c \). This conjugate prior density for \( c \) is the Inverted-Gamma density (Bernardo and Smith, 1994, p. 438)

\[
f(c) = IG(c|n_c, \gamma \lambda_c),
\]

where \( \gamma = \Gamma(2)(\frac{N_c - 2}{N_c - 1}) \), and \( n_c, \lambda_c \) and \( N_c \) are hyperparameters which have to be specified. I will show in Section 5.3 that \( N_c \) is the number of bidders in a typical auction. It is distinct from \( N_j \), the number of bidders in auction \( j \), which is data.

The marginal likelihood, \( P(w|m = cv) \), is derived in appendix A. It is given by

\[
P(w|m = cv) = \delta_c GG(t_c|n_c, \gamma \lambda_c, n);
\]

\( GG(\cdot) \) is a Gamma-Gamma density function (Bernardo and Smith, 1994, p. 304) with parameters \( n_c, \gamma \lambda_c \) and \( n \), the number of auctions. This Gamma-Gamma density function represents the conjugate prior predictive density, \( f(t_c) \), of the sufficient statistic \( t_c \) indicated generically by \( f(t_m) \) in equation (12).

### 2.5.2 The Independent-Private-Values Paradigm

From equation (7), the density function of the winning bids follow an exponential distribution. The likelihood, using equation (11) is given by

\[
\ell(\theta_1; w, N) = \delta_p GA(t_p|n, \theta_1), \quad w_j > \frac{1}{\theta_1(N_j - 1)} \quad \forall j;
\]

\( t_p \) is the sufficient statistic for the likelihood specified by (17). \( GA(t_p|\cdot) \) is the sampling distribution of the sufficient statistic \( t_p \), with parameters \( n \) and \( \theta_1 \). \( GA(t_p|\cdot) \) is a Gamma distribution truncated from below, with \( t_p > \frac{1}{\theta_1} \sum_{j=1}^{n} \frac{N_j}{N_j - 1}; \) it is given in appendix B. \( \delta_p \) is determined by data only and is specified in appendix B.

Prior elicitation requires specifying the distribution of the parameter \( \theta_1 \). \( \theta_1 \) is the precision
of the private cost, \( C^i_j \), the latent variable.\(^{16}\) The precision reflects the uncertainty of a bidder about the private cost of other bidders. Thus if \( \theta_1 \) is large, then each bidder knows not just his own cost of procuring crude oil for IOC, but the cost of other bidders as well.

A conjugate prior density with respect to the likelihood given by equation (17) is specified for \( \theta_1 \). This conjugate density is a Gamma density truncated from above (see appendix B),

\[
f(\theta_1) = GA(\theta_1|\theta_p, N_\alpha \lambda_p), \quad 0 < \theta_1 < \theta_1, \tag{18}
\]

where \( \theta_p, N_\alpha, \lambda_p \) and \( \theta_1 \), the upper bound of \( \theta_1 \), are the hyperparameters, and have to be specified. Since \( C^i_j \) has been assumed to follow an exponential distribution with parameter \( \theta_1 \), \( \theta_1 \) is the upper limit to the precision of the private cost, \( C^i_j \).

I obtain the marginal likelihood, \( P(w|m = pv) \), in appendix B. It is given by

\[
P(w|m = pv) = \delta_p GG(t_p|\theta_p, N_\alpha \lambda_p, \eta), \quad w_j > \frac{1}{\theta_1(N_\alpha - 1)} \quad \forall j. \tag{19}
\]

\( GG(t_p|\cdot) \) is the density function of the Gamma-Gamma variate, \( t_p \), that is truncated from below; that is \( t_p > \frac{K}{\theta_1} \). It is the conjugate prior predictive density of the sufficient statistic \( t_p \).

### 2.5.3 Hyperparameter Specification

The posterior odds ratio of the common-value paradigm versus the private-values paradigm, is obtained by dividing \( P(w|m = cu) \) (see (16)) by \( P(w|m = pv) \) (see (19)), as demonstrated in equation (9). It is given by

\[
P_{cu,pv} = \frac{\left[ \frac{\Gamma(n+2)}{\Gamma(n+2)} \prod_j \left[ \frac{N_j}{\left( N_j - 1 \right)} \Gamma(2) \lambda_c \right] \right]^n \left[ \frac{t_p + N_\alpha \lambda_p}{t_c + \left( \frac{N_\alpha - 2}{N_\alpha - 1} \right) \Gamma(2) \lambda_c} \right]^{n+n}}{\left[ \frac{N_j}{\left( N_j - 1 \right)} \Gamma(2) \lambda_c \right]^{n+n}}
\]

\[
\left[ \frac{\Gamma(n+n)}{\Gamma(t_p + \lambda_p N_\alpha \theta_1)(n+n)} \right] \frac{\Gamma_{\lambda_p N_\alpha \theta_1}(n)}{\Gamma(n)} \right]. \tag{20}
\]

The posterior odds ratio consists of terms that involve the hyperparameters under the common-value paradigm, the private-values paradigm and data.

\(^{16}\) \( \theta_1 \) is measured in U.S. dollars per metric tonne.
The posterior odds ratio given by equation (20) indicates how the prior opinion about which paradigm describes the low-price sealed-bid auctions conducted by the IOC, changes, once the data on the winning bids is observed. It is possible that the revision of the prior odds after observing the data is sensitive to the prior information used. Hence, a sensitivity analysis of the posterior odds ratio to prior information is required; i.e., it is vital to ascertain whether the revision of prior information, on observing the winning bids, is robust across the prior information. In this paper, the sensitivity of the posterior odds ratio is studied by keeping the prior distribution, \( f(\theta|m) \), of the parameter \( \theta \), unchanged under both models, and varying the value of the hyperparameters that govern the distribution of \( \theta \). As noted above, \( \theta = c \) in the common-value paradigm, and \( \theta \equiv \theta_1 \) under the private-values paradigm.

Therefore, to obtain the posterior odds ratio and to examine its sensitivity to the prior information, the next step is to specify a range of values for the hyperparameters under the two models. These hyperparameters are \( n_c, \lambda_c, \lambda_p, \lambda_p \xi_1 \) and \( N_\ast \), the number of bidders in a typical auction.

To specify the hyperparameters, it is instructive to visualize a hypothetical auction before observing the data. Let \( W_\ast \) be some single, yet to be observed winning bid, and \( N_\ast \), the number of bidders in this hypothetical auction. The specification of hyperparameters is then, in part, based on three assumptions. First, the expected values of the prior predictive under the common-value paradigm, \( (E_c(W_\ast)) \), and the private-values paradigm, \( (E_p(W_\ast)) \), for some single, yet to be observed, winning bid, \( W_\ast \), are assumed to be identical. This means that if the aim was to predict a yet to be observed winning bid, \( \bar{w}_\ast \), for some hypothetical auction, prior to data on the winning bid being observed; and the loss entailed in specifying \( \bar{w}_\ast \) for \( w_\ast \) is the quadratic loss,\(^{17}\) then the two paradigms, would predict the same value for the yet to be observed winning bid, prior to data on the winning bid being observed. Thus, \( \bar{w}_\ast = E_c(W_\ast) = E_p(W_\ast) \). The density function and the expected value of the prior predictive under the common-value and the private-values paradigms is derived in appendix C and D, respectively. Equating the expected value of \( W_\ast \) under the two paradigms and assuming that \( n_c = n_p = n \).

\(^{17}\)The loss function is given by \( \kappa(\bar{w}_\ast - \bar{w}_\ast)^2 \).
gives the following condition restricting the hyperparameters in the priors for the two models

$$\lambda_c = \left[ \frac{1}{\theta_1(N_\ast - 1) + \lambda p \frac{\Gamma(N_b \theta_1 (n - 1))}{\Gamma_p(N_b \theta_1) + \Gamma(2)}} \right] \left[ \frac{(N_\ast - 1) (n - 1)}{(N_\ast - 2) \Gamma(2)} \right].$$

Equating $\eta_c$ and $\eta_p$ to $\eta$ implies using the same amount of prior information under both paradigms. I now demonstrate that $\eta_c$ and $\eta_p$ are the weights with which the prior information is combined with sample information under the common-value paradigm and the private-values paradigm, respectively; and these weights can be interpreted as the size of the "fictitious" prior sample.

To gain insight into what the hyperparameter $\eta$ reflects, it is instructive to look at the posterior predictive distribution of some single out-of-sample winning bid $W_{n+i}$. For example, if $i = 1$, then the distribution of interest is the distribution of the winning bid for the unobserved 38-th auction, $W_{38}$. Henceforth, the out-of-sample winning bid that has to be predicted after observing the data will be indicated by $W_*$ instead of $W_{n+i}$. The number of bidders in this out-of-sample auction will be indicated by $N_\ast$, the number of bidders in the hypothetical auction that was visualized before observing the data. Note the difference between the prior predictive winning bid, $W_\ast$, and the posterior predictive winning bid, $W_*$. Both variables indicate the winning bid in an out-of-sample auction. The difference is that the former refers to an out-of-sample winning bid before the sample is observed; and the latter refers to an out-of-sample winning bid after the winning bids, $w$, and the number of bidders, $N_j$, $j = 1, \ldots, 37$, for the 37 auctions have been observed. The density function of the prior and posterior predictive winning bid is, as a result different. The posterior predictive density for $W_*$ under the common-value and the private-values paradigms is derived in appendix C and D, respectively. If the loss incurred while predicting $W_*$ is the quadratic loss function, then the optimal predictor, $\hat{w}_*|w$, of the yet to be observed winning bid, $W_*$, is just the posterior mean of $W_*|w$.

Under the common-value paradigm, the optimal predictor, $\hat{w}_*|w$, of the yet to be observed,
out-of-sample winning bid, $W_*$, is the expected value of $W_*|w$, whose density function is given by (C.3) in appendix C. It is

$$
\hat{w}_*^C|w = \left[ \frac{N - 1}{(n - 1) + n} \right] \hat{w}_* + \left[ \frac{N - 1}{(n - 1) + n} \right] E(W_*),
$$

(22)

where $\hat{w}_*$ is the optimal prior predictor of the yet to be observed winning bid $W_*$, prior to data on the winning bids and the number of bidders in the 37 auctions being observed. As mentioned above, this is just the expected value of the prior predictive, $E_c(W_*)$, given in appendix C, equation (C.2). $E(W_*)$ is the expected value of the out-of-sample winning bid, $W_*$, whose density function is the exponential density function specified by equation (13). Equation (22) indicates that the optimal predictor of the out-of-sample winning bid, $\hat{w}_*^C|w$, is just a weighted combination of the prior predictor of the winning bid, $\hat{w}_*$, and the expected value of $W_*$. Since the prior is a natural conjugate prior, it has a "fictitious" sample interpretation (Poirier, 1994, p. 292). Thus, the prior can be viewed as arising from a fictitious sample from the same underlying population that generated the sample of winning bids, $w = w_1, ..., w_{37}$. The optimal predictor, given by (22), can then be viewed as having been obtained by combining sample information with additional "fictitious" sample information (the prior information). The proportion in which these two types of information is combined is given by the weights $\frac{N}{N_* - 1} n$ and $n - 1$, respectively. Since $n$ is the size of the actual sample ($n = 37$), the weight attached to the sample information is related to the sample size. Thus $n$ can be interpreted as the size of the "fictitious" sample. This means that the weights in which the two types of information, $\hat{w}_*$ and $E(W_*)$ are combined, are just the relative sizes of the fictitious sample and the actual sample, respectively. Thus, varying $n$ changes the proportion in which the "fictitious" sample information (prior information) and the actual sample information are combined to produce the optimal predictor of the out-of-sample winning bid. If $n = 1$, then no prior information is used to generate the optimal predictor. The optimal predictor represents only sample information; i.e., the prior for $W_*$ is noninformative or diffuse. As $n$ increases, the size of the "fictitious" sample increases.

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20 The expected value is given by

$$
E(W_*) = \hat{c}^{ml} \frac{N_* - 2}{N_* - 1} \frac{1}{\Gamma(2)}
$$

where $\hat{c}^{ml}$ is the maximum-likelihood estimate of $c$. 

24
sample increases, and therefore more prior information is being used to predict the unobserved, out of sample winning bid, \( W_s \); i.e., the prior is becoming informative as \( n \) increases. What has to be ensured is that the prior does not become "too" informative (that is, more informative than the data). In view of the actual sample size being 37, the size of the "fictitious" prior sample was set at \( n = 3, 5, 7 \).

The third assumption made to specify the hyperparameters under the two models is to make the uncertainty faced by a bidder regarding the private signal of other bidders comparable under the two models. This is done since the low-price sealed-bid auctions have been modeled as a static, noncooperative game of incomplete information. The incomplete information is reflected in each bidder knowing his own private value, but is uncertain concerning the private value of other bidders. Under the common-value model, each bidder is uncertain about the common cost, \( c \), of performing the task for IOC, as well as about other bidders' estimate of cost. Under the private-values paradigm, each bidder is uncertain about the cost at which other bidders will procure crude oil for IOC.

Uncertainty regarding the private values is captured by the precision of the latent variables \( X_j^t | c, Q \) and \( C_j^t | \theta_1 \) under the common-value and the private-values paradigm, respectively.\(^{21}\) Greater precision implies less uncertainty. The uncertainty facing a bidder about the private signal of other bidders is important as it affects the cost at which IOC procures oil through the low-price, sealed-bid auctions. The greater is this uncertainty, the higher is the padding of the bidders when they submit the sealed bids; the result is a higher cost of procuring oil for IOC. Thus, a meaningful comparison of the two paradigms requires that the uncertainty about private value is comparable under the two models. The specification of hyperparameters ensures this comparability. This is clarified in the discussion below.

Under the common-value paradigm, the parameter \( Q \) was directly related to the precision of the estimator of cost (see footnote 10). In equation (13) I assumed that \( Q = 1 \). Under the private-values paradigm, a truncated Gamma distribution has been assumed for \( \theta_1 \), with \( 0 < \theta_1 < \theta_1 \). Thus \( \theta_1 \) is the upper limit to the precision of the private cost, \( C_j^t \), since \( C_j^t | \theta_1 \) follows an Exponential distribution. To ensure comparability of the precision that the bidders face under the two models, \( \theta_1 \) is set to take values, \( \theta_1 = 1.00, 1.01, 1.10 \). The latter two values

\(^{21}\) Precision is being referred to as the inverse of the standard deviation here.
of $\theta_1$ examine the sensitivity of the posterior odds ratio, when the uncertainty about private values under the two paradigms is different. Specifically, the uncertainty under the private-values paradigm is less than under the common-value paradigm. Or the precision of the private value under the private-values paradigm is 1% ($\theta_1 = 1.01$), and 10% ($\theta_1 = 1.10$) higher than that under the common-value paradigms. These are reasonable values of $\theta_1$, with which to conduct a sensitivity analysis; larger values would mean a significant advantage to bidders under the private-values paradigm, in terms of amount of information available about private values, compared to the information available under the common-value paradigm.

Since $N_*$ is the number of bidders in some typical auction, it has been varied to take values $N_* = 5, 7, 9$. $\hat{w}_* = E_c(W_*) = E_p(W_*)$ is just the predicted bid, prior to observing data, in a hypothetical auction. This has been set to take values 100, 125 and 150. The higher values of the predicted bid, $\hat{w}_*$, in the hypothetical auction indicates that the bidders are bidding more conservatively in the hypothetical auction. The hyperparameter $\theta_1$ has been specified to take values 1.00, 1.01 and 1.1; I have already explained why this has been done. For a specific value of $N_*$, $\hat{w}_*$ and $\eta$, the hyperparameter $\lambda_c$ is determined from the relation, $\hat{w}_* = \Gamma(2) \frac{N_*-2}{N_*-1} \frac{\lambda_c}{\eta-1}$, the latter being the expected value of the prior predictive under the common-value paradigm, $E_c(W_*)$, given by equation (C.2) in appendix C. Then $\lambda_p$ is obtained from (21). Note that for specific values of $\eta$, $\theta_1$ and $N_*$, specifying $\hat{w}_*$, the predicted winning bid in the hypothetical auction, specifies both $\lambda_c$ and $\lambda_p$.

2.6 Results

In tables 4-6, I report the posterior odds ratio, $P_{cv,pv}$, or the Bayes factor in favor of the common-value paradigm. Each table reports the posterior odds ratio for a specific value of $\eta$. Within each table, the columns report the posterior odds ratio for different values of $N_*$, the number of bidders in some hypothetical auction. A row in a table reports the posterior odds ratio for a particular value of $\hat{w}_*$ (or $\lambda_c, \lambda_p$) and $\theta_1$.

For all values of $\hat{w}_*, \eta, \theta_1$, or $N_*$, the posterior odds ratio, $P_{cv,pv} < 1$. This means that the prior odds of 1:1 for the common-value paradigm versus the private-values paradigm, are

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Conservative behavior of the bidders in the hypothetical auction lead to high winning bids.
updated in favor of the private-values paradigm, after the data is observed. In other words, the posterior odds ratio indicates that the learning from data that is going on under the two paradigms is different. This difference in the learning from data is evident from the plots of the prior predictive density function and the posterior predictive density function of some out-of-sample winning bid $W_*$, under the two models (equations (C.1) and (C.3), appendix C for the common-value paradigm, and (D.1) and (D.3), appendix D for the private-values paradigm). In figures 1-2, I plot the prior and posterior predictive density function for $W_*$ for the common-value and the private-values paradigms, for $\theta_* = 150$, $n= 3$, $\theta_1 = 1$ and $N_*= 9$; similar plots were obtained for different values of these hyperparameters. The plot for the private-values paradigm (figure 1) reveals considerable divergence between the prior and the posterior predictive density function of some out-of-sample winning bid. This divergence indicates that beliefs about the out-of-sample winning bid formed before observing the data are different from the beliefs about the out-of-sample winning bid after the data is observed under the private-values paradigm. On the other hand, the prior and posterior predictive density functions almost coincide for the common-value paradigm (figure 2); i.e. if I start out with the prior belief that the out-of-sample winning bid is a bid under the common-value paradigm, then this belief gets revised marginally once data is observed. In short, prior beliefs about which paradigm generated the out-of-sample winning bid are revised when I start off with the prior belief that the private-values paradigm generated this out-of-sample winning bid.

The explanation for this result is institutional and based on anecdotal evidence. I have explained in Section 2, that the bidders in these auctions procure crude-oil through informal means too; and it is this informal market that the IOC is targeting through these auctions to meet its short-term demand-supply imbalances. The IOC is able to obtain crude-oil through these auctions at prices below the world market prices of crude-oil. This can be seen by noting that maximum winning bid in the auctions for which I have data, reported in table 1, is below the world crude-oil prices for the years 1991-95 that I have reported in table 2, rows 1-2. The quantity and the price at which a bidder is able to procure crude oil through these informal means is exclusively the private information of the bidder, making the private-values paradigm a more probable explanation of these auctions conducted by the IOC.

Despite the private-values paradigm being more probable once the winning bids are ob-
served, certain shifts in the posterior odds ratio are worth noting. First, consider the effect of increasing $\mathcal{A}_1$ (compare rows 1-3, rows 4-6, rows 7-9 of tables 4-6) for specific values of $N_*$ and $n$. As long as the number of bidders in the hypothetical auction is 5, $N_*$ = 5 (see column 1 of tables 4-6 and compare rows 1-3, 4-6 and 7-9), as $\mathcal{A}_1$ increases, the posterior odds favor the private-values paradigm more strongly (posterior odds ratio is decreasing). This is in accordance with the earlier discussion regarding the decrease in uncertainty about private values that bidders face under the private-values paradigm as $\mathcal{A}_1$ increases. However, the effect of increasing $\mathcal{A}_1$ on the posterior odds is marginal for $N_*$ = 7 (see column 2 of tables 4-6 and compare rows 1-3, 4-6 and 7-9). And for $N_*$ = 9, a moderate shift of the posterior odds towards the common-value paradigm is noticed in some cases (see column 3 of tables 5 and 6, and compare rows 4-6 and 7-9; note that these rows correspond to $\mathcal{W}_* = 125$ and 150 respectively).

Second, consider the effect on posterior odds of an increase in $N_*$, the number of bidders in a hypothetical auction, for specific values of $n$, $\mathcal{A}_1$, and $\mathcal{W}_*$ (compare columns 1-3 of tables 4-6 and look across rows). The posterior odds favor the private-values paradigm more strongly ($P_{cv,pv}$ is decreasing) for $\mathcal{W}_* = 100$ as $N_*$ increases (see across rows 1-3 of tables 4-6) from 5 to 9. For $\mathcal{W}_* = 125$ and 150 (see across rows 4-6 and 7-9 respectively, of tables 4-6), the posterior odds continue favoring the private-values paradigm more strongly when $N_*$ is increased from 5 to 7. But when $N_*$ is increased to 9, the posterior odds shift moderately in favor of the common-value paradigm (that is, $P_{cv,pv}$ is increasing).

The slight increase in $P_{cv,pv}$ when $N_*$ = 9 and $\mathcal{W}_* = 125, 150$ indicate what is going on in the hypothetical auction visualized to specify the hyperparameters. Increasing $N_*$ to 9 and $\mathcal{W}_*$ to 125 and 150 indicates that in the hypothetical auction the bidders are bidding more conservatively as the number of potential bidders in the hypothetical auction, $N_*$, is increasing. This accords with the winner's curse argument in the common-value paradigm, where bidders bid aggressively at first when the number of potential bidders increases; i.e., the bids decrease with the number of potential bidders. But then recognizing the winner's curse, their bidding behavior becomes conservative; i.e., the bids increase with the number of potential bidders. Thus the hypothetical auction is apriori making the common-value paradigm more probable than the private-value paradigm when $N_*$ = 9 and $\mathcal{W}_* = 125, 150$. This gets reflected in posterior odds ratio shifting moderately in favor of the common-value paradigm, even though it is still
strictly less than 1.

Third, consider the effect on the posterior odds of increasing the size of the "fictitious" prior sample, \( \eta \), holding \( \xi_1 \), \( \psi \), and \( N_* \) fixed at specific values (compare corresponding row-column entries of tables 4-6). The posterior odds favor the common-value paradigm moderately as \( \eta \) is increased. This reflects the effect of increasing prior information which equates the prior predicted value of some yet to be observed winning bid under the common-value and the private-values paradigm.

2.7 The Affiliated-Values Paradigm

The symmetric common-value paradigm and the symmetric independent-private-values paradigm discussed above can be interpreted as the polar cases of the symmetric affiliated-values paradigm (McAfee and McMillan (1987, pp. 705-706), Milgrom and Weber (1982) and Laffont and Vuong (1996, p. 415)). The symmetric affiliated-value paradigm has components of both the common-value and the private-values paradigms; there is a common component that is unknown to all bidders, and each bidder also has a cost that is private to this bidder. The distribution that is common knowledge, in an auction \( j \) is now the joint distribution \( c, c_1^j, ..., c_N^j | \theta \). Also the utility of a potential bidder \( i \), in the \( j \)th auction, is now given by \( u(c, c_i^j) \). Bidder \( i \) in the \( j \)th auction will form her equilibrium bid knowing the number of bidders \( N_j \), her own private cost \( c_i^j \), the joint distribution \( c, c_1^j, ..., c_N^j | \theta \), \( u(c, c_i^j) \), and that all this is common knowledge. When \( c = 0 \), \( u(c, c_i^j) = c_i^j \), and the \( c_i^j \) are independently distributed across all \( i \) bidders in the \( j \)th auction, then the affiliated-values paradigm reduces to the private-values paradigm. If \( c_i^j = 0 \) for all \( i \) bidders in the \( j \)th auction and \( u(c, c_i^j) = c \), then the affiliated-values paradigm reduces to the common-value paradigm.

The advantage of comparing models through the posterior odds ratio is that it is possible to ascertain whether the auction conducted by IOC have components of both the common-values and the private-values paradigms, and are as a result, better explained by the symmetric affiliated-values paradigm. Since, the common-value and the private-values paradigm are interpreted as polar cases of the symmetric affiliated-value paradigm, a posterior odds ratio that indicates the common-value and the private-values to be equally probable, points to the auctions
having components of both the common-value and the private-values paradigm. A posterior odds ratio close to one will imply that the symmetric common-value and the symmetric private-value paradigms are equally probable explanations of the low-price sealed-bid auctions held by the IOC. Comparison of the affiliated-values paradigm with the common-value paradigm and the private-values paradigm is then a meaningful exercise.

The results discussed in Section 6 indicate that posterior odds of the common-value versus the private-values paradigm, are not close to one, implying that the symmetric affiliated-value model is not supported by the data.

2.8 Conclusions

I conclude that for the low-price sealed-bid auctions conducted by IOC, the 1 : 1 prior odds for the common-value paradigm versus the private-values paradigm, are updated towards the private-values paradigm, after the winning bids are observed. Further this conclusion is robust across prior specification; specifically across the specification of the hyperparameters $\bar{w}, u, N_*, \theta_1$. Since the world price of crude-oil is relatively stable over the period in which these auctions were held, so that to an approximation, the bidders know the world crude-oil price, the underlying bidding model assumes that the bids are a function of the cost of procuring crude-oil oil, which is an independently and identically distributed shock across auctions. The explanation for the data favoring the independent private-values paradigm is the access of the bidders to the informal nature of the market for crude oil that the IOC is targeting through these auctions; this access is private information of the bidders. The fact that low-price, sealed-bid auctions are being conducted by IOC, also supports my result informally. IOC presumably knows these are not common-value auctions since procuring crude oil through low-price, sealed-bid auctions would yield the lowest expected revenue to IOC, compared to other auction forms, under the common-value paradigm following the results of Milgrom and Weber.
APPENDIX A
Marginal likelihood for the common-value paradigm

In this appendix I specify the quantities \( \delta_c \), \( t_c \) and \( GA(t_c|n, \frac{1}{c}) \) given in equation (14), the likelihood function under the common-value paradigm. \( \delta_c \) is

\[
\delta_c = \Gamma^n(2) \frac{\Gamma(n)}{t_c^{n-1}} \prod_j \left( \frac{N_j - 2}{N_j - 1} \right).
\]

\( t_c \), the sufficient statistic for the likelihood function under the common-value paradigm given by equation (14), is

\[
t_c = \Gamma(2) \sum_{j=1}^n \frac{N_j - 2}{N_j - 1} w_j.
\]

\( GA(t_c|n, \frac{1}{c}) \), the density function of a Gamma distribution is given by

\[
GA(t_c|n, \frac{1}{c}) = \left( \frac{1}{c} \right)^n \Gamma(n) t_c^{n-1} e^{-\frac{1}{c} t_c}.
\]

The marginal likelihood under the common-value paradigm, \( P(w|m = cv) \), is obtained from equations (14) and (15) using equation (12). It is given by

\[
P(w|m) = \delta_c \int f(c^{-1}) f(t_c|c^{-1}) dc^{-1} = \delta_c GG(t_c|\mu_c, \gamma \lambda_c, n),
\]

\[
= \delta_c \left[ \frac{(\gamma \lambda_c)^{\mu_c}}{\Gamma(\mu_c)} \right] \left[ \frac{\Gamma(\mu_c + n)}{\Gamma(n)} \right] \left[ \frac{t_c^{\mu_c - 1}}{(\gamma \lambda_c + t_c)^{\mu_c + n}} \right],
\]

where \( GG(\cdot) \) is a Gamma-Gamma density function (Bernardo and Smith, 1994, p. 304) with parameters \( \mu_c \), \( \gamma \lambda_c \) and \( n \), the number of auctions. \( \gamma = \Gamma(2)(N_\star - 2)/(N_\star - 1) \); and I have specified \( \delta_c \), \( t_c \) and \( f(t_c|c^{-1}) \) in (A.1), (A.2) and (A.3), respectively. \( f(c^{-1}) \) is obtained from the prior density function for \( C \) specified in (15); \( C^{-1} \sim GG(\mu_c, \gamma \lambda) \).

APPENDIX B
Marginal likelihood for the private-values paradigm
In equation (17), $GA(t_p|n, \theta_1)$ is the density function of a Gamma distribution truncated from below. It is given by

$$GA(t_p|n, \theta_1) = \left[ \frac{\theta_1^n \Gamma_p - e^{-\theta_1 t_p}}{\Gamma(n) - \Gamma_{\nu_1}(n)} \right], \quad t_p > \frac{1}{\theta_1} \sum_{j=1}^{n} \frac{N_j}{N_j - 1}. \quad (B.1)$$

$\Gamma_{\nu_1}(n)$ is an incomplete gamma function. $\Gamma(n)$ is a gamma function specified in footnote 4.

$\nu$ is determined by the number of bidders, $N_j$ in the $j = 1, \ldots, 37$ auctions; it is given by

$$\nu = \sum_{j=1}^{n} \frac{N_j}{N_j - 1}. \quad (B.2)$$

Further, $\delta_p$ is

$$\delta_p = \left[ \Gamma(n) - \Gamma_{\nu_1}(n) \right] \left[ \frac{1}{\theta_1} \right] \left[ \prod_{j=1}^{n} N_j e^{\frac{N_j}{N_j - 1}} \right]. \quad (B.3)$$

$GA(\theta_1|\omega, N, \lambda_p)$, the prior density function given in equation (18) is

$$GA(\theta_1|\omega, N, \lambda_p) = \frac{(N_p \lambda_p)^n}{\Gamma_p \Gamma_{\nu_1}(n)} \theta_1^{n-1} e^{-\lambda_p N \theta_1}, \quad 0 < \theta_1 < \theta_1. \quad (B.4)$$

$P(\omega|m)$ for the private-values paradigm in equation (19) is obtained by substituting (17) and (18) in equation (12). It is given by

$$P(\omega|m) = \delta_p \int_0^{\theta_1} f(t_p|\theta_1) f(\theta_1) d\theta_1 = \delta_p GG(t_p|\omega, N, \lambda_p, n),$$

$$= \delta_p \int_0^{\theta_1} \left[ \frac{\theta_1^n \Gamma_p - e^{-\theta_1 t_p}}{\Gamma(n) - \Gamma_{\nu_1}(n)} \right] \frac{(N_p \lambda_p)^n}{\Gamma_p \Gamma_{\nu_1}(n)} \theta_1^{n-1} e^{-\lambda_p N \theta_1} d\theta_1,$$

$$= \delta_p \left[ \frac{(N_p \lambda_p)^n}{(t_p + \lambda_p N)^{n+n}} \right] \left[ \frac{\Gamma_p \Gamma_{\nu_1}(n+n)}{\Gamma_p \Gamma_{\nu_1}(n)} \right], \quad (B.5)$$

$^{23}$ A generically specified incomplete gamma function, $\Gamma_{\text{im}}(\alpha)$, is given by

$$\Gamma_{\text{im}}(\alpha) = \int_0^{\text{im}} t^{\alpha-1} e^{-t} dt.$$
where \( w_j > \frac{1}{2} (N_j - 1) \) \( \forall j \). \( f(t_p|\theta_1), \delta_p \) and \( f(\theta_1) \) are given by (B.1), (B.3) and (B.4), respectively.

APPENDIX C

Prior and posterior predictive for the common-value paradigm

The density function of the prior predictive, \( W_* \), under the common-value paradigm is

\[
f(W_*) = \int f(c^{-1})f_c(\gamma w_*|c^{-1})dc^{-1}
= GG(w_*|n_c, \gamma \lambda_c, 1),
= \left[ \frac{(\gamma \lambda_c)^{n_c}}{\Gamma(n_c)} \right] \left[ \frac{1}{\Gamma(n_c + 1)} \right] \left[ \frac{1}{(\gamma \lambda_c + w_*)^{n_c + 1}} \right].
\]

(C.1)

\( f_c(\gamma w_*|c^{-1}) \) is obtained from the density function of the winning bid given by equation (13); \( \gamma w_* \sim EXP(c^{-1}) \). \( f(c^{-1}) \) is obtained from the prior density function for \( c \) given by equation (15); it is a Gamma density function with the same parameters. \( GG(w_*|\beta) \) is the density function of a Gamma-Gamma variate. Thus, the prior predictive follows a Gamma-Gamma distribution with parameters \( n_c, \gamma \lambda_c \) and 1. \( \gamma \) is given by \( \Gamma(2) \frac{N_* - 2}{n_c - 1} \).

The expected value of the prior predictive, \( E_c(W_*) \), is

\[
E_c(W_*) = \Gamma(2) \frac{N_* - 2}{n_c - 1} \frac{\lambda_c}{n_c - 1}.
\]

(C.2)

The posterior predictive density for \( W_* \) under the common-value paradigm is

\[
f(w_*|w) = \int f_c(\gamma w_*|c^{-1})f(c^{-1}|w)dc^{-1},
= GG(w_*|n + n, (\gamma \lambda_c + t_c), 1),
= \left[ \frac{(\gamma \lambda_c + t_c)^{n+n}}{\Gamma(n + n)} \right] \left[ \frac{\Gamma(n + n + 1)}{\Gamma(n + n)} \right] \left[ \frac{1}{(\gamma \lambda_c + t_c + w_*)^{n+n+1}} \right].
\]

(C.3)
where \( GG(w_\ast \mid \bullet) \) is the density function of a Gamma-Gamma distribution, and here is the density function of \( W_\ast \). \( f_c(\gamma w_\ast \mid c^{-1}) \) is obtained from the likelihood of an out-of-sample winning bid given by equation (13); \( \gamma w_\ast \sim EXP(c^{-1}) \). \( f(c^{-1} \mid w) \) is the posterior density function of \( c^{-1} \); \( c^{-1} \mid w \sim (n + n, \gamma \lambda_c + t_c) \). \( n = 37 \) is the sample size; that is, the number of auctions on which data is available. And \( N_\ast \) is the number of bidders in the yet to be observed auction.

**APPENDIX D**

**Prior and Predictive for the private-values paradigm**

The density function of the prior predictive, \( W_\ast \), under the private-values paradigm is a truncated Gamma-Gamma density function, \( GG(w_\ast \mid \bullet) \), with \( w_\ast > \frac{1}{\theta_1(N_\ast - 1)} \). It is given by

\[
f(w_\ast) = \int_{0}^{N_\ast \theta_1} f_p(w_\ast \mid N_\ast \theta_1) f(N_\ast \theta_1) dN_\ast \theta_1, \\
= GG(w_\ast \mid \lambda_p, \lambda_p, 1), \\
= \left[ \frac{(\lambda_p)^n_p}{(w_\ast - \frac{1}{\theta_1(N_\ast - 1)} + \lambda_p)^{n_p + 1}} \right] \left[ \frac{\Gamma N_\ast \theta_1(n_p + 1)}{\Gamma \theta_1 N_\ast \theta_1(n_p)} \right], \\
(D.1)
\]

where \( \tau = (w_\ast - \frac{1}{\theta_1(N_\ast - 1)} + \lambda_p) \). The density function of the winning bid under the private-values paradigm, \( f_p(w_\ast \mid \theta_1) \), is given by equation (7), with \( w_\ast > \frac{1}{\theta_1(N_\ast - 1)} \). And \( f(N_\ast \theta_1) \) can be obtained from (18).

The expected value of the prior predictive, \( E_p(W_\ast) \), is

\[
E_p(W_\ast) = \frac{1}{\theta_1(N_\ast - 1)} + \lambda_p \left[ \frac{\Gamma N_\ast \theta_1(n_p - 1)}{\Gamma \theta_1 N_\ast \theta_1(n_p)} \right]. \\
(D.2)
\]

The posterior predictive density for \( W_\ast \) under the private-values paradigm is a truncated Gamma-Gamma density function with \( w_\ast > \frac{\nu}{\theta_1} \). It is given by

\[
f(w_\ast \mid w) = \int_{0}^{w_\ast \frac{\nu}{\theta_1}} f_p(w_\ast \mid N_\ast \theta_1) f(N_\ast \theta_1 \mid w) dN_\ast \theta_1, \\
= GG(w_\ast \mid n + n, (\lambda_p + t_p), 1), \\
34
where \( f_p(w_*|N_0|\theta_1) \) is the likelihood of an out-of-sample winning bid. It is truncated exponential, with \( w_* > \frac{\nu}{\theta_i} \). \(^{24}\) \( f(N_0|\theta_1|w) \) is the posterior density function of \( N_0|\theta_1 \). It is a truncated Gamma density function with \( N_0|\theta_1 < \nu \frac{N_0}{\theta_i} \). \(^{25}\) As specified in equation (B.2), \( \nu = \sum_{j=1}^{n} \frac{N_j}{N_j-1} \).

\(^{24}\) It is given by \( f(w_*|N_0, \theta_1) = (N_0, \theta_1)e^{-N_0, \theta_1(w_* - \theta_i^*)} \).

\(^{25}\) It is given by \( f(N_0, \theta_1|w) = \left( \frac{(\lambda_p + \theta_i)^{(n+n)}}{\Gamma(\lambda_p + \theta_i)N_0, \theta_1(n+n)} \right) (N_0, \theta_1)^{n+n-1}e^{-N_0, \theta_1(\lambda_p + \theta_i)} \).
Table 1

Sample descriptive statistics; sample size = 37

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. dev.</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winning bid (U.S. dollar per metric tonne)</td>
<td>128.84</td>
<td>22.9</td>
<td>95.51</td>
<td>160.12</td>
</tr>
<tr>
<td>Number of bidders</td>
<td>6.16</td>
<td>2.08</td>
<td>3.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>

Table 2

Sufficient statistic

<table>
<thead>
<tr>
<th>Model (m)</th>
<th>$t_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>common-value</td>
<td>3689.78</td>
</tr>
<tr>
<td>private-value</td>
<td>30218.33</td>
</tr>
</tbody>
</table>
Table 3
Economic indicators of the Indian economy\(^1\)
and world crude oil prices\(^2\)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Average(^3) crude oil spot price index (1990=100)</td>
<td>84.2</td>
<td>82.9</td>
<td>73.2</td>
<td>69.2</td>
<td>74.7</td>
</tr>
<tr>
<td>Average crude spot oil price (U.S. dollar per metric tonne(^4))</td>
<td>193.3</td>
<td>190.3</td>
<td>168.2</td>
<td>159.0</td>
<td>171.6</td>
</tr>
<tr>
<td>Gross domestic product, 1990 prices (Billions of Rupees)</td>
<td>5,378.0</td>
<td>5,670.1</td>
<td>5,944.0</td>
<td>6,393.9</td>
<td>6,862.4</td>
</tr>
<tr>
<td>GDP deflator (1990=100)</td>
<td>114.7</td>
<td>124.5</td>
<td>136.2</td>
<td>149.2</td>
<td>160.1</td>
</tr>
<tr>
<td>Exports of goods and services (Billions of Rupees)</td>
<td>562.5</td>
<td>673.1</td>
<td>861.1</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>Imports of goods and services (Billions of Rupees)</td>
<td>-562.5</td>
<td>-730.0</td>
<td>-857.0</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>Gross fixed capital formation (Billions of Rupees)</td>
<td>1,365.0</td>
<td>1,588.6</td>
<td>1,750.0</td>
<td>2,140.0</td>
<td>2,702.6</td>
</tr>
</tbody>
</table>

3. The international energy agency quotes an average over 11 different types of crude oil.
4. Conversion factors reported in the White Paper, IOC, 1994 have been used to convert the crude oil price from U.S Dollar per barrel to U.S dollar per metric tonne, since the IEA does not report these for crude-oil.
### Table 4

$P_{cv,pv}: n=3$

<table>
<thead>
<tr>
<th>$\tilde{w}_* = 100$</th>
<th>$\theta_1 = 1.00$</th>
<th>$N_*=5$</th>
<th>$N_*=7$</th>
<th>$N_*=9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.23896$</td>
<td>$0.16679$</td>
<td>$0.13763$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23895$</td>
<td>$0.16678$</td>
<td>$0.13763$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23886$</td>
<td>$0.16676$</td>
<td>$0.13762$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.22254$</td>
<td>$0.16838$</td>
<td>$0.15322$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.22253$</td>
<td>$0.16838$</td>
<td>$0.15322$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.22248$</td>
<td>$0.16836$</td>
<td>$0.15321$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20761$</td>
<td>$0.17001$</td>
<td>$0.17016$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20760$</td>
<td>$0.17001$</td>
<td>$0.17016$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20757$</td>
<td>$0.17000$</td>
<td>$0.17016$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 5

$P_{cv,pv}: n=5$

<table>
<thead>
<tr>
<th>$\tilde{w}_* = 100$</th>
<th>$\theta_1 = 1.00$</th>
<th>$N_*=5$</th>
<th>$N_*=7$</th>
<th>$N_*=9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.26766$</td>
<td>$0.16588$</td>
<td>$0.13821$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.26764$</td>
<td>$0.16587$</td>
<td>$0.13820$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.26751$</td>
<td>$0.16584$</td>
<td>$0.13819$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23320$</td>
<td>$0.16897$</td>
<td>$0.16915$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23319$</td>
<td>$0.16897$</td>
<td>$0.16915$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23313$</td>
<td>$0.16896$</td>
<td>$0.16916$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20427$</td>
<td>$0.17208$</td>
<td>$0.20517$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20426$</td>
<td>$0.17208$</td>
<td>$0.20518$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.20423$</td>
<td>$0.17208$</td>
<td>$0.20519$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 2-1: Prior and Posterior Predictive Density Functions

Private-Values Paradigm,
\( N_s = 9, \, \omega = 150, \, n = 3, \, \Theta_1 = 1 \)

\[ \text{prior} \quad \text{posterior} \]

Figure 2-2: Prior and Posterior Predictive Density Functions

Common-Value Paradigm,
\( N_s = 9, \, \omega = 150, \, n = 3, \, \Theta_1 = 1 \)

\[ \text{prior} \quad \text{posterior} \]
Bibliography


Chapter 3

Evaluating Data in Structural Parametric Auction Models

3.1 Introduction

Typically auction data is available in one of two forms; either data on all bids within and across auctions or just winning bids across auctions is available. For example, Paarsch (1992) studies first-price, sealed-bid tree planting auctions using data on winning bids. Laffont, Ossard and Vuong use winning bids to study a Dutch auction for eggplants. Bajari (1997) and Porter and Zona (1983) use data on all bids to study procurement auctions. While sometimes data on all bids is not available due to the form of the auction, at other times it is the result of an empirical researcher's inability to get data on all bids in an auction. Thus, in a first-price, sealed-bid auction to procure crude-oil studied in Chapter 2, only the winning bid was made public; other bids were not available even though these bids were observed by the authority that conducted these auctions. On the other hand in a Dutch auction, where an auctioneer calls out an initial high and continuously lowers this price till one bidder accepts the current price, only the winning bid is observed.

In this scenario, where an empirical researcher has access to data on the winning bids only, a concern of an empirical researcher is whether the winning bids are sufficient for the parameters of the distribution of the private signals of the bidders. That is, is she always losing expected
information about the parameters of the distribution of private signals of the bidders when she has data on winning bids only compared to when she would have data on all bids within an auction.\(^1\) I show that for certain specifications of the private signals of the bidders for a structural low-price, sealed-bid auction under the independent-private-values paradigm, the minimum winning bid is sufficient for the scalar parameter that characterizes the distribution of private signals of the bidders. In this framework the minimum winning bid contains the same expected information about this scalar parameter as all the bids within and across auctions. Inference about this scalar parameter is unaffected by the available data comprising of all the bids or just the winning bids. Once the distribution of the private signals of the bidders is not characterized by a scalar parameter, the winning bids (or the minimum winning bid) are just members of the set of sufficient statistics, so that the expected information about the parameters is strictly less if just the winning bids are used.

A subjective Bayesian carrying out inference about the parameters of the distribution of private signals of the bidders will specify the same priors for the parameters irrespective of the available data comprising of all the bids or just the winning bids. Inference about the parameters, since it is based on the posterior distribution of the parameters, will be affected through the likelihood function. If the winning bids are not sufficient for the parameters of the distribution of the private signals of the bidders the likelihood function, and as a result the posterior distribution of the parameters will be different depending on whether data on all the bids or just the winning bids is available.

Specifying subjective priors in a low-price, sealed-bid auction under the independent private-values paradigm is a difficult exercise. One reason is that the support of the data depends on the parameters of the distribution of the private signals of the bidders in structural auction models. For example, in a Dutch auction under the independent-private-values paradigm the bids or the winning bid are less than the unconditional expected value of the second highest private

\(^1\)A similar concern arises in the study of natural phenomena in the statistics literature. For example the distribution of the highest wave is important in the design of sea structures, the distribution of the largest flood is important in designing dams, etc. The concern over the loss of information in using the largest observation has lead to the generalized Pareto Distribution (Castillo and Hadi, 1997 and Pickands, 1975) which is used to model exceedences over a threshold, thereby considering several of the largest observations instead of just the largest one.
value. In a low-price, sealed-bid auction under the independent-private-values paradigm, the bids or the winning bid are greater than the equilibrium bid that corresponds to a zero cost of procuring the object being auctioned. Specifying subjective priors for the parameters of the distribution of private signals of the bidders implies a prior on the support of the data since the support of the data is a function of these parameters. This implied prior has to be taken into consideration in any prior elicitation about the parameters of the distribution of the private signals of the bidders.²

An alternative to specifying subjective priors is to find structural rules that determine priors that are noninformative. In a pioneering paper, Bernardo (1979) has introduced the so-called reference prior to represent the idea of a noninformative prior. Reference priors are appealing to both nonsubjective Bayesians and frequentists since the posterior probabilities agree with sampling probabilities to a certain order (Ghosh (1994) and Ghosh and Mukherjee (1992)). They are appealing to subjective Bayesians as well since they serve as a reference point in a prior sensitivity analysis. The reference prior emerges from maximizing an asymptotic expansion of Lindley's measure of information. Lindley's (1965) measure of information is defined as the expected Kulback-Liebler divergence between the posterior and the prior; the larger the measure, more informative the data and hence less informative the prior. When there are no nuisance parameters and certain regularity conditions are satisfied, Bernardo's reference prior is the Jeffreys' prior. Ghosh (1994), Ghosal and Samanta (1995) and Ghosal (1997), extending Bernardo's (1979) work, have obtained the reference prior when the support of the data depends on the parameters. In general, in this scenario, the reference prior is not the Jeffreys' prior.

In this paper I examine reference priors for a structural parametric low-price, sealed-bid auction under the independent-private-values paradigm using parametric assumptions made by Paarsch (1992) for the distribution of private values as illustrations. The parameters in the support of the distribution function of the bids or the winning bid will be called "nonregular"

²This was one reason why conjugate priors were used in Chapter 2. Since priors are conjugate with respect to a likelihood function, the nonregular feature of the likelihood function in that the support of the data depends on the parameters of the distribution of the private signal of the bidders, is taken account of in prior elicitation of these parameters.
parameters; the rest of the parameters are "regular" parameters. Thus, given "nonregular" parameters, the distribution function of the bids or winning bid satisfies standard regularity conditions with respect to the "regular" parameters. Since the reference prior depends on the sampling density of the data, I study the reference priors under two frameworks.

In the first framework, the minimum winning bid is sufficient for a scalar parameter. I show that the reference prior as well as the exact reference posterior distributions, whether obtained from the bids or the minimum winning bid, are identical. Hence inference about the parameters is unaffected by whether the available data comprises of all the bids within and across auctions or just the winning bids across auctions.

In the second framework the winning bids (and hence, the minimum winning bid) are members of the set of sufficient statistics so that expected information about the parameters is less if just the winning bids are used. In this scenario, I show that the reference posterior distribution of the "regular" parameter obtained from the winning bids is more dispersed than that obtained from the bids; the reference prior reflects the same pattern as long as it is not proportional to a constant. This follows from the difference in the Fisher information for the "regular" parameter obtained from all the bids compared with the winning bids being a non-null psd matrix. Inference about the "regular" parameter will differ depending on whether data on all the bids or just the winning bids is available through both the reference prior and the likelihood function. For the "nonregular" parameter, conditional on the "regular" parameter, the reference prior and the posterior are identical, whether obtained from the bids or the minimum winning bid. This reflects the sufficiency of the minimum winning bid for the "nonregular" parameter conditional on the "regular" parameter.

When I integrate out the "nonregular" parameter from the exact joint reference posterior for the "regular" and the "nonregular" parameters, I obtain the marginal reference posterior for the "regular" parameter as the product of the concentrated likelihood function and the reference prior based on this concentrated likelihood. This bears similarity to the classical procedure of "concentrating out" the "nonregular" parameter from the likelihood function with the minimum winning bid, since the minimum winning bid is a superconsistent estimator of the "nonregular" parameter, and using this concentrated likelihood function to carry out inference
about the "regular" parameter. I note that this similarity follows from the Bayesian procedure of "integrating out" the "nonregular" parameter being comparable with the classical procedure of "concentrating out" the "nonregular" parameter.

These results get repeated when there is a partitioning of parameters into "nuisance parameters" and "parameters of interest". In addition, I note a new feature. The marginal reference posterior for the "regular parameter" is the product of the prior and the concentrated likelihood function of all the bids in the absence of "nuisance parameters". In contrast, the marginal reference posterior for the "regular parameter", when it is the "parameter of interest", is the product of the same prior and the concentrated likelihood of one less than all the bids. The difference in the case where there is a partitioning of parameters into "nuisance parameters" and "parameters of interest" and where there is no such partitioning, is in the reference prior for the "regular" parameter; the likelihood function is identical in the two scenarios. I find the reference prior for the "regular" parameter to be "more informative" when there is a partitioning of parameters in the sense described above.

An outline of the paper is as follows.

Section 2 describes the statistical framework used in the paper. The assumptions under which the minimum winning bid is sufficient for a scalar parameter are also stated. When the winning bids are not sufficient for the parameter vector, I prove that the difference in the Fisher information about the parameters obtained from the bids compared with the winning bids is a non-null psd matrix. A brief review of the concept of reference prior is given in Section 3. In Section 4, I describe a low-price, sealed-bid auction under the specification of the distribution of private values as in Paarsch (1992). Section 5 derives the reference priors and posteriors for a scalar parameter when the minimum winning bid is sufficient for this parameter. Section 6 is devoted to the vector parameter case when the winning bids are not sufficient for the parameters. Reference analysis in the presence of "nuisance parameters" and "parameters of interest" is discussed in Section 7. Section 8 concludes.
3.2 Statistical Framework

In the discussion that follows, random variables will be indicated by capital letters, and the realization of a random variable by a lower case letter; in addition, matrices will be indicated by bold faced letters. The quantities used in this paper are defined in Table 1; all quantities defined in Table 1 are evaluated at a value $\theta_0$ of $\theta$; I will indicate when $\theta_0$ stands for the "true" value of $\theta$. Given a random variable $Z \in \mathbb{R}^+$, $E_{Z|\theta}(\cdot)$ and $V_{Z|\theta}(\cdot)$ will indicate the expected value and the variance, respectively, with respect to the distribution $Z | \theta$. The support of the density function of a random variable $Z$ is denoted by $\text{supp}(Z)$; if the support depends on $\theta$ it will be indicated by $\text{supp}_Z(\theta)$. $\pi(\theta)$ and $\pi(\theta|z)$ will indicate the prior and the posterior density functions of $\theta$, respectively.

Assumption 1: Given data $B_j = [B_{j1}, ..., B_{jN}]'$ on $n$ bids in each auction $j$ and $j = 1, ..., T$ auctions, and the $k \times 1$ parameter vector $\theta \in \Theta$, $B_j|\theta$ is independently and identically distributed across the bidders $i = 1, ..., n$ and the auctions $j = 1, ..., T$ with density function $f_\theta(b_j^i | \theta)$. Note that I have made the simplifying assumption that the number of bidders participating in each auction is identical; further discussion of this assumption is provided in Section 4, footnote 6. I will also refer to the $n \times T$ matrix of all the data $B = [B_1, ..., B_T]$, where $B_j$ is an $n$ dimensional column vector of $n$ bids in auction $j$. $W_j$ is the winning bid in auction $j$. This could be either the highest or the lowest order statistic for a sample of size $n$ from the distribution of $B_j^i|\theta$; I will assume it to be the lowest order statistic in the examples in this paper. $W_*$ is the minimum winning bid across auctions $j = 1, ..., T$, $W_* = \min_{j=1, ..., T} \{W_j\}$. The $T$ dimensional vector of winning bids, $[W_1, ..., W_T]'$, will be indicated by $W$.

When standard regularity conditions (see Poirier, 1995, p. 259) are satisfied so that the support of the data does not depend on $\theta$, I will refer to it as the regular case. If the support of the data depends on the parameter $\theta$, but other regularity conditions are satisfied, I will refer to it as the nonregular case. For the nonregular case I make the following assumption if $\theta$ is a vector.

Assumption 2: Assuming $\theta$ is a vector and can always be partitioned into $\theta = (\eta, \varphi)$, where $\eta$ is a scalar and $\varphi$ is a $k - 1$ dimensional vector,
(a) the support of the density function of the random variable \( B_j \), \( f_b(b_j | \theta) \), depends on \( \theta \), \( B_j \geq l(\eta, \varphi) \);
(b) \( l(\eta, \varphi) \) is a strictly monotonic function of \( \eta \) for every \( \varphi \).

The importance of Assumption 2 is that it enables me to rewrite \( f_b(b_j | \eta, \varphi) \) as \( f_b(b_j | l^{-1}(W_*), \varphi) \).

Then evaluating all quantities at the "true" value of \( \varphi \), the expected value of the score function of the log likelihood of bids with respect to \( \varphi \) is zero, \( E_{Z_j|\theta}[S_Z(\varphi; Z_j)] = 0 \); and the Fisher information for \( \varphi \) equals the negative of the sampling expectation of the Hessian matrix \( E_{Z_j|\theta}[S_Z(\varphi; Z_j)S_Z'(\varphi; Z_j)] = -E_{Z_j|\theta}[H_Z(\varphi; B_j)] \).

These results hold for the entire parameter vector \( \theta \) when standard regularity conditions hold since regularity allows the interchange of the operations of differentiation and integration. They break down when the support of the data depends on \( \theta \). By fixing the lower boundary of the distribution of bids, the dependence of the support of the density function of the bids on \( \theta \) is removed, making the likelihood function "regular" with respect to \( \varphi \) again.

For the scalar parameter nonregular case, Huzurbazar (1976, pp. 158, 174-186) establishes that the sample minimum or maximum is a sufficient statistic for \( \theta \) if and only if \( \theta \) is a scalar and if the density function for the data and the support of this density function satisfy certain conditions. These necessary and sufficient conditions are given in Assumption 3 which follows.

I will be using the Huzurbazar results in context of a low-price, sealed-bid auction in which the winning bid is the lowest bid in the auction. Hence the minimum winning bid across auctions, \( W_* \), is the sufficient statistic for \( \theta \).

Assumption 3: Assuming \( \theta \) is a scalar, and given that the support of the data, \( B \), depends on \( \theta \), it is assumed that \( B_j \geq l(\theta) \), where \( l(\theta) \) is a strictly monotonic, continuous and differentiable function of \( \theta \), and that the form of \( f_b(b_j | \theta) \) is

\[
f_b(b_j | \theta) = m(b_j)q(\theta);
\]

\( m(b_j) \), \( q(\theta) \) are strictly positive functions of \( b_j \) and \( \theta \), respectively.

\(^3\)For a proof see Hong (1998).
When Assumptions 1 and 3 hold, the *minimum winning bid is sufficient* for $\theta$, and the Fisher information from all the bids in an auction about $\theta$, $J_B(\theta)$, equals the Fisher information about $\theta$ in the minimum winning bid across auctions, $J_W(\theta)$. Once $\theta$ is a vector, irrespective of whether or not the support of the data depends on $\theta$, the *minimum winning bid* will not be *sufficient* for $\theta$ because a single sufficient statistic cannot exist for a vector of parameters. I prove below that the difference in the Fisher information about $\theta$ from all the bids within and across auctions compared with the minimum winning bid across auctions is a *non-null psd matrix*. I also show that the difference in the Fisher information about $\theta$ from all the bids within and across auctions, $B$, compared with the winning bids across auctions, $W$, is a *non-null psd matrix*, so that the vector of winning bids across auctions is *not sufficient* for $\theta$.

I first express the Fisher information matrix of $nT$ bids $B = [B_1, ..., B_T]'$, as the sum of two terms in the Lemma below. The first term is the Fisher information of the $T$ winning bids. The second term is the average Fisher information of $n-1$ ordered bids in each of the $j$ auctions, $B_{n-1:n} = [B_{1:n-1:n}, ..., B_{T:n-1:n}]$, where $B_{j,n-1:n} = [B^{1:n}_j, ..., B^{n-1:n}_j]'$ is the $n-1$ dimensional vector of ordered bids in auction $j$ given $W_j$; the average is over the sampling distribution of the winning bid across the $j$ auctions.

**Lemma 1:** Under Assumption 1, using quantities defined in Table 1, and either

(a) $\theta$ is a $k$ dimensional vector and standard regularity conditions hold; or

(b) $\theta = \varphi$ is a $(k-1)$ dimensional vector and Assumption 2 holds:

\[
J_B(\theta_0) = J_W(\theta_0) + E_{W_j}[J_{B_{n-1:n}}(\theta_0)],
\]

where all quantities have been evaluated at $\theta = \theta_0$. $J_B(\theta_0)$ is the Fisher information from $nT$ bids, $B$. $J_{B_{n-1:n}}(\theta_0)$ is the Fisher information of the log likelihood function of the distribution of $n-1$ ordered bids, $B_{j,n-1:n}$ given $W_j$ across all auctions $j = 1, ..., T$, and $J_W(\theta_0)$ is the Fisher information from the winning bids across the $T$ auctions.

**Proof.** : See Appendix A.

If $V_p$ is a $p$ dimensional vector of statistics with $p < nT$ and $p \geq k$, then the Fisher
information, $J_{V_p}(\theta)$, about $\theta$ given $V_p$, is identical to the Fisher information in the sample of $nT$ bids, $J_B(\theta)$, if and only if $V_p$ are jointly sufficient statistics for $\theta$; this follows from Lemma 1.4

Using the result in Lemma 1, I now prove that the difference in the Fisher information about $\theta$ from the bids within and across the $T$ auctions compared with the winning bids across the $T$ auctions is a non-null psd matrix. Thus the winning bids provide strictly less expected information about $\theta$ than all the bids within and across auctions.

**Corollary 1:** From Lemma 1, and that Assumption 3 does not hold,

$$J_B(\theta_o) - J_W(\theta_o)$$

is positive semidefinite but not a null matrix,

where all quantities have been evaluated at $\theta = \theta_o$.

**Proof.** I will prove this by contradiction. Since the $T$ auctions are independent and identical, and the likelihood function is "regular" with respect to $\theta$ under Assumption 2, I can write Lemma 1 as,

$$TJ_B(\theta_o) = TJ_W(\theta_o) + TE_{W_j(\theta)} [J_{B_{j,n-1:n}}(\theta_o)].$$

Then proving Corollary 1 is equivalent to proving

$$J_B(\theta_o) - J_W(\theta_o)$$

is positive semidefinite but not a null matrix.

Since $J_{B_{j,n-1:n}}(\theta_o)$ is psd, $E_{W_j(\theta)} [J_{B_{j,n-1:n}}(\theta_o)]$ is psd too. I want to rule out the possibility that $E_{W_j(\theta)} [J_{B_{n_j-1:n_j}}(\theta_o)]$ is a null matrix. Assume that it is a null matrix. Then $W_j$ is sufficient for $\theta$; but a single sufficient statistic cannot exist for a vector of parameters. Hence $E_{W_j(\theta)} [J_{B_{n_j-1:n_j}}(\theta_o)]$ cannot be a null matrix.5

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4A proof of this is available from the author. It follows from a more generalized version of Lemma 1,

$$J_B(\mu) = J_{V_p}(\mu) + E_{V_p(\theta)} [J_{V_{nT-p}|V_p}(\mu)],$$

where I have partitioned $nT$ jointly sufficient statistics arbitrarily into a $p$ dimensional vector of statistics, $V_p$, and an $nT - p$ dimensional vector of statistics, $V_{nT-p}$. $J_{V_{nT-p}|V_p}(\mu)$ is the Fisher information of the loglikelihood function of the distribution of $V_{nT-p}$ conditional on $V_p$, and $J_{V_p}(\mu)$ is the Fisher information from the $V_p$ statistics.

5For the scalar parameter case $J_{B_j}(\theta_o) - J_{W_j}(\theta_o) \geq 0$. I want to rule out the possibility that $J_{B_j}(\theta_o) - J_{W_j}(\theta_o) = 0$. Suppose it is zero. Then $W_j$ is sufficient for $\theta$; but the necessary and sufficient conditions for $W_j$
From Lemma 1 and Corollary 1 it follows that the minimum winning bid across auctions will not be sufficient for $\theta$ or that the Fisher information from the minimum winning bid across auctions will be strictly less than the Fisher information about $\theta$ from all the bids within and across auctions. This is now stated.

**Corollary 2:** From Lemma 1 and Corollary 1,

(a) $J_B(\theta_0) = J_{W_\ast}(\theta_0) + E_{W_\ast|\theta}[J_{B - W_\ast}(\theta_0)]$,

(b) $J_B(\theta_0) - J_{W_\ast}(\theta_0)$ is a non-null psd matrix.

$J_B(\theta_0)$ is the Fisher information from all the bids within and across auctions. $J_{W_\ast}(\theta_0)$ is the Fisher information from the minimum winning bid across auctions. $J_{B - W_\ast}(\theta_0)$ is the Fisher information from all the bids within and across auctions except the minimum winning bid, $W_\ast$; $W_\ast|\theta$ is the sampling distribution of the minimum winning bid across auctions.

**Proof.** The proof for (a) is along similar lines to the proof for Lemma 1. The logic for the proof again comes from footnote 4. The proof of (b) is along lines similar to Corollary 1.

### 3.3 Review of Reference Priors

In this Section I provide an overview of the reference prior idea of Bernardo (1979). I begin by defining Lindley's measure of information which provides a measure of the information about the parameter in a sample and forms the basis for obtaining reference priors. The first two definitions are from Kass and Wasserman (1996, pp. 1345-1351).

**Definition 1:** **Lindley's measure of information** is the expected Kullback-Leibler distance between the posterior density, $\pi(\theta|z)$, and the prior density, $\pi(\theta)$,

$$I(\pi(\theta), Z) = \int K(\pi(\theta|z), \pi(\theta)) f(z)dz,$$

where $K(\pi(\theta|z), \pi(\theta)) = \int_\Theta \pi(\theta|z) \log(\pi(\theta|z)/\pi(\theta)) d\theta$, is the Kullback-Leibler distance between the posterior and the prior density. The expectation in Lindley's measure of information is with respect to the marginal density of the data, $f(z) = \int_\Theta f_z(z|\theta) \pi(\theta) d\theta$.

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*to be sufficient for $\theta$ are given by Assumption (3), and this is ruled out by assumption in this corollary.*
Given \( T \) independently and identically random variables, \( \mathbf{Z}_T = (Z_1, \ldots, Z_T) \), where \( Z_j \) is an \( n \) dimensional column vector, Lindley's measure of information given in Definition 1 is the asymptotic expected information that an experiment provides about \( \theta \). Hence as \( T \to \infty \), 
\[ I_\infty^T = \lim_{T \to \infty} I_\infty^T = I(\pi(\theta), \mathbf{Z}_T) \]
provides a measure of missing information as a function of the prior density function, \( \pi(\theta) \) (Lindley, 1956 and Good, 1960, 1966); the larger the measure, the more informative the data and less informative the prior. Bernardo's idea was to maximize \( I_\infty^T \) to obtain the reference prior. However \( I_\infty^T \) involves the sample size and is, as a result, typically infinite. Bernardo got around this problem by first finding the sequence of priors \( \pi_T(\theta) \) which maximize \( I(\pi(\theta), \mathbf{Z}_T) \). Corresponding to this sequence of priors is a sequence of posterior density functions, \( \pi_T(\theta|\mathbf{Z}_T) \) with a limiting posterior \( \pi(\theta|\mathbf{Z}) \); the reference prior is that positive function of \( \theta \) that produces this limit posterior via Bayes' theorem.

The key idea behind the reference prior, whether in the regular or the nonregular case, is that Lindley's measure admits an asymptotic expansion with the leading term, free of \( \pi(\theta) \), going to infinity with \( T \), second term a function of \( \pi(\theta) \) but free of \( T \), and the remainder going to zero. Maximization of the second term, with respect to \( \pi(\theta) \), gives the reference prior.

In the \( k \) dimensional regular case, Ibragimov and Has'minskii (1973) have obtained the asymptotic expansion of Lindley's measure of information,
\[
\frac{k}{2} \log \frac{T}{2\pi e} + \int \pi(\theta) \frac{\sqrt{\det \mathbf{J}_Z(\theta)}}{\pi(\theta)} d\theta + o(1),
\]
where \( \mathbf{J}_Z(\theta) \) is the Fisher information from \( Z \). Maximization with respect to \( \pi(\theta) \) leads to Jeffreys' prior which is defined now.

**Definition 2: Jeffreys' prior.** Given data on some random variable \( Z \), with density function \( f_z(z|\theta) \), and \( \mathbf{J}_Z(\theta) \) being the Fisher information matrix, Jeffreys' prior for \( \theta \), \( \pi_{\text{jeff}}(\theta) \), is
\[
\pi_{\text{jeff}}(\theta) \propto \sqrt{\det \mathbf{J}_Z(\theta)},
\]
where “det” indicates the determinant of the matrix.

Jeffreys' prior is often used as a noninformative prior when the support of the data *does*...
not depend on $\theta$. As $T \to \infty$ and the support of the data $Z$ does not depend on $\theta$, the posterior density for $\theta$ can be approximated by a Normal distribution with a variance-covariance matrix that equals the inverse of the Fisher information matrix evaluated at $\hat{\theta}_{ml}$, the maximum likelihood estimator of $\theta$, $\sqrt{T} (\theta - \hat{\theta}_{ml}) \sim N \left( 0_k, [J_Z (\hat{\theta}_{ml})]^{-1} \right)$. The sampling distribution of $\hat{\theta}_{ml}$ as $T \to \infty$ is $\sqrt{T} (\hat{\theta}_{ml} - \theta_0) \sim N \left( 0_k, [J_Z (\theta_0)]^{-1} \right)$, where $\theta_0$ is the "true" parameter value. Thus Jeffreys' prior is related to the variance-covariance matrix of the asymptotic distribution of the MLE or the asymptotic posterior distribution.

Corresponding to the asymptotic expansion of Lindley's measure of information in equation (3) for the regular case, Ghosal and Samanta (1997) have obtained an expansion of Lindley's measure of information for the one-parameter nonregular case,

$$\log \frac{T}{e} + \int \pi(\theta) \frac{|c(\theta)|}{\pi(\theta)} d\theta + o(1),$$

where

$$c(\theta) = E_{\theta_0} \left[ \frac{\partial}{\partial \theta} \log f_z (Z_j | \theta) \right] = t'(\theta) f_z (l(\theta) | \theta) - u'(\theta) f_z (u(\theta) | \theta).$$

$\zeta_Z (\theta) = [l(\theta), u(\theta)]$ defines the support of the sampling density of the data, $f_z (Z_j | \theta)$. Maximization of the second term in equation (5), with respect to $\pi(\theta)$, gives the reference prior for the scalar boundary parameter $\theta$. The definition below is from Ghosal and Samanta (1997).

**Definition 3:** Reference prior for scalar parameter nonregular case.

(a) If $\theta$ is a scalar;

(b) $f_z (z | \theta)$, the sampling density of $Z$, is positive only on an interval $\zeta_Z (\theta)$ depending on $\theta$, where $\zeta_Z (\theta) = [l(\theta), u(\theta)]$ and it is permitted that one of the end points need not depend on $\theta$ and may be plus or minus infinity; and

(c) $\zeta_Z (\theta)$ is monotone in $\theta$;

the reference prior for $\theta$, $\pi_{\text{ref}} (\theta)$, is

$$\pi_{\text{ref}} (\theta) \propto |c(\theta)|.$$  

$\big| \big|$ is the absolute value of $c(\theta)$ which I have defined in equation (6).
I now explain the role of $c(\theta)$. In a manner analogous to the regular case, as $T \to \infty$, the posterior distribution of $\theta$ is $T(\theta - \hat{\theta}_{ml}) \sim \exp(c(\theta))$ with variance $[c(\theta)]^{-2}; \hat{\theta}_{ml} = \min\{l^{-1}(W_\ast), u^{-1}(B^{n:m})\}$, where $W_\ast$ and $B^{n:m}$ are the minimum and maximum bid within and across auctions, respectively. The sampling distribution of $\hat{\theta}_{ml}$ as $T \to \infty$ is $T(\hat{\theta}_{ml} - \theta_0) \sim \exp(c(\theta_0))$. Thus, like the regular case in Definition 2, the noninformative prior in the nonregular case is proportional to the square root of the precision of a suitably normalized function of the maximum likelihood estimator.

From Ghosal (1997), the reference prior when $\theta$ is a vector is defined next.

**Definition 4: Reference prior for the multi-parameter non-regular case.**

(a) Suppose $\theta$ is a vector, and $\theta = [\eta, \varphi]$, where $\eta$ is a scalar, and $\varphi$ a $k - 1$ dimensional vector;

(b) further suppose $f_z(z|\theta)$ is positive only on an interval $\zeta_z(\eta) = [l(\eta), u(\eta)]$ depending on $\eta$, with $\zeta_z(\eta)$ being monotone in $\eta$ and one of the end points need not depend on $\eta$.

Then the reference prior for $[\eta, \varphi], \pi_{ref}^z(\eta, \varphi)$, is defined to be

$$\pi_{ref}^z(\eta, \varphi) \propto |c(\eta, \varphi)| \sqrt{\det J^\varphi_Z(\eta, \varphi)}.$$ (8)

$c(\eta, \varphi)$ corresponds to the quantity given in equation (3) for the multi-parameter case,

$$c(\eta, \varphi) = E_{z|\eta, \varphi} \left[ \frac{\partial}{\partial \eta} \log f_z(Z_j|\eta, \varphi) \right] = l'(\eta)f_z(l(\eta)|\eta, \varphi) - u'(\eta)f_z(u(\eta)|\eta, \varphi);$$ (9)

and $J^\varphi_Z(\eta, \varphi)$ is the lower right hand block of $J_Z(\eta, \varphi)$, the Fisher information matrix about $\eta, \varphi$ from $Z$ :

$$J^\varphi_Z(\eta, \varphi) = E_{z|\eta, \varphi} \left[ \frac{\partial}{\partial \varphi} \log f_z(Z_j|\eta, \varphi) \right] \left[ \frac{\partial}{\partial \varphi} \log f_z(Z_j|\eta, \varphi) \right]' .$$ (10)

Since given $\eta$, the density function $f_z(z|\theta)$, is regular with respect to $\varphi$, I will refer to $\eta$ as the "nonregular" parameter and $\varphi$ as the "regular" parameter. The reference prior in the multi-parameter nonregular case discussed in Definition 4, is obtained by maximizing the third
term in the asymptotic expansion of Lindley's measure of information with respect to \( \pi(\eta, \varphi) \),

\[
\log \frac{T}{e} + \frac{k}{2} \log \frac{T}{2\pi e} + \int \pi(\eta, \varphi) \frac{|c(\eta, \varphi)| \sqrt{\det J_2^{\infty}(\eta, \varphi)}}{\pi(\eta, \varphi)} d\eta d\varphi + o(1),
\]

where \( c(\eta, \varphi) \) and \( J_2^{\infty}(\eta, \varphi) \) are given in Definition 4. Note that the expression in equation (11) is the counterpart for the asymptotic expansion of Lindley's measure of information for the regular case given in equation (3), and the one-parameter nonregular case given in equation (5).

Suppose interest centres on reporting inference about either \( \eta \) or \( \varphi \); let \( \varphi \) be the "nuisance parameter" and \( \eta \) the "parameter of interest". The problem is then to identify a reference prior for \( \theta = (\eta, \varphi) \) when interest centres on reporting the marginal inference for \( \eta \). Assuming that the conditional reference prior for \( \varphi \) given \( \eta \) is \( \pi_{\text{ref}}(\varphi|\eta) = \sqrt{\det J_2^{\infty}(\eta, \varphi)} \), Ghosal (1997, p. 173) maximizes the asymptotic expansion of Lindley's measure of information given above with respect to \( \pi(\eta) \) to obtain the marginal reference prior for \( \eta \),

\[
\pi_{\text{ref}}^*(\eta) = \exp \int \pi_{\text{ref}}^*(\varphi|\eta) \log c(\eta, \varphi) d\varphi.
\]

The problem is that typically \( \pi_{\text{ref}}^*(\varphi|\eta) \) is improper. As long as the parameter space for \( \varphi \), indicated by \( \Psi \), is compact, Ghosal (1997) suggests, following Berger and Bernardo (1989), of finding a sequence of compact subsets of \( \Psi \), \( \Psi_1 \subset \Psi_2 \subset \ldots \), such that \( \bigcup_{r=1}^{\infty} \Psi_r = \Psi \) and \( \pi_{\text{ref}}^*(\varphi|\eta) \) is proper for each \( \Psi_r \). Then \( \pi_{\text{ref},r}(\eta) \) will be obtained for each \( \Psi_r \), and the limit of this sequence is the marginal reference prior for \( \eta \) when \( \varphi \) is nuisance. A similar procedure is adopted when \( \eta \) is nuisance with the roles of \( \sqrt{\det J_2^{\infty}(\eta, \varphi)} \) and \( c(\eta, \varphi) \) reversed. When \( \eta \) and \( \varphi \) are a priori independent according to the reference prior, the reference prior when either \( \eta \) or \( \varphi \) is nuisance, takes the special form given in Definition 5 which follows.

**Definition 5**: Reference prior for the multi-parameter non-regular case in the presence of nuisance parameters.

(a) If \( \theta \) is a vector, and \( \theta = [\eta, \varphi] \), where \( \eta \) is a scalar, and \( \varphi \) a \( k-1 \) dimensional vector;

(b) \( f_z(z|\theta) \) is positive only on an interval \( \zeta_2(\eta) = [l(\eta), u(\eta)] \) depending on \( \eta \), with \( \zeta_2(\eta) \) being monotone in \( \eta \);
(c) either $\eta$ or $\varphi$ is nuisance; and

(d) $c(\eta, \varphi) = c_1(\eta)c_2(\varphi)$ and $\sqrt{\det J^{\varphi}_{\varphi} (\eta, \varphi)} = \lambda_1(\eta)\lambda_2(\varphi)$, where $c_1(\eta)$, $\lambda_1(\eta)$ and $c_2(\varphi)$, $\lambda_2(\varphi)$ are functions of $\eta$ and $\varphi$, respectively. Then the reference prior for $[\eta, \varphi]$, $\pi^{*}_{\text{ref}}(\eta, \varphi)$, is defined to be

$$\pi^{*}_{\text{ref}}(\eta, \varphi) \propto c_1(\eta)\lambda_2(\varphi).$$

(12)

The implication of assumption (d), whether or not nuisance parameters are present, is that $\eta$ and $\varphi$ are apriori independent according to the reference prior.

From the discussion above, several points are worth noting.

First, neither the reference prior $\pi^{*}_{\text{ref}}(\theta)$ nor Jeffreys' prior need be proper; it is just a positive function or a tool for deriving the reference posterior distribution via Bayes' theorem. It is only the reference posterior distributions that have a probabilistic interpretation (Bernardo and Smith, 1994, p. 306). When the reference prior is proportional to a constant, I am unable to study the impact of the sampling density of the data on the reference prior; the exact reference posteriors will be different, capturing the difference in the likelihood function of the bids and the winning bid.

Second, the reference prior provides a method for handling "nuisance parameters" when there is a partitioning of parameters into "parameters of interest" and "nuisance parameters". Jeffreys' prior does not distinguish between "nuisance parameters" and "parameters of interest". From Definitions 2, 3 and 4, and the exposition above, it can be seen that, in general, Jeffreys' prior and the reference prior are different when the support of the data depends on the parameters. Of course, the two coincide when standard regularity conditions are satisfied so that the support of the data does not depend on the parameters and there is no partitioning of the parameters into "nuisance parameters" and "parameters of interest".

Third, the reference prior, like the Jeffreys' prior, is invariant to the following kind of reparametrization; this result holds irrespective of the partitioning of the parameters into "nuisance parameters" and the "parameter of interest". If $(\eta, \varphi)$ are reparametrized to $(\alpha, \beta)$, where $\alpha = \alpha(\eta)$ and $\beta = \beta(\varphi)$ are one-to-one monotonic functions, then the reference prior, $\pi^{*}(\alpha, \beta)$,
for \((\alpha, \beta)\) is related to the reference prior for \((\eta, \varphi)\) by

\[
\pi^*(\alpha, \beta) = \left| \frac{\partial \eta}{\partial \alpha} \right| \det \left( \frac{\partial \varphi}{\partial \beta} \right) \pi(\eta(\alpha), \varphi(\beta)),
\]

where \(\eta(\alpha)\) and \(\varphi(\beta)\) are the inverse transformations, and \(\frac{\partial \eta}{\partial \beta}\) and \(\frac{\partial \varphi}{\partial \alpha}\) are the Jacobian of the transformation from \(\varphi \rightarrow \beta\) and \(\eta \rightarrow \alpha\), respectively.

### 3.4 Low-Price, Sealed-Bid Auction

In this Section I specify two different parametric assumptions about the distribution of the private costs of the bidders and consequently the distribution of the bids and the winning bid for a low-price, sealed-bid auction under the symmetric independent-private-values paradigm. Paarsch (1992) has previously studied similar tree planting auctions in British Columbia under these parametric assumptions. Similar parametric assumptions have been made in the auctions studied in Chapter 2. I will indicate an auction by the subscript \(j\) and bidders in an auction by \(i\). \(C_j^i\) is the private cost of bidder \(i\) in auction \(j\); the density function of \(C_j^i\) is indicated by \(f_c(c_j^i \mid \theta)\), and the distribution function by \(F_c(c_j^i \mid \theta)\), where \(\theta = (\delta, \varphi)\) is the parameter vector that characterizes the distribution of \(C_j^i\). I have reparametrized the lower bound of the support of the density function of bids, winning bids and the minimum winning bid, \(l(\delta, \varphi, n)\), as \(\eta(\delta, \varphi, n)\), where \(n\) is the number of potential bidders, which I am assuming is the same as the number of participants in an auction.\(^6\) In both examples I will indicate the regular

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\(^6\)In Chapter 2 I had assumed that the number of potential bidders was equal to the number of participants which varies across auctions. Hence a subscript \(j\) would be appended to the number of potential bidders, \(n_j\). Then the lower bound of the density function of \(Z_j^i = B_j^i, W_j^i, W_*\) depends on \(n_j\), \(Z_j^i \leq l_j(\delta, \varphi, n_j)\), where \(l_j(\delta, \varphi, n_j)\) is the lower bound of the density function of \(Z_j^i\) that varies across the \(T\) auctions. To obtain the reference prior for \((\delta, \varphi)\) or \((\eta, \varphi)\), I need the asymptotic distribution of the maximum likelihood estimators of \((\delta, \varphi)\). This involves maximizing the likelihood function subject to the \(T\) constraints imposed by the lower bound varying across the \(T\) auctions. All constraints will not be binding at any time; suppose \(q < T\) constraints are binding. Donald and Paarsch (1996, pp. 529-546) and Hong (1998, pp. 13-16) have derived the asymptotic distribution theory for similar class of problems in the context of discrete covariates. They distinguish three cases. When the number of parameters to be estimated, \(k\), is greater than the number of constraints, \(q\), then \(\sqrt{T}(\hat{\mu} - \mu_0)\) is asymptotically normally distributed with a singular covariance matrix of rank \(k - q\), where \(\mu_0\) is the true value of \(\mu\). When \(k = q\), \(T(\hat{\mu} - \mu_0)\) has a nondegenerate distribution as a linear combination of \(k\) independent exponential distributions. \(k < q\) leads to an asymptotic distribution of \(T(\hat{\mu} - \mu_0)\) that is a mixture of distributions, where the mixture is taken over the different \(k\)-element subsets, and each component of the mixture is a linear combination of the \(k\) independent exponential distributions.

In this Chapter, I have made a further assumption in addition to Chapter 2: that \(n_j\) is the same across the \(T\) auctions and denoted by \(n\). I have made this assumption to be within the framework of Ghosal (1997) and
parameter by $\varphi$ and the nonregular parameter by $\eta$ or $\delta$.

3.4.1 Bids Additive in Costs: Truncated Exponential Case

If bids are additive in cost, then Paarsch (1992, p. 204) assumes that $C_j^i$ follows an exponential distribution with parameter $\theta$. I will assume $C_j^i$ follows a truncated exponential distribution with parameters $(\delta, \varphi)$; $C_j^i \sim TEXP(\frac{1}{\varphi}, \delta)$, $C_j^i > \delta$. $\delta$ is some minimum private cost of a bidder. For example, in the auctions for crude-oil studied in Chapter 2, $\delta$ is the minimum cost at which a bidder can procure crude-oil. One of the aims of IOC, though not officially stated, in conducting these auctions is to target the informal market for crude oil that coexists with the formal crude oil market. By assuming that $C_j^i$ follows a truncated exponential distribution with parameters $(\delta, \varphi)$, I am assuming that the cost of procuring oil by a bidder has two components. Since crude oil being traded internationally is a relatively homogeneous commodity, and in the period these auctions were held crude oil price was relatively stable, there is a common cost of procuring crude oil that is known to all bidders; $\delta$ is this common known cost. The second component $\varphi$ is an idiosyncratic cost that is private to each bidder; this private cost will depend on the accessibility of the bidder to the informal crude-oil market.

The equilibrium bid follows a truncated exponential distribution, $B_j^i \sim TEXP(\frac{1}{\varphi}, \eta)$, where

$$\eta(\delta, \varphi, n) = \delta + \frac{1}{(n-1) \varphi}$$

is the lower bound of the support of the density function of the bids,

$$f_b(b_j^i | \eta, \varphi) = \varphi e^{-\varphi (b_j^i - \eta)}, \quad b_j^i > \eta.$$  \hspace{1cm} (13)

I have suppressed the dependence of $\eta(\delta, \varphi, n)$ on $\delta$, $\varphi$ and $n$ for notational convenience.

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Ghosal and Samanta (1997). In the above terminology, their framework would be equivalent to the case where only one constraint is binding. Deriving reference priors with multiple binding constraints is part of an agenda for future research.

7If bids are assumed to be additive in cost, then $C_j^i$ could follow a truncated exponential distribution too. For a low-price, sealed-bid symmetric independent-private-values auction, Paarsch (1992, p. 195) has shown that

$$b_j^i = c_j^i + \frac{\int_{c_j^i}^{\infty} [F_c(\xi)]^{n-1} d\xi}{[F_c(c_j^i)]^{n-1}},$$

where $F_c(\cdot)$ is the distribution function of $C_j^i$. If $C_j^i$ follows a truncated exponential distribution with parameters $\delta, \varphi$, then $F_c(c_j^i | \delta, \varphi) = e^{-\varphi (c_j^i - \delta)}$. Hence $b_j^i = c_j^i + \frac{1}{(n-1) \varphi}$, as when $C_j^i$ follows an exponential distribution with parameter $\varphi$.

8Since $C_j^i$ is measured in U.S. dollar per metric tonne, both $\delta$ and $(\varphi)^{-1}$ will be measured in the same units.
The winning bid in auction $j$, $W_j$ follows a truncated exponential distribution too, $W_j \sim TEX\left(\frac{1}{n\varphi}, \eta\right)$; the density function of the winning bid is

$$f_w(w_j|\eta, n\varphi) = n\varphi e^{-n\varphi(w_j-\eta)}, \quad w_j > \eta.$$  

The minimum winning bid across the $T$ auctions, $W_*$, follows a truncated exponential distribution, $W_* \sim TEX\left(\frac{1}{nT\varphi}, \eta\right)$ with density function

$$f_{w_*}(w_*|\eta, nT\varphi) = nT\varphi e^{-nT\varphi(w_*-\eta)}, \quad w_* > \eta.$$  

### 3.4.2 Bids Proportional to Costs: Pareto Case

When bids are assumed to be proportional to costs, then Paarsch (1992, p. 203) assumes that $C^i_j$ follows a Pareto distribution; $C^i_j \sim Pa(\delta, \varphi)$ with $C^i_j > \delta$.\footnote{$\delta, \eta$ is measured in U.S. dollar per metric tonne since $C^i_j$ is measured in this unit. $\varphi$ is a unitless quantity; Johnson, Kotz and Balakrishnan (1994, p. 573) refer to it as Pareto’s constant or a shape parameter.} Let $\eta(\delta, \varphi, n) = \frac{\delta^{(n-1)}}{\varphi^{n-1}}$ indicate a function of $\delta$ and $\varphi$; the dependence of $\eta(\delta, \varphi, n)$ on $\delta$ and $\varphi$ will be suppressed for notational convenience. The equilibrium bid follows a Pareto distribution, $B^i_j \sim Pa (\eta, \varphi)$, $B^i_j > \eta$, with density function

$$f_b(b^i_j|\eta, \varphi) = \frac{\varphi n \varphi}{(b^i_j)^{\varphi+1}}, \quad b^i_j > \eta.$$  

The winning bid follows a Pareto distribution too; $W_j \sim Pa (\eta, n\varphi)$, $W_j > \eta$ with density function

$$f_w(w_j|\eta, n\varphi) = \frac{n\varphi \eta^{n\varphi}}{(w_j)^{n\varphi+1}}, \quad w_j > \eta.$$  

The minimum winning bid across the $T$ auctions, $W_*$, follows a Pareto distribution, $W_* \sim Pa (\eta, nT\varphi)$ with density function

$$f_{w_*}(w_*|\eta, nT\varphi) = \frac{nT\varphi \eta^{n\varphi}}{(w_*)^{nT\varphi+1}}, \quad w_* > \eta.$$  

$\varphi$ is the shape parameter of the distributions of $C^i_j$, $B^i_j$, $W_j$, and $W_*$. $\delta$ is some minimum cost; it has an interpretation similar to $\delta$ in the previous case for the low-price, sealed-bid auctions. Thus $\delta$ is some minimum common cost of procuring crude oil that is known to all the
bidders. \( \frac{\varphi(n-1)}{\varphi(n-1)-1} \) is the per unit increase in a bidder's equilibrium bid as the cost of procuring oil increases.\(^{10} \frac{\varphi(n-1)}{\varphi(n-1)-1} \delta \) or \( \eta \) is the winning bid when the bidder's cost of procuring crude oil is the minimum cost known to all the bidders, \( \delta \).

### 3.5 Winning Bid Is Sufficient

In Section 2, following Huzurbazar (pp. 158, 174-186), I have stated in Assumption 2, the necessary and sufficient conditions under which the minimum winning bid across auctions is sufficient for the scalar parameter \( \theta \). In the propositions that follow, assuming that the necessary and sufficient conditions under which the minimum winning bid is sufficient for \( \theta \), I relate the reference prior obtained from bids within and across auctions with that obtained from the minimum winning bid, the Jeffreys' prior obtained from bids within and across auctions with that obtained from the minimum winning bid and the reference prior with Jeffreys' prior.

**Proposition 1:** Under Assumption 1 and 3, using Definition 3 and quantities defined in Table 1,

\[
\pi^b_{\text{ref}}(\theta) = \pi^w_{\text{ref}}(\theta) \propto |u'(\theta)m(\theta)q(\theta)|,
\]

where \( \pi^b_{\text{ref}}(\theta) \) is the reference prior obtained from data on \( nT \) bids and \( \pi^w_{\text{ref}}(\theta) \) is the reference prior obtained from the minimum winning bid across the \( T \) auctions.

**Proof.** Since \( f_\theta(l(\theta) | \theta) = m(l(\theta))q(\theta) \), the reference prior from the bids, \( \pi^b_{\text{ref}}(\theta) \propto |l'(\theta)m(l(\theta))q(\theta)| \), follows from Definition 3. The density function of the winning bid in auction \( j \) is \( f_w(w_j | \theta) = nm(w_j)[q(\theta)][1 - F_b(w_j)]^{n-1} \), where \( F_b(w_j) = \int_{l(\theta)}^{w_j} f_b(b_j | \theta) db_j \) is the distribution function of \( B^j \); the density function of the minimum winning bid, \( f_w(w_\ast | \theta) \), is \( nTm(w_\ast)[q(\theta)][1 - F_b(w_\ast)]^{n-1}[1 - F_w(w_\ast | \theta)]^{T-1} \), where \( F_w(w_\ast | \theta) = \int_{l(\theta)}^{w_\ast} f_w(w_j | \theta)dw_j \) is the distribution function of a winning bid, \( W_j \). The density of the minimum winning bid evaluated at \( l(\theta) \) is \( nTm(l(\theta))[q(\theta)] \) since \( F_b(l(\theta)) \) and \( F_w(l(\theta) | \theta) = 0 \). Hence Proposition 1 follows.

The intuition for this result follows from the reference prior being obtained by maximizing an asymptotic expansion of Lindley's measure of information given in Definition 1. With the

\(^{10}\)Since \( C^j_\delta \sim Pa(\delta, \varphi) \), the distribution function of \( C^j_\delta \) is \( F_\varphi(c^j_\delta | \delta, \varphi) = 1 - (\frac{\delta}{c^j_\delta})^{\varphi} \). Hence from Footnote 8, \( b_j = \frac{\varphi(n-1)}{\varphi(n-1)-1} c^j_\delta \).
minimum winning bid across auctions being a sufficient statistic for \( \theta \), the asymptotic posterior density function in Lindley’s measure of information is the same and hence the reference prior that results from bids within and across auctions and the minimum winning bid is identical as well. In the two examples that follow, I show that the exact reference posterior from the bids and the minimum winning bid is identical.

**Proposition 2:** Under Assumption 1 and 3, and using quantities defined in Table 1,

\[
\pi_{\text{Jeff}}^{\text{b}}(\theta) = \pi_{\text{Jeff}}^{\text{w}}(\theta) \propto \left| \frac{\partial \log q(\theta)}{\partial \theta} \right|
\]

where \( \pi_{\text{Jeff}}^{\text{b}}(\theta) \) is the Jeffreys’ prior obtained from the bids within and across auctions and \( \pi_{\text{Jeff}}^{\text{w}}(\theta) \) the Jeffreys’ obtained from the minimum winning bid.

**Proof.** From Definition 2, \( \pi_{\text{Jeff}}^{\text{b}}(\theta) \propto \sqrt{J_2(\theta)} \), where \( J_2(\theta) \) is the Fisher information from \( Z \) about \( \theta \). Since the minimum winning bid is sufficient for \( \theta \), \( J_\theta(\theta) = J_{\text{w}}(\theta) \). From equation (1), \( J_\theta(\theta) = J_{\text{w}}(\theta) = (nT)^2 \left( \frac{\partial \log q(\theta)}{\partial \theta} \right)^2 \). Since Jeffreys’ prior is defined up to a constant of proportionality, Proposition 2 follows.

**Proposition 3:** Under Assumption 1 and 2, using Definitions 2 and 3 and quantities defined in Table 1,

\[
\pi_{\text{Jeff}}^{\text{b}}(\theta) = \pi_{\text{Jeff}}^{\text{r}}(\theta).
\]

**Proof.** \( J_2(\theta) = E_{\theta|\theta} \left[ \left( \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \right)^2 \right] \) by Definition 2. From Definition 3, \( c(\theta) = (E_{\theta|\theta} \left[ \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \right])^2 \). Then from Jensen’s inequality,

\[
E_{\theta|\theta} \left[ \left( \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \right)^2 \right] \geq \left( E_{\theta|\theta} \left[ \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \right] \right)^2.
\]

Taking the positive square-root of the above expression, gives Jeffreys’ prior defined in Definition 2 on the LHS and the reference prior defined in Definition 3 on the RHS. From Jensen’s inequality, the above equation holds with equality if \( \frac{\partial}{\partial \theta} [\log f_2(Z_j^i|\theta)] \) is not a function of \( Z_j^i \), so that \( E_{\theta|\theta} \left[ \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \right] \) equals \( \frac{\partial}{\partial \theta} \log f_2(Z_j^i|\theta) \). Under Assumption 2, this is the case; hence Corollary 3 follows.

Intuitively, Proposition 3 follows from the score function not depending on \( Z_j^i = B_j^i, W_j, W_\ast \).
Since both the reference prior and Jeffreys’ prior involves the sampling expectation of the score, whether the sampling density \(Z_j^T|\theta\) is regular or nonregular, is irrelevant.

I now demonstrate the results in Propositions 1, 2 and 3 with two examples below. The exact reference posterior distributions are derived as well. In Table 2, Row 1, I summarize the results of this Section. I assume \(\varphi\) is given and derive the reference prior for \(\delta\). To use the terminology of Definition 3, \(\theta = \delta\) will refer to the nonregular parameter in both examples. The lower bound of the density function of the bid and the minimum winning bid, \(l(\theta) = l(\delta)\); of course \(l(\delta)\) will be different in the two examples.

3.5.1 Bids Additive in Cost: Truncated Exponential Case

Assuming bids are additive in cost as in Section 4.1, I derive the reference prior from the bids within and across auctions and the minimum winning bid across auctions. The lower bound of the density function of the bid and the minimum winning bid, \(l(\delta) = \delta + \frac{1}{(n-1)\varphi}\), and \(\frac{\partial l(\delta)}{\partial \delta} = 1\). First, the reference prior for \(\delta\) given \(\varphi\) from both the bids and the minimum winning bid is the improper uniform (Lebesgue measure on \((0, \infty))\). Second, Jeffreys’ prior and the reference prior coincide as proved in Proposition 3. Third, the exact reference posterior for \(\delta\) given \(\varphi\) is proper and the same from the bids and the minimum winning bid which follows from the latter being sufficient for \(\delta\) given \(\varphi\).

Example 1a: Reference prior from bids

From Section 4.1, assuming \(\varphi\) is given, and using the terminology of Definition 3, the reference prior for \(\delta\) given \(\varphi\) from the bids is

\[
\pi^b_{\text{ref}}(\delta|\varphi) \propto \left| \frac{\partial l(\delta)}{\partial \delta} f_b\left(\delta + \frac{1}{(n-1)\varphi} \delta, \varphi\right) \right|, \quad \delta > 0, \\
\propto \text{constant,} \tag{19}
\]

where \(f_b(\delta|\delta, \varphi)\) from equation (13) is the density function of the bids evaluated at the lower bound and \(\frac{\partial l(\delta)}{\partial \delta} = 1\). Since \(\varphi\) is given, the reference prior for \(\delta\) is proportional to a constant.
and is hence improper. The exact reference posterior for \( \delta \) given \( \varphi \) is

\[
\pi_{\text{ref}}^b(\delta|\varphi, b) \propto \varphi \prod_{j=1}^{n} \left[ e^{-\varphi(b_j - \delta - \frac{1}{n-1})} \right],
\]

\[
\propto e^{nT\varphi\delta}, \quad 0 < \delta < w_* - \frac{1}{(n-1)\varphi},
\]

(20)

where \( w_* = \min_{j=1,\ldots,T} w_j \). Since \( \delta \) is bounded given \( \varphi \), the exact reference posterior density for \( \delta \) is proper. Identical results are obtained with the minimum winning bid across auctions in the next example.

Example 1b: Reference prior from minimum winning bid

Since the minimum winning bid across auctions is sufficient for \( \delta \) given \( \varphi \), the reference prior for \( \delta \) given \( \varphi \) is identical to (19) from Corollary 1. Further from equation (B.1), Appendix B, since \( J_B(\delta) = J_W(\delta) = (nT)^2\varphi^2 \), Jeffreys' prior for \( \delta \) given \( \varphi \) is identical to the reference prior for \( \delta \) given \( \varphi \); that is, it is proportional to a constant. The exact reference posterior is

\[
\pi_{\text{ref}}^w(\delta|\varphi, w_*) \propto \varphi \left[ nT\varphi^{-nT\varphi(w_* - \delta - \frac{1}{n-1})} \right], \quad \delta > 0,
\]

\[
\propto e^{nT\delta\varphi}, \quad 0 < \delta < w_* - \frac{1}{(n-1)\varphi},
\]

(21)

which is identical to the result obtained from (20).

3.5.2 Bids Proportional to Cost

The reference prior from bids within and across auctions and the minimum winning bid across auctions is obtained assuming that bids are proportional to cost as in Section 4.2. Using the terminology of Definition 3, with \( \varphi \) given, I set \( \theta = \delta \), \( l(\theta) = l(\delta) = \frac{\delta \varphi (n-1)}{\varphi (n-1) - 1} \) and as a result \( \frac{\partial l(\delta)}{\partial \delta} = \frac{\varphi (n-1)}{\varphi (n-1) - 1} \). The reference prior for \( \delta \) given \( \varphi \) is proportional to \((\delta)^{-1}\). Unlike the previous case, when bids were additive in cost, it is not the improper uniform even though \( \delta \) varies between \((0, \infty)\) in both cases. The dissimilarity in the reference prior in the two cases follows from the reference prior depending on the sampling distribution of the data which is different in the two cases. This difference in the likelihood function is picked up by the Jeffreys' prior for \( \delta \) given \( \varphi \) which again is not improper uniform like in the previous subsection. Second,
like the previous case when bids were additive in cost, the Jeffreys' prior and the reference prior coincide in this case too, where bids are proportional to cost. Third, the exact reference posterior belong to the power-function distribution. This is similar to the results obtained by Arnold and Press (1983, pp. 192-193) where they arrive at a noninformative prior as a special case of the conjugate prior for the scale parameter of a Pareto likelihood when the shape parameter is known; since this noninformative prior is the same as Jeffreys' prior or the reference prior for $\delta$ given $\varphi$, that I have derived below, I obtain a power-function distribution for $\delta$ given $\varphi$ as well.

**Example 2a: Reference prior from bids**

Given $\varphi$, the reference prior for $\delta$ can be obtained from Definition 3 and equation (16) in Section 4.2. Since $\frac{\partial n(\delta)}{\partial \delta} = \frac{\varphi(n-1)}{\varphi(n-1) - 1}$ and the density function of the bids in equation (16) evaluated at the lower bound, $f_b(\eta|\delta, \varphi) = \frac{\varphi(n-1)}{\eta}$, the reference prior for $\delta$ given $\varphi$ from Definition 3 is

$$
\pi^{\delta}_{ref}(\delta|\varphi) \propto \left[ \frac{\varphi(n-1)}{\varphi(n-1) - 1} \right] \frac{\varphi}{\eta},
$$

$$
\propto \frac{1}{\delta}, \quad \delta > 0. \quad (22)
$$

The exact reference posterior when data on $nT$ bids is available is

$$
\pi^{\delta}_{ref}(\delta|\varphi, b) \propto \frac{1}{\delta} \prod_{j=1}^{T} \prod_{i=1}^{n} \left[ \frac{\varphi \eta^\delta}{(b_j^\delta \eta)^{\delta+1}} \right],
$$

$$
\propto \delta^{nT\varphi - 1}, \quad 0 < \delta < \frac{\varphi(n-1) - 1}{\varphi(n-1)} w_*,
$$

$$
= PF\left(nT \varphi, \frac{\varphi(n-1) - 1}{\varphi(n-1)} w_* \right). \quad (23)
$$

$PF$ indicates the density function of the power-function distribution (Johnson and Kotz, p. 607). Note that if $0 < nT \varphi < 1$, then the posterior density of $\delta$ given $\varphi$ will have an asymptote at $\delta = 0$; but $0 < nT \varphi < 1$ would imply that $\delta < 0$. Hence $0 < nT \varphi < 1$ is ruled out.

**Example 2b: Reference prior from minimum winning bid**

Since the minimum winning bid across auctions is sufficient for $\delta$ given $\varphi$, the reference prior for $\delta$ given $\varphi$, given data on the minimum winning bid across auctions, is identical to that obtained
in (22) for the \( nT \) bids in auction \( j \). The reference posterior is

\[
\pi^{w_r}_{\text{ref}}(\delta | \varphi, w_r) \propto \frac{1}{\delta} \left[ \frac{nT \varphi (nT \varphi + 1)}{(w_r)^{nT \varphi + 1}} \right], \]

\[
\propto \delta^{nT \varphi - 1}, \quad 0 < \delta < \frac{\varphi(n - 1) - 1}{\varphi(n - 1)}w_r, \]

\[
= PF \left( nT \varphi, \frac{\varphi(n - 1) - 1}{\varphi(n - 1)}w_r \right), \tag{24}
\]

which is identical to the exact reference posterior from bids in (23). From equation (C.1), Appendix C, since \( J_B(\delta) = J_W(\delta) = \frac{(nT)^2 \delta^2}{\delta^2} \), the reference prior for \( \delta \) given \( \varphi \) is identical to Jeffreys' prior for \( \delta \) given \( \varphi \).

The conclusion that emerges from Section 5 is that when the minimum winning bid is sufficient for a parameter, the reference prior for that parameter obtained from data on all the bids is identical to that obtained from just the minimum winning bid across auctions. Further, the reference prior and Jeffreys' prior are identical. For the two examples of sealed-bid auctions that I have, two results stand out. First, the reference prior and Jeffreys' prior are both improper. Second, even though \( \delta \in (0, \infty) \) in both examples, the prior, whether it is Jeffreys' or the reference prior, is proportional to a constant when bids are additive in cost, and proportional to \( \delta^{-1} \) when bids are proportional to cost, reflecting the dependence of the reference prior and the Jeffreys' prior on the sampling density of the data.

### 3.6 Winning Bid Not Sufficient

Once \( \theta \) is not a scalar and \( \theta = (\eta, \varphi) \) where \( \eta \) is the "nonregular" parameter and \( \varphi \) the "regular" parameter, two features of the sampling density of the data become important in constructing reference priors.

**First**, Huzurbazar (pp. 174-187) shows that a single sufficient statistic does not exist once Assumption 3 is relaxed and \( \theta \) is not a scalar. Then \( J_B(\theta) \neq J_{W_r}(\theta) \); I have proved in Section 2 that \( J_B(\theta) - J_{W_r}(\theta) \) is a non-null psd matrix. The key point is that once the density function of bids is characterized by a distribution with two or more parameters, a single minimal sufficient statistic cannot exist for a vector of parameters, \( \theta \). Hence if \( \theta \) is a vector,
the minimum winning bid cannot summarize all the expected information in the \( nT \) bids for \( \theta \). Further, the \( T \) winning bids across auctions are also not sufficient for \( \theta \), so that \( J_B(\theta) - J_W(\theta) \) is a non-null psd matrix as well; this has been proved in Section 2. This feature of the sampling density gets reflected in the following two results on reference priors and posteriors for the two examples described in Section 4.

The first result pertains to the "regular parameter". The reference prior, when it is not proportional to a constant, and the exact reference posterior for the "regular" parameters \( \varphi \) is more dispersed when it is obtained from the winning bids across auctions compared to when it is obtained from all the bids within and across auctions. This result follows from the winning bids not being sufficient for the "regular" parameter \( \varphi \), or the difference in the Fisher information for \( \varphi \) obtained from all the bids and the winning bids, \( J_B^{\varphi} (\eta, \varphi) - J_W^{\varphi} (\eta, \varphi) \), being a non-null psd matrix.

The second result pertains to the "nonregular parameter". The reference prior for \( \eta \) and the reference posterior for the "nonregular" parameter \( \eta \) given \( \varphi \), whether obtained from all the bids in an auction or just the winning bid, is identical to the reference prior for the "nonregular" parameter \( \eta \) given \( \varphi \) obtained from the minimum winning bid. This follows from the minimum winning bid being sufficient for \( \eta \) given \( \varphi \). Given \( \varphi \), in both examples described in Section 4, the density function of the bids satisfies the necessary and sufficient conditions given in Assumption 3 under which the minimum winning bid is sufficient for a parameter; in essence, \( \eta \) given \( \varphi \) reduces to the single parameter case of Section 5. Note that in both examples \( \eta \) and \( \varphi \) are independent according to the reference prior; this follows from the sampling expectation of the score function, \( c(\eta, \varphi) \), and \( \sqrt{\det J_Z^{\varphi}} (\eta, \varphi), Z = B, W, W_\ast \), factorizing into functions of \( \eta \) alone and \( \varphi \) alone.

The second important feature of the sampling density of the data is that the MLE of the "regular" and the "nonregular" parameter asymptotically converge to different distributions at different rates. The MLE of the "regular" parameter after "concentrating" out the "nonregular" parameter from the likelihood function converges in distribution at rate \( \sqrt{T} \) to a Normal distribution; \( \sqrt{T} (\theta_{ml} - \theta_\ast) \sim N (0, J_Z^{\varphi}(\tilde{\eta}, \varphi)) \), where \( \tilde{\eta} = w_\ast \). The MLE of the "nonregular" parameter is the minimum winning bid, \( W_\ast \). It converges in distribution at rate \( T \) to an Expo-
nential distribution; \( T(W_\star - \eta_\star) \sim EXP(c(\eta, \varphi)) \), where \( c(\eta, \varphi) \) is the sampling expectation of the score function given in equation (9). Thus \( W_\star \) is a superconsistent estimator of \( \eta \).

The difference in the distributions to which the estimator of the "regular" and the "non-regular" parameter converge is captured by the reference prior through the difference in the contribution made by the "regular" and the "nonregular" parameter to the reference prior for \((\eta, \varphi)\) given in Definition 4. Both the "regular" and the "nonregular" parameter contribute the standard deviation of the asymptotic distribution of the relevant estimator. Since the asymptotic distribution of the estimator of the "regular" and the "nonregular" parameter are different, the "regular" parameter \( \varphi \) contributes \( \sqrt{J^2_2(\eta, \varphi)} \) and the "nonregular" parameter \( \eta \) contributes \( c(\eta, \varphi) \) to the reference prior.

The superconsistency of \( W_\star \), the estimator of the "nonregular" parameter, or the faster rate of convergence of the estimator of the "nonregular" compared with the "regular" parameter, shows up in the reference prior for the "regular" parameter obtained from the bids or the winning bids as follows.

Integrating out \( \eta \) from the exact reference posterior for \((\eta, \varphi)\), I obtain the marginal reference posterior for the "regular" parameter \( \varphi \). An objective Bayesian would use this posterior to proceed with inference about the "regular" parameter \( \varphi \). In classical literature, as in Christensen and Kiefer (1991) and Hong (1998), \( \eta \) is "concentrated out" of the likelihood function; that is, \( \eta \) is substituted with \( w_\star \) in the likelihood function. Inference about \( \varphi \), the "regular" parameter proceeds by maximizing this "concentrated" likelihood function with respect to \( \varphi \). Christensen and Kiefer (1991) argue that because \( W_\star \) is a superconsistent estimator of \( \eta \), it is almost as if \( w_\star = \eta \); hence, it is appropriate to substitute \( w_\star \) for \( \eta \) in the likelihood, and maximize it with respect to the other parameters.

Integrating out the "nonregular" parameter, \( \eta \), from the exact reference posterior for \((\eta, \varphi)\), to obtain the marginal reference posterior for the "regular" parameter \( \varphi \) in Bayesian inference,

---

11 I obtain, in the two examples in this paper, the same marginal reference prior for \( \varphi \) if I use the uniform-integrated likelihood as in Berger and Liseo (1998, p. 2, 23). The uniform-integrated likelihood is obtained by integrating out \( \eta \) with respect to the Lebesgue measure, from the likelihood function; then this uniform-integrated likelihood is used to construct the reference prior for \( \varphi \). Inference about \( \varphi \) is based on the posterior which is a product of these two quantities. Also note that in both examples in my paper, the "concentrated" likelihood and the uniform-integrated likelihood are identical.
is similar to the classical procedure of "concentrating out" the "nonregular" parameter and maximizing the concentrated likelihood function with respect to the "regular" parameter, \( \varphi \). The difference is that the classical procedure "concentrates out" the parameter \( \eta \) with respect to a distribution that puts all the mass of the distribution on one point, \( l^{-1}(\omega_*, \varphi) \).\(^{12}\) On the other hand, the exact reference posterior for the "regular" parameter is obtained by concentrating out \( \eta \), the "nonregular" parameter with respect to the exact reference posterior for \( \eta | \varphi \).\(^{13}\) Hence I should observe some similarity; in terms of the superconsistency of \( \omega_* \) for \( \eta \) showing up, whether I do classical or Bayesian inference about the "regular" parameter \( \varphi \).

The marginal reference posterior for the "regular" parameter \( \varphi \) obtained by integrating out \( \eta \) from the exact reference posterior for \( (\eta, \varphi) \) is proportional to the product of the marginal prior for the "regular" parameter and the likelihood function. The likelihood function of the bids and the winning bids is as if the bids or the winning bids were truncated at \( \omega_* \) instead of \( \eta \); this is the concentrated likelihood function in the classical literature. \( \sqrt{\det J^\varphi_2(\omega_*, \varphi)} \), \( Z = B, W \), is the marginal reference prior for the "regular" parameter \( \varphi \) that I obtain. But this is Jeffreys' prior for \( \varphi \) obtained from the concentrated likelihood, the likelihood function of the bids and the winning bids is as if the bids or the winning bids were truncated at \( \omega_* \) instead of \( \eta \). This is intuitive since I have already noted in Section 3 that the reference prior is the Jeffreys' prior for the "regular" parameter. I illustrate these points through the two examples below.

Lancaster (1997) makes a similar point in context of the likelihood function in job search models with a difference. He obtains the posterior distribution for the "regular" parameters by integrating out the "nonregular" parameter from the joint posterior distribution of the "regular"

\(^{12}\)Since \( l(\eta) \) is a linear function of \( \eta \), \( l(\eta) = \eta, l^{-1}(\omega_*, \varphi) = \omega_* \) here.

\(^{13}\)This point becomes clearer if I use the asymptotic posterior distribution of \( \eta \) given \( \varphi \) to integrate out \( \eta \). Ghosal and Samanta (1997, pp 183-184) show that, given \( \varphi \), the asymptotic posterior distribution of \( \eta \) converges to an exponential distribution at rate \( T \);

\[ T(\eta - \omega_*) \sim \text{EXP}(c(\eta, \varphi)) \text{ as } T \rightarrow \infty. \]

Note that the mode of this posterior distribution is at \( \omega_* \). If I use this distribution to integrate out \( \eta \), then the similarity to "concentrating out" \( \eta \) from the likelihood function with \( \omega_* \) becomes evident. "Concentrating out" \( \eta \) amounts to using a prior for \( \eta \) that puts all the mass on \( \eta = \omega_* \); integrating out \( \eta \) with respect to the asymptotic posterior distribution for \( \eta \) given \( \varphi \) amounts to using a prior that puts the maximum but not all mass at \( \eta = \omega_* \). Since the mode of the truncated exponential density function, \( \eta \geq \omega_* \), that I have given above is at \( \eta = \omega_* \).
parameters and the “nonregular” parameter.¹⁴ Like the examples that follow, this is the product of the prior and the likelihood of the data as if the data was truncated at the sample minimum rather than at the nonregular parameter due to the superconsistency of the sample minimum for the “nonregular” parameter. However he is unable to obtain this integral analytically and as a result obtains a Laplace approximation of the integral as the total number of search periods goes to infinity. He then comments on the similarity of this Laplace approximation and the classical procedure of “concentrating” out $\eta$. In the examples in this paper, I too can take a Laplace approximation to integrate out $\eta$ from the joint posterior density function of $(\eta, \varphi)$; similar to Lancaster (1997), the posterior for $\varphi$ is the product of the concentrated likelihood function and the reference prior based on this likelihood.

But in both the examples in this paper, I can obtain this integral analytically too. Unlike Lancaster (1997), when bids are proportional to cost, I see the superconsistency still showing up. The marginal posterior for the “regular” parameters is the product of the likelihood of bids or the winning bids as if they were truncated at $w_*$ instead of $\eta$, and the reference prior based on this likelihood. The message is that the superconsistency of the sample minimum for the “nonregular” parameter showing up in the marginal posterior distribution for the “regular” parameters could be an exact result; I observe it in my sample of $nT$ bids or $T$ winning bids when bids are proportional to cost. The key point is the functional form of the integrand since the Laplace approximation to an integral is obtained by examining the integral in the vicinity of the maximum of the integrand. I elaborate on this in the examples below.

The main results of the examples that follow are summarized in Table 2, Row 2.

3.6.1 Bids Additive in Cost

To apply Definition 4, I reparametrize the density function in equations (13) and (14) with the nonregular parameter, $\eta = \delta + \frac{1}{(n-1)\varphi}$, and the regular parameter is $\varphi$. The lower bound of the density function of the bid or the winning bid is $l(\eta) = \eta$.

Example 1a: Reference prior from bids

¹⁴In Lancaster (1997), the reservation wage is the “nonregular” parameter and the parameters of the distribution of wage offers are the “regular” parameters.
Now in the terminology of Definition 4, \( c(\eta, \varphi) = t'(\eta) f_b(\eta|\eta, \varphi) = \varphi \). The Fisher information matrix for \( \boldsymbol{\theta} \) from the bids is given in equation (B.1), Appendix B. The element in the second row and column of this matrix gives \( \sqrt{\text{det} \mathbf{J}_{\varphi \varphi}^B(\eta, \varphi)} = \frac{1}{\varphi} \sqrt{nT + \frac{(nT)^2}{(n-1)^2}} \). Then from Definition 4 the reference prior for \( [\eta, \varphi]' \) is

\[
\pi_{\text{ref}}^b(\eta, \varphi) \propto \varphi \left[ \frac{1}{\varphi} \sqrt{nT + \frac{(nT)^2}{(n-1)^2}} \right], \quad \eta, \varphi > 0, \\
\propto \text{constant.} \tag{25}
\]

The reference posterior is

\[
\pi_{\text{ref}}^b(\eta, \varphi|b) = \pi_{\text{ref}}^b(\varphi|b) \pi_{\text{ref}}^b(\eta|\varphi, b), \quad 0 < \eta < w_* ,
\]

\[
\propto \left[ \varphi^n \exp \left( -\varphi \sum_{j=1}^T \sum_{i=1}^n b_i^j \right) \right] \left\{ \exp \left( nT \varphi \eta \right) \right\},
\]

\[
\propto GA \left[ nT + 1, \left( \sum_{j=1}^T \sum_{i=1}^n b_i^j \right)^{-1} \right] \left\{ \exp \left( nT \varphi \eta \right) \right\}. \tag{26}
\]

The first term in (26) is proportional to the reference posterior density function for \( \varphi \), which is a Gamma density function that I have indicated by GA; and the second term is proportional to the reference posterior density function for \( \eta|\varphi \). Since \( 0 < \eta < w_* \), the kernel of the reference posterior, \( \eta|\varphi, b \), is bounded, and the posterior is proper.

Example 1b: Reference prior from winning bids

Now \( c(\eta, \varphi) = t'(\eta) f_w(\eta|\eta, \varphi) = n \varphi \). The Fisher information matrix, \( \mathbf{J}_w(\eta, \varphi) \), from the winning bids is given in equation (B.2), Appendix B; the element in the second row and column of this matrix gives \( \sqrt{\text{det} \mathbf{J}_{\varphi \varphi}^w(\eta, \varphi)} = \frac{1}{\varphi} \sqrt{T + \frac{(nT)^2}{(n-1)^2}} \). The reference prior from Definition 4 is again proportional to a constant,

\[
\pi_{\text{ref}}^w(\eta, \varphi) \propto \text{constant}, \quad \eta, \varphi > 0. 
\]

The reference posterior is

\[
\pi_{\text{ref}}^w(\eta, \varphi|w) = \pi_{\text{ref}}^w(\varphi|w) \pi_{\text{ref}}^w(\eta|\varphi, w), \quad 0 < \eta < w_*, 
\]
The first term in (27) is proportional to the reference posterior density function for \( \varphi \); it is a Gamma density function. The second term is the kernel of the reference posterior for \( \eta | \varphi \).

Since the reference prior from the bids and the winning bids is a constant, it does not capture the fact that the winning bids are not sufficient for \( \theta \), but are just members of the set of sufficient statistics. This feature gets reflected in the exact reference posteriors for \( \varphi \). From equations (26) and (27), the reference posterior for \( \varphi \), from all the bids in an auction and the winning bid, is a Gamma distribution with different parameters. Since \( nT + 1 > T + 1 \), and \( \sum_{j=1}^{T} \sum_{i=1}^{n} b_{ij} w_j \), the reference posterior for the regular parameter, \( \varphi \), is more dispersed when data on winning bids is available (equation (27)) compared to when data on all the bids within and across auctions is available (equation (26). Thus inference about \( \varphi \) differs depending on the available data comprising of all bids within and across auctions or just the winning bids across auctions due to the difference in the likelihood functions of the two data sets; the reference prior, since it is proportional to a constant, plays no role in inference about \( \varphi \). The kernel of the conditional posterior for \( \eta | \varphi \) is the same in equation (26) and (27) and is identical to the kernel of the conditional posterior for \( \eta | \varphi \) obtained from the minimum winning bid, \( \pi_{ref}^{w_{*}} (\eta | \varphi, w_{*}) \). This reflects the conditional sufficiency of the minimum winning bid for \( \eta \). Further, the Jeffreys' prior \( \pi_{eff}^{j} (\eta, \varphi | z) \propto constant \), as well, where \( Z = B, W, W_{*} \). This result is specific to this example. It follows from the determinant of the Fisher information matrix from the bids or the winning bids, \( J_{Z} (\eta, \varphi) \), being a function of the number of potential bidders, \( n \), which gets absorbed in the constant of proportionality.

\[
\alpha \propto \varphi^{T} \exp \left( -\varphi n \sum_{j=1}^{T} w_{j} \right) [\exp (nT \varphi \eta)] \\
\alpha \propto GA \left[ T + 1, \left( n \sum_{j=1}^{T} w_{j} \right)^{-1} \right] [\exp (nT \varphi \eta)] .
\] (27)

---

\(^{15}\)From equation (13), \( c(\eta, \varphi) = nT \varphi \), and \( \sqrt{\text{det} J_{W_{*}}^{\varphi}} (\eta, \varphi) = \frac{1}{\varphi} \sqrt{1 + \frac{(nT \eta)^{2}}{(n-1)}} \) from equation (B.3), Appendix B; then the reference prior from the minimum winning bid is again proportional to a constant. The reference posterior for \( (\eta, \varphi) \) is

\[
\pi_{ref}^{w_{*}} (\eta, \varphi | w_{*}) \propto GA \left[ 2, (nT \varphi \eta)^{-1} \right] [\exp (nT \varphi \eta)] ,
\] where the term in the second square brackets is the reference posterior kernel for \( \eta | \varphi \) obtained from the minimum winning bid.
Integrating out \( \eta \) from the posterior density function of \((\eta, \varphi)\) in equation (26) by taking a Laplace approximation to the integral (Murray 1974, pp. 28-35) as the number of auctions \( T \to \infty \), and retaining the lead term of the approximation, I obtain

\[
\pi^b_{\text{ref}} (\varphi|b) \propto \left[ \frac{1}{\varphi} \right] \left[ \prod_{j=1}^{T} \prod_{i=1}^{n} \varphi e^{-\varphi(b_j - w_i)} \right];
\]

(28)

this is derived in Appendix D. The term in the second square brackets is the concentrated likelihood function; that is, the likelihood of the \( nT \) bids as if the truncated exponential density function of the bids given in equation (13) is truncated not at \( \eta \) but rather at \( w_* \). The first term in the square brackets is \( \sqrt{\det J^\varphi_B (w_*, \varphi)} \), the reference prior or the Jeffreys' prior for \( \varphi \) based on this concentrated likelihood function. I have mentioned earlier that classical inference proceeds by substituting \( \eta \) with \( w_* \) in the likelihood function since the minimum winning bid is a superconsistent estimator of the "nonregular" parameter; inference about the remaining "regular" parameters is then carried out by maximizing this concentrated likelihood function. Objective Bayesian inference is proceeding in a similar manner in (28). Inference about \( \varphi \) is based on the concentrated likelihood function and the Jeffreys' prior based on this concentrated likelihood function.

I obtain similar results when I integrate out \( \eta \) from the posterior density function of \((\eta, \varphi)\) in equation (27); this is demonstrated in Appendix D.

### 3.6.2 Bids Proportional to Cost

Reparameterizing the density function of bids in equations (16) and (17) such that the non-regular parameter in Definition 4 is \( \eta = \frac{\delta^{(n-1)}}{\varphi (n-1)-1} \), and the regular parameter is \( \varphi \). The lower bound of the density function for a bid and the winning bid in (16) and (17), respectively, is \( l(\eta) = \eta \).

**Example 2a: Bids proportional to cost, reference prior from bids**

From (16), \( c(\eta, \varphi) = l'(\eta) f_\eta(\eta|\eta, \varphi) = \frac{\xi}{\eta} \). I obtain \( J^\varphi_B (\eta, \varphi) \) as the element in the second row and column of the Fisher information matrix from bids, \( J_B (\eta, \varphi) \); this is given in equation (C.1), Appendix C. \( \sqrt{\det J^\varphi_B (\eta, \varphi)} = \frac{1}{\varphi} \sqrt{nT + \frac{(nT)^2 \varphi^2}{(\varphi n-1)^2}} \), where \( m = n - 1 \). Then from Definition 4
the reference prior for \( \eta, \varphi \) is

\[
\pi_{\text{ref}}^b (\eta, \varphi) \propto \frac{\varphi}{\eta} \left\{ 1 + \frac{T_n \varphi^2}{(\varphi m - 1)^2} \right\}, \quad \eta, \varphi > 0,
\]

(29)

The exact reference posterior for \( \eta, \varphi \) is

\[
\pi_{\text{ref}}^b (\eta, \varphi|b) = \pi_{\text{ref}}^b (\varphi|b) \pi_{\text{ref}}^b (\eta|\varphi, b), \quad 0 < \eta < w_\star;
\]

\[
\propto \left[ \sqrt{1 + \frac{nT \varphi^2}{(\varphi m - 1)^2}} \varphi^m e^{-\varphi (\sum \log b_j)} \right]^{\eta nT \varphi - 1},
\]

\[
\propto \left[ \sqrt{1 + \frac{nT \varphi^2}{(\varphi m - 1)^2}} \right] GA \left( nT + 1, \left( \sum_{j=1}^{n} \log b_j \right)^{-1} \right) \left[ PF (nT \varphi, w_\star) \right].
\]

(30)

The first term in the square brackets of (30) is the kernel of the reference posterior for \( \varphi, \pi_{\text{ref}}^b (\varphi|b) \).

The term in the second square brackets is the posterior kernel of \( \eta|\varphi \); it belongs to the power-function distribution.

**Example 2b: Reference prior from winning bids**

From equation (17), \( c(\eta, \varphi) = l'(\eta) f_{\omega}(\eta|\eta, \varphi) = \frac{m \varphi}{\eta} \). I obtain \( J_{\text{ref}}^{\varphi} (\eta, \varphi) \) as the element in the second row and column of the Fisher information matrix from the winning bids, \( J_{\text{w}} (\eta, \varphi) \); this is given in equation (C.2), Appendix C. \( \sqrt{\det J_{\text{ref}}^{\varphi} (\eta, \varphi)} = \frac{1}{\varphi} \sqrt{1 + \frac{T_n \varphi^2}{(\varphi m - 1)^2}} \), where \( m = n - 1 \).

Then from Definition 4 the reference prior for \( \eta, \varphi \) is

\[
\pi_{\text{ref}}^w (\eta, \varphi) \propto \frac{\varphi}{\eta} \left\{ 1 + \frac{T_n \varphi^2}{(\varphi m - 1)^2} \right\}, \quad \eta, \varphi > 0;
\]

\[
\propto \left\{ \frac{1}{\eta} \right\} \left\{ 1 + \frac{T_n \varphi^2}{(\varphi m - 1)^2} \right\}.
\]

(31)
The exact reference posterior for \([\eta, \varphi]\) is

\[
\pi^w_{\text{ref}}(\eta, \varphi|w) = \frac{\pi^w_{\text{ref}}(\varphi|w) \pi^w_{\text{ref}}(\eta|\varphi, w)}{\pi^w_{\text{ref}}(\eta|\varphi, w)}, \quad 0 < \eta < w^*_s;
\]

\[
\propto \left[ \frac{Tn^2\varphi^2}{(\varphi m - 1)^2} \varphi^T e^{-\varphi m(\sum \log w_j)} \right] \left[ \eta^{nT\varphi - 1} \right],
\]

\[
\propto \left[ \frac{Tn^2\varphi^2}{(\varphi m - 1)^2} GA \left( T + 1, \left( n \sum_{j=1}^T \log w_j \right)^{-1} \right) \right] \left[ PF(nT\varphi, w_s) \right].
\]

(32)

The first term in the square brackets of (32) is the kernel of the reference posterior for \(\varphi, \pi^\text{ref}_b(\varphi|w)\). The term in the second square brackets is the posterior kernel of \(\eta|\varphi\).

There are several points worth noting in Example 2. First, from equations (29) and (31), the reference priors for \((\eta, \varphi)\) from the bids and the winning bids are not identical.

Second, both the reference prior and the reference posterior for \(\eta|\varphi\), whether they are obtained from all the bids or just the winning bids across auctions, are identical to the reference prior and posterior for \(\eta|\varphi\) obtained from the minimum winning bid; this follows from the conditional sufficiency of the minimum winning bid for \(\eta|\varphi\). Since \(\eta\) and \(\varphi\) are \textit{apriori} independently distributed, from equations (29) and (31), I find that the reference prior for \(\eta\), obtained from the bids and the winning bids, respectively, is identical and improper. It is given by the first term in the curly brackets in the two equations and is proportional to \((\eta)^{-1}\). The reference prior from the minimum winning bid is the same as well. The reference posterior for \(\eta|\varphi\) obtained from the bids and the winning bids is given by the second term in the square brackets in equations (30) and (32), respectively. It is identical, belonging to the family of power-function distribution, and is the same as the reference posterior for \(\eta|\varphi\) obtained from the minimum

\textit{From equation (18), }c(\eta^*, \varphi) = \frac{\pi^w_{\text{ref}}}{\eta} \text{ and } \sqrt{\text{det} J^w_{\text{ref}}(\eta^*, \varphi)} = \frac{1}{\eta} \sqrt{1 + \frac{(nT)^2\varphi^2}{\omega_m(\omega_m - 1)^2}} \text{ from equation (C.3), Appendix C. The reference prior for } (\eta, \varphi) \text{ obtained from the minimum winning bid is}

\[
\pi^w_{\text{ref}}(\eta, \varphi) \propto \left\{ \frac{1}{\eta} \right\} \left\{ \sqrt{1 + \frac{(nT)^2\varphi^2}{\omega_m(\omega_m - 1)^2}} \right\}.
\]

The reference prior for \(\eta\) is \((\eta)^{-1}\).
winning bid, $\pi_{\text{ref}}^{w*}(\eta|\varphi, w_*)$.\(^{17}\)

Third, the reference prior for $\varphi$ from all the bids and the winning bids is improper; it is given by the second term in the curly brackets in equations (29) and (31), respectively. This impropriety follows from $\lim_{\varphi \to \infty} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} = \sqrt{1 + \frac{Tn^2}{m^2}}$ and $\lim_{\varphi \to \infty} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} = \sqrt{1 + \frac{Tn^2}{m^2}}$, so that the kernel of the reference prior is not integrable. I have plotted the kernel of the reference prior for $\varphi$ obtained from the bids, $\sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}}$, and the winning bids, $\sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}}$, in Figure 1, to make this point obvious.

Fourth, the winning bids are not sufficient for the regular parameter $\varphi$, or the Fisher information about $\varphi$ obtained from bids within and across auctions is strictly greater than the Fisher information about $\varphi$ obtained from the winning bids across auctions. The reference prior, as well as, the exact reference posterior for $\varphi$ picks up this feature of the sampling density of the data. From the plot of the reference prior kernel for $\varphi$ in Figure 1, I observe that the reference prior from the winning bids lies above that obtained from the bids and the right tail of the kernel for $\varphi$ obtained from the winning bids is thicker than the right tail of the kernel from all the bids. This behavior of the reference prior reflects the feature of the sampling density of the data that I mentioned earlier; all the bids contain strictly more Fisher information than the winning bids for the regular parameter $\varphi$. The exact reference posterior reflects this feature too; the first term in the square brackets in equations (30) and (32) gives the reference prior for $\varphi$, obtained from all the bids and the winning bids, respectively. The first term in the posterior for $\varphi$ is contributed by the reference prior for $\varphi$; this has been plotted in Figure 1. The second term is a Gamma density function. This is more dispersed for the winning bids than all the bids since the winning bid is the lowest bid in an auction $j$, so that $\sum_{j=1}^{T} \sum_{i=1}^{n} \log b_j > \sum_{j=1}^{T} \log w_j$, and $nT + 1 > T + 1$.

Another feature of the reference prior that I observe from Figure 1 is that most of the mass

\(^{17}\)I have obtained the reference prior for $(\eta, \varphi)$ from the minimum winning bid in the last footnote. The reference posterior is

$$\pi_{\text{ref}}^{w*}(\eta, \varphi|w_*) \propto \left[\sqrt{1 + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} GA \left(2, (nT\log w_j)^{-1}\right)\right],$$

$$[PF(nT\varphi, w_*)]$$

The second term is the kernel of the reference posterior for $\eta|\varphi$. 

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of the kernel of the reference prior for \( \varphi \), whether obtained from the bids or the winning bids, is in its right tail. Further, the right tail of the reference prior kernel for \( \varphi \) obtained from the bids is proportional to the right tail of the reference prior kernel for \( \varphi \) from the winning bids.  

Since reference priors are defined up to a constant of proportionality the right tails of the two reference priors will get absorbed in the constant of proportionality. Hence as \( \varphi \to \infty \) it is immaterial whether the reference prior is obtained from all the bids or just the winning bids. To illustrate this point, I have normalized the improper reference prior in Figures 2 and 3 to make it proper and plotted the density function of normalized proper prior by truncating \( \varphi \) at \( \varphi = 20, 100 \). As I increase the truncation from 20 to 100, I find that the normalized prior density function shifts mass towards the right tail in an effort to make the improper reference prior proper. Since the right tail of the normalized reference prior whether obtained from all the bids or just the winning bids almost coincide as I increase the truncation point for \( \varphi \), the normalized reference prior density functions, whether obtained from the bids or the winning bids will be similar. Large values of \( \varphi \) are however uninteresting for the auctions that I am studying; for example, the MLE of \( \varphi \) that I obtain from my data set on winning bids is 0.53.  

Johnson, Kotz and Balakrishnan (1994, Vol. 2, p. 579) point out that \( \varphi \) can be interpreted as the elasticity of the distribution function of the cost, \( F_c(\tau|\eta, \varphi) \), with respect to \( \tau \), where \( \tau \) is a realized cost. Since bids are proportional to cost, if cost increases by 10\%, \( \varphi \) is the answer to the question, if bids increase by 10\% what is the probability that the cost had increased by 10\%?  

The answer is clearly less than one since bidders submit bids that are padded above cost.

In contrast to the previous example where bids are additive in cost, here inference about \( \varphi \) differs depending on whether data is available on bids or just the winning bids due to both

---

18. Note that the latter is just a scaled-up version of the former where the constant by which it is scaled is \( n \). From the expressions for the reference prior for \( \varphi \) given in the second curly brackets in equations (29) and (31) as \( \varphi \to \infty \), the ratio \( \frac{\varphi^2}{(\varphi n - 1)^2} \to 1 \). Hence the right tail behaviour of the reference priors is just a function of \( n \).

19. This is also the reference prior for \( \varphi \) that emerges from maximizing Lindley's measure of information in equation (11) with respect to \( \pi(\eta, \varphi) \) when \( \varphi \in [0, 20] \) or \( [0, 100] \). The fact that \( \varphi \) is bounded ensures that this reference prior is proper.

20. If a variable \( Z_j \sim PA(\eta, \varphi) \), then \( \varphi \) can be interpreted as the following elasticity,

\[
\varphi = \frac{d \log F_c(z|\eta, \varphi)}{d \log z},
\]

where \( F_c(z|\eta, \varphi) \) is the distribution function of \( Z_j \).
the reference prior and the likelihood function being different. I illustrate the difference in inference about \( \varphi \) from data on all bids compared with data on winning bids by considering low-price, sealed-bid auctions described in Section 4 assuming that the number of potential bidders in each auction is six, \( n = 6 \).\(^{21}\) I do not have data on bids for individual auctions. Hence I use an approximation for \( \sum_{j=1}^{n} \log b_j \) in equation (30).\(^{22}\) The marginal posterior for \( \varphi \) is given by the term in square brackets in equations (30) and (32) from all the bids in the auctions and the winning bids, respectively. The first term (square-root term) in the square brackets in both equations is the contribution of the reference prior and the second term \( (GA(\varphi)) \) the contribution of the likelihood function to the marginal posterior for \( \varphi \). I have plotted the kernel of the marginal reference posterior for \( \varphi \) given by the term in the square brackets in equation (30) and (32) in Figure 4 and 5, respectively. The kernel of the Gamma density functions, the contribution of the likelihood function to the posterior for \( \varphi \), has also been plotted. An objective Bayesian would end up with different point estimates for \( \varphi \) if she had data on all the bids compared with the winning bids and used the quadratic loss function since the posterior mean is different between Figures 4 and 5, respectively. Interval estimates would be different since the tail behavior of the two posteriors is different. One scenario were same point estimates would be obtained was if the all-or-nothing loss function was used; the mode of the two posterior distributions in Figures 4 and 5 is the same.

Integrating out \( \eta \) from the posterior density function of \( (\eta, \varphi) \) given in equation (30), I obtain

\[
\pi^b_{\text{ref}}(\varphi|b) \propto \frac{1}{\varphi} \sqrt{1 + \frac{Tn\varphi^2}{(\varphi m - 1)^2}} \prod_{j=1}^{T} \prod_{i=1}^{n} \left[ \frac{\varphi w_i}{(b_j)^{\varphi+1}} \right];
\]

(33)

this is derived in Appendix E. Like the previous example, the term in the second square brackets

\(^{21}\) The average number of participants in the auctions conducted by IOC is 6.

\(^{22}\) Using the MLE of \( \eta \) and \( \varphi \), I draw five bids above the winning bid in an auction, from the Pareto distribution \( PA(\eta^{\text{MLE}}, \varphi^{\text{MLE}}) \). \( \eta^{\text{MLE}} = 95.51 \) U.S. dollar per metric tonne, is the minimum winning bid in my data set. \( \varphi^{\text{MLE}} = 0.53 \) and has been obtained as follows

\[
\varphi^{\text{MLE}} = T \sum_{j=1}^{T} n_j \log \left( \frac{w_j}{\eta^{\text{MLE}}} \right),
\]

where \( T = 37 \) is the number of auctions on which data is available. I do this for each of the \( T = 37 \) auctions and obtain \( \sum_{j=1}^{T} \sum_{i=1}^{n} \log b_j \).
is the Pareto density function of the bids given in equation (15) with a difference; it is as if the bids are truncated at \( w_* \) instead of \( \eta \) following the superconsistency of \( W_* \) as an estimator of \( \eta \).

The first term in the square brackets is the square-root of the Fisher information for \( \varphi \) based on this concentrated likelihood, \( \sqrt{\det \mathbf{J}_{\mathbf{B}}^{\varphi} (w_*, \varphi)} \). In Appendix E, I show that similar results are obtained when I do this exercise with the winning bids across auctions.

Unlike the previous example where bids are additive in cost, the result in (33) is exact.

The question is why I could obtain an exact result in (33) whereas when bids are additive in cost, I needed a Laplace approximation to obtain (28); I have shown in the Appendix D that I cannot obtain (28) by taking the exact integral of the posterior density function of \( (\eta, \varphi) \) with respect to \( \eta \). The answer lies in the basic idea that drives the Laplace approximation: the largest contribution to an integral comes from the neighborhood of the point where the integrand reaches its maximum within the range of integration, and this largest contribution becoming progressively dominant as \( T \) becomes large. For both examples the integrand reaches a maximum at \( w_* \) since the relevant integration for the case when bids are additive in cost is \( \int_0^{w_*} e^{nT\varphi \eta} d\eta \), and for the case where bids are proportional to cost \( \int_0^{w_*} \eta^n T \varphi - 1 d\eta \); see Appendix D and E, respectively for details. The dominating effect of the integrand will be clearly achieved for a smaller \( T \) in the case where bids are proportional to cost. The result is that I see the superconsistency of \( w_* \) for \( \eta \) showing up in a finite sample when bids are proportional to cost:

the marginal posterior for \( \varphi \) is the product of the concentrated likelihood and the reference prior based on this likelihood.

There are several interesting features to note about the Jeffreys' prior in this example. Jeffreys' prior, whether obtained from the bids, winning bids and the minimum winning bid is the same, \( \pi_{\text{Jeff}}^*(\eta, \varphi) \propto \frac{1}{\eta} \). The similarity of the Jeffreys' prior obtained from the bids and the winning bids with the Jeffreys' prior obtained from the minimum winning bid captures the sufficiency of the minimum winning bid for \( \eta \) given \( \varphi \). Jeffreys' prior does not reflect that the minimum winning bid or the winning bids across auctions are not sufficient for the regular parameter, \( \varphi \). Since a noninformative prior is supposed to "let the data speak for themselves", the reference prior for \( (\eta, \varphi) \) does a better job of reflecting features of the sampling density of the data than the Jeffreys' prior. Second, unlike the case when bids are additive in cost,
it is not proportional to a constant. This results from the Jeffreys’ prior depending on the sampling density, which is different under the two models. The off-diagonal elements of the Fisher information matrix from the bids or the winning bid, given in Appendix C, equations (C.1) and (C.2), respectively, are functions of both parameters, \( \eta, \varphi \). Third, similar to Example 1, when bids are additive in cost, Jeffreys’ prior obtained from the bids is identical to the Jeffreys’ prior obtained from the winning bids. In both cases, the only difference in the Fisher information matrix from the bids and the Fisher information matrix from the winning bid, is in the element on the principal diagonal corresponding to the regular parameter, \( J_{x}^{\varphi}(\eta, \varphi) \). This element differs only by the factor \( n \). Since Jeffreys’ prior is defined up to a constant of proportionality, and \( n \) gets absorbed in this constant of proportionality, in both examples the Jeffreys’ prior is the same whether it is obtained from the bids or the winning bid.

Finally, Arnold and Press (1982, pp. 293-297), obtained an improper bivariate posterior for \((\eta, \varphi)\) with a Pareto likelihood and independent, noninformative priors for the two parameters. The reason for this was that the likelihood function becomes unbounded at \( \eta = 0 \); with both the noninformative prior for \( \eta \) and the likelihood putting an infinite mass at \( \eta = 0 \), the marginal posterior density function for \( \eta \) becomes unbounded for \( \eta \) near 0.\(^{23}\) As a result they recommend that this noninformative prior not be used to represent diffuseness. I do not observe the kind of anomalous behavior noted by Arnold and Press when I use the reference prior to represent diffuseness. From equations (30) or (32), the posterior kernel of \((\eta, \varphi)\) would become unbounded near \( \eta = 0 \) if \( 0 < nT \varphi < 1 \); however, \( 0 < nT \varphi < 1 \) violates the positive support of the density of \( \eta|\varphi \) since \( \eta = \frac{\varphi(n-1)}{\varphi(n-1)^{-1}} \). Thus even though the likelihood function of the bids or the winning bids given in equations (15) and (16) respectively, puts an infinite mass near \( \eta = 0 \), the reference prior puts no mass near \( \eta = 0 \). Hence, I do not observe the impropriety in the posterior for \((\eta, \varphi)\) near \( \eta = 0 \) like Arnold and Press. The message is that it is attractive to use the reference prior to represent diffuseness in the Pareto case since the exact reference posterior is proper.

\(^{23}\)Arnold and Press assume that the two parameters are a priori independent with \( \eta \) having a power function distribution, \( PF(a, b) \) and \( \varphi \) a Gamma distribution, \( GA(c, d) \). Their noninformative prior emerges by setting \( c = d = 0 \) and \( a = 0, b = \infty \); \( a = 0, b = \infty \) leads to the asymptote in the posterior for \((\eta, \varphi)\) in the direction of \( \eta \) near \( \eta = 0 \).
3.7 Reference Analysis With Nuisance Parameters

In this Section there is a partitioning of the parameters into “nuisance parameters” and “parameters of interest”, and the focus is on inference about the “parameter of interest”; the results of this Section are summarized in Table 2, Row 3. Since reference priors capture features of the sampling density of the data, and the likelihood function of all the bids (or the winning bids) is identical irrespective of whether “nuisance parameters” are present, as in this section, or the entire parameter vector is of interest, as in the last section, the two main results of the previous Section go through. First, if \( \eta \) is the “parameter of interest”, the reference prior for \( \eta \) obtained from all the bids and the winning bids is identical to the reference prior for \( \eta \) obtained from the minimum winning bid reflecting the conditionally sufficiency of the minimum winning bid for \( \eta | \varphi \). The exact reference posterior for \( \eta | \varphi \) obtained from the bids and the winning bids is identical to that obtained from the minimum winning bid as well. Second, when \( \varphi \) is the “parameter of interest”, the reference prior and the posterior for \( \varphi \) obtained from the winning bids across auctions are more dispersed than those obtained from bids within and across auctions reflecting that the Fisher information from the bids about \( \varphi \) is strictly greater than that obtained from the winning bids. Intuitively, the reference prior for the “parameters of interest” is obtained by maximizing Lindley’s measure of information with respect to the “parameters of interest” having “integrated out” the “nuisance parameter” by specifying some noninformative prior for the “nuisance parameter” conditional on the “parameters of interest”.\(^{24}\) Hence the reference prior for the “parameter of interest” will reflect features in the likelihood function of the bids or the winning bids exclusive to the “parameter of interest”.

Inference about the “parameter of interest” will differ when there is a partitioning of parameters, as in this Section, compared with no partitioning, as in the last Section, since the reference prior is different in the two cases. The reference prior for the “parameters of interest” is obtained by integrating out the “nuisance parameter” with respect to some noninformative prior; the product of this reference prior for the “parameter of interest” and the noninformative prior with respect to which the “nuisance parameter” has been integrated out, gives the joint

\(^{24}\)If \( \varphi \) is the parameter of interest, then the noninformative prior specified for the “nuisance” parameter, conditional on the “parameters of interest”, \( \eta | \varphi \), is the sampling expectation of the score function, \( c(\eta, \cdot) \).
reference prior for $\theta = (\eta, \varphi)$. When there is no partitioning of parameters, as in the last section, this kind of integration is not done to obtain the reference prior for $\theta$. I elaborate in Section 7.3 that if $\varphi$ is the "parameter of interest", the reference prior for $\varphi$, seems "more informative" than the reference prior for $\varphi$ when there is no partitioning of parameters as in Section 6.

3.7.1 Bids Additive in Cost

Following the analysis in the last section, I reparametrize the density functions in (13) and (14), with the nonregular parameter, $\eta = \delta + \frac{1}{(n-1)\varphi}$, and the regular parameter is $\varphi$. Then the lower bound of the support of the density functions given in (13) and (14), $l(\eta) = \eta$.

Example 1a: Reference prior from bids

To apply Definition 5, I need to factorize $c(\eta, \varphi)$ and $\sqrt{\det J_{\mathcal{B}}^{\varphi\varphi}(\eta, \varphi)}$ into functions of $\eta$ and $\varphi$, $c_1(\eta)$, $c_2(\varphi)$ and $\lambda_1(\eta)$, $\lambda_2(\varphi)$, respectively. As mentioned before, $c(\eta, \varphi) = l'(\eta) f_b(\eta|\eta, \varphi)$, where $f_b(\eta|\eta, \varphi)$ is the density function of a bid given in (13) evaluated at the lower bound, and $\sqrt{\det J_{\mathcal{B}}^{\varphi\varphi}(\eta, \varphi)}$ is the square root of the element in the second row and column of the Fisher information matrix from the bids since $\varphi$ is a scalar. Since $c(\eta, \varphi) = \varphi$, $c_1(\eta) = 1$, and $c_2(\varphi) = \varphi$. Further, from equation (B.1), Appendix B, $\sqrt{\det J_{\mathcal{B}}^{\varphi\varphi}(\eta, \varphi)} = \frac{1}{\varphi} \sqrt{nT + \frac{(nT)^2}{(n^2-1)}}$, so that $\lambda_1(\eta) = 1$ and $\lambda_2(\varphi) = \frac{1}{\varphi}$. Then from Definition 5, the reference prior when either $\eta$ or $\varphi$ is the "nuisance parameter," is

$$
\pi_{\text{ref}}^b(\eta, \varphi) \propto c_1(\eta) \lambda_2(\varphi), \quad \eta, \varphi > 0,
$$

$$
\propto \frac{1}{\varphi}.
$$

(34)

The exact reference posterior is

$$
\pi_{\text{ref}}^b(\eta, \varphi|b) = \pi_{\text{ref}}^b(\varphi|b_j) \pi_{\text{ref}}^b(\eta|\varphi, b_j), \quad 0 < \eta < w_j,
$$

$$
\propto \left[ \varphi^{nT-1} \exp \left( -\varphi \sum_{j=1}^{T} \sum_{i=1}^{n} b_{ij} \right) \right] \left[ e^{nT\eta} \right],
$$

$$
\propto GA \left[ nT, \left( \sum_{j=1}^{T} \sum_{i=1}^{n} b_{ij} \right)^{-1} \right] \left[ e^{nT\eta} \right].
$$

(35)
The first term is the reference posterior kernel for \( \phi \); it is the kernel of a Gamma density function. The second term is the reference posterior kernel for \( \eta|\phi \).

**Example 1b: Reference prior from winning bids**

Now \( c(\eta, \phi) = l'(\eta)f_w(\eta|\eta, \phi) = n\phi \), where I obtain \( f_w(\eta|\eta, \phi) \) from (14). Also from equation (B.2), Appendix B, \( \sqrt{\text{det } J^\phi(\eta, \phi)} = \frac{1}{\phi} \sqrt{T + \frac{(nT)^2}{(n+1)}} \). Then \( c_1(\eta) = 1 \), \( c_2(\phi) = \phi \), and \( \lambda_1(\eta) = 1 \), \( \lambda_2(\phi) = \frac{1}{\phi} \). Thus the reference prior from the winning bids is identical to that obtained from the bids, which is given in equation (34). At first sight it may look peculiar that the marginal reference for \( \phi \), whether obtained from the bids or the winning bids, is not constant but identical; in both cases it is proportional to \((\phi)^{-1}\). This follows from \( \lambda_2(\phi) \), the part of the Fisher information that contributes to the reference prior for \( \phi \), differing by a factor that is a function of the sample size \( nT \) between the two cases. Since reference priors are defined up to a constant of proportionality, this factor gets absorbed in this constant. Hence it does not show up in the reference prior for \( (\eta, \phi) \) whether obtained from all the bids or the winning bid. The reference posterior is different; it is

\[
\pi^w_{\text{ref}}(\eta, \phi|w) = \pi^w_{\text{ref}}(\phi|b_j) \pi^w_{\text{ref}}(\eta|\phi, b_j), \quad 0 < \eta < w_j, \\
\propto \left[ \frac{1}{\phi} \right] \left[ \prod_{j=1}^{T} n\phi \exp(-n\phi w_j) \right] \left[ \exp(nT \phi \eta) \right], \\
\propto GA \left[ T, \left( n \sum_{j=1}^{T} \log w_j \right)^{-1} \right] \left[ e^{nT \phi \eta} \right]. \quad (36)
\]

The first term is proportional to the reference posterior density function for \( \phi \) which is a Gamma or an Exponential density function. The second term is the kernel of the reference posterior density function for \( \eta|\phi \). Comparing equations (34) and (35), the reference posterior for \( \phi \) obtained from the winning bids is more dispersed than that obtained from the bids and the reference posterior for \( \eta|\phi \) is identical to the reference posterior for \( \eta|\phi \) obtained from the minimum winning bid as expected.\(^{25}\) Since \( (\phi)^{-1} \) is the reference prior for \( \phi \) irrespective of data on all the bids or just the winning bids being available, inference about \( \phi \) differs through the *likelihood* function.

\(^{25}\)The reference prior for \( (\eta, \phi) \) from the minimum winning bid is \((\phi)^{-1}\) since \( c(\eta, \phi) = nT \phi \) and
3.7.2 Bids Proportional to Cost

I reparametrize the density functions in equations (16) and (17) as in the last section. The nonregular parameter, \( \eta = \frac{\phi(n-1)}{\phi(n-1)-1} \), and the regular parameter is \( \varphi \). The lower limit of the support of the density functions given in (16) and (17), \( \ell(\eta) = \eta \).

**Example 2a: Reference prior from bids**

Now from (16), I obtain, \( c(\eta, \varphi) = l(\eta)f_{\eta}(\eta|\eta, \varphi) = \frac{\varphi}{\eta} \), so that \( c_1(\eta) = \frac{1}{\eta} \) and \( c_2(\varphi) = \varphi \). From equation (C.1), Appendix C, \( \sqrt{\text{det} J_B^{\varphi}(\theta, \varphi)} = \frac{1}{\varphi} \sqrt{nT + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} \), where \( m = n - 1 \); hence \( \lambda_1(\eta) = 1 \) and \( \lambda_2(\varphi) = \frac{1}{\varphi} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \). The reference prior for \( [\eta, \varphi] \) from Definition 5, when either \( \eta \) or \( \varphi \) is the "nuisance parameter," is

\[
\pi^b_{\text{ref}}(\eta, \varphi) \propto c_1(\eta)\lambda_2(\varphi), \quad \eta, \varphi > 0,
\]

\[
\propto \frac{1}{\eta} \left[ \frac{1}{\varphi} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \right].
\]

(37)

The exact reference posterior is

\[
\pi^b_{\text{ref}}(\eta, \varphi|b) = \pi^b_{\text{ref}}(\varphi|b) \pi^b_{\text{ref}}(\eta|\varphi, b), \quad 0 < \eta < \infty,
\]

\[
\propto \frac{1}{\eta^2} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \left[ \varphi^{nT} \eta^{nT\varphi} \prod_{j=1}^{n} \prod_{i=1}^{m} \left[ \frac{1}{nT} \right]^{\varphi+1} \right]
\]

\[
\propto \frac{1}{\eta^2} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \left[ \left( \sum_{j=1}^{n} \log b_j \right)^{-1} \right] \left[ \prod_{j=1}^{n} \prod_{i=1}^{m} \left[ \frac{1}{nT} \right] \right] \left[ \varphi^{nT} \eta^{nT\varphi} \right]
\]

\[
\propto \frac{1}{\eta^2} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \left[ \left( \sum_{j=1}^{n} \log b_j \right)^{-1} \right] \left[ \prod_{j=1}^{n} \prod_{i=1}^{m} \left[ \frac{1}{nT} \right] \right] \left[ \varphi^{nT} \eta^{nT\varphi} \right]
\]

\[
\propto \frac{1}{\eta^2} \sqrt{1 + \frac{nT\varphi^2}{(\varphi m - 1)^2}} \left[ \left( \sum_{j=1}^{n} \log b_j \right)^{-1} \right] \left[ \prod_{j=1}^{n} \prod_{i=1}^{m} \left[ \frac{1}{nT} \right] \right] \left[ \varphi^{nT} \eta^{nT\varphi} \right]
\]

(38)

where the first term in the square brackets is the kernel of the posterior for \( \varphi \) and the second term is the kernel for \( \eta|\varphi \).

\[
\sqrt{\text{det} J_{\omega(\varphi)}^{\varphi}(\eta, \varphi)} = \frac{1}{\varphi} \sqrt{1 + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} \text{ from (B.3), Appendix B. The reference posterior is}
\]

\[
\pi^w_{\text{ref}}(\eta, \varphi|w) \propto \varphi^{nT} \eta^{nT\varphi} \left[ \left( \sum_{j=1}^{n} \log b_j \right)^{-1} \right] \left[ \prod_{j=1}^{n} \prod_{i=1}^{m} \left[ \frac{1}{nT} \right] \right] \left[ \varphi^{nT} \eta^{nT\varphi} \right],
\]

where the term in the second square brackets is the reference posterior for \( \eta|\varphi \) when either \( \eta \) or \( \varphi \) is the "nuisance parameter".
Example 2b: Reference prior from winning bids

From equation (17), \( c(\eta, \varphi) = \ell'(\eta) f_\omega(\eta|\eta, \varphi) = \frac{n\varphi}{\eta} \); hence \( c_1(\eta) = \frac{1}{\eta} \) and \( c_2(\varphi) = \varphi \). From equation (C.2), Appendix C, \( \sqrt{\det J^{\omega\varphi}(\theta, \varphi)} = \frac{1}{\varphi} \sqrt{T + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} \), where \( m = n - 1 \); hence \( \lambda_1(\eta) = 1 \) and \( \lambda_2(\varphi) = \frac{1}{\varphi} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} \). The reference prior for \([\eta, \varphi]\) from Definition 5, when either \( \eta \) or \( \varphi \) is the “nuisance parameter,” is

\[
\pi^w_{\text{ref}}(\eta, \varphi) \propto c_1(\eta)\lambda_2(\varphi), \quad \eta, \varphi > 0,
\]

\[
\propto \frac{1}{\eta} \left[ \frac{1}{\varphi} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} \right].
\]  

(39)

The exact reference posterior is

\[
\pi^w_{\text{ref}}(\eta, \varphi|w) = \pi^w_{\text{ref}}(\varphi|w) \pi^w_{\text{ref}}(\eta|\varphi, w), \quad 0 < \eta < w,
\]

\[
\propto \frac{1}{\eta\varphi} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} \left[ \prod_{j=1}^{T} n\varphi\eta^{n\varphi} e^{-\log w_j(n\varphi + 1)} \right],
\]

\[
\propto \left[ \sqrt{1 + \frac{n^2\varphi^2}{(\varphi m - 1)^2}} GA \left( T, \left( n \sum_{j=1}^{T} \log w_j \right)^{-1} \right) \right] \left[ PF(nT, \varphi, w) \right].
\]  

(40)

Comparing equations (37) and (39), the marginal reference prior for \( \eta \) is identical; it is improper being proportional to \((\eta)^{-1}\). The marginal reference for \( \eta \) obtained from the minimum winning bid is also \((\eta)^{-1}\).\(^{26}\) From equations (38) and (40), the reference posterior for \( \eta|\varphi \) obtained from all the bids and the winning bids, respectively, is identical to the reference

\(^{26}\) From equation (18), \( c(\eta, \varphi') = \frac{nT\varphi}{\eta} \), and from equation (C.3), Appendix C, \( \sqrt{\det J^{\omega\varphi}(\theta, \varphi')} = \frac{1}{\varphi} \sqrt{1 + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} \). The reference prior for \([\eta, \varphi]\) from the minimum winning bid is

\[
\pi^w_{\text{ref}}(\eta, \varphi) \propto \frac{1}{\eta} \left[ \frac{1}{\varphi} \sqrt{1 + \frac{(nT)^2\varphi^2}{(\varphi m - 1)^2}} \right],
\]

so that the reference prior for \( \eta \) is \((\eta)^{-1}\).
posterior for $\eta|\varphi$ obtained from the minimum winning bid;\textsuperscript{27} it belongs to the power-function distribution. This again reflects the sufficiency of the minimum winning bid for $\eta|\varphi$. Now compare the marginal reference priors for $\varphi$ given in the square brackets in equations (37) and (39). In both cases, it is the product of two functions of $\varphi$; the first function is $(\varphi)^{-1}$. The second function is the kernel of the marginal reference prior for $\varphi$ when both $(\eta, \varphi)$ are "parameters of interest"; these are given in the second term in curly brackets in equations (29) and (31) in the last section. But I have already shown in the last Section that the marginal reference prior for $\varphi$ is more dispersed when it is obtained from the winning bids compared to when it is obtained from the bids when both $(\eta, \varphi)$ are the "parameters of interest" (see Figures 1-3). From equations (38) and (40), the reference posterior for $\varphi$ obtained from the winning bids is more dispersed than that obtained from the bids as expected and hence inference about $\varphi$ will differ as in the last section. Inference about $\varphi$ is different when data on all the bids compared with just the winning bids is available on account of both the likelihood function and the reference prior being different. Further, since the marginal reference kernel for $\varphi$ in equations (37) and (39) is the product of two functions, each of which is not integrable, it is not integrable as well; hence the marginal reference prior for $\varphi$ is improper.

3.7.3 Comparing reference priors in the presence and absence of "nuisance parameters"

An interesting question at this point is how the reference prior for $(\eta, \varphi)$ differs when parameters are partitioned into "nuisance" parameters and "parameters of interest", as in this Section, and when there is no partitioning of this kind, as in the last Section. The likelihood function of all the bids (or just the winning bids) will be identical irrespective of whether "nuisance parameters" are present, as in this section, or the entire parameter vector is of interest, as in the last section. As a result, inference about the "parameter of interest" will differ between

\textsuperscript{27}The posterior for $(\eta, \varphi)$ from the minimum winning bid is

$$
\pi^\omega_{\text{ref}}(\eta, \varphi|w.) \propto \sqrt{1 + \frac{(nT)^2 \varphi^2}{(p - 1)^2}} \frac{[GA(1, (nT \log w.)^{-1})]}{[PF(nT \varphi, w.)]},
$$

where the term in the second square brackets is the reference posterior for $\eta|\varphi$ obtained from the minimum winning bid.
the two cases only if the reference prior is different. I will confine myself to bringing out the difference between Section 6 and 7 through the reference prior obtained from all bids and the likelihood function of the bids. Discussion proceeds along similar lines when data on winning bids is available.

Suppose the "parameter of interest" is the "regular" parameter, \( \varphi \). If bids are additive in cost, I compare (25) with (34); I find that the marginal reference prior for \( \varphi \) is different. While it is proportional to a constant when there is no partitioning, as in (25), it is proportional to \( (\varphi)^{-1} \) when there is a partitioning of parameters, as in (34), and \( \eta \) is the "nuisance parameter". Similarly when bids are proportional to cost, I find the marginal reference prior for \( \varphi \) to be different when I compare (29) with (37).

These results are not surprising since there is a difference in how the reference prior for \( (\eta, \varphi) \) is obtained in Section 6 and 7. Details of this are given in Section 3, Definitions 4 and 5, and at the beginning of this Section.

What is interesting to note is the marginal reference posterior for \( \varphi \) obtained by integrating out \( \eta \) from the reference posterior for \( (\eta, \varphi) \). Whether bids are additive or proportional to cost, it is the product of the "concentrated" likelihood of \( nT - 1 \) bids and the Jeffreys' prior based on this "concentrated" likelihood.

When bids are additive in cost, integrating out \( \eta \) from the reference posterior for \( (\eta, \varphi) \) in equation (35), I obtain the marginal reference posterior for \( \varphi \):

\[
\pi_{ref}^b (\varphi | b) \propto \left[ \frac{1}{\varphi} \right] \left[ \prod_{j=1}^{T-1} \prod_{i=1}^{n} \varphi e^{-\varphi (b_j - w_i)} \right].
\]

(41)

The first term is the prior for \( \varphi \) and the second term is the likelihood of \( nT - 1 \) bids each of which follows an exponential distribution given in equation (13) but truncated at \( w \). When bids are proportional to cost, I obtain the marginal reference posterior for \( \varphi \) by integrating out...

---

28 When bids are proportional to cost, and data on all the bids is available, the marginal reference prior for \( \varphi \) is proportional to \( \frac{\sqrt{1 + \frac{nT\psi^2}{\omega}}}{\psi} \) in the absence of "nuisance parameters"; this follows from (29). From (37), the marginal reference prior for \( \varphi \) is proportional to \( \frac{1}{\varphi} \sqrt{1 + \frac{nT\psi^2}{\omega}} \) when \( \eta \) is the "nuisance parameter".
Again, the first term is the prior for \( \varphi \) and the second term the likelihood of \( nT - 1 \) bids each of which now follows a Pareto distribution given by equation (15) but with \( b_j^i > w_* \).

I now compare the marginal reference posterior for \( \varphi \) in equations (41) and (42) with the marginal reference posterior for \( \varphi \) from all the bids within and across auctions in the absence of "nuisance parameters" given in the last Section by equations (28) and (33), respectively. I find that it is the product of the same prior and the "concentrated" likelihood of \( nT \) bids instead of \( nT - 1 \) bids as in equations (41) and (42). It is almost as if \( nT - 1 \) bids, when there is a partitioning of parameters into "nuisance parameters" and "parameters of interest", provide the same information about \( \varphi \) as \( nT \) bids when there is no partitioning. It is in this sense that the reference prior for \( \varphi \) is "more informative" when there partitioning of parameters with the regular parameter being the "parameter of interest", and "less informative" when there is no partitioning.\(^{29} \)

### 3.8 Conclusion

I obtain reference priors and the exact reference posteriors for a low-price, sealed-bid auction under the independent-private-values paradigm under specifications of private values as in Paarsch (1992). For the scalar parameter case, when the minimum winning bid is sufficient for the scalar parameter, the reference prior and the posterior obtained from all the bids within and across auctions and the minimum winning bid are identical. The reference prior and the Jeffreys' prior are identical as well. When the minimum winning bid is not sufficient for the parameters, then for the specific examples in this paper, the minimum winning bid is sufficient for the nonregular parameter conditional on the regular parameter; hence the reference priors and the posteriors, whether obtained from the bids or winning bids, are identical to the reference prior and posterior obtained from the minimum winning bid. Since the winning bids are not

\(^{29}\)Similar results are obtained when I conduct this comparison with the winning bids across the \( T \) auctions.
sufficient for the regular parameter, I obtain reference priors and posteriors from the winning bids that are more dispersed than when they are obtained from all the bids.
APPENDIX A

Appendix for Lemma 1

Since the likelihood function is regular under Assumption 2, and the auctions are all identical, I can write the expression in equation (2) as

$$TJ_{B_j} (\theta_o) = TJ_{W_j} (\theta_o) + TE_{W_j|a} \left[ J_{B_{j,n-1:n}} (\theta_o) \right].$$  \hspace{1cm} (A.1)

Given data on bids in an auction $j$, I can write the density function of the $n$ ordered bids in auction $j$, $B_{j,n:n} = [B_j^1, ..., B_j^n]$, as

$$f_{n:n}(b_{j,n:n}|\theta) = f_w(w_j|\theta)f_{n-1:n}(b_{j,n-1:n}|\theta).$$  \hspace{1cm} (A.2)

$f_{n:n}(b_{j,n:n}|\theta)$ is the density function of the $n$ ordered bids in auction $j$, $f_{n-1:n}(b_{j,n-1:n}|\theta)$ is the density function of $n-1$ ordered bids in auction $j$, and $f_w(w_j|\theta)$ is the density function of the winning bid in auction $j$; the winning bid is the smallest of the $n$ ordered bids, $B_{j,n:n}$, in auction $j$. Then I can write the Hessian of the log likelihood of the $n$ dimensional statistics, $B_{j,n:n}$, as

$$H_{B_{j,n:n}} (\theta; B_{j,n:n}) = \frac{\partial^2 \log f_{n:n}(b_{j,n:n}|\theta)}{\partial \theta \partial \theta'} + \frac{\partial^2 \log f_{n-1:n}(b_{j,n-1:n}|\theta)}{\partial \theta \partial \theta'}$$

Then taking the expectation of both sides with respect to the density function $f_{n:n}(b_{j,n:n}|\theta) = f_w(w_j|\theta)f_{n-1}$ and multiplying by -1, I obtain

$$J_{B_{j,n:n}} (\theta) = J_{W_j} (\theta) + E_{W_j|a} \left[ J_{B_{j,n-1:n}} (\theta) \right].$$

$J_{B_{j,n:n}} (\theta)$ is the Fisher information of the log likelihood function of the distribution of $n$ ordered bids, $B_{j,n:n}$ in auction $j$. $J_{B_{j,n-1:n}} (\theta)$ is the Fisher information of the log likelihood function of the distribution of $n-1$ ordered bids in auction $j$, $B_{j,n-1:n}$, given $W_j$, and $J_{W_j} (\theta)$ is the Fisher information from the winning bid, $W_j$, in auction $j$. Recognizing that $J_{B_j} (\theta) = J_{B_{j,n:n}} (\theta)$ since the $n$ ordered bids, $B_{j,n:n}$, contain the same Fisher information about $\theta$ as the $n$ bids in...
auction $j$, $B_j$, and multiplying both sides by $T$ completes the proof.

**APPENDIX B**

**Bids additive in cost**

The Fisher information matrix for $(\eta, \varphi)$ from bids within and across auctions and winning bids across auctions, when the bids are additive in cost as in Section 4.1 is given by equations (B.1) and (B.2), respectively,

$$J_B(\eta, \varphi) = \begin{bmatrix} (nT)^2 \varphi^2 & -\frac{(nT)^2}{n-1} \\ -\frac{(nT)^2}{n-1} & \frac{nT}{\varphi^2} + \frac{(nT)^2}{\varphi^2(n-1)^2} \end{bmatrix}, \quad \text{(B.1)}$$

and

$$J_W(\eta, \varphi) = \begin{bmatrix} (nT)^2 \varphi^2 & -\frac{(nT)^2}{n-1} \\ -\frac{(nT)^2}{n-1} & \frac{T}{\varphi^2} + \frac{(nT)^2}{\varphi^2(n-1)^2} \end{bmatrix}. \quad \text{(B.2)}$$

The difference between the two is

$$J_B(\eta, \varphi) - J_W(\eta, \varphi) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{T(n-1)}{\varphi^2} \end{bmatrix},$$

which is psd.

The Fisher information for $(\eta, \varphi)$ from the minimum winning bid across auctions is

$$J_{W*}(\eta, \varphi) = \begin{bmatrix} (nT)^2 \varphi^2 & -\frac{(nT)^2}{n-1} \\ -\frac{(nT)^2}{n-1} & \frac{1}{\varphi^2} + \frac{(nT)^2}{\varphi^2(n-1)^2} \end{bmatrix}.$$

**APPENDIX C**

**Bids proportional to cost**
When bids are proportional to cost as in Section 4.2, the Fisher information matrix for \((\eta, \varphi)\) from the bids and the winning bid are given in equation (C.1) and (C.2), respectively,

\[
\mathbf{J_B(\eta, \varphi)} = \begin{bmatrix}
(nT)^2 \varphi^2 \eta^2 & -(nT)^2 \varphi \eta (\varphi m - 1) \\
-(nT)^2 \varphi \eta (\varphi m - 1) & \frac{nT(\varphi m - 1)^2 + (nT)^2 \varphi^2}{\varphi (\varphi m - 1)^2}
\end{bmatrix},
\]

(C.1)

and \(\mathbf{J_W(\eta, \varphi)} = \begin{bmatrix}
(nT)^2 \varphi^2 \eta^2 & -(nT)^2 \varphi \eta (\varphi m - 1) \\
-(nT)^2 \varphi \eta (\varphi m - 1) & \frac{T(\varphi m - 1)^2 + (nT)^2 \varphi^2}{\varphi (\varphi m - 1)^2}
\end{bmatrix},
\]

(C.2)

where \(m = n - 1\). Hence

\[
\mathbf{J_B(\eta, \varphi)} - \mathbf{J_W(\eta, \varphi)} = \begin{bmatrix}
0 & 0 \\
0 & \frac{T(\varphi m - 1)}{\varphi}
\end{bmatrix},
\]

(C.3)

which is \(psd\).

The Fisher information for \((\eta, \varphi)\) from the minimum winning bid across auctions is

\[
\mathbf{J_{W*}(\eta, \varphi)} = \begin{bmatrix}
(nT)^2 \varphi^2 \eta^2 & -(nT)^2 \varphi \eta (\varphi m - 1) \\
-(nT)^2 \varphi \eta (\varphi m - 1) & \frac{(\varphi m - 1)^2 + (nT)^2 \varphi^2}{\varphi (\varphi m - 1)^2}
\end{bmatrix}.
\]

**APPENDIX D**

Superconsistency of minimum winning bid, bids additive in cost

From equation (25) the reference prior for \((\eta, \varphi)\) is proportional to a constant. Hence the reference posterior is proportional to the likelihood function of the \(nT\) bids which I obtain from equation (13);

\[
\pi^b_{ref}(\eta, \varphi | b) \propto \prod_{j=1}^{T} \prod_{i=1}^{n} \varphi e^{-\varphi (b^i_j - \eta)}.
\]

Integrating out \(\eta\) from the posterior,

\[
\pi^b_{ref}(\varphi | b) \propto \int_0^w \prod_{j=1}^{T} \prod_{i=1}^{n} \varphi e^{-\varphi (b^i_j - \eta)} d\eta,
\]

\[
\]
\[ \alpha = \varphi^T e^{-\varphi (\sum \sum b_j)} \int_0^w e^{nT \varphi \eta} d\eta, \]
\[ (D.1) \]
\[ \alpha = \left[ \varphi^T e^{-\varphi \sum \sum b_j} \right] \left[ \frac{1}{\varphi} \left( e^{nT \varphi w_*} - 1 \right) \right], \]
\[ \alpha = \frac{1}{\varphi} \left[ \varphi^T e^{-\varphi \sum (b_j - w_*)} - \varphi^T e^{-\varphi \sum \sum b_j} \right], \]
\[ \alpha = \frac{1}{\varphi} \left[ T \prod_{j=1}^n \varphi e^{-\varphi (b_j - w_*)} \right] - \frac{1}{\varphi} \left[ \prod_{j=1}^n \varphi e^{-\varphi b_j} \right]. \]
\[ (D.2) \]

Hence taking the exact integral I cannot obtain (28); I obtain (D.2) where the term in the second square bracket is the density function of an untruncated exponential distribution with parameter \( \varphi \).

Now I take a Laplace approximation (Murray 1974, p. 35) to the integral in (D.1) as \( T \to \infty \); I obtain
\[ \int_0^w e^{nT \varphi \eta} d\eta = \frac{1}{nT \varphi} e^{nT \varphi w_*} + e^{nT \varphi w_*} O(T^{-2}). \]
\[ (D.3) \]

Substituting the lead term of this integral in (D.1), I obtain (28).

I obtain similar results when I integrate \( \eta \) from the posterior density function of \( (\eta, \varphi) \) in equation (27) by taking a Laplace approximation of the integral;
\[ \tau^w_{ref}(\varphi|w) \propto \left[ \frac{1}{\varphi} \right] \left[ \prod_{j=1}^T \varphi e^{-n \varphi (w_j - w_*)} \right]. \]

The second term is the concentrated likelihood function of \( T \) winning bids; each winning bid following a Truncated Exponential distribution given by equation (14) with \( \eta = w_* \). The first term is \( \sqrt{\det J_{\varphi \varphi}(w_*, \varphi) \right), the reference prior for \( \varphi \) based on this concentrated likelihood function.

**APPENDIX E**

Superconsistency of minimum winning bid, bids proportional to cost

I have obtained the reference prior for \( (\eta, \varphi) \) from all the bids in equation (29). The reference posterior is proportional to the product of the likelihood function of the \( nT \) bids which I obtain
from equation (16) and the reference prior for \((\eta, \varphi)\):

\[
\pi^b_{\text{ref}} (\eta, \varphi|b) \propto \left[ \frac{1}{\eta} \sqrt{1 + \frac{Tn\varphi^2}{(\varphi m - 1)^2}} \right] \prod_{j=1}^{n} \left[ \frac{\varphi \eta^\varphi}{(b_j^\varphi)^{\varphi + 1}} \right].
\]

Integrating out \(\eta\) I obtain

\[
\pi^b_{\text{ref}} (\varphi|b) \propto \left[ \sqrt{1 + \frac{Tn\varphi^2}{(\varphi m - 1)^2}} \right] \prod_{j=1}^{n} \left[ \frac{\varphi}{(b_j^\varphi)^{\varphi + 1}} \right] \int_0^\infty \eta^{nT\varphi - 1} d\eta,
\]

(E.1)

which is the result in equation (33). A Laplace approximation for the integral in (E.1), as \(T \to \infty\) would be

\[
\int_0^\infty \eta^{nT\varphi - 1} d\eta = \left[ \frac{w_\ast}{T(n/w_\ast)} \right] \frac{1}{\varphi} e^{nT\varphi \log w_\ast} + e^{nT\varphi \log w_\ast}O(T^{-1}).
\]

(E.2)

Substituting the lead term in (E.1) and noting that the term in the square brackets will get absorbed in the constant of proportionality, I obtain (33) again.

I obtain similar results when I do this exercise with the winning bids across auctions. Integrating out \(\eta\) from equation (32), I get

\[
\pi^w_{\text{ref}} (\varphi|w) \propto \left[ \frac{1}{\varphi} \sqrt{1 + \frac{Tn^2\varphi^2}{(\varphi m - 1)^2}} \right] \prod_{j=1}^{T} \left[ \frac{n\varphi w_\ast^{n\varphi}}{(w_j)^{n\varphi + 1}} \right]
\]

The first term is \(\sqrt{\det J_{w_\ast, \varphi}^{\text{ref}} (w_\ast, \varphi)}\), the reference prior for \(\varphi\) obtained from the winning bids across auctions evaluated at \(\eta = w_\ast\). The second term is the likelihood of \(T\) winning bids, with each winning bid following a Pareto distribution given by equation (17) with \(\eta = w_\ast\).
Table 1: Definition of Quantities

- \( n \): number of participants in an auction.
- \( T \): number of auctions in the sample.
- \( B = [B_1, ..., B_j, ..., B_T] = n \times T \) dimensional matrix of bids.
- \( B_j = [B_j^1, ..., B_j^n] = n \) dimensional vector of bids in auction \( j \).
- \( W_j = \min_{i=1,...,n} B_j^i \): winning bid in auction \( j \).
- \( W_* = \min_{j=1,...,T} W_j \): minimum winning bid across the \( T \) auctions.

\[ B_{n:n} = [B_1_{n:n}, ..., B_{T,n:n}] = n \times T \] matrix of \( n \) ordered bids in \( T \) auctions.

\[ B_{j,n:n} = [B_j^{1:n}, ..., B_j^{i:n}, ..., B_j^{n-1:n}, W_j] \]
- \( = n \) dimensional vector of ordered bids in auction \( j \), \( B_j^{1:n} \leq ... \leq B_j^{n-1:n} \leq W_j \).

\[ B_{n-1:n} = [B_{i,n-1:n}, ..., B_{j,n-1:n}, ..., B_{T,n-1:n}] \]
- \( = n - 1 \times T \) dimensional matrix of \( n - 1 \) ordered bids in \( T \) auctions.

\[ B_{j,n-1:n} = [B_j^{1:n}, ..., B_j^{i:n}, ..., B_j^{n-1:n}] \]
- \( = n - 1 \) dimensional vector of ordered bids in auction \( j \), \( B_j^{1:n} \leq ... \leq B_j^{n-1:n} \).

\( f_{z}(z|\theta) \): density function of \( Z \) with support \( \zeta_{Z} \).

\( \zeta_{Z}(\theta) \): support of \( f_{z}(z|\theta) \) if it depends on \( \theta \).

\( F_{z}(z|\theta) = \int_{-\infty}^{z} f(\tau|\theta)d\tau \): distribution function of \( Z \).
\[ f_{n:n}(b_{j:n} | \theta) = n! \prod_{i=1}^{n} f(b_{j}^{i:n} | \theta), -\infty < b_{j}^{1:n} \leq \cdots \leq w_{j} < \infty; \]
\[ = \text{joint density of } n \text{ ordered bids, } B_{n:n}, \text{ in auction } j. \]

\[ f_{n-1:n}(b_{j:n-1:n} | \theta) = \frac{(n-1)! \prod_{i=1}^{n-1} f(b_{j}^{i:n} | \theta)}{f(w_{j} | \theta)}; -\infty < b_{j}^{1:n} \leq \cdots \leq b_{j}^{n-1:n} \leq w_{j}; \]
\[ = \text{joint density of } n-1 \text{ ordered bids, } B_{n-1:n}, \text{ in auction } j. \]

\[ f_{w}(w_{j} | \theta) = n f_{b}(w_{j} | \theta)[1 - F_{b}(w_{j} | \theta)]^{n-1}, \]
\[ = \text{density function of the lowest order statistics from } f_{b}(b_{j} | \theta). \]

\[ V_{i} = \nu_{i}(B_{1}, ..., B_{j}, ..., B_{T}) = \text{an arbitrary statistic.} \]

\[ V = [V_{1}, ..., V_{nT}]' = nT \text{ jointly sufficient statistics for } \theta. \]

\[ \ell(\theta; Z_{j}) = \sum_{i=1}^{T} \log f(z | \theta) = \text{log-likelihood of } T \text{ observations, } Z_{j}. \]

\[ S_{Z}(\theta; Z) = \frac{\partial \ell(\theta; Z)}{\partial \theta} = \frac{1}{f_{z}(z | \theta)} \frac{\partial f_{z}(z | \theta)}{\partial \theta} = \text{score function of } \ell(\theta; Z). \]

\[ S_{Z}(\theta; Z) = \sum_{j=1}^{n} S_{Z}(\theta; Z), \]
\[ = \text{score function of } \ell(\theta; Z), \ Z \text{ being } nT \text{ dimensional.} \]

\[ J_{Z}(\theta) = E_{Z | \theta} [S_{Z}(\theta; Z)S_{Z}'(\theta; Z)] = \text{Fisher information from } Z. \]

\[ J_{Z}(\theta) = E_{Z | \theta} [S_{Z}(\theta; Z)S_{Z}'(\theta; Z)], \]
\[ = \text{Fisher information from } nT \text{ dimensional vector } Z, \]
\[ = nTJ_{Z}(\theta) \text{ under standard regularity conditions.} \]

\[ H_{Z}(\theta; Z) = \frac{\partial^{2} \ell(\theta; Z)}{\partial \theta \partial \theta'} = \text{Hessian matrix of } Z. \]

\[ m(\cdot), q(\cdot) = \text{strictly positive functions.} \]
<table>
<thead>
<tr>
<th>Winning Bid Sufficient for δ (scalar case)</th>
<th>Winning Bid Not Sufficient for η, ϕ;</th>
<th>( \eta, \varphi ) &quot;Parameters of Interest&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>B, W, ( W_\ast )</td>
<td>B</td>
<td>W</td>
</tr>
<tr>
<td>( \pi^b_{ref}(\delta</td>
<td>\varphi) \propto \text{constant}, )</td>
<td>( \pi^b_{ref}(\eta, \varphi</td>
</tr>
<tr>
<td>( 0 &lt; \delta &lt; w_\ast - \frac{1}{m_\Phi} )</td>
<td>( GA\left[ nT + 1, \left( \sum_{j=1}^{n} b_j \right)^{-1} \right] )</td>
<td>( GA\left[ T + 1, \left( \sum_{j=1}^{n} w_j \right)^{-1} \right] )</td>
</tr>
<tr>
<td>( \pi^b_{ref}(\delta</td>
<td>\varphi, b) = PF\left( nT\varphi, \frac{\omega_\ast - 1}{\varphi} w_\ast \right) ), ( 0 &lt; \delta &lt; \frac{\omega_\ast - 1}{\varphi} w_\ast ).</td>
<td>( \pi^b_{ref}(\eta, \varphi</td>
</tr>
</tbody>
</table>

\( \pi^b_{ref}(\eta, \varphi) \propto \left\{ \frac{1}{\eta^\alpha} \right\} \left\{ \sqrt{1 + \frac{T n \varphi^2}{(\omega_\ast - 1)^2}} \right\} \), \( \eta, \varphi > 0 \).
Figure 3-1: Reference Prior Kernel for \( \varphi \), Bids Proportional to Cost

Figure 3-2: Normalized Reference Prior Density for \( \varphi \), Truncation at \( \varphi = 20 \)
Figure 3-3: Normalized Reference Prior Density for $\varphi$, Truncation at $\varphi = 100$

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Figure 3-4: Reference Posterior Kernel from Bids for $\varphi$, Bids Proportional to Cost
Figure 3-5: Reference Prior Kernel from Winning Bids for $\varphi$, Bids Proportional to Cost
Bibliography


Chapter 4

The Economics of Timber Auctions in Southern Ontario

4.1 Introduction

The aim of this paper is to provide a statistical analysis of the bidding decision of the sawmills participating in the first-price, sealed-bid auctions conducted by the county of Simcoe in Southern Ontario for standing timber in the woodlots that it owns.1 Ascertaining key aspects of the bidding decision of the sawmills is of interest to the county of Simcoe since it will aid the county in designing an auction that satisfies its two-fold aim in conducting these auctions. One is to get a fair market value for the standing timber on the woodlots that it owns. Second, is to support a viable forest industry since the benefit to the rural community in terms of economic development and environmental protection is significant. The auctions can be viewed as an insurance for the sawmills since they ensure the availability of timber on a periodic basis.

The bidding behavior of the sawmills participating in the sales conducted by the county of Simcoe is interesting for at least three reasons.

First, the auctions conducted by the county of Simcoe are multi-unit auctions, with the standing timber on multiple woodlots being put up for sale in each auction. Several papers in recent years have pointed out the additional issues that arise on account of the multi-unit

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1 Prior to 1996, the Ministry of Natural Resources was conducting these auctions on behalf of the county of Simcoe.
aspect of the auction. The key issue is that the auction of a unit, a woodlot in the current context, cannot be regarded to be independent of the auction of the other units within a sale. The bidders bid strategically across woodlots within a sale. For example, Donald, Paarsch and Robert (1997), in the context of a sequential, oral, ascending-price, open-exit auction with multi-unit demand, point out that the winning prices in an auction form an increasing sequence in the case where potential bidders demand more than one unit and no resale is allowed.

Second, I identify three types of asymmetries in bidder's valuations: asymmetries due to harvesting technology, location of the sawmill of a bidder, and the frequency of participation. As a result, I cannot assume that the bidders' valuations are drawn from the same distribution; the bidders are not identical. The key issue now is that in a first-price, sealed-bid auction the winner may not be the bidder with the highest valuation. In view of the literature on asymmetric auctions, it is straightforward to handle asymmetries in bidder's valuations due to the harvesting technology and location of the sawmill. The asymmetry in bidder's valuations due to the frequency of participation adds an additional dimension to the bidding strategies of the sawmills. The equilibrium strategy of a bidder is now affected by not just the potential competition she expects, but by the probability of participation of bidders, as well.

Third, the bidders who are bidding frequently seem to be coordinating their bidding strategies so that they avoid competing with each other. I find the frequent bidders bidding on woodlots with different average volume per tree.

Given the fact that our knowledge of the bidding strategy of the bidders in a multi-unit, asymmetric setting, with noncompetitive bidding is limited, commenting on an optimal auction form or ranking auction forms in terms of expected revenue of the seller becomes even more difficult. The auctions conducted by the county are a classic example of the inappropriateness of the existing results on ranking of auction forms when a single object is being auctioned. For

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2 The effect of asymmetry on the bidding strategies is minimal in the case of the English auction or the second-price auction. It is still optimal for the bidder to bid her valuation; the revenue to the seller is still the second-highest valuation. In a first-price, sealed-bid auction, the equilibrium strategy of a bidder is again to bid the expected value of the second-highest valuation conditional on his valuation being the highest. Since valuations are not drawn from the same distribution, the high bidder need not be the high type bidder.

3 Assuming the independent-private-values model, with bidders drawing their valuations independently from different distributions, LeBrun, 1999, and Bajari, 1998 prove the existence and uniqueness of a pure strategy equilibrium for a first-price, sealed-bid auction of a single object. They also assume that the bidders are risk neutral.

4 Excellent survey of this literature is provided in Milogram and Weber (1982) and McAfee and McMillan
example, the Ontario Ministry of Natural Resources has experimented with English auctions in the past; this practice was given up since the bids received were lower than the usual bids. This is contrary to the result obtained for a single-unit auction by Milogram and Weber (1982, p. 1095) that an English auction generates higher revenue than a first-price, sealed-bid auction if the bidder’s value estimates are statistically dependent; this is the case in the auctions conducted by the county as I will show below.\(^5\)

Empirical work which takes into account the additional issues that arise from the multi-unit aspect of the auction is limited. Two notable exceptions are the work by Donald, Paarsch and Robert (1997) and Hendricks, Porter and Boudreau (1987) and Porter (1995). The former paper studies sequential, oral, ascending-price, open-exit auctions of Siberian timber export permits within an independent private-values framework. The latter set of papers studies US government first-price, sealed-bid auctions of oil and gas leases for the Outer Continental Shelf (OCS) from 1954 through 1969. In the terminology of Weber (1983), these are simultaneous-independent auctions with multiple tracts up for lease in each auction or sale. Since the value of the tract is unknown till drilling and production of oil and gas commences, Hendricks, Porter and Boudreau justify a common-value model for these auctions.

The auctions conducted by the county of Simcoe are similar to the OCS auctions in that they are simultaneous-independent, first-price, sealed-bid auctions, where multiple woodlots are put up for sale in each auction. Unlike the OCS auctions, I cannot justify the common-value model as an appropriate bidding model. I will attempt to justify that, for the auctions conducted by the county of Simcoe, the private-values model seems more appropriate. Further, I will try to establish that the private estimates of the bidders for the standing timber on the woodlots are independent after taking into account the observable variations between the bidders, as well as, the woodlots.

I emphasize again that the attempt here is neither to construct a theoretical model of bidding nor carry out any estimation; these will be attempted in future research. Rather the

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\(^5\)In the English auction described by Milogram and Weber (1982, p. 1104) the price is posted on an electronic display; this price is raised continuously. A bidder who wishes to be active at the current price depresses a button; once he has released the button, he has withdrawn from the auction and does not re-enter the auction. In this "button" auction, the bidder observes the price level, as well as, the number of active bidders since these appear on the electronic display continuously.
aim is to isolate key components of the participation and bidding decision in order to proceed with this future research.

I describe the auction mechanism in section 2. In section 3 a description of the data and important summary statistics are given. I also motivate why woodlots should be classified as hardwood and softwood lots. Explanation is also provided for why no bids were received on some woodlots even though the right to harvest timber on these woodlots was auctioned off more than once. At the same time there were woodlots on which no bids were submitted the first time they were part of an auction, but got sold the second time they were part of an auction.

In section 4, I categorize the participating sawmills in three ways corresponding to the three asymmetries noted earlier. Since the technology to harvest and process hardwood and softwood is different, I classify the sawmills into hardwood and softwood sawmills. Of the 109 sawmills participating in these auctions, the sawmill of only four bidders is located within the county of Simcoe, giving them a distinct cost advantage over the bidders who have their sawmills outside the county of Simcoe. On this basis, I classify the sawmills as insiders and outsiders. Finally, I see three sawmills bidding consistently throughout this period, and the rest bidding sporadically, leading to a possible incentive on the part of the former to coordinate their strategies. On this basis I classify the sawmills as fringe and nonfringe sawmills. The three nonfringe sawmills will be called X, Y and Z. A preliminary bidding model is also proposed.

I am able to recover some evidence of strategic bidding across auctions within a sale on the basis of the money left on the table and the preliminary bidding model proposed. As a result, it is inappropriate to ignore the multi-unit aspect of the sales conducted by the county of Simcoe.

Section 5 concentrates on three nonfringe sawmills, X, Y and Z. I suggest in this section that these three bidders seem to be coordinating their bidding strategies by bidding on woodlots with different volume per tree to avoid competing with each other. Competition for these nonfringe sawmills comes from the fringe sawmills; this is discussed in Section 6. Section 7 concludes.

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6This is being done to ensure that the identity of these sawmills is not made public.
4.2 Description Of The Auction Mechanism

The county of Simcoe organizes up to three sales each year to transfer the harvesting rights to the standing timber on the woodlots that it owns. A sale involves auctioning off multiple woodlots; a first-price, sealed-bid auction is conducted separately for each woodlot. Thus each sale consists of the simultaneous but independent right to harvest the standing timber from multiple woodlots.

A sale begins with the county officials marking down the trees that are to be harvested in different woodlots across the various townships of Simcoe and some townships in the adjoining county of Dufferin. Thereafter, all sawmills on the mailing list maintained by the county are informed about the sale through a tender notice that is mailed to them. The information provided by the county to the sawmills in the tender notice consists of the time and date on which the tender will be closed, and the time at which the tender will be opened, which is normally 1-2 hours after the tender is closed. All bids have to be submitted before the time and date at which the tender is to be closed. For each woodlot in an auction, the township in which the woodlot is located, the number, species, average diameter of the trees of a particular species, and an estimate of the standing timber on the woodlot is also provided in the tender notice. Prior to the sale, the county conducts an information session and a non-mandatory site tour of all the woodlots for which the right to harvest timber is being transferred in the sale. The time of this site tour is included in the tender notice mailed by the county to the sawmill.

The sawmill submits a sealed bid individually for the woodlots on which it wants to bid prior to the closing of the tender. It can bid on more than one woodlot which introduces the multi-unit aspect of these auctions. Twenty five percent of the sealed bid submitted by the sawmill for a woodlot has to be deposited by the sawmill as downpayment with the sealed bid. This amount is returned by the county to the sawmill if the sawmill does not have the highest bid on a woodlot. At the specified time, the county opens the sealed bids for each woodlot. The sawmill that submits the highest bid on a woodlot is awarded the contract for harvesting timber on that woodlot; it has to pay the balance of the bid that it submitted at this point. The identity and the bid of all the bidders on each woodlot is made public by the county.

7The mailing list is revised periodically by the county; it was last revised in January, 1995.
When a sawmill wins the right to harvest the standing timber on a tract, it has a year to harvest the timber. Resale of the right to harvest timber by the winning sawmill to another sawmill is strictly forbidden by the county. Note that southern Ontario is well-developed in terms of transportation and communication so that the winning sawmills do not have to expend time and resources on building roads to access the woodlots. The county does provide special instructions as to how the felled timber is to be removed from the woodlot so that other trees on the woodlot are not damaged; fines are issued if these instructions are not met.

While there is no announced reservation price, the county participates in the auctions by retaining the right to reject a bid if it finds the bid to be too low. The county bases this assessment on a private estimate of the value of the woodlot. In my sample, there were 15 woodlots on which the right to harvest timber was not transferred to the bidder with the highest bid because the county found the bid too low. This is just 4% of the 339 woodlots that were auctioned across these years. It seems that the secret reserve price set by the county is non-binding.

### 4.3 Description Of The Data

My study focuses on the sales conducted by the county of Simcoe between the years 1987-1998. In this time period 29 sales that involved the right to harvest the standing timber on 379 woodlots were held; a total of 704 bids were submitted. Of these 379 woodlots, no bids were submitted on 91 woodlots; I examine why bids were not submitted in these woodlots at the end of this section.

For each woodlot, I have data on the number of trees, species and average diameter of the trees, an estimate of the volume of standing timber, and the identity and bid of the sawmills that submitted bids for this woodlot. The location of each woodlot is also known. For bidders who own sawmills within the county of Simcoe, I know the exact location of their sawmills. For bidders who have sawmills located outside the township of Simcoe, I know the township in which the sawmill is located.

Table 1 lists some of the variables.

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8 For the years 1994 and 1995 I have data on only a single sale even though three sales were held in each year.
9 All monetary variables have been expressed in real Canadian dollars. The lumber and timber industrial price
Table 1

List of Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Unit of Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of trees</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>Volume</td>
<td>V</td>
<td>cubic metre</td>
</tr>
<tr>
<td>Real Bid on a Woodlot</td>
<td>B</td>
<td>$ (canadian)</td>
</tr>
<tr>
<td>Real Winning Bid on a Woodlot</td>
<td>W</td>
<td>$ (canadian)</td>
</tr>
<tr>
<td>Real Second Highest Bid on a Woodlot</td>
<td>B2</td>
<td>$ (canadian)</td>
</tr>
<tr>
<td>Money Left on the Table on a Woodlot</td>
<td>W-B2</td>
<td>$ (canadian)</td>
</tr>
<tr>
<td>(Volume)/(Number of Trees)</td>
<td>V/T</td>
<td>cubic metre per tree</td>
</tr>
</tbody>
</table>

In Table 2, I report some summary statistics. Since the county of Simcoe is known for its good quality softwood and I know the species of the trees on a woodlot, I have categorized all the woodlots into three categories, softwood, hardwood and mixedwood. I have included pines of all kinds under softwood and the rest of the species as hardwood. When a woodlot contains only softwood trees, I call that a softwood lot and similarly for hardwood lots. When a woodlot contains both softwood and hardwood trees I call it a mixedwood lot. On an average each sale consisted of 6 softwood lots, 3 hardwood lots and 5 mixedwood lots. Of the 288 tracts on which bids were submitted, I observe that on an average 3 sawmills submitted bids on each tract, the average bid per cubic meter per tree, B/V/T, being 0.0163 $/m^3/tree.\(^\text{10}\)

Mixedwood lots do not lend themselves to easy interpretation. The reason is that there were just two mixedwood lots that contained roughly the same volume of hardwood and softwood trees. Further, there is just one key sawmill, Y, which every once in a while, bids for woodlots with large diameter, hardwood trees; I find that 20% of the bids submitted by sawmill Y are on large diameter hardwood lots.\(^\text{11}\) Other sawmills seem to be targeting either softwood or hardwood in these auctions.

\(^\text{index, with 1986 as the base year, has been used to convert all nominal variables into real variables.}\)
\(^\text{10} m^3 \text{ indicates cubic metre, the unit in which volume is measured.}\)
\(^\text{11} \text{County officials inform me that sawmill Y is supplying processed lumber for electricity poles which require large diameter, hardwood trees.}\)
SAMPLE DESCRIPTIVE STATISTICS: ALL TRACTS

<table>
<thead>
<tr>
<th>Variable</th>
<th>Average</th>
<th>St.Dev.</th>
<th>Min</th>
<th>Max</th>
<th>Sample size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Woodlots</td>
<td>14</td>
<td>7.04</td>
<td>5</td>
<td>29</td>
<td>29</td>
</tr>
<tr>
<td>Softwood Lots</td>
<td>6</td>
<td>3.18</td>
<td>1</td>
<td>13</td>
<td>29</td>
</tr>
<tr>
<td>Hardwood Lots</td>
<td>3</td>
<td>3.06</td>
<td>0</td>
<td>11</td>
<td>29</td>
</tr>
<tr>
<td>Mixedwood Lots</td>
<td>5</td>
<td>3.03</td>
<td>1</td>
<td>10</td>
<td>29</td>
</tr>
<tr>
<td>Participants</td>
<td>2.448</td>
<td>1.605</td>
<td>1</td>
<td>9</td>
<td>288</td>
</tr>
<tr>
<td>V/T</td>
<td>0.3383</td>
<td>0.2066</td>
<td>0.0494</td>
<td>1.626</td>
<td>379</td>
</tr>
<tr>
<td>B/V/T</td>
<td>0.01627</td>
<td>0.034</td>
<td>0.0072</td>
<td>0.3714</td>
<td>704</td>
</tr>
<tr>
<td>W/V/T</td>
<td>0.01807</td>
<td>0.0407</td>
<td>8.35E-05</td>
<td>0.3714</td>
<td>288</td>
</tr>
<tr>
<td>(W-B2)/V/T</td>
<td>0.00775</td>
<td>0.0221</td>
<td>0</td>
<td>0.2525</td>
<td>288</td>
</tr>
</tbody>
</table>

On this basis I categorized the mixedwood lots further into primarily softwood lots and primarily hardwood lots if more than 60% of the volume on a woodlot came from softwood or hardwood species, respectively. By this categorization, 198 primarily softwood lots and 181 primarily hardwood lots were put up for sale in this period. Table 3 presents summary statistics with this categorization. There are three points to note here.

First, the average number of participants is the same on both types of woodlots. Given the fact that the county of Simcoe is known for its softwood, I had expected competition to be more intense on the softwood lots.

Second, the average bid/m3/tree is higher on the hardwood lots than the softwood lots, again contrary to the fact that Simcoe is known for it's softwood. A possible explanation could be the average volume of timber per tree, which is higher in the former than the latter since hardwood trees typically have larger diameter. However, though the average volume/tree is only 27% higher in hardwood lots compared with softwood lots, the average bid/volume/tree is more than double in the former than the latter.
Table 3  
SAMPLE DESCRIPTIVE STATISTICS BY TYPE OF WOODLOT

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>St.Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Primarily Softwood Lots</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participants</td>
<td>2.46</td>
<td>1.52</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>V/T</td>
<td>0.3063</td>
<td>0.1405</td>
<td>0.0441</td>
<td>0.8444</td>
</tr>
<tr>
<td>B/V/T</td>
<td>0.0100</td>
<td>0.0311</td>
<td>0.0008</td>
<td>0.3714</td>
</tr>
<tr>
<td>W/V/T</td>
<td>0.0129</td>
<td>0.0419</td>
<td>8.35E-05</td>
<td>0.3714</td>
</tr>
<tr>
<td>(W-B2)/V/T</td>
<td>0.00551</td>
<td>0.0227</td>
<td>0.2525</td>
<td>0</td>
</tr>
<tr>
<td>(2) Primarily Hardwood Lots</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participants</td>
<td>2.43</td>
<td>1.73</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>V/T</td>
<td>0.3736</td>
<td>0.2434</td>
<td>0.0494</td>
<td>1.6261</td>
</tr>
<tr>
<td>B/V/T</td>
<td>0.0259</td>
<td>0.0362</td>
<td>0.0064</td>
<td>0.2235</td>
</tr>
<tr>
<td>W/V/T</td>
<td>0.0263</td>
<td>0.0374</td>
<td>0.0004</td>
<td>0.2235</td>
</tr>
<tr>
<td>(W-B2)/V/T</td>
<td>0.01128</td>
<td>0.0207</td>
<td>0.1702</td>
<td>0</td>
</tr>
</tbody>
</table>

Hardwoods *typically* have higher market value than softwoods. While this could explain both features of the data mentioned in the last two points, it is not consistent with what I observe for the maximum volume/tree and bid/volume/tree when I compare the two types of woodlots. I find that the maximum bid/m3/tree is 40% higher in the softwood lots than the hardwood lots. The corresponding volume/tree is 15% lower on the softwood lot compared with the hardwood lot; it is 0.59 m3/tree on the former and 0.68 m3/tree on the latter. It is also important to note that the primarily hardwood and softwood lots on which the maximum bid/volume/tree was submitted contained *exclusively* hardwood and softwood trees, respectively.\(^\text{12}\)

These observations seem to indicate that there are some sawmills whose valuation of the woodlot is affected by factors, some of which may be *observable*, other than the volume of timber per tree on a woodlot and the market price of lumber. This becomes clear in the next section, where I show that there is heterogeneity amongst the bidders at least in terms of the

\(^{12}\)This was not the case for the woodlots which had the maximum volume/tree; the bid submitted on these woodlots, as a result, was low. Thus the highest bid submitted on the primarily hardwood lot with the largest volume/tree, 1.63 m3/tree, was only 0.07 $/m3/tree because 80% of the trees in this woodlot were large diameter, damaged hardwood trees which could be used only for fuelwood.
kind of woodlot they prefer to bid on, and whether their sawmills are located within the county of Simcoe or not.

The third point concerns the money left on the table or the difference between the winning bid and the second-highest bid. I find that for both type of woodlots, the average money left on the table, $(W-B2)/V/T$, is approximately half of the average winning bid, $W/V/T$; in other words, the winning bid is twice the second-highest bid. Engelbrecht-Wiggins and Weber (1979) describe an equilibrium in aggressive/nonaggressive strategies for a multi-unit auction with budgetary restrictions or capacity constraints and without resale, where the spread between the winning and the second-highest bid is similar to the one that I have noted. Hendricks, Porter and Boudreau (1987, p. 524) also note a similar spread in the OCS wildcat auctions. As Weber (1983, pp. 179-181) points out, even when the different units in the multi-unit auction are similar, bidders bid aggressively on some units and less aggressively on others. The aggressive bidding is done in an attempt to ensure that at least some woodlots are won in an auction. This could be either because the sawmill has an order for processed lumber of a particular type, or to ensure that some minimum timber is available to keep the sawmill running. I have mentioned above that the auctions are like an insurance for the sawmills in that they promise a regulated supply of timber. The less aggressive bidding is done because as sawmill, knowing that resale is not allowed by the county, wants to ensure that it does not win too many woodlots. A possible reason is that the sawmill does not want to overcommit harvesting resources on one auction within a year. That is, capacity constraints in the form of limited local resources to harvest timber is a possible reason. This gets support from the fact that I see firms with sawmills outside the county bringing in their own equipment to harvest the timber. The key point is that the sawmill bids less aggressively on these woodlots, expecting to win these woodlots if the other sawmills also bid unaggressively, and as a result win them at a bargain price.

Engelbrecht-Wiggins and Weber (1979) and Weber (1983) assume simultaneous, but independent, auction of identical multiple units, with no resale, to arrive at the equilibrium in aggressive/nonaggressive strategies that I have described above. The following assumptions are made about the environment in which the buyers and sellers interact. The bidders are assumed to be risk-neutral and identical and face budgetary restrictions and capacity constraints. The bidders face no uncertainty about the “true” value of the object being auctioned; further, the
private-values of the bidders are independent.

I will show below that two key assumptions which deliver the equilibrium in aggressive/nonaggressive strategies described by Engelbrecht-Wiggans and Weber (1979) are violated in the auctions conducted by the county of Simcoe; the woodlots in a sale are not identical and the bidders are asymmetric. The large difference in the winning bid and the second-highest bid could, as a result, be on account of asymmetries in bidders valuations and not the aggressive/nonaggressive strategy of bidders in multi-unit auctions with capacity constraints.

I now analyze the 91 woodlots on which bids were not submitted. 20 of these woodlots were part of only one auction; the county withdrew them from future auctions. 40 woodlots were part of 75 auctions but still received no bids.\textsuperscript{13} Finally there were 16 woodlots on which no bids were submitted the first time they were part of an auction. Bids were received on these woodlots the second time they were part of an auction with the highest bidders winning the woodlot. Why was it that the county was successful in auctioning off some of these woodlots the second time they were part of an auction and not the others.

First, the woodlots which received no bids are scattered all over the county of Simcoe; there does not seem to be any particular geographical area where the sawmills are systematically not submitting bids.

Second, I observe that the average volume per tree is 43% higher in the woodlots that eventually got auctioned-off compared with the ones that did not; the average volume per tree was 0.2369 m\textsuperscript{3}/tree, which is below the average given in Table 2. Thus bidders systematically rejected poor quality woodlots or woodlots yielding low volume per tree.\textsuperscript{14}

What still needs to be explained is why sawmills bid on the 16 woodlots the second time they were part of an auction, while they did not bid the first time. 12 bidders won these 16 woodlots the second time they were put up for sale. 9 of them were inactive in the first auction when the woodlot was put up for sale and no sawmill bid on it. The remaining three were

\textsuperscript{13}A woodlot, if it receives no bid the first time it is part of an auction, is in most instances part of three consecutive auctions. If no bid is received even then, the county fells these trees itself or arranges an across the counter sale. It may even pay a woodlot operator to fell these trees.

\textsuperscript{14}This is also consistent with the practice in auctions held before 1992 of having stand improvement sales. The county auctioned off the right to harvest trees on these woodlots for silvicultural reasons. That is, selective thinning improved the health of the forest even though the felled trees had marginal commercial value. Considerable operational dollars were saved by this process as the alternative was for the county to hire private operators to carry out this selective thinning. This was common knowledge.
however active, with the sawmill of two of these bidders located outside the county of Simcoe. Their behavior seems to be consistent with the aggressive/nonaggressive bidding strategy in multi-unit auctions with capacity constraints that I have discussed earlier; the bidders do not bid on these woodlots the first time they are put up for sale because they do not want to overcommit capital. An additional explanation for the two bidders who have sawmills located outside the county of Simcoe is the high fixed costs of bringing in non-local resources to harvest timber; once these fixed costs are incurred, it is in their interest to bid on as many woodlots subject to the constraint imposed by these non-local resources.

4.4 Categorizing Asymmetries in Sawmills

I observe three types of asymmetries in the valuations of the participating sawmills. First, there is technological asymmetry on account of the difference in the harvesting technology. Second, there are locational asymmetries depending on whether the sawmill of the participant is located within the county of Simcoe or outside it. Third, there are asymmetries in valuation of the bidders on account of the frequency with which they participate in the auctions. I now discuss these three asymmetries.

Hardwood trees typically have larger diameter than softwood trees thereby requiring different kinds of technology to harvest the trees from the woodlots, as well as, to process the lumber from the two types of species. Hardwood trees are felled manually with chain-saws while mechanical harvesters are typically used to fell softwood trees. Harvesters cannot be used for felling hardwood trees because they are scattered over a woodlot making it impossible to use a harvester without damaging other trees in the woodlot; softwood trees, on the other hand, are grown in rows, making them more amenable to mechanical harvesters.

On this basis, I have classified all sawmills into hardwood firms and softwood firms. If 80% of the bids of a sawmill are on primarily softwood lots, I categorize it as a softwood firm; if 80% of the bids of a sawmill are on primarily hardwood lots, I classify it as a hardwood firm. I report, in Table 4, statistics on number of bids submitted, based on this classification. Even though the number of hardwood firms is twice the number of softwood firms, the average number of bids submitted by the latter is approximately seven times more than the former. Given the
fact that approximately the same number of primarily softwood and hardwood lots were put up for sale, Table 4 clearly indicates that bidding on the softwood lots was more intense than the hardwood lots. Since hardwood trees have to be felled manually while softwood trees are felled with mechanical harvesters, technology makes it feasible for a softwood sawmill to bid frequently compared to a hardwood sawmill.

Table 4
Analysis of Number of Bids Submitted by Softwood and Hardwood Firms

<table>
<thead>
<tr>
<th>Variable</th>
<th>Softwood</th>
<th>Hardwood Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Sawmills</td>
<td>33</td>
<td>76</td>
</tr>
<tr>
<td>Average # of Bids</td>
<td>13.58</td>
<td>2.74</td>
</tr>
<tr>
<td>Maximum # of Bids</td>
<td>99</td>
<td>19</td>
</tr>
<tr>
<td>St. Dev. of # of bids</td>
<td>24.24</td>
<td>2.84</td>
</tr>
</tbody>
</table>

I have indicated in Section 3 that the sawmills of some bidders are located within the county of Simcoe, and of other bidders outside of the county of Simcoe. I will indicate the former as insider firms and the latter as outsider firms. This distinction is important because the insiders hire local resources to harvest timber. These local resources comprise the harvesting equipment, which the bidder may own or rent locally, and qualified cutter-skidder operators who may either be the employees of the bidding sawmills or contracted by them locally. The outsiders, on the other hand, have to bring in both the harvesting equipment and the operators from outside the county of Simcoe since local resources are limited and hired by the bidders who have sawmills located in the county of Simcoe. Hence, this “outsideness” gives the insiders a distinct cost advantage over the outsiders. Further, having incurred the sunk or fixed cost of bringing in harvesting equipment and qualified cutter-skidder operators from outside the county of Simcoe, it may be in the interest of an outsider to bid frequently in the year she is “in”. In Tables 7 and 8 below, I will show the existence of fringe-nonfringe dynamics described above.

There are 109 sawmills participating in these auctions in the period under study. There are four insider firms, that is, bidders who have sawmills within the county; while one of them is a hardwood firm, the other three are softwood firms. The rest are outsiders and again could be either softwood or hardwood firms.
I next wanted to ascertain whether the sawmills who participated in these auctions were bidding equally frequently. If they were not, then were the sawmills who bid more frequently learning from their past bidding decisions and as a result winning more frequently. The next Table indicates that the number of times a sawmill participates varies substantially across sawmills and there is some evidence that firms who bid more frequently also win more frequently. In Table 5, I have categorized the sawmills in terms of the number of bids they submit. The second column of the table gives the number of firms in that category. In the third and the fourth columns I have the total number of bids and wins of these sawmills respectively; the last column is the hit rate, which is the ratio of the number of wins to the number of bids. As an example, 9 sawmills submitted between 11-30 bids. These 9 sawmills submitted a total of 178 bids and won 83 woodlots on which they submitted bids.

Table 5
Analysis by Number of Bids

<table>
<thead>
<tr>
<th>Categories</th>
<th># of Sawmills</th>
<th># of Bids</th>
<th># of Wins</th>
<th># of Wins/ # of Bids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>46</td>
<td>46</td>
<td>12</td>
<td>0.26</td>
</tr>
<tr>
<td>2-10</td>
<td>51</td>
<td>169</td>
<td>47</td>
<td>0.28</td>
</tr>
<tr>
<td>11-30</td>
<td>9</td>
<td>178</td>
<td>83</td>
<td>0.47</td>
</tr>
<tr>
<td>&gt; 30</td>
<td>3</td>
<td>251</td>
<td>138</td>
<td>0.55</td>
</tr>
</tbody>
</table>

A number of points are worth noting in Table 5. First, about 42% of the 109 participants in the auctions submitted just one bid. Another 56% submitted more than one but less than thirty bids. Except for two, these 98% participants who submitted less than 30 bids, are outsiders. What is to be noted about the participation of these firms is that they bid sporadically across auctions and years.

Second, three sawmills were bidding consistently throughout the sample period; these are the sawmills who submitted more than 30 bids. These sawmills were X, Y and Z.; all three are softwood firms. While the former was an outsider firm, the latter two were insiders with their sawmills located in the township of Springwater in the county of Simcoe. While sawmills Y and Z participated in all sales every year, sawmill X participated in all sales every alternate year till 1995.
A possible explanation for sawmill X bidding every alternative year till 1995 and the sporadic bidding pattern mentioned in the last point is the feature of the “outsideness” of these sawmills that I have discussed previously. Another explanation for the sporadic bidding of the outsider sawmills is that since their sawmills are not located in the county of Simcoe, they find it cost-effective to obtain hardwood from other counties in Southern Ontario that are known for their valuable hardwood. I will elaborate on this point in Section 6.

Third, I find some evidence of bidders who bid more frequently also winning more frequently if I consider the softwood sawmills. Softwood sawmills submitting one bid have a hit rate of 26%. The wins of these sawmills are attributed either to “luck”, or to their bidding experience in the counties surrounding Simcoe. The hit rate increases to 38% for sawmills submitting between 2-30 bids. Note that all these sawmills, with the exception of two, are outsiders. The key point about the sawmills in the 2-30 category is that they are outsiders who bid in sales once in a while, and not consistently through the sample period.

Finally there are three softwood sawmills who submit more than 30 bids; they have a hit rate of 55%. These sawmills are the three key players whom I have mentioned before. They bid consistently through the sample period. The fact that they are winning more frequently will get further support in the sections below, when I show that after the first couple of sales, these three firms seem to be coordinating their bidding strategies perhaps on the basis of the learning from the first couple of sales.

The observation that three sawmill bidding consistently throughout the period for which this study is being done, and the rest bidding sporadically across the sales, motivates a further classification of the sawmills who participated in these auctions as fringe and nonfringe firms, with sawmills X, Y and Z being the nonfringe firms. The key feature of “fringeness” is that nonfringe firms have an incentive to coordinate their strategies since they are active in the auctions all the time. The fringe firms, on the other hand, may find it impossible to coordinate their bidding since a large number of them bid just once. The ones who bid more than once, just go in and out without any systematic pattern.

In Table 6, I summarize the different ways in which the participating sawmills are classified. There are 103 outsider firms and 4 insider firms, with no hardwood, nonfringe sawmill. All three key players are softwood, nonfringe sawmills, with two being insiders and one being an
outsider. For softwood lots, I will henceforth concentrate on the bidding behavior of the 3 softwood, nonfringe firms and the softwood, fringe firms; for hardwood lots, I need to consider only fringe firms since there are no hardwood sawmills bidding consistently in each sale for the period under study.

Table 6

<table>
<thead>
<tr>
<th>Classification of Sawmills</th>
<th>Softwood Firms</th>
<th>Hardwood Firms</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonfringe</td>
<td>Insider</td>
<td>Y,Z</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Outsider</td>
<td>X</td>
<td>0</td>
</tr>
<tr>
<td>Fringe</td>
<td>Insider</td>
<td>A*</td>
<td>B*</td>
</tr>
<tr>
<td></td>
<td>Outsider</td>
<td>29</td>
<td>75</td>
</tr>
</tbody>
</table>

* A and B are two fringe insider sawmills.

In Table 7, I have given the number of bids, wins, the ratio of wins to bids, average V/T, average B/V/T, across the years 1987-98 for the sawmills classified according to Table 6.

An important message of this table is that pays to be an insider. Sawmill Y and Z, the two insiders, have the highest win/bid ratio,\(^{15}\) both are winning more than half the woodlots on which they are bidding. What could be the reason for this? Two explanations come to mind.

First, being insiders, sawmills Y and Z are better informed about the value of the woodlot than the outsiders. In other words, the auctions under study are common-value auctions with bidders not knowing the "true" value of the woodlot till they have actually harvested the woodlot, and the insiders have a less noisy signal of the "true" value of the woodlot than the outsiders. Since there is little uncertainty about the cost of harvesting the woodlots, this argument implies that the estimate of the volume of timber and other information about the woodlot provided by the county in the tender notice is a noisy signal of the value of the tract. I have mentioned in Section 2, that the county participates in the auction as a bidder by having a secret reserve price which is based on the information about the tract that it provides to all

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\(^{15}\)I have noted that there are two more insider firms. Both are however fringe firms and do not bid very often. Hence I do not focus on them separately but study them with the fringe firms because they seem to be exhibiting features of "fringeness" rather than "insideness".
bidders. In the entire sample period, only 4% of the woodlots put up for sale were withdrawn by the county on account of the highest bid submitted being very low. On this basis it is reasonable to conclude that the information provided by the county about the woodlot gives the bidders a fairly accurate idea about the "true" value of the tract.

Table 7

<table>
<thead>
<tr>
<th>Sawmill</th>
<th># Bids</th>
<th># Wins</th>
<th>Average* V/T</th>
<th>Average* B/V/T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>97</td>
<td>61</td>
<td>0.629</td>
<td>0.4364</td>
</tr>
<tr>
<td>Z</td>
<td>102</td>
<td>57</td>
<td>0.559</td>
<td>0.3249</td>
</tr>
<tr>
<td>Fringe Softwood</td>
<td>198</td>
<td>59</td>
<td>0.298</td>
<td>0.4057</td>
</tr>
<tr>
<td>Fringe Hardwood</td>
<td>223</td>
<td>73</td>
<td>0.327</td>
<td>0.4646</td>
</tr>
</tbody>
</table>

* The average is over the number of sawmills on which the bidder bids.

A second and more plausible reason is that being insiders gives them a distinct cost advantage; I have discussed earlier that the insiders hire local resources to harvest timber.

Another message of Table 7 is that the three key nonfringe bidders, X, Y and Z are bidding on woodlots with different average volume per tree; this is reflected in their average bid/m3/tree too. In the next Section, I examine this in further detail to see if these three sawmills are coordinating their bidding strategy on the basis of the diameter of the trees on a woodlot.

4.4.1 A Preliminary Bidding Model

In the discussion below, I will indicate a sale by the subscript $t$, an auction within a sale by the subscript $j$, and a bidder by the superscript $i$. To remind the reader, each sale, $t$, organized by the county of Simcoe involves a first-price, sealed-bid auction for each of the multiple woodlots that are being auctioned off in that sale. A bidder $i$ who is participating in the $t$-th sale can submit bids on as many woodlots, $j$, that he wants, thereby introducing the multi-unit aspect of these auctions.

A sawmill will submit a bid to harvest the trees on a woodlot in a sale on the basis of it's valuation of the woodlot. This valuation is the sum of two components. The first component
is the difference in the market price of the timber on that woodlot and the cost a sawmill will incur in harvesting the timber from the woodlot. From the discussion above it is reasonable to assume that the cost of harvesting timber from a woodlot is known to the bidders; I will indicate the harvesting cost incurred by bidder \( i \), on woodlot \( j \) in the \( t \)-th sale by \( c_{j,t}^i \). \( mp_{j,t} \) is the market price of timber available on woodlot \( j \) in sale \( t \). I am assuming this is known to all bidders since it will be a function of past lumber prices and the bids submitted on similar woodlots in the past sales.\(^{16}\) The second component of a bidder's valuation is the demand that this bidder faces in the market for processed lumber; this is private information of a bidder. The first component of a bidder's valuation is nonstochastic. Orders for processed lumber, which is the second component of the value of a woodlot to a bidder, is stochastic. I will indicate the stochastic component by \( \eta_{j,t}^i \); this is the demand that bidder \( i \) faces in the secondary market for the kind of lumber that is available on woodlot \( j \) in sale \( t \). The equilibrium bidding rule is then given by:

\[
\begin{align*}
    b_{j,t}^i &= e^i(v_{j,t}^i), \quad \text{with} \\
    v_{j,t}^i &= (mp_{j,t} - c_{j,t}^i) + \eta_{j,t}^i 
\end{align*}
\]

where \( b_{j,t}^i, v_{j,t}^i \) and \( c_{j,t}^i \) are the bid, valuation, and cost, respectively, of bidder \( i \) in the \( j \)-th auction held in sale \( t \). \( e^i(\cdot) \) is a strictly increasing function.

On the basis of the analysis in Sections 3 and 4, I assume the log-linear bidding rule in equation (3) below describes the equilibrium bidding behavior of only softwood firms in the sample. I consider only softwood firms because from Table 6, I observe that all hardwood firms are fringe firms. All of them, with the exception of one, are also outsiders; but this firm is, for all practical purposes, an outsider since it's sawmill is located on the boundary of the county of Simcoe and the adjoining county of Dufferin. Hence the hardwood firms in the sample are all outsider, fringe firms. They will, as a result, not display the insider-outsider and fringe-nonfringe dynamics that I am expecting to capture through this regression.\(^{17}\)

\(^{16}\) There is a reasonably well documented cycle in the Canadian lumber and timber industrial prices.

\(^{17}\) The hardwood firms will exhibit the aggressive-nonaggressive bidding strategy that characterizes the multi-unit aspect of the auction.
\[ \log(b^i_{j,t}) = \beta_0 + \beta_1 (\text{PRICE})_t + \beta_2 (V/T)_{j,t} + \beta_3 (\text{TYPE})_{j,t} + \beta_4 (\text{PTP})_{j,t} \\
+ \beta_5 (\text{OutFr})^t + \beta_6 (\text{InNFr})^t + \beta_7 (\text{InFr})^t + \\
\beta_8 (\text{NWINS})^t_{-j,t} + \beta_9 (\text{CAPACITY})^t_{-j,t} + \epsilon^i_{j,t}. \]  

(1)

\( \log(b^i_{j,t}) \) is the nominal bid submitted by a bidder \( i \) in auction \( j \) in the sale held at time period \( t \). The results of a least squares regression for equation (3) are summarized in Table 8.\(^{18} \)

| Variable | Coefficient | Estimate | Standard Error | t-value | Prob > |t|
|----------|-------------|----------|----------------|---------|---------|
| constant | \( \beta_0 \) | 5.795 | 0.292 | 19.819 | 0 |
| PRICE    | \( \beta_1 \) | 1.601 | 0.154 | 10.408 | 0 |
| \( V/T \) | \( \beta_2 \) | 1.029 | 0.396 | 2.599 | 0.010 |
| TYPE     | \( \beta_3 \) | -0.404 | 0.128 | -3.163 | 0.002 |
| PTP      | \( \beta_4 \) | 0.199 | 0.036 | 5.465 | 0 |
| OutFr    | \( \beta_5 \) | -0.243 | 0.171 | -1.425 | 0.155 |
| InNFr    | \( \beta_6 \) | 0.115 | 0.165 | 0.699 | 0.485 |
| InFr     | \( \beta_7 \) | -0.041 | 0.253 | -0.162 | 0.872 |
| NWINS    | \( \beta_8 \) | 0.047 | 0.026 | 1.769 | 0.078 |
| CAPACITY | \( \beta_9 \) | 0.028 | 0.034 | 0.180 | 0.418 |

There are four types of explanatory variables in the regression given in (3): variables specific to a sale, variables specific to an auction or a woodlot in a sale, variables specific to a bidder, and variables specific to a bidder within a sale \( t \). The only variable in the first category is PRICE; this is the lumber and timber industrial price index with 1986 as the base year and available on a monthly basis. I observe that an increase in the price of lumber leads to more aggressive bidding, with a 1% price increase leading to a 1.6% increase in the nominal bid.

\(^{18}\)The coefficients in the regression are rates of change in the nominal bid due to a change in the exogeneous variable, ceterus paribus.
There are three auction or woodlot specific variables within a sale; I have indexed them by the subscript \( j, t \) indicating aspects of the \( j \)-th woodlot in the \( t \)-th sale. \((V/T)_{j,t}\) is the volume per tree, expressed in m\(^3\)/tree, on woodlot \( j \) in sale \( t \). Woodlots with higher volume per tree receive higher bids; bidders increase their nominal bid by 1\% with an additional cubic metre per tree.

\((\text{TYPE})_{j,t}\) is a dummy variable which takes values 0 or 1. Since I have included only softwood firms in this regression, I consider only primarily softwood lots; these were woodlots where more than 60\% of the timber, in terms of volume, was from softwood trees. \((\text{TYPE})_{j,t}\) takes value 0 if the woodlot \( j \) in sale \( t \) is \emph{exclusively} softwood and 1 otherwise. The coefficient for the variable \( \text{TYPE} \) is negative indicating that softwood firms bid more aggressively on woodlots that are \emph{exclusively} softwood lots. Softwood firms value \emph{exclusively} softwood lots since the hardwood on the primarily softwood lots is of limited commercial value to these firms in view of the fact that resale is not allowed by the county of Simcoe.

\((\text{PTP})_{j,t}\) is the number of participants in the auction \( j \) in sale \( t \); it is being used as a proxy for potential competition. This seems reasonable since there is no announced reserve price. Further, as I mentioned in Section 3, only 4\% of the woodlots were withdrawn from the sale because the county found the bid too low, leading me to conclude that the secret reserve price is non-binding. Sawmills submit bids that are 0.2\% higher with potential competition from an additional bidder.

There are three bidder specific variables. All three are dummy variables, which together with the constant capture the fringe-nonfringe and insider-outsider dynamics of the bidding sawmills; these are indexed by the superscript \( i \) for the bidder. \((\text{OutFr})_i^i\) is 1 if the bidder \( i \) is an outsider, fringe firm and 0 otherwise. \((\text{InNFr})_i^i\) is 1 if the bidder \( i \) is an insider, nonfringe firm and 0 otherwise; and \((\text{InFr})_i^i\) is 1 if the bidder \( i \) is an insider, fringe firm and 0 otherwise. \( \beta_6 \) and the difference, \( \beta_7 - \beta_5 \), are the insider-outsider difference in log bids of the nonfringe and the fringe sawmills, respectively. Both have positive signs indicating that the insider firms bid more aggressively than the outsiders because the former have a cost advantage in employing local resources to harvest timber.\(^{19} \) \( \beta_5 \) and \( \beta_7 - \beta_6 \) are the fringe-nonfringe difference in log bids of the outsider and the insider firms, respectively; I will analyze these in the next two

\(^{19}\)The F-statistics for the null hypotheses, \( \beta_7 - \beta_5 = 0 \) is 0.93 and the tail-area probability, \( Pr > F \), is 0.505.
Sections.

There are two variables specific to a bidder $i$ in a sale $t$. $(\text{NWINS})_{i-jt}^{i}$ is the number of auctions that a sawmill $i$ wins in sale $t$ excluding the current auction $j$; it can be viewed as a proxy for the number of auctions, excluding the current auction, bidder $i$ expects to win in sale $t$. This variable captures two effects. First is the demand effect; this is private information of the bidder. A bidder that has a large order for processed lumber will want to win a large number of woodlots in a sale. She will, as a result, bid aggressively on woodlot $j$, even if she is expecting to win a large number of woodlots, other than the $j$-th woodlot, in sale $t$. The second effect is the aggressive-nonaggressive bidding by a sawmill that characterizes the multi-unit aspect of these auctions as described in Engelbrecht-Wiggans and Weber (1979). If a bidder is expecting to win a large number of woodlots in sale, she will want to obtain an additional woodlot at only a bargain price; hence she will bid nonaggressively on this additional woodlot. That is, if in sale $t$, a bidder has won a large number of woodlots, she will want the $j$-th woodlot at only a bargain price; she will hence bid nonaggressively on woodlot $j$. These two effects work in opposite directions. It is interesting to observe that the sign of the coefficient for the variable $(\text{NWINS})_{i-jt}^{i}$ is positive indicating that the demand effect is dominating the aggressive/nonaggressive bidding effect; as I have explained the latter follows from the multi-unit aspect of the sales. A bidder will increase her bid by 0.05% for each additional woodlot that she wants to win.

Finally, $(\text{CAPACITY})_{i-t}^{i}$ is the number of woodlots won by bidder $i$ in the sales held in the previous calendar year excluding the current sale. Let $t$ be a sale held in April 1998; then $(\text{CAPACITY})_{i-t}^{i}$ is the number of woodlots won by a bidder $i$ in sales held between April 1997 and April 1998, excluding the sale held in April, 1998. The county allows a harvesting time of one year from the date of the sale. A bidder who has won a large number of woodlots in April 1997, will bid less aggressively in the subsequent sales in 1997 since she is constrained by the harvesting resources available to her; in the sale held in April 1998 she will bid aggressively since she will not face a capacity constraint from the resources she had committed in April 1997. In other words, the harvesting resources committed in April 1997, will be available in April 1998, and as a result, she will bid more aggressively. Similarly, a bidder who wins a small number of woodlots in the sale held in April 1997, will bid aggressively in the subsequent sales held in
1997 to avoid being shut out. Having won a large number of woodlots in these subsequent sales in 1997, she will face a capacity constraint in April 1998; hence she will bid less aggressively in the sale held in April 1998. I observe a positive sign for the coefficient on this variable in Table 8 supporting this conjecture.

4.5 Analysis of Nonfringe Softwood Sawmills

In this Section, I analyze the bidding strategies of the 3 nonfringe softwood firms, X, Y and Z, the latter being an outsider firm.

In Figure 1, I plot the average volume per tree, V/T, for the four sawmills, X, Y, Z and the softwood, fringe sawmills for the years 1987-98.20 Since all these are softwood firms, I am trying to ascertain whether V/T characterizes a woodlot to some extent and helps me to distinguish between these sawmills on this basis. Since 20% of the bids submitted by sawmill Y are on primarily hardwood lots, and hardwood lots typically have trees with large diameters, I am computing the average V/T for sawmill Y on only the primarily softwood lots that it bids on.

I observe that prior to year 5, sawmills X, Y and Z are bidding many times on the woodlots with the same average V/T. Examples are X and Z in years 2 and 4, and Y and Z in year 3. After year 5, I see that these three sawmills are bidding on woodlots with different V/T, with sawmill Y bidding on the woodlots which have trees with large diameter, X coming next and Z bidding on woodlots with trees with the smallest diameter. It is as if on the basis of their bidding experience in the first four years, the three sawmills have coordinated their bidding strategies and are not bidding against each other. Several examples of coordination games of this kind are discussed in Osborne and Rubinstein (1998, pp 15-16) and Tirole (1988, 406-408).

Visualize the following scenario. There are three types of woodlots, Type 1, Type 2 and Type 3, corresponding to woodlots with trees that have large, medium and small diameter, respectively. In Table 9, I have made a matrix of payoffs of the three sawmills, X, Y and Z, with (i, j, k) standing for their payoffs respectively. In this matrix, I have assumed that sawmill X bids on woodlots with trees of medium diameter or Type 2 woodlots.21 The diagonal elements

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20 I have excluded the years 1994 and 1995 from this sample since I have information only about the first auction for these two years.

21 There will be two more matrices in addition to the one in Table 6. One will correspond to Y bidding on
correspond to the two pure strategy Nash equilibria. These two equilibria correspond to the situation where the main concern of sawmills X, Y and Z is not to bid against each other, even though sawmill Y prefers woodlots with high diameter softwood trees since he is bidding on woodlots with large diameter, hardwood trees. Off-diagonal elements are zero corresponding to the situation where at least two sawmills compete with each other by bidding on the same type of woodlot. Clearly, once they decide to coordinate their bidding strategies, competing against each other is a Pareto inferior bidding strategy for each of the three sawmills.

Table 9
Payoffs in a Coordination Game

<table>
<thead>
<tr>
<th></th>
<th>Type 3 Woodlot</th>
<th>Type 1 Woodlot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1 Woodlot</td>
<td>(2,1,1)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>Type 3 Woodlot</td>
<td>(0,0,0)</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>

Here Y and Z are bidding on Type 1 or Type 3 woodlots and X on Type 2 woodlot. \((i,j,k)\) are the payoffs of \((Y,X,Z)\).

Support for the conjecture that sawmills X, Y and Z are coordinating their bidding strategies after period 5 comes from two additional sources. In Figure 2 I have plotted the average real bid/volume/number of trees for these three nonfringe sawmills and the fringe sawmills. Figures 1 and 2 together support the hypothesis of bidders coordinating their strategies after period 5 as follows. Volume per tree is a key variable that affects the bid submitted by a bidder once account is taken of the heterogeneity of the bidders in that they are softwood or hardwood firms and insiders or outsiders. This is evident from the peaks and the troughs in Figures 1 and 2 for all sawmills occurring in the same years; all bidders submit high or low bids depending on whether the trees to be harvested on a woodlot have high or low volume per tree. The positive coefficient for the variable \((V/T)\) in equation (3) supports this hypothesis too.\(^{22}\) Hence, like Figure 1, I find that after year 5, sawmill Y submits the highest bid/m3/tree, with X coming

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\(^{22}\) The value of the t-statistic is in the tails of the t-distribution indicating that \((V/T)\) is statistically significant.
next, and then sawmill Z. On the other hand, in years 2 and 4, I find sawmills X and R, submitting similar bid/m3/trees. This supports the conjecture that these three sawmills are coordinating their bidding strategies after year 5 and bidding on softwood lots with different V/T.

As further evidence of the hypothesis that sawmills X, Y and Z are coordinating their bidding strategies after period 5, I carried out a Chow test. The Chow test involves two regressions. In the first regression, given by equation (3), I assume that no coordination of strategies is taking place in period 5. The fringe-nonfringe difference in bids will then be due to the demand effect that I have mentioned earlier. The fact that the nonfringe firms are bidding consistently in these auctions, which motivated this classification of the sawmills, is because they have orders for processes lumber that is their private information. Hence I should observe the nonfringe firms bidding more aggressively than the fringe firms. I see in Table 8, that the estimates of $\beta_5$ and $\beta_7 - \beta_6$ are negative. This regression will be called the restricted regression.

In the second regression, which I call the unrestricted regression, the fringe-nonfringe difference in bids, up to year 5, will be due to the demand effect discussed above. After year 5, when these three sawmills start coordinating their strategies, I expect that on an average, the bids of the nonfringe sawmills will be lower than the bids of the fringe firms. In effect since they do not compete with each other, the nonfringe sawmills benefit in terms of being able to win with lower bids. Sawmills Y and Z are nonfringe insiders and sawmill X is a nonfringe outsider firm; I am expecting the following to change, prior to, and post coordination. The expected bid of the outsider nonfringe firm, X will change; this is given by the coefficient $\beta_6$. Further, the fringe-nonfringe difference in bids of insiders, $\beta_7 - \beta_6$, the fringe-nonfringe difference in the bids of the outsiders, $\beta_5$, and the insider-outsider difference in bids of nonfringe firms, $\beta_6$, will be different. The fringe-nonfringe difference in the bids of the outsiders or insiders should decrease after coordination since the fringe will continue to bid as before, but the nonfringe will bid less aggressively. The insider-outsider difference in the bids of the nonfringe firms is the difference in the bids of the insiders, Y and Z, and the outsider, X. Prior to coordination, the difference in their bids was due to the fact that X's sawmill was outside the county of Simcoe and the fact that the three were competing against each other. Once they start coordinating their bidding strategies, the difference in their bids is only on account of the former. Hence,
the insider-outsider difference of the nonfringe firms should decrease after coordination.\footnote{I am not expecting a change in the insider-outsider difference in bids of fringe firms, \( \beta_7 - \beta_5 \), once coordination by the nonfringe firms commences. The expected bids of the insider and outsider fringe firms, \( \beta_o + \beta_7 \), and \( \beta_o + \beta_5 \), will be unchanged as well.}

### Table 10

**Results from Unrestricted Regression Prior to Coordination**

| Variable | Estimate | Standard Error | t-value | Prob > |t| |
|----------|----------|----------------|---------|---------|---|
| constant | 5.782    | 0.292          | 19.798  | 0.000   |   |
| OutFr    | -0.470   | 0.332          | -1.416  | 0.160   |   |
| InNFr    | 0.247    | 0.323          | 0.765   | 0.450   |   |
| InFr     | -0.365   | 0.613          | -0.595  | 0.550   |   |

I will indicate the coefficients before coordination by the three nonfringe sawmills starts by "nc" and after coordination starts by "cr". Thus the unrestricted regression can be split into two parts. Upto year 5 when the three sawmills are not coordinating their bidding strategies, I will run the regression in equation (3) with \( \beta_o, \beta_5, \beta_6, \beta_7, e_{ij,t} \) replaced with \( \beta_o^{nc}, \beta_5^{nc}, \beta_6^{nc}, \beta_7^{nc}, e_{ij,t}^{nc} \), respectively. Note that the constant term in these regression is the log bid of the outsider nonfringe firm, \( X \); and I am expecting sawmill \( X \) to bid differently once he starts coordinating his bidding with the other two nonfringe sawmills. After year 5, when the three nonfringe sawmills coordinate their bidding strategies, the regression in equation (3) is run with \( \beta_o, \beta_5, \beta_6, \beta_7, e_{ij,t} \) replaced with \( \beta_o^{cr}, \beta_5^{cr}, \beta_6^{cr}, \beta_7^{cr}, e_{ij,t}^{cr} \), respectively. I have reported the results of these two regressions in Tables 10 and 11, respectively. The results confirm the conjectures made above about the differences in bids, prior to, and post coordination.
Results from Unrestricted Regression Post Coordination

| Variable | Estimate | Standard Error | t-value | Prob >|t| |
|----------|----------|----------------|---------|--------|
| constant | 5.786    | 0.139          | 41.38   | 0      |
| OutFr    | -0.114   | 0.160          | -0.713  | 0.476  |
| InNFr    | 0.008    | 0.168          | -0.049  | 0.039  |
| InFr     | -0.045   | 0.225          | 0.198   | 0.843  |

The basic idea of the Chow test is that if the three bidders, X, Y and Z are coordinating their bidding strategies after year 5, the residuals from the restricted and the unrestricted regression will be different. The null hypothesis under which the Chow test is performed is that there is no coordination in bidding strategies by the outsider, nonfringe sawmills from year 5 onwards; that is,

\[ H_0: \beta^{nc}_o = \beta^c_o, \beta^{nc}_5 = \beta^c_5, \beta^{nc}_6 = \beta^c_6, \beta^{nc}_7 = \beta^c_7. \]

(4)

The value of the F-statistic for the Chow test is 2.054. The tail-area probability, Prob >F, is 0.0862. Hence the data supports the coordination hypothesis, and as a result the fringe-nonfringe dynamics.

While volume per tree and the observable bidder heterogeneity explain the difference in the bids submitted by the different bidders, Figures 1 and 2 show that these observable factors cannot explain all the variation in the bids. Several examples can be given to illustrate this point; I give two examples here.

First, compare sawmills X and Z in year 4 in Figure 1 and Figure 2. Both are softwood, nonfringe sawmills. I have also indicated that they are coordinating their bidding only after year 5. From Figure 1, I see that in year 4 they bid on woodlots with similar average volume per tree; their average bid/m3/tree is also identical from Figure 2. This seems surprising since sawmill X being an outsider has to obtain harvesting resources from outside of Simcoe and then transport them to his sawmills that are located outside the county of Simcoe. I have observed in the last section that insider-outsider difference in log bids from the nonfringe firms is given by the

\[ \text{Since } \beta^o_6 \text{ is the same, prior to, and post coordination, } \beta^c_7 - \beta^o_5 \text{ will be unchanged with coordination if and only if } \beta^c_7 \text{ is the same prior to, and post coordination.} \]
coefficient $\beta_6$ in equation (3), and Table 8 provides support for the existence of insider-outsider dynamics for nonfringe firms.

Second, compare sawmills Y and Z in Figures 1 and 2. Both are softwood, insider, nonfringe sawmills. In year 4 their average bid/m3/tree is identical (see Figure 2) even though they are bidding on woodlots with different volume per tree (see Figure 1).

The $R^2$ for the regression in equation (3) is 0.445 pointing to about 50% variation in log bids that is unexplained by the observable woodlot, sale or bidder specific characteristics.

On the basis of the discussion above, I can make the following assumptions about the bidding environment for the nonfringe firms, X, Y and Z, which to a first approximation describe their bidding behavior. First, the three bidders draw their valuations from different distributions; this asymmetry in the bidders is on account of the location of their sawmills. While X's sawmill is located outside the county of Simcoe, the sawmills of the other two softwood nonfringe sawmills is located in the county of Simcoe. Second, there is residual variation in the bid per cubic meter per tree which cannot be accounted for by observable bidder, woodlot and sale specific variations. I have also explained why the bidders having a fairly good idea about the "true" value of the tract. Further, the coefficient on the variable PTP is positive indicating that bidding is more aggressive as potential competition increases, an observation consistent with the private-values model for the bidders. On the basis of these observations, a private-values environment for these auctions seems a reasonable assumption. Third, I am assuming that the private-values of the bidders are independent. Correlations in valuations could be possible if there is a general state of excess supply or demand for lumber; thus, all bidders would face a high demand in the market for processed lumber if there was a general state of excess demand. Since there is a well-documented cycle of lumber prices, bidders do not face any uncertainty in their valuation from this source. Hence in equation (2), the stochastic part of a bidder's valuation can be assumed to be drawn independently of other bidder's valuations. Fourth, prior to period 5, sawmill X, Y and Z bid competitively; after period 5, they appear to be coordinating their bidding strategies.
4.6 Competition From Fringe Sawmills

The above discussion still leaves me with the question as to what the fringe sawmills are doing? There are 30 softwood, fringe firms, all of whom, with the exception of one sawmill, are outsiders. I have indicated earlier that the difference in the coefficients, $\beta_7 - \beta_5$, in equation (3) is the insider-outsider difference in log bids of fringe softwood sawmills, and that data supports insider-outsider dynamics. The 76 hardwood, fringe firms, are all outsiders except for one. The sawmill of this bidder is located in the township of Adjala-Tosorontio, on the border of the county of Simcoe and Dufferin. Hence for all practical purposes, this bidder can be considered to be an outsider as well.

I first focus on the softwood, fringe sawmills. Both Figure 1 and 2 show that the fringe softwood sawmills are not bidding on softwood lots of any particular type in terms of average volume per tree. I see them bidding on woodlots which have trees with high, medium or low diameter. It is the fringe softwood mills that are competing with the nonfringe softwood mills as the latter have coordinated their bidding strategies in the manner described above. This is obvious from Figure 3 where I have plotted on the y-axis on the left, the number of bids, and on the y-axis on the right, the number of wins to number of bids across the years for the softwood, fringe and the three softwood, nonfringe firms, X, Y and Z. The hypothesis is that the years in which the softwood, fringe bidders submit a small number of bids are the years in which the softwood, nonfringe wins a large proportion of the woodlots on which they bid; year 2 and years 4-7 are examples. One year that stands out in terms of the large number of bids submitted by the softwood, fringe is year 9; that year the softwood, nonfringe won just 35% of the woodlots on which they bid.

I have mentioned before that the one softwood, nonfringe firm that bids, once in a while, on large diameter hardwood lots is sawmill Y. Then is it the case that in these years sawmill Y has a high win/bid ratio only if the hardwood, fringe firms submit a small number of bids? Figure 4 is a counterpart of Figure 3 for hardwood, fringe sawmills and sawmill Y in his role as a hardwood, nonfringe sawmill; thus I am considering only the hardwood lots on which sawmill Y bid in this sample. I see that when sawmill Y bids on hardwood lots, his bidding is so aggressive that he wins almost all woodlots on which he bids. Further the years when the fringe is submitting the maximum number of bids, years 2, 7 and 8, also happen to be the years when
sawmill Y has a win/bid ratio close to 1. It seems that while bidding on hardwood lots, sawmill Y is not exhibiting the kind of fringe-nonfringe dynamics that I noted for softwood sawmills. Hence, to a first approximation, I can assume that there are only hardwood, fringe sawmills in my sample and no hardwood, nonfringe sawmill.

The fact that fringe firms do not coordinate their strategies and compete with each other, as well as, the nonfringe firm gets support from the win/bid ratio in column 3 of Table 7 for fringe firms. Comparing rows 4 and 5 with rows 1-3, I find that the win/bid ratio for the fringe firms is the lowest.

I next examine why the fringe firms are bidding sporadically. Further, is there any link between the sporadic bidding of these 106 fringe firms and the fact that 105 of them are outsiders.

Three bidding patterns are noticeable amongst the nonfringe firms. 23 fringe sawmills who submit more than one bid participate in just one auction in the entire time period that I am examining. For these fringe sawmills it is is reasonable to assume that they obtain orders for processed lumber infrequently. At the point of time when these orders arrive, they bid aggressively in that auction and then exit the sample.

Of the remaining fringe firms who submit more than one bid, and participate in more than one auction in the sample, I find two kinds of bidding patterns.

<table>
<thead>
<tr>
<th>Sawmills</th>
<th># of Bids</th>
<th>Bidding History</th>
</tr>
</thead>
<tbody>
<tr>
<td>Softwood</td>
<td>23</td>
<td>96.1(5), 96.2(3), 97.1(6), 97.2(6), 97.3(3)</td>
</tr>
<tr>
<td>Hardwood</td>
<td>8</td>
<td>92.1(2), 92.2(1), 93.1(2), 93.2(2), 93.3(1)</td>
</tr>
<tr>
<td>Softwood</td>
<td>16</td>
<td>88.1(5), 88.2(3), 89.1(3), 93.2(2), 96.1(1), 98.2(2)</td>
</tr>
<tr>
<td>Hardwood</td>
<td>12</td>
<td>89.1(2), 90.2(1), 93.4(1), 96.2(8)</td>
</tr>
</tbody>
</table>

Some of them bid quite frequently within the span of a year once they are “in” the county of Simcoe to bid. Again, having obtained orders for processed lumber, these sawmills enter the county of Simcoe. Once they have bid and won in an auction, they have committed harvesting resources that have been obtained from outside the county for a year; I have elaborate on
this feature of "outsideness" earlier. Having committed these sunk costs, it is possible they bid according to the aggressive/nonaggressive bidding strategy that has been discussed by Engelbrecht-Wiggans and Weber (1979) and Weber (1983, pp. 179-181). I present in the first and second row of Table 12 the bidding history of a softwood, fringe, outsider and a hardwood, fringe outsider, respectively, as examples to substantiate this point. An entry 98.1(8) indicates that the bidder submitted bids in the first sale held in the year 1998; the number in the brackets is the number of bids submitted by this bidder.

The rest of the fringe sawmills just go "in" and "out" of auctions throughout the sample period; examples of this bidding pattern are given in the last two rows of Table 12, where again I have a softwood and a hardwood sawmill. An explanations for this follows. Around the county of Simcoe, are other counties known for their valuable hardwood and softwood. These fringe, outsider sawmills could be bidding in these counties; in fact they may no longer be fringe firms in these counties, and as a result may have considerable harvesting resources engaged here. This would make it profitable for them to bid every once-in-a-while in the county of Simcoe contingent on obtaining orders for processed lumber.

On the basis of this discussion above I suggest the following bidding model for the fringe sawmills. First, the fringe bidders are not identical. They exhibit two types of asymmetries, technological and locational. Technological asymmetry follows from the fringe bidders comprising of both softwood and hardwood sawmills. Locational asymmetry follows from the presence of both outsiders and insiders amongst the fringe bidders. Second, the relationship between the bidders valuations is best described by the independent-private-values model. The justification for this is similar to the one presented in the last section for the softwood nonfringe sawmills. Third, the fringe bidders bid competitively throughout the sample period. They arrive to participate in an auction in Simcoe with some probability that is determined by the following factors: (1) the type of woodlots, softwood or hardwood, that have been put up for sale in the auction; (2) the kind of order for processed lumber that the sawmill faces; (3) whether it is already "in" the county of Simcoe and as a result has harvesting resources employed in the county for the period of a year; (4) whether it has been active in counties neighboring Simcoe. Once it is "in", the bids that it submits are affected by the same factors as the nonfringe sawmills; I have discussed these in Section 4.
4.7 Conclusion

For the auctions conducted by the county of Simcoe, it seems reasonable to assume that the independent-private-values model describes the bidding behavior of the participating sawmills. The bidders are not identical; they exhibit asymmetries on account of technology, location, and frequency of participation. While the fringe firms bid competitively, the nonfringe firms seem to be coordinating their bidding strategies on the basis of their bidding experience in the first five years of the sample period.

Some support is provided by the data for aggressive/nonaggressive bidding strategies that characterize the bidding behavior of bidders with multi-unit demand facing budgetary restrictions or capacity constraints that has been discussed by Engelbrecht-Wiggans and Weber (1979); the evidence is however, not conclusive. The large differences in the winning bid and the second-highest bid noted by Engelbrecht-Wiggans and Weber (1979) could be due to asymmetries in bidders valuations. Since the county allows a year's harvesting time, and resources for harvesting timber are limited, I cannot assume that the sales are independent. Neither the woodlots in a sale, nor the environment across sales is homogenous.

While it is difficult to make any policy conclusions from the analysis in this paper, the fact that most sawmills target either softwood or hardwood, and that one of the aims of the county of Simcoe is to ensure the viability of the forest industry, it may want to reconsider its policy of selling mixed woodlots. This follows from sawmills targeting either hardwood or softwood on a woodlot that has both. With resale not being allowed, the hardwood trees on the woodlot would be of marginal commercial value to a softwood sawmill, and the softwood trees of marginal commercial value to a hardwood sawmill.

I conclude by saying that the analysis in this paper is suggestive. Detailed work on modelling the bidding strategies of the bidders in the manner described above will clarify further the bidding strategies and allow comparison of auctions forms and the manner in which woodlots should be put up for sale in an auction.
Figure 4-1: Average Volume/# Trees Across Years
Figure 4-2: Real Bid/Volume/# Trees Across Years
Figure 4-3: # of Bids and # Wins/# Bids Across Years
Figure 4-4: # of Bids and # Wins/# Bids Across Years
Figure 4-5: County of Simcoe
Bibliography


