INCLUSIVE RARE $B$ DECAYS USING EFFECTIVE FIELD THEORIES

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Physics
University of Toronto

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Abstract

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In this thesis we will discuss several properties of rare decays of $B$ mesons. First we discuss properties of the inclusive radiative decay $\bar{B} \to X_s \gamma$, where $X_s$ stands for any hadronic state containing an $s$ quark. We extend previous studies of this decay, which included perturbative corrections to order $\alpha_s$ and nonperturbative contributions up to order $(\Lambda_{\text{QCD}}/m_b)^2$ and calculate the $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3$ contributions to this decay. The values of the nonperturbative parameters entering at this order are unknown, leading to uncertainties in the standard model prediction of this decay. We estimate the size of these nonperturbative uncertainties by varying these parameters in the range suggested by dimensional analysis. We also estimate uncertainties arising from a cut on the photon energy which is required experimentally.

Another decay mode investigated is $\bar{B} \to X_s \ell^+ \ell^-$. We study the $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3$ contributions to the leptonic invariant mass spectrum, the forward-backward asymmetry and hadronic invariant mass moments and estimate the resulting uncertainties. We calculate how the size of these uncertainties depend on the value of an experimental cut that has to be applied to eliminate the large background from other $B$ decays.

A model independent way to determinate the CKM matrix element $|V_{ub}|$ from the dilepton invariant mass spectrum of the inclusive decay $\bar{B} \to X_u \ell^+ \bar{\nu}$ is presented next. We show that cuts required to eliminate the charm background still allow for a theoretically clean way to determine the CKM matrix element $|V_{ub}|$. We also discuss the
utility of the $\bar{B} \to X_s \ell^+\ell^-$ decay rate above the $\psi(2S)$ resonance to reduce the resulting uncertainties.

Finally, we introduce a novel effective theory valid for highly energetic particles. In decays where the phase space is sufficiently restricted such that final state particles have very high energies compared to their mass, the perturbative as well as nonperturbative series diverge. The effective theory presented allows to sum perturbative Sudakov logarithms in a framework that also incorporates the nonperturbative physics in such limits of phase space.
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Chapter 1

Introduction

Our current understanding of matter and its interactions is based on the identification of a few elementary constituents and four fundamental forces acting upon them. The constituents are the six point-like quarks, named up (u), down (d), strange (s), charm (c), bottom (b) and top (t), in combination with the leptons, the electron (e), muon (μ) and tau (τ) with corresponding neutrinos ν_e, ν_μ and ν_τ. The standard model describes only three of nature’s four interactions, with the strong, weak and electromagnetic forces specified by SU(3)_{\text{colour}} \times SU(2)_{\text{weak}} \times U(1)_{\text{Hypercharge}} gauge transformations [1, 2] and gravity not being part of the standard model.

The standard model has withstood stringent experimental tests. The electromagnetic part of this theory, quantum electrodynamics (QED), was developed many years ago and experimental tests of QED are numerous. The quantity most precisely measured [3] and calculated [4] is the anomalous magnetic moment of the electron, with the agreement between theory and experiment at eight significant digits. This makes QED one of the best tested theories in physics. The weak interaction, especially the properties of the Z boson, has been tested very well in recent experiments at CERN in Geneva [5]. These experiments have measured the coupling of the Z boson to quarks and leptons thoroughly and agree with the standard model predictions at one part in a thousand.
The theory of the strong interaction, quantum chromodynamics (QCD), is asymptotically free, meaning that the interaction gets weaker towards higher energies. At high energies, where theoretical predictions can be calculated perturbatively, direct tests of QCD can be performed. Once again, all experiments confirm the predictions of the standard model [6].

Despite the success of the standard model, many questions remain to be answered. One of the most important questions is the mechanism of electroweak symmetry breaking. Although the Higgs mechanism [7] successfully describes the spontaneous symmetry breaking, the Higgs boson has yet to be discovered. There are also several shortcomings, such as the fact that the Higgs boson mass is unstable against radiative corrections. Many different scenarios of physics beyond the standard model have been proposed, which try to solve some of these problems. Some models, such as technicolour [8], provide a method of electroweak symmetry breaking without a fundamental Higgs; others, such as supersymmetry [9], still need a fundamental Higgs, but stabilize the mass of the Higgs boson.

Another unanswered question is the origin of fermion masses and the quark mixing matrix. Several proposals can be found in the literature. In the framework of grand unified theories (GUT) [10], the pattern of fermion masses and the mixing matrix can take a very simple form at the GUT scale. The hierarchy in these parameters is introduced in the evolution to present day energy scales [11]. A very different approach has been suggested recently [12], in which the left and right handed fermions are stuck to different points on a thick 3+1-dimensional brane embedded in a higher dimensional space. The separation of the left and right handed fermions gives rise to an exponentially suppressed overlap of the wave functions, which explains the hierarchy of fermion masses.

The flavour sector of the standard model involving the two heaviest quarks, the top and bottom quark, has not yet been investigated in great detail. Several experiments are dedicated to exploring the flavour sector of the b quark system. The CESR storage
ring at Cornell with the CLEO detector was the first experiment built to measure the properties of $B$ mesons, particles containing $b$ quarks. Since the summer of 1999 two more experiments are online: BaBar at SLAC [14] and Belle at KEK [15]. CLEO is designed to investigate $B$ decays in general and provides, amongst many other things, the most accurate measurements of the CKM matrix elements $V_{ub}$ and $V_{cb}$ to date [16, 17, 18, 19, 20]. The main goal of the experiments at SLAC and KEK is to measure CP violation in the $b$ quark system. The standard model predicts that all observed CP violation is due to a single phase in the quark mixing matrix [21]. Since many models of new physics give rise to new sources of CP violation [22], these new $B$ factories are an excellent testing ground for physics beyond the standard model.

Exploring the mechanism of electroweak symmetry breaking requires testing the standard model at energy scales of a few hundred GeV. A direct approach will be pursued at the large hadron collider (LHC) at CERN [23], which will collide protons at a centre of mass energy of 14 TeV. At this energy the Higgs boson, and possibly particles not present in the standard model, should be produced directly. A more indirect approach is to perform low energy precision measurements of processes that are suppressed in the standard model. New physics can give sizable corrections to some of these rare processes, making them very sensitive to physics beyond the standard model.

### 1.1 The flavour sector of the standard model

The leptons and quarks in the standard model are organized into three families each. The left handed up and down type quarks form doublets under weak isospin $SU(2)_L$

\[
\begin{pmatrix}
    u \\
    d
\end{pmatrix}_L,
\begin{pmatrix}
    c \\
    s^\prime
\end{pmatrix}_L,
\begin{pmatrix}
    t \\
    b^\prime
\end{pmatrix}_L,
\]

while the leptons form doublets with their corresponding neutrino

\[
\begin{pmatrix}
    e \\
    \nu_e
\end{pmatrix}_L,
\begin{pmatrix}
    \mu \\
    \nu_\mu
\end{pmatrix}_L,
\begin{pmatrix}
    \tau \\
    \nu_\tau
\end{pmatrix}_L.
\]
Each fermion has a right handed counterpart that transforms as a singlet under this $SU(2)_L^1$. Focusing on the quark sector of the standard model, the weak interaction allows for charged current transitions between left handed up and down type quarks via the emission of a $W^+$ or $W^-$ boson. The mass eigenstates of the down type quarks, $(d, s, b)$, are not identical to the weak interaction eigenstates, $(d', s', b')$, but are related to them via the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\text{CKM}}$ [25]

$$
\begin{pmatrix}
d' \\
s' \\
b'
\end{pmatrix} = V_{\text{CKM}} \begin{pmatrix}
d \\
s \\
b
\end{pmatrix}.
$$

(1.3)

In terms of these mass eigenstates the charged current (CC) of the weak interactions is given by

$$
J_{\mu}^{\text{CC}} = \frac{e}{\sqrt{2} \sin \theta_W} (\bar{u}, \bar{c}, \bar{t}) \gamma_\mu L V_{\text{CKM}} \begin{pmatrix}
d \\
s \\
b
\end{pmatrix},
$$

(1.4)

where $L = (1 - \gamma_5)/2$ denotes the left-handed projection operator. The weak mixing angle $\sin^2 \theta_W = 0.23124 \pm 0.0002$ is a parameter of the standard model which is measured with high accuracy [26].

The CKM matrix is a unitary matrix, with the individual matrix elements conventionally written as

$$
V_{\text{CKM}} \equiv \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}.
$$

(1.5)

Each element in this matrix determines the strength of the various charged current transitions. Unitarity of this matrix can be used to relate several matrix elements, and it is well known that a $3 \times 3$ unitary matrix depends only on three real parameters and

---

1In its purest definition, the standard model contains no right handed neutrinos, which implies that neutrinos are massless. Recent results indicate [24] that neutrinos might have a small mass.
one complex phase. A useful parameterization of the CKM matrix has been proposed by Wolfenstein [27]

$$V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix},$$  \hspace{1cm} (1.6)

which is based on the observation that the diagonal elements in the CKM matrix, leading to CC transitions within one family, are of order unity. The more off diagonal the matrix elements are, the smaller their size, and the parameter $\lambda = 0.220 \pm 0.003$ [26] has been measured accurately.

Checking the consistency of the observed mixing matrix with the standard model predictions is one of the main goals in particle physics today. Unitarity of the CKM matrix gives rise to six relations between its matrix elements, $\sum_{j=1}^{3} V_{ij}V_{jk}^* = 0$, five of which are independent of one another. These relations can be represented graphically as triangles in the complex plane, so called unitarity triangles. The most commonly used unitarity triangle is obtained from the unitarity constraint

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$  \hspace{1cm} (1.7)

which is illustrated in Fig. 1.1. The $b$ system provides an excellent ground for measuring many aspects of this unitarity triangle [28]. The length of two of the sides can be
determined by the measurement of $|V_{ub}|$ [16, 17, 18] and $|V_{cb}|$ [19, 20]. A theoretically clean way to determine the angle $\beta$ is by measuring CP violation in the decay $B \to \psi K_s$ [29], and methods to determine the angles $\alpha$ [30] and $\gamma$ [31] from $B$ decays have been proposed in the literature.

1.2 Rare $B$ decays

The quark level transition $b \to c$ accounts for approximately 99% of $\bar{B}$ mesons decays. In this thesis we investigate decays with no charmed particles in the final state, which are strongly suppressed compared to $b \to c$ decay modes.

Decays due to the transition $b \to u$ are suppressed relative to $b \to c$ modes by the CKM factor $|V_{ub}/V_{cb}|^2 \approx 0.006$. An example is the exclusive mode $\bar{B} \to \rho^- \ell^+ \nu_\ell$, with a branching ratio of $(2.5^{+0.8}_{-1}) \times 10^{-4}$. A precise determination of the CKM matrix element $|V_{ub}|$ from such exclusive decay modes [16] is difficult, however, because it requires knowledge of the nonperturbative physics responsible for binding the quarks into the initial and final state hadrons. Currently, such measurements rely on quark models to predict the relevant form factors [32] and the uncertainty in $|V_{ub}|$ extracted this way is dominated by this model dependence. An alternative is to obtain $|V_{ub}|$ from the inclusive decay $\bar{B} \to X_u \ell^+ \bar{\nu}$ [17, 18], where $X_u$ denotes any hadronic final state of $B$ decays originating from a $b \to u$ quark level transition. As will be explained in Section 2.1, such inclusive decay modes can be calculated model independently with small theoretical uncertainties. Unfortunately, it is impossible to measure the fully inclusive $\bar{B} \to X_u \ell^+ \bar{\nu}$ decay rate due to the contamination from the decay $\bar{B} \to X_c \ell^+ \bar{\nu}$. However, since the $c$ quark is heavier than the $u$ quark, the phase space for the $b \to u$ decay mode is larger than that for the $b \to c$ mode, making it possible to measure $\bar{B} \to X_u \ell^+ \bar{\nu}$ in a corner of the available phase space. For most observables, this window of phase space is too small for the theoretical tools to apply [33, 34] and only recently a method to extract
model independently from the decay $\bar{B} \to X_s \ell^+ \bar{\nu}$ [35] has become available. This will be presented in Chapter 5.

A second class of rare decay modes is due to flavour changing neutral currents (FCNC). The standard model does not contain any FCNC’s at tree level. They arise at the one loop level and are therefore strongly suppressed. Physics beyond the standard model contains new particles and interactions that can give contributions to these decays comparable to those of the standard model [37, 38]. This makes FCNC decays sensitive to physics beyond the standard model, and precise standard model predictions of the observables are required. For the same reasons as before, only inclusive FCNC decays are calculable model independently and there are two decay modes, the radiative decay $\bar{B} \to X_s \gamma$ and the leptonic decay $\bar{B} \to X_s \ell^+ \ell^-$, that have received most of the attention in recent years [39, 40, 41, 42]. Recently, the radiative decay $\bar{B} \to K^* \gamma$ has been observed by the CLEO collaboration with a branching ratio of $(4.0 \pm 1.9) \times 10^{-5}$ [36]. The inclusive branching fraction for $\bar{B} \to X_s \gamma$ has also been measured [43] with the result $\mathcal{B}(\bar{B} \to X_s \gamma) = (2.3 \pm 0.7) \times 10^{-4}$, whereas the decay $\bar{B} \to X_s \ell^+ \ell^-$ is suppressed by an additional factor of the fine structure constant $\alpha \simeq 1/137$ and has not yet been observed. The current limit on the inclusive branching fraction is $\mathcal{B}(\bar{B} \to X_s \ell^+ \ell^-) < 4.2 \times 10^{-5}$ [44], with the standard model prediction of this branching fraction being an order of magnitude lower. Experimental cuts to eliminate the background from other $B$ decays are again required, restricting the available phase space considerably. The effects of these cuts on theoretical uncertainties have been calculated recently [45, 46, 47], and the results are presented in Chapters 3 and 4.

1.3 Organization of this thesis

In the next Chapter, we will introduce the tools required for the calculations presented in this thesis. First, we introduce the operator product expansion (OPE), which allows
us to calculate inclusive $B$ decays model independently. At leading order in the OPE this reproduces the parton model results, where quarks are treated as free particles. Nonperturbative effects are parameterized by matrix elements of higher dimensional operators, suppressed by inverse powers of $m_b$. Next we show how to use heavy quark effective theory (HQET) to express these matrix elements in terms of a few universal parameters. Finally, we introduce the effective Hamiltonian that mediates the FCNC transition $b \rightarrow s X$.

In Chapter 3 we study the inclusive radiative decay $\bar{B} \rightarrow X_s \gamma$. We extend previous studies of this decay which included nonperturbative corrections up to $\mathcal{O}(\Lambda_{QCD}/m_b)^2$, and calculate the $\mathcal{O}(\Lambda_{QCD}/m_b)^3$ contributions. Varying the unknown parameters arising at that order we estimate nonperturbative uncertainties in the standard model prediction of this decay. In addition to these nonperturbative uncertainties there are theoretical uncertainties due to experimental cuts required to eliminate the strong background from other $B$ decay modes. We derive model independent bounds on these uncertainties, in addition to estimating them using a simple model for the nonperturbative physics of the $B$ meson.

In Chapter 4 we study the decay $\bar{B} \rightarrow X_s \ell^+\ell^-$. We calculate $\mathcal{O}(\Lambda_{QCD}/m_b)^3$ corrections to the leptonic invariant mass spectrum, the forward-backward asymmetry and hadronic invariant mass moments and estimate the corresponding uncertainties. We also calculate how the size of these uncertainties depend on the value of an experimental cut that has to be applied to eliminate the large background from other decay modes.

A model independent way to determine the CKM matrix element $|V_{ub}|$ from the dilepton invariant mass spectrum of the inclusive decay $\bar{B} \rightarrow X_u \ell^+\bar{\nu}$ is presented in Chapter 5. We investigate the effect of the cuts required to discriminate against the overwhelming background from the decay $\bar{B} \rightarrow X_c \ell^+\bar{\nu}$ for various differential decay rates. We show that while these cuts cause the OPE to break down in the case of the electron energy and the hadronic invariant mass spectrum, this is not the case for the
dilepton invariant mass spectrum. We also discuss the utility of the $\bar{B} \rightarrow X_s\ell^+\ell^-$ decay rate above the $\psi(2S)$ resonance.

In Chapter 6 of this thesis we return to the the behaviour of the inclusive decay $\bar{B} \rightarrow X_s\gamma$ close to the endpoint of maximal photon energy, a very interesting problem in its own right. Not only does the nonperturbative expansion in inverse powers of the heavy quark mass break down, but the perturbative series contains a logarithmic enhancement in this endpoint region as well. These large endpoint logarithms, known as Sudakov logarithms, must be resummed to render the perturbative expansion well behaved. Solutions to both problems are known: the breakdown of the $1/m_b$ expansion is due to the fact that the energy of the $s$ quark jet is much greater than its invariant mass in the endpoint under consideration. This leads to a particular class of terms, suppressed by inverse powers of $m_b$ in the fully inclusive process, being enhanced by kinematic factors. These terms have to be summed into a nonperturbative structure function. The perturbative logarithms were summed using a factorization formalism developed many years ago. We present a novel effective theory that combines both of these approaches. The theory contains a nonlocal operator whose matrix element is the structure function discussed above. The Sudakov logarithms arise in this approach in matrix elements of this non-local operator and can be summed using standard techniques in field theory.
Chapter 2

Inclusive rare $B$ decays and effective theories

2.1 The operator product expansion

The strong interaction binds quarks into colour singlet particles called hadrons. Hadrons containing one quark and one anti-quark are called mesons, whereas hadrons with three quarks are called baryons\(^1\). Table 2.1 show the quark content and mass of some common hadrons. This strong interaction is too strong to be treated perturbatively. Hadronic decay rates to any particular final hadronic state are therefore notoriously difficult to calculate, even though the underlying parton level rate is easily determined. Such exclusive decays, in comparison with inclusive decays where one sums over all final hadronic states, require knowledge of matrix elements of the quark level transition between the initial and final hadronic states. These matrix elements are conventionally parameterized by several form factors, which depend on the nonperturbative structure of QCD. So far, the only way to determine these form factors from first principles is using lattice QCD [48].

\(^1\)When referring to the quark content of a hadron, we mean the content of valence quarks. The number of valence quarks determines the flavour quantum numbers of a hadron, a conserved quantity in QCD.
Table 2.1: The quark content and mass of a few selected hadrons.

<table>
<thead>
<tr>
<th>Hadron</th>
<th>Quark content</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$uud$</td>
<td>0.938</td>
</tr>
<tr>
<td>$n$</td>
<td>$udd$</td>
<td>0.939</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>$\bar{d}s$</td>
<td>0.498</td>
</tr>
<tr>
<td>$D$</td>
<td>$c\bar{u}$</td>
<td>1.865</td>
</tr>
<tr>
<td>$J/\psi$</td>
<td>$c\bar{c}$</td>
<td>3.097</td>
</tr>
<tr>
<td>$\bar{B}$</td>
<td>$b\bar{d}$</td>
<td>5.279</td>
</tr>
</tbody>
</table>

However, due to restrictions in computer resources, most of the required calculations can only be carried out in the quenched approximation, where fermion loops are neglected. In addition, the small Compton wavelength of the heavy quarks require a lattice spacing too small for current simulations. This introduces significant uncertainties into the calculations performed to date. Thus, these form factors are commonly obtained from model calculations [32]. For certain decays of hadrons containing a single heavy quark $Q$ into a hadron containing a different heavy quark $Q'$, all the form factors can be related to one single function, the Isgur-Wise function [49], with corrections being suppressed by powers of $\Lambda_{QCD}/m_Q$ and $\Lambda_{QCD}/m_{Q'}$. However, since in this thesis we are only interested in the decay of a $b$ quark to a light quark, these methods are not applicable.

Inclusive decays of hadrons, corresponding to the underlying quark level transition

$$b \to qP,$$  \hspace{1cm} (2.1)

where $P$ denotes any final state that does not interact strongly, are much easier to deal with theoretically [50, 51, 52, 53, 54]. Examples of such decays are the inclusive decay modes

$$\bar{B} \to X_{(u,c)}\ell^+\bar{\nu}_\ell, \quad \bar{B} \to X_s\gamma, \quad \bar{B} \to X_s\ell^+\ell^-,$$  \hspace{1cm} (2.2)

where $X_q$ denotes any final state mediated by the quark level transition (2.1). Focusing
on the decays of $B$ mesons, the Hamiltonian mediating the decay of a $b$ quark to a light quark $q$ can be factorized in the form

$$H = -\frac{4G_F}{\sqrt{2}} J_H^\mu J_P^\mu,$$

(2.3)

where $J_H^\mu$ is the hadronic current mediating the $b \to q$ transition and is of the form

$$J_H^\mu = \bar{q} \Gamma^\mu b,$$

(2.4)

with $\Gamma^\mu$ depending on the operator mediating the transition. The current coupling to the colour singlet particles, such as photons or leptons, is denoted by $J_L^\mu$. When considering inclusive decay modes, one sums over all possible final hadronic states. This allows us to write the differential decay rate as the product of a tensor $P_{\mu\nu}$ and a hadron tensor $W_{\mu\nu}$

$$d\Gamma = \frac{4G_F^2}{m_B} d\Pi P_{\mu\nu} W_{\mu\nu}.$$  

(2.5)

In this expression the symbol $d\Pi$ denotes the appropriate phase space differential and the two tensors are given by

$$P_{\mu\nu} = \langle 0| J_P^{\mu*}|P\rangle \langle P| J_P^\nu|0 \rangle$$

(2.6)

$$W_{\mu\nu} = (2\pi)^4 \delta^{(4)}(p_B - p_X - q) \sum_{X_\mu} \langle B| J_H^{\mu*}|X_\mu\rangle \langle X_\mu| J_H^\nu|B \rangle.$$  

(2.7)

The tensor $P_{\mu\nu}$ can easily be calculated at leading order in perturbation theory, with corrections suppressed by powers of the fine structure constant. The hadron tensor $W_{\mu\nu}$ depends on the Lorentz invariants $u \cdot q$ and $q^2$, where we have defined the four-velocity of the $B$ meson

$$u^\mu = \frac{p_B^\mu}{m_B}.$$  

(2.8)

$W_{\mu\nu}$ is related via the optical theorem to the discontinuity across a cut of the forward scattering amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = \left(-\frac{1}{\pi}\right) \text{Im} T_{\mu\nu}.$$  

(2.9)

Here

$$T_{\mu\nu} = -i \int d^4x e^{-i q \cdot x} \langle B | T\{J_\mu^*(x), J_\nu(0)\} | B \rangle.$$  

(2.10)
The analytic structure of the forward scattering amplitude $T^{\mu\nu}$ in the complex $\nu \cdot q$ plane at fixed $q^2$ contains branch cuts corresponding to the possible physical processes. The process $\bar{B} \rightarrow X_f P$ has the kinematics

$$ (p_B - q)^2 = p_X^2 \implies m_B^2 - 2m_B \nu \cdot q + q^2 = m_X^2, \quad (2.11) $$

therefore the corresponding cut of $T^{\mu\nu}$ exists for

$$ \nu \cdot q < \frac{m_B^2 + q^2 - m_{X_{\text{min}}}^2}{2m_B}. \quad (2.12) $$

In the above equation $m_X$ is the invariant mass of the final hadronic state with $m_{X_{\text{min}}}$ being the lowest possible invariant mass. There is a second physical process $\bar{B} + P \rightarrow X_{bb}$, with the final hadronic state containing two $b$ quarks. The kinematics for this process is given by

$$ (p_B + q)^2 = p_X^2 \implies m_B^2 + 2m_B \nu \cdot q + q^2 = m_X^2, \quad (2.13) $$

thus corresponding to a cut at

$$ \nu \cdot q > \frac{m_{X_{bb}}^2 - m_B^2 - q^2}{2m_B}. \quad (2.14) $$

This analytic structure is illustrated in Fig. 2.1. Since we are interested in decays of $B$ mesons, the hadron tensor is related to the discontinuity across the first cut which we will call the physical cut. The discontinuity can be calculated by integrating $T^{\mu\nu}$ along the path labelled $C_D$ in Fig. 2.1. By Cauchy's theorem, one can also obtain this discontinuity by integrating along the path $C$. Since this path stays away from the cuts, except for the region where it pinches the physical cut, the correlations

$$ T \{ J^+(x), J(0) \} \quad (2.15) $$

along this path are short range and can be expanded in terms of local operators. Taking the Fourier transform as in (2.10) we can thus write [50]

$$ -i \int d^4 x e^{-ixq} T \{ J^+(x), J(0) \} \sim \frac{1}{m_b} \left[ O_0 + \frac{1}{2m_b} O_1 + \frac{1}{4m_b^2} O_2 + \frac{1}{8m_b^3} O_3 + \ldots \right], \quad (2.16) $$
where $O_n$ represents a set of local operators of dimension $d = (3 + n)$. This expansion in terms of local operators is called operator product expansion (OPE). For a generic current $J_{\mu}^H$, the expressions for these operators are quite lengthy. The complete set of operators for $d \leq 5$ [39] and $d = 6$ [45] is given in Appendix A.1. In this thesis we include operators up to and including dimension $d = 6$. It is important to note that at the point where the contour pinches the physical cut the OPE is not valid. However, if this pinch is in a region far away from the endpoint of the physical cut the kinematic restrictions are on the production of high mass states ($m_X \gg m_{X_{\min}}$). It is expected that the spectrum obtained via the OPE can still be compared to the physical spectrum after smearing it over an adequate region. The size of the required smearing region can be estimated by the difference of the partonic and hadronic endpoints for the kinematic variables under consideration.

For transitions of a $b$ quark into a light quark, as we are considering in this work, the lowest possible masses of the final hadronic states corresponding to the two different cuts are approximately

$$m_{X_{\min}} \approx 0 \quad m_{X_{\min}}^{\text{unphysical}} \approx 2m_B.$$  \hspace{1cm} (2.17)
With this, the two cuts can be written as
\[ v \cdot q < \frac{m_B^2 + q^2}{2m_B}, \quad v \cdot q > \frac{3m_B^2 - q^2}{2m_B}. \] (2.18)

Thus, as \( q^2 \) approaches the endpoint of the physical phase space for \( B \) decays \( q^2 \to m_B^2 \), the gap between the two cuts goes to zero. This implies that the contour can not be deformed away from the endpoint of the physical cut and one thus expects the results obtained via the OPE to fail for \( q^2 \) too close to \( m_B^2 \) [52]. This will be seen in more detail in later sections.

It is convenient to decompose the tensor \( T^{\mu\nu} \) into the general tensor structure consistent with the symmetries. It has to be formed out of the four-vectors \( u' \) and \( q' \), the metric tensor \( g^{\mu\nu} \) and the totally antisymmetric Levi-Civita tensor \( \varepsilon^{\mu\nu\rho\sigma} \):
\[
T^{\mu\nu} = -T_1(v \cdot q, q^2)g^{\mu\nu} + T_2(v \cdot q, q^2)v^\mu v^\nu - T_3(v \cdot q, q^2)i\varepsilon^{\mu\nu\rho\sigma}v_\alpha q_\beta \\
+ T_4(v \cdot q, q^2)q^\mu q^\nu + T_5(v \cdot q, q^2)(v^\mu q^\nu + q^\mu v^\nu).
\] (2.19)

The differential decay rate (2.5) can then be written in terms of the discontinuities across the form factors \( T_i(v \cdot q, q^2) \), with the exact expression depending on the currents \( J_H^\mu \) and \( J_P^\mu \).

### 2.2 Higher dimensional operators and their forward matrix elements

It is clear from (2.10) and (2.16) that to calculate the forward scattering tensor we have to determine matrix elements of the higher dimensional operators \( O_n \). These matrix elements can be parameterized within the framework of heavy quark effective theory (HQET) [49, 55, 56]. HQET uses the fact that the typical momentum of the light degrees of freedom is of order \( \Lambda_{\text{QCD}} \sim 300 \text{ MeV} \). Since the Compton wave length of the heavy quark is much smaller than this momentum scale, the light degrees of freedom are
insensitive to the mass and the spin configuration of the heavy quark. Thus, the required
matrix elements do not depend on the mass and the spin of the heavy quark.

To construct HQET, one splits the momentum of the heavy quark $p_{\mu}$ into a large
component $P_{\mu} \sim m_b$ and a small component $k_{\mu} \sim \Lambda_{\text{QCD}}$

$$p_{\mu} = P_{\mu} + k_{\mu}. \quad (2.20)$$

The large component is unchanged by the interactions with soft gluons and can therefore
be written as

$$P_{\mu} = m_b v_{\mu}, \quad (2.21)$$

whereas $k_{\mu}$ is the residual momentum of the effective heavy quark field. Next, the spinors
of the low energy theory are defined. Since they should only describe heavy particles,
not antiparticles, they are two component spinors. The Dirac equation implies

$$(m_b \not{\!p} - m_b) h_v = 0 \implies \frac{1 + \not{\!p}}{2} h_v = h_v. \quad (2.22)$$

Thus, the projection operator $P_+ = 1/2(1 + \not{\!p})$ projects onto the spinors $h_v$ of the low
energy theory. With these ingredients one can write down the most general Lagrangian
describing the interaction of a heavy $b$ quark with soft gluons

$$\mathcal{L}_{\text{eff}} = \bar{h}_v (i \not{\!v} \cdot \not{D}) h_v + \frac{c_k}{2m_b} \bar{h}_v (i \not{D})^2 h_v + \frac{c_m}{2m_b} \bar{h}_v i \sigma_{\mu\nu} h_v G^{\mu\nu} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_b^2}, \alpha_s\right). \quad (2.23)$$

The coefficients of the higher dimensional operators $c_k$ and $c_m$ can be determined by
matching amplitudes in the full and effective theory, which leads at tree level to

$$c_k = 1; \quad c_m = \frac{1}{2}. \quad (2.24)$$

Finally, the physical $B$ meson states $|B(v)\rangle$ can be expanded in terms of the infinite
mass states of HQET [57]. This expansion can be performed by using the Gell-Mann-
Low theorem (see e.g. [58]), which implies that at first order in $\Lambda_{\text{QCD}}/m_b$

$$|B(v)\rangle = \left\{ 1 + i \int d^3x \int_{-\infty}^{0} dt \mathcal{L}_1(x) - \frac{1}{V} \left\langle H(v) \left| i \int d^3x \int_{-\infty}^{0} dt \mathcal{L}_1(x) \right| H(v) \right\rangle \right\} |H(v)\rangle, \quad (2.25)$$
where \( V \) is the normalization volume and \( \mathcal{L}_f \) contains only the terms of the HQET Lagrangian (2.23) suppressed by powers of \( m_b \). Using these definitions, we can now proceed to parameterize the matrix elements of higher dimensional operators appearing in the OPE (2.16).

We will first consider operators of the form
\[
O^{(\Gamma)}_{\mu_1, \mu_2, \ldots, \mu_n} = \bar{h}_v (iD_{\mu_1}) (iD_{\mu_2}) \cdots (iD_{\mu_n}) \Gamma h_v,
\]
where \( \Gamma \) is an arbitrary Dirac matrix. A general Dirac matrix can be expanded in the basis set of 16 independent Dirac matrices, \( I, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5 \) and \( \sigma_{\mu\nu} \). Since in operators of the form (2.26) the arbitrary Dirac structure \( \Gamma \) is sandwiched between the projection operators
\[
P_+ = \frac{(1 + \gamma_5)}{2},
\]
it can be shown that the Dirac structure in the operator (2.26) can be expanded in only four independent matrices \( I \) and \( s_\mu \) [59], where \( s_\mu \) satisfies the relations
\[
s_\mu s_\nu = (-g_{\mu\nu} + v_\mu v_\nu)P_+ + i \epsilon_{\alpha\mu\nu\beta} v^\alpha s^\beta, \quad \text{with} \quad v \cdot s = 0.
\]
Sandwiching the sixteen fundamental Dirac matrices between the projection operator \( P_+ \), one can relate them to these four matrices via
\[
1 \rightarrow P_+, \quad \gamma_\mu \rightarrow P_+ \gamma_\mu P_+ = v_\mu P_+,
\]
\[
\gamma_\mu \gamma_5 \rightarrow P_+ \gamma_\mu \gamma_5 P_+ = s_\mu, \quad \gamma_5 \rightarrow P_+ \gamma_5 P_+ = 0,
\]
\[
\sigma_{\mu\nu} \rightarrow P_+ \sigma_{\mu\nu} P_+ = v^\alpha \epsilon_{\alpha\mu\nu\beta} s^\beta.
\]
A general formula to sandwich any Dirac structure between \( P_+ \) is
\[
P_+ \Gamma P_+ = \frac{1}{2} P_+ \text{Tr} \{ P_+ \Gamma \} - \frac{1}{2} s_\mu \text{Tr} \{ s_\mu \Gamma \}.
\]

These simplifications of the Dirac structure in heavy quark operators allow us to consider only the two operators
\[
O^{(1)}_{\mu_1, \mu_2, \ldots, \mu_n} = \bar{h}_v (iD_{\mu_1}) (iD_{\mu_2}) \cdots (iD_{\mu_n}) h_v
\]
\[
O^{(s)}_{\mu_1, \mu_2, \ldots, \mu_n, \lambda} = \bar{h}_v (iD_{\mu_1}) (iD_{\mu_2}) \cdots (iD_{\mu_n}) s_\lambda h_v,
\]
being even and odd under parity, respectively. To obtain matrix elements of these two operators, we write down the most general tensor structure consistent with the symmetries. This tensor has to be constructed from the four-velocity $v^\mu$, the metric tensor $g^{\mu\nu}$ and the Levi-Civita tensor $\varepsilon_{\mu\nu\alpha\beta}$, the only available tensors after taking the matrix elements.

For $d = 3$ we find

$$
\langle H(v)|\bar{h}_v h_v|H(v)\rangle = 2m_b \tag{2.34}
$$

$$
\langle H(v)|\bar{h}_v s_\lambda h_v|H(v)\rangle = 0, \tag{2.35}
$$

where the value of the first matrix element is determined by the normalization of the states. In fact, because the vector current in full QCD is a symmetry current, its matrix element between the spinors of full QCD is

$$
\langle B(v)|\bar{b}\gamma^\mu b|B(v)\rangle = 2p_B^\mu = 2m_B v^\mu. \tag{2.36}
$$

For $d = 4$, the matrix element of the parity odd operator vanishes trivially, whereas the matrix element of the parity even operator can be written as

$$
\langle H(v)|\bar{h}_v (iD_\mu) h_v|H(v)\rangle = A v_\mu. \tag{2.37}
$$

Contracting both sides with the four-velocity $v^\mu$, one finds

$$
A = \langle H(v)|\bar{h}_v (i v \cdot D) h_v|H(v)\rangle. \tag{2.38}
$$

The equation of motion for the heavy quark field implies

$$
\bar{h}_v (i v \cdot D) h_v = -\frac{1}{2m_b}\bar{h}_v (iD)^2 h_v - \frac{1}{4m_b}\bar{h}_v i\sigma_{\mu\nu} h_v G^{\mu\nu} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_b^2}, \alpha_s\right). \tag{2.39}
$$

Thus, to leading order in the $1/m_b$ expansion this matrix element vanishes by the HQET equation of motion [50, 60] and higher order corrections can be systematically incorporated [39] by using Eq. (2.39).
For matrix elements of dimension five or greater, we can use parity to write down the most general tensor possible. For the parity even operator we find

$$\langle H(v)|O^{(1)}_{\alpha,\nu_1,\ldots,\nu_{n-2},\beta}|H(v)\rangle = 2m_B \left[ g_{\alpha\beta} - v_\alpha v_\beta \right] A_{\nu_1,\ldots,\nu_{n-2}},$$  \hspace{1cm} (2.40)

and for the parity odd operator

$$\langle H(v)|O^{(s)}_{\alpha,\nu_1,\ldots,\nu_{n-2},\beta,\lambda}|H(v)\rangle = 2m_B d_H \left\{ i\varepsilon_{\rho\alpha\beta\lambda}\nu^\rho B_{\nu_1,\ldots,\nu_{n-2}} + [g_{\alpha\beta} - v_\alpha v_\beta] C^{(1)}_{\nu_1,\ldots,\nu_{n-2};\lambda} \right.$$  
$$+ [g_{\alpha\lambda} - v_\alpha v_\lambda] C^{(2)}_{\nu_1,\ldots,\nu_{n-2};\beta} + [g_{\beta\lambda} - v_\beta v_\lambda] C^{(3)}_{\nu_1,\ldots,\nu_{n-2};\alpha} \right\},$$  \hspace{1cm} (2.44)

where $d_H = 3$ for pseudoscalar mesons and $d_H = -1$ for vector mesons. The tensors $A$, $B$ and $C^{(i)}$ are composed of the metric tensor and the four-velocity. The tensors $C^{(i)}$ are odd under parity and vanish upon contracting the last index with the four-velocity. For operators of arbitrary dimension these tensors are quite complicated, but for $d = 5$ and $d = 6$ the most general tensor can easily be written down. For $d = 5$ the tensors $A$ and $B$ are just scalars and the $C^{(i)}$ are zero because of parity. Conventionally the parameters $\lambda_1$ and $\lambda_2$ are used for these two scalars by defining [61, 62]

$$\langle H(v)|\bar{h}_v(iD_\alpha)(iD_\beta)h_v|H(v)\rangle = \frac{2m_B}{3} [g_{\alpha\beta} - v_\alpha v_\beta] \lambda_1,$$  \hspace{1cm} (2.45)

$$\langle H(v)|\bar{h}_v(iD_\alpha)(iD_\beta)\delta_\lambda h_v|H(v)\rangle = 2m_B \frac{d_H}{6} i\varepsilon_{\rho\alpha\delta\lambda}\nu^\rho \lambda_2.$$  \hspace{1cm} (2.46)

With this definition the matrix element of a general dimension five operator can be written as

$$\langle B(v)|\bar{h}_v\Gamma iD_\alpha iD_\beta h_v|B(v)\rangle = \frac{m_B}{3} \text{Tr} \left\{ \Gamma P_+ \left( \lambda_1 (g_{\alpha\beta} - v_\alpha v_\beta) + \frac{d_H}{2} \lambda_2 i\sigma_{\alpha\beta} \right) P_+ \right\},$$  \hspace{1cm} (2.47)

where $P_+$ was defined in (2.22), and $\Gamma$ is an arbitrary Dirac structure.

For dimension six operators all the tensors have only one free index and can therefore only be proportional to the four-velocity. The $C^{(i)}$ tensors vanish when contracted with
the four-velocity, thus parity requires these tensors to be zero again. The two matrix elements can thus be written as [57, 59]

\[ \langle H(v)|\tilde h_v(iD_\alpha)(iD_\mu)(iD_\beta)h_v|H(v)\rangle = \frac{2m_B}{3} [g_{\alpha\beta} - v_\alpha v_\beta] v_\mu \rho_1 \]

\[ \langle H(v)|\tilde h_v(iD_\alpha)(iD_\mu)(iD_\beta)s_\lambda h_v|H(v)\rangle = 2m_B \frac{d_H}{6} i \varepsilon_{\rho\alpha\beta\lambda} v^\rho v_\mu \rho_2 . \]

This leads to

\[ \langle H(v)|\tilde h_v(iD_\alpha)(iD_\mu)(iD_\beta)\Gamma h_v|H(v)\rangle = \frac{m_B}{3} \text{Tr} \left\{ \Gamma P_+ \left( \rho_1 (g_{\alpha\beta} - v_\alpha v_\beta) + \frac{d_H}{2} \rho_2 i \sigma_{\alpha\beta} \right) P_+ \right\} v_\mu . \]

Starting at order \( \Lambda_{\overline{\text{QCD}}}/m_b^3 \), matrix elements of time-ordered products are required as well [57], which are due to the expansion of the physical B meson state in terms of the HQET states (2.25). Introducing the notation

\[ \frac{1}{2m_B} \left\langle H(v)|\tilde h_v(iD)^2 h_v i \int d^3x \int_{-\infty}^0 dt \mathcal{L}_I(x)|H(v)\right\rangle = \frac{T_1 + d_H T_2}{m_b} \]

\[ \frac{1}{2m_B} \left\langle H(v)|\tilde h_v \frac{1}{2} (-i \sigma_{\mu\nu})G^{\mu\nu} h_v i \int d^3x \int_{-\infty}^0 dt \mathcal{L}_I(x)|H(v)\right\rangle = \frac{T_3 + d_H T_4}{m_b} , \]

these new nonlocal matrix elements can be absorbed by redefining the dimension five matrix elements according to

\[ \lambda_1 \rightarrow \lambda_1 + \frac{T_1 + d_H T_2}{m_b} \]

\[ \lambda_2 \rightarrow \lambda_2 + \frac{T_3 + d_H T_4}{3m_b} . \]

Four-fermion operators have dimension six or greater, thus they contribute to the OPE (2.16) at \( \mathcal{O}(\Lambda_{\overline{\text{QCD}}}/m_b)^3 \) as well [63, 64]. In this work, only one combination of four-fermion operators is relevant, whose matrix element we define as \( f_1 \)

\[ \left\langle H_v \right| \frac{16\pi^2}{m_b^3} \tilde h_v \gamma^\mu L \bar{s} \bar{s} \gamma^\nu L h_v (g_{\mu\nu} - v_\mu v_\nu) \left| H_v \right\rangle = f_1 . \]

The prefactors in this operator are chosen such that the size of \( f_1 \) is comparable to the size of matrix elements of the other dimension six operators.
There is one more parameter in addition to these matrix elements, the mass of the heavy quark. Conventionally one re-expresses this quark mass in terms of the corresponding meson mass

\[
m_b = m_H - \bar{\Lambda} + \frac{\lambda_1 + d_H \lambda_2}{2m_b} - \frac{\rho_1 + d_H \rho_2}{4m_b^2} + \frac{T_1 + T_3 + d_H(T_2 + T_4)}{4m_b^2} + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right)^3,
\]

where \( d_H = 3 \) for pseudoscalar mesons \( m_H = (m_B, m_D) \) and \( d_H = -1 \) for vector mesons \( m_H = (m_{B^*}, m_{D^*}) \). This introduces the parameter \( \bar{\Lambda} \), which characterizes the energy of the light degrees of freedom inside the heavy meson.

Although the rough size of the parameters introduced in this Section can be obtained from naive dimensional analysis

\[
\begin{align*}
\bar{\Lambda} &\sim \Lambda_{QCD} \\
\lambda_i &\sim \Lambda_{QCD}^2 \\
\rho_i \sim T_i \sim f_i &\sim \Lambda_{QCD}^3,
\end{align*}
\]

their exact values are a priori unknown and have to be determined from experiment. Some information can be obtained from the mass splitting between the pseudoscalar and vector \( B^{(*)} \) and \( D^{(*)} \) mesons. Up to \( \mathcal{O}(\Lambda_{QCD}/m_b) \) corrections, the matrix element \( \lambda_2 \) is directly proportional to the mass splitting of the heavy mesons

\[
\lambda_2 \approx \frac{m_{B^*}^2 - m_B^2}{4} \approx \frac{m_{D^*}^2 - m_D^2}{4} \approx 0.12 \text{ GeV}^2.
\]

Combining the mass splitting of the \( B \) mesons with that of the \( D \) mesons, a relation between the matrix elements of the local operators \( \rho_i \) and the nonlocal operators \( T_i \) can be obtained [57]

\[
\rho_2 - T_2 - T_4 = \frac{\left(\frac{a_S(m_c)}{a_S(m_b)}\right)^{3/30} m_B^2 \Delta m_B (m_D + \bar{\Lambda}) - m_D^2 \Delta m_D (m_B + \bar{\Lambda})}{m_B + \bar{\Lambda} - \left(\frac{a_S(m_c)}{a_S(m_b)}\right)^{3/30} (m_D + \bar{\Lambda})},
\]

where \( \Delta m_H = m_{H^*} - m_H \). For the other matrix elements no such simple relations exist and a considerable amount of effort has gone into extracting the parameters \( \bar{\Lambda} \) and \( \lambda_1 \) from experiment. One possibility for measuring these parameters is using the rare decay \( \bar{B} \to X_s \gamma \) and will be presented in the next Chapter.
2.3 The $b \to s$ effective Hamiltonian

In the standard model, FCNC $B$ decays corresponding to the quark level transition $b \to sX$ occur via penguin and box diagrams with virtual electroweak bosons and up-type quarks in the loops. Examples of such graphs are shown in Fig. 2.2. If all up type quarks had identical masses, the only difference between graphs with different up type quarks would be the coupling to the $W$ boson via the CKM matrix elements. The sum of the three possible diagrams would be proportional to

$$V_{tb}^* V_{ts} + V_{cb}^* V_{cs} + V_{ub}^* V_{us} = 0, \quad (2.59)$$

which vanishes because of the unitarity of the CKM matrix. This vanishing of the sum of the three graphs is known as the GIM mechanism [65]. The result of these graphs is therefore proportional to the mass difference of the up-type quarks. Since the top quark is so much heavier than the up and the charm quark, the graphs with the top quark in the loop dominate the process.

Diagrams such as those shown in Fig. 2.2 contain logarithmically enhanced terms of the form $(\alpha_s(m_W) \log (m_b/m_W))^n$. Since $(\alpha_s(m_W) \log (m_b/m_W)) \sim 0.35$, these leading logarithmic terms reduce the convergence of the perturbative series and should therefore be resummed. This resummation can be performed by using renormalization group equations (RGE). This is most easily done in an effective theory where the top quark
and the weak bosons have been integrated out of the theory. The most general effective Hamiltonian mediating the quark level \( b \rightarrow sX \) transition at energies far below the mass of the \( W \) boson contains ten operators [66]

\[
\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts} \sum_{i=1}^{10} C_i(\mu) O_i(\mu),
\]

where

\[
\begin{align*}
O_1 &= (\bar{s}_L \gamma_\mu b_L)(\bar{c}_L \gamma^\mu c_L), \\
O_2 &= (\bar{s}_L \gamma_\mu b_L)(\bar{c}_L \gamma^\mu c_L), \\
O_3 &= (\bar{s}_L \gamma_\mu b_L) \sum_{q=u,d,s,c,b} (\bar{q}_L \gamma^\mu q_L), \\
O_4 &= (\bar{s}_L \gamma_\mu b_L) \sum_{q=u,d,s,c,b} (\bar{q}_L \gamma^\mu q_L), \\
O_5 &= (\bar{s}_L \gamma_\mu b_L) \sum_{q=u,d,s,c,b} (\bar{q}_R \gamma^\mu q_R), \\
O_6 &= (\bar{s}_L \gamma_\mu b_L) \sum_{q=u,d,s,c,b} (\bar{q}_R \gamma^\mu q_R), \\
O_7 &= \frac{e}{16\pi^2} \bar{s}_a \sigma_{\mu\nu}(m_b R + m_s L) b_\alpha F_{\mu\nu}, \\
O_8 &= \frac{g}{16\pi^2} \bar{s}_a T^{a}_{\alpha\beta} \sigma_{\mu\nu}(m_b R + m_s L) b_\beta G^{a\mu\nu}, \\
O_9 &= \frac{e^2}{16\pi^2} \bar{s}_a \gamma^\mu L b_\alpha \bar{\tau}_\mu, \\
O_{10} &= \frac{e^2}{16\pi^2} \bar{s}_a \gamma^\mu L b_\alpha \bar{\gamma}_\mu g_{5}. 
\end{align*}
\]

Here \( L/R = \frac{1}{2} (1 \pm \gamma^5) \) are the usual left and right handed chiral projection operators and \( \alpha \) and \( \beta \) are \( SU(3) \) colour indices. The top quark and the weak bosons are integrated out at the same scale. This makes the matching onto the effective theory considerably easier, with the effect of introducing logarithmic dependence on the ratios \( m_W/m_t \) in the matching coefficients. Since these two scales only differ roughly by a factor of two, a summation of these logarithms is not required. Performing the matching calculation at the scale \( \mu = m_W \) results in the Wilson coefficients [67, 68]

\[
C_2(m_W) = 1
\]
\[ C_7(m_W) = -\frac{1}{2}A(x) \]

\[ C_8(m_W) = -\frac{1}{2}B(x) \]

\[ C_9(m_W) = \frac{\alpha}{2\pi} \left[ \frac{4}{9} + \frac{C(x)}{\sin^2 \Theta_W} - D(x) \right] \]

\[ C_{10}(m_W) = -\frac{\alpha}{2\pi \sin^2 \Theta_W}, \quad (2.63) \]

with

\[ A(x) = \frac{x(7 - 5x - 8x^2)}{12(1 - x)^3} + \frac{x^2(2 - 3x)}{2(1 - x)^4} \log x \]

\[ B(x) = \frac{x(2 + 5x - x^2)}{4(1 - x)^3} + \frac{3x^2}{2(1 - x)^4} \log x \]

\[ C(x) = \frac{x(4 - x)}{8(1 - x)} + \frac{3x^2}{8(1 - x)^2} \log x \]

\[ D(x) = \frac{x(6 - x)}{2(1 - x)} + \frac{x(2 + 3x)}{2(1 - x)^2} \log x - \frac{x^2(25 - 19x)}{36(1 - x)^3} \]

\[ -\frac{x^2(6 + 2x - 5x^2)}{18(1 - x)^4} \log x - \frac{4}{9} \log x \quad (2.64) \]

where \( x = m_W/m_t \) and all other coefficients are zero. In the effective theory the only remaining relevant scale in loops is the mass of the \( b \) quark, leading to logarithms of the form \( \log (\mu/m_b) \), where \( \mu \) is the renormalization scale. Running the operators from \( \mu = m_W \) to \( \mu = m_b \) using the RGE’s eliminates this scale dependence and sums the logarithms of the form \( \alpha_s^n(m_b) \log^m (m_b/m_W) \) in the coefficient functions. This running has been calculated in the next to leading logarithmic approximation [67, 68], for which the two exponents satisfy \( n = m + 1 \), and the resulting Wilson coefficients are given in Table 2.3. Notice that operators with vanishing matching coefficient at \( \mu = m_W \)

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_{7\text{eff}} )</th>
<th>( C_9 )</th>
<th>( C_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.240</td>
<td>1.103</td>
<td>0.011</td>
<td>-0.025</td>
<td>0.007</td>
<td>-0.030</td>
<td>-0.311</td>
<td>4.153</td>
<td>-4.546</td>
</tr>
</tbody>
</table>

Table 2.2: The Wilson coefficients \( C_i(m_b) \) in the next-to-leading log approximation.

According to convention we give the value for \( C_{7\text{eff}} = C_7 - C_5/3 - C_6 \).
contribute at the scale $\mu = m_b$. It is a general feature that all operators of a given dimension that are not forbidden by any symmetries get induced by the RGE, even if their coefficient at the matching scale is zero.

This effective Hamiltonian now serves as the starting point for calculating differential decay distributions of the inclusive processes $\bar{B} \rightarrow X_s \gamma$ and $\bar{B} \rightarrow X_s \ell^+ \ell^-$. 
Chapter 3

The decay $\bar{B} \to X_s \gamma$

3.1 Introduction

In this Chapter we will investigate moments of the photon energy spectrum of the inclusive FCNC decay $\bar{B} \to X_s \gamma$. We will find that the first moment and the variance of the photon energy spectrum are sensitive to the universal parameters $\lambda_1$ and $\bar{\Lambda}$ and can thus be used to measure these parameters [69]. A precise knowledge of these parameters is crucial for pushing the precision of heavy quark physics, since many observables, such as $|V_{cb}|$, depend on these parameters. Using the tools developed in Chapter 2 we calculate the $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3$ nonperturbative contributions to photon energy moments of this decay [45]. As mentioned in Section 2.2 the size of the matrix elements arising at this order are unknown. This leads to uncertainties in the theoretical predictions which we estimate by varying the size of the dimension six matrix elements in the range expected from dimensional analysis (2.56). Experimentally, a cut is required to reject large backgrounds, and the CLEO collaboration only included photons with energy above 2.2 GeV in its original analysis [43]. This restricts the phase space considerably, causing the OPE used to calculate these moments to break down. We estimate the resulting theoretical uncertainties using a model independent approach as well as using a very simple model.
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Figure 3.1: The Feynman diagrams contributing to the parton level decay $b \rightarrow s\gamma$ at one loop. The square represents the operator $O_7$.

The starting point of the calculation is the $b \rightarrow sX$ effective Hamiltonian presented in (2.60). At leading order in the strong coupling constant $\alpha_s$, the dominant operator contributing to the $b \rightarrow s\gamma$ transition is the operator $O_7$. The decay at the parton level can be calculated perturbatively, with the Feynman diagrams to order $\alpha_s$, shown in Fig. 3.1. The square in these diagrams represents the $b \rightarrow s$ transition operator obtained from $O_7$

$$\bar{s}\sigma_{\mu\nu} \frac{1 + \gamma_5}{2} bq_{\nu}. \quad (3.1)$$

Defining $x = 2E_\gamma/m_b$ in the rest frame of the $b$ quark, the differential decay rate at one loop is [69]

$$\frac{d\Gamma}{dx} = \Gamma_0 \left\{ \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( 5 + \frac{4}{3} \pi^2 \right) \right] \delta(1 - x) \\
+ \frac{\alpha_s C_F}{4\pi} \left[ 7 + x - 2x^2 - 2(1 + x) \log(1 - x) \\
- \left( 4 \log(1 - x) \right) \frac{1}{1 - x} + \frac{7}{1 - x} \right] \right\}. \quad (3.2)$$

Here

$$\Gamma_0 = \frac{G_F^2 |V_{tb} V_{ts}^*|^2 \alpha |C_7^{\text{eff}}|^2}{32\pi^4} m_b^5, \quad (3.3)$$

and the subscript "+" denotes the usual plus distribution,

$$\frac{1}{(1 - x)_+} \equiv \lim_{\beta \rightarrow 0} \left\{ \frac{1}{1 - x} \theta(1 - x - \beta) + \log(\beta) \delta(1 - x - \beta) \right\}$$
In accordance with convention, we have defined the effective Wilson coefficient

\[ C_7^{\text{eff}} = C_7 - \frac{C_5}{3} - C_6. \]  

(3.5)

Nonperturbative contributions to this inclusive decay can be calculated using the tools explained in the Section 2.1. With the tensor \( P_{\mu\nu} \) for a radiative decay given by

\[ P_{\mu\nu} = \langle 0|\epsilon_\mu|q\rangle \langle q|\epsilon^\dagger_\nu|0\rangle = -g_{\mu\nu}, \]  

(3.6)

we can use Eqs. (2.5), (2.9) and (2.19) to write the differential decay rate as

\[ \frac{d\Gamma}{dx} = \frac{\Gamma_0}{m_B} \frac{1-x}{2} \text{Im} \{4T_1(x) - T_2(x) - 2T_5(x)\}, \]  

(3.7)

with \( \Gamma_0 \) given in (3.3). Using the general expressions for the operators given in Appendix A.1. together with the Dirac structure (see Appendix A.1) corresponding to the operator \( O_7 \).

\[ \Gamma_1 = \frac{e}{8\pi^2} m_b \sigma^{\mu\nu} \frac{1-\gamma_5}{2} q^\nu, \quad \Gamma_2 = \frac{e}{8\pi^2} m_b \sigma^{\mu\nu} \frac{1+\gamma_5}{2} q^\nu, \]  

(3.8)

the form factors \( T_i(x) \) can be calculated and are given in Appendix A.2.

Inserting these expressions up to \( \mathcal{O}(\Lambda_{\text{QCD}}/m_b)^2 \) into (3.7) we find [39]

\[ \frac{d\Gamma}{dx} = \Gamma_0 \left( 1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2} \right) \left[ \delta(1-x) - \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \delta'(1-x) - \frac{\lambda_1}{6m_b^2} \delta''(1-x) \right]. \]  

(3.9)

Notice that, apart from the radiative corrections (3.2), this decay receives contributions only at the endpoint of the photon energy spectrum \( x = 1 \). This is easy to understand, since the final state contains only two particles. In the rest frame of the \( b \) quark, the photon and the \( s \) quark recoil back to back, both with energy \( E = m_b/2 \). This ensures a contribution only at \( x = 1 \). Nonperturbative corrections give rise to derivatives of delta functions, also evaluated at the endpoint. To compare this singular spectrum with the experimentally measured spectrum, it has to be smeared with a weight function, as explained in Section 2.1.
We can calculate moments of the photon energy spectrum by choosing this weight function to be of the form $x^n$. For $n = 0$ this yields the total decay rate

$$\Gamma = \Gamma_0 \left(1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2}\right),$$

(3.10)

and $n = 1$ and $n = 2$ result in the first and second moment of the photon energy spectrum, respectively [69]

$$\langle E_\gamma \rangle = \frac{m_b}{2} \left(1 - \frac{\lambda_1 + 3\lambda_2}{2m_b^2}\right) = \frac{m_B - \bar{\Lambda}}{2}$$

(3.11)

$$\langle E_\gamma^2 \rangle = \frac{m_b^2}{4} \left(1 - \frac{4\lambda_1 + 9\lambda_2}{3m_b^2}\right).$$

(3.12)

From these two moments it is easy to calculate the variance of the photon energy spectrum with the result

$$\text{var}(E_\gamma) = \langle E_\gamma^2 \rangle - \langle E_\gamma \rangle^2 = -\frac{\lambda_1}{12}.$$  

(3.13)

Corrections to the parton model results are suppressed by two powers of the $b$ quark mass [50], as is expected from the fact that all matrix elements of dimension four vanish.

Knowledge of these leading nonperturbative corrections, in combination with the perturbative corrections given in (3.2), gives a standard model prediction of the rate of this decay model independently at the 1% level. This is much better than current experimental uncertainties. A reduction of the experimental errors, which should be possible with the upgrade at CLEO and the $B$ factories at SLAC and KEK, will therefore strongly enhance the sensitivity to new physics contributing to this rare decay mode.

The first moment (3.11) and the variance (3.13) are directly sensitive on the HQET parameters $\bar{\Lambda}$ and $\lambda_1$, respectively. New physics only affects the value of the Wilson coefficient $C_7^{\text{eff}}$ and cancels in these moments [38]. Thus, the moments of the photon spectrum are insensitive to physics beyond the standard model and can be used to measure these HQET parameters [69]. A similar approach, using moments of the semileptonic decay $\bar{B} \rightarrow X_c e\bar{\nu}$ to measure these HQET parameters has been proposed in [62, 70]. The $O(\Lambda_{QCD}/m_b)^3$ corrections to these semileptonic moments have been calculated in [57, 71].
Since the matrix elements of the dimension six operators are unknown, their contributions result in uncertainties in the standard model predictions. Such uncertainties have been estimated for moments of the semileptonic decay $\bar{B} \rightarrow X_c\ell\bar{\nu}$ [57, 71], and we perform a similar analysis for the decay $\bar{B} \rightarrow X_s\gamma$.

### 3.2 Higher Order nonperturbative corrections

To calculate the contributions from dimension six operators, suppressed by three powers of the $b$ quark mass, we will repeat the steps of the last subsection, but now including the $(\Lambda_{QCD}/m_b)^3$ contributions to the form factors given in Appendix A.2. We find [45]

\[
\frac{d\Gamma}{dx} = \Gamma [\delta(1-x) + A_1\delta'(1-x) + A_2\delta''(1-x) + A_3\delta'''(1-x)],
\] (3.14)

with

\[
\begin{align*}
\Gamma &= \Gamma_0 \left(1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2} - \frac{11\rho_1 - 27\rho_2}{6m_b^3} + \frac{T_1 + 3T_2 - 3(T_3 + 3T_4)}{2m_b^3}\right) \quad (3.15) \\
A_1 &= -\frac{\lambda_1 + 3\lambda_2}{2m_b^2} - \frac{5\rho_1 - 21\rho_2}{6m_b^3} - \frac{T_1 + 3T_2 + T_3 + 3T_4}{2m_b^3} \quad (3.16) \\
A_2 &= -\frac{\rho_1}{6m_b^3} - \frac{T_1 + 3T_2}{6m_b^3} \\
A_3 &= -\frac{\rho_1}{18m_b^3}.
\end{align*}
\]

Choosing the weighting function to be $x^n$, ($n = 0, 1, 2$) as before we obtain the first few moments, now including $\Lambda_{QCD}/m_b^3$ contributions:

\[
\begin{align*}
\Gamma &= \Gamma_0 \left(1 + \frac{\lambda_1 - 9\lambda_2}{2m_b^2} - \frac{11\rho_1 - 27\rho_2}{6m_b^3} + \frac{T_1 + 3T_2 - 3(T_3 + 3T_4)}{2m_b^3}\right) \quad (3.17) \\
\langle E_\gamma \rangle &= \frac{m_b}{2} \left(1 - \frac{\lambda_1 + 3\lambda_2}{2m_b^2} - \frac{5\rho_1 - 21\rho_2}{6m_b^3} - \frac{T_1 + 3T_2 + T_3 + 3T_4}{3m_b^3}\right) \\
&= \frac{m_B - \Lambda}{2} - \frac{13\rho_1 - 33\rho_2}{24m_b^2} - \frac{T_1 + 3T_2 + T_3 + 3T_4}{8m_b^3} \quad (3.18) \]
\[\langle E_\gamma^2 \rangle = \frac{m_b^2}{4} \left(1 - \frac{4\lambda_1 + 9\lambda_2}{3m_b^2} - \frac{7\rho_1 - 24\rho_2}{6m_b^3} - \frac{4T_1 + 12T_2 + 3T_3 + 9T_4}{3m_b^3}\right) \quad (3.19) \]
\[\text{var}(E_\gamma) = -\frac{\lambda_1}{12} - \frac{2\rho_1 - 3\rho_2}{12m_b} - \frac{T_1 + 3T_2}{12m_b} \quad (3.20)\]
The matrix elements of the $1/m_b^3$ operators are expected by dimensional analysis to be of the order $\Lambda_{\text{QCD}}^3 \sim (0.5 \text{GeV})^3$. This allows us to estimate the uncertainty due to these matrix elements in the moments presented above. The uncertainties of the total decay rate are suppressed by three powers of the heavy quark mass and are below the 1% level. Given the current experimental precision, they therefore are negligible when using this decay mode to search for new physics. If the moments of this decay are used to measure the parameters $\lambda_1$ and $\bar{\lambda}$, however, the $(\Lambda_{\text{QCD}}/m_b)^3$ contributions are only suppressed by one or two powers of the heavy quark mass, respectively. Hence, they can introduce significant theoretical uncertainties in these measurements.

Although the CLEO collaboration has published a measurement of the photon energy spectrum [43], they have not quoted results for the first two moments. Without having additional information, such as the acceptance as a function of the energy, we will not attempt to extract the values for these moments from [43]. Instead we will use the central values $\lambda_1 = -0.19 \text{GeV}^2$ and $\bar{\lambda} = 0.39 \text{GeV}$, extracted from semileptonic $B$ decays [62], together with Eqs. (3.11) and (3.13), to obtain the representative data

$$\langle E_\gamma \rangle = 2.45 \text{ GeV}, \quad \text{var}(E_\gamma) = 0.016 \text{ GeV}^2.$$  

Although obviously no information on the values $\lambda_1$ and $\bar{\lambda}$ can be obtained this way, we can still estimate the $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3$ uncertainties. To compare these uncertainties with the ones from semileptonic decays, it is useful to estimate them in the same way as in [57, 71]. We use Eqs. (3.18) and (3.20) to extract $\bar{\lambda}$ and $\lambda_1$ from (3.21) and randomly vary the magnitude of the parameters $\rho_2$ and $T_1 - T_1$ in between $(-0.5 \text{GeV})^3$ and $(0.5 \text{GeV})^3$. From a vacuum saturation approximation [72] $\rho_1$ is predicted to be positive, thus we vary $\rho_1$ only between 0 and $(0.5 \text{GeV})^3$. All random numbers are drawn from a flat distribution. Finally, we impose the constraint (2.58). In Fig. 3.2 we show an ellipse in the $\bar{\lambda} - \lambda_1$ plane which is centered about the mean value of the obtained distribution and which contains 68% of the points. This should give a reasonable estimate of the theoretical uncertainty in the extraction of $\bar{\lambda}$ and $\lambda_1$ due to higher order corrections. As a comparison we also
Figure 3.2: The estimated size of the uncertainties due to $\Lambda_{QCD}/m_b^3$ corrections on $\lambda_1$ and $\bar{\lambda}$ from $B \to X_s\gamma$ (solid) and from semileptonic decay (dashed) using the method described in the text. The position of the two ellipses has no meaning, only the relative sizes.

show the corresponding ellipse obtained from the hadron invariant mass spectrum of the inclusive decay $\bar{B} \to X_c\bar{\nu}$ [57]. Of course, the relative position of the two ellipses has no meaning since we have not used experimental values for the moments of the photon spectrum. Only the relative size can be compared. It is interesting to note that the two ellipses are roughly orthogonal to one another. This implies that moments of the decay $\bar{B} \to X_s\gamma$ should provide useful information in determining the heavy quark parameters $\bar{\lambda}$ and $\lambda_1$.

3.3 Uncertainties from the photon energy cut

In the previous Section we estimated the theoretical uncertainties in moments of the inclusive decay $\bar{B} \to X_s\gamma$. To measure this decay, however, all events with photon energy below a certain cut have to be rejected to suppress the strong background. This cut reduces the available phase space considerably with the result that the measured decay rate is far from being fully inclusive. In its measurement of the inclusive photon energy
spectrum \cite{43}, the CLEO collaboration imposed a cut at 2.2 GeV, about 500 MeV below the maximum photon energy. As we explained in Section 2.1, the differential decay rate obtained from the OPE can only be compared to experimentally observable quantities after smearing it over a region much larger than the difference of partonic and hadronic endpoints, \( m_B - m_b \sim \Lambda_{\text{QCD}} \). Since for the cut imposed by the CLEO collaboration the available phase space is of comparable size as the resonance region, there could be large corrections to the OPE results when comparing them to the measured moments. For a smearing region of order \( \Lambda_{\text{QCD}} / m_b \), the breakdown of the OPE is due to a class of higher dimensional terms, all becoming equally important for that smearing region \cite{73, 74}. At each order in \( \Lambda_{\text{QCD}} / m_b \), these terms contain the maximum number of derivatives of the delta function and are of the form

\[
A_n \left( \frac{m_b}{2} \right)^n \delta^{(n)} \left( \frac{m_b}{2} - E_\gamma \right).
\]  

These terms are called the "most singular" terms. The coefficients \( A_n \) contain only the leading term in the \( \Lambda_{\text{QCD}} / m_b \) expansion and can be obtained from (3.15)

\[
A_0 = 1, \quad A_1 = 0, \quad A_2 = -\frac{\lambda_1}{6m_b^2}, \quad A_3 = -\frac{\rho_1}{18m_b^2}.
\]  

Summing these most singular terms results in the photon energy spectrum

\[
\frac{d\Gamma_s}{dE_\gamma} = \Gamma_0 \sum_{n=0}^{\infty} \left( \frac{m_b}{2} \right)^n A_n \delta^{(n)} \left( \frac{m_b}{2} - E_\gamma \right),
\]  

which can also be interpreted as the singular expansion of the "shape function" \cite{73}

\[
\frac{d\Gamma_s}{dE_\gamma} = \Gamma_0 f(E_\gamma).
\]  

The expression for the differential decay rate obtained by performing an OPE contains only delta functions and their derivatives which contribute at \( E_\gamma = m_b/2 \). A photon energy cut affecting the lower bound on the range of integration does not change the results for the moments as long it is below \( E_\gamma = m_b/2 \). The effect of the cut is hidden in the fact that the OPE for the differential decay rate gets worse as the smearing
region diminishes. The experimental value of the higher moments of the photon energy spectrum measured in the presence of a cut will be larger than the value which determines the parameters $\bar{\lambda}$ and $\lambda_1$ as defined in (3.11) and (3.13). Therefore, a cut will shift the values of the two parameters extracted in this way, leading to uncertainties on their values.

In this Section we will take two different approaches to investigate the size of those uncertainties. The first will use the singular expansion of the shape function to yield model independent bounds on the uncertainties and in the second we will estimate them using a simple version of the ACCMM model [75], explained later.

### 3.3.1 Model independent bounds on the uncertainties

The $n$'th moment ($n \geq 1$) with a cut on the photon energy $E_0$ is given by

$$
M_n^{E_0} = \frac{\int_{E_0}^{E_{max}} E_0^n \frac{d\Gamma}{dE_\gamma} dE_\gamma}{\Gamma^{E_0}} = \frac{\int_0^{E_{max}} \theta(E_\gamma - E_0) E_\gamma^n \frac{d\Gamma}{dE_\gamma} dE_\gamma}{\Gamma^{E_0}},
$$

(3.26)

where

$$
\Gamma^{E_0} = \int_{E_0}^{E_{max}} \frac{d\Gamma}{dE_\gamma} dE_\gamma
$$

(3.27)

is the total decay rate with a cut. The positive powers of $E_\gamma$ in the integrand of the numerator of Eq. (3.26) weight the higher energy part of the spectrum more, therefore

$$
M_n^{E_0} \geq M_n,
$$

(3.28)

where $M_n$ is the moment without a photon energy cut.

In [33] a method was proposed to obtain model independent bounds on the total decay rate. In this approach one replaces the step function in (3.26) with a smooth function $P(E_\gamma, E_0)$, obeying

$$
P(E_\gamma, E_0) \leq 1 \quad \text{for} \quad E_0 \leq E_\gamma \leq E_{max}
$$

$$
P(E_\gamma, E_0) \leq 0 \quad \text{for} \quad 0 \leq E_\gamma \leq E_0,
$$

(3.29)
Figure 3.3: The model independent bound on the total decay rate for $\rho_1 = (0.3\text{ GeV})^3$ and different values of $\lambda_1$ and $m_b$.

to find the inequality

$$\Gamma^E_0 \geq \Gamma^E_P = \int_0^{E_{\text{max}}} P(E_\gamma, E_0) \frac{d\Gamma}{dE_\gamma} dE_\gamma. \quad (3.30)$$

If $P$ is a polynomial of order $k$, only the first $k$ terms in the singular expansion of the shape function (3.24) contribute. In the following analysis we will use the polynomial

$$P(E_\gamma, E_0) = 1 - \left(\frac{E_1 - E_\gamma}{E_1 - E_0}\right)^s \quad E_0 < E_1. \quad (3.31)$$

In order to have a monotonic function which satisfies the requirements (3.29), $E_0$ has to coincide with the value of the photon energy cut and $E_1$ with the maximum photon energy, $E_{\text{max}}$. Choosing $s = (n - 3)$ for the n’th moment ensures that only the known coefficients $A_n$ (3.23) with $n \leq 3$ contribute to the bounds. The resulting bounds on the total decay rate are shown in Fig. 3.3, for $\rho_1 = (0.3\text{ GeV})^3$ and two sets of values for $\lambda_1$ and $m_b$. One can see that depending on their values, a cut at 2.2 GeV can have a significant effect on the measurement of the total decay rate. This is in agreement with the analysis done using the ACCMM model [40] which has been used by the CLEO collaboration [43]. Since $P(E_\gamma, E_0)$ is a monotonically increasing function, this analysis
Figure 3.4: The model independent bound on the uncertainty of $\bar{\Lambda}$ for $\rho_1 = (0.3\ GeV)^3$ and different values of $\lambda_1$ and $m_b$.

can also be used to obtain a bound on the moments as measured by experiment

$$\frac{\int_{0}^{E_{\text{max}}} P(E_\gamma, E_0) E_\gamma^a dE_\gamma dE_\gamma}{\Gamma_{P_\gamma}^{E_0}} = M_{nP}^{E_0} \geq M_n^{E_0} \geq M_n.$$  \hspace{1cm} (3.32)

For the first moment we find using Eqs. (3.31) and (3.32)

$$M_{1}^{E_0} \leq \frac{3 (2 E_0 - m_b) m_b (2 E_0 - 4 E_{\text{max}} + m_b) + (-4 E_{\text{max}} + 3 m_b) \lambda_1 + \rho_1}{2 (3 (2 E_0 - m_b) (2 E_0 - 4 E_{\text{max}} + m_b) + \lambda_1)}.$$  \hspace{1cm} (3.33)

From this result it is trivial to obtain a bound on the uncertainty on $\bar{\Lambda}$. A plot of this bound for $\rho_1 = (0.3\ GeV)^3$ and two different sets of values $\lambda_1$ and $m_b$ is presented in Fig. 3.4. For a cut at 2.2 GeV the bound on the uncertainty on $\bar{\Lambda}$ is between 100 MeV and 400 MeV, thus it can be of the same order as $\Lambda_{\text{QCD}}$, leading to large errors. In order to be able to determine a precise value of $\bar{\Lambda}$, the cut would have to be lowered to around 2 GeV. For this value the bound on the uncertainty is between 20 MeV and 100 MeV. Since $\lambda_1$ is related to the variance of the spectrum, which is the difference of two moments, the $P$-function analysis can not be used to bound its uncertainty. This is due to the fact that the variance is sensitive to the tails of the spectrum.

An inequality such as derived in (3.30) holds true only for a smooth, positive definite shape function. In general, there is nothing that guarantees the validity of this assump-
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Although models such as the ACCMM model [75] predict a positive shape function, it should also be pointed out that the inequality (3.30) is true rigorously only for smooth functions $d\Gamma_s/dE_\gamma$. In this approach we use a singular expansion of the differential decay rate, so the convergence of this expansion is essential. As said above, the shape function analysis is only valid if the smearing region is of order $\Lambda_{\text{QCD}}$, a scale which also sets the size of matrix elements of higher dimensional operators. We therefore expect the bounds to break down for values of the matrix elements which exceed the available phase space considerably. The breakdown of the inequality (3.30) can be seen by using $E_{\text{max}} = 2.6$ GeV, $E_0 = 2$ GeV, $\lambda_1 = -0.2$ GeV$^2$ and $m_b = 4.8$ GeV. Inserting those values into (3.30) we find

$$0.193 \frac{\rho_1}{\text{GeV}^3} + 0.917 = \frac{\Gamma_{\rho_1}^{E_0}}{\Gamma} \leq 1$$

which breaks down for $\rho_1 \geq (0.76 \text{ GeV})^3$.

Since we could not obtain a bound on the uncertainty on $\lambda_1$ and we are in no position to rigorously justify the assumptions made in this Section, we will now use a simple model to calculate the effect of the photon energy cut on the first two moments.

3.3.2 Uncertainties using the ACCMM model

In this Section we will use a simplified version of the ACCMM model [75] to estimate the value of the moments as measured in the presence of a cut on the photon energy. The ACCMM model parameterizes the shape function (3.25) by assuming a Gaussian distribution of the relative momentum of the $b$ quark inside the $B$ meson. To make the analysis even simpler, we will neglect the mass of the $s$ quark and the momentum dependence of the $b$ quark mass. This leads to the spectral function [73]

$$\frac{1}{\Gamma} \frac{d\Gamma}{dE_\gamma} = \frac{1}{\sqrt{2\pi}\sigma_E} \exp \left\{ -\frac{(E_\gamma - m_b)^2}{2\sigma_E^2} \right\} ,$$

where $\sigma_E^2 = -\frac{\lambda_1}{12}$ is a tunable parameter in the model.
Varying $\sigma_E$ in the first moment of this spectral function gives an estimate of the uncertainty on the value of $\bar{\Lambda}$ and the result is shown in Fig. 3.5. For a cut at 2.2 GeV, the uncertainty on the parameter $\bar{\Lambda}$ in this model $\sim$ 180 MeV. Since the real effect of the photon energy cut could easily exceed this estimate by a factor of two, a cut on the photon energy at 2.2 GeV could destroy the possibility of accurately determining the value of $\bar{\Lambda}$. If the cut could be lowered to $\approx$ 2 GeV, then an accurate extraction should be possible. This is in agreement with the model independent results obtained in the last Section.

The result for a similar calculation for the variance of the spectrum is shown in Fig. 3.6. For a cut of 2.2 GeV, the uncertainty on $\lambda_1$ in this model is $\sim 0.3$ GeV$^2$. Again, this indicates that an extraction of $\lambda_1$ from the present CLEO measurement is unreliable. Lowering the cut to $\approx 1.9$ GeV should enable a precise determination of this parameter from the decay $\bar{B} \to X_s\gamma$.

### 3.4 Conclusions

We have analyzed the uncertainties on the extraction of the two nonperturbative matrix elements $\bar{\Lambda}$ and $\lambda_1$ from the mean photon energy and its variance in the inclusive
Figure 3.6: The uncertainty on $\lambda_1$ due to a photon energy cut from the ACCMM model.

decay $\bar{B} \to X_s\gamma$. Besides perturbative corrections which have not been considered here, uncertainties arise from matrix elements of higher dimensional operators which are suppressed by additional powers of $\Lambda_{QCD}/m_b$. We calculated the first two moments up to order $(\Lambda_{QCD}/m_b)^3$ and estimated the effect of the unknown matrix elements by varying their values in their expected range of magnitude. We compared the resulting uncertainties on $\bar{\Lambda}$ and $\lambda_1$ with the corresponding ones obtained from the semileptonic decay $\bar{B} \to X_s\ell\bar{\nu}$. We found that combining the two methods of extracting $\bar{\Lambda}$ and $\lambda_1$ yields small nonperturbative uncertainties.

A more serious uncertainty arises from the effect of a cut on the photon energy which has to be imposed in order to reduce the large background from other processes. The differential decay rate as calculated via the OPE is given in terms of a delta function and its derivatives contributing at $E_\gamma = m_b/2$. The effect of a cut on the photon energy therefore does not affect the results for the moments. It is hidden in the fact that the OPE breaks down as the cut approaches the endpoint of the photon energy. We have used an approach suggested in [33] to obtain a model independent bound on the uncertainty on $\bar{\Lambda}$, whereas no such bound could be derived for the uncertainty on $\lambda_1$. The bound indicates that an accurate extraction of $\bar{\Lambda}$ is definitely possible for a cut at 2 GeV, whereas for the
present cut at 2.2 GeV the errors might be large. We have also used a simplified version of the ACCMM model to estimate the effect the photon energy cut. We again find that an accurate determination of $\bar{\Lambda}$ should be possible if the cut can be lowered to about 2 GeV. The cut might have to be lowered even further to allow a precise determination of $\lambda_1$. 
Chapter 4

The decay $\bar{B} \rightarrow X_s \ell^+ \ell^-$

4.1 Introduction and Kinematics

The decay $\bar{B} \rightarrow X_s \ell^+ \ell^-$ is suppressed relative to $\bar{B} \rightarrow X_s \gamma$ by an additional factor of the electromagnetic coupling constant $\alpha \simeq 1/137$, and has not yet been observed. As mentioned in chapter 1, the current experimental limits on its branching ratio are about an order of magnitude above the standard model expectations. Despite its small branching fraction, this decay has the appeal of being sensitive to the signs and magnitudes of the Wilson coefficients $C_7^{\text{eff}}, C_9$ and $C_{10}$, which can all be affected by physics beyond the standard model. The decay $\bar{B} \rightarrow X_s \gamma$ depends only on the absolute value of $C_7^{\text{eff}}$ as can be seen from (3.3) and is therefore only sensitive to new physics that affects this one parameter in the standard model. The dependence of the decay $\bar{B} \rightarrow X_s \ell^+ \ell^-$ on the additional parameters makes it valuable for the detection of new physics, even if suppressed by an additional factor of $\alpha$.

In this Chapter we will calculate the nonperturbative uncertainties in the lepton invariant mass spectrum, hadronic invariant mass moments as well as the forward backward asymmetry [46, 47]. It has been shown that from this information all the signs and magnitudes of the coefficients $C_7^{\text{eff}}, C_9$ and $C_{10}$ can be determined [76]. Experimental studies
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of this process [44] impose cuts on the available phase space. This is primarily due to the necessity of removing the background from sequential decays and the long distance contribution from $\bar{B} \rightarrow (J/\psi, \psi')X_s$ with the $(J/\psi, \psi')$ decaying into two leptons. We incorporate representative cuts into the theoretical analysis.

For this decay mode there are three operators contributing, $O_7$, $O_9$ and $O_{10}$. This leads to the amplitude mediating this decay

$$A(b \rightarrow s \ell^+ \ell^-) = \frac{G_F \alpha}{\sqrt{2\pi}} V_{ts}^* V_{tb} \left[ \left(C_9^{\text{eff}} - C_{10}\right)(\bar{s}\gamma_\mu L)(\bar{\ell}\gamma^\mu L\ell) + \left(C_9^{\text{eff}} + C_{10}\right)(\bar{s}\gamma_\mu L)(\bar{\ell}\gamma^\mu R\ell) - 2C_7^{\text{eff}} \left(\bar{s}\sigma_{\mu\nu} \frac{q^\nu}{Q^2} (m_sL + m_bR)b\right)(\bar{\ell}\gamma^\mu \ell) \right]. \quad (4.1)$$

This can be written in a more compact form

$$A(b \rightarrow s \ell^+ \ell^-) = \frac{G_F \alpha}{\sqrt{2\pi}} V_{ts}^* V_{tb} \left( H^L_\mu P^L_\mu + H^R_\mu P^R_\mu \right), \quad (4.2)$$

with

$$P^L/R_\mu = \bar{\ell}\gamma_\mu L(R)\ell$$

$$H^L/R_\mu = \bar{s} \left[ R\gamma_\mu \left(C_9^{\text{eff}} + C_{10} + 2C_7^{\text{eff}} \frac{\hat{q}}{Q^2}\right) + 2m_s C_7^{\text{eff}} \gamma_\mu \frac{\hat{q}}{Q^2} L \right] b, \quad (4.3)$$

where $q = (p_+ + p_-)$ is the dilepton momentum. In the above expressions we have followed convention and defined $C_9^{\text{eff}} \equiv C_9^{\text{eff}}(s)$, which contains the operator mixing of $O_{1-6}$ into $O_9$ as well as the one loop matrix elements of $O_{1-6,9}$. The full analytic expression is quite lengthy and may be found in [68]. A plot of the real and imaginary parts of $C_9^{\text{eff}}$ as a function of $\hat{s}$ is shown in Fig. 4.1. Notice that the amplitude (4.1) reduces to a $(V - A) \times (V - A)$ amplitude for $C_7^{\text{eff}} = 0$, $C_9^{\text{eff}} = 1/2$, and $C_{10} = -1/2$. This provides some useful cross-checks with known results for semileptonic $B$ decays. The calculation of the differential decay rate follows the general procedure explained in Chapter 2 leading to the triple differential decay rate

$$\frac{d^3B}{d\hat{u} d\hat{s} dv \cdot \hat{q}} = \frac{4 B_0}{2m_B} \left(-\frac{1}{\pi}\right) \text{Im} \left\{ 2 \hat{s} \left(T^L_1(v \cdot \hat{q}, \hat{s}) + T^R_1(v \cdot \hat{q}, \hat{s})\right) \right\}$$
Figure 4.1: The effective Wilson coefficient $C_{9}^\text{eff}$ as a function of $\hat{s}$. The real part is shown in (a) and the imaginary part in (b).

$$
+ \left( (v \cdot \hat{q})^2 - \hat{s} - \frac{\hat{u}^2}{4} \right) \left( T_2^L(v \cdot \hat{q}, \hat{s}) + T_2^R(v \cdot \hat{q}, \hat{s}) \right)
+ \hat{u} \hat{s} \left( T_3^L(v \cdot \hat{q}, \hat{s}) - T_3^R(v \cdot \hat{q}, \hat{s}) \right),
$$

(4.4)

Here we used the same conventions as in [41, 42] and normalized the $\bar{B} \to X_s \ell^+ \ell^-$ branching ratio to the semileptonic branching ratio $B_{st}$, a quantity well known experimentally to

$$dB(\bar{B} \to X_s \ell^+ \ell^-) = B_{st} \frac{d\Gamma(\bar{B} \to X_s \ell^+ \ell^-)}{\Gamma(\bar{B} \to X_c \ell \nu)}. \quad (4.5)$$

This introduces the normalization constant

$$B_0 = B_{st} \frac{3\alpha^2 |V_{ts}V_{tb}|^2}{16\pi^2 |V_{cb}|^2} \frac{1}{f(\hat{m}_c)\kappa(\hat{m}_c)}. \quad (4.6)$$

In this expression $f(\hat{m}_c)$ is the well known phase space factor for the parton decay rate $b \to ce\bar{\nu}$

$$f(\hat{m}_c) = 1 - 8\hat{m}_c^2 + 8\hat{m}_c^6 - \hat{m}_c^8 - 24\hat{m}_c^4 \log \hat{m}_c, \quad (4.7)$$

and $\kappa(\hat{m}_c)$ includes the $O(\alpha_s)$ QCD corrections as well as the nonperturbative corrections up to $O(\Lambda/m_b)^3$

$$\kappa(\hat{m}_c) = 1 + \frac{\alpha_s(m_b)}{\pi} g(\hat{m}_c) + \frac{h_1(m_c)}{2m_b^2} + \frac{h_2(m_c)}{6m_b^2}, \quad (4.8)$$

where

$$g(\hat{m}_c) = \frac{A_0(\hat{m}_c)}{f(\hat{m}_c)}$$
The analytic expression for the perturbative function $A_\theta(\hat{m}_c)$ has been calculated in [70] and is given by

$$A_\theta(\hat{m}_c) = \frac{25}{6} - \frac{2}{3} \pi^2 - \frac{478}{9} \hat{m}_c^2 + \frac{64}{3} \pi^2 (1 + \hat{m}_c^2) \hat{m}_c^3 - \frac{32}{3} \pi^2 \hat{m}_c^4 + \frac{478}{9} \hat{m}_c^6$$

$$- \left( \frac{25}{6} + \frac{2}{3} \pi^2 \right) \hat{m}_c^4 - \frac{2}{3} (36 + \hat{m}_c^4) \hat{m}_c^4 \ln \hat{m}_c^2$$

$$+ \left( - \frac{40}{3} \hat{m}_c^2 + \frac{256}{3} (1 + \hat{m}_c^2) \ln (1 + \hat{m}_c) \hat{m}_c^3 - 60 \hat{m}_c^4 + \frac{8}{9} \hat{m}_c^6 - \frac{34}{9} \hat{m}_c^8 \right) \ln \hat{m}_c^2$$

$$+ \left( - \frac{34}{9} + \frac{128}{9} \hat{m}_c^2 - \frac{128}{9} \hat{m}_c^4 + \frac{34}{9} \hat{m}_c^8 \right) \ln (1 - \hat{m}_c^2)$$

$$+ \left( \frac{8}{3} - \frac{128}{3} (1 + \hat{m}_c^2) \hat{m}_c^3 + 80 \hat{m}_c^4 + \frac{8}{3} \hat{m}_c^6 \right) \ln \hat{m}_c^2 \ln (1 - \hat{m}_c^2)$$

$$+ \left( \frac{4}{3} + \frac{128}{3} (1 + \hat{m}_c^2) \hat{m}_c^3 + 64 \hat{m}_c^4 + 4 \hat{m}_c^8 \right) \text{Li}_2(\hat{m}_c^2)$$

$$- \frac{512}{3} \hat{m}_c^2 (1 + \hat{m}_c^2) \text{Li}_2(\hat{m}_c).$$

The form factors $T_i^{L/R}$ are given in Appendix A.3 and we have defined kinematic variables $v \cdot \tilde{q} = v \cdot q / m_b$, $\hat{s} = q^2 / m_b^2$, and $\hat{u} = [(p_b - p_-)^2 - (p_b - p_+)^2] / m_b^2$. In terms of these leptonic variables the limits of phase space are given by

$$-\sqrt{\hat{s} + \hat{u}^2 / 4} \leq v \cdot \hat{q} \leq \sqrt{\hat{s} + \hat{u}^2 / 4}$$

$$-\hat{u}(\hat{s}, \hat{m}_s) \leq \hat{u} \leq \hat{u}(\hat{s}, \hat{m}_s)$$

$$\frac{4 \hat{m}_s^2}{\hat{s}_0} \leq \hat{s} \leq (1 - \hat{m}_s)^2,$$

where $\hat{u}(\hat{s}, \hat{m}_s) = \sqrt{[\hat{s} - (1 + \hat{m}_s)^2] [\hat{s} - (1 - \hat{m}_s)^2]}$.

For the calculation of the hadron invariant mass moments it will be convenient to express the phase space in terms of the parton energy fraction $x_0 = E_q / m_b$ and the parton invariant mass fraction $\hat{s}_0 = p_q^2 / m_b^2$. They are related to the leptonic variables introduced above via

$$v \cdot \hat{q} = 1 - x_0.$$
\[ \hat{s} = 1 + \hat{s}_0 - 2x_0. \quad (4.12) \]

The phase space can then be expressed as

\[
\begin{align*}
-2\sqrt{x_0^2 - \hat{s}_0} & \leq \hat{u} \leq 2\sqrt{x_0^2 - \hat{s}_0} \\
\hat{m}_s^2 & \leq \hat{s}_0 \leq x_0^2 \\
\hat{m}_s & \leq x_0 \leq \frac{1}{2}(1 + \hat{m}_s^2).
\end{align*}
\]

Since the form factors \(T_i\) are independent of \(\hat{u}\), this first integration is trivial and we arrive at

\[
\frac{d^2B}{dx_0\,d\hat{s}_0} = \frac{32}{2m_B} B_0 \left( -\frac{1}{\pi} \right) \sqrt{x_0^2 - \hat{s}_0} \text{Im} \left\{ \left[ (1 - 2x_0 + \hat{s}_0) \left( T_1^L(x_0, \hat{s}_0) + T_1^R(x_0, \hat{s}_0) \right) \right. \right.
\]

\[
\left. \left. + \frac{x_0^2 - \hat{s}_0}{3} \left( T_2^L(x_0, \hat{s}_0) + T_2^R(x_0, \hat{s}_0) \right) \right] \right\}. \quad (4.14)
\]

This doubly differential decay rate serves as the starting point for calculating the dilepton invariant mass distribution and hadronic energy and invariant mass moments.

### 4.2 The partially integrated branching ratio

An interesting experimentally accessible quantity is the dilepton invariant mass spectrum. Inserting the expressions for \(T_i^{(L/R)}\) into the double differential decay rate and integrating over \(v\cdot\hat{q}\) by picking out the residues, we find for the dilepton invariant mass spectrum

\[
\frac{dB}{d\hat{s}} = 2 B_0 \left\{ \left[ \frac{1}{3} (1 - \hat{s})^2 \left( 1 + 2\hat{s} \right) \left( 2 + \frac{\lambda_1}{m_b^2} \right) + \left( 1 - 15\hat{s}^2 + 10\hat{s}^3 \right) \left( \frac{\lambda_2}{m_b^2} - \frac{\rho_2}{m_b^3} \right) \right. \right.
\]

\[
\left. \left. - \frac{10\hat{s}^4 + 23\hat{s}^3 - 9\hat{s}^2 + 13\hat{s} + 11 \frac{\rho_1}{m_b^3}}{9(1 - \hat{s})} \right] \left| C_{10} \right|^2 + C_{10}^2 \right\}
\]

\[
+ \left[ \frac{4}{3} (1 - \hat{s})^2 (2 + \hat{s}) \left( 2 + \frac{\lambda_1}{m_b^2} \right) + 4 \left( -6 - 3\hat{s} + 5\hat{s}^3 \right) \left( \frac{\lambda_2}{m_b^2} - \frac{\rho_2}{m_b^3} \right) \right.
\]

\[
\left. \left. - \frac{4(5\hat{s}^4 + 19\hat{s}^3 + 9\hat{s}^2 - 7\hat{s} + 22) \frac{\rho_1}{m_b^3}}{9(1 - \hat{s})} \right] \frac{C_{10}^2}{\hat{s}} \right\}
\]

\[
+ \left[ 4(1 - \hat{s})^2 \left( 2 + \frac{\lambda_1}{m_b^2} \right) + 4 \left( -5 - 6\hat{s} + 7\hat{s}^2 \right) \left( \frac{\lambda_2}{m_b^2} - \frac{\rho_2}{m_b^3} \right) \right] \frac{C_{10}^2}{\hat{s}} \right\}. \quad (4.15)
\]
The dependence on $\mathcal{T}_{1-4}$ can be obtained by making the replacements (2.53) in (4.15). The matrix element of the kinetic energy operator, $\lambda_1$, only enters the $q^2$ spectrum in a very simple form, because the unit operator and the kinetic energy operator are related by reparameterization invariance [77]. Any quantity which can be written independent of the heavy quark velocity $v$ must depend only on the combination $(1 + \lambda_1/2m_b^2)$. The $q^2$ spectrum (and the total rate written in terms of $m_b$) are invariant under a redefinition of $v$, but, for example, the lepton energy spectrum is not since $E_\ell = v \cdot p_\ell$. (Equivalently, the $\lambda_1$ term is a time-dilation effect, and hence is universal in any quantity that is independent of the rest frame of the $B$ meson [53, 78].)

Figure 4.2: The differential decay spectrum $\frac{1}{B_0} \frac{d\hat{B}}{d\hat{s}}$ for the decay $\hat{B} \rightarrow X_s \ell^+ \ell^-$. The solid line shows the free quark prediction, the long-dashed line includes the $\mathcal{O}(\Lambda/m_b^2)$ corrections and the short-dashed line contains all corrections up to $\mathcal{O}(\Lambda/m_b^4)$. The negative branching ratio at large $\hat{s}$ is a theoretical artifact and shows that the OPE breaks down for large values of $\hat{s}$.

A plot of this distribution is shown in Fig. 4.2, where we have used the representative
values
\[ \lambda_1 = -0.19 \text{ GeV}^2, \quad \lambda_2 = 0.12 \text{ GeV}^2, \quad \bar{\lambda} = 0.39 \text{ GeV}. \] (4.16)

For the matrix elements of the dimension six operators we use the generic size \((\Lambda_{\text{QCD}})^3 \sim (0.5 \text{ GeV})^3\) as suggested by dimensional analysis (2.56). One immediately notices divergences at both endpoints of this spectrum. The divergence at the \(\hat{s} \to 0\) endpoint is due to the intermediate photon going on-shell and becoming a real photon. This is a well known feature of the decay \(\bar{B} \to X_s \ell^+ \ell^-\) [41, 66]. In this limit one therefore expects this spectrum to reduce to the \(\bar{B} \to X_s \gamma\) rate with an on-shell photon in the final state, convoluted with the fragmentation function giving the probability for a photon to fragment into a lepton pair. This correspondence is explicitly verified by the analytic form of the divergent term

\[
\frac{1}{B_0} \left. \frac{d\mathcal{B}}{d\hat{s}} \right|_{\hat{s} \to 0} \sim \frac{32}{3} \frac{|C_{7}^{\text{eff}}|^2}{\hat{s}} \left( 1 + \frac{\lambda_1 - 9 \lambda_2}{2m_b^2} - \frac{11 \rho_1 - 27 \rho_2}{6m_b^3} \right. \\
\left. + \frac{T_1 + 3T_2 - 3(T_3 + 3T_4)}{2m_b^3} \right) ,
\] (4.17)

where the term multiplying \(1/\hat{s}\) is proportional to the total rate for \(\bar{B} \to X_s \gamma\) (3.17). This divergence is regulated by the mass of the leptons and will not be considered any further.

The divergence at the \(\hat{s} \to 1\) endpoint is entirely due to the \(1/m_b^3\) operators as can be seen from Fig. 4.2. In this case the analytic form of the divergent term is

\[
\frac{1}{B_0} \left. \frac{d\mathcal{B}}{d\hat{s}} \right|_{\hat{s} \to 1} \sim \frac{32}{3m_b^3} \left( C_{10}^2 + (2 C_7^{\text{eff}} + C_9^{\text{eff}})^2 \right) \frac{\rho_1}{1 - \hat{s}} .
\] (4.18)

This leads, upon integration, to an unphysical logarithmic divergence in the expression for the total rate that is regulated by the mass of the \(s\) quark. This divergence can be understood by considering a similar effect in the semileptonic decay \(\bar{B} \to X_c \ell \bar{\nu}_\ell\) [57, 63], which contains a term \(\sim \rho_1/(1 - \hat{s})\), leading to a term \(\sim \log(m_c/m_b)\) upon integration. In that context, the origin of this logarithmic term can be clarified by performing an OPE for the total rather than the differential rate [63, 64]. The OPE then contains a
four-fermion operator of the form
\[ \frac{16\pi^2}{m_b^3} \bar{b} \gamma^\mu L c \bar{c} \gamma^\nu L b \left( g_{\mu\nu} - v_{\mu} v_{\nu} \right), \] (4.19)
whose matrix element can be calculated perturbatively at the scale \( \mu = m_b \)
\[ \rho_1 \log(m_c). \] (4.20)

Thus, the logarithmic dependence on the mass of the final quark is due to the matrix element of this four-fermion operator\(^1\). To calculate this matrix element it is essential that the mass of the \( c \) quark be large compared to the QCD scale \( \Lambda_{\text{QCD}} \). Consequently, for the decay \( \bar{B} \to X_s \ell^+ \ell^- \) these methods are not applicable because the \( s \) quark is too light.

Including higher orders in perturbation theory, the matrix element of the four-fermion operator contains terms of the form \( \alpha_s^n \log^{n+1}(\hat{m}_s) \) which are of the same size as \( \log(m_s) \), making a perturbative calculation of this matrix element impossible. Thus, a seventh nonperturbative matrix element \( f_1 \) defined in (2.54) is required. It contributes only at the \( \hat{s} \to 1 \) endpoint of the spectrum and cancels the logarithmic divergence proportional to \( \rho_1 \log(m_s) \) in the total rate:
\[ \frac{dB}{ds} \to \frac{dB}{ds} - \frac{32}{3m_b^3} B_0 \left( C_{10}^2 + \left( 2C_7^{\text{eff}} + C_9^{\text{eff}} \right)^2 \right) \delta (1 - \hat{s}) \left( \rho_1 \log(m_s) - f_1 \right). \] (4.21)

Another noticeable feature in the dilepton invariant mass spectrum is the cusp due to the \( c\bar{c} \) threshold. Near this value of \( \hat{s} \) the methods used to calculate the physical spectrum fail because of long distance contributions from the resonant decay of the \( B \) meson into \( c\bar{c} \) bound states, such as the \( \psi \) and the \( \psi' \). These two bound states may decay into two leptons and fake the signal. Experimentally one deals with this resonance region by simply cutting it out. Thus, to compare reliably to experiment we have to include such a cut in the calculation. Defining the partially integrated branching ratio by
\[ B_x = \frac{1}{B_0} \int_x^1 ds \frac{dB}{ds}, \] (4.22)

\(^1\)In [64] the logarithmic dependence on the charm quark mass was summed in the leading logarithmic approximation by running the operators occurring in the OPE for the total rate to the scale \( \mu = m_c \), at which the matrix elements do not contain a logarithmic dependence on the charm quark mass. All the dependence has been moved and summed in the coefficient functions.
we plot the contribution of the individual matrix elements relative to the leading order 

\[ B \xrightarrow{\text{parton}} X, \rho_1, \rho_2, \lambda_2, \lambda_1 \]

Figure 4.3: The fractional contributions to \( B_x \) with respect to the parton model result from the \( O(1/m_b^2) \) operators. The solid, dashed and dotted lines correspond to the contributions from \( \lambda_2, \rho_1 \) and \( \rho_2 \), respectively. The contribution from \( \lambda_1 \) is too small to be seen. The two vertical lines illustrate the positions of the \( J/\psi \) and the \( \psi' \) resonance.

The parton result in Figure 4.3. One can see that the nonperturbative contributions become more important as the accessible phase space is decreased\(^2\). For \( x \sim 0.75 \), about 6 GeV\(^2\) below the endpoint, the contribution from the \( \rho_1 \) matrix element is of the same size as the parton model prediction. This is not surprising, since the size of the resonance regime where the OPE fails is of the order \( \Delta q^2 \approx 2\Lambda_{\text{QCD}}m_b \approx 5 \text{ GeV}^2 \).

A lower cut of \( x = (14.33 \text{ GeV}^2/m_b^2) = 0.59 \) has been suggested by the CLEO collaboration in order to eliminate the \( c\bar{c} \) resonance region and the background from sequential decays [44]. For this value the partially integrated rate is

\[
B_{0.59} = 3.8 + 1.9 \left( \frac{\lambda_1}{m_b^2} + \frac{T_1 + 3T_2}{m_b^3} \right) - 134.7 \left( \frac{\lambda_2}{m_b^2} + \frac{T_1 + 3T_2}{3m_b^3} \right) \\
+ 614.9 \frac{\rho_1}{m_b^3} + 134.7 \frac{\rho_2}{m_b^3} + 560.2 \frac{f_1}{m_b^3}.
\]

\(^2\)We emphasize that the sizes of the \( \rho_1 \) contributions shown here should not be taken as accurate indications of the actual size of the corrections, but rather as estimates of the uncertainty in the prediction.
Even at this value of the cut $\chi$ the coefficients of the non-perturbative matrix elements indicate a poorly converging OPE. One can estimate the uncertainty induced by the $O(\Lambda_{QCD}/m_b)^3$ parameters by the same method as for the decay $\bar{B} \to X_s\ell^+\ell^-$. We fix $\lambda_i$ to the values given in (4.16) and then randomly vary the magnitudes of the parameters $\rho_i$, $T_i$ and $f_i$ between $-(0.5\text{ GeV})^3$ and $(0.5\text{ GeV})^3$, imposing positivity of $\rho_1$ and enforcing the constraint (2.58). Taking the 1$\sigma$ deviation as a reasonable estimate of the uncertainties from $O(\Lambda_{QCD}/m_b)^3$ contributions, we find the uncertainty to be at the 10% level. It is clear from (4.23) that the $\rho_1$ contribution is large, and relaxing the positivity constraint on $\rho_1$ enlarges the uncertainty to about 20%. Since the cut on $q^2$ cannot be lowered because of the $\psi'$ resonance, these uncertainties are intrinsic to this approach in the large dilepton invariant mass region. This poor convergence of the OPE implies that the branching ratio above the $\psi'$ can not be predicted reliably.

It is important to notice that in the invariant mass region below the $J/\psi$ resonance, the uncertainties from these matrix elements are much smaller. For example, integrating the differential spectrum up to the cut specified in the CLEO analysis [36] $\hat{s} = (m_{J/\psi} - 0.1\text{ GeV})^2/m_b^2 = 0.35$ we find

$$
\int_{0.01}^{0.35} d\hat{s} \frac{d\mathcal{B}}{d\hat{s}} = 22.0 \left[ 1 + 0.5 \left( \frac{\lambda_1}{m_b^2} + \frac{T_1 + 3T_2}{m_b^3} \right) + 1.2 \left( \frac{\lambda_2}{m_b^2} + \frac{T_2 + 3T_4}{3m_b^3} \right) - 3.7 \frac{\rho_1}{m_b^3} - 1.2 \frac{\rho_2}{m_b^3} \right].
$$

(4.24)

It is still true that the coefficient of the $\rho_1$ term is $\sim 10$ times larger than that of the $\lambda_1$ term, but all the coefficients are of order unity. Thus, we conclude that in this region the $O(\Lambda/m_b)^3$ nonperturbative corrections are not a significant source of theoretical uncertainty.
4.3 The Forward-Backward Asymmetry

The differential forward-backward (FB) asymmetry is defined by

\[ \frac{dA}{ds} = \int_0^1 dz \frac{dB}{dz ds} - \int_{-1}^0 dz \frac{dB}{dz ds}, \]  

where

\[ z = \cos \theta = \frac{\hat{u}}{u(s, m_\ell)}, \]  

parameterizes the angle between the b quark and the positively charged lepton in the dilepton CM frame. It has been shown [79] that new physics can modify this spectrum, so it is interesting to see how \( \mathcal{O}(\Lambda_{QCD}/m_b)^3 \) terms contribute to the SM prediction.

Integrating the triple differential decay rate (4.4) we find

\[ \frac{dA}{ds} = C_{10}^{\text{eff}} \left( -8(1 - \hat{s})^2 - \frac{4(3 + 2\hat{s} + 3\hat{s}^2)}{3m_b^2} \lambda_1 + \frac{(28 + 40\hat{s} - 36\hat{s}^2)}{m_b^2} \lambda_2 \right. \]
\[ + \frac{4(5 + 2\hat{s} + \hat{s}^2)}{3m_b^2} \rho_1 - \frac{4(7 + 10\hat{s} - 9\hat{s}^2)}{m_b^3} \rho_2 \)
\[ + C_{10}^{\text{eff}} \left( -4\hat{s}(1 - \hat{s})^2 - \frac{2\hat{s}(3 + 2\hat{s} + 3\hat{s}^2)}{3m_b^2} \lambda_1 + \frac{2\hat{s}(9 + 14\hat{s} - 15\hat{s}^2)}{m_b^2} \lambda_2 \right. \]
\[ - \frac{2\hat{s}(1 + 2\hat{s} + 5\hat{s}^2)}{3m_b^2} \rho_1 - \frac{2\hat{s}(1 + 6\hat{s} - 15\hat{s}^2)}{m_b^3} \rho_2 \right), \]  

Here we have again omitted the trivial dependence on \( T_{1-4} \). It is clear from this expression that the third order terms do not have abnormally large coefficients, and therefore introduce only small variations relative to the second order expression.

An experimentally more useful quantity is the normalized FB asymmetry defined by

\[ \frac{d\bar{A}}{ds} = \frac{dA}{ds} \frac{dB}{ds}. \]  

Unfortunately, the spectrum has inherited the poor behavior of the differential branching ratio in the endpoint region. In Fig. 4.4 we illustrate the uncertainties of the normalized FB asymmetry originating from the matrix elements of the dimension six operators. The three curves show the mean value and the 1\( \sigma \) uncertainty of the FB asymmetry, obtained in a way similar to that explained in Section 4.2. One can see that up to a value of
4.4 Extracting $\bar{A}$ and $\lambda_1$ from the hadron invariant mass moments

It has been suggested by Ali and Hiller [42] that one can use the first two moments of the hadron invariant mass spectrum defined by

$$\langle s_H^n \rangle = \int (s_H - m_H^2)^n \frac{d\mathcal{B}}{ds_H} ds_H$$

(4.29)

to extract the parameters $\bar{A}$ and $\lambda_1$. This idea is similar to the approaches used for the rare radiative $\bar{B} \to X_s\gamma$ decay in Chapter 3, though the experimental task is more challenging due to the small size of the branching ratio. To calculate these hadronic
moments we relate them to calculable partonic moments via

\[
\langle s_H \rangle = \frac{\bar{\Lambda}^2 - \frac{\bar{\Lambda}(\lambda_1 + 3\lambda_2)}{m_B}}{m_B} \\
+ \left( m_B^2 - 2m_B\bar{\Lambda} + \frac{\bar{\Lambda}^2 + \lambda_1 + 3\lambda_2 - \frac{\rho_1 + 3\rho_2}{2m_B}}{\frac{\tau_1 + \tau_3 + 3(\tau_2 + \tau_4)}{2m_B}} \right) \langle \hat{s}_0 \rangle \\
+ \left( 2m_B\bar{\Lambda} - 2\bar{\Lambda}^2 - \lambda_1 - 3\lambda_2 + \frac{\bar{\Lambda}(\lambda_1 + 3\lambda_2)}{m_B} + \frac{\rho_1 + 3\rho_2}{2m_B} \right. \\
\left. - \frac{\tau_1 + \tau_3 + 3(\tau_2 + \tau_4)}{2m_B} \right) \langle x_0 \rangle
\]

(4.30)

\[
\langle s_H^2 \rangle = \left( m_B^2 - 4m_B^2\bar{\Lambda} + 6m_B^2\bar{\Lambda}^2 + 2m_B^2(\lambda_1 + 3\lambda_2) - 4m_B\bar{\Lambda}^3 - 4m_B\bar{\Lambda}(\lambda_1 + 3\lambda_2) \\
- m_B(\rho_1 + 3\rho_2) + m_B(\tau_1 + \tau_3 + 3(\tau_2 + \tau_4)) \right) \langle \hat{s}_0^2 \rangle \\
+ 4 \left( m_B^2\bar{\Lambda}^2 - 2m_B\bar{\Lambda}^3 - m_B\bar{\Lambda}(\lambda_1 + 3\lambda_2) \right) \langle x_0^2 \rangle \\
+ \left( 4m_B^3\bar{\Lambda} - 12m_B^2\bar{\Lambda}^2 - 2m_B^2(\lambda_1 + 3\lambda_2) + 12m_B\bar{\Lambda}^3 + 10m_B\bar{\Lambda}(\lambda_1 + 3\lambda_2) \\
+ m_B(\rho_1 + 3\rho_2) - m_B(\tau_1 + \tau_3 + 3(\tau_2 + \tau_4)) \right) \langle x_0 \hat{s}_0 \rangle \\
+ 2 \left( m_B\bar{\Lambda}^2 - 2m_B\bar{\Lambda}^3 - (\lambda_1 + 3\lambda_2)m_B\bar{\Lambda} \right) \langle \hat{s}_0 \rangle \\
+ 4m_B\bar{\Lambda}\bar{\Lambda} \langle x_0 \rangle.
\]

(4.31)

where we have used the mass relation (2.55). We therefore have to calculate the first two moments of the parton energy \( \langle x_0 \rangle, \langle x_0^2 \rangle \) and parton invariant mass \( \langle \hat{s}_0 \rangle, \langle \hat{s}_0^2 \rangle \), as well as the mixed moment \( \langle x_0 \hat{s}_0 \rangle \). Defining

\[
M^{(m,n)} = \langle x_0^m (\hat{s}_0 - \hat{m}_0)\rangle^n = \frac{1}{B_0} \int_{m_s}^X dx_0 \int_{\hat{m}_0}^{\hat{s}_0} d\hat{s}_0 \int_{\hat{m}_0}^{\hat{s}_0} \frac{d^2B}{dx_0 d\hat{s}_0},
\]

(4.32)

we give the results for the required partonic moments in Appendix A.4. As before, we have included the dependence on the cut on the lepton invariant mass in these results. It is important to note that the results for the partonic moments given in Appendix A.4 are expressed in terms of the b quark mass \( m_b \) and must be re-expressed in terms of the \( B \) meson mass \( m_B \) using the mass relation (2.55). Using again for the cut on the invariant mass the value proposed by the CLEO collaboration, \( q_0^2 = 14.33 \text{ GeV}^2 \), we find for the two moments

\[
\langle s_H \rangle = m_B^2 \left[ 0.36 \frac{\bar{\Lambda}}{m_B} + 0.64 \frac{\lambda_1}{m_B^2} + 0.67 \frac{\lambda_2}{m_B^2} - 0.09 \frac{\bar{\Lambda}^2}{m_B^2} + 8.48 \frac{\rho_1}{m_B^3} + 3.79 \frac{\rho_2}{m_B^3} \right]
\]
Consider first the expression for $\langle s_H^2 \rangle$. The $\lambda_1$ term has a small coefficient and tends to cancel against higher order corrections, making this moment particularly insensitive to $\lambda_1$. One can see the problem another way by solving this equation for $\lambda_1$: the solution exhibits a pole near $\tilde{\Lambda} = 0.4$, close to the expected value of $\tilde{\Lambda}$ [61, 62]. As a result, the extracted value of $\lambda_1$ is extremely sensitive to the values of the higher order parameters. Since the presence of this pole persists as the value of the cut is changed, we conclude that this observable is unsuitable for extracting $\lambda_1$. For the first moment $\langle s_H \rangle$ the convergence of the OPE is much better. Estimating the uncertainties from the unknown values of the dimension six operators by the method explained in Section 3.2, we present the resulting

$$
\langle s_H^2 \rangle = m_B^4 \left[ -0.05 \frac{\lambda_1}{m_B^2} + 0.14 \frac{\tilde{\Lambda}^2}{m_B^2} - 0.53 \frac{\rho_1}{m_B^2} - 0.21 \frac{\rho_2}{m_B^2} + 0.63 \frac{\tilde{\Lambda} \lambda_1}{m_B^3} 
+ 0.46 \frac{\tilde{\Lambda} \lambda_2}{m_B^3} - 0.05 \frac{T_1 + 3T_2}{m_B^3} \right].
$$

(4.34)
constraint in the \( \bar{\lambda} - \lambda_1 \) plane in Fig. 4.5. Superimposed on this figure is the ellipse obtained in [71] from an analogous study of moments of \( \bar{B} \to X_c \ell \nu_\ell \). Unfortunately, the bound from our analysis constrains a similar linear combination of the two parameters, meaning that this moment does not provide much additional information about the values of \( \bar{\lambda} \) or \( \lambda_1 \).

4.5 Conclusions

In this Chapter of the thesis we studied nonperturbative uncertainties in the rare inclusive decay \( \bar{B} \to X_s \ell^+ \ell^- \). We have parameterized the corrections arising at \( \mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3 \) in terms of two matrix elements of local operators \( \rho_{1,2} \), four matrix elements of time-ordered products \( T_{-4} \), and one matrix element of a four-fermion operator \( f_1 \).

The numerical values of these parameters are unknown, yet even so knowledge of the analytic form of the corrections allows one to study the convergence properties of the operator product expansion in various regions of phase space. Furthermore, the assumption that these parameters, being of nonperturbative origin, should be \( \mathcal{O}(\Lambda_{\text{QCD}}^3) \) permits making numerical estimates of theoretical uncertainties in observable quantities.

We first considered the corrections to the differential spectrum \( d\mathcal{B}/d\hat{s} \). The experimental spectrum contains two prominent resonances due to intermediate \( J/\psi \) and \( \psi' \) production, and the necessity of cutting these resonances out divides the accessible spectrum into two parts: the region of low dilepton invariant mass below the \( J/\psi \) resonance, and the region of high dilepton invariant mass above the \( \psi' \) resonance. In the first region, we find that the parton level calculation dominates, and that nonperturbative corrections are small. The operator product expansion appears to be converging according to expectation and, should it be possible to take experimental data in this region, the results will not suffer from significant nonperturbative uncertainties.

On the other hand we do find that, as expected, the nonperturbative uncertainties
increase as one moves into the high dilepton invariant mass region. We also showed that the rate obtained by integrating over the entire region above the $\psi'$ resonance contains uncertainties from dimension six matrix elements at the 10% level, the uncertainties being dominated by those from the $\rho_1$ matrix element. This result will impact the potential for doing precise searches for new physics using data from this region of phase space.

We also studied the contributions from dimension six operators to the forward-backward asymmetry. This quantity probes different combinations of Wilson coefficients than the rate, and has been proposed as a complementary source of information about possible new physics effects. As with the rate, the spectrum $dA/ds$ is divided into two regions by the $J/\psi$ and $\psi'$ resonances. We find that the dimension six contributions are not unduly large anywhere in the phase space, suggesting that this observable has a well behaved OPE. However if the differential forward-backward asymmetry is normalized to the differential rate, the resulting spectrum contains large uncertainties in the high dilepton invariant mass region. In the region of phase space below the $J/\psi$ resonance, however, we find the nonperturbative corrections to be small.

Finally we addressed a recent proposal suggesting that hadron invariant mass moments of the differential spectrum for $\bar{B} \to X_s \ell^+ \ell^-$ could, due to the sensitivity of these moments to nonperturbative effects, be used to constrain the values of the HQET parameters $\lambda_1$ and $\bar{\lambda}$. Restricting the phase space to be above the $\psi'$ resonance, we found that the first of these moments $\langle s_H \rangle$ provided a constraint in the $\bar{\lambda} - \lambda_1$ plane, but that this constraint was nearly the same as those derived from other, more experimentally promising, processes, and therefore seems to be of limited interest for this purpose. As for the second invariant mass moment $\langle s^2_H \rangle$, we found that the nonperturbative uncertainties were such that it was not possible to extract a stable constraint on the values of $\bar{\lambda}$ or $\lambda_1$. From these results, we conclude that these moments are not well suited to the extraction of these parameters.
Chapter 5

A model independent determination
of $|V_{ub}|$

5.1 Introduction

As explained in Chapter 1, $|V_{ub}|$ corresponds to the length of one of the sides of the unitarity triangle, shown in Fig. 1.1. Thus, a precise and model independent determination of the CKM matrix element $V_{ub}$ is important for testing the standard model at $B$ factories via the comparison of the angles and the sides of this unitarity triangle. The first extraction of $|V_{ub}|$ from experimental data relied on a study of the lepton energy spectrum in inclusive charmless semileptonic $B$ decay [17]. Recently $|V_{ub}|$ was also measured from exclusive semileptonic $\bar{B} \to p\ell\bar{\nu}$ and $\bar{B} \to \pi\ell\bar{\nu}$ decay [16], and from inclusive decays using the reconstruction of the invariant mass of the hadronic final state [18].

These determinations suffer from large model dependence. The exclusive $|V_{ub}|$ measurements rely on form factor models or quenched lattice calculations at the present
Inclusive $B$ decay rates are currently on a better theoretical footing, since they can be computed model independently using the OPE described in Chapter 2. However, the predictions of the OPE are only model independent for sufficiently inclusive observables, while the $\bar{B} \to X_u \ell \bar{\nu}$ decay rate can only be measured by imposing severe cuts on the phase space to eliminate the $\sim 100$ times larger $\bar{B} \to X_c \ell \bar{\nu}$ background. For both the charged lepton and hadronic invariant mass spectra, these cuts spoil the convergence of the OPE, and the most singular terms must be resummed into a nonperturbative $b$ quark distribution function[73, 74]. While it may be possible to extract this from the photon spectrum in $\bar{B} \to X_s \gamma$ [73, 82] (see Eq. (3.25)), it would clearly be simpler to find an observable for which the OPE did not break down in the region of phase space free from charm background. In this Chapter we show that this is the situation for the lepton invariant mass spectrum.

5.2 The importance of the shape function

As discussed in Chapter 2, the OPE $\propto \dy$ converges for sufficiently inclusive observables, and breaks down when the phase space is restricted such that the final hadronic state is dominated by only a few resonances. This simply reflects the fact that an inclusive treatment based on local duality is no longer appropriate. However, when the kinematic cut is in a region of phase space dominated by highly energetic, low invariant mass final states, the OPE breaks down even for quantities smeared over a larger region of phase space, where the decay is not resonance dominated. In the latter case, a certain class of terms in the OPE, called the "most singular terms" in Section 3.3, all become of order unity and have to be resummed into a nonperturbative light-cone distribution function $f(k_+)$ [73]. The size of the smearing region required for the OPE to converge has to be

---

1A model independent determination of $|V_{ub}|$ from exclusive decays is possible without first order heavy quark symmetry or chiral symmetry breaking corrections [80], but it requires data on $\bar{B} \to K^* \ell^+ \ell^-$. A model independent extraction is also possible from decays to wrong-sign charm [81], but this is very challenging experimentally.
larger than the difference of the partonic and hadronic endpoint. For the electron energy spectra we find

$$\Delta E^{(\text{smear})}_\ell = \frac{m_B}{2} - \frac{m_b}{2} \sim \Lambda_{\text{QCD}}. \quad (5.1)$$

In inclusive semileptonic $B$ decay, for a particular hadronic final state $X$, the maximum lepton energy is $E^{(\text{max})}_\ell = (m_B^2 - m_X^2)/2m_B$ (in the $B$ rest frame). To eliminate charm background one must impose a cut $E_\ell > (m_B^2 - m_D^2)/2m_B$, leaving a region

$$\Delta E^{(\text{endpoint})}_\ell = m_D^2/2m_B \simeq 0.33 \text{ GeV} \sim \Lambda_{\text{QCD}} \quad (5.2)$$

to be integrated over. Since this is of the same size as (5.1), the distribution function is important. It may be included by convoluting the parton-level spectrum $d\Gamma_p/dE_\ell$ with this distribution function by replacing $m_b$ by $m^*_b \equiv m_b + k_+$, and integrating over the light-cone momentum

$$\frac{d\Gamma}{dE_\ell} = \int dk_+ f(k_+) \frac{d\Gamma_p}{dE_\ell |_{m_b \to m^*_b}}. \quad (5.3)$$

Analogous formulae hold for other differential distributions [83]. For purposes of illustration, we will use a simple model for the structure function given by the one-parameter ansatz [84]

$$f(k_+) = \frac{32}{\pi^2 \Lambda} (1 - x)^2 \exp \left[ -\frac{4}{\pi} (1 - x)^2 \right] \Theta(1 - x), \quad x \equiv \frac{k_+}{\Lambda}, \quad (5.4)$$

taking the model parameter $\Lambda = 0.48 \text{ GeV}$, corresponding to $m_b = 4.8 \text{ GeV}$.

The charm background can also be eliminated by reconstructing the invariant mass of the hadronic final state, $s_H$, since decays with $s_H < m_D^2$ must arise from the $b \to u$ transition. While this analysis is challenging experimentally, the $s_H < m_D^2$ cut allows a much larger fraction of $b \to u$ decays than the $E_\ell > (m_B^2 - m_D^2)/2m_B$ constraint. This is expected to result in a reduction of the theoretical uncertainties [33, 34], although both the lepton endpoint region, $E_\ell > (m_B^2 - m_D^2)/2m_B$, and the low hadronic invariant mass region, $s_H < m_D^2$, receive contributions from the same set of hadronic final states (but with very different weights). However, the same nonperturbative effects which lead to the
Figure 5.1: The lepton energy and hadronic invariant mass spectra. The dashed curves are the $b$ quark decay results to $O(\alpha_s)$, while the solid curves are obtained by smearing with the model distribution function $f(k_+)$ in Eq. (5.4). The unshaded side of the vertical lines indicate the region free from charm background. The area under each curve has been normalized to one.

breakdown of predictive power in the lepton endpoint region also give large uncertainties in the hadron mass spectrum [33] since the available phase space

$$\Delta s_H^{(endpoint)} = m_B^2$$

is numerically ($m_B^2 \sim 2m_B\Lambda_{QCD}$) of the same size as the difference of the partonic and hadronic endpoints

$$\Delta s_H^{(smear)} = m_B^2 - m_b^2 \approx 2m_B\Lambda_{QCD}.$$  

The situation is illustrated in Fig. 5.1.

The situation is very different for the lepton invariant mass spectrum. Decays with $q^2 \equiv (p_\ell + p_\nu)^2 > (m_B - m_D)^2$ must arise from $b \to u$ transition. Such a cut forbids the hadronic final state from moving fast in the $B$ rest frame, and so the light-cone expansion which gives rise to the shape function is not relevant in this region of phase space. This has first been pointed out in [85] in the context of $\bar{B} \to X_s\ell^+\ell^-$ decay and in [34] for semileptonic $\bar{B} \to X_u$ decay. This is clear from the kinematics: the difference of partonic
Figure 5.2: The lepton invariant mass spectrum to $\mathcal{O}(\alpha_s)$. The meaning of the curves is the same as in Fig. 5.1.

and hadronic endpoints is

$$\Delta q^2_{\text{(smear)}} = m_B^2 - m_b^2 \sim 2\Lambda_{QCD} m_b,$$

(5.7)

while we are integrating over the parametrically larger region

$$\Delta q^2_{\text{(endpoint)}} = 2m_B m_D - m_D^2 \approx 2m_B m_D,$$

(5.8)

which is numerically of order $2m_B^{3/2}\sqrt{\Lambda_{QCD}}$.

The better behavior of the $q^2$ spectrum than the $E_\ell$ and $s_H$ spectra is also reflected in the perturbation series. There are Sudakov double logarithms near the phase space boundaries in the $E_\ell$ and $s_H$ spectra, whereas there are only single logarithms in the $q^2$ spectrum. The connection between the convergence of OPE and the appearance of Sudakov logarithms will be discussed in Chapter 6.

The effect of smearing the $q^2$ spectrum with the model distribution function in Eq. (5.4) is illustrated in Fig. 5.2. In accord with our previous arguments, it is easily seen to be subleading over the region of interest. Table I compares qualitatively the utility of the lepton energy, the hadronic invariant mass, and the lepton invariant mass spectra for the determination of $|V_{ub}|$. 


<table>
<thead>
<tr>
<th>Decay distribution</th>
<th>Width of region without charm background</th>
<th>Nonperturbative region near endpoint</th>
<th>Fraction of $b \to u$ events included</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\Gamma/dE_\ell$</td>
<td>$\Delta E_\ell = m_B^2/2m_B$</td>
<td>$\Delta E_\ell \sim \Lambda_{QCD}$</td>
<td>$\sim 10%$</td>
</tr>
<tr>
<td>$d\Gamma/ds_H$</td>
<td>$\Delta s_H = m_D^2$</td>
<td>$\Delta s_H \sim \Lambda_{QCD}m_b$</td>
<td>$\sim 80%$</td>
</tr>
<tr>
<td>$d\Gamma/dq^2$</td>
<td>$\Delta q^2 = 2m_Bm_D - m_D^2$</td>
<td>$\Delta q^2 \sim \Lambda_{QCD}m_b$</td>
<td>$\sim 20%$</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison between the lepton energy, hadronic invariant mass, and lepton invariant mass spectra for the determination of $|V_{ub}|$. The region dominated by non-perturbative effects is parametrically smaller than the region without charm background only for the $q^2$ spectrum in the last row (viewing $m_B^2 \sim m_c^2 \sim \Lambda_{QCD}m_b$). The last column gives rough numbers corresponding to the plots in Figs. 1 and 2.

5.3 $|V_{ub}|$ from the dilepton invariant mass spectrum of inclusive semileptonic $B$ decays

We now proceed to calculate the $\bar{B} \to X_u \ell\bar{\nu}$ decay rate with lepton invariant mass above a given cutoff, working to a fixed order in the OPE (i.e., ignoring the light-cone distribution function, which is irrelevant for our analysis). The lepton invariant mass spectrum including the leading perturbative and nonperturbative corrections is given by

$$
\frac{1}{\Gamma_0} \frac{d\Gamma}{dq^2} = \left(1 + \frac{\lambda_1}{2m_b^2}\right) 2 (1 - \tilde{q}^2)^2 (1 + 2\tilde{q}^2) + \frac{\lambda_2}{m_b^2} (3 - 45\tilde{q}^4 + 30\tilde{q}^6)
+ \frac{\alpha_s(m_b)}{\pi} X(\tilde{q}^2) + \left(\frac{\alpha_s(m_b)}{\pi}\right)^2 \beta_0 Y(\tilde{q}^2) + \ldots,
$$

(5.9)

where $\tilde{q}^2 = q^2/m_b^2$, $\beta_0 = 11 - 2n_f/3$, and

$$
\Gamma_0 = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192 \pi^3}
$$

(5.10)

is the tree level $b \to u$ decay rate. The ellipses in Eq. (5.9) denote terms of order $(\Lambda_{QCD}/m_b)^3$ and order $\alpha_s^2$ terms not enhanced by $\beta_0$. The function $X(\tilde{q}^2)$ has been
Figure 5.3: The numerical result of the BLM corrections as a function of $q^2$. $Y(q^2)$, obtained from [87].

calculated analytically in Ref. [86]

$$X(q^2) = -\frac{2}{3} \left[ 2(1 - q^2)^2(1 + 2q^2)(\pi^2 + 2 \text{Li}_2(q^2) - 2 \text{Li}_2(1 - q^2)) 
+ 4q^2(1 - q^2 - 2q^4) \log(q^2) + 2(1 - q^2)^2(5 + 4q^2) \log(1 - q^2) 
- (1 - q^2)(5 + 9q^2 - 6q^4) \right], \quad (5.11)$$

whereas $Y(q^2)$ was computed numerically in Ref. [87] and is shown in Fig. 5.3. The order $(\Lambda_{QCD}/m_b)^3$ nonperturbative corrections can be obtained from the dilepton invariant mass spectrum for the decay $\bar{B} \to X_u \ell^+ \ell^-$ given in (4.15) by setting $C_9^{\text{eff}} = -C_{10} = 1/2$, $C_7 = 0$.

The relation between the total $\bar{B} \to X_u \ell^+ \ell^-$ decay rate and $|V_{ub}|$ can be calculated reliably using the OPE [88, 89] and is known including the full $\alpha_s^2$ corrections [90]. Thus, we shall only compute the fraction of $\bar{B} \to X_u \ell^+ \ell^-$ events with $q^2 > q_0^2$, $F(q_0^2)$. After integrating the spectrum in Eq. (5.9), we can eliminate the dependence on the $b$ quark mass in favor of the spin averaged meson mass [70, 91] $\bar{m}_B = (m_B + 3m_{B^*})/4 \approx 5.313$ GeV, using (2.55). We find

$$F(q_0^2) = 1 - 2Q_0^2 + 2Q_0^6 - Q_0^8 - \frac{4\Lambda}{m_B} \left( Q_0^2 - 3Q_0^6 + 2Q_0^8 \right) - \frac{6\Lambda^2}{m_B^2} \left( Q_0^2 - 7Q_0^6 + 6Q_0^8 \right) 
+ \frac{2\alpha_1}{\bar{m}_B^2} \left( Q_0^2 - 3Q_0^6 + 2Q_0^8 \right) - \frac{12\alpha_2}{\bar{m}_B^2} \left( Q_0^2 - 2Q_0^6 + Q_0^8 \right)$$
where $Q_0 \equiv q_0/\bar{m}_B$. The functions $\tilde{X}(Q_0^2)$ and $\tilde{Y}(Q_0^2)$ can be calculated from $X(q^2)$ and $Y(q^2)$. Converting to the physical $B$ meson mass has introduced a strong dependence on the parameter $\bar{\Lambda}$, the mass of the light degrees of freedom in the $B$ meson.

For $q_0^2 = (m_B - m_D)^2 \simeq 11.6 \text{ GeV}^2$, we find $F(11.6 \text{ GeV}^2) = 0.287 + 0.027\alpha_s(m_b) - 0.016\alpha_s^2(m_b)\beta_0 - 0.20\bar{\Lambda}/(1 \text{ GeV}) - 0.02\bar{\Lambda}^2/(1 \text{ GeV}^2) + 0.02\lambda_1/(1 \text{ GeV}^2) - 0.13\lambda_2/(1 \text{ GeV}^2) + \ldots$. The order $\alpha_s\bar{\Lambda}$ term is negligible and has been omitted. Using $\bar{\Lambda} = 0.4 \text{ GeV}$, $\lambda_1 = -0.2 \text{ GeV}^2$, we obtain $F(11.6 \text{ GeV}^2) = 0.186$. There are several sources of uncertainties in the value for $F$. The perturbative uncertainties are negligible, as can be seen from the size of the $O(\alpha_s^2\beta_0)$ contributions. At the present time there is a sizable uncertainty since $\bar{\Lambda}$ is not known accurately. In the future, a ±50 MeV error in $\bar{\Lambda}$ will result in a ±5% uncertainty in $F$. Finally, uncertainties from $1/m_b^2$ operators can be estimated by varying the matrix elements of the dimension six operators within the range expected by dimensional analysis, as discussed in detail in Chapter 3. This results in an additional ±4% uncertainty in $F$. As mentioned before, this is a somewhat ad hoc procedure, since there is no real way to quantify the theoretical error due to unknown higher order terms. Therefore, these estimates should be treated as nothing more than (hopefully) educated guesses. They do allow, however, for a consistent comparison of the uncertainties in different quantities.

If $q_0^2$ has to be chosen larger, then the uncertainties increase. For example, for $q_0^2 = 15 \text{ GeV}^2$, we obtain $F(15 \text{ GeV}^2) = 0.158 + 0.024\alpha_s(m_b) - 0.012\alpha_s^2(m_b)\beta_0 - 0.18\bar{\Lambda}/(1 \text{ GeV}) + 0.01\bar{\Lambda}^2/(1 \text{ GeV}^2) + 0.02\lambda_1/(1 \text{ GeV}^2) - 0.13\lambda_2/(1 \text{ GeV}^2) + \ldots \simeq 0.067$, using the previous values of $\bar{\Lambda}$ and $\lambda_1$. The perturbative uncertainties are still negligible, while the uncertainty due to a ±50 MeV error in $\bar{\Lambda}$ and unknown dimension six matrix elements increase to ±14% and ±13%, respectively.

Another possible method to compute $F(q_0^2)$ uses the upsilon expansion [88]. By expressing $q^2$ in terms of the $\Upsilon$ mass instead of $\bar{m}_B$, the dependence of $F(q_0^2)$ on $\bar{\Lambda}$ and $\lambda_1$
is eliminated. Instead, the result is sensitive to unknown nonperturbative contributions to $m_T$. The uncertainty related to these effects can be systematically taken into account and has been estimated to be small [88]. One finds,

$$|V_{ub}| = (3.04 \pm 0.06 \pm 0.08) \times 10^{-3} \left( \frac{B(\bar{B} \to X_u e\nu)_{q^2>q_0^2}}{0.001 \times F(q_0^2)} \frac{1.6 \text{ ps}}{\tau_B} \right)^{1/2} .$$

(5.13)

The errors explicitly shown in Eq. (5.13) are the estimates of the perturbative and non-perturbative uncertainties in the upsilon expansion, respectively.

For $q_0^2 = (m_B - m_D)^2$ we find $F(11.6 \text{ GeV}^2) = 0.168 + 0.016 \epsilon + 0.014 \epsilon_{\text{BLM}}^2 - 0.17 \lambda_2/1 \text{ GeV}^2 + \ldots \simeq 0.178$, where $\epsilon \equiv 1$ shows the order in the upsilon expansion. This result is in good agreement with 0.186 obtained from Eq. (5.12). The uncertainty due to $\Lambda$ is absent in the upsilon expansion, however the size of the perturbative corrections has increased. The uncertainties due to $(\Lambda_{\text{QCD}}/m_b)^3$ operators is estimated to be $\pm 7\%$. For $q_0^2 = 15 \text{ GeV}^2$, we obtain $F(15 \text{ GeV}^2) = 0.060 + 0.011 \epsilon + 0.011 \epsilon_{\text{BLM}}^2 - 0.14 \lambda_2/1 \text{ GeV}^2 + \ldots \simeq 0.064$, which is again in good agreement with 0.067 obtained earlier. For this value of $q_0^2$, the $(\Lambda_{\text{QCD}}/m_b)^3$ uncertainties increase to $\pm 21\%$. $F(q_0^2)$ calculated in the upsilon expansion is plotted in Fig. 5.4, where the shaded region shows our estimate of the uncertainty due to the $(\Lambda_{\text{QCD}}/m_b)^3$ corrections.

Concerning experimental considerations, measuring the $q^2$ spectrum requires reconstruction of the neutrino four-momentum, just like measuring the hadronic invariant mass spectrum. Imposing a lepton energy cut, which may be required for this technique, is not a problem. The constraint $q^2 > (m_B - m_D)^2$ automatically implies $E_\ell > (m_B - m_D)^2/2m_B \sim 1.1 \text{ GeV}$ in the $B$ rest frame. Even if the $E_\ell$ cut has to be slightly larger than this, the utility of our method will not be affected, but a dedicated calculation including the affects of arbitrary $E_\ell$ and $q^2$ cuts may be warranted.
Figure 5.4: (a) The fraction of $\bar{B} \to X_u \ell \bar{\nu}$ events with $q^2 > q^2_0$, $F(q^2_0)$, to order $\epsilon^2_{\text{BLM}}$ and $\Lambda_{\text{QCD}}^3/m_b^3$ in the upsilon expansion. The shaded region is the uncertainty due to $\Lambda_{\text{QCD}}^3/m_b^3$ terms, as discussed in the text. The dashed line indicates the lower cut $q^2_0 = (m_B - m_D)^2 \approx 11.6 \text{ GeV}^2$, which corresponds to $F = 0.178 \pm 0.012$. (b) The estimated uncertainty in $F(q^2_0)$ due to $\Lambda_{\text{QCD}}^3/m_b^3$ terms as a percentage of $F(q^2_0)$.

5.4 Including information of the $q^2$ spectrum of $B \to X_s \ell^+ \ell^-$

If experimental resolution on the reconstruction of the neutrino momentum necessitates a significantly larger cut than $q^2_0 = (m_B - m_D)^2$, then the uncertainties in the OPE calculation of $F(q^2_0)$ increase. In this case, it may instead be possible to obtain useful model independent information on the $q^2$ spectrum in the region $q^2 > m_{\psi(2S)}^2 \approx 13.6 \text{ GeV}^2$ from the $q^2$ spectrum in the rare decay $\bar{B} \to X_s \ell^+ \ell^-$ given in (4.15), which may be measured in the upcoming Tevatron Run-II. The $\bar{q}^2$-dependence of $C_9^{\text{eff}}$ yields negligible uncertainties if we use a mean $\tilde{C}_9$ obtained by averaging it in the region $0.5 < \bar{q}^2 < 1$ weighted with the $b$ quark decay rate $(1-\bar{q}^2)^2 (1+2\bar{q}^2)$. This results in the numerical value $\tilde{C}_9 = 4.47 + 0.44 i$. In the $q^2 > m_{\psi(2S)}^2$ region the $C_7^T$ contribution is negligible, and the $C_7 \tilde{C}_9$ term makes about a 20% contribution to the rate. For the $\tilde{C}_9^2 + C_{10}^2$ contributions nonperturbative effects are identical to those which occur in $\bar{B} \to X_u \ell \bar{\nu}$ decay, up to
corrections suppressed by $|\bar{C}_9 + C_{10}|/|\bar{C}_9 - C_{10}| \sim 0.02$. Therefore, the relation

$$\frac{d\Gamma(\bar{B} \to X_u \ell \bar{\nu})/dq^2}{d\Gamma(\bar{B} \to X_s \ell^+ \ell^-)/dq^2} = \frac{|V_{ub}|^2}{|V_{tb}|^2} \frac{8\pi^2}{\alpha^2} \frac{1}{|\bar{C}_9|^2 + |C_{10}|^2 + 12 \text{ Re}(C_7 \bar{C}_9)/(1 + 2q^2)},$$  \tag{5.14}

is expected to hold to a very good accuracy. There are several sources of corrections to this formula which need to be estimated: i) nonperturbative effects that enter the $C_7 \bar{C}_9$ term differently, ii) mass effects from the strange quark and muon, iii) higher $c\bar{c}$ resonance contributions in $\bar{B} \to X_s \ell^+ \ell^-$, and iv) scale dependence. Of these, i) and ii) are expected to be small unless $q^2$ is very close to $m_B^2$. The effects of iii) have also been estimated to be at the few percent level [85], although these uncertainties are very hard to quantify and could be comparable to the $\pm 8\%$ scale dependence [68] of the $\bar{B} \to X_s \ell^+ \ell^-$ rate. Integrating over a large enough range of $q^2$, $q_0^2 < q^2 < m_B^2$ with $m_{\psi(2S)}^2 < q_0^2 \leq 17 \text{ GeV}^2$, the result implied by Eq. (5.14),

$$\frac{B(\bar{B} \to X_u \ell \bar{\nu})|_{q^2 > q_0^2}}{B(\bar{B} \to X_s \ell^+ \ell^-)|_{q^2 > q_0^2}} = \frac{|V_{ub}|^2}{|V_{tb}|^2} \frac{8\pi^2}{\alpha^2} \frac{1}{|\bar{C}_9|^2 + |C_{10}|^2 + 12 \text{ Re}(C_7 \bar{C}_9) B(q_0^2)},$$  \tag{5.15}

is expected to hold at the $\sim 15\%$ level. Here $Q_0 \equiv q_0^2/m_B^2$, and $B(q_0^2) = 2/[3(1 + Q_0^2)] - 4(\tilde{\Lambda}/\tilde{m}_B) Q_0^2/[3(1 + Q_0^2)^2] + \ldots$. For $q_0^2$ significantly above $(m_B - m_D)^2$, this formula may lead to a determination of $|V_{ub}|$ with smaller theoretical uncertainty than the one using the OPE calculation of $F(q_0^2)$.

### 5.5 Conclusions

In conclusion, we have shown that the $q^2$ spectrum in inclusive semileptonic $\bar{B} \to X_u \ell \bar{\nu}$ decay gives a model independent determination of $|V_{ub}|$ with small theoretical uncertainty. Nonperturbative effects are only important in the resonance region, and play a parametrically suppressed role when $d\Gamma/dq^2$ is integrated over $q^2 > (m_B - m_D)^2$, which is required to eliminate the charm background. This is a qualitatively better situation than the extraction of $|V_{ub}|$ from the endpoint region of the lepton energy spectrum, or from the hadronic invariant mass spectrum.
Chapter 6

Sudakov Logarithms from Effective Field Theory

6.1 Introduction

As discussed in the previous Chapter, if the final state phase space in an inclusive decay is sufficiently restricted, the conventional OPE breaks down. This is due to the decay being dominated by only a few resonances, invalidating the inclusive treatment. However, in cases where the remaining phase space is dominated by highly energetic final state particles with low invariant mass, this breakdown occurs before the resonance regime dominates the decay. In this situation, a certain class of terms, formally suppressed by powers of $m_b$, is enhanced by a kinematic factor causing the breakdown of the OPE. However, it was shown in [73, 74] that these terms can be resummed resulting in the introduction of a universal structure function $f(k_+)$, as in (3.25). This structure function gives the probability of finding a $b$ quark with residual light-cone momentum $k_+$ inside the $\bar{B}$ meson [73, 74]. Once this resummation has been performed, the inclusive treatment is expected to be valid.

In addition to these large nonperturbative corrections, there are perturbative contri-
butions containing logarithmic terms which become large in the same regions of restricted phase space. These terms are of the form $\alpha_s^n \log^m \Delta$, with $n \leq m \leq 2n$ and $\Delta$ being the size of the restricted phase space. Such logarithms, known as Sudakov logarithms [92] are more singular than the single logarithmic terms occurring in standard perturbative expansions and only occur in such restricted regions of phase space. Since they spoil the convergence of the perturbative series, they must be summed to all orders as well. This may be done using an approach based on the factorization formalism developed by Collins, Soper, and Sterman [93].

Recently there has been some discussion in the literature of summing Sudakov logarithms using effective field theory (EFT) techniques [94, 95, 96]. Such an approach would have several advantages over the conventional method; in particular, while factorization formulas are based solely on perturbation theory, EFTs, by construction, include nonperturbative physics. Furthermore, it should be straightforward (if tedious) in an EFT approach to go beyond the leading twist approximation. In the various versions of the EFT approach which have been suggested, the structure function is a nonperturbative operator in a large energy effective theory (LEET) [97], which describes light-like particles coupled to soft degrees of freedom having momenta of $\mathcal{O}(\Lambda_{QCD})$. Sudakov logarithms that arise in matrix elements of this structure function are then summed using the renormalization group equations (RGEs). However, a difficulty with the approaches presented to date is that, as pointed out in Refs. [96], in the modified minimal subtraction ($\overline{MS}$) scheme logarithms arising at one-loop in LEET do not match logarithms arising at one-loop in QCD for any choice of the matching scale $\mu$; hence these logarithms may not be summed using the RGEs.

In this chapter we present an EFT containing both soft and collinear modes as physical degrees of freedom, the light-cone effective theory (LCET), and apply it to inclusive $\bar{B} \to X_s \gamma$ decays. We show that LCET can be matched onto QCD at the scale $\mu = m_b$ and that it can also be matched onto LEET at the scale $\mu = m_b \sqrt{y-x}$, where $x = 2E_/m_b$.
is the rescaled photon energy, and \(1 - y = k_+ / m_b\) is the residual light-cone momentum fraction of the \(b\)-quark in the \(B\)-meson. This makes the connection between QCD and LEET. We then calculate the RGEs in these two theories and show that they resum the Sudakov logarithms into an expression identical to that derived in Refs [98, 99].

### 6.2 Scales in \(\bar{B} \to X_s \gamma\) and LEET

As discussed in Chapter 3, the dominant contribution to the decay \(\bar{B} \to X_s \gamma\), arises from the magnetic penguin operator [66]

\[
O_7 = \frac{e}{16\pi^2} m_b \bar{s} \sigma^{\mu\nu} \frac{1}{2} (1 + \gamma_5) b F_{\mu\nu}, \tag{6.1}
\]

where the strange quark mass has been set to zero.\(^1\) It is helpful to write the kinematics of the decay as shown in Fig. 6.1, with the momenta of the \(b\) quark, photon, and light \(s\) quark given by

\[
p_b^\mu = m_b v^\mu + k^\mu, \quad q^\mu = \frac{m_b}{2} x \bar{n}^\mu, \quad p_s^\mu = \frac{m_b}{2} n^\mu + l^\mu + k^\mu \tag{6.2}
\]

where, in the rest frame of the \(\bar{B}\) meson,

\[
v^\mu = (1, 0, 0, 0), \quad n^\mu = (1, 0, 0, -1), \quad \bar{n}^\mu = (1, 0, 0, 1). \tag{6.3}
\]

Here \(k^\mu\) is a residual momentum of order \(\Lambda_{\text{QCD}}\), and \(l^\mu = \frac{m_b}{2} (1 - x) \bar{n}^\mu\), where \(x = 2E_\gamma / m_b\). The invariant mass of the light \(s\)-quark jet

\[
p_s^2 \approx m_b n \cdot (l + k) = m_b^2 (1 - x + \hat{k}_+) , \tag{6.4}
\]

where \(\hat{k}_+ = k_+ / m_b\), is \(O(m_b^2)\) except near the endpoint of the photon energy spectrum, where \(x \to 1\). In the usual OPE, inclusive quantities are calculated by taking the imaginary part of the graphs in Fig. 6.1 and expanding in powers of \(k^\mu / \sqrt{p_s^2}\). As long as \(x\) is not too close to the endpoint, this is an expansion in powers in \(k^\mu / m_b\), which
matches onto local operators. As shown in detail in Chapter 3 this leads to an expansion for the photon energy spectrum as a function of $x$ in powers of $\alpha_s$ and $1/m_b$ [69]:

\[
\frac{d\Gamma}{dx} = \Gamma_0 \left\{ \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( 2 \log \frac{\mu^2}{m_b^2} + 5 + \frac{4}{3} \pi^2 \right) \right] \delta(1 - x) \right. \\
+ \frac{\alpha_s C_F}{4\pi} \left[ 7 + x - 2x^2 - 2(1 + x) \log(1 - x) - \left( 4 \frac{\log(1 - x)}{1 - x} + \frac{7}{1 - x} \right) \right] \left. \\
+ \frac{1}{2m_b^2} \left[ \lambda_1 \delta(1 - x) - \left( \lambda_1 + 3\lambda_2 \right) \delta'(1 - x) - \frac{\lambda_1}{3} \delta''(1 - x) \right] \right\} \\
+ O(\alpha_s^2, 1/m_b^3),
\]  

(6.5)

where

\[
\Gamma_0 = \frac{G_F^2 |V_{tb}V_{ts}^*|^2 \alpha |C_7(\mu)|^2}{32\pi^4} m_b^5 \left[ \frac{m_b(\mu)}{m_b} \right]^2,
\]  

(6.6)

and the subscript"+" denotes the usual plus distribution,

\[
\frac{1}{(1 - x)_+} \equiv \lim_{\beta \to 0} \left\{ \frac{1}{1 - x} \theta(1 - x - \beta) + \log(\beta) \delta(1 - x - \beta) \right\}
\]

\[
\left( \frac{\log(1 - x)}{1 - x} \right)_+ \equiv \lim_{\beta \to 0} \left\{ \frac{\log(1 - x)}{1 - x} \theta(1 - x - \beta) + \frac{1}{2} \log^2(\beta) \delta(1 - x - \beta) \right\}.
\]  

(6.7)

The parameters $\lambda_1$ and $\lambda_2$ are matrix elements of local dimension five operators.

Near the endpoint of the photon spectrum, $x \to 1$, both the perturbative and non-perturbative corrections are singular and the OPE breaks down. The severity of the breakdown is most easily seen by integrating the spectrum over a region $1 - \Delta < x < 1$.

\footnote{As in Chapter 3 we will ignore the contribution of operators other than $O_7$ to the decay.}
In this region the most singular terms in the $1/m_b$ expansion sum up into a nonperturbative shape function of characteristic width $\Lambda_{\text{QCD}}/m_b$, and hence must be resummed for $\Delta \leq \Lambda_{\text{QCD}}/m_b$ [73]. The perturbative series is of the form

$$\frac{1}{\Gamma_0} \int_{1-\Delta}^{1} \frac{dT}{dx} = 1 + \frac{\alpha_s C_F}{4\pi} \left(-2 \log^2 \Delta - 7 \log \Delta + \ldots\right) + \mathcal{O} \left(\alpha_s^2\right)$$

(6.8)

where the ellipses denote terms that are finite as $\Delta \to 0$. The presence of these large double logarithmic terms (Sudakov logarithms) can spoil the convergence of perturbation theory. The full series has been shown to exponentiate [98, 99] and terms of the form

$$\exp \left[ \sum_n \left(a_n \alpha_s^n \log^{n+1} \Delta + b_n \alpha_s^n \log^n \Delta\right) \right]$$

(6.9)

must be resummed for $\Delta \leq \exp \left(-\sqrt{\pi / \alpha_s (m_b)}\right)$, which is parametrically larger than $\Lambda_{\text{QCD}}/m_b$ in the $m_b \to \infty$ limit [100].

Such large logs are to be expected, since near the endpoint of the photon energy spectrum the invariant mass of the light quark jet scales as $m_b \sqrt{1-x}$, while the OPE is performed at $\mu = m_b$. In order to sum logarithms of $\Delta$ (or the more complicated plus distributions in the differential spectrum, Eq. 6.5) we would expect to have to switch to a new effective theory at $\mu = m_b$, use the renormalization group to run down to a scale of order $m_b \sqrt{1-x}$, at which point the OPE is be performed. (In fact, we will see that the situation is slightly more complicated than this).

We are then left with the question of the appropriate theory below the scale $m_b$. The simplest possibility is to expand the theory in powers of $k^\mu/m_b$ and $\ell^\mu/m_b$. The heavy quark is therefore treated in the heavy quark effective theory (HQET), while the light quark propagator is treated in the large energy effective theory (LEET) proposed many years ago by Dugan and Grinstein[97]. Expanding the $s$ quark propagator, we find the LEET propagator

$$\frac{i\not{p}_s}{p_s^2} \simeq \not{p} \frac{i}{2n \cdot (\ell + k)} + \ldots$$

(6.10)

In LEET, we have an effective theory of lightlike Wilson lines, much as HQET is an effective theory of timelike Wilson lines[94]. The hope would then be to match QCD
onto LEET and then run down to the scale $m_b \sqrt{1 - x}$ where the OPE is performed; this is the approach taken in [96]. However, a simple attempt at matching shows that this does not sum the appropriate logarithms.

Consider the one-loop matching of the operator $\hat{O}_\tau$ from QCD to LEET. We regulate ultraviolet (UV) divergences with dimensional regularization ($d = 4 - 2\epsilon$). We introduce a small invariant mass $p_s^2$ for the $s$ quark which regulates all infrared (IR) divergences except that in the heavy-quark wave function diagram, Fig. 6.2(b). This IR divergence is regulated using dimensional regularization. The vertex diagram, Fig. 6.2(a) yields

$$A_{QCD}^{(a)} = -\bar{s}\Gamma^\mu m_b \frac{\alpha_s C_F}{4\pi} \left[ \log \frac{p_s^2}{m_b^2} + 2 \log \frac{p_s^2}{m_b^2} + \pi^2 \right],$$

(6.11)

where

$$\Gamma^\mu = \frac{e}{8\pi^2} m_b \sigma^{\mu \nu} \frac{(1 + \gamma_5)}{2} q_\nu.$$

(6.12)

According to the LSZ reduction formula we have to multiply this result by a factor of $\sqrt{Z}$ for each external field

$$Z_b = 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{3}{\epsilon} + 3 \log \frac{\bar{\mu}^2}{m_b^2} + 4 \right],$$

(6.13)

$$Z_s = 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{\epsilon} - \log \frac{p_s^2}{m_b^2} + \log \frac{\bar{\mu}^2}{m_b^2} - 1 \right],$$

(6.14)

where

$$\bar{\mu}^2 = 4\pi \mu^2 e^{-\gamma_E}.$$  

(6.15)

Adding the counterterm required to subtract off the UV divergence

$$Z_\tau = 1 + \frac{\alpha_s C_F}{4\pi} \frac{1}{\epsilon},$$

(6.16)

we find

$$A_{QCD} = \bar{s}\Gamma^\mu m_b \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \log \frac{p_s^2}{m_b^2} + \frac{3}{2} \log \frac{p_s^2}{m_b^2} + \frac{1}{\epsilon} + 2 \log \frac{\bar{\mu}^2}{m_b^2} + \pi^2 + \frac{3}{2} \right) \right].$$

(6.17)
Figure 6.2: One-loop corrections to the matrix element of \( \hat{O}_7 \) in QCD.

The corresponding LEET diagrams are shown in Fig. 6.3. Neither of the wave function graphs give a contribution, since the light quark wave function is proportional to \( n^2 = 0 \) and the heavy quark wave function vanishes in dimensional regularization. Thus the only contribution is from the vertex graph, and we find

\[
A_{\text{LEET}} = -\xi_n \Gamma^\mu \Gamma_5 \frac{\alpha_s C_F}{4\pi} \left( 4\pi \frac{\mu^2 m_h^2}{p_s^4} \right) \frac{\Gamma(1 + 2\epsilon)\Gamma(1 - 2\epsilon)\Gamma(1 + \epsilon)}{\epsilon^2}
\]

\[
= -\xi_n \Gamma^\mu \Gamma_5 \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{\epsilon^2} - \frac{2}{\epsilon} \log \frac{p_s^2}{\mu^2} - \frac{1}{\epsilon} \log \frac{\mu^2}{m_h^2}
+ 2 \log^2 \frac{p_s^2}{m_h^2} + \frac{1}{2} \log^2 \frac{\mu^2}{m_h^2} - 2 \log \frac{p_s^2}{m_h^2} \log \frac{\mu^2}{m_h^2} + \frac{3\pi^2}{4} \right], \quad (6.18)
\]

The difference between the calculations in the two theories must be absorbed into the coefficient of operators in the effective theory. We immediately notice two problems:

1. The LEET graph contains a divergent term proportional to \( \frac{1}{\epsilon} \log p_s^2 \). Since \( p_s^2 \) is an IR scale in the problem, it is not clear how to sensibly renormalize this term. In Refs. [96] this nonlocal divergence was cancelled by a nonlocal counterterm; however, this matching still leaves large logarithms in the coefficient of this operator.
2. There is no matching scale $\mu$ where all the logarithmic dependence on $p^2/m^2_b$ cancels. Thus, there are large logarithms in the matching coefficients, which cannot be summed using the renormalization group.

The problem is that LEET only describes the coupling of light-like particles to soft gluons, but does not describe the splitting of an energetic particle into two almost collinear particles. Thus, by matching onto LEET one is integrating out the collinear modes which also contribute to infrared physics. As we will show below, once collinear degrees of freedom are included, both of the above problems are resolved.

### 6.3 The Light-cone Effective Theory

#### 6.3.1 Collinear and soft modes

The importance of collinear modes is of course well known, and is reflected in the standard perturbative QCD approach by the factorization of decay amplitudes into hard, soft and jet (collinear) pieces [93]. To perform a similar factorization in an effective field theory, we first identify the various important momentum modes in the problem. For the moment, let us neglect the residual momentum $k$. It is convenient to introduce light-cone coordinates $p^\mu = (p^+, p^-, p_1^\perp)$, where $p^+ = n \cdot p$ and $p^- = \bar{n} \cdot p$. In these coordinates the three momenta given in (6.2) are

$$p_b^\mu = (m_b, m_b, 0), \quad q^\mu = (x m_b, 0, 0), \quad p'_\perp^\mu = (m_b(1 - x), m_b, 0).$$

(6.19)

We can identify three relevant scales which are widely separated as $x \to 1$. The first is the mass of the $b$ quark, $m_b$, the second is the invariant mass of the jet of light hadrons, $\sqrt{p_\perp^2} = m_b \sqrt{1 - x}$, and the third is the projection of the jet momentum onto the direction of the photon momentum, $p'_\perp^+ = m_b(1 - x)$.

The situation here is similar to that encountered in non-relativistic QCD (NRQCD), in which loop graphs dominated by widely separated scales occur in a single process. In
NRQCD the important scales are known as hard, soft, ultrasoft and potential, depending on how the energy and momentum scale with \( v \), the magnitude of the three-velocity of the heavy quark. These scales may be dealt with either by breaking the integrals up by hand in a threshold expansion [101], or in an effective field theory by introducing separate fields for each relevant low-energy momentum scale [102]. We will follow the second procedure: to make the power counting manifest and to obtain the Feynman rules in the intermediate theory we introduce separate fields to describe light-like soft and collinear fields. Let us define a parameter \( \lambda \sim \sqrt{1-x} \) that becomes small in the limit \( x \to 1 \). Collinear modes have momentum \( p \sim m_\text{b}(1, \lambda^2, \lambda) \) and have a characteristic invariant mass of \( \sqrt{p^2} = \mathcal{O}(\lambda) \), while soft modes have momentum \( p \sim m_\text{b}(\lambda^2, \lambda^2, \lambda^2) \), and have a smaller characteristic invariant mass of \( \mathcal{O}(\lambda^2) \).

Between the scales \( m_\text{b} \) and \( m_\text{b}\sqrt{1-x} \) the effective theory contains both collinear and soft modes. We call this theory light-cone effective theory (LCET). Below the scale \( \mu = m_\text{b}\sqrt{1-x} \) the collinear modes have large invariant mass, and are integrated out of the theory. As we will show in the next section, this corresponds to LEET, restricted to inclusive processes. To maintain manifest power counting in \( \lambda \), the various interactions in the theory must be expanded in powers of \( \lambda \) in the Lagrangian; this is analogous to the multipole expansion which must be performed in NRQCD [103].

The Feynman rules in LCET are obtained by expanding the interactions of the various modes in powers of \( \lambda \). The propagators are shown in Fig. 6.4. The soft sector of LCET

\[
\begin{align*}
  i \frac{\bar{p} \cdot p}{2p^2} & \quad \text{collinear quark} \quad -i \frac{g^{uv}}{k^2} \quad \text{collinear gluon} \\
  \frac{i P_+}{\nu p} & \quad \text{heavy quark} \quad -i \frac{g^{uv}}{k^2} \quad \text{soft gluon}
\end{align*}
\]

Figure 6.4: Propagators in the light-cone effective theory.

is equivalent to LEET. To see this, consider the emission of a soft line of momentum \( k \)
from a collinear line of momentum \( p \). The momentum of the collinear particle after the interaction is

\[
p - k \sim (p^+ - k^+, p^-, p^\perp),
\]

where only the leading powers of \( \lambda \) are kept. Thus, collinear propagators in soft loops reduce to LEET propagators:

\[
\frac{\not{p} \cdot (p - k)}{2 (p - k)^2} \sim \frac{\not{p}}{2} \frac{p^-}{(p - k)^+ p^- - (p^\perp)^2} = -\frac{\not{p}}{2} \frac{1}{n \cdot k},
\]

since \( p^2 = 0 \) from the equation of motion. (This is analogous to NRQCD, where in ultrasoft loops the Feynman rules reduce to those for HQET.)

The Feynman rules for the leading interactions in LCET are shown in Fig. 6.5. There

\[
\includegraphics{fig6_5.png}
\]

Figure 6.5: Leading order interactions in the light-cone effective theory.

is, however, a subtlety regarding the coupling of collinear gluons to collinear quarks. Naively this coupling scales as \( 1/\lambda \), which appears to violate power counting; however, graphs which scale like \( 1/\lambda^n \) vanish in the effective theory. Note also that there is no vertex between two heavy quarks and a collinear gluon, since a heavy quark cannot emit a collinear gluon and stay on its mass shell. Instead, one can see by expanding the corresponding QCD amplitude that the coupling of the collinear gluon to the heavy quark occurs through the nonlocal vertex shown in Fig. 6.6. This vertex is \( \mathcal{O}(\lambda) \), but it does not vanish when coupled to the \( \mathcal{O}(1/\lambda) \) collinear - collinear vertex so it gives an \( \mathcal{O}(1) \) contribution.

The spinors in LCET are related, at leading order, to the QCD spinors via

\[
h_v = P_+ u, \quad \xi_n = P_n u, \quad \xi_n = P_n u,
\]

(6.22)
where we have defined the projection operators
\[ P_+ = \frac{Q + 1}{2}, \quad P_n = \frac{p_n}{4}, \quad P_\bar{n} = \frac{p_{\bar{n}}}{4}, \]
(6.23)
which project out the heavy quark spinor, a massless spinor in the \(n\) direction, and a massless spinor in the \(\bar{n}\) direction respectively.

### 6.3.2 Matching onto LCET

At tree level, the matching of \(\hat{O}_7\) onto LCET is trivial. Defining the current in LCET by
\[ V^\mu = \bar{\xi}_n \Gamma^\mu h_0, \]
(6.24)
where \(\Gamma^\mu\) is given in (6.12), the Wilson coefficient \(C_V\) at tree level is
\[ C_V = 1 + O(\alpha_s). \]
(6.25)

To perform this matching at one-loop, we repeat the one-loop matching calculation discussed in Section 6.2, but now using LCET instead of LEET, hence including collinear modes. The calculation is simplest if we set the invariant mass of the \(s\) quark to zero; this introduces additional infrared divergences to the calculation which cancel in the matching conditions. The one loop matrix element of \(\hat{O}_7\) in full QCD can be calculated from the diagrams in Fig. 6.2, and we find the amplitude
\[ A_{QCD} = \bar{s} \Gamma^\mu b \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \frac{1}{\epsilon^2} + \frac{5}{2\epsilon} + \frac{1}{2} \log \frac{\mu^2}{m_s^2} + \frac{7}{2} \log \frac{\mu^2}{m_b^2} + \frac{\pi^2}{12} + 6 \right) \right]. \]
(6.26)
where all the $1/\varepsilon$ divergences are infrared in origin.

The one-loop correction in LCET can be calculated from the Feynman diagrams in Figs. 6.3 and 6.7. In pure dimensional regularization all graphs are zero, as there is no scale present in the loop integrals. Thus, we find the matching condition

$$C_V Z_V = \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \frac{\pi^2}{12} + 6 \right) \right] \left[ 1 + \frac{\alpha_s C_F}{4\pi} \left( \frac{1}{\varepsilon^2} + \frac{\log(\mu^2/m_b^2)}{\varepsilon} + \frac{5}{2\varepsilon} \right) \right],$$

(6.27)

where $Z_V$ is the counterterm required to subtract the UV divergences in LCET. This derivation of course assumes that LCET reproduces the infrared behaviour of QCD. We can check this by instead introducing a small invariant mass for the $s$ quark, as in Section 6.2, and explicitly verifying that the dependence on the invariant mass in LCET is identical to that in full QCD given in Eq. (6.17).

The soft gluon contribution in LCET is identical to the LEET result, given in (6.18)

$$A_s = -C_V \xi_n \alpha_s C_F \frac{4\pi}{4\pi} \left[ \frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} \log \frac{p_s^2}{\mu^2} - \frac{1}{\varepsilon} \log \frac{\mu^2}{m_b^2} \right. + 2 \log \frac{p_s^2}{m_b^2} + \frac{1}{2} \log \frac{m_b^2}{\mu^2} - 2 \log \frac{p_s^2}{m_b^2} \log \frac{\mu^2}{m_b^2} + \frac{3\pi^2}{4} \left. - \frac{2}{\varepsilon} \right] .$$

(6.28)

The collinear vertex diagram, Fig. 6.7(a), gives

Figure 6.7: The one-loop collinear gluon corrections to the vertex $V^\mu$.

$$A_c^{(v)} = C_V \xi_n \alpha_s C_F \frac{4\pi}{2\pi} \left( \frac{\mu^2}{p_s^2} \right)^\varepsilon \left[ 1 - \frac{1 - \varepsilon}{2} \Gamma(1 + \varepsilon) \frac{1}{\Gamma(1 - 2\varepsilon)} \frac{1}{\mu^2} \right]$$

$$= -C_V \xi_n \alpha_s C_F \frac{4\pi}{4\pi} \left[ -\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon^2} \log \frac{p_s^2}{\mu^2} - \log \frac{p_s^2}{m_b^2} + 2 \log \frac{p_s^2}{m_b^2} \log \frac{\mu^2}{m_b^2} - \log \frac{\mu^2}{m_b^2} + 2 \log \frac{p_s^2}{m_b^2} - 2 \log \frac{\mu^2}{m_b^2} + \frac{\pi^2}{6} - 4 \right].$$

(6.29)
The wave function graph, Fig. 6.7(b), gives the same result as in full QCD (6.14), thus we obtain for the contribution of the collinear gluons

$$A_c = -C_V \xi_n \Gamma^\mu h \frac{\alpha_s C_F}{4\pi} \left[ -\frac{2}{\epsilon^2} - \frac{3}{2\epsilon} + \frac{2}{\epsilon} \log \frac{p_s^2}{\mu^2} - \log^2 \frac{p_s^2}{m_b^2} + 2 \log \frac{p_s^2}{m_b^2} \log \frac{\mu^2}{m_b^2} - \log^2 \frac{\mu^2}{m_b^2} + \frac{3}{2} \log \frac{p_s^2}{m_b^2} - \log \frac{\mu^2}{m_b^2} + \frac{\pi^2}{6} - \frac{9}{2} \right] \quad (6.30)$$

Adding the soft and collinear contributions, as well as the counterterm given in (6.27), we obtain

$$A_{cs} = -C_V \xi_n \Gamma^\mu h \frac{\alpha_s C_F}{4\pi} \left[ \log^2 \frac{p_s^2}{m_b^2} + \frac{3}{2} \log \frac{p_s^2}{m_b^2} + \frac{1}{\epsilon} - \frac{1}{2} \log^2 \frac{\mu^2}{m_b^2} - \frac{3}{2} \log \frac{\mu^2}{m_b^2} + \frac{11\pi^2}{12} - \frac{9}{2} \right] \quad (6.31)$$

Note that the troublesome divergence $\sim \frac{1}{\epsilon} \log p_s^2$ cancels once the two contributions (6.28) and (6.30) are added. Thus, both collinear and soft modes are required for the theory to be renormalized sensibly. Comparing to the full theory result (6.17), we see that LCET reproduces the IR physics of QCD, and that at the scale $\mu = m_b$ all nonanalytic terms vanish. This determines the matching scale to be $m_b$, and the matching coefficient at one loop is

$$C_V(m_b) = 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{\pi^2}{12} + 6 \right] \quad (6.32)$$

confirming the result found by calculating in pure dimensional regularization (6.27).

### 6.3.3 Renormalization group equations

From the counterterm given in (6.27) it is simple to extract the anomalous dimension of the operator $V^\mu$ in LCET. From the definition,

$$\gamma_V = Z_V^{-1} \left( \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) Z_V \quad (6.33)$$

we have

$$\frac{\partial}{\partial \mu} Z_V = \frac{\alpha_s(\mu) C_F}{2\pi \epsilon}$$

$$\frac{\beta}{\partial g} Z_V = -\frac{\alpha_s(\mu) C_F}{2\pi} \left( \frac{1}{\epsilon} + \log \frac{\mu^2}{m_b^2} + \frac{5}{2} \right) \quad (6.34)$$
where we have used $\beta = -g\epsilon + \mathcal{O}(g^2)$ in the $\overline{\text{MS}}$ scheme. This gives the anomalous dimension

$$\gamma_V = -\frac{\alpha_s(\mu)C_F}{2\pi} \left( \log \frac{\mu^2}{m_b^2} + \frac{5}{2} \right).$$

(6.35)

Note that the divergent piece of the anomalous dimension cancels between the two terms [96]. The RGE for the coefficient of the operator $V^\mu$ is therefore

$$\mu \frac{d}{d\mu} C_V(\mu) = \gamma_V C_V(\mu).$$

(6.36)

Solving this RGE we obtain

$$C_V(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s} \right)^{\frac{C_F}{2\beta_0}} \left( \frac{s - \frac{4}{3\alpha_s}}{s_0} \right)^{-\frac{C_F}{2\beta_0}} C_V(m_b),$$

(6.37)

where $\alpha_s \equiv \alpha_s(m_b)$. $\beta_0 = 11 - 2/3n_f$, and $C_V(m_b) = 1 + \mathcal{O}(\alpha_s(m_b))$.

### 6.4 The soft theory

#### 6.4.1 Matching

The light-cone effective theory is valid down to $\mu \approx m_b\sqrt{1-x}$, the typical invariant mass of the light $s$-quark jet. At this scale we perform an OPE; however, instead of expanding in powers of $k/m_b$ (which leads to an OPE in terms of local operators), we expand in powers of both $k/m_b$ and $l/m_b$. This leads to an expansion in terms of nonlocal operators where two vertices are separated along a light-cone direction. The leading term in this OPE is the nonlocal operator

$$O(y) = \bar{h}_u \delta(1 - y + i\hat{D}_+) h_v,$$

(6.38)

where $\hat{D}_+ = D_+/m_b$. The hadronic matrix element of this operator is the well known structure function [73]

$$f(y) = \frac{\langle B|\bar{h}_u\delta(1 - y + i\hat{D}_+)h_v|B\rangle}{\langle B|\bar{h}_u h_v|B\rangle},$$

(6.39)
while between quark states we have

\[ \langle b(k) | \tilde{h}_v \delta(1 - y + i \hat{D}_+) h_v | b(k) \rangle = \delta(1 - y + \hat{k}_+) + O(\alpha_s). \]  

(6.40)

Thus, LEET consists of a continuous set of operators labeled by \( y \). Each operator has a coefficient that depends on the kinematic variable \( x \), and the differential rate for \( \bar{B} \to X_s \gamma \) is given by the integral

\[ \frac{d\Gamma}{dx} = \Gamma_0 \int dy C(y, x; \mu) f(y; \mu). \]  

(6.41)

To match onto LEET at one-loop we compare the differential decay rate in the parton model, \( b \to X_s \gamma \), which in LEET is

\[ \frac{d\Gamma}{dx} \bigg|_{\hat{k}_+} = \Gamma_0 \int dy C(y, x; \mu) \langle b(k) | O(y; \mu) | b(k) \rangle. \]  

(6.42)

We therefore need the one-loop matrix element of \( O(y) \) between quark states. This may be calculated from the diagrams shown in Fig. 6.8, using the Feynman rules given in Fig. 6.9 \([104]\). Again all divergences are regulated in dimensional regularization. As an example, Fig. 6.8(a) gives

\[ \langle b(k) | O^{(a)}(y) | b(k) \rangle = -C_F g^2 \left( \frac{\mu}{m_b} \right)^{4-d} \times \]

\[ \times \int \frac{d^{d-2} \hat{q}_- \ dq_+ \ dq_{\perp} \ \delta(\hat{k}_+ + 1 - y) - \delta(\hat{k}_+ + \hat{q}_+ + 1 - y)}{(2\pi)^{d-2} 2\pi (\hat{q}_- \hat{q}_- - \hat{q}_-^2 + i\epsilon)(\hat{q}_+ + \hat{q}_- + i\epsilon)\hat{q}_+}. \]  

(6.43)

Figure 6.8: Feynman diagrams contributing to the one-loop matrix element of the structure function.
The first term is proportional to

\[
\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\epsilon)(q \cdot n)(q \cdot n)}
\]

\[
= 8 \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \int_0^\infty d\lambda \frac{\lambda}{(q^2 + 2\lambda q \cdot (u(1-x) + x n))^3}
\]

\[
= -\frac{4i}{(4\pi)^{d/2}} \Gamma(3-d/2) \int_0^1 dx \int_0^\infty d\lambda \lambda^{d-5} ((1-x)^2 + 2(1-x))^{d/2-3}. \quad (6.44)
\]

The \(\lambda\) integral vanishes in dimensional regularization, and so this term vanishes. After performing the trivial \(\hat{q}_+\) integral in the second term, we are left with

\[
\langle b(k)|O^{(a)}(y)|b(k)\rangle = -\frac{C_F g^2}{2\pi} \left( \frac{\mu}{m_b} \right)^{4-d} \frac{1}{k_++1-y} \times
\]

\[
\times \int \frac{d^{d-2}\hat{q}_-}{(2\pi)^{d-2}} \frac{1}{2\pi} \frac{1}{\hat{q}_-(k_++1-y) + \hat{q}_-^2 - i\epsilon \hat{q}_- - (k_++1-y) + i\epsilon}
\]

\[
= i \frac{C_F g^2}{2\pi} \left( \frac{\mu}{m_b} \right)^{4-d} \frac{\theta(k_++1-y)}{k_++1-y} \int \frac{d^{d-2}\hat{q}_+}{(2\pi)^{d-2}} \frac{1}{\hat{q}_+^2 + (k_++1-y)^2}
\]

\[
= i \frac{C_F g^2}{8\pi^2} \left( \frac{\mu^2}{m_b^2} \right)^\epsilon \frac{\theta(k_++1-y)}{(k_++1-y)^{1+2\epsilon}}. \quad (6.45)
\]

Using the identity

\[
\frac{\theta(y-x)}{(y-x)^{1+2\epsilon}} = -\frac{1}{2\epsilon} \delta(y-x) + \theta(y-x) \left[ \frac{1}{(y-x)_+} - 2\epsilon \left( \frac{\log(y-x)}{(y-x)} \right)_+ + O(\epsilon^2) \right]. \quad (6.46)
\]

we find

\[
\langle b(k)|O^{(a)}(y)|b(k)\rangle = -i \frac{\alpha_s C_F}{4\pi} \left\{ \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\hat{m}_b^2}{m_b^2} + \frac{1}{2} \log^2 \frac{\hat{m}_b^2}{m_b^2} + \frac{\pi^2}{12} \right) \delta(1-y + \hat{k}_+)
\]

\[
- \theta(1-y + \hat{k}_+) \left[ \left( \frac{2}{\epsilon} + 2\log \frac{\hat{m}_b^2}{m_b^2} \right) \frac{1}{(1-y + \hat{k}_+)_+} \right]
\]

Figure 6.9: The Feynman rules for the LEET operator \(O(y)\).
The diagram in Fig. 6.8(b) gives the same result as (a), while the diagram in Fig. 6.8(c) gives

\[ \langle b(k)|O^{(c)}(y)|b(k) \rangle = -i \frac{\alpha_s C_F}{4\pi} \left( \frac{2}{\epsilon} - 2 \log \frac{\mu^2}{m_b^2} \right) \delta(1 - y + \hat{k}_+) + 4 \frac{\theta(1 - y + \hat{k}_+)}{(1 - y + \hat{k}_+)_+} \].

(6.48)

In dimensional regularization the wavefunction diagrams vanish. Since the decay rate is infrared finite, including the wavefunction graphs simply converts an infrared $1/\epsilon$ divergence to an ultraviolet divergence. Therefore, we may neglect the wavefunction counterterm. and combining all graphs we find the bare matrix element

\[ \langle b(k)|O^{\text{bare}}(y)|b(k) \rangle = i \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log \frac{\mu^2}{m_b^2} - \frac{2}{\epsilon^2} \log \frac{\mu^2}{m_b^2} + \log^2 \frac{\mu^2}{m_b^2} + \frac{\pi^2}{12} \right) \right] \delta(1 - y + \hat{k}_+)
+ i \frac{\alpha_s C_F}{4\pi} \theta(1 - y + \hat{k}_+) \left[ \left( \frac{4}{\epsilon} - 4 \log \frac{\mu^2}{m_b^2} + 4 \right) \frac{1}{(1 - y + \hat{k}_+)_+}
+ 8 \left( \log \frac{\mu^2}{m_b^2} \right)_+ \right], \]

(6.49)

where all divergences are ultraviolet. The renormalized operator $O(y; \mu)$ is related to the bare operator by

\[ O^{\text{bare}}(y) = \int dy' Z(y', y; \mu) O(y'; \mu). \]

(6.50)

Renormalizing in MS (generalized in the obvious way to cancel the $1/\epsilon^2$ divergences), we find

\[ Z(y', y; \mu) = \left\{ 1 - \frac{\alpha_s(\mu) C_F}{2\pi} \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\mu^2}{m_b^2} - \frac{1}{\epsilon} \right) \right\} \delta(y' - y)
+ \frac{\alpha_s(\mu) C_F}{\pi} \frac{1}{\epsilon (y' - y)_+} \theta(y' - y) \right\}.

(6.51)

Note that the counterterm consists of a diagonal piece which is proportional to $\delta(y' - y)$, and an off-diagonal piece proportional to $\theta(y' - y)$. This latter terms mixes the operator $O(y)$ with all operators $O(y')$ with $y' > y$. 
Inserting the one-loop matrix element of the renormalized operator into (6.42) we find the differential decay rate in the parton model $b \to X_s \gamma$

$$\left. \frac{d\Gamma}{dx} \right|_{k_+} = \Gamma_0 \int dy \, C(y, x; \bar{\mu})(O(y; \bar{\mu}))$$

$$= \Gamma_0 \int dy \, C(y, x; \bar{\mu}) \left\{ \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \log^2 \frac{\bar{\mu}^2}{m_b^2} - 2 \log \frac{\bar{\mu}^2}{m_b^2} + \frac{\pi^2}{12} \right) \right] \delta(1 - y + k_+) \right.$$}

$$- \frac{\alpha_s C_F}{4\pi} \theta(1 - y + \hat{k}_+) \left[ \left( 4 - 4 \log \frac{\bar{\mu}^2}{m_b^2} \right) \frac{1}{(1 - y + \hat{k}_+)_+} 
+ 8 \left( \frac{\log(1 - y + \hat{k}_+)_+}{1 - y + \hat{k}_+}_+ \right) \right]. \quad (6.52)$$

The short-distance coefficient $C(y, x; \mu)$ is determined by matching LCET onto LEET. In LCET, the Feynman diagrams for the forward scattering matrix element are shown in Fig. 6.10. As with LEET, all divergences are regulated in dimensional regularization. Expanding the expression for the forward scattering amplitude obtained from these graphs in powers of $(1 - x + \hat{k}_+)$, we find for the differential decay rate

$$\left. \frac{d\Gamma}{dx} \right|_{k_+} = C^2_{\bar{\mu}}(\bar{\mu}) \left\{ \left[ 1 + \frac{\alpha_s C_F}{4\pi} \left( \log^2 \frac{\bar{\mu}^2}{m_b^2} + 5 \log \frac{\bar{\mu}^2}{m_b^2} + \frac{\pi^2}{6} + 7 \right) \right] \delta(1 - x + \hat{k}_+) 
$$

$$- \frac{\alpha_s C_f}{4\pi} \theta(1 - y + \hat{k}_+) \left[ 4 \left( \frac{\log(1 - x + \hat{k}_+)}{1 - x + \hat{k}_+}_+ \right) + 7 \left( \frac{1}{1 - x + \hat{k}_+}_+ \right) \right]. \quad (6.53)$$

Figure 6.10: Collinear–soft theory Feynman diagrams which contribute to the forward scattering amplitude through $O(\alpha_s)$. 
Comparing Eqs. (6.53) and (6.52) gives the short-distance coefficient $C(y, x; \mu)$. At tree level, the matching is trivial, and we write

$$
C(y, x; \mu) = C^2_F(\mu) \left[ \delta(y - x) + \frac{\alpha_s C_F}{4\pi} C^{(1)}(y, x; \mu) + O(\alpha_s^2) \right]
$$

(6.54)

where $\mu_0$ is the matching scale. At one loop, we find

$$
C^{(1)}(y, x; \mu) = \left( 2 \log^2 \frac{\mu^2}{m_b^2} + 3 \log \frac{\mu^2}{m_b^2} + \frac{\pi^2}{4} + 7 \right) \delta(y - x) - \left( 3 + 4 \log \frac{\mu^2}{m_b^2} \right) \frac{\theta(y - x)}{(y - x)_+}
$$

$$
+ 4 \theta(y - x) \left( \frac{\log(y - x)}{y - x} \right)_+
$$

$$
= \left( 2 \log^2 \frac{\mu^2}{m_b^2(y - x)} + 3 \log \frac{\mu^2}{m_b^2(y - x)} + \frac{\pi^2}{4} + 7 \right) \delta(y - x)
$$

$$
- 4 \frac{\theta(y - x)}{y - x} \log \frac{\mu^2}{m_b^2(y - x)} - 3 \frac{\theta(y - x)}{y - x}.
$$

(6.55)

At the scale $\mu = m_b \sqrt{y - x}$ the logarithmic terms vanish, and we find

$$
C^{(1)}(y, x; m_b \sqrt{y - x}) = \left( \frac{\pi^2}{4} + 7 \right) \delta(y - x) - 3 \frac{\theta(y - x)}{y - x}.
$$

(6.56)

The matching scale is therefore different for each operator $O(y)$.

### 6.4.2 Renormalization group

The differential decay rate in LEET given in (6.52) may be written as

$$
\frac{d \Gamma}{dx} = \Gamma_0 \int dy C(y, x; \mu) \left\{ \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( \log \frac{\mu^2}{m_b^2(1 - y + \hat{k}_+)^2} \right. \right.
$$

$$
\left. \left. - 2 \log \frac{\mu^2}{m_b^2(1 - y + \hat{k}_+)^2} \right] \delta(1 - y + \hat{k}_+) \right.
$$

$$
+ \frac{\alpha_s C_F}{4\pi} \left( \frac{4}{1 - y + \hat{k}_+} \log \frac{\mu^2}{m_b^2(1 - y + \hat{k}_+)^2} - \frac{1}{1 - y + \hat{k}_+} \right) \right\},
$$

(6.57)

and so the large logarithms in the matrix element of $O(y, \mu)$ vanish at the scale $\mu = m_b(1 - y + \hat{k}_+)$. (This expression looks highly singular, but as can be seen from (6.52), the delta functions combine with the other terms to form plus functions.) Thus, in order to sum all logarithms of $\mu$ we must run the operator $O(y)$ from the matching
scale $\bar{\mu} = m_b \sqrt{y - x}$ to $\bar{\mu} = m_b (1 - y + \bar{k}_+)$). From (6.50) and (6.51) we obtain the renormalization group equation

$$\bar{\mu} \frac{d}{d \bar{\mu}} C(y, x; \bar{\mu}) = \int dy' \gamma(y, y'; \bar{\mu}) C(y', x; \bar{\mu}), \quad (6.58)$$

where $\gamma(y, y'; \mu)$ is the continuous anomalous dimension matrix

$$\gamma(y, y'; \bar{\mu}) = \frac{\alpha_s C_F}{\pi} \left[ \left( \log \frac{\bar{\mu}^2}{m_b^2} - 1 \right) \delta(y' - y) - \frac{2}{(y' - y)_+} \theta(y' - y) \right]. \quad (6.59)$$

Solving (6.58) analytically, however, is nontrivial and beyond the scope of this work [82]. Instead, we may diagonalize the anomalous dimension matrix by taking high moments of the spectrum. This will allow us to compare our results to those of Refs. [98, 99].

To calculate the moments we set the residual momentum $k$ to zero. (This residual momentum can easily be incorporated by boosting from the rest frame of the $b$ quark, $p_b = m_b v$, to the frame $p = m_b v + k$). Taking moments unconvolutes the expression for the differential decay rate in LEET (6.42) and we obtain

$$\Gamma(N) = \int_0^1 dx x^{N-1} \frac{d\Gamma}{dx} = \Gamma_0 \int_0^1 dx x^{N-1} \int_{-\infty}^{\infty} dy C(y - x; \bar{\mu}) \langle O(y; \bar{\mu}) \rangle$$

$$= \Gamma_0 \int_0^1 dx x^{N-1} C(1 - z; \bar{\mu}) \int_0^1 dy y^{N-1} \langle O(y; \bar{\mu}) \rangle$$

$$= \Gamma_0 C(N; \bar{\mu}) \langle O(N; \bar{\mu}) \rangle, \quad (6.60)$$

where $C'(N; \bar{\mu})$ and $C(N; \bar{\mu})$ are defined by

$$O(N; \bar{\mu}) = \int_0^1 dy y^{N-1} O(y; \bar{\mu})$$

$$C(N; \bar{\mu}) = \int_0^1 dx x^{N-1} C(x; \bar{\mu}). \quad (6.61)$$

Thus, the operator $O(N; \bar{\mu})$ is just a linear combination of the set of operators $O(y; \mu)$.

The matching from LCET onto LEET at tree level is trivial and we find

$$C(N; \mu) = C^2_f(\mu) \left[ 1 + \frac{\alpha_s C_F}{4\pi} C^{(1)}(N; \bar{\mu}) \right] + O(\alpha_s^2). \quad (6.62)$$
Determining $C^{(1)}(N; \mu_0)$ requires the one-loop expression of $\Gamma(N)$ in LCET and the one-loop matrix element of $O(N; \mu)$ between partonic states. The one-loop expression for the differential decay rate in LCET is given in (6.53). Setting $k_+$ to zero and taking moments we obtain

$$
\Gamma(N; \mu) = \int_0^1 x^{N-1} \frac{d\Gamma}{dx} = \Gamma^0 C_V^2(\mu) \left\{ 1 - \frac{\alpha_s C_F}{4\pi} \left[ 2 \log^2 \frac{N}{n_0} - 7 \log \frac{N}{n_0} 
- \log^2 \frac{\mu^2}{m_b^2} - 5 \log \frac{\mu^2}{m_b^2} \right] \right\} + \ldots , \quad (6.63)
$$

where $n_0 = e^{-\gamma_E}$ and the ellipses denote terms not enhanced logarithmically. This needs to be compared to the one-loop matrix element of $\langle O(N; \mu) \rangle$, which can be obtained from (6.52):

$$
\langle O(N; \mu) \rangle = \int_0^1 dy y^{N-1} \langle O(y; \mu) \rangle = 1 - \frac{\alpha_s C_F}{4\pi} \left[ 4 \log^2 \frac{\mu N}{m_b n_0} - 4 \log \frac{\mu N}{m_b n_0} \right] + \ldots . \quad (6.64)
$$

The one loop matching coefficient is now easily determined using (6.60), (6.63) and (6.64) and we find

$$
C^{(1)}(N; \mu) = \frac{\alpha_s C_F}{4\pi} \left[ 2 \log^2 \frac{\mu^2 N}{m_b^2 n_0} + 3 \log \frac{\mu^2 N}{m_b^2 n_0} \right] + \ldots . \quad (6.65)
$$

At the matching scale $\mu = m_b \sqrt{n_0/N}$ all logarithms in this matching coefficient vanish. Furthermore, from (6.64) the matrix element $\langle O(N; \mu) \rangle$ contains no large logarithms of $N$ at the scale $\mu = m_b n_0/N$. Thus we run in LCET from $m_b$ to $m_b \sqrt{n_0/N}$, perform the OPE, and run $C(N; \mu)$ from $m_b \sqrt{n_0/N}$ to $m_b n_0/N$.

The running of the coefficient $C_V$ in LCET from the scale $m_b$ to the scale $m_b \sqrt{n_0/N}$ is obtained by setting $\mu = m_b \sqrt{n_0/N}$ in (6.37). The running in LEET is determined by the RGE for $C(N; \mu)$

$$
\frac{d}{d\mu} C(N; \mu) = \gamma(N; \mu) C(N; \mu), \quad (6.66)
$$

where the anomalous dimension is given by

$$
\gamma(N; \mu) = \int_0^1 dz z^{N-1} \gamma(z; \mu) = -\frac{\alpha_s(\mu) C_F}{\pi} \left[ 1 - 2 \log \left( \frac{\mu N}{m_b n_0} \right) \right] . \quad (6.67)
$$
The solution to this equation is

$$C(N; \frac{m_b n_0}{N}) = C_V^2 \left( m_b \sqrt{\frac{n_0}{N}} \right) \left( \frac{\alpha_s(m_b \sqrt{\frac{n_0}{N}})}{\alpha_s(m_b \sqrt{\frac{n_0}{N}})} \right)^{\frac{2 C_F}{\frac{3}{2} \alpha_s} \left( \frac{5}{3} \log \frac{N}{n_0} \right)} \left( \frac{n_0}{N} \right)^{\frac{2 C_F}{\alpha_s}}. \quad (6.68)$$

This sums perturbative logarithms of $N$ into the coefficient $C(N)$. We can then substitute this into (6.60) to obtain an expression for the resummed moments of the differential decay rate.

Using the result for $C_V(\mu)$ given in (6.37) and taking the matrix element of $O(N; \mu)$ between hadronic states, we find

$$\Gamma(N) = \Gamma_0 f(N; m_b n_0/N) \left( \frac{\alpha_s(m_b \sqrt{\frac{n_0}{N}})}{\alpha_s} \right)^{\frac{2 C_F}{\frac{3}{2} \alpha_s} \left( \frac{5}{3} \log \frac{N}{n_0} \right)} \left( \frac{\alpha_s(m_b \sqrt{\frac{n_0}{N}})}{\alpha_s} \right)^{\frac{2 C_F}{\frac{3}{2} \alpha_s} \left( \frac{5}{3} \log \frac{N}{n_0} \right)}.$$

Logarithms are explicitly summed in this expression and only long distance physics is contained in the function $f(N; m_b n_0/N)$.

We can easily compare our results to those in the literature. A resummed expression for $\Gamma(N)$ is given in Ref. [98]:

$$\Gamma(N) = \Gamma_0 f(N; m_b / N) \exp \left[ -\int_{m_b y}^{1} \frac{dy}{y} \left( 2 \int_{m_b y}^{m_b \sqrt{y}} \frac{d\mu}{\mu} \Gamma_c(\mu) + \Gamma(m_b y) + \gamma(m_b \sqrt{y}) \right) \right], \quad (6.70)$$

where

$$\Gamma_c(\mu) = \frac{\alpha_s(\mu) C_F}{\pi}, \quad \Gamma(\mu) = -\frac{\alpha_s(\mu) C_F}{\pi}, \quad \gamma(\mu) = -\frac{3\alpha_s(\mu) C_F}{4\pi}. \quad (6.71)$$

Performing the integrals in the exponent we reproduce (6.69). Thus the approach presented here, based on an effective field theory, is in agreement with the factorization formalism approach for summing perturbative logarithms.
6.5 Conclusions

In the specific case of $\bar{B} \to X_s \gamma$ we have shown how Sudakov logarithms can be summed within an effective field theory framework. First we construct LCET, a theory which includes both collinear and soft degrees of freedom. By performing a one-loop calculation we show that LCET can be matched onto QCD at the scale $m_b$ without introducing nonanalytic terms into the short-distance coefficient. In addition we determine the one-loop anomalous dimension and solve the RGE. Next we integrate out collinear modes at the scale $m_b\sqrt{y-x}$ by switching to LEET. We perform an OPE in powers of $(y-x)$ which leads to the appearance of a nonlocal operator where two vertices are separated along the light-cone. The matrix element of this operator between $\bar{B}$ meson states is the structure function. We perform the OPE at one-loop in LCET and match onto the light-cone operator in LEET. At the scale $m_b\sqrt{y-x}$ no nonanalytic terms are introduced into the short-distance coefficient. Furthermore at the scale $m_b(y-x)$ all logarithms in the structure function vanish. Therefore, to sum all logarithms of $x-y$ we must run to the scale $m_b(y-x)$.

In order to compare the factorization formalism results in the literature we repeat our analysis for large moments of the decay rate. In this case we find that LCET matches onto LEET at the scale $m_b\sqrt{n_0/N}$, and that there are no large logarithms in the structure function at the scale $m_bn_0/N$. Using the LCET RGE we sum logarithms of $N$ between the scales $m_b$ and $m_b\sqrt{n_0/N}$. We then switch to LEET and sum logarithms of $N$ between the scales $m_b\sqrt{n_0/N}$ and $m_bn_0/N$. This sums all perturbative logarithms of $N$. We find that our result agrees with results presented in the literature. This gives us confidence that we have constructed the correct effective field theories.

Though we have presented this work entirely in the context of $\bar{B} \to X_s \gamma$ our approach is general. It should be straightforward to apply LCET and LEET to other processes in which Sudakov logarithms arise. As noted before, there are several features of this effective theory that are very different from other, well accepted effective theories. It will
take a more thorough investigation of this theory to understand these features in more
detail.
Chapter 7

Conclusions

In this thesis we used effective field theories to calculate various properties of inclusive rare $B$ decays. Rare $B$ decays are decays that do not occur via the $b \to c$ quark level transition responsible for $\sim 99\%$ of all $B$ decays. Two different types of rare $B$ decays were considered in this thesis: neutral flavour changing decays mediated by a $b \to s$ transition, which occur only at one loop in the standard model and are therefore suppressed by $\alpha/(4\pi)^2 \sim 10^{-4}$ and decays occurring via the $b \to u$ quark level transition, suppressed by the CKM factor $|V_{ub}/V_{cd}|^2 \sim 0.006$.

The first decay under consideration was the radiative FCNC decay $\bar{B} \to X_s \gamma$, the branching ratio of which has recently been measured to be $\mathcal{B}(\bar{B} \to X_s \gamma) = (2.3 \pm 0.7) \times 10^{-4}$. Using the OPE we calculated the photon energy spectrum of this decay up to $\mathcal{O}(\Lambda_{QCD}/m_b)^3$. This resulted in a singular spectrum, contributing only at $E_\gamma = m_b/2$. Since this spectrum contains derivatives of delta functions, it has a nonzero mean and variance. We calculated the mean and the variance of this spectrum, which are sensitive to the universal HQET parameters $\hat{\Lambda}$ and $\lambda_1$, respectively. This allows the determination of these HQET parameters from the observed photon energy spectrum, similar to the determination using the lepton energy and hadron invariant mass moments of the semileptonic $B \to X_c \ell \nu$ decay. The matrix elements of the dimension six operators,
contributing to the photon energy spectrum at $\mathcal{O}(\Lambda_{QCD}/m_b)^3$, are unknown and therefore give rise to uncertainties in the determination of the parameters $\tilde{\Lambda}$ and $\lambda_1$. Such nonperturbative uncertainties have been estimated for the lepton energy and hadron invariant mass moments of the semileptonic decay rate. We performed a similar analysis for the photon energy moments of $\bar{B} \to X_s\gamma$ and estimated the size of the uncertainties by varying the matrix elements of the $1/m_b^3$ operators in the range expected by dimensional analysis. We compared the result to the corresponding uncertainties obtained from semileptonic decays and found that combining the two methods of determining $\tilde{\Lambda}$ and $\lambda_1$ results in small nonperturbative uncertainties. An experimental cut on the photon energy has to be applied to suppress the strong background from other $B$ decays. This leads to the measured decay spectrum not being fully inclusive and therefore to additional uncertainties when comparing to the theoretical calculation. We derived model independent bounds on these uncertainties and estimated them using a simple model for the nonperturbative physics of the $B$ meson. We found that these uncertainties can be large for the cut $E_\gamma^0 = 2.2 \text{ GeV}$ used in the original CLEO analysis. This cut might have to be lowered to about 2 GeV to get a precise determination of $\tilde{\Lambda}$, and even further to measure $\lambda_1$.

The next decay considered was the FCNC decay $\bar{B} \to X_s\ell^+\ell^-$. Although this decay is suppressed with respect to the radiative FCNC decay $\bar{B} \to X_s\gamma$ by an additional factor of the fine structure constant and has yet to be observed, it has the appeal of providing more information about physics beyond the standard model than $\bar{B} \to X_s\gamma$. The presence of long distance effects due to $c\bar{c}$ resonances divides the accessible phase space into two regions: the region below the $\psi(1S)$ resonance and the region above the $\psi(2S)$ resonance. We estimated the nonperturbative uncertainties from dimension six operators in the region of phase space above the $\psi(2S)$ resonance, which is free from sequential $B$ decay background. We found that the convergence of the OPE in this restricted region of phase space is very poor, indicating that the rate above the $\psi(2S)$
resonance can not be predicted reliably from the OPE. We also calculated $\mathcal{O}(\Lambda_{QCD}/m_b)^3$ contributions to the FB asymmetry and showed them to be small over the entire region of phase space. The normalized FB asymmetry, however, inherits the poor convergence of the decay rate, and can not be predicted well above the $\psi(2S)$ resonance. Finally, we addressed a recent proposal to use hadronic invariant mass moments to determine the HQET parameters $\bar{\Lambda}$ and $\lambda_1$. We calculated the $\mathcal{O}(\Lambda_{QCD}/m_b)^3$ contributions to the first two hadron invariant mass moments and showed that the second moment has too little sensitivity on these parameters, due to small numerical coefficients, to be useful. The first moment does constrain a linear combination of $\bar{\Lambda}$, however this linear combination is very similar to the one constrained by moments of the experimentally much more promising decay $\bar{B} \rightarrow X_c \ell \bar{\nu}$. Thus we concluded that hadron invariant mass moments of the decay $\bar{B} \rightarrow X_s \ell^+ \ell^-$ do not provide useful information on the HQET parameters.

The decay rate of the inclusive decay $\bar{B} \rightarrow X_u \ell \bar{\nu}$ is proportional to $|V_{ub}|^2$ and can therefore be used to determine this CKM matrix element. Suppression of the overwhelming background of the decay $\bar{B} \rightarrow X_c \ell \bar{\nu}$ requires a kinematic cut which can spoil the applicability of the OPE. We showed that while suppressing this background with a cut on the lepton energy $E^0_l = (m_B^2 - m_D^2)/2m_B$ or the hadronic invariant mass $s^0_H = m_D^2$ spoils the convergence of the OPE, this is not the case for a cut on the lepton invariant mass $q^2_0 = (m_B - m_D)^2 = 11.6\,\text{GeV}^2$. We explained the reason for the better behaviour of the lepton invariant mass spectrum and calculated the fraction of $b \rightarrow u$ events above such a cut on $q^2$. We showed that the uncertainties on $|V_{ub}|^2$ are below 20% if the value of the cut $q^2_0$ does not exceed 15\,\text{GeV}^2. If experimental conditions force the value of the cut above $\approx 15\,\text{GeV}^2$, we showed that it is possible to combine information from the decays $\bar{B} \rightarrow X_c \ell \bar{\nu}$ and $\bar{B} \rightarrow X_s \ell^+ \ell^-$ to obtain a model independent determination of $|V_{ub}|^2$. This is due to the fact that the current mediating the $b \rightarrow s \ell^+ \ell^-$ transition at large lepton invariant mass is very close to the $(V - A) \times (V - A)$ current which mediates the semileptonic $b \rightarrow u \ell \bar{\nu}$ decay. This allowed us to relate $|V_{ub}|^2$ to the ratio of branching
ratios of $\bar{B} \to X_s \ell \bar{\nu}$ and $\bar{B} \to X_s \ell^+ \ell^-$, both with a lower cut on the dilepton invariant mass.

In the final chapter of this thesis we presented an effective theory describing inclusive decays containing highly energetic final hadronic states. Such a theory is relevant for inclusive decays where experimental cuts restrict the available phase space to a region where the energy of the final state hadron is much greater than its invariant mass. In such restricted regions of phase space both the nonperturbative expansion in powers of $\Lambda_{\text{QCD}}/m_b$ and the perturbative expansion in powers of $\alpha_s$ break down. This requires the introduction of a nonperturbative structure function as well as a resummation of perturbative Sudakov logarithms using RGEs. Previous attempts at constructing an effective theory describing this region of phase space tried to match QCD directly onto LEET, a theory containing only soft gluons. In LEET the nonperturbative structure function is given by the matrix element of a nonlocal operator. Using the endpoint of the photon energy spectrum, $E_\gamma \to m_b/2$, of the radiative FCNC decay $\bar{B} \to X_s \gamma$ as an example, we showed that it is not possible to perform this matching without introducing large logarithms into Wilson coefficients. Therefore, such an approach does not allow summing Sudakov logarithms. Introducing an intermediate effective theory, LCET, containing both soft and collinear gluons, we then showed that it is possible to match QCD onto LCET, which can then be matched onto LEET. Since this matching can be performed without requiring logarithmic terms in the short distance coefficients, these two effective theories can be used to sum Sudakov logarithms using the standard RGE. We solved the RGEs for large moments of the photon energy spectrum and showed that we reproduce well known results in the literature.
Appendix A

Appendix

A.1 The general expressions for the first three operators in the OPE

In this appendix we will present the first three terms in the OPE for general operators

\[ T\{\bar{b}\Gamma_1 s, s\Gamma_2 b\} = \frac{1}{m_b} \left[ O_0 + \frac{1}{2m_b} O_1 + \frac{1}{4m_b^2} O_2 + \frac{1}{8m_b^3} O_3 + \ldots \right]. \quad (A.1) \]

With the conventions

\[ D_\mu = \partial_\mu - igT^a A^a_\mu \]
\[ G_{\mu\nu} = [iD_\mu, iD_\nu] \quad (A.2) \]

and defining

\[ \Delta = 1 - 2\nu \cdot \hat{q} \quad . \quad (A.3) \]

they are given by

\[ O_0 = -\frac{1}{\Delta} \delta \Gamma_1 (\not{\hat{q}} - \not{\hat{q}} + \hat{m}_s) \Gamma_2 b \quad (A.4) \]
\[ O_1 = -\frac{2}{\Delta} \bar{h} \Gamma_1 \gamma^a \Gamma_2 iD_\alpha h + \frac{4}{\Delta^2} (\nu - \hat{q})^a \bar{h} \Gamma_1 (\not{\hat{q}} - \not{\hat{q}} + \hat{m}_s) \Gamma_2 iD_\alpha h \quad (A.5) \]
\[ O_2 = -\frac{16}{\Delta^3} (\nu - \hat{q})^a (\nu - \hat{q})^\beta \bar{h} \Gamma_1 (\not{\hat{q}} - \not{\hat{q}} + \hat{m}_s) \Gamma_2 iD_\alpha iD_\beta h \]
\[O_3 = \frac{8}{3\Delta^2} \tilde{h} \Gamma_1 \gamma^\alpha \Gamma_2 (iD\alpha (iD)^2 + (iD)^2 iD\alpha + iD\beta iD\alpha iD\beta) h \]

\[-\frac{32}{3\Delta^3} (v - q) \rho (v - q) \tilde{h} \Gamma_1 \gamma^\alpha \Gamma_2 (iD\rho iD\sigma iD\sigma + iD\rho iD\sigma iD\alpha + iD\rho iD\alpha iD\sigma) h \]

\[-\frac{32}{3\Delta^3} (v - q) \rho \tilde{h} \Gamma_1 (\psi - \hat{q} + \hat{m}_s) \Gamma_2 (iD\alpha (iD)^2 + (iD)^2 iD\alpha + iD\beta iD\alpha iD\beta) h \]

\[+ \frac{64}{3\Delta^4} (v - q) \rho (v - q) \tilde{h} \Gamma_1 (\psi - \hat{q} + \hat{m}_s) \Gamma_2 (iD\alpha iD\rho iD\sigma + iD\rho iD\sigma iD\alpha + iD\rho iD\alpha iD\sigma) h \]

\[-\frac{16}{3\Delta^3} (v - q) \rho (v - q) \tilde{h} \Gamma_1 \gamma^\beta \Gamma_2 (iD^\mu G_{\rho \beta}) h \]

\[-\frac{16}{3\Delta^3} (v - q) \rho \tilde{h} \Gamma_1 (\psi - \hat{q} + \hat{m}_s) \Gamma_2 (iD^\mu G_{\rho \beta}) h \]

\[+ \frac{16}{3\Delta^2} \tilde{h} \Gamma_1 \gamma^\beta \Gamma_2 (iD^\mu G_{\rho \beta}) h \]

\[+ \frac{2}{\Delta^2} i\varepsilon^\beta \rho \sigma \tilde{h} \Gamma_1 \gamma_\rho \gamma^5 \Gamma_2 \Gamma_2 (iD_\sigma, G_{\rho \beta}) h \]

\[+ \frac{8}{3\Delta^3} \tilde{m}_s (v - q) \sigma \tilde{h} \Gamma_1 i\sigma^\alpha \Gamma_2 (iD_\sigma, G_{\rho \beta}) h \]

\[+ \frac{8}{3\Delta^3} i\varepsilon^\beta \rho \sigma (v - q) \sigma (v - q) \tilde{h} \Gamma_1 \gamma_\rho \gamma^5 \Gamma_2 \Gamma_2 (iD^\lambda, G_{\rho \beta}) h \]

\[+ \frac{1}{\Delta^2} \tilde{h} \Gamma_1 \gamma^\alpha \Gamma_2 (iD_\mu G_{\rho \beta}) h \]

\[+ \frac{1}{\Delta^2} \tilde{h} \Gamma_1 \gamma^\alpha \Gamma_2 i\sigma^\rho \{iD_\rho, G_{\rho \beta}\} h \]

\[+ \frac{2}{\Delta^2} (1 - v \cdot q) \tilde{h} \Gamma_1 (\psi - \hat{q} + \hat{m}_s) \Gamma_2 (iD_\mu G_{\rho \beta}) h \]

\[+ \frac{2}{\Delta^2} (1 - v \cdot q) \tilde{h} \Gamma_1 (\psi - \hat{q} + \hat{m}_s) \Gamma_2 i\sigma^\rho \{iD_\rho, G_{\rho \beta}\} h \]

\[-\frac{1}{\Delta^2} \tilde{h} \Gamma_1 \gamma^\rho \Gamma_2 \gamma^\rho \Gamma_2 (iD_\rho iD_\alpha iD_\beta) h \]
\[ -\frac{3}{2} \tilde{h} \Gamma_1 \gamma^a \Gamma_2 i D_a (iD)^2 h \\
+ \frac{3}{2} \tilde{h} \Gamma_1 \gamma^a \Gamma_2 i \sigma^\alpha \Gamma_3 i D_\alpha i D_\rho i D_\sigma h \\
+ \frac{3}{2} (v - \hat{q})^a \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 \gamma^\rho i D_\rho i D_\alpha i D_\sigma h \\
+ \frac{6}{\lambda^2} (v - \hat{q})^a \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 i D_\alpha (iD)^2 h \\
+ \frac{6}{\lambda^2} (v - \hat{q})^a \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 i \sigma^\alpha i D_\alpha i D_\rho i D_\sigma h \\
+ \frac{1}{\lambda^2} \tilde{h} \gamma^a \Gamma_1 \gamma^\rho \Gamma_2 i D_\rho i D_\alpha i D_\sigma h \\
- \frac{3}{\lambda^2} \tilde{h} \Gamma_1 \gamma^a \Gamma_2 (iD)^2 i D_\alpha h \\
+ \frac{3}{\lambda^2} \tilde{h} i \sigma^\alpha \Gamma_1 \gamma^\rho \Gamma_2 i D_\alpha i D_\beta i D_\rho h \\
+ \frac{2}{\lambda^2} (v - \hat{q})^\alpha \tilde{h} \gamma^\rho \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 i D_\rho h i D_\alpha i D_\beta \\
+ \frac{6}{\lambda^2} (v - \hat{q})^\alpha \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 (iD)^2 i D_\alpha h \\
- \frac{6}{\lambda^2} (v - \hat{q})^\alpha \tilde{h} i \sigma^\alpha \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 i D_\rho i D_\sigma i D_\alpha h \\
- \frac{2}{\lambda^2} \tilde{h} \gamma^\rho \Gamma_1 \gamma^a \Gamma_2 \gamma^\beta i D_\rho i D_\alpha i D_\beta h \\
+ \frac{1}{\lambda^2} (v - \hat{q})^\alpha \tilde{h} \gamma^\rho \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 \gamma^\beta i D_\rho i D_\alpha i D_\beta h \\
+ \frac{1}{\lambda^2} (v - \hat{q})^\beta \tilde{h} \gamma^\rho \Gamma_1 \gamma^a \Gamma_2 i D_\rho (iD)^2 h \\
- \frac{16}{\lambda^3} (v - \hat{q})^a (v - \hat{q})^\beta \tilde{h} \gamma^\rho \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 i D_\rho i D_\alpha i D_\beta h \\
+ \frac{2}{\lambda^2} \tilde{m}_s \tilde{h} \gamma^\rho \Gamma_1 i \sigma^\alpha \Gamma_2 i D_\rho G_{\alpha\beta} h \\
- \frac{2}{\lambda^2} i \sigma^\alpha \Gamma_2 \gamma^\rho \Gamma_1 \Gamma_2 \gamma^a \gamma^\beta \Gamma_2 i D_\alpha G_{\rho\sigma} h \\
+ \frac{4}{\lambda^2} (v - \hat{q})^\beta \tilde{h} \Gamma_1 \gamma^a \Gamma_2 \gamma^\rho (iD_\alpha i D_\beta + i D_\beta i D_\alpha) i D_\rho h \\
+ \frac{4}{\lambda^2} \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 \gamma^\rho (iD_\rho) i D_\rho h \\
- \frac{16}{\lambda^3} (v - \hat{q})^a (v - \hat{q})^\beta \tilde{h} \Gamma_1 (\hat{p} - \hat{q} + \hat{m}_s) \Gamma_2 \gamma^\rho i D_\alpha i D_\beta i D_\rho h \\
- \frac{2}{\lambda^2} \tilde{m}_s \tilde{h} \Gamma_1 i \sigma^\alpha \Gamma_2 \gamma^\rho G_{\alpha\beta} i D_\rho h \\
- \frac{2}{\lambda^2} i \sigma^\alpha \Gamma_2 \gamma^\rho \Gamma_1 \gamma^a \gamma^\beta \Gamma_2 \gamma^\lambda G_{\rho\sigma} i D_\lambda h \] 

(A.8)
A.2 The form factors contributing to the decay $\bar{B} \to X_s \gamma$

In this appendix we present the form factors $T_i(x)$ contributing to the decay $B \to X_s \gamma$.

\begin{align*}
T_1(x) &= 2m_B(1-x)^2A(x) \quad \text{(A.9)} \\
T_2(x) &= 0 \quad \text{(A.10)} \\
T_3(x) &= 2m_B(1-x)A(x) \quad \text{(A.11)}
\end{align*}

where

\begin{equation}
A(x) = \frac{1}{1-x} - \left( \frac{5-3x}{6(1-x)^3} \hat{\lambda}_1 - \frac{3}{2(1-x)^2} \hat{\lambda}_2 \right) + \left( \frac{5x^2 - 12x + 9}{6(1-x)^4} \hat{\rho}_1 + \frac{1+x}{2x^3} \hat{\rho}_2 \right) \quad \text{(A.12)}
\end{equation}

A.3 The contribution of dimension six operators to the form factors contributing to the decay $\bar{B} \to X_s \ell^+ \ell^-$

In this appendix we present the form factors $T_i$. These form factors have been calculated previously up to $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^2$ [41], and we do not reproduce those results here. We decompose the new contributions arising at $\mathcal{O}(\Lambda_{\text{QCD}}/m_b)^3$ as

\begin{equation}
T_i^{L/R} = 2M_B \left[ T_i^{(C_9 \pm C_{10})^2} \left( C_{10} \pm C_9^{\text{eff}} \right)^2 + T_i^{C_7^2 |C_7^{\text{eff}}|^2 + T_i^{C_7(C_9 \pm C_{10})C_7^{\text{eff}}(C_9 \pm C_{10})} \right] \right. \\
\left. \text{(A.13)} \right.
\end{equation}

For completeness we have included the full $\hat{m}_s$ dependence in these expressions, though in our analysis we set $\hat{m}_s = 0$. Defining $x = 1 - 2v \cdot \hat{q} + \hat{s} + i\epsilon$ with $\hat{q} = \frac{q}{m_b}$, $\hat{s} = \hat{q}^2$ and $\hat{\rho}_i = \rho_i/m_b^3$, we find that the third order contributions are

\begin{align*}
T_i^{(C_9 \pm C_{10})^2} &= \frac{1}{12x}(\hat{\rho}_1 + 3\hat{\rho}_2) \\
&\quad - \frac{1}{6x^2} \left( (2 + \hat{s}) \hat{\rho}_1 - 3(2 - \hat{s}) \hat{\rho}_2 + (\hat{\rho}_1 + 3\hat{\rho}_2) v \cdot \hat{q} - (\hat{\rho}_1 + 3\hat{\rho}_2) (v \cdot \hat{q})^2 \right)
\end{align*}
\[ T_2^{(C_9 \pm C_{10})^2} = -\frac{1}{6 \mathcal{x}} (\hat{\rho}_1 + 3 \hat{\rho}_2) \]
\[ + \frac{1}{3 \mathcal{x}^2} (4 \hat{\rho}_1 + 6 \hat{\rho}_2 - (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q}) \]
\[ + \frac{2}{3 \mathcal{x}^3} (3 \hat{s} \hat{\rho}_2 - 2 (2 \hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q} + 2 (2 \hat{\rho}_1 + 3 \hat{\rho}_2) (v \cdot \hat{q})^2) \]
\[ + \frac{8}{3 \mathcal{x}^4} \hat{\rho}_1 (1 - v \cdot \hat{q}) (\hat{s} - (v \cdot \hat{q})^2) \]  
(A.14)

\[ T_3^{(C_9 \pm C_{10})^2} = -\frac{1}{6 \mathcal{x}^2} (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q} \]
\[ + \frac{2}{3 \mathcal{x}^3} (1 - v \cdot \hat{q}) (3 \hat{\rho}_2 - (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q}) \]
\[ + \frac{4}{3 \mathcal{x}^4} \hat{\rho}_1 (1 - v \cdot \hat{q}) (\hat{s} - (v \cdot \hat{q})^2) \]  
(A.16)

\[ T_1^{C_9^2} = \frac{1}{3 \tilde{s}^2 \mathcal{x}} \left( (\hat{\rho}_1 + 3 \hat{\rho}_2) \left( (1 - 5 \tilde{m}_s^2) \hat{s} - 2 (1 + \tilde{m}_s^2) (v \cdot \hat{q})^2 \right) \right) \]
\[ - \frac{2}{3 \tilde{s}^2 \mathcal{x}^2} \hat{s} \left( 4 \tilde{m}_s^2 + \hat{s} + \tilde{m}_s^2 \hat{s} - 4 \right) \hat{\rho}_1 + 3 \left( 8 \tilde{m}_s^2 + \hat{s} + \tilde{m}_s^2 \hat{s} \right) \hat{\rho}_2 \]
\[ + \left( 1 - 5 \tilde{m}_s^2 \right) \hat{s} (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q} + \left( 1 + \tilde{m}_s^2 \right) \left( (8 - \hat{s}) \hat{\rho}_1 - 3 \hat{s} \hat{\rho}_2 \right) (v \cdot \hat{q})^2 \]
\[ - 2 \left( 1 + \tilde{m}_s^2 \right) (\hat{\rho}_1 + 3 \hat{\rho}_2) (v \cdot \hat{q})^3 \]
\[ - \frac{8}{3 \tilde{s}^2 \mathcal{x}^3} \left( 1 + \tilde{m}_s^2 \right) (1 - v \cdot \hat{q}) (\hat{s} (\hat{\rho}_1 + 3 \hat{\rho}_2) - 2 (2 \hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q}) (\hat{s} - (v \cdot \hat{q})^2) \]
\[ - \frac{16}{3 \tilde{s}^2 \mathcal{x}^4} \hat{\rho}_1 (1 - v \cdot \hat{q}) (\hat{s} - (v \cdot \hat{q})^2) \]
\[ + \left( \hat{s} + 3 \tilde{m}_s^2 \hat{s} + \left( 1 + \tilde{m}_s^2 \right) \hat{s} v \cdot \hat{q} - 2 \left( 1 + \tilde{m}_s^2 \right) (v \cdot \hat{q})^2 \right) \]  
(A.17)

\[ T_2^{C_9^2} = \frac{2}{3 \tilde{s} \mathcal{x}} \left( 1 + \tilde{m}_s^2 \right) (\hat{\rho}_1 + 3 \hat{\rho}_2) \]
\[ + \frac{4}{3 \tilde{s} \mathcal{x}^2} \left( 1 + \tilde{m}_s^2 \right) (4 \hat{\rho}_1 - (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q}) \]
\[ + \frac{8}{3 \tilde{s} \mathcal{x}^3} \left( 1 + \tilde{m}_s^2 \right) \left( 3 \hat{s} \hat{\rho}_2 + 2 (2 \hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q} - 2 (2 \hat{\rho}_1 + 3 \hat{\rho}_2) (v \cdot \hat{q})^2 \right) \]
\[ - \frac{32}{3 \tilde{s} \mathcal{x}^4} \left( 1 + \tilde{m}_s^2 \right) \hat{\rho}_1 (1 - v \cdot \hat{q}) (\hat{s} - (v \cdot \hat{q})^2) \]  
(A.18)

\[ T_3^{C_9^2} = -\frac{2}{3 \tilde{s} \mathcal{x}^2} \left( 1 - \tilde{m}_s^2 \right) (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q} \]
\[ - \frac{2}{3 \tilde{s} \mathcal{x}^2} \left( 1 - \tilde{m}_s^2 \right) v \cdot \hat{q} ((8 - \hat{s}) \hat{\rho}_1 - 3 \hat{s} \hat{\rho}_2 - 2 (\hat{\rho}_1 + 3 \hat{\rho}_2) v \cdot \hat{q}) \]
\[ + \frac{8}{3 \tilde{s} \mathcal{x}^3} \left( 1 - \tilde{m}_s^2 \right) (1 - v \cdot \hat{q}) \]
\[ T_1^{C_1(C_9 \equiv C_{10})} = -\frac{1}{x} (\hat{\rho}_1 + 3\hat{\rho}_2) \]
\[ \frac{1}{3} \hat{s}^2 x^2 \left( 2 \left( 3 + \hat{m}^2_r \right) (\hat{\rho}_1 + 3\hat{\rho}_2) - 6 (4\hat{\rho}_2 + \hat{s} (\hat{\rho}_1 + 3\hat{\rho}_2)) (\hat{s} \cdot \hat{q}) \right) \]
\[ + 2 \left( 3 - \hat{m}^2_r \right) (\hat{\rho}_1 + 3\hat{\rho}_2) (\hat{s} \cdot \hat{q})^2 \]
\[ + \frac{8}{3} \frac{1}{x^2} \left( (1 - \hat{m}^2_r) (\hat{\rho}_1 - 3\hat{\rho}_2) (1 - \hat{s} \cdot \hat{q}) (\hat{s} - (\hat{s} \cdot \hat{q})^2) \right) \]
\[ - \frac{16}{3} \frac{1}{x^2} \hat{\rho}_1 (1 - \hat{s} \cdot \hat{q}) \left( \hat{s}^2 - (1 - \hat{m}^2_r) \hat{s} \cdot \hat{q} - (\hat{s} \cdot \hat{q})^2 \right) + (1 - \hat{m}^2_r) (\hat{s} \cdot \hat{q})^3 \] \[ (A.19) \]

\[ T_2^{C_1(C_9 \equiv C_{10})} = \frac{4}{3} \frac{1}{x^2} (\hat{\rho}_1 + 3\hat{\rho}_2) + \frac{8}{3} \frac{1}{x^2} \hat{\rho}_2 \left( \hat{m}^2_r + \hat{s} \cdot \hat{q} \right) \] \[ (A.20) \]

\[ T_3^{C_1(C_9 \equiv C_{10})} = \frac{2}{3} \frac{1}{x^2} \left( 12\hat{\rho}_2 - \left( 3 + \hat{m}^2_r \right) (\hat{\rho}_1 + 3\hat{\rho}_2) \right) \]
\[ - \frac{8}{3} \frac{1}{x^2} \left( \hat{\rho}_1 + \hat{m}^2_r \hat{\rho}_1 + 3\hat{m}^2_r \hat{\rho}_2 \right) (1 - \hat{s} \cdot \hat{q}) \hat{s} \cdot \hat{q} \]
\[ + \frac{16}{3} \frac{1}{x^2} \left( 1 + \hat{m}^2_r \right) \hat{\rho}_1 (1 - \hat{s} \cdot \hat{q}) \left( \hat{s} - (\hat{s} \cdot \hat{q})^2 \right) \] \[ (A.21) \]

\[ T_4^{C_1(C_9 \equiv C_{10})} = \frac{2}{3} \frac{1}{x^2} \left( 12\hat{\rho}_2 - \left( 3 + \hat{m}^2_r \right) (\hat{\rho}_1 + 3\hat{\rho}_2) \right) \]
\[ - \frac{8}{3} \frac{1}{x^2} \left( \hat{\rho}_1 + \hat{m}^2_r \hat{\rho}_1 + 3\hat{m}^2_r \hat{\rho}_2 \right) (1 - \hat{s} \cdot \hat{q}) \hat{s} \cdot \hat{q} \]
\[ + \frac{16}{3} \frac{1}{x^2} \left( 1 + \hat{m}^2_r \right) \hat{\rho}_1 (1 - \hat{s} \cdot \hat{q}) \left( \hat{s} - (\hat{s} \cdot \hat{q})^2 \right) \] \[ (A.22) \]

### A.4 The moments up to $O(1/m_b^3)$ with a cut on the dilepton invariant mass

We write the moments in the form

\[ M^{(m,n)} = \frac{B_0}{B_\chi} \left( C_{10}^2 M_{10,10}^{(m,n)} + |C_7^{\text{eff}}|^2 M_{7,7}^{(m,n)} + M_{9,9}^{(m,n)} + C_7^{\text{eff}} M_{7,9}^{(m,n)} \right) \] \[ (A.23) \]

where $B_\chi$ is the partially integrated rate. The coefficient $C_9^{\text{eff}}$ depends on the parameters $x_0$ and $s_0$ as explained in section 4.1, so we express the moments $M_{9,9}^{(m,n)}$ and $M_{7,9}^{(m,n)}$ as integrals which we evaluate numerically.

\[ M_{9,9}^{(m,n)} = \frac{16}{2M_B} \left( -\frac{1}{\pi} \right) \int_{\hat{m}_e}^{\hat{m}_t} dx_0 \int_{s_0^2}^{s_0^2} d\delta_0 x_0^m s_0^n \left[ \sqrt{x_0^2 - \delta_0} \right] \]
\[ \text{Im} \left[ 2(1 - 2x_0 + \delta_0) T_1^{(C_9 \equiv C_{10})^2} + \frac{x_0^2 - \delta_0^2}{3} T_2^{(C_9 \equiv C_{10})^2} \right] |C_9^{\text{eff}}(x_0, \delta_0)|^2 \]
\[ M_{1,9}^{(m,n)} = \frac{16}{2M_B} \left( -\frac{1}{\pi} \right) \int_{\tilde{m}_s}^{\tilde{x}_0} x_0^{\frac{1}{2}(1-x)} \int_{\tilde{\delta}_0}^{\tilde{x}_o} d\tilde{x}_0 \tilde{d}_0 \tilde{x}_0^m \tilde{\delta}_0^n \left[ \sqrt{x_0^{2} - \tilde{\delta}_0^{2}} \text{Im} \left\{ \frac{2(1 - 2x_0 + \tilde{\delta}_0)}{T_1^{C_{9}\pm C_{10}}} + \frac{x_0^{2} - \tilde{\delta}_0^{2}}{3} T_2^{C_{9}\pm C_{10}} \right\} \right] \right) \]

(A.24)

For the other contributions we find

\[ M_{10,10}^{(1,0)} = \frac{(1 - \chi)^4 (7 + 8 \chi)}{30} + \frac{\lambda_1}{m_b^2} \left( \frac{(1 - \chi)^3 (1 + \chi)}{3} \right) \]

\[ - \frac{\lambda_2}{m_b^2} \left( \frac{2(1 - \chi)^2 \chi (1 + 6 \chi - 4 \chi^2)}{3} \right) \]

\[ - \frac{\rho_1}{m_b^3} \left( \frac{67 - 30 \chi + 30 \chi^3 - 35 \chi^4 - 32 \chi^5}{45} \right) \]

\[ + \frac{\rho_2}{m_b^3} \left( \frac{1 + 10 \chi - 50 \chi^3 + 55 \chi^4 - 16 \chi^5}{5} \right) \]

(A.25)

\[ M_{7,7}^{(1,0)} = -\frac{2(41 - 60 \chi + 18 \chi^2 + 4 \chi^3 - 3 \chi^4 + 24 \log(\chi))}{9} \]

\[ - \frac{\lambda_1}{m_b^2} \left( \frac{8(8 - 9 \chi + \chi^3 + 6 \log(\chi))}{9} \right) \]

\[ + \frac{\lambda_2}{m_b^2} \left( \frac{4(7 - 2 \chi - 10 \chi^3 + 5 \chi^4 + 12 \log(\chi))}{3} \right) \]

\[ + \frac{\rho_1}{m_b^3} \left( \frac{8(2 - 27 \chi + 19 \chi^3 + 6 \chi^4 + 18 \log(\chi))}{27} \right) \]

\[ + \frac{\rho_2}{m_b^3} \left( \frac{8(13 - 27 \chi + 23 \chi^3 + 9 \chi^4 + 6 \log(\chi))}{9} \right) \]

(A.26)

\[ M_{10,10}^{(2,0)} = \frac{(1 - \chi)^5 (4 + 5 \chi)}{45} + \frac{\lambda_1}{m_b^2} \left( \frac{(1 - \chi)^4 (43 + 67 \chi + 25 \chi^2)}{270} \right) \]

\[ + \frac{\lambda_2}{m_b^2} \left( \frac{(1 - \chi)^3 (13 + 24 \chi - 222 \chi^2 + 125 \chi^3)}{90} \right) \]

\[ - \frac{\rho_1}{m_b^3} \left( \frac{(1 - \chi)^2 (24 - 27 \chi - 3 \chi^2 + 191 \chi^3 + 175 \chi^4)}{270} \right) \]

\[ - \frac{\rho_2}{m_b^3} \left( \frac{(1 - \chi)^3 (14 - 63 \chi - 306 \chi^2 + 175 \chi^3)}{90} \right) \]

(A.27)

\[ M_{7,7}^{(2,0)} = -\frac{2(119 - 210 \chi + 120 \chi^2 - 20 \chi^3 - 15 \chi^4 + 6 \chi^5 + 60 \log(\chi))}{45} \]

\[ - \frac{\lambda_1}{m_b^2} \left( \frac{127 - 150 \chi - 12 \chi^2 + 44 \chi^3 - 3 \chi^4 - 6 \chi^5 + 84 \log(\chi)}{27} \right) \]

\[ + \frac{\lambda_2}{m_b^2} \left( \frac{9 - 2 \chi + 4 \chi^2 - 36 \chi^3 + 35 \chi^4 - 10 \chi^5 + 12 \log(\chi)}{3} \right) \]
\[
\left( \frac{\zeta}{\zeta^{100} + 270} \right)^{\frac{\omega}{\zeta^{100}}} + \left( \frac{\zeta}{270} \right)^{\frac{\omega}{\zeta^{100}}} + \left( \frac{\zeta}{270} \right)^{\frac{\omega}{\zeta^{100}}} = 0
\]
\begin{align*}
\mathcal{M}^{(1,1)}_{1,7} &= -\frac{\rho_2}{m_b^3} \left( \frac{(1 - \chi)^3 (13 + 9 \chi - 252 \chi^2 + 110 \chi^3)}{90} \right) \\
&\quad - \frac{\lambda_1}{m_b^2} \left( \frac{4 \left( 2 + 15 \chi - 33 \chi^2 + 16 \chi^3 + 3 \chi^4 - 3 \chi^5 + 6 \log(\chi) \right)}{27} \right) \\
&\quad - \frac{\lambda_2}{m_b^2} \left( \frac{4 \left( 10 - 13 \chi - \chi^2 + 8 \chi^3 - 5 \chi^4 + \chi^5 + 6 \log(\chi) \right)}{3} \right) \\
&\quad + \frac{\rho_1}{m_b^3} \left( \frac{4 \left( 62 - 75 \chi - 195 \chi^2 + 280 \chi^3 - 15 \chi^4 - 57 \chi^5 - 30 \log(\chi) \right)}{135} \right) \\
&\quad + \frac{\rho_2}{m_b^3} \left( \frac{4 \left( 257 - 285 \chi - 195 \chi^2 + 400 \chi^3 - 210 \chi^4 + 33 \chi^5 + 150 \log(\chi) \right)}{45} \right)
\end{align*}
(A.33)
Bibliography


