A Non-Linear Aeroelastic Model for the Study of Flapping-Wing Flight

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Institute for Aerospace Studies
University of Toronto

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Abstract

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A non-linear aeroelastic model for the study of flapping-wing flight is presented. This model has been developed to simulate the fully stalled and attached aerodynamic behaviour of a flapping wing and can account for any forcing function. An implicit unconditionally-stable time-marching method known as the Newmark method is used to accurately model the non-linear stalled and attached flow regimes. An iteration procedure is performed at each time step to eliminate any errors associated with the temporal discretization process. A finite element formulation is used to model the elastic behaviour of the wing which is composed of a leading edge composite spar and light-weight rigid ribs covered with fabric. A viscous damping model is used to simulate the structural damping of the wing.

The Newmark code generates instantaneous lift and thrust values as well as torsional and bending moments along the wing span. Average lift values are in good agreement with experimental results obtained from tests performed on a scaled down model of the ornithopter at the NRC wind tunnel in Ottawa. Furthermore, bending and twisting moments obtained from strain gages embedded in the full-scale ornithopter’s wing spar show that the predicted instantaneous moments are also quite accurate. Also, comparisons with experimental data show that the Newmark code can accurately predict the twisting behaviour of the wing for zero forward speed as well as cruise conditions.
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Nomenclature

Roman Symbols

\( a \)  Mass-proportional damping constant
\( A \)  Element cross-sectional area
\( AR \)  Aspect Ratio
\( b \)  Stiffness-proportional damping constant
\( c \)  Wing segment chord length
\( C_d \)  Drag coefficient
\( C_f \)  Friction coefficient
\( C_{mac} \)  Airfoil moment coefficient about its aerodynamic center
\( C_n \)  Normal force coefficient
\( D_c \)  Drag due to camber
\( D_f \)  Friction drag
\( E \)  Modulus of Elasticity
\( F_z \)  Total chordwise force
\( F'(k) \)  Terms for modified Theodorsen function
\( g_r \)  Acceleration due to gravity
\( G \)  Shear modulus of elasticity
\( h \)  Total plunging displacement
\( \tilde{h} \)  Elastic component of plunging displacement
\( \Delta \tilde{h} \)  Non-flapping plunging displacement
\( h_0 \)  Imposed displacement
\( I \)  Moment of inertia
\( J \)  Polar moment of inertia
\( k \)  Reduced frequency
\( l_e \)  Length of element
\( L \)  Total Lift
\( m \)  Distributed moment per unit length
\( M \)  Total twisting moment acting on a wing segment
\( \bar{m} \)  Mass per unit length
\( n \)  Distributed normal force per unit length
\( N \)  Total normal force acting on a wing segment
\( p \)  Contact pressure
$P_1, P_2$ Element boundary conditions for torsion
$Q_1 \ldots Q_4$ Element boundary conditions for bending
$R$ Total thrust
$T_s$ Leading edge suction force
$U$ Free stream velocity
$v_r$ Relative velocity of upper versus lower surface of the wing
$V$ Relative velocity at $\frac{1}{4}$ chord location
$V_x$ Relative velocity tangential to a wing segment
$w$ Weighting function
$x$ Distance from flapping axis to middle of segment
$y$ Distance from the leading edge

**Greek Letters**

$\alpha$ Relative angle of attack at $\frac{3}{4}$ chord point due to wing segment motion
$\alpha'$ The flow's relative angle of attack at $\frac{3}{4}$ chord point
$\alpha_0$ Wing segment's angle of zero-lift line
$\alpha_{stall}$ Segment stall angle
$\beta$ Newmark method parameter
$\Gamma$ Flapping dihedral angle
$\Gamma_0$ Maximum flapping dihedral angle
$\delta$ Newmark method parameter
$\Delta x$ Length of an element
$\eta_s$ Leading edge suction efficiency
$\theta$ Total segment twist angle with respect to $U$
$\tilde{\theta}$ Elastic twist angle
$\Delta \tilde{\theta}$ Non-flapping elastic twist
$\tilde{\theta}_n$ Angle of flapping axis with respect to $U$
$\tilde{\theta}_{wash}$ Built in pre-twist
$\rho$ Atmospheric density
$\mu$ Mean dihedral angle
$\omega$ Flapping frequency
$\phi$ Interpolation function
$\zeta$ Decay constant
$\psi$ Transverse twist angle (along local $y$-axis)
Subscripts

\[ a \] Apparent mass
\[ ac \] Aerodynamic center
\[ aero \] Aerodynamic
\[ ave \] Average
\[ c \] Circulatory
\[ cf \] Crossflow
\[ cs \] Center section
\[ con \] Contact
\[ damp \] Damping
\[ dsr \] Shear-rate-dependent damping
\[ ea \] Elastic axis
\[ ef \] Effective
\[ f \] Friction
\[ fr \] Fabric and rib
\[ inertia \] Inertial
\[ rs \] Rigid section
\[ s \] Spar
\[ sep \] Separated flow
\[ sr \] Super rib
\[ st \] Static
\[ te \] Trailing edge

Superscripts

\[ \text{—} \] Mean value
\[ \sim \] Elastic component
\[ n, n+1 \] Time level
\[ \cdot \] Time derivative
Chapter 1

Introduction

Human beings have always been fascinated and inspired by the flight of birds. For this reason many of the early attempts at flight involved direct imitation of the flapping motion of a bird's wings. Although the flapping motions of birds seem natural and effortless, the aerodynamic and structural difficulties of implementing flapping-wing propulsion are considerable. The lack of success in many of the initial attempts was mainly due to a poor knowledge of flapping wing aerodynamics as well as a lack of light-weight and high-strength engineering materials such as composites.

1.1 Origins of Flapping Wing Flight

It is important to point out that flapping wing flight is recognized as such when a flapping wing produces lift as well as thrust necessary to sustain flight. This definition is necessary to distinguish an ornithopter from a plane which uses flapping technology only partially to achieve flight.

Alphonse Penaud's rubber powered ornithopter of 1874 [1] which was characterized by having a single-membrane airfoil flew for about 50 feet (Figure 1.1). A current version of this design is sold as a toy under the name "Tim Bird". Other researchers who have contributed to this field are Alexander Lippisch and E. von Holst. In the 1940's von Holst built a more sophisticated rubber band powered model in which flapping forces varied through the flapping cycle [2]. In 1985 AeroVironment of Monrovia California built a motorized robot ptesaur in order to study the flight mechanics of these prehistoric creatures [3]. This aircraft however did not achieve sustained flapping flight.

Another aspect of flapping-wing flight has been the attempts at designing a human-powered ornithopter which was also investigated at the University of Toronto by Fowler
One of the milestones in the history of flapping wing flight was when Lippisch tested a human-powered ornithopter in 1929 which was capable of extended glides [5]. His tests have been the most successful human-carrying flapping-wing flights (Figure 1.2). Another human-powered ornithopter of notable success was built by Emil Hartman in 1959. Hartman's machine had a span of eleven meters and made extended hops during towing trials at Cranfield. Hartman's aircraft was constructed primarily of wood and fabric, had conventional tail surfaces and weighed about 125 kg without a pilot [6].

1.2 Flapping-Wing Research at UTIAS

Initial studies into the feasibility of a human-carrying ornithopter at University of Toronto Institute for Aerospace Studies (UTIAS) focused on the design of an aircraft with weight and performance characteristics of a general aviation aircraft. Because of the high cost and difficulties involved in certifying such an aircraft, the research team at UTIAS instead concentrated on the development of a human-carrying aircraft in the ultralight category. One of the first proposals for a human-carrying ornithopter was given by Skarsgard [6] in his Master’s thesis in 1991. Skarsgard proposed a cable-actuated flapping system (Figure 1.3) which would drive a single-membrane wing known as the Mark 6. However, it was found in subsequent studies that a root driven, double-membrane wing with a larger leading edge radius would provide a much more efficient platform for flapping wing propulsion. These conclusions lead to the design of the Mark 8 wing which was used on the first successful flapping-wing radio-controlled aircraft.

In September 1991 the first engine-powered and radio-controlled ornithopter flew successfully for 2 minute and 46 seconds. (Figure 1.4). This quarter-scale proof-of-concept model was built jointly by Dr. J.D. DeLaurier of University of Toronto’s Institute for
Aerospace Studies and Mr. J. Harris who is a Principal Research Engineer at the Batelle Memorial Institute in Columbus, Ohio. A synopsis of the work leading to the quarter-scale ornithopter's flight is described in Reference [7].

A second model ornithopter was built for the Canada Pavilion at the Expo 1992 in Seville Spain. This aircraft, appropriately named Expotopter, was similar to the quarter-scale model in that it was completely flight-worthy but due to the success of the quarter-scale model it was not necessary to fly the Expotopter. It was used as a test bed for wind tunnel testing at NRC's wind tunnel in Ottawa. The quarter-scale model ornithopter and Expotopter have served as proof-of-concept aircrafts which have aided the research team at UTIAS in the development of a full-scale piloted ornithopter.

1.3 Flapping-Wing Aircraft Design

Using the knowledge gained during the construction and testing of the quarter-scale model and Expotopter, a full-scale human-carrying ornithopter was designed and built at UTIAS. Figure 1.5 shows a 3 view drawing of the full-scale ornithopter in its current (2000) configuration. Details of the design, construction and initial testing are described in Reference [8]. Results from the initial static-flapping (no forward speed) tests as well as some of the low speed taxi tests are presented by Mehler [9]. Furthermore, the 1997 and 1998 taxi tests and the main bulk of the experimental data are discussed in detail by Fenton [10].
1.3.1 General Characteristics

It can be seen from Figure 1.5 that the fuselage and tail of the aircraft are fairly conventional. Wings of the aircraft however are anything but conventional. The wing is actually made up of three panels which are hinged together. The three-panel feature is patented because it not only reduces the unbalanced oscillatory loads applied to the fuselage, but it also evens out the instantaneous power required throughout the flapping cycle. The engine sizing for an ornithopter is determined by the peak power required during the flapping cycle. For a two-panel design the power required during the downstroke is several times higher than that for the upstroke. Therefore the required engine must be much larger than average power considerations would dictate. For bird and bats some storage may be provided by their tendons or ornithopter designers have used elastic media such as bungee cord to store energy during the upstroke. However, an ornithopter is not required to be a scale model of an animal, and the three-panel wing thus provides a convenient way for
Figure 1.5: 3 View Drawing of Full-Scale Ornithopter (Drawing By Dave Loewen)
The front view shows that the wing is supported at four points. The center section is connected to the drive train by two pylons. The other supports constitute the flapping axis for each wing. The engine is a three-cylinder two-stroke Konig 4130 generating about 24 horsepower at 4000 RPM. The drive train provides a reduction of about 55 which results in 1.1 to 1.2 Hz flapping frequency (Figure 1.6). This is achieved with a three-stage reduction using chains and sprockets, terminating in a scotch yoke mechanism (Figure 1.7) which converts the rotary motion of the engine to an oscillatory up-down motion driving the vertical pylons which are connected to the center section.

Mounted directly on the engine shaft is a flywheel/fan unit to even out engine back-loads due to flapping. In addition, the internal location of the engine means that a fan is required for forced cooling. Both the flywheel and cooling functions are accomplished with an annular flywheel with spokes shaped as fan blades.

The aircraft was designed and built at University of Toronto Institute for Aerospace studies except for the carbon fiber landing gear which is from Grove aviation of California and the wooden vertical tail and the rear fuselage which were built by members of the Recreational Aircraft Association of Brampton, Ontario. The empty weight of the aircraft is 550 pounds and the gross weight is about 700 pounds. This gross weight is consistent with the weight of a non-flapping propeller-driven aircraft which would use the same engine and was designed by a team including the author as part of the Aircraft Design course at the University of Toronto [11].

1.3.2 Wing Design

A single surface flapping wing such as the one used by Penaud consists of a stiff leading edge spar attached to a thin flexible material. Such a wing has very little leading edge
suction. Therefore thrusting is solely generated by the forces normal to the wing chord. It has been shown that the efficiency of such an arrangement is fairly low [2]. Since most birds do have thin and cambered wings it was unclear how birds can achieve sufficient lift and thrust over the flapping cycle. Rayner [12] for example suggested that neither lift nor thrust are generated when birds "collapse" their wings during the upstroke. Subsequent research showed that if the upstroke was not loaded the average lift will not equal the weight of certain example birds which means that the original assumption is wrong. Lighthill [13] points out that lift is generated both on the upstroke as well as the downstroke. This can only be achieved if the birds fold their wings on the upstroke, effectively producing a lower wing span. Therefore the upstroke produces negative thrust and positive lift, but in smaller magnitudes than those during the downstroke (Figure 1.8). Net positive thrust is thus obtained. Since the problems associated with designing wings that mimic a bird's flapping and folding motion are tremendous, a thick double surfaced airfoil was used instead.

![Diagram](image)

Figure 1.8: Forces in Upstroke and Downstroke for Animal Flight

A S1020 airfoil used in the quarter-scale model ornithopter wing was designed by Professor Michael Selig of University of Illinois at Urbana-Champaign. This thick airfoil provides very high leading edge suction efficiency as well as considerable structural depth. The full-scale ornithopter wing also incorporates the S1020 airfoil. The inner portion of the wing uses this airfoil up to the "knuckle" (Figure 1.9), and the outer tapered portion linearly transforms the S1020 to a Selig and Donovan SD8020 symmetrical airfoil at the tip. Figure 1.10 shows the airfoil shapes along the span of the full-scale ornithopter wing. The main reason for using the S1020 airfoil for the inner portion of the wing is that this part of the wing is primarily responsible for lift generation, whereas the outer portion of the wing is designed to provide good thrusting performance during both the upstroke and downstroke.
A rigid section, which is also known as the “SuperBox” (Figure 1.9), is a closed structure made up of thin composite panels, internal ribs, a D-nose spar and a rear shear web. Its function is to support the wing and transfer the loads to the outrigger struts. The outer rib of the SuperBox is referred to as the “SuperRib”. The rest of the wing consists of 20 full-length ribs, made with a foam core and capped with basswood strips. The wing is covered with lightweight polyester fabric.

The center section is made up of composite panels, fore and aft shear webs, and nine internal ribs. The aerodynamic shape is created with a hollow blue foam D-nose, and nine half-length ribs.
Figure 1.11 shows a cross section of the Mark 8 wing spar at the flapping axis. The composite spar design consists of a plywood and carbon fiber shear web for adequate bending stiffness and the torsional characteristics are determined by the cross-sectional shape which is made up of a Kevlar-Epoxy surface surrounding a foam core.

The cross section of the full-scale ornithopter wing spar (Figure 1.12) is similar to that of the quarter-scale model. The two spars are similar because it was felt that this configuration is the lightest and most efficient spar for a flapping wing.

The only major difference is the way unidirectional carbon layers are applied. The quarter-scale model uses a sheet of carbon covering the entire surface of the shear web, whereas the full-scale spar uses large carbon "spar caps" on the inside edges of the shear web as shown in Figure 1.12. Carbon is not necessary on the middle portion of the shear web as it coincides with the spar's neutral axis and is not highly stressed [14]. The carbon spar at the leading edge of the D-nose section is designed to support the chordwise moments and help prevent buckling at the leading edge. It serves the same purpose as the Kevlar threads used in the leading edge of the quarter-scale model.

![Figure 1.11: Quarter-Scale Wing Spar](image1)

![Figure 1.12: Full-Scale Wing Spar](image2)

1.3.3 Wing Twist and Shearflexing

As mentioned by Dr. J.D. DeLaurier in his 1993 paper [15] "efficient flapping is characterized by pitching angles lagging plunging by approximately 90°". A special feature of this wing is that it is able to provide the required torsional compliance for this particular "phasing" while incorporating the efficient double-surface airfoil designed specifically for this project by Professor Selig of the University of Illinois. This feature which is known as "shearflexing" allows a thick double surface wing to behave basically as two single-surface
wings joined at the spar. Figure 1.13 shows how this is accomplished where the closed "torsion box" formed by the thick airfoil is opened by splitting the trailing edge [2].

The shearflexing principle allows the designer to "tailor" the elastic properties of the wing to achieve the desired dynamic twist which determines the wing thrusting and lifting performance. Table 1.1 shows the as-built bending and torsional stiffness properties of the full-scale ornithopter wing. A more detailed summary of the structural and aerodynamic properties of the Exoptoher, quarter-scale model and the Full-Scale Ornithopter are presented in Appendix A.

![Figure 1.13: The Shearflexing Principle](image)

Previous studies showed that the shearflexing action required that the twisting vary linearly with span. Furthermore linear twisting was shown to be desirable for efficient flapping. Some more aerodynamic help was needed to get the desired linear twisting and this lead to the addition of the delta-shaped wing tip (bat tip) in order to give linear twisting to the tip (Figure 1.9).

### 1.4 The Fullwing Code

Analytical work done for this project is based on a program called "Fullwing" which was written in order to predict the performance of a flapping wing in steady flight. The analytical work has consisted of a combination of theoretical and experimental work as several spars were built and tested in order to determine the torsional and bending properties of
each wing. Fullwing was originally written in BASIC by Dr. J.D. DeLaurier and was later converted to the C language by Fowler [14].

Fullwing breaks up the wing into several segments. Each segment has its own structural and aerodynamic characteristics. The motion and displacement of each section is described in Figure 1.14. The wing bending displacement is measured by coordinate $h$ which is perpendicular to the flapping axis and its twist is measured by the coordinate $\hat{\theta}$ where subscript $i$ denotes the $i$th segment. The wing coordinates at any point in the flapping cycle are given by column vectors $\{h\}$ and $\{\hat{\theta}\}$. Furthermore the spar is assumed to have

<table>
<thead>
<tr>
<th>Segment Number</th>
<th>$EI$ $lb$-$ft^2$</th>
<th>$GJ_{SPAR}$ $lb$-$ft^2/rad$</th>
<th>$GJ_{FR}$ $lb$-$ft^2/rad$</th>
</tr>
</thead>
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<td>1</td>
<td>235503</td>
<td>14000</td>
<td>386</td>
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<tr>
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<td>13318</td>
<td>358</td>
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<td>3</td>
<td>148046</td>
<td>12664</td>
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<tr>
<td>5</td>
<td>82454</td>
<td>11417</td>
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<td>46813</td>
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</tr>
<tr>
<td>11</td>
<td>2000</td>
<td>427</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1.1: Elastic Properties of Full Scale Ornithopter Wing

Figure 1.14: A Representative section
an imposed time-varying displacement \( \{h_0\} \) from mechanical forcing, such that

\[
\{h\} = \{h_0\} + \{\dot{h}\} \tag{1.1}
\]

where \( \{\dot{h}\} \) is the elastic deformation. The elastic deformations are related to the imposed loads through the structural influence coefficients by:

\[
\begin{bmatrix}
C_{hh} & C_{h\dot{\theta}} \\
C_{\dot{h}h} & C_{\dot{h}\dot{\theta}}
\end{bmatrix}
\begin{bmatrix}
-\mathcal{N} \\
\mathcal{M}
\end{bmatrix} =
\begin{bmatrix}
\ddot{h} \\
\dot{\theta}
\end{bmatrix} \tag{1.2}
\]

Noting that the elastic deformations consist of static and dynamic parts:

\[
\begin{align*}
\dot{h} &= \Delta\ddot{h} + \delta h \\
\dot{\theta} &= \Delta\dot{\theta} + \delta \theta
\end{align*} \tag{1.3}
\]

Where \( \Delta\ddot{h} \) and \( \Delta\dot{\theta} \) are the static non-flapping elastic deformations and \( \delta h \) and \( \delta \theta \) are the time varying elastic deformations Equation 1.2 can now be broken up into static and dynamic components. The static equations are:

\[
\begin{bmatrix}
C_{hh} & C_{h\dot{\theta}} \\
C_{\dot{h}h} & C_{\dot{h}\dot{\theta}}
\end{bmatrix}
\begin{bmatrix}
-\mathcal{N}_{static} \\
\mathcal{M}_{static}
\end{bmatrix} =
\begin{bmatrix}
\Delta\ddot{h} \\
\Delta\dot{\theta}
\end{bmatrix} \tag{1.4}
\]

and the dynamic equations are given by:

\[
\begin{bmatrix}
C_{hh} & C_{h\dot{\theta}} \\
C_{\dot{h}h} & C_{\dot{h}\dot{\theta}}
\end{bmatrix}
\begin{bmatrix}
-\mathcal{N}_{dynamic} \\
\mathcal{M}_{dynamic}
\end{bmatrix} =
\begin{bmatrix}
\delta h \\
\delta \theta
\end{bmatrix} \tag{1.5}
\]

Equation 1.4 reduces to a set of simultaneous linear equations and these are solved in the static module of Fullwing. Equation 1.5 is a non-linear second-order differential equation in terms of \( \delta h \) and \( \delta \theta \). It is reduced to a set of linear second-order differential equations by assuming a fully attached flow model where \( \mathcal{N}_{dynamic} \) and \( \mathcal{M}_{dynamic} \) are linear functions of \( \delta h \) and \( \delta \theta \). The equations are then solved by assuming that the dynamic response of the wing is harmonic with a frequency which is the same as the imposed motion but which is shifted by a phase angle \( \Phi \). This is done in Fullwing's dynamic module. The aerodynamic module then re-calculates the aerodynamic loads while checking whether each wing segment is in a stalled or attached flow regime.

The most significant assumption made in Fullwing is that of fully-attached flow in order to linearize the equations. This gives inaccurate results as stalling dominates in the low forward speed, high flapping-frequency regime, which became apparent during the static tests in 1996. It was noted then that the phase angle between the pitching and plunging
was nowhere near the design value of -90 degrees. A "full-stall" formulation should give a more accurate representation of the problem. A second assumption is that the response is assumed to have a simple harmonic motion. Although this is a reasonable description of the actual response, the presence of non-harmonic motion cannot be predicted. Also, its use of structural influence coefficients instead of a stiffness formulation has the disadvantage that for every new geometry the coefficients have to be re-derived. An algorithm that uses transformation matrices can handle various geometric configurations more easily.

1.5 Objectives

The primary goal of this study is to develop a non-linear aeroelastic model of a flapping wing. Within this framework this project has the following objectives:

- to implement a "full-stall" model for solving the aeroelastic equations.
- to develop a non-linear time-marching method which would enable us to study the effect of any forcing function.
- to develop a finite element structural model that can accommodate variations in wing properties and geometry.
- to validate the code using experimental data obtained from taxi tests as well as data from wind tunnel tests.
Chapter 2

Structural Analysis

Since the beginning of the ornithopter project some twenty years ago, extensive experimental effort has been devoted to determining the bending and torsional properties of sample spars. This effort culminated in the building and testing of eight sample spars which were built and some were loaded to destruction in the course of the full-scale ornithopter's design and construction. Because of this effort, elastic properties of the spars used in the construction of the two quarter-scale models and the full-scale ornithopter have been determined quite accurately. The structural model presented here is designed to take advantage of the elastic properties ($EI$'s and $GJ$'s) which were determined experimentally.

This section presents the linear ordinary differential equations governing the flapping motion of a wing and describes the matrices and vectors associated with them. The two principal modes of motion for a flapping wing are bending and torsion. A finite element discretization breaks the wing into spar elements with bending and torsional degrees of freedom and fabric and rib elements which have torsional degrees of freedom only (Figure 2.1).

For bending, the computational domain is divided into a set of elements each having two nodes (Figure 2.2). A single element is then isolated and the Galerkin method is applied in order to derive the finite element formulation of the problem. The Galerkin method is based on the method of weighted residuals (MWR) and is described in detail by Reddy [16]. For bending, a set of Hermite cubic interpolation functions are used. The equilibrium equation for a beam in pure bending is given by:

$$EI \frac{d^4 h}{dx^4} + \bar{m} \frac{d^2 \dot{h}}{dt^2} = -n(x,t)$$  \hspace{1cm} (2.1)

where $\bar{m}$ is mass per unit length and $n(x,t)$ is a distributed transverse force over the length...
of the beam. Application of the Galerkin method transforms the first term on the left hand side of Equation 2.1 into the stiffness matrix and the second term becomes the mass matrix.

For torsion, the domain is also divided into a set of elements, with each element having two nodes as shown in Figure 2.3. The Galerkin method is applied using a set of linear interpolation functions. The system equilibrium equation for a member in pure torsion is given by:

$$-GJ \frac{d^2 \theta}{dx^2} + \frac{Jm}{A} \frac{d^2 \theta}{dt^2} = m(x,t)$$ \hspace{1cm} (2.2)

where \( m(x,t) \) is a distributed torque per unit length. Just as with bending the first and second terms on the left hand side of Equation 2.2 are transformed into the stiffness and
mass matrices respectively. Appendix B shows the complete procedure for obtaining the mass and stiffness matrices using the Galerkin method.

2.1 System Equilibrium Equations

The system equilibrium equations, including damping and neglecting external reactions which are accounted for by boundary conditions, are:

\[
[K] \{y\} + [D] \{\dot{y}\} + [M] \{\ddot{y}\} = \{F\} \tag{2.3}
\]

The stiffness matrix for an element having both bending and torsional degrees of freedom is given by:

\[
[K] = \begin{bmatrix}
\frac{12EI}{\Delta x^3} & 0 & -\frac{6EI}{\Delta x^2} & -\frac{12EI}{\Delta x^2} & 0 & -\frac{6EI}{\Delta x^2} \\
0 & \frac{GJ}{\Delta x} & 0 & 0 & -\frac{GJ}{\Delta x} & 0 \\
-\frac{6EI}{\Delta x^2} & 0 & \frac{4EI}{\Delta x^2} & \frac{6EI}{\Delta x^2} & 0 & \frac{2EI}{\Delta x} \\
-\frac{12EI}{\Delta x^3} & 0 & \frac{6EI}{\Delta x^2} & \frac{12EI}{\Delta x^2} & 0 & \frac{6EI}{\Delta x} \\
0 & -\frac{GJ}{\Delta x} & 0 & 0 & \frac{GJ}{\Delta x} & 0 \\
-\frac{6EI}{\Delta x^2} & 0 & \frac{2EI}{\Delta x} & \frac{6EI}{\Delta x^2} & 0 & \frac{4EI}{\Delta x}
\end{bmatrix} \tag{2.4}
\]

where $EI$ is the bending stiffness parameter and $GJ$ is the corresponding torsional stiffness parameter for an element. $\Delta x$ is the length of a particular element. The overall consistent
mass matrix for an element with total mass per unit length $\bar{m}$ is given by:

$$[M] = \bar{m}\Delta x = \begin{bmatrix}
\frac{156}{420} & 0 & -\frac{22\Delta x}{420} & \frac{54}{420} & 0 & \frac{13\Delta x}{420} \\
0 & \frac{J}{3A} & 0 & 0 & \frac{J}{6A} & 0 \\
-\frac{22\Delta x}{420} & 0 & \frac{4\Delta x^2}{420} & -\frac{13\Delta x}{420} & 0 & -\frac{3\Delta x^2}{420} \\
\frac{54}{420} & 0 & -\frac{13\Delta x}{420} & \frac{156}{420} & 0 & \frac{22\Delta x}{420} \\
0 & \frac{J}{6A} & 0 & 0 & \frac{J}{3A} & 0 \\
\frac{13\Delta x}{420} & 0 & -\frac{3\Delta x^2}{420} & \frac{22\Delta x}{420} & 0 & \frac{4\Delta x^2}{420}
\end{bmatrix}$$  \hspace{1cm} (2.5)

Assuming that the normal force and moment are constant over an element, (i.e. $n(x,t) = n(t)$ and $m(x,t) = m(t)$) the force vector and the vector of independent variables are given by:

$$\{F\} = \begin{bmatrix}
-N(t) \\
M(t) \\
\frac{N(t)\Delta x}{12} \\
-N(t) \\
M(t) \\
\frac{N(t)\Delta x}{12}
\end{bmatrix} \quad \{y\} = \begin{bmatrix}
\dot{h}_1 \\
\dot{\theta}_1 \\
\nu_1 \\
\dot{h}_2 \\
\dot{\theta}_2 \\
\nu_2
\end{bmatrix}$$  \hspace{1cm} (2.6)

where $N(t) = n(t)\Delta x$ and $M(t) = m(t)\Delta x$.

The aerodynamic forces and moments associated with a flapping wing in steady flight are non-linear functions of the twisting and plunging deflections of the wing and their first and second derivatives. Equation 2.3 is a second-order non-linear equation that is solved by using a Taylor series expansion to approximate the non-linear components of the force vector. Several time-marching methods were considered and the non-linear Newmark method was chosen because of its stability and ease of use. The non-linear Newmark method is described in detail in Reference [17]. In the present study, since a temporal approximation has been used to obtain a set of linear second-order equations, an iteration must be performed at each time step to ensure that the equilibrium equations are satisfied. The performance of the Newmark algorithm has been studied extensively and it is known to be unconditionally stable [18].
2.2 Coordinate Transformation

The system matrices are not very useful in their current form as any structure consists of members that are not aligned with the global coordinate system. The objective of this section is to present a transformation matrix which would allow an element to be transformed back and forth between the global and local (element) coordinate systems. The ability to use transformation matrices is one of the main advantages of the stiffness formulation as opposed to the flexibility (structural influence coefficient) approach.

2.2.1 Rotational Transformation

The approach for deriving the necessary rotational transformation matrix is to first derive the matrix for a full space frame member with three displacements and three angular deflections at each node which is a $12 \times 12$ matrix. This matrix is then simplified by eliminating the rows and columns that are not needed.

A $12 \times 12$ rotational transformation matrix for a space frame has the following general form [19]:

$$
[T_r] = \begin{bmatrix}
[T_r] & 0 & 0 \\
0 & [T_r] & 0 \\
0 & 0 & [T_r]
\end{bmatrix}
$$

(2.7)

where $T_r$ is a $3 \times 3$ matrix of direction cosines. The direction cosines for the $y$ and $z$ could be found by direct geometrical consideration. but are more easily determined by using two successive transformations. The objective here is to express displacement or force quantities in the local $x, y, z$ directions in terms of the corresponding quantities in the global $x_g, y_g, z_g$ directions. The result is [19]:

$$
[T_r] = \begin{bmatrix}
\frac{e}{\sqrt{e^2 + g^2}} & \frac{f}{\sqrt{e^2 + g^2}} & \frac{-g}{\sqrt{e^2 + g^2}} \\
\frac{-e}{\sqrt{e^2 + g^2}} & \frac{-f}{\sqrt{e^2 + g^2}} & 0 \\
0 & \frac{-g}{\sqrt{e^2 + g^2}} & \frac{e}{\sqrt{e^2 + g^2}}
\end{bmatrix}
$$

(2.8)

e, f and $g$ are given by:

$$
e = \frac{x_2 - x_1}{l_e} \quad f = \frac{y_2 - y_1}{l_e} \quad g = \frac{z_2 - z_1}{l_e}
$$

(2.9)

where $l_e$ is the length of an element in the global coordinate system which is given by:

$$
l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
$$

(2.10)
Up to this point all matrices have been presented for a fully three-dimensional space frame with 6 degrees of freedom per node. In the case of the ornithopter wing however only 3 degrees of freedom per node are considered. They are \( \dot{h}_1, \dot{\theta}_1, \omega_1 \) at node 1 and \( \dot{h}_2, \dot{\theta}_2, \omega_2 \) at node 2. The 12x12 matrix of Equation 2.7 is therefore reduced to:

\[
[\tau_r] = \begin{bmatrix}
\frac{e}{\sqrt{e^2 + g^2}} & 0 & 0 & 0 & 0 & 0 & \dot{h}_1 \\
0 & e & f & 0 & 0 & 0 & \dot{\theta}_1 \\
0 & \frac{-e f}{\sqrt{e^2 + g^2}} & \sqrt{e^2 + g^2} & 0 & 0 & 0 & \omega_1 \\
0 & 0 & 0 & \frac{e}{\sqrt{e^2 + g^2}} & 0 & 0 & \dot{h}_2 \\
0 & 0 & 0 & 0 & e & f & \dot{\theta}_2 \\
0 & 0 & 0 & 0 & \frac{-e f}{\sqrt{e^2 + g^2}} & \sqrt{e^2 + g^2} & \omega_2
\end{bmatrix}
\] (2.11)

The above expression for \([\tau_r]\) is used in the Newmark code. One of the main advantages of a transformation matrix formulation is that various geometrical configurations can be handled with ease.

### 2.2.2 Translational Transformation

A translational transformation matrix is required to transform the spar and fabric and rib elements to the elastic axis which is parallel to them in the \( x - y \) plane. The translational transformation matrix \([\tau_t]\) is given by [20]:

\[
[\tau_t] = \begin{bmatrix}
1 & d & 0 & 0 & 0 & 0 & \dot{h}_1 \\
0 & 1 & 0 & 0 & 0 & 0 & \dot{\theta}_1 \\
0 & 0 & 1 & 0 & 0 & 0 & \omega_1 \\
0 & 0 & 0 & 1 & d & 0 & \dot{h}_2 \\
0 & 0 & 0 & 0 & 1 & 0 & \dot{\theta}_2 \\
0 & 0 & 0 & 0 & 0 & 1 & \omega_2
\end{bmatrix}
\] (2.12)

where \( d \) is the distance between any element and the elastic axis. For a spar element \( d = y_s - y_{ea} \) and for a fabric and rib element \( d = y_{fr} - y_{ea} \). Translational transformation is essentially the matrix equivalent of the parallel axis theorem for moments of inertia. In this particular case it is convenient to use a translational transformation and no cross connecting elements (rigid links) are required. This decreases the number of elements which makes the structural part of the code much more efficient.
2.3 System Matrix Transformation

In the previous section a displacement and force transformation matrix was derived using coordinate transformations. However we need to know the corresponding transformation matrix or procedure for transforming the stiffness, mass or other system matrices. In the local (element) coordinate system we have:

\[ \{ \mathbf{F} \} = [\mathbf{K}] \{ \mathbf{y} \} \]  \hspace{1cm} (2.13)

where

\[ \{ \mathbf{y} \} = [\tau] \{ \mathbf{y}_g \} \]  \hspace{1cm} (2.14)

\[ \{ \mathbf{F} \} = [\tau] \{ \mathbf{F}_g \} \]  \hspace{1cm} (2.15)

Substituting the above expressions into Equation 2.13 we have:

\[ [\tau] \{ \mathbf{F}_g \} = [\mathbf{K}] [\tau] \{ \mathbf{y}_g \} \]  \hspace{1cm} (2.16)

Premultiplying Equation 2.16 by \([\tau]^{-1}\) we get:

\[ \{ \mathbf{F}_g \} = [\tau]^{-1} [\mathbf{K}] [\tau] \{ \mathbf{y}_g \} \]  \hspace{1cm} (2.17)

Equation 2.17 can also be written as:

\[ [\mathbf{K}_g] \{ \mathbf{y}_g \} = \{ \mathbf{F}_g \} \]  \hspace{1cm} (2.18)

where the global stiffness matrix is given by:

\[ [\mathbf{K}_g] = [\tau]^{-1} [\mathbf{K}] [\tau] \]  \hspace{1cm} (2.19)

Similarly the global mass and damping matrices are given by:

\[ [\mathbf{M}_g] = [\tau]^{-1} [\mathbf{M}] [\tau] \]  \hspace{1cm} (2.20)

\[ [\mathbf{D}_g] = [\tau]^{-1} [\mathbf{D}] [\tau] \]  \hspace{1cm} (2.21)

The transformation procedure involves transforming the local system matrices into the global coordinates using two successive transformations as shown above. The order of transformation is not important but the Newmark code transformation procedure involves a translation to the elastic axis followed by a rotation into the global coordinate system.

Once the global stiffness, mass and damping element matrices have been obtained the system matrices are assembled using a standard finite element assembly procedure. Reddy [16] provides a very good explanation of the finite element assembly procedure.
Boundary conditions associated with the “Shearflex” portion of the wing are fairly straight forward. The “Shearflex” portion is essentially modeled as a cantilevered beam for both bending and torsion. Therefore all boundary conditions at the SuperRib are taken to be zero. This assumes that the relative displacement of the SuperBox as compared to the rest of the wing is negligible.

### 2.4 Non-Linear Time Marching Algorithm

The system differential equations (Equation 2.3) including damping and neglecting external reactions which are accounted for by boundary conditions are:

\[
[K] \{ \dot{y} \} + [D] \{ \ddot{y} \} + [M] \{ \dddot{y} \} = \{ F \} \]  \hspace{1cm} (2.22)

The aerodynamic forces and moments associated with a flapping wing in steady flight are non-linear functions of the twisting and plunging deflections of the wing and their first and second derivatives. Equation 2.22 is a second order non-linear equation which is solved directly using a Taylor series approximation in time to approximate the non-linear components of the force vector. Several time marching method were considered and the Newmark method was chosen because of its versatility and ease of use. The procedure for implementing the Newmark method is outlined here. The following assumptions are made in implementing the Newmark method [17]:

\[
\{ \dot{y} \}^{n+1} = \{ \dot{y} \}^n + [(1 - \delta) \{ \dot{y} \} + \delta \{ \dot{y} \}^{n+1}] \Delta t \]  \hspace{1cm} (2.23)

\[
\{ y \}^{n+1} = \{ y \}^n + \{ \dot{y} \}^n \Delta t + [(0.5 - \beta) \{ \ddot{y} \}^n + \beta \{ \dddot{y} \}^{n+1}] \Delta t^2 \]  \hspace{1cm} (2.24)

where \( \beta \) and \( \delta \) are parameters that can be determined to obtain integration accuracy and stability. When \( \delta = \frac{1}{2} \) and \( \beta = \frac{1}{6} \) relations 2.23 and 2.24 correspond to the linear acceleration method. Newmark originally proposed an unconditionally stable scheme known as the constant average acceleration method (Figure 2.4) in which \( \delta = \frac{1}{2} \) and \( \beta = \frac{1}{4} \). In addition to Equations 2.23 and 2.24, in order to solve for displacements, velocities and accelerations at time step \( n+1 \) the equilibrium equations 2.22 at time step \( n+1 \) are also required:

\[
[K] \{ y \}^{n+1} + [D] \{ \dot{y} \}^{n+1} + [M] \{ \ddot{y} \}^{n+1} = \{ F \}^{n+1} \]  \hspace{1cm} (2.25)

Solving Equation 2.24 for \( \{ \dddot{y} \}^{n+1} \) in terms of \( \{ y \}^{n+1} \) and then substituting for \( \{ \dddot{y} \}^{n+1} \) in Equation 2.23 we obtain equations for \( \{ \dddot{y} \}^{n+1} \) and \( \{ \dot{y} \}^{n+1} \), each in terms of the unknown displacements \( \{ y \}^{n+1} \) only. The two relations for \( \{ \dddot{y} \}^{n+1} \) and \( \{ \dot{y} \}^{n+1} \) are then substituted
Figure 2.4: Newmark’s Constant Average Acceleration Method

into 2.25 to solve for \( \{y\}^{n+1} \). Finally using Equations 2.23 and 2.24 \( \{\ddot{y}\}^{n+1} \) and \( \{\dot{y}\}^{n+1} \) can also be calculated. Substituting these expressions for \( \{\ddot{y}\}^{n+1} \) and \( \{\dot{y}\}^{n+1} \) into Equation 2.25 gives a system of simultaneous equations which can be solved to give \( \{y\}^{n+1} \):

\[
(K + a_0[M] + a_1[D]) \{y\}^{n+1} = \{F\}^{n+1} + [M] (a_0 \{y\}^n + a_2 \{\dot{y}\}^n + a_3 \{\ddot{y}\}^n) + [D] (a_1 \{y\}^n + a_4 \{\dot{y}\}^n + a_5 \{\ddot{y}\}^n)
\]  

(2.26)

where the constants \( a_0 \ldots a_5 \) are functions of the time step (\( \Delta t \)) and parameters \( \delta \) and \( \beta \).

In this case since the equations are non-linear and a temporal approximation has been used to obtain a set of linear second order equations, an iteration must be performed at each time step to ensure that the equilibrium equations are satisfied. The non-linear Newmark method is implemented using the following procedure [21]:

1. Specify initial conditions: \( \{y\}^0, \{\dot{y}\}^0 \) and \( \{\ddot{y}\}^0 \).

2. Select time step \( \Delta t \) and parameters \( \beta \) and \( \delta \) and calculate integration constants:

\[
\delta \geq 0.5 \quad \beta \geq 0.25(0.5 + \delta)^2
\]

\[
a_0 = \frac{1}{3\Delta t^2} \quad a_1 = \frac{\delta}{3\Delta t} \quad a_2 = \frac{1}{3\Delta t} \quad a_3 = \frac{1}{2\beta} - 1
\]

\[
a_4 = \frac{\delta}{\beta} - 1 \quad a_5 = \frac{\Delta t}{2} \left( \frac{\delta}{\beta} - 2 \right)
\]  

(2.27)

3. Form the stiffness matrix \([K]\), mass matrix \([M]\) and the damping matrix \([D]\) at time step \( n \).
4. Form effective stiffness matrix $[\hat{K}]$:

$$[\hat{K}] = [K] + a_0 [M] + a_1 [D] \quad (2.28)$$

5. At time step $n$ calculate the effective load vector $\{\hat{F}\}^{n+1}$:

$$\{\hat{F}\}^{n+1} = \{F\}^{n+1} + [M] (a_0 \{y\}^n + a_2 \{\dot{y}\}^n + a_3 \{\ddot{y}\}^n)$$

$$+ [D] (a_1 \{y\}^n + a_4 \{\dot{y}\}^n + a_5 \{\ddot{y}\}^n) \quad (2.29)$$

6. Solve for deflections at time step $n + 1$:

$$[\hat{K}] \{y\}^{n+1} = \{\hat{F}\}^{n+1} \quad (2.30)$$

and the incremental change in displacement is given by:

$$\{\Delta y\} = \{y\}^{n+1} - \{y\}^n \quad (2.31)$$

7. Iterate for dynamic equilibrium. Find displacements, velocities and accelerations for iteration loop $i + 1$ and decide if another iteration loop needs to be performed.

(a) Iteration loop $i$.

(b) Evaluate the $i$'th iteration approximation for accelerations and velocities:

$$\{y\}^{n+1}_i = \{y\}^n + \{\Delta y\}_i \quad (2.32)$$

$$\{\dot{y}\}^{n+1}_i = a_1 \{\Delta y\}_i - a_4 \{\dot{y}\}^n - a_5 \{\ddot{y}\}^n \quad (2.33)$$

$$\{\ddot{y}\}^{n+1}_i = a_0 \{\Delta y\}_i - a_2 \{\dot{y}\}^n - a_3 \{\ddot{y}\}^n \quad (2.34)$$

(c) Evaluate the $i$'th iteration residual forces:

$$\{F_{res}\}^{n+1}_i = \{F\}^{n+1} - [M] \{\dot{y}\}^{n+1}_i - [D] \{\ddot{y}\}^{n+1}_i - [K] \{y\}^{n+1}_i \quad (2.35)$$

(d) Solve for the $i + 1$'st residual displacement increments:

$$[\hat{K}] \{\delta y\}_{i+1} = \{F_{res}\}^{n+1}_i \quad (2.36)$$

(e) Evaluate the corrected displacement increments:

$$\{\Delta y\}_{i+1} = \{\Delta y\}_i + \{\delta y\}_{i+1} \quad (2.37)$$
(f) Check for convergence:

\[ \frac{|\{\delta y\}_{i+1}|}{|\{y\}^n + \{\Delta y\}_{i+1}|} \leq \epsilon \]  

(2.38)

where \( \epsilon \) is a tolerance factor and norm of a vector \( |\{v\}| \) is defined as:

\[ |\{v\}| = \sqrt{\sum_{i=1}^{N} v_i^2} \]  

(2.39)

8. If the convergence criterion is satisfied then return to step 3 to process the next time step. Otherwise return to step 7(a) for the next iteration.

The performance of the Newmark algorithm has been studied extensively and it is known to be unconditionally stable [18].

Due to the presence of the stalled and attached flow regimes, the implicit Newmark method is the only time marching technique suitable for this problem. Initially several very sophisticated explicit methods as well as other implicit time marching techniques were used but a diverging solution was obtained in every case. Table 2.1 shows the time integration techniques that were tried and the resulting outcome.

<table>
<thead>
<tr>
<th>Method</th>
<th>Implicit or Explicit</th>
<th>Result</th>
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<tbody>
<tr>
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<td>Fourth order Runge-Kutta Method (RK4)</td>
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<td>Fourth order Runge-Kutta with Step Size Adjustment</td>
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<tr>
<td>Wilson ( \theta ) Method</td>
<td>Implicit</td>
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<td>Houbolt Method</td>
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</tr>
<tr>
<td>Newmark Constant Average Acceleration Method</td>
<td>Implicit</td>
<td>Converges</td>
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</table>

Table 2.1: Time Integration Procedures
Chapter 3

Aerodynamic and Inertial Forces and Moments

The unsteady aerodynamic model for the study of a flapping wing is based on a modified strip theory approach. Within its constraints, vortex wake effects are accounted for as well as partial leading edge suction and post-stall behaviour. Moreover, the contributions of sectional mean angle of attack, camber and friction drag are added which allow this model to be used for the calculation of average lift and thrust, power required and propulsive efficiency of a flapping wing in equilibrium flight. This model is treated in more detail in References [15] and [22].

The starting point for the aerodynamic analysis is to determine the normal force \( N \) and moment \( M \) in Equation 2.6 acting on a wing segment. For any element, \( M \) and \( N \) consist of aerodynamic and inertial terms:

\[
N = N_{aero} + N_{inertia} \tag{3.1}
\]
\[
M = M_{aero} + M_{inertia} + M_{damp} \tag{3.2}
\]

3.1 Aerodynamic Forces and Moments

The aerodynamic forces and moments are introduced in this section and the damping moment is treated separately in the next chapter along with the spar structural damping. Figure 3.1 shows the aerodynamic forces on a representative segment (element) of the wing. Aerodynamic loads depend on whether the flow over a segment is attached or stalled.
3.1.1 Attached Flow

In the attached flow regime the aerodynamic normal force is given by:

\[ N_{\text{aero}} = N_c + N_a \]  \hspace{1cm} (3.3)

Where \( N_c \) is the circulatory normal force and \( N_a \) is the apparent mass normal force. The circulatory normal force on the wing segment is given by:

\[ N_c = \frac{1}{2} \rho U^2 V C_n c \Delta x \]  \hspace{1cm} (3.4)

where \( U \) is free stream air speed, \( c \) is the chord length of a particular segment and \( \Delta x \) is the segment's width. \( V \) is the flow velocity at the \( \frac{1}{4} \) chord location. The Fullwing code uses a linear version of the above expression where the velocity at the \( \frac{1}{4} \) chord location is replaced with the free stream velocity \( U \). The normal force coefficient is shown to be [15]:

\[ C_n = 2\pi (\alpha' + \alpha_0 + \bar{\theta}_a + \bar{\theta}_{\text{wash}}) \]  \hspace{1cm} (3.5)

where \( \alpha_0 \) is the zero lift angle. \( \bar{\theta}_a \) is the flapping axis angle of attack and \( \bar{\theta}_{\text{wash}} \) is the built-in pre-twist of a particular wing segment. \( \alpha' \) is the relative angle of attack at the \( \frac{3}{4} \) chord point and is given by [15]:

\[ \alpha' = \frac{AR}{2 + AR} \left[ F'(k) \alpha + \frac{c}{2U} \frac{G''(k)}{k} \dot{\alpha} \right] - \frac{2}{2 + AR} \left[ \alpha_0 + \bar{\theta}_a + \bar{\theta}_{\text{wash}} \right] \]  \hspace{1cm} (3.6)
Equation 3.6 is predicated on simple harmonic motion for $\alpha$. In this case the motion can be periodic but not necessarily simple harmonic. Therefore, the use of this equation is considered to be an approximation to the unsteady shed wake effects. $AR$ is the wing's aspect ratio and $k$ is the reduced frequency which is given by:

$$k = \frac{\omega}{2U}$$

Upon using a simplified formulation of the modified Theodorsen function, $F'(k)$ and $G'(k)$ are given by:

$$F'(k) = 1 - \frac{C_1 k^2}{k^2 + C_2^2} \quad G'(k) = -\frac{C_1 C_2 k}{k^2 + C_2^2}$$

$$C_1 = \frac{0.5AR}{2.32 + AR} \quad C_2 = 0.181 + \frac{0.772}{AR}$$

The equations for $\alpha$ and $\dot{\alpha}$ are from [15]:

$$\alpha = \left[ \frac{\hat{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) + (0.75c - y_{ea}) \dot{\theta}}{U} \right] \left( \dot{\theta} \right) + \bar{\theta}$$

$$\dot{\alpha} = \left[ \left( \hat{h}_0 + \hat{h} \right) \cos(\dot{\theta} + \bar{\theta}_{wash}) - \hat{h} \dot{\theta} \sin(\dot{\theta} + \bar{\theta}_{wash}) + (0.75c - y_{ea}) \dot{\theta} \right] \left( \dot{\theta} \right) + \bar{\theta}$$

The fullwing code uses a linear version of the above expression for $\dot{\alpha}$ by simply ignoring the second term in the numerator. The non-linear Newmark code includes the complete expression.

It is appropriate at this point to introduce the total twist angle $\theta$ of a segment about its elastic axis, which is a combination of elastic and constant parts:

$$\theta = \dot{\theta} + \bar{\theta}_a + \bar{\theta}_{wash}$$

where $\dot{\theta}$ is the elastic deflection of a wing segment about its elastic axis and $\bar{\theta}_a$ is the flapping axis angle of attack with respect to the free stream velocity and $\bar{\theta}_{wash}$ is the built-in pretwist of a particular segment. Similarly, the total plunging displacement is a combination of an imposed motion $\hat{h}_0$ and an elastic component $\hat{h}$:

$$h = \hat{h}_0 + \hat{h}$$

The imposed motion for a given wing segment is defined as:

$$h_0 = \Gamma_0 x \cos(\omega t)$$

where $\Gamma_0$ is the maximum flapping amplitude (which for the full-scale ornithopter is about 31 degrees) and $x$ is the distance from the center of a given wing segment to the flapping.
Returning to equation 3.4, note that the flow velocity $V$ must include the downwash as well as the wing's motion relative to the free stream velocity $U$. This is done by including $\alpha'$ along with the kinematic parameters:

$$V = \sqrt{\left[U \cos \theta - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{\text{wash}})\right]^2 + \left[U (\alpha' + \bar{\theta}_a + \bar{\theta}_{\text{wash}}) - (0.5c - y_{es}) \dot{\theta}\right]^2} \quad (3.14)$$

An additional normal-force contribution comes from the apparent-mass effect which acts at the mid-chord location (Figure 3.1) and is given by:

$$N_a = \frac{1}{4} \rho \pi c^2 (U \dot{\alpha} - \frac{1}{4} c \bar{\theta}) \Delta x \quad (3.15)$$

A section's circulation distribution generates forces in the chordwise direction as shown in Figure 3.1. From DeLaurier [23], the chordwise force due to camber is given by:

$$D_c = -2\pi \alpha_0 (\alpha' + \bar{\theta}_a + \bar{\theta}_{\text{wash}}) \frac{\rho U V}{2} c \Delta x \quad (3.16)$$

The leading edge suction force is obtained from Garrick [24]:

$$T_s = \eta_s 2\pi \left(\alpha' + \bar{\theta}_a + \bar{\theta}_{\text{wash}} - \frac{c \dot{\theta}}{4U}\right)^2 \frac{\rho U V}{2} c \Delta x \quad (3.17)$$

The only change to Garrick's formulation is the addition of the $\eta_s$ term which is referred to as the leading-edge suction efficiency factor and is determined experimentally. This efficiency factor is required since Garrick's formulation is based on ideal potential flow.

Viscous drag on the airfoil due to skin friction is found by using the skin-friction drag coefficient $C_d$ for which an expression may be found in Hoerner [25]:

$$D_f = C_d \frac{\rho V_x^2}{2} c \Delta x \quad (3.18)$$

where $V_x$ is the relative flow speed tangent to the section, which can be approximated by:

$$V_x = U \cos \theta - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{\text{wash}}) \quad (3.19)$$

Therefore the total chordwise force is:

$$F_x = T_s - D_c - D_f \quad (3.20)$$

### 3.1.2 Stalled Flow

When the attached flow range is exceeded, totally separated flow is assumed to abruptly occur, for which all chordwise forces are negligible:

$$T_s = D_c = D_f = 0 \quad (3.21)$$
and the normal force is given by:

\[
(N_{aero})_{sep} = (N_c)_{sep} + (N_a)_{sep}
\]  \hspace{1cm} (3.22)

\((N_c)_{sep}\) is assumed to act at the mid-chord point due to crossflow drag and is given by:

\[
(N_c)_{sep} = (C_d)_{cf} \frac{\rho \dot{V} V_c}{2} c \Delta x
\]  \hspace{1cm} (3.23)

where

\[
\dot{V} = \sqrt{V_r^2 + V_a^2}
\]  \hspace{1cm} (3.24)

\(V_n\) is the mid-chord normal velocity component due to the wing's motion given by:

\[
V_n = \dot{h} \cos(\dot{\theta} + \bar{\theta}) + \frac{1}{2}(0.5c - y_{ea}) \dot{\theta} + U \sin \theta
\]  \hspace{1cm} (3.25)

and \(V_x\) is given by Equation 3.19. It is evident from Equation 3.24 that \(\dot{V}\) is a non-linear function of the independent variables \(\dot{\theta}\) and \(\dot{h}\). This shows that both the stalled and attached-flow aerodynamic formulations are indeed non-linear. Also, \((N_a)_{sep}\) is assumed to be half of the attached-flow value given by Equation 3.15:

\[
(N_a)_{sep} = \frac{N_a}{2} = \frac{1}{8} \rho \pi c^3 (U' \dot{\alpha} - \frac{1}{4} c \dot{\theta}) \Delta x
\]  \hspace{1cm} (3.26)

### 3.1.3 Aerodynamic Moments

The attached-flow aerodynamic moment about the elastic axis is a function of the circulatory and apparent mass normal forces and is given by:

\[
M_{aero} = M_{ac} - N_c(0.25c - y_{ea}) - N_a(0.5c - y_{ea}) - \frac{1}{16} \rho \pi c^3 U' \Delta x \dot{\theta} - \frac{1}{128} \rho c^4 \Delta x \ddot{\theta}
\]  \hspace{1cm} (3.27)

\(N_c\) and \(N_a\) are given by Equations 3.4 and 3.15 respectively. The fourth and fifth terms in the above equation account for the apparent camber and apparent inertia moments respectively [15]. The moment about the aerodynamic center is given by [26]:

\[
M_{ac} = C_{mac} \frac{\rho U V}{2} c^2 \Delta x
\]  \hspace{1cm} (3.28)

The chordwise forces do not contribute to these moments as they essentially pass through the elastic axis of a wing segment.

The stalled aerodynamic moment is given by:

\[
(M_{aero})_{sep} = -[(N_c)_{sep} + (N_a)_{sep}](0.5c - y_{ea})
\]  \hspace{1cm} (3.29)

where \((N_c)_{sep}\) and \((N_a)_{sep}\) are given by Equations 3.23 and 3.26 respectively. The moment about the aerodynamic center due to apparent camber and apparent inertia effects are negligible because these quantities are defined for attached flow only.
3.1.4 Stall Criterion

Prouty [27] has shown that a pitching airfoil can retain attached flow at angles greatly exceeding the static stall angle. An advantage of a strip-theory model is that it allows for an approximation to the localized stall behaviour. Prouty uses a dynamic stall-delay effect, represented by an angle $\Delta \alpha$ to account for the difference between the static and effective stall angles:

$$ (\alpha_{\text{stall}})_{s} - (\alpha_{\text{stall}})_{e} = \Delta \alpha = \xi \sqrt{\frac{c}{2l}} \left( \frac{\hat{\alpha}}{\hat{\alpha}_{\text{mag}}} \right) \sqrt{\frac{c}{2l}} $$

where $\xi$ is determined experimentally and depends on the local Mach number. In this case it was determined that $\Delta \alpha$ is given by [15]:

$$ \Delta \alpha = 0.51 \left( \frac{\hat{\alpha}}{\hat{\alpha}_{\text{mag}}} \right) \sqrt{\frac{c}{2l}} \left( \frac{\hat{\theta}}{3} \right) $$

where $\hat{\alpha}_{\text{mag}} = \text{abs}(\hat{\alpha})$. The magnitude of $\hat{\alpha}$ is used to ensure that the term under the square root is positive and the term in the brackets ensures that the correct sign is used. Therefore the criterion for attached flow over a wing segment is:

$$ (\alpha_{\text{stall}})_{\text{min}} \leq \left[ \alpha' + \bar{\alpha} + \bar{\alpha}_{\text{wash}} - \frac{3}{4} \left( \frac{c}{2l} \right) \right] \leq (\alpha_{\text{stall}})_{\text{max}} $$

![Figure 3.2: Inertial Loads and Moments on a Wing Segment](image)
3.2 Inertial Forces and Moments

Figure 3.2 shows the inertial forces and moments acting on a wing segment. The Fullwing code treats the masses as inertial loads but the Newmark code breaks down the inertial loads into reactions that involve the elastic components and external forces that are a result of the imposed motion and gravity. The inertial reactions have already been considered by the consistent mass matrix formulation and the external inertial normal force is given by [28]:

\[ N_{\text{inertia}} = (m_{\text{spar}} + m_{fr}) (g_r - \ddot{h}_0) \]  \hspace{1cm} (3.33)

Further, the external inertial moment is given by [28]:

\[ M_{\text{inertia}} = [(y_s - y_e) m_{\text{spar}} + (0.5c - y_e) m_{fr}] (g_r - \ddot{h}_0) \]  \hspace{1cm} (3.34)

where \( g_r \) is acceleration due to gravity.

3.3 Temporal Approximation for Non-Linear Terms

The most straightforward solution procedure is to use a non-linear time marching algorithm such as the fourth-order Runge Kutta method. This would avoid the requirement of finding a temporal approximation to the non-linear terms. An attempt was made to use this method to solve the equilibrium equations. However, due to the presence of two distinct flow regimes and inherent instability, the solution would diverge after a few time steps.

It was noted in the previous section that the circulatory and apparent-mass normal forces and moment about the aerodynamic center are non-linear functions of \( \hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \) and \( \ddot{h} \):

\[ N_c = f(\hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \ddot{h}) \quad N_a = f(\hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \ddot{h}) \quad M_{ac} = f(\hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \ddot{h}) \]

\[ (N_c)_{\text{sep}} = f(\hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \ddot{h}) \quad (N_a)_{\text{sep}} = f(\hat{\theta}, \dot{\hat{\theta}}, \ddot{\hat{\theta}}, \hat{h} \ddot{h}) \]  \hspace{1cm} (3.35)

The value of the circulatory normal force for attached flow at time \( n+1 \) is found by using a Taylor series approximation:

\[ N_c^{n+1} = N_c^n + \frac{\partial N_c^n}{\partial \hat{\theta}} (\hat{\theta}^{n+1} - \hat{\theta}^n) + \frac{\partial N_c^n}{\partial \dot{\hat{\theta}}} (\dot{\hat{\theta}}^{n+1} - \dot{\hat{\theta}}^n) + \frac{\partial N_c^n}{\partial \ddot{\hat{\theta}}} (\ddot{\hat{\theta}}^{n+1} - \ddot{\hat{\theta}}^n) \]

\[ + \frac{\partial N_c^n}{\partial \hat{h}} (\hat{h}^{n+1} - \hat{h}^n) + \frac{\partial N_c^n}{\partial \dot{\hat{h}}} (\dot{\hat{h}}^{n+1} - \dot{\hat{h}}^n) + \frac{\partial N_c^n}{\partial \ddot{\hat{h}}} (\ddot{\hat{h}}^{n+1} - \ddot{\hat{h}}^n) \]  \hspace{1cm} (3.36)
CHAPTER 3. AERODYNAMIC AND INERTIAL FORCES AND MOMENTS

Similarly the apparent mass normal for attached flow is given by:

\[ N_{a}^{n+1} = N_{a}^{n} + \frac{\partial N_{a}^{n}}{\partial \theta} [\dot{\theta}^{n+1} - \dot{\theta}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{\theta}} [\ddot{\theta}^{n+1} - \ddot{\theta}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{\bar{\theta}}} [\dddot{\theta}^{n+1} - \dddot{\theta}^{n}] \\
+ \frac{\partial N_{a}^{n}}{\partial \bar{\bar{\bar{\theta}}}} [\dddot{\bar{\theta}}^{n+1} - \dddot{\bar{\theta}}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{\bar{\bar{\bar{\theta}}}}} [\dddot{\bar{\bar{\theta}}}^{n+1} - \dddot{\bar{\bar{\theta}}}^{n}] \\
+ \frac{\partial N_{a}^{n}}{\partial \bar{h}} [\dot{h}_{0}^{n+1} - \dot{h}_{0}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{h}} [\ddot{h}^{n+1} - \ddot{h}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{\bar{h}}} [\dddot{h}_{0}^{n+1} - \dddot{h}_{0}^{n}] + \frac{\partial N_{a}^{n}}{\partial \bar{\bar{h}}} [\dddot{h}^{n+1} - \dddot{h}^{n}] \] (3.37)

The stalled flow circulatory normal does not include \( \dddot{\bar{\theta}} \) and \( \dddot{\bar{h}} \) terms. It is given by:

\[ (N_{c})_{n+1}^{n} = (N_{c})_{n}^{n} + \frac{\partial (N_{c})_{n}^{n}}{\partial \dot{\theta}} [\dot{\bar{\theta}}^{n+1} - \dot{\bar{\theta}}^{n}] + \frac{\partial (N_{c})_{n}^{n}}{\partial \ddot{\theta}} [\ddot{\bar{\theta}}^{n+1} - \ddot{\bar{\theta}}^{n}] \\
+ \frac{\partial (N_{c})_{n}^{n}}{\partial \dddot{\theta}} [\dddot{\bar{\theta}}^{n+1} - \dddot{\bar{\theta}}^{n}] + \frac{\partial (N_{c})_{n}^{n}}{\partial \bar{h}} [\dot{\bar{h}}^{n+1} - \dot{\bar{h}}^{n}] + \frac{\partial (N_{c})_{n}^{n}}{\partial \ddot{h}} [\ddot{\bar{h}}^{n+1} - \ddot{\bar{h}}^{n}] + \frac{\partial (N_{c})_{n}^{n}}{\partial \dddot{h}} [\dddot{\bar{h}}^{n+1} - \dddot{\bar{h}}^{n}] \] (3.38)

The stalled flow apparent mass normal is similar to the attached flow expression:

\[ (N_{a})_{n+1}^{n} = (N_{a})_{n}^{n} + \frac{\partial (N_{a})_{n}^{n}}{\partial \dot{\theta}} [\dot{\bar{\theta}}^{n+1} - \dot{\bar{\theta}}^{n}] + \frac{\partial (N_{a})_{n}^{n}}{\partial \ddot{\theta}} [\ddot{\bar{\theta}}^{n+1} - \ddot{\bar{\theta}}^{n}] + \frac{\partial (N_{a})_{n}^{n}}{\partial \dddot{\theta}} [\dddot{\bar{\theta}}^{n+1} - \dddot{\bar{\theta}}^{n}] \\
+ \frac{\partial (N_{a})_{n}^{n}}{\partial \bar{h}} [\dot{\bar{h}}^{n+1} - \dot{\bar{h}}^{n}] + \frac{\partial (N_{a})_{n}^{n}}{\partial \ddot{h}} [\ddot{\bar{h}}^{n+1} - \ddot{\bar{h}}^{n}] + \frac{\partial (N_{a})_{n}^{n}}{\partial \dddot{h}} [\dddot{\bar{h}}^{n+1} - \dddot{\bar{h}}^{n}] \] (3.39)

The aerodynamic moments (Equations 3.27 and 3.29) at the \( n + 1 \)st time step are also expanded using the procedure outlined above, where \( M_{ax}^{n+1} \) is expressed using a Taylor series approximation:

\[ M_{ax}^{n+1} = M_{ax}^{n} + \frac{\partial M_{ax}^{n}}{\partial \dot{\theta}} [\dot{\bar{\theta}}^{n+1} - \dot{\bar{\theta}}^{n}] + \frac{\partial M_{ax}^{n}}{\partial \ddot{\theta}} [\ddot{\bar{\theta}}^{n+1} - \ddot{\bar{\theta}}^{n}] + \frac{\partial M_{ax}^{n}}{\partial \dddot{\theta}} [\dddot{\bar{\theta}}^{n+1} - \dddot{\bar{\theta}}^{n}] \\
+ \frac{\partial M_{ax}^{n}}{\partial \bar{h}} [\dot{\bar{h}}^{n+1} - \dot{\bar{h}}^{n}] + \frac{\partial M_{ax}^{n}}{\partial \ddot{h}} [\ddot{\bar{h}}^{n+1} - \ddot{\bar{h}}^{n}] + \frac{\partial M_{ax}^{n}}{\partial \dddot{h}} [\dddot{\bar{h}}^{n+1} - \dddot{\bar{h}}^{n}] \] (3.40)

Partial derivatives given in the above equations are shown explicitly in Appendix C.

Terms involving the \( n \)th time step are moved to the right hand side of equilibrium equations (Equation 2.3) and terms involving the \( n + 1 \)st time step are moved to the left hand side of Equation 2.3. Essentially, stiffness, mass and damping matrices are augmented by terms that account for the nonlinearity of the forcing functions. Some references (References [18] and [20]) refer to the augmenting matrices as tangent stiffness and tangent mass matrices.
3.4 Rigid and Center Sections

Aerodynamic forces and moments acting on the rigid and center sections of the wing are identical to the outer portion of the wing, using the same equations and the same criterion for stall. The only difference is that the elastic variables are taken to be zero ($\dot{\theta} = \dot{\tilde{h}} = 0$). Furthermore the forcing function for the rigid section is given by:

$$\begin{align*}
(h_0)_{rs} &= -\frac{(\Delta x)_{rs}}{2} \Gamma_0 \cos(\omega t) \\
(\Delta x)_{rs} &= -\frac{(\Delta x)_{rs}}{2} \Gamma_0 \cos(\omega t)
\end{align*}$$

(3.41)

where $(\Delta x)_{rs}$ is the width of the rigid section and $\omega$ is the flapping frequency. Likewise, the center section forcing function is given by:

$$
(h_0)_{cs} = -(\Delta x)_{rs} \Gamma_0 \cos(\omega t)
$$

(3.42)

$\Gamma_0$ is the flapping amplitude angle which is about 31° for the full-scale ornithopter.

3.5 Average Lift and Thrust

The total lift and thrust of a flapping wing at any given time is the sum of the contributions from the rigid, center and flexible sections. The loads outlined in the previous section are normal $(N)$ and tangential $(F_x)$ to a given segment. Lift $L$ and thrust $R$ for a particular segment of the flexible wing are calculated as follows:

$$
R = F_x \cos(\theta) - N \sin(\theta) \quad L = N \cos(\theta) + F_x \sin(\theta)
$$

(3.43)

Lift and thrust produced by the rigid and center sections are calculated in a similar way. The average thrust and lift generated by the whole wing over $N_t$ time intervals is:

$$
R_{ave} = \frac{1}{N_t} \sum_{i=1}^{N_t} \left( 2 \sum_{j=1}^{N_e} R_j + 2 R_{rs} + R_{cs} \right) \quad L_{ave} = \frac{1}{N_t} \sum_{i=1}^{N_t} \left( 2 \sum_{j=1}^{N_e} L_j + 2 L_{rs} + L_{cs} \right)
$$

(3.44)

Where $N_e$ is the total number of elements per wing.
3.6 Bending and Twisting Moments

The bending moments encountered at each wing segment are calculated by transforming the forces in the wing's frame of reference to the flapping axis frame of reference. The flapping axis angle of attack with respect to free stream velocity can thus be accounted for. Hence, a new normal force $F_{\text{normal}}$ must be defined [14]:

$$F_{\text{normal}} = N \cos(\theta - \bar{\theta}) + (N_{\text{spar}} + N_{fr}) \cos(\bar{\theta}) + (T_s + D_f + D_c) \sin(\theta - \bar{\theta})$$  \hspace{1cm} (3.45)

where normal forces acting on the spar and fabric and rib components are [28]:

$$N_{\text{spar}} = m_{\text{spar}} \left[ \dot{h} + (y_s - y_{ce}) \ddot{\theta} - g_r \right]$$

$$N_{fr} = m_{fr} \left[ \ddot{h} + (0.5c - y_{ce}) \ddot{\theta} - g_r \right]$$  \hspace{1cm} (3.46)

The bending moment at the flapping axis is thus calculated by adding up the contributions of all of the segments on the flexible portion:

$$\left( M_{\text{bend}} \right)_{sr} = \sum_{i=1}^{N_s} (F_{\text{normal}})_i x_i$$  \hspace{1cm} (3.47)

where $x_i$ is the distance of a given segment to the flapping axis. Similarly the twisting moment is given by:

$$\left( M_{\text{twist}} \right)_{sr} = \sum_{i=1}^{N_s} (M_{ac})_i - (F_{\text{normal}})_i y_i$$  \hspace{1cm} (3.48)

where $y_i$ is the distance of the elastic axis for a given segment from a line perpendicular to the flapping axis and passing through the elastic axis at the flapping axis location.

3.7 Power Requirement and Propulsive Efficiency

The instantaneous power required to move an element against its aerodynamic loads for attached flow is given by [15]:

$$P_{in} = F_x \dot{h} \sin(\theta - \bar{\theta}) + N_{aero} \left[ \dot{h} \cos(\theta - \bar{\theta}) + \frac{1}{4} c \dot{\theta} \right] - M_{ac} \ddot{\theta}$$

$$+ \frac{1}{16} \rho \pi c^2 \Delta x \left( U \dot{\alpha} - \frac{1}{4} c \ddot{\theta} \right)$$  \hspace{1cm} (3.49)

for stalled flow, the input power expression becomes:

$$P_{in} = (N_{aero})_{sep} \left[ \dot{h} \cos(\theta - \bar{\theta}) + \frac{1}{2} c \dot{\theta} \right]$$  \hspace{1cm} (3.50)
The average input power over $N_t$ time intervals is given by:

$$\langle P_{in} \rangle_{ave} = \frac{1}{N_t} \sum_{i=1}^{N_t} \left( 2 \sum_{j=1}^{N_r} (P_{in})_j + 2(P_{in})_{rs} + (P_{in})_{cs} \right)$$  \hspace{1cm} (3.51)

The average power put out by the wings in thrust is the average thrust times the flight speed:

$$\langle P_{out} \rangle_{ave} = R_{ave} U$$  \hspace{1cm} (3.52)

therefore the average propulsive efficiency is the ratio of the average output and input powers:

$$\eta_p = \frac{\langle P_{out} \rangle_{ave}}{\langle P_{in} \rangle_{ave}}$$  \hspace{1cm} (3.53)

Although the propulsive efficiency calculated above is low compared to a standard propeller, it must be noted that the power input includes both the lifting and thrusting power inputs. If the lifting portion of the input power is taken out, propulsive efficiency would be comparable to that of a standard propeller.

### 3.8 Driving and Supporting Structure Loads

![Figure 3.3: Outer Wing Panel Free-Body Diagram](image)

Although the strength of the outriggers and attachment points are not directly related to the wing design or performance, they are important parameters in the overall ornithopter
design. Figure 3.3 shows the loads acting on one of the wing's outer panels. The two unknown reactions are determined by considering equilibrium in the vertical direction as well as rotation about the flapping axis. Upon solving the resulting two equations and two unknowns $F_{\text{drive}}$ and $F_{\text{pivot}}$ are found to be:

$$F_{\text{drive}} = \left[ \left( \sum_{i=1}^{N_x} (F_{\text{normal}})_i x_i \right) - (F_{\text{normal}})_r \frac{\Delta x}{2} \right] \frac{1}{\cos(\Gamma) \Delta x}$$

(3.54)

where $(F_{\text{normal}})_r$ is the normal force acting on the rigid section. and $\Gamma = \Gamma_0 \cos(\omega t) + \mu$ where $\mu$ is the mean dihedral angle which for the full-scale ornithopter is about $7.6^\circ$. $F_{\text{pivot}}$ is then given by:

$$F_{\text{pivot}} = F_{\text{drive}} + \left( \sum_{i=1}^{N_x} (F_{\text{normal}})_i \right) \cos(\Gamma) + (F_{\text{normal}})_r \cos(\Gamma)$$

(3.55)

The force at the driving hinge $F_{\text{drive}}$ can subsequently be used to calculate the force required to drive the center section.
Chapter 4

Damping

Early in the development of the Newmark code it became apparent that a reliable damping model is required. Without a damping model and when using the Expothopter data file the solution would diverge. The reason was found to be that the undamped differential equations had positive eigenvalues. This showed that the initial theoretical model did not represent the actual system. However, this was in contradiction to what the Fullwing code predicted, as it did not have a damping model either. Furthermore, the Expothopter did perform reasonably well in wind tunnel tests conducted in Ottawa in 1995. At first it was believed that the cause was a set of stiff differential equations. However, examination of the system eigenvalues showed that the equations are not stiff. Finally it was concluded that a damping model was required as that is the only component not accounted for by Fullwing. Furthermore, the shearflexing action of the wing creates damping forces along the trailing edge that must be accounted for.

The reason for Fullwing’s predictions lies in the assumptions that were made when the code was developed. The most critical assumption is that the output is assumed to be harmonic (sinusoidal) with a frequency equal to that of the forcing function. This basically guarantees that the output will not diverge. Therefore, when using a time-marching method, one must account for damping in order to obtain a stable solution.

Besides aerodynamic damping, there are two other forms of damping associated with a flapping wing:

1. Fabric and rib damping which is a result of the frictional losses due to the rubbing of the wing’s fabric covering against the ribs caused by the shearflexing action. This action also contributes to damping as the trailing edge strips rub against one another.

2. Structural damping which is due to the viscoelastic losses in the spar. In homoge-
neous materials such as metals this type of damping is relatively well understood and damping coefficients are known. In the case of the ornithopter wing, which is a non-homogeneous material, experiments were conducted to determine the free-decay constant $\zeta$ for the overall wing.

### 4.1 Fabric and Rib Friction Damping

Experiments carried out in 1997 showed that the total damping moment is a combination of a constant value ($M_{d0}$) and a moment which is a function of the shear rate ($M_{dar}$):

$$M_{\text{damp}} = M_{d0} + M_{dar}$$

(4.1)

In terms of the friction force:

$$F_f = C_{f0} p A_{con} + C_f p A_{con} v_r$$

(4.2)

The first term on the right-hand side represents the shear-rate-independent friction force and the second term represents the shear-rate-dependent friction force. $C_{f0}$ is the friction coefficient for the shear-rate-independent motion and $C_f$ is the friction coefficient for the shear-rate-dependent motion. $p$ is the contact pressure and $A_{con}$ is the contact area between the surfaces. Also, $v_r$ is the spanwise velocity of the upper surface relative to the lower surface of the wing.

Considering the shear-rate-independent term and referring to Figure 4.1. we can split the total friction force into a uniformly distributed fabric and rib force $f_0$ and a trailing edge force $f_{ote}$:

$$f_0 = C_{f0} p w_{rib} dy$$

(4.3)

$$f_{ote} = C_{f0} p A_{te}$$

(4.4)

For a single wing segment the total shear-rate-independent damping moment is found by summing the moments about the elastic axis which in Figure 4.1. is shown to be the back of the spar. $A_{te}$ is contact area of the trailing edge surface and $w_{rib}$ is the width of a single rib. One thus has:

$$M_{d0} = C_{f0} p A_{te}(c - y_{ea}) + C_{f0} p w_{rib} \int_0^{c-y_{ea}} dy$$

(4.5)
where \( c \) is the chord length of a particular wing section and \( y_{ea} \) is the distance from the leading edge to the elastic axis. \( M_{d0} \) is therefore given by:

\[
M_{d0} = C_f \rho (c - y_{ea}) \left[ \frac{v_{te}}{2} \left( c - y_{ea} \right) \right] \text{sgn} \left( \frac{\dot{\theta}}{v_{te}} \right)
\]  

(4.6)

For the shear-rate-dependent damping moment, it is assumed that the friction force is related to velocity as shown by the second term on the right hand side of Equation 4.2. For each segment of the wing there is a twist/shear rate expressed as \( \frac{\partial v}{\partial x} \) which is shown in Figure 4.2. At this point we assume that the relative velocity of the upper surface to the lower surface \( v_r \) has a linear distribution, going from zero at the elastic axis to \( v_r = v_{te} \) at the trailing edge. This gives rise to a friction force distribution as shown in Figure 4.3.

The two force components \( f(y) \) and \( f_{te} \) are given by:

\[
f(y) = C_f \rho v_r(y) w_{rib} dy
\]  

(4.7)

\[
f_{te} = C_f \rho v_{te} A_{te}
\]  

(4.8)
The shear-rate-dependent damping moment is thus given by:

\[ M_{dar} = F_{te} (c - y_{ea}) + \int_{0}^{c-y_{ea}} f(y) y \, dA \]

\[ = C_f p \, v_{te} \left[ A_{te} (c - y_{ea}) + \frac{w_{rib}}{(c - y_{ea})} \int_{0}^{c-y_{ea}} y^2 \, dy \right] \tag{4.9} \]

Performing the integral on the right hand side and simplifying gives:

\[ M_{dar} = C_f p \, v_{te} (c - y_{ea}) \left[ A_{te} + \frac{w_{rib}}{3} (c - y_{ea}) \right] \tag{4.10} \]

The velocity at the trailing edge can be written as:

\[ v_{te} = \frac{\partial x}{\partial t} = \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{\partial x}{\partial \theta} \dot{\theta} \tag{4.11} \]

So that \( M_{dar} \) is finally given by:

\[ M_{dar} = C_f p \, (c - y_{ea}) \left[ A_{te} + \frac{w_{rib}}{3} (c - y_{ea}) \right] \frac{\partial x}{\partial \theta} \dot{\theta} \tag{4.12} \]

### 4.2 Friction Damping Parameters

This section presents typical numerical input values for the full-scale ornithopter. A typical element used here is from an 11-element wing which was originally used in the Fullwing code and is shown in Figure 4.1.

Width of a single rib is 1\( \frac{1}{2} \) inches and since an element includes 2 full rib \( w_{rib} = 0.0883 \text{ ft} \). Trailing edge area \( A_{te} \) is composed of 2 full trailing edge clips plus the contact area of the trailing edge strip: \( A_{te} = 2 \times A_{clip} + A_{strip} = 0.015 \text{ ft}^2 \). The value of \( p \) is difficult to estimate because the fabric tension can vary due to several effects. An order-of-magnitude approximation may be the wing loading which is about 4.3\( \text{ lb/ft}^2 \). The value of \( \frac{\partial x}{\partial \theta} \) is also estimated to be about 2 ft/rad. These values along with the friction coefficients \( C_f \) and \( C_{f0} \) are listed in Table 4.1.
Figure 4.4: Typical Element Dimensions

<table>
<thead>
<tr>
<th>Input Variable</th>
<th>Typical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{rib} )</td>
<td>0.0833 ft</td>
</tr>
<tr>
<td>( A_{te} )</td>
<td>0.015 ( ft^2 )</td>
</tr>
<tr>
<td>( p )</td>
<td>4.3 ( lb/ft^2 )</td>
</tr>
<tr>
<td>( \frac{\partial^2}{\partial \theta^2} )</td>
<td>2.0 ( ft/rad )</td>
</tr>
<tr>
<td>( C_{fo} )</td>
<td>0.26</td>
</tr>
<tr>
<td>( C_f )</td>
<td>0.124 ( sec/ft )</td>
</tr>
</tbody>
</table>

Table 4.1: Full-Scale Ornithopter Damping Properties

4.3 Structural Damping

Structural damping is primarily due to mechanisms such as hysteresis in the material and slip in connections. These mechanisms are not well understood and they are awkward to incorporate into the equilibrium equations. Therefore, the actual mechanism is usually approximated by viscous damping. Comparisons of theory and experiment show that this approach is sufficiently accurate in most cases [20]. With such approximate methods, experimental observations of the vibratory response of structures are used to assign a fraction of critical damping as a function of frequency, or, more commonly, a single decay constant \( \zeta \) for the frequency range.

A popular damping scheme, called Rayleigh or proportional damping, is to form a
damping matrix $[D]$ as a linear combination of the stiffness and mass matrices [17]:

$$[D] = [M] \sum_{k=0}^{r-1} a_k ([M^{-1}] [K])^k$$  \hspace{1cm} (4.13)

where the coefficients $a_k$, $k=0\ldots r-1$ are calculated from the $r$ simultaneous equations:

$$\zeta_i = \frac{1}{2} \left( \frac{a_0}{\omega_i} + a_1 \omega_i + a_2 \omega_i^3 + \cdots + a_{r-1} \omega_i^{2r-3} \right)$$  \hspace{1cm} (4.14)

For a two degree of freedom system:

$$[D] = a[M] + b[K]$$  \hspace{1cm} (4.15)

where $a$ and $b$ are called the mass and stiffness-proportional damping constants respectively. The damping matrix given above is orthogonal because it permits modes to be uncoupled by eigenvectors associated with the undamped eigenvalues. The relationship between $a$, $b$ and decay constants $\zeta$ at a frequency $\omega$ is given by:

$$\zeta = \frac{a + \omega^2 b}{2\omega}$$  \hspace{1cm} (4.16)

Damping constants $a$ and $b$ are determined by choosing two distinct decay constants ($\zeta_1$ and $\zeta_2$) at two different frequencies ($\omega_1$ and $\omega_2$) and solving simultaneous equations for $a$ and $b$. Thus:

$$a = 2\omega_1 \omega_2 \frac{\zeta_1 \omega_2 - \zeta_2 \omega_1}{\omega_2^2 - \omega_1^2} \quad b = 2 \frac{\zeta_2 \omega_2 - \zeta_1 \omega_1}{\omega_2^2 - \omega_1^2}$$  \hspace{1cm} (4.17)

As part of the work leading to the design of the full-scale ornithopter, several sample spars were constructed. One of these, known as Spar 6, was used to perform fatigue testing (Figure 4.5). Spar 6’s bending and torsional stiffness characteristics are identical to actual wing spar’s characteristics at the flapping-axis location.

In 1998, Spar 6 was used to determine its bending and torsional decay constants. Figure 4.6 shows Spar 6 in its bending configuration and Figure 4.7 shows Spar 6 in its torsional configuration.

The testing procedure involved adding weights to the spar and striking it with a hammer while its vibratory response was measured using an accelerometer. The weights were varied in order to produce a range of natural frequencies. The recorded signals were then band pass filtered in order to filter out the noise contribution. Figure 4.8 shows the recorded and filtered signals associated with a torsional decay test. It is noteworthy that the major noise component has a frequency of exactly 60 Hz which corresponds to the frequency of the ambient lighting.
Figure 4.9 shows the experimental results as well as a theoretical decay constant obtained using Rayleigh constants $a=0.0136$ and $b=0.0008$. It is evident from the graph that decay constants for bending and torsion are fairly constant over the range of frequencies tested. Basically, the Rayleigh model assumes a damping ratio that is within the limits prescribed by the two main damping modes. The Rayleigh model it is also reasonable considering that the decay constants are on the order of $6 \times 10^{-3}$, or 0.6% of critical damping.
Figure 4.6: Bending Configuration

Figure 4.7: Torsional Configuration
Torsion Decay Signal
Spar 6 (T17000A)

Decay Constant Variation
Spar 6

Figure 4.8: Original and Filtered Decay Signals

Figure 4.9: Spar 6 Decay Constant Variation
Chapter 5

Results and Discussion

A large quantity of experimental results have been obtained during the design and testing of the two quarter-scale ornithopters as well as the full-scale ornithopter. The results of tests on the Expothopter are presented in a thesis by Fowler [14] and the bulk of experimental data for the full-scale ornithopter is contained in two master theses by Mehler [9] and Fenton [10]. The objective of this chapter is to present a comparison between the non-linear Newmark code and experimental results in all flight regimes, as well as a comparison between the Newmark and Fullwing codes at near-flight conditions. The initial step for any finite element code validation is a convergence study.

5.1 Convergence Study

A traditional convergence study for a linear static problem usually involves the evaluation and examination of potential energy as the mesh is refined. In a non-linear dynamic problem involving non-conservative forces such as friction damping that approach can not be used.

The approach used here is to study the effect of mesh refinement on a flapping wing's performance parameters such as average lift and thrust as well as maximum and minimum bending and twisting moments. Computer performance limits the maximum number of elements to about 100. This means that at each iteration a system of $300 \times 300$ equations must be solved. Since the Fullwing code used a set of 11 elements, meshes with 11, 22, 35, 77 and 110 elements were considered.

Figures 5.1 and 5.2 show the convergence trend for static flapping twisting and bending moments respectively. Furthermore average static lift and thrust convergence curves are shown in Figures 5.3 and 5.4 respectively. It is clear that convergence is quite good considering the presence of non-conservative forces and a non-linear aerodynamic model.
The convergence for flight conditions may also be investigated in a similar way. Since the full-scale ornithopter's take off is designed to be at a flapping frequency of 1.2 Hz and a forward speed of about 80 ft/s the cruise condition convergence plot are produced for a range of 0.6 to 1.3 Hz and corresponding to a flight speed of 80 ft/s and an angle of attack of zero.

Figures 5.5 and 5.6 show the convergence trends for twisting and bending moments respectively. Furthermore average flight condition lift and thrust convergence curves are shown in Figures 5.7 and 5.8 respectively. Convergence for twisting moment (Figure 5.5) occurs at a much slower rate which is due to the presence of non-conservative fabric and rib friction forces which are only associated with the twisting degree of freedom. It is clear however that with 55 elements convergence has been achieved and for this reason a minimum of 55 elements have been used in producing all the graphs in this document.
Figure 5.1: Static Flapping Twisting Moment Convergence

Figure 5.2: Static Flapping Bending Moment Convergence
Figure 5.3: Average Static Lift Convergence

Figure 5.4: Average Static Thrust Convergence
CHAPTER 5. RESULTS AND DISCUSSION

Figure 5.5: Flight Condition Twisting Moment Convergence

Figure 5.6: Flight Condition Bending Moment Convergence
Figure 5.7: Flight Condition Average Lift Convergence

Figure 5.8: Flight Condition Average Thrust Convergence
5.2 Quarter-Scale Model Wing Tests

One of the wings of the quarter-scale model was attached to a flapping mechanism in the subsonic wind tunnel at the University of Toronto's Institute for Aerospace Studies [22]. The wing was also attached to scales that measured the generated average lift and thrust values at different flapping frequencies. Figures 5.10 and 5.11 show a comparison between theoretical results from Newmark and Fullwing codes as well as experimental data obtained during the tests. The discrepancy between theoretical and experimental lift is mainly due to the difference between the actual bending and torsional stiffness properties of the wing and the theoretical values used in the analysis.

5.3 Expothopter Wind Tunnel Tests

Earlier in the project it had become evident that that more experimental data was required in order to validate the results of the Fullwing code. The wind tunnel facilities at the University of Toronto did not permit the testing of even a single panel of the Expothopter. Therefore in 1995, the Expothopter was fitted with an electric motor and taken to the National Research Council of Canada's 9-meter wind tunnel in Ottawa.

During the two weeks spent at the NRC's facilities in Ottawa an extensive amount of data was obtained. However, due to problems with strain gages in the thrust-measuring configuration, accurate thrust data is limited. The lift performance data however, is quite good. Figures 5.12 to 5.17 shows Expothopter's lift and thrust performance as well as the Fullwing and Newmark predictions. It is clear from the lift graphs that there is very good agreement between Newmark and Fullwing lifting results and experimental data (Figures 5.12, 5.14 and 5.16).

5.4 1996 Full-Scale Ornithopter Static-Flapping Tests

In 1996, a series of static-flapping tests were conducted. Three very important parameters were measured during these tests. They were:

1. Static thrust

2. Bending and twisting moments along the wing span

3. Wing tip twist angle
Figure 5.9: Expothopter in NRC’s Wind Tunnel

Video footage of a scale attached to the ornithopter was used to determine the instantaneous static thrust for different flapping frequencies. Figure 5.20 shows a sample run where static thrust is measured over a 40 second interval. Each run is broken up into quasi-steady segments such that the average thrust does not vary significantly over a 3 to 4 second period. Average static thrust and the corresponding flapping frequency are thus determined for that particular segment. The results of this analysis as well as Newmark and Fullwing predictions are shown in Figure 5.21. The Newmark code clearly provides a far better estimate for the static thrust.

Strain gages were attached to the wing at four locations as shown in Figure 5.18 and twisting and bending moments were measured at different flapping frequencies. Figures 5.22 and 5.23 show a summary of maximum and minimum twisting and bending moment values for flapping frequencies from 0.4 to 1.1 Hz. Newmark’s prediction is also shown. It should be recalled that the aerodynamic model is based on a strip theory. Such an approximation is limited, particularly in a fully-stalled flow regime.

The strip theory aerodynamic model discussed in Chapter 3 does not account for the presence of an induced velocity component for static flapping. A first estimate for this induced velocity $\nu$ can be found using an actuator disk formulation used for propellers. The expression for induced velocity is given by [34]:

$$\nu = \sqrt{\frac{R_{ave}}{2\rho A_d}} \quad (5.1)$$

where $R_{ave}$ is the average static thrust, $\rho$ is the density of air and $A_d$ is the area swept by the flapping motion of the wings which for the full-scale ornithopter is about 240 $ft^2$. 
Using the above equation is an iterative process as the thrust changes for a given induced velocity. Several iteration were done and the induced velocity was found to be about 10 ft/s. This however did not alter the magnitude of the bending and twisting moment for static flapping shown in Figures 5.22 and 5.23.

Video footage of the wing tip motion was used to assess the twisting behaviour of the wing and compare it to Newmark's predictions. Figures 5.24 and 5.25 show the wing tip twist at 0.91 and 0.97 Hz, respectively. It is evident from these figures that the non-linear aerodynamic formulation is capable of predicting the wing's twisting behaviour at conditions of zero forward speed. Newmark output shows that the twist angle is quite "flat" in the mid-downstroke position which corresponds approximately to the \( \frac{3}{4} \) position in each cycle (i.e. at 0.75 seconds, 1.75 seconds, 2.75 seconds...).

### 5.5 1997 and 1998 Taxi Tests

In the summer of 1997 and 1998, extensive taxi trials were conducted at Downsview airport in Toronto. Figure 5.19 shows the ornithopter during a lift-off test in 1999 with the author's car in the background. Unfortunately, all throughout 1997 and 1998 some strain gages failed progressively and only a limited amount of experimental data was collected. This is sufficient, however, to provide an opportunity for comparing the Newmark predictions with experimental data.

Figure 5.26 presents the data for the complete run on September 15 1997. The purpose of this figure is to illustrate which part of a run would be suitable for comparison with analytical results from the Newmark code. On this particular run it is obvious from the throttle curve that the pilot was trying to increase power incrementally. This was due to the fact that at that time the first stage chain connecting the engine to the drive train was coming loose and the reason was thought to be that the throttle was increased abruptly. Actually the reason was later found to be the relative motion of the engine with respect to the drive train. The period between 40 to 50 seconds into the run the throttle as well as the air speed and the angle of attack are fairly constant and that particular segment was chosen for comparison with Newmark predictions.

The strain gages provided a significant quantity of instantaneous twisting and bending-moment data during the 1997-1998 testing season. Figures 5.27 and 5.28 show the instantaneous twisting and bending moments during run 1 on September 15 1997 from 44 to 50 seconds into the run. There is a maximum error of 15% associated with the twisting-moment data and about 10% for the bending-moment data.
Figures 5.29 and 5.30 show a summary of the maximum and minimum twisting and bending moments during the 1997 and 1998 seasons. It should be noted that the reason for the limited amount of twisting-moment data is that all twisting-moment strain gages had failed by the end of the 1997 season. Furthermore, the results presented here represent only portions of taxi tests where steady-state conditions were present. A steady-state condition in a taxi test would occur only when the throttle was maintained at a constant level and air speed was not rapidly changing. Instantaneous bending and twisting moment data for the entire 1997-1998 testing seasons as well as the Newmark predictions are presented in Appendix D and Appendix E.

As a tool, the non-linear Newmark code is very valuable for predicting lifting and thrusting performance of the wings, as shown in Figures 5.31 and 5.32 for a flapping frequency of 1.2 Hz and an angle of attack $\bar{\phi}_a$ of zero. Although there is good agreement for average lifting performance, the static thrusting values are considerably different. As part of the 1996 static tests the value of average thrust at 1 Hz was measured to be about 25 lb. This, and the fact that in the 1997 and 1998 taxi trials the ornithopter was able to start its ground roll under its own power, shows that the Newmark thrust predictions are more accurate.

Furthermore Figures 5.31 and 5.32 illustrate that for a flapping frequency of 1.2 Hz the take-off speed for a 700 lb aircraft is about 80 ft/s. Maximum average thrust which in this case occurs at 70 ft/s is a function of the torsional stiffness of the wing. Torsional stiffness of the wing has been chosen such that the take-off airspeed is on the downward sloping part of the thrust curve. This is a highly desirable feature of the wing because if the airspeed is decreased the thrust will increase. This phenomenon is referred to as "thrust stability".
**CHAPTER 5. RESULTS AND DISCUSSION**

**Figure 5.10: Quarter-Scale Lift Performance**

**Average Lift vs. Flapping Frequency**
Quarter-Scale Model's Wing. UTIAS Wind Tunnel Tests

Average Lift (lb)

<table>
<thead>
<tr>
<th>Flapping Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

- **Experimental Data**
- **Fullwing**
- **Newmark**

**Figure 5.11: Quarter-Scale Thrust Performance**

**Average Thrust vs. Flapping Frequency**
Quarter-Scale Model's Wing. UTIAS Wind Tunnel Tests

Average Thrust (lb)

<table>
<thead>
<tr>
<th>Flapping Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-0.3</td>
</tr>
</tbody>
</table>

- **Experimental Data**
- **Fullwing**
- **Newmark**
Figure 5.12: Exporthopter Lift versus Angle of Attack

Figure 5.13: Exporthopter Thrust versus Angle of Attack
**Figure 5.14:** Exopthopter Lift versus Flapping Frequency

**Figure 5.15:** Exopthopter Thrust versus Flapping Frequency
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Average Lift vs. Air Speed
Exophopter NRC Test
\( \theta = 0 \text{ deg. } \omega = 2 \text{ Hz} \)

Figure 5.16: Exophopter Lift versus Air Speed

Average Thrust vs. Air Speed
Exophopter NRC Test
\( \theta = 0 \text{ deg. } \omega = 3 \text{ Hz} \)

Figure 5.17: Exophopter Thrust versus Air Speed
Figure 5.18: Full-Scale Ornithopter Strain Gage Locations

Figure 5.19: Lift-off Test in 1999
Figure 5.20: Instantaneous Static Thrust

Figure 5.21: Newmark and Fullwing Static Thrust Comparison
CHAPTER 5. RESULTS AND DISCUSSION

Maximum & Minimum Twisting Moments
1996 Static Tests
\( \theta_s = 0 \text{ deg.} \ U = 0.01 \text{ ft/sec} \)

- Experimental Data
- Newmark

Figure 5.22: Static Twisting Moment versus Flapping Frequency

Maximum & Minimum Bending Moments
1996 Static Tests
\( \theta_s = 0 \text{ deg.} \ U = 0.01 \text{ ft/sec} \)

- Experimental Data
- Newmark

Figure 5.23: Static Bending Moment versus Flapping Frequency
Figure 5.24: Wing Tip Twist Angle Variation (0.91 Hz)

Figure 5.25: Wing Tip Twist Angle Variation (0.97 Hz)
Complete Run Data
15 September 1997, Run 1

Twisting Moment
Air Speed
Angle of Attack
Throttle Position

Right Wing Twisting Moment at #3 (ft-lb) vs. Time (sec)

Throttle Position (%), Air Speed (ft/s), Angle of Attack (deg)
Instantaneous Twisting Moment Variation

15 September 1997, Run 1

$\bar{\theta}_w = 3$ deg, $U=44$ ft/sec, $\omega=0.96$ Hz

Figure 5.27: Instantaneous Twisting Moment

Instantaneous Bending Moment Variation

15 September 1997, Run 1

$\bar{\theta}_w = 3$ deg, $U=44$ ft/sec, $\omega=0.96$ Hz

Figure 5.28: Instantaneous Bending Moment
**Maximum & Minimum Twisting Moments**

1997 Taxi Tests

(θ₁₈ deg., U ft/sec)

- Experimental Data
- Newmark

---

**Figure 5.29:** Taxi Test Twisting Moment versus Flapping Frequency

---

**Maximum & Minimum Bending Moments**

1997 and 1998 Taxi Tests

(θ₁₈ deg., U ft/sec)

- Experimental Data
- Newmark

---

**Figure 5.30:** Taxi Test Bending Moment versus Flapping Frequency
Figure 5.31: Average Lift vs. Air Speed

Figure 5.32: Average Thrust vs. Air Speed
Chapter 6

Conclusions

This study has presented an updated numerical method for predicting the performance of a flapping wing in steady flight. This is a design-oriented analysis which uses a modified strip theory aerodynamic model which accounts for vortex-wake effects using a modified Theodorsen function approach. Furthermore localized aerodynamic characteristics of the flow such as post-stall behaviour, partial leading edge suction, and camber and friction drag are also accounted for.

The finite-element structural model that is used is both practical and versatile enough to handle different geometrical configurations with little or no modifications. A translational matrix formulation eliminates the need for cross members that would have otherwise been required to connect the spar and fabric and rib elements to one another. The Rayleigh damping model is a convenient way for modeling the damping characteristics of the spar and experimental data was used to determine approximate values for bending and torsional decay constants associated with a sample spar. The Rayleigh model is also suitable considering that the spar's decay constant is about 0.6% of critical damping. The fabric and rib friction-damping model accounts for the energy loss due to the shearflexing action of the wing. This damping model is necessary since shearflexing is a very unique feature of the ornithopter's wing. Over the past twenty years a considerable effort has been made in order to determine the elastic properties of the spars used in the two quarter-scale model ornithopters and the full-scale ornithopter. This structural formulation has been chosen in order to take advantage of the elastic properties which have been determined experimentally.

The constant average acceleration technique which is a subset of the Newmark Beta implicit time-marching method is, in the author's opinion, the only time integration technique suitable for this problem. Several sophisticated explicit time-marching methods were
investigated before the implicit Newmark method was chosen. Presence of two distinct flow regimes, namely attached and stalled flows, requires that the time integration method be unconditionally stable.

Because of the first order temporal approximation used to evaluate the non-linear aerodynamic forces at the n+1'st time step an iteration must be performed. This iteration is required in order to ensure that the dynamic equilibrium equations are satisfied.

Convergence rate for the Newmark code is excellent. Despite the presence of non-conservative forces such as damping the convergence rate for average quantities such as average lift and average thrust are very good. There is a slower convergence rate for maximum and minimum twisting moments associated with flight condition results. This is mainly due to the dominant effect of the damping at higher flow speeds.

There is very good agreement between Newmark code predictions and results obtained from testing the Exopothopter in NRC's 9 meter wind tunnel in Ottawa. The error associated with the experimental average thrust is the due to problems with the load measuring equipment in the thrust measuring configuration.

Newmark predictions for maximum and minimum twisting moment during static flapping are quite accurate and much better than expected considering the relatively simple aerodynamic model used. There is a maximum error of about 30% associated with the maximum and minimum bending moments at the flapping-axis location for static flapping. One of the issues raised during the 1996 static-flapping tests was the unexpected twisting behaviour of the wings. This was particularly worrisome at the time because the phase angle between flapping and twisting was not close to the optimum value of -90 degrees. By properly modeling the fully-stalled characteristics of the wing it became possible to predict the wing's correct twisting behaviour at near-static conditions. This basically shows that severe stalling at near-static conditions produces a twisting behaviour that is far from optimum. This is possibly the most important contribution from this study.

There is better agreement between theory and experimental data for near-flight conditions. The maximum error for maximum and minimum bending and torsion is about 15%. This is quite reasonable considering that by 1998 the strain gages still functioning were more than two years old. The average age of strain gages as recommended by the manufacturer is two years. Furthermore, the strain gages were attached to the spar using Cyanoacrylate (CA), which becomes brittle faster than epoxy or other types of adhesives.

In addition to the aerodynamic loads, the procedure for obtaining several other important parameters are presented in this document. and they are also calculated by the Newmark code. Power required, propulsive efficiency, drive and pivot loads are signifi-
cant in that they provide vital first approximations for the design or selection of other components of the plane which are affected by the aerodynamic loads.

This analysis has not only helped the research team at Institute for Aerospace Studies better understand the aerodynamic characteristics of the ornithopter's wings, but it will also provide a valuable tool for future design of flapping-wing aircraft.
References


REFERENCES


REFERENCES


Appendix A

Properties

This Appendix includes the latest structural and aerodynamic properties of the quarter-scale ornithopter, expothopter and Full-Scale Ornithopter wings. This data is essentially what is included in a Newmark input file.

A.1 Quarter-Scale Ornithopter

Tables A.1 through A.7 show the aeroelastic and geometric properties of an 11-segment quarter-scale ornithopter wing.

A.2 Expothopter

Tables A.8 through A.14 show the aeroelastic and geometric properties of an 11-segment Expothopter wing. The major difference between the Expothopter and the quarter-scale model ornithopter wing is that the Expothopter has a double-tapered wing because it was believed that such an arrangement would have a better aeroelastic performance.

A.3 Full-Scale Ornithopter

Tables A.15 through A.21 show the aeroelastic and geometric properties of an 11-segment full-scale ornithopter wing.
### Table A.1: Quarter-Scale Ornithopter General Configuration

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inboard Taper Angle (deg.)</td>
<td>0</td>
</tr>
<tr>
<td>Outboard Taper Angle (deg.)</td>
<td>15.38</td>
</tr>
<tr>
<td>Flapping Amplitude (deg.)</td>
<td>27.26</td>
</tr>
<tr>
<td>Mean Dihedral Angle (deg.)</td>
<td>6.83</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>9.40</td>
</tr>
</tbody>
</table>

### Table A.2: Quarter-Scale Ornithopter Damping Properties

<table>
<thead>
<tr>
<th>Variable &amp; Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{rib}$ (ft)</td>
<td>0.02083</td>
</tr>
<tr>
<td>$A_{le}$ ($ft^2$)</td>
<td>0.00215</td>
</tr>
<tr>
<td>$p$ (lb/ft^2)</td>
<td>2.0</td>
</tr>
<tr>
<td>$\frac{\Delta x}{\Delta y}$ (ft/rad)</td>
<td>0.03</td>
</tr>
<tr>
<td>$C_f/0$</td>
<td>0.26</td>
</tr>
<tr>
<td>$C_f$ (sec/ft)</td>
<td>0.124</td>
</tr>
<tr>
<td>$a$</td>
<td>0.05</td>
</tr>
<tr>
<td>$b$</td>
<td>0.005</td>
</tr>
</tbody>
</table>

### Table A.3: Quarter-Scale Ornithopter Rigid and Center Sections

<table>
<thead>
<tr>
<th>Property and Units</th>
<th>Rigid Section</th>
<th>Center Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord Length $c$ (ft)</td>
<td>0.92</td>
<td>0.92</td>
</tr>
<tr>
<td>Section Width $\Delta x$ (ft)</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td>Built-in Pre-Twist $\bar{\theta}_{wash}$ (deg.)</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Zero-Lift-Line Angle $\alpha_0$ (deg.)</td>
<td>5.33</td>
<td>5.33</td>
</tr>
<tr>
<td>Maximum Stall Angle $\alpha_{max}$ (deg.)</td>
<td>9.0</td>
<td>9.0</td>
</tr>
<tr>
<td>Minimum Stall Angle $\alpha_{min}$ (deg.)</td>
<td>-9.0</td>
<td>-9.0</td>
</tr>
<tr>
<td>Cross-Flow Drag Coefficient $(C_d)_{cf}$</td>
<td>1.98</td>
<td>1.98</td>
</tr>
<tr>
<td>Friction Drag Coefficient $C_d$</td>
<td>0.012</td>
<td>0.012</td>
</tr>
<tr>
<td>Moment Coefficient about Aerodynamic Center $C_{mac}$</td>
<td>-0.175</td>
<td>-0.175</td>
</tr>
<tr>
<td>Leading Edge Suction Coefficient $\eta_s$</td>
<td>0.915</td>
<td>0.915</td>
</tr>
<tr>
<td>Mass (slugs)</td>
<td>0.003636</td>
<td>0.018120</td>
</tr>
</tbody>
</table>
### Table A.4: Quarter-Scale Ornithopter Wing Elastic Properties

<table>
<thead>
<tr>
<th>Seg. No.</th>
<th>EI ( \text{lb} - \text{ft}^2 )</th>
<th>GJ\text{SPAR} ( \text{lb} - \text{ft}^2/\text{rad} )</th>
<th>GJ\text{FR} ( \text{lb} - \text{ft}^2/\text{rad} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>118.29</td>
<td>11.04</td>
<td>3.73</td>
</tr>
<tr>
<td>2</td>
<td>118.29</td>
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<td>3.73</td>
</tr>
<tr>
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<td>118.29</td>
<td>11.04</td>
<td>3.73</td>
</tr>
<tr>
<td>4</td>
<td>118.29</td>
<td>11.04</td>
<td>3.73</td>
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<tr>
<td>5</td>
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<td>11.04</td>
<td>3.73</td>
</tr>
<tr>
<td>6</td>
<td>28.39</td>
<td>2.65</td>
<td>3.2</td>
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<td>7</td>
<td>18.32</td>
<td>1.71</td>
<td>2.29</td>
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<td>8</td>
<td>12.30</td>
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<td>9</td>
<td>8.01</td>
<td>0.75</td>
<td>1.03</td>
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<td>10</td>
<td>5.35</td>
<td>0.5</td>
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<tr>
<td>11</td>
<td>1.0</td>
<td>0.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### Table A.5: Quarter-Scale Ornithopter Wing Geometric Properties

<table>
<thead>
<tr>
<th>Seg. No.</th>
<th>Chord Length ( \text{ft} )</th>
<th>Seg. Width ( \text{ft} )</th>
<th>Elastic Axis Loc. ( \text{ft} )</th>
<th>Spar Loc. ( \text{ft} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.92</td>
<td>0.33</td>
<td>0.077</td>
<td>0.062</td>
</tr>
<tr>
<td>2</td>
<td>0.92</td>
<td>0.33</td>
<td>0.077</td>
<td>0.062</td>
</tr>
<tr>
<td>3</td>
<td>0.92</td>
<td>0.33</td>
<td>0.077</td>
<td>0.062</td>
</tr>
<tr>
<td>4</td>
<td>0.92</td>
<td>0.33</td>
<td>0.077</td>
<td>0.062</td>
</tr>
<tr>
<td>5</td>
<td>0.92</td>
<td>0.33</td>
<td>0.077</td>
<td>0.062</td>
</tr>
<tr>
<td>6</td>
<td>0.87</td>
<td>0.33</td>
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<td>0.78</td>
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<td>0.042</td>
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<td>0.33</td>
<td>0.032</td>
<td>0.025</td>
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<td>0.33</td>
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<tr>
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<td>0.44</td>
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<td>-0.062</td>
<td>0.153</td>
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</table>

Table A.5: Quarter-Scale Ornithopter Wing Geometric Properties
### Table A.6: Quarter-Scale Ornithopter Wing Inertial Properties

<table>
<thead>
<tr>
<th>Seg. No.</th>
<th>Mass of Spar slugs</th>
<th>Mass of Fabric &amp; Rib slugs</th>
<th>Spar Moment of Inertia (\text{slug} - \text{ft}^2)</th>
<th>Fabric &amp; Rib Moment of Inertia (\text{slug} - \text{ft}^2)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000571</td>
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<tr>
<td>2</td>
<td>0.000571</td>
<td>0.00054</td>
<td>0.0000006</td>
<td>0.0000479</td>
</tr>
<tr>
<td>3</td>
<td>0.000571</td>
<td>0.00054</td>
<td>0.0000006</td>
<td>0.0000479</td>
</tr>
<tr>
<td>4</td>
<td>0.000571</td>
<td>0.00054</td>
<td>0.0000006</td>
<td>0.0000479</td>
</tr>
<tr>
<td>5</td>
<td>0.000571</td>
<td>0.00054</td>
<td>0.0000006</td>
<td>0.0000479</td>
</tr>
<tr>
<td>6</td>
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<td>0.00000</td>
<td>0.0000023</td>
<td>0.0000000</td>
</tr>
</tbody>
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### Table A.7: Quarter-Scale Ornithopter Wing Aerodynamic Properties

<table>
<thead>
<tr>
<th>Seg. No.</th>
<th>(C_{mac})</th>
<th>(\alpha_0) deg.</th>
<th>(\alpha_{max}) deg.</th>
<th>(\alpha_{min}) deg.</th>
<th>((C_d)_{ef})</th>
<th>(C_d)</th>
<th>(\eta_s)</th>
<th>(\theta_{wash}) deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1750</td>
<td>5.33</td>
<td>9.00</td>
<td>-9.0</td>
<td>1.98</td>
<td>0.012</td>
<td>0.915</td>
<td>3.62</td>
</tr>
<tr>
<td>2</td>
<td>-0.1750</td>
<td>5.33</td>
<td>9.00</td>
<td>-9.0</td>
<td>1.98</td>
<td>0.012</td>
<td>0.915</td>
<td>3.59</td>
</tr>
<tr>
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<td>-0.1750</td>
<td>5.33</td>
<td>9.00</td>
<td>-9.0</td>
<td>1.98</td>
<td>0.012</td>
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</tr>
<tr>
<td>4</td>
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<td>5.33</td>
<td>9.00</td>
<td>-9.0</td>
<td>1.98</td>
<td>0.012</td>
<td>0.915</td>
<td>3.52</td>
</tr>
<tr>
<td>5</td>
<td>-0.1750</td>
<td>5.33</td>
<td>9.00</td>
<td>-9.0</td>
<td>1.98</td>
<td>0.012</td>
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<td>0.012</td>
<td>0.911</td>
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</tr>
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<td>9.00</td>
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<td>0.012</td>
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<td>10</td>
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<td>-9.0</td>
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<td>0.00</td>
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<td>-29.0</td>
<td>1.98</td>
<td>0.012</td>
<td>0.000</td>
<td>1.15</td>
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</table>
### Table A.8: Expothopter General Configuration

<table>
<thead>
<tr>
<th>Inboard Taper Angle (deg.)</th>
<th>2.87</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outboard Taper Angle (deg.)</td>
<td>15.40</td>
</tr>
<tr>
<td>Flapping Amplitude (deg.)</td>
<td>26.13</td>
</tr>
<tr>
<td>Mean Dihedral Angle (deg.)</td>
<td>7.5</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>11.61</td>
</tr>
</tbody>
</table>

### Table A.9: Expothopter Damping Properties

<table>
<thead>
<tr>
<th>Variable &amp; Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{rib}$ (ft)</td>
<td>0.02083</td>
</tr>
<tr>
<td>$A_{le}$ ($ft^2$)</td>
<td>0.00215</td>
</tr>
<tr>
<td>$p$ ($lb/ft^2$)</td>
<td>2.0</td>
</tr>
<tr>
<td>$\frac{E}{2g}$ (ft/rad)</td>
<td>0.03</td>
</tr>
<tr>
<td>$C_{fo}$</td>
<td>0.26</td>
</tr>
<tr>
<td>$C_f$ (sec/ft)</td>
<td>0.124</td>
</tr>
<tr>
<td>$a$</td>
<td>0.05</td>
</tr>
<tr>
<td>$b$</td>
<td>0.005</td>
</tr>
</tbody>
</table>

### Table A.10: Expothopter Rigid and Center Sections

<table>
<thead>
<tr>
<th>Property and Units</th>
<th>Rigid Section</th>
<th>Center Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord Length $c$ (ft)</td>
<td>1.11</td>
<td>1.12</td>
</tr>
<tr>
<td>Section Width $\Delta x$ (ft)</td>
<td>0.52</td>
<td>1.80</td>
</tr>
<tr>
<td>Built-in Pre-Twist $\bar{\theta}_{wash}$ (deg.)</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>Zero-Lift-Line Angle $\alpha_0$ (deg.)</td>
<td>5.33</td>
<td>5.33</td>
</tr>
<tr>
<td>Maximum Stall Angle $\alpha_{max}$ (deg.)</td>
<td>9.0</td>
<td>9.0</td>
</tr>
<tr>
<td>Minimum Stall Angle $\alpha_{min}$ (deg.)</td>
<td>-9.0</td>
<td>-9.0</td>
</tr>
<tr>
<td>Cross-Flow Drag Coefficient $(C_d)_{cf}$</td>
<td>1.98</td>
<td>1.98</td>
</tr>
<tr>
<td>Friction Drag Coefficient $C_d$</td>
<td>0.012</td>
<td>0.012</td>
</tr>
<tr>
<td>Moment Coefficient about Aerodynamic Center $C_{mac}$</td>
<td>-0.175</td>
<td>-0.175</td>
</tr>
<tr>
<td>Leading Edge Suction Coefficient $\eta_s$</td>
<td>0.915</td>
<td>0.915</td>
</tr>
<tr>
<td>Mass (slugs)</td>
<td>0.003636</td>
<td>0.018120</td>
</tr>
</tbody>
</table>
### Table A.11: Expothopter Wing Elastic Properties

<table>
<thead>
<tr>
<th>Seg. No.</th>
<th>EI $lb \cdot ft^2$</th>
<th>$GJ_{SPAR}$ $lb \cdot ft^2/\text{rad}$</th>
<th>$GJ_{FR}$ $lb \cdot ft^2/\text{rad}$</th>
</tr>
</thead>
<tbody>
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### Table A.12: Expothopter Wing Geometric Properties

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### Table A.14: Expothopter Wing Aerodynamic Properties

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<th>( \alpha_{max} ) deg.</th>
<th>( \alpha_{min} ) deg.</th>
<th>( (C_d)_{cf} )</th>
<th>( C_d )</th>
<th>( \eta_s )</th>
<th>( \theta_{wash} ) deg.</th>
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</thead>
<tbody>
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<td>0.012</td>
<td>0.915</td>
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### Table A.15: Full-Scale Ornithopter General Configuration

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<td>Outboard Taper Angle (deg.)</td>
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<tr>
<td>Flapping Amplitude (deg.)</td>
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<td>Mean Dihedral Angle (deg.)</td>
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### Table A.16: Full-Scale Ornithopter Damping Properties

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<td>( A_{te} ) (( ft^2 ))</td>
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<tr>
<td>( p ) (( lb/ft^2 ))</td>
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<tr>
<td>( \frac{\pi}{30} ) (( ft/rad ))</td>
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<td>( C_f ) (sec/ft)</td>
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### Table A.17: Full-Scale Ornithopter Rigid and Center Sections

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### Table A.18: Full-Scale Ornithopter Wing Elastic Properties

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### Table A.19: Full-Scale Ornithopter Geometric Properties

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### Table A.20: Full-Scale Ornithopter Wing Inertial Properties

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### Table A.21: Full-Scale Ornithopter Wing Aerodynamic Properties

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<th>$\alpha_{max}$ deg.</th>
<th>$\alpha_{min}$ deg.</th>
<th>$(C_d)_{cf}$</th>
<th>$C_d$</th>
<th>$\eta_s$</th>
<th>$\bar{\theta}_{wash}$ deg.</th>
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<td>0.012</td>
<td>0.915</td>
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<td>0.012</td>
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<td>0.915</td>
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<td>0.012</td>
<td>0.915</td>
<td>6.25</td>
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Appendix B

System Matrix Derivation

The method of weighted residuals is a technique for obtaining approximate solutions to linear and non-linear partial differential equations. Applying the method of weighted residuals involves two steps. The first step is to assume a function which approximately satisfies the given differential equation and boundary conditions. Substitution of this approximate solution into the original differential equation and boundary conditions then results in some error called a residual. This residual is then required to vanish in some average sense over the entire solution domain.

The second step is to solve the equations resulting from the first step and thereby specialize the general functional form to a particular function, which then becomes the approximate solution sought. As an example consider that an approximate functional representation for a variable \( u \) governed by the differential equation

\[
L(u) - f = 0 \tag{B.1}
\]

in the domain \( D \) bounded by the surface \( \Sigma \). The symbol \( L \) denotes a differential operator and the function \( f \) is a known function of the independent variables. The method of weighted residual is applied in two steps as follows.

First the unknown exact solution \( u \) is approximated by \( \hat{u} \) where the functional behaviour of \( \hat{u} \) is completely specified in terms of the unknown parameters. In other words:

\[
u \approx \hat{u} = \sum_{i=1}^{m} \phi_i c_i \tag{B.2}\]

where \( \phi_i \) are the assumed functions and there are \( m \) unknown parameters \( c_i \). When \( \hat{u} \) is substituted into Equation B.1, it is unlikely that the equation will be satisfied, that is:

\[
L(\hat{u}) - f \neq 0 \tag{B.3}\]
in fact in most cases the following is true:

\[ L(\tilde{u}) - f = R \]  \hspace{1cm} (B.4)

where \( R \) is the residual or error that results from approximating \( u \) by \( \tilde{u} \). The method of residuals seeks to determine the \( m \) unknowns \( c_i \) such that the error \( R \) over the entire solution domain is small. This is accomplished by forming a weighted average of the error and specifying that this weighted average vanish over the solution domain. Hence \( m \) linearly independent weighting functions \( \tilde{\omega} \) are chosen such that:

\[ \int_D [L(\tilde{u}) - f] \tilde{\omega} dD = \int_D R \tilde{\omega} dD = 0 \hspace{1cm} i = 1, 2, \ldots, m \]  \hspace{1cm} (B.5)

then \( R \approx 0 \) in some sense. The above equations are a set of \( m \) algebraic equations which are solved for \( c_i \).

There are a broad range of weighted residual techniques because of the large choice of weighting functions. When the weighting functions are the same as the approximating functions used to represent \( u \) (i.e. \( \tilde{\omega} = \omega_i \)) the technique is known as the Galerkin technique.

This Appendix presents the derivation of the linear ordinary differential equations governing the flapping motion of a wing in air and shows how these equations can be discretized using the Galerkin technique. The two principal modes of motion for a flapping wing are bending and torsion.

### B.1 Bending

Figure B.1 shows a beam with flexural rigidity \( EI(x) \) and mass \( m(x) \) both functions of the spatial coordinate \( x \). The beam is undergoing transverse motion in the plane of the paper under the action of a distributed force \( n(x, t) \). The transverse displacement at any point along the beam is represented by \( \hat{h}(x, t) \), which is a function of both the spatial coordinate \( x \) and time \( t \).

A small element of the beam of length \( dx \) is shown in Figure B.2 with the left image showing the external forces acting on the element and the right image showing the resultant inertia of the element. The inertial moment caused by the angular acceleration of the element has been neglected. The infinitesimal element is in equilibrium under the forces and moment identified in Figure B.2. For equilibrium in the vertical direction we have:

\[ V - \left( V + \frac{dV}{dx} dx \right) - n(x, t) dx = (\bar{m} dx) \frac{d^2 \hat{h}}{dt^2} \]  \hspace{1cm} (B.6)
where \( \bar{m} \) is mass per unit length and it is constant over the element. Rearranging we get:

\[
\frac{dV}{dx} + \bar{m} \frac{d^2 \dot{h}}{dt^2} = -n(x,t)dx
\]  

(B.7)

dividing both sides by \( dx \) we get:

\[
\frac{dV}{dx} + \bar{m} \frac{d^2 \dot{h}}{dt^2} = -n(x,t)
\]  

(B.8)

from elementary strength of materials:

\[
M = EI \frac{d^2 \dot{h}}{dx^2} \\
V = \frac{dM}{dx} \\
V = EI \frac{d^3 \dot{h}}{dx^3}
\]  

(B.9)

Substituting \( V \) from above equation into Equation B.8 we get:

\[
EI \frac{d^4 \dot{h}}{dx^4} + \bar{m} \frac{d^2 \dot{h}}{dt^2} = -n(x,t)
\]  

(B.10)
B.1.1 Finite Element Discretization

The domain is divided into a set of line elements each element having two nodes as shown in Figure B.3 and referred to as a beam element. A single element is then isolated and the Galerkin method is applied in order to derive the finite element formulation of the problem.

![Figure B.3: Finite Element Discretization of a Beam Element](image)

B.1.2 The Galerkin Formulation for Bending

The system equations are now multiplied by the interpolation function \( \tilde{w} \) and they are integrated over the domain of the element. Equation B.10 therefore becomes:

\[
EI \int_{x_e}^{x_{e+1}} \frac{d^4\hat{h}}{dx^4} \tilde{w}(x) dx + \bar{m} \int_{x_e}^{x_{e+1}} \frac{d^2\hat{h}}{dt^2} \tilde{w}(x) dx = - \int_{x_e}^{x_{e+1}} n(x,t) \tilde{w}(x) dx \tag{B.11}
\]

To continue the derivation it is necessary to integrate by parts in order to reduce the power of terms inside the first integral:

\[
EI \left[ \frac{d^3\hat{h}}{dx^3} \tilde{w}(x) \right]_{x_e}^{x_{e+1}} - EI \int_{x_e}^{x_{e+1}} \frac{d^3\hat{h}}{dx^3} \tilde{\ddot{w}}(x) dx + \bar{m} \int_{x_e}^{x_{e+1}} \frac{d^2\hat{h}}{dt^2} \tilde{w}(x) dx = - \int_{x_e}^{x_{e+1}} n(x,t)\tilde{w}(x) dx \tag{B.12}
\]

The power of the second integral can now be reduced by integrating it by parts once more:

\[
EI \left[ \frac{d^2\hat{h}}{dx^2} \tilde{w}(x) \right]_{x_e}^{x_{e+1}} - EI \int_{x_e}^{x_{e+1}} \frac{d^2\hat{h}}{dx^2} \tilde{\dddot{w}}(x) \frac{dx}{dx} + EI \int_{x_e}^{x_{e+1}} \frac{d^2\hat{h}}{dx^2} \tilde{\ddot{w}}(x) \frac{dx}{dx} + \bar{m} \int_{x_e}^{x_{e+1}} \frac{d\hat{h}}{dt^2} \tilde{\dddot{w}}(x) dx = - \int_{x_e}^{x_{e+1}} n(x,t)\tilde{w}(x) dx \tag{B.13}
\]
Rearranging we get:

\[
EI \int_{x_e}^{x_{e+1}} \frac{d^3 \bar{h}}{dx^3} \frac{d^2 \bar{w}}{dx^2} dx + \bar{m} \int_{x_e}^{x_{e+1}} \frac{d^2 \bar{h}}{dt^2} \ddot{w}(x) dx = - \int_{x_e}^{x_{e+1}} f(x,t) \hat{w}(x) dx
\]

\[
+ EI \left[ \frac{d^3 \bar{h}}{dx^3} \right]_{x_{e+1}}^{x_e} \left[ \frac{d \bar{w}}{dx} \right]_{x_{e+1}}^{x_e} - EI \left[ \frac{d^2 \bar{h}}{dx^2} \right]_{x_e} \left[ \frac{d \bar{w}}{dx} \right]_{x_e} - EI \left[ \frac{d^3 \bar{h}}{dx^3} \right]_{x_{e+1}} \hat{w}(x_{e+1}) + EI \left[ \frac{d^3 \bar{h}}{dx^3} \right]_{x_e} \hat{w}(x_e) \quad (B.14)
\]

In order to simplify the above equation we introduce:

\[
\frac{d \bar{h}}{dx} = -v'
\]

\[
Q_1 = \bar{h}_1 = EI \left[ \frac{d^3 \bar{h}}{dx^3} \right]_{x_e}
\]

\[
Q_2 = v_1 = EI \left[ \frac{d^2 \bar{h}}{dx^2} \right]_{x_e}
\]

\[
Q_3 = \bar{h}_2 = -EI \left[ \frac{d^3 \bar{h}}{dx^3} \right]_{x_{e+1}}
\]

\[
Q_4 = v_2 = -EI \left[ \frac{d^2 \bar{h}}{dx^2} \right]_{x_{e+1}}
\]

Here \(Q_1\) and \(Q_3\) denote the shear forces and \(Q_2\) and \(Q_4\) denote the bending moments. Substituting the above equations into Equation B.14 we get:

\[
EI \int_{x_e}^{x_{e+1}} \frac{d^3 \bar{h}}{dx^3} \frac{d^2 \bar{w}}{dx^2} dx + \bar{m} \int_{x_e}^{x_{e+1}} \frac{d^2 \bar{h}}{dt^2} \ddot{w}(x) dx = - \int_{x_e}^{x_{e+1}} n(x,t) \hat{w}(x) dx
\]

\[
+ Q_1 \left[ \frac{d \bar{w}}{dx} \right]_{x_{e+1}} + Q_2 \left[ \frac{d \bar{w}}{dx} \right]_{x_e} + Q_3 \hat{w}(x_{e+1}) + Q_1 \hat{w}(x_e) \quad (B.15)
\]

which is the variational formulation of the problem described by Equation B.10.

### B.1.3 Interpolation Functions for Bending

The variational form of Equation B.15 requires that the interpolation functions be continuous with continuous derivatives up to order 3 (so that \(Q_1\) and \(Q_3\) are nonzero), and that they satisfy interpolation properties so that the approximation for \(\bar{h}\) satisfies the end conditions.
conditions of an element. In other words:

\[
\begin{align*}
\hat{h}(x_e) &= \hat{h}_1 \\
\hat{h}(x_{e+1}) &= \hat{h}_2 \\
\left[ \frac{d\hat{h}}{dx} \right]_{x_e} &= -\psi_1 \\
\left[ \frac{d\hat{h}}{dx} \right]_{x_{e+1}} &= -\psi_2
\end{align*}
\]

(B.16)

Since there is a total of four conditions in an element, a four-parameter polynomial must be selected for \( \hat{h} \):

\[
\hat{h}(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3
\]

(B.17)

The continuity conditions are automatically met for \( c_4 \neq 0 \). The next step involves expressing \( c \) in terms of \( \hat{h}_1, \hat{h}_2, \psi_1 \) and \( \psi_2 \) such that the boundary conditions are satisfied:

\[
\begin{align*}
\hat{h}_1 &= \hat{h}(x_e) = c_1 + c_2 x_e + c_3 x_e^2 + c_4 x_e^3 \\
\psi_1 &= \left( -\frac{d\hat{h}}{dx} \right)_{x_e} = -c_2 - 2c_3 x_e - 3c_4 x_e^2 \\
\hat{h}_2 &= \hat{h}(x_{e+1}) = c_1 + c_2 x_{e+1} + c_3 x_{e+1}^2 + c_4 x_{e+1}^3 \\
\psi_2 &= \left( -\frac{d\hat{h}}{dx} \right)_{x_{e+1}} = -c_2 - 2c_3 x_{e+1} - 3c_4 x_{e+1}^2
\end{align*}
\]

(B.18)

or in matrix form:

\[
\begin{bmatrix}
\hat{h}_1 \\
\psi_1 \\
\hat{h}_2 \\
\psi_2
\end{bmatrix} =
\begin{bmatrix}
1 & x_e & x_e^2 & x_e^3 \\
0 & -1 & -2x_e & -3x_e^2 \\
1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3 \\
0 & -1 & -2x_{e+1} & -3x_{e+1}^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
\]

(B.19)

Inverting the matrix equation above to write the \( c \) in terms of \( \hat{h}_i \) and \( \psi_i \) and substituting the result into Equation B.17 we get:

\[
\hat{h}_e(x) = \phi_1 \hat{h}_1 + \phi_2 \psi_1 + \phi_3 \hat{h}_2 + \phi_4 \psi_2
\]

(B.20)

where

\[
\begin{align*}
\phi_1 &= 1 - 3 \left( \frac{x - x_e}{\Delta x} \right)^2 + 2 \left( \frac{x - x_e}{\Delta x} \right)^3 \\
\phi_2 &= -(x - x_e) \left( 1 - \frac{x - x_e}{\Delta x} \right)^2 \\
\phi_3 &= 3 \left( \frac{x - x_e}{\Delta x} \right)^2 - 2 \left( \frac{x - x_e}{\Delta x} \right)^3 \\
\phi_4 &= -(x - x_e) \left[ \left( \frac{x - x_e}{\Delta x} \right)^2 - \frac{x - x_e}{\Delta x} \right]
\end{align*}
\]

(B.21)
In terms of the local (or element) coordinate \( s = \frac{x-x_e}{\Delta x} \) the interpolation functions take the simple form:

\[
\begin{align*}
\phi_1 &= 1 - 3s^2 + 2s^3 \\
\phi_2 &= -s\Delta x (1 - s)^2 \\
\phi_3 &= 3s^2 - 2s^3 \\
\phi_4 &= -s\Delta x (s^2 - s)
\end{align*}
\]  

(B.22)

Figure B.4: Hermite Cubic Interpolation Functions

Figure B.4 shows the approximate shape of these interpolation functions over a typical element. The first two conditions of Equation B.16 imply that:

\[
\begin{align*}
\phi_1(x_e) &= 1 & \phi_i(x_e) &= 0 & i \neq 1 \\
\phi_3(x_{e+1}) &= 1 & \phi_i(x_{e+1}) &= 0 & i \neq 3
\end{align*}
\]  

(B.23)

The last two conditions of Equation B.16 imply that:
\[\begin{align*}
-d\phi_2|_{x_+} &= 1 \\
-d\phi_i|_{x_+} &= 0 \quad i \neq 2 \\
-d\phi_4|_{x_+} &= 1 \\
-d\phi_i|_{x_+} &= 0 \quad i \neq 4
\end{align*}\]  
\text{Equation B.24}

It is easy to verify that the interpolation functions do satisfy the above conditions. It should be pointed out that order of the interpolation functions derived above is the minimum required for the variational formulation in Equation B.15. If higher order approximations of \( \hat{h} \) is desired, additional unknowns at each of the nodes must be specified.

Now in order to change the lower integration limit in Equation B.15 from \( x_+ \) to \( s = 0 \) and to change the upper limit from \( x_+ \) to \( s = 1 \) we need the following transformations:

\[
\begin{align*}
\frac{d\phi_i}{dx} &= \frac{d\phi_i}{ds} \frac{ds}{dx} = \frac{1}{\Delta x} \frac{d\phi_i}{ds} \\
\frac{d^2\phi_i}{dx^2} &= \frac{d^2\phi_i}{ds^2} \left( \frac{ds}{dx} \right)^2 = \frac{1}{\Delta x^2 ds^2} \\
dx = \frac{dx}{ds} ds = \Delta x ds
\end{align*}
\text{Equation B.25}

Equation B.15 therefore becomes:

\[
\frac{E l}{\Delta x^3} \int_0^1 \frac{d^2\hat{h}}{d^2s} \frac{d^2\hat{w}}{ds^2} ds ds + \bar{m} \Delta x \int_0^1 \frac{d^2\hat{w}}{dt^2} \hat{w}(s) ds = -\Delta x \int_0^1 n(x,t) \hat{w}(s) ds \\
+ Q_4 \left[ -\frac{d\hat{w}}{ds} \right]_1 + Q_2 \left[ -\frac{d\hat{w}}{ds} \right]_0 + Q_3 \hat{w}(1) + Q_1 \hat{w}(0) \tag{B.26}
\]

The essence of the Galerkin method is that after obtaining the variational formulation as it was done above we plug in the approximate form of \( \hat{h} \) and let test function equal to the four interpolation functions successively. In other words in Equation B.26 we let:

\[
\hat{h} = \phi_1 \hat{h}_1 + \phi_2 \phi'_1 + \phi_3 \hat{h}_2 + \phi_4 \phi'_2 \\
\hat{w} = \phi_i \quad i = 1 \ldots 4 \tag{B.27}
\]

**B.1.4 Bending Stiffness Matrix**

The first term on the left hand side of Equation B.26 represents the stiffness matrix for a beam in bending. Each term of the four by four matrix is given by:

\[
K_{ij} = \frac{E l}{\Delta x^3} \int_0^1 \frac{d^2\phi_i}{ds^2} \frac{d^2\phi_j}{ds^2} ds \quad \text{for } i,j = 1 \ldots 4 \tag{B.28}
\]
Each of the terms in the stiffness are then evaluated and the complete stiffness matrix is given by:

$$[K] = \frac{2EI}{\Delta x^3} \begin{bmatrix} 6 & -3\Delta x & -6 & -3\Delta x \\ -3\Delta x & 2 \Delta x^2 & 3\Delta x & \Delta x^2 \\ -6 & 3\Delta x & 6 & 3\Delta x \\ -3\Delta x & \Delta x^2 & 3\Delta x & 2\Delta x^2 \end{bmatrix} \quad (B.29)$$

**B.1.5 Consistent Mass Matrix for Bending**

Similarly the second term on the left hand side of the variational formulation (Equation B.26) represents the consistent mass matrix and each term of the four by four matrix is given by:

$$M_{ij} = m\Delta x \int_0^1 \phi_i \phi_j ds \quad \text{for } i, j = 1 \ldots 4 \quad (B.30)$$

Substituting the appropriate interpolation functions from Equation B.22 and integrating over the element we get:

$$[M] = \frac{m\Delta x}{420} \begin{bmatrix} 156 & -22\Delta x & 54 & 13\Delta x \\ -22\Delta x & 4\Delta x^2 & -13\Delta x & -3\Delta x^2 \\ 54 & -13\Delta x & 156 & 22\Delta x \\ 13\Delta x & -3\Delta x^2 & 22\Delta x & 4\Delta x^2 \end{bmatrix} \quad (B.31)$$

Inertial effects can be modeled using a consistent mass matrix as shown above or a they can be modeled as forces which is equivalent to using a lumped mass approach. The Fullwing code uses a lumped mass approach whereas the Newmark code uses the consistent mass matrix shown above.

**B.1.6 Bending Force Vector**

Terms on the right hand side of Equation B.26 denote the force vector for a beam element. Assuming that $n(x, t)$ is constant over the element we can take it out of the integral and get:

$$-\Delta x \int_0^1 n(x, t) \ddot{w}(s) ds = -\Delta x f(t) \int_0^1 \ddot{w}(s) ds \quad (B.32)$$

Therefore the force vector is given by:

$$F_i = -\Delta x f(t) \int_0^1 \phi_i ds + Q_i \quad \text{for } i = 1 \ldots 4 \quad (B.33)$$
Again substituting the appropriate interpolation functions from Equation B.22 and integrating over the element the force vector for a beam element is given by:

\[ \{F\} = -\frac{n(t)\Delta x}{12} \begin{pmatrix} 6 \\ -\Delta x \\ 6 \\ \Delta x \end{pmatrix} \left[ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \right] \]  
(B.34)

### B.2 Torsion

Figure B.5 shows a torsion member with torsional stiffness \(GJ(x)\) and mass moment of inertia \(I(x)\) both functions of the spatial coordinate \(x\). The member is undergoing torsional motion in the plane of the paper under the action of a distributed torque \(m(x,t)\). The twist angle at any point along the member is represented by \(\hat{\theta}(x,t)\), which is a function of both the spatial coordinate \(x\) and time \(t\).

![Figure B.5: Torsional Member Under Distributed Torque](image)

A small element of the torsional member of length \(dx\) is shown in Figure B.6 with the left image showing the external moments acting on the element and the right image showing the resultant inertia of the element. The infinitesimal element is in equilibrium under the moments identified in Figure B.6. For equilibrium we have:

\[-T + \left( T + \frac{dT}{dx} dx \right) + m(x,t)dx = I_x \frac{d^2\hat{\theta}}{dt^2} dx\]  
(B.35)

where \(I_x\) is mass moment of inertia per unit length of the element and is given by

\[ I_x = \frac{Jm}{A} \]  
(B.36)
\( J \) is the polar moment of inertia, \( \bar{m} \) is the mass per unit length and \( A \) is the cross-sectional area of the element. Rearranging and dividing by \( dx \) we get:

\[
\frac{dT}{dx} + m(x, t) = \frac{J\bar{m}}{A} \frac{d^2\dot{\theta}}{dt^2} \tag{B.37}
\]

Assuming that the torsional stiffness of the element is constant over its length we have:

\[
T = GJ \frac{d\theta}{dx} \tag{B.38}
\]

therefore we can write the system equations for torsion as:

\[
-GJ \frac{d^2\dot{\theta}}{dx^2} + \frac{J\bar{m}}{A} \frac{d^2\dot{\theta}}{dt^2} = m(x, t) \tag{B.39}
\]

### B.2.1 Finite Element Discretization

The domain is divided into a set of line elements each element having two nodes as shown in Figure B.7 and referred to as a torsional element. Just as in the bending case a single element is then isolated and the Galerkin method is applied in order to derive the finite element formulation of the problem.

### B.2.2 The Galerkin Formulation for Torsion

As with bending we select an interpolation (test) function \( \tilde{u} \) which is differentiable at least once with respect to \( x \). The system equations are then multiplied by the interpolation function and they are integrated over the domain of the element. Equation B.39 therefore becomes:

\[
-GJ \int_{x_e}^{x_{e+1}} \frac{d^2\dot{\theta}}{dx^2} \tilde{u}(x)dx + \frac{J\bar{m}}{A} \int_{x_e}^{x_{e+1}} \frac{d^2\dot{\theta}}{dt^2} \tilde{u}(x)dx = -\int_{x_e}^{x_{e+1}} m(x, t) \tilde{u}(x)dx \tag{B.40}
\]
To continue the derivation it is necessary to integrate by parts in order to reduce the power of terms inside the first integral:

\[-GJ \left[ \frac{d\theta}{dx} \hat{\omega}(x) \right]^{x_{e+1}}_{x_e} + GJ \int_{x_e}^{x_{e+1}} \frac{d\theta}{dx} \frac{d\hat{\omega}}{dx} dx + \frac{Jm}{A} \int_{x_e}^{x_{e+1}} \frac{d^2\hat{\omega}}{dx^2} \hat{\omega}(x) dx\]

\[= \int_{x_e}^{x_{e+1}} m_t(x,t) \hat{\omega}(x) dx \quad (B.41)\]

Introducing

\[P_1 = -GJ \left[ \frac{d\theta}{dx} \right]_{x_e} \quad P_2 = GJ \left[ \frac{d\theta}{dx} \right]_{x_{e+1}} \quad (B.42)\]

Therefore Equation B.41 becomes:

\[GJ \int_{x_e}^{x_{e+1}} \frac{d\theta}{dx} \frac{d\hat{\omega}}{dx} dx + \frac{Jm}{A} \int_{x_e}^{x_{e+1}} \frac{d^2\hat{\omega}}{dx^2} \hat{\omega}(x) dx\]

\[= \int_{x_e}^{x_{e+1}} m_t(x,t) \hat{\omega}(x) dx + P_1 \hat{\omega}(x_e) + P_2 \hat{\omega}(x_{e+1}) \quad (B.43)\]

### B.2.3 Interpolation Functions for Torsion

Equation B.43 requires that the interpolation function be continuous with continuous derivatives up to order one and must also satisfy the boundary conditions which are \(\hat{\theta}(x_e) = \hat{\theta}_1\) at the left node and \(\hat{\theta}(x_{e+1}) = \hat{\theta}_2\). Therefore a linear two-parameter function is chosen:

\[\hat{\theta}(x) = c_1 + c_2 x \quad (B.44)\]

In order to satisfy the boundary conditions we must have:

\[\hat{\theta}(x_e) = c_1 + c_2 x_e\]

\[\hat{\theta}(x_{e+1}) = c_1 + c_2 x_{e+1} \quad (B.45)\]
which in matrix form is:

\[
\begin{pmatrix}
\tilde{\theta}_1 \\
\tilde{\theta}_2
\end{pmatrix} =
\begin{bmatrix}
1 & x_e \\
1 & x_{e+1}
\end{bmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
\]

\[\text{(B.46)}\]

c_1 and c_2 are therefore given by:

\[
c_1 = \frac{\tilde{\theta}_1 x_{e+1} - \tilde{\theta}_2 x_e}{x_{e+1} - x_e}, \quad c_2 = \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{x_{e+1} - x_e}
\]

\[\text{(B.47)}\]

substituting c_1 and c_2 from the above expression into Equation B.44 and collecting the coefficients of \(\tilde{\theta}_1\) we get:

\[
\tilde{\theta}(x) = \phi_1 \tilde{\theta}_1 + \phi_2 \tilde{\theta}_2
\]

\[\text{(B.48)}\]

where

\[
\phi_1 = \frac{x_{e+1} - x}{x_{e+1} - x_e}, \quad \phi_2 = \frac{x - x_e}{x_{e+1} - x_e}
\]

\[\text{(B.49)}\]

The shape functions have been selected so that at \(x = x_e\) \(\phi_1 = 1\) and \(\phi_2 = 0\) and at \(x = x_{e+1}\) \(\phi_2 = 1\) and \(\phi_1 = 0\). In terms of the local (element) coordinate \(s = \frac{x-x_e}{\Delta x}\) the interpolation functions become:

\[
\phi_1 = 1 - s, \quad \phi_2 = s
\]

\[\text{(B.50)}\]

![Figure B.8: Interpolation Functions for a Torsion Element](image)

Figure B.8 shows the linear interpolation function used for torsion.

In order to change the interpolation limits in Equation B.43 we make the following transformations:

\[
\frac{d\phi_i}{dx} = \frac{d\phi_i}{ds} \frac{ds}{dx} = \frac{1}{\Delta x} \frac{d\phi_i}{ds}
\]

\[
dx = \frac{dx}{ds} ds = \Delta x ds
\]

\[\text{(B.51)}\]
Equation B.43 therefore becomes:

\[
\frac{GJ}{\Delta x} \int_0^1 \frac{d\tilde{\theta}}{ds} \frac{d\tilde{w}}{ds} ds + \frac{J\bar{m}\Delta x}{A} \int_0^1 \frac{d^2\tilde{\theta}}{dt^2} \tilde{w}(s) ds = \Delta x \int_0^1 m(s, t)\tilde{w}(s) ds + P_1 \tilde{w}(0) + P_2 \tilde{w}(1)
\]  
(B.52)

Equation B.52 is the variational formulation of the torsion problem and just as it was done with bending we substitute:

\[
\tilde{\theta} = \phi_1 \tilde{\theta}_1 + \phi_2 \tilde{\theta}_2 = (1 - s)\tilde{\theta}_1 + s\tilde{\theta}_2
\]  
(B.53)

into Equation B.52 and let \( \tilde{w} = \phi_1 \) and \( \tilde{w} = \phi_2 \) successively.

The first term in Equation B.52 is the stiffness matrix for torsion and it is given by:

\[
[K] = \frac{GJ}{\Delta x} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]  
(B.54)

Similarly the second term on the left hand of the variational formulation (Equation B.52) represents the consistent mass matrix and each term of the two by two matrix is given by:

\[
M_{ij} = \frac{J\bar{m}\Delta x}{A} \int_0^1 \phi_i \phi_j ds \quad \text{for } i, j = 1 \ldots 2
\]  
(B.55)

Substituting the appropriate interpolation functions from Equation B.50 and integrating over the element we get:

\[
[M] = \frac{J\bar{m}\Delta x}{A} \begin{bmatrix}
\frac{1}{3} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{3}
\end{bmatrix}
\]  
(B.56)

Just as with the bending case rotational inertia effects can be modeled using a consistent mass matrix as shown above or a they can be modeled as forces which is equivalent to using a lumped mass approach. The Newmark code uses a consistent mass matrix whereas the Fullwing code uses a lumped mass formulation.

Terms on the right hand side of Equation B.52 denote the force vector for a torsional element. Assuming that \( m(x, t) \) is constant over the element we can take it out of the integral and get:

\[
\Delta x \int_0^1 m(x, t)\tilde{w}(s) ds = \Delta x m(t) \int_0^1 \tilde{w}(s) ds
\]  
(B.57)
Therefore the force vector is given by:

\[ F_i = \Delta x \int_0^1 \sigma_i ds + P_i \quad \text{for } i = 1 \ldots 2 \]  

(B.58)

Again substituting the appropriate interpolation functions from Equation B.50 and integrating over the element the force vector for a torsional element is given by:

\[
\{ F \} = m(t) \Delta x \left\{ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right\} + \left\{ \begin{array}{c}
P_1 \\
P_2
\end{array} \right\}
\]  

(B.59)

which is basically a lumped formulation as it indicates that half of element’s torsional load is concentrated at each node.

### B.3 Combined Matrices

In the previous two sections the bending and torsion matrices were derived separately using the Galerkin technique. In this section the combined stiffness, mass and forces matrices are presented.

Up to this point the overall non-damped system differential equations are given by:

\[
[K] \{ \dot{y} \} + [M] \{ \ddot{y} \} = \{ F \} + \{ Q \}
\]  

(B.60)

The overall stiffness matrix is given by:

\[
[K] = \begin{bmatrix}
\frac{12EI}{\Delta x^3} & 0 & -\frac{6EI}{\Delta x^2} & -\frac{12EI}{\Delta x^3} & 0 & -\frac{6EI}{\Delta x^2} \\
\frac{6EI}{\Delta x^2} & 0 & 0 & -\frac{GJ}{\Delta x} & 0 & -\frac{GJ}{\Delta x} \\
\frac{12EI}{\Delta x^3} & 0 & \frac{6EI}{\Delta x^2} & 0 & -\frac{6EI}{\Delta x^2} & \frac{6EI}{\Delta x^3} \\
0 & -\frac{GJ}{\Delta x} & 0 & 0 & \frac{GJ}{\Delta x} & 0 \\
\frac{6EI}{\Delta x^2} & 0 & \frac{2EI}{\Delta x} & 0 & \frac{4EI}{\Delta x} & 0 \\
\frac{6EI}{\Delta x^2} & 0 & \frac{2EI}{\Delta x} & 0 & \frac{4EI}{\Delta x} & 0
\end{bmatrix}
\]  

(B.61)

and the overall consistent mass matrix is given by:

\[
[M] = m \Delta x
\]  

\[
\begin{bmatrix}
\frac{156}{420} & 0 & -\frac{22 \Delta x}{420} & \frac{54}{420} & 0 & \frac{13 \Delta x}{420} \\
0 & \frac{J}{3A} & 0 & 0 & \frac{J}{6A} & 0 \\
-\frac{22 \Delta x}{420} & 0 & \frac{4 \Delta x^2}{420} & -\frac{13 \Delta x}{420} & 0 & -\frac{3 \Delta x^2}{420} \\
\frac{54}{420} & 0 & -\frac{13 \Delta x}{420} & \frac{156}{420} & 0 & \frac{22 \Delta x}{420} \\
0 & \frac{J}{6A} & 0 & 0 & \frac{J}{3A} & 0 \\
\frac{13 \Delta x}{420} & 0 & -\frac{3 \Delta x^2}{420} & \frac{22 \Delta x}{420} & 0 & \frac{4 \Delta x^2}{420}
\end{bmatrix}
\]  

(B.62)
The force vector including the reaction forces are:

\[
\{ \mathbf{F} \} = \Delta x \begin{bmatrix} -\frac{n(t)}{2} \\ m(t) \\ \frac{n(t)\Delta x}{12} \\ -\frac{n(t)}{2} \\ m(t) \\ -\frac{n(t)\Delta x}{12} \end{bmatrix} + \begin{bmatrix} Q_1 \\ P_1 \\ Q_2 \\ Q_3 \\ P_2 \\ Q_4 \end{bmatrix} \tag{B.63}
\]

and the vector of independent variables and their second derivatives are:

\[
\{ \mathbf{y} \} = \begin{bmatrix} \dot{h}_1 \\ \dot{\theta}_1 \\ \ddot{h}_1 \\ \ddot{\theta}_1 \\ \dot{h}_2 \\ \dot{\theta}_2 \\ \ddot{h}_2 \\ \ddot{\theta}_2 \end{bmatrix} \quad \{ \ddot{\mathbf{y}} \} = \begin{bmatrix} \dddot{h}_1 \\ \dddot{\theta}_1 \\ \dddot{h}_1 \\ \dddot{\theta}_1 \\ \dddot{h}_2 \\ \dddot{\theta}_2 \\ \dddot{h}_2 \\ \dddot{\theta}_2 \end{bmatrix} \tag{B.64}
\]
Appendix C

Partial Derivatives

This Appendix presents the expanded form of the partial derivatives required to express the non-linear aerodynamic forces and moments (Equation 3.36 to Equation 3.40). There are two sets of derivatives, one for attached flow and another for stalled flow conditions.

C.1 Attached Flow

Three term that are needed for attached flow are the circulatory normal $N_c$, apparent mass normal $N_a$ and moment about the aerodynamic center $M_{ac}$.

C.1.1 $N_c$

Attached flow circulatory normal force at time step $n$ is given by:

\[ N_c^n = c \Delta x \pi \rho U (\alpha_0 + \theta_a + \theta_{wash} - 2(\alpha_0 + \theta_a + \theta_{wash}))/((2 + AR) +
\]

\[ AR\left( F'(k)(\dot{\theta} + (0.75c\dot{\theta} - \dot{\theta} yea + \dot{h} \cos(\theta + \theta_{wash}))/\right)
\]

\[ + c G'(k)(0.75c\ddot{\theta} + \dot{\theta} U - \ddot{\theta} yea + \ddot{h} \cos(\theta + \theta_{wash}) -
\]

\[ \dot{h} \dot{\theta} \sin(\theta + \theta_{wash}))/((2k U^2)/(2 + AR))((U \cos(\theta_a + \theta_a + \theta_{wash}) - \dot{h} \sin(\theta_a + \theta_{wash})) +
\]

\[ (-0.5c\dot{\theta} + U \cos(\theta_a + \theta_a + \theta_{wash}) - 2(\alpha_0 + \theta_a + \theta_{wash}))/((2 + AR) +
\]

\[ AR\left( F'(k)(\dot{\theta} + (0.75c\dot{\theta} - \dot{\theta} yea + \dot{h} \cos(\theta + \theta_{wash}))/\right)
\]

\[ + c G'(k)(0.75c\ddot{\theta} + \dot{\theta} U - \ddot{\theta} yea +
\]

\[ \ddot{h} \cos(\theta + \theta_{wash}) - \dot{h} \dot{\theta} \sin(\theta + \theta_{wash}))/((2k U^2))/(2 + AR))\right)^2)}^{\frac{1}{2}} \quad (C.1)\]
partial derivative of $N^a_c$ with respect to $\hat{\theta}$ is given by:

$$
\frac{\partial N^a_c}{\partial \hat{\theta}} = A R c \Delta x \pi \rho \mathcal{U} \left( c G'(k) \left( - \left( \dot{h} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right) \right) + F'(k) \left( 1 - \dot{h} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right) \right) \left( \mathcal{U} \cos(\bar{\theta}_a + \hat{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right)/(2 k \mathcal{U}^2) + \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + AR \left( F'(k) \left( \dot{\theta} + 0.75 c \dot{\theta} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right)/\mathcal{U} \right) \right) \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + \left( c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} \mathcal{U} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right) + \dot{h} \sin(\hat{\theta} + \bar{\theta}_{wash})/(2 k \mathcal{U}^2) \right) \left( 2 + AR \right) \right) + \left( 2(\mathcal{U} \cos(\bar{\theta}_a + \hat{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right) \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + AR \left( F'(k) \left( \dot{\theta} + 0.75 c \dot{\theta} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right)/\mathcal{U} \right) \right) \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + \left( c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} \mathcal{U} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right) + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash})/(2 k \mathcal{U}^2) \right) \left( 2 + AR \right) \right) + 2(\mathcal{U} \cos(\bar{\theta}_a + \hat{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right) \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + AR \left( F'(k) \left( \dot{\theta} + 0.75 c \dot{\theta} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right)/\mathcal{U} \right) \right) \left( c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} \mathcal{U} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right) + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash})/(2 k \mathcal{U}^2) \right) \left( 2 + AR \right) \right) \right) + 2(\mathcal{U} \cos(\bar{\theta}_a + \hat{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \sin(\hat{\theta} + \bar{\theta}_{wash}) \right) \left( -0.5 c \dot{\theta} + \mathcal{U} \left( \bar{\theta}_a + \bar{\theta}_{wash} \right) - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + AR \left( F'(k) \left( \dot{\theta} + 0.75 c \dot{\theta} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right)/\mathcal{U} \right) \right) \left( c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} \mathcal{U} - \dot{\theta}_{yca} + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash}) \right) + \dot{h} \cos(\hat{\theta} + \bar{\theta}_{wash})/(2 k \mathcal{U}^2) \right) \left( 2 + AR \right) \right) \right) \right)}^{1/2} \quad \text{(C.2)}
partial derivative of $N^n_{\varepsilon}$ with respect to $\dot{\theta}$ is given by:

$$\frac{\partial N^n_{\varepsilon}}{\partial \dot{\theta}} = c \Delta x \pi \rho U (-0.5 c + A R U \left( F'(k) (0.75 c - y_{ea}) / U \right) +$$

$$c G'(k) \left( U - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}) \right) / \left( 2 k U^2 \right) - (2 + A R) \left( \alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash} - 2 (\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash}) \right) / \left( 2 + A R \right) +$$

$$c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} U - \dot{h} y_{ea} + \dot{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \sin(\dot{\theta} + \bar{\theta}_{wash}) \right) / \left( 2 k U^2 \right)$$

$$+ (2 + A R) \left( \alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash} - 2 (\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash}) \right) / \left( 2 + A R \right) +$$

$$A R \left( F'(k) \left( \dot{\theta} + 0.75 c \dot{\theta} - \dot{h} y_{ea} + \dot{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) \right) / U \right)$$

$$+ c G'(k) \left( 0.75 c \dot{\theta} + \dot{\theta} U - \dot{h} y_{ea} + \dot{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) \right) / \left( 2 k U^2 \right)$$

$$1. A R c \Delta x \pi \rho U \left( F'(k) \left( 0.75 c - y_{ea} \right) / U \right) + c G'(k) \left( \dot{\theta} - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}) \right) / \left( 2 k U^2 \right)$$

$$\left( U \cos(\dot{\theta}_a + \bar{\theta} + \bar{\theta}_{wash}) - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}) \right)^2 + (0.75 \dot{c} + \dot{U} - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}) \right) / \left( 2 k U^2 \right)$$

(C.3)
partial derivative of $N_c^n$ with respect to $\ddot{\theta}$ is given by:

\[
\frac{\partial N_c^n}{\partial \ddot{\theta}} = 0.5 \, AR \, c^2 \, \Delta x \, G'(k) \, \pi \, \rho \, (0.75 \, c - y_{ea}) \\
(\alpha_0 + \ddot{\theta} + \ddot{\omega}_w) - 2(\alpha_0 + \ddot{\theta} + \ddot{\omega}_w)/(2 + AR) + AR(F'(k)(\ddot{\theta} + (0.75 \, c \, \ddot{\theta} - \dot{\theta} \, y_{ea} + \\
\dot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w))/L' + c \, G'(k)(0.75 \, c \, \ddot{\theta} + \dot{\theta} \, L' - \ddot{\theta} \, y_{ea} + \\
\ddot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w) - \dot{\theta} \, \sin(\ddot{\theta} + \ddot{\omega}_w))/(2 \, k \, L'^2)/(2 + AR))(-0.5 \, c \, \ddot{\theta} + \\
L'(\ddot{\theta} + \ddot{\omega}_w) - 2(\alpha_0 + \ddot{\theta} + \ddot{\omega}_w)/(2 + AR) + AR(F'(k)(\ddot{\theta} + \\
(0.75 \, c \, \ddot{\theta} - \dot{\theta} \, y_{ea} + \ddot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w))/L' + c \, G'(k) \\
(0.75 \, c \, \ddot{\theta} + \dot{\theta} \, L' - \ddot{\theta} \, y_{ea} + \ddot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w) - \dot{\theta} \, \sin(\ddot{\theta} + \ddot{\omega}_w))/ \\
(2 \, k \, L'^2)/(2 + AR)^2)^{1/2} + 0.5 \, AR \, c^2 \, \Delta x \, G'(k) \, \pi \, \rho \, (0.75 \, c - y_{ea}) \\
((L' \, \cos(\ddot{\theta} + \ddot{\omega}_w) - \dot{\theta} \, \sin(\ddot{\theta} + \ddot{\omega}_w))^2 + (-0.5 \, c \, \ddot{\theta} + L' \\
(\ddot{\theta} + \ddot{\omega}_w) - 2(\alpha_0 + \ddot{\theta} + \ddot{\omega}_w)/(2 + AR) + AR(F'(k)(\ddot{\theta} + \\
(0.75 \, c \, \ddot{\theta} - \dot{\theta} \, y_{ea} + \ddot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w))/L' + c \, G'(k) \\
(0.75 \, c \, \ddot{\theta} + \dot{\theta} \, L' - \ddot{\theta} \, y_{ea} + \ddot{\theta} \cos(\ddot{\theta} + \ddot{\omega}_w) - \dot{\theta} \, \sin(\ddot{\theta} + \ddot{\omega}_w))/ \\
(2 \, k \, L'^2)/(2 + AR)^2)^{1/2}/((2 + AR) \, k \, L)) \quad (C.4)
partial derivative of \( N^n_c \) with respect to \( \dot{h} \) is given by:

\[
\frac{\partial N^n_c}{\partial h} = 0.5 \Delta x \pi \rho U (\alpha_0 + \bar{\theta} + \bar{\theta}_{wash} - 2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash}))/ (2 + AR) + AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
((L' \cos(\bar{\theta}_a + \bar{\theta} + \bar{\theta}_{wash}) - h \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/ (2 + AR) \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
((L' \cos(\bar{\theta}_a + \bar{\theta} + \bar{\theta}_{wash}) - h \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/ (2 + AR) \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
((L' \cos(\bar{\theta}_a + \bar{\theta} + \bar{\theta}_{wash}) - h \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/ (2 + AR) \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
1. \Delta x \pi \rho U (F'(k) \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k) \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash})/(2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]

\[
(2 + AR)) \]

\[
2 \]

\[
1 \]

\[
2 + AR \]

\[
AR(F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash}))/L -\]

\[
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} y_{ea} + h \cos(\bar{\theta} + \bar{\theta}_{wash})))/L -\]

\[
\cos(\bar{\theta} + \bar{\theta}_{wash}) - h \dot{\theta} \sin(\bar{\theta} + \bar{\theta}_{wash}))/ (2 + AR)) \]
Partial derivative of \( \mathcal{N}_c^n \) with respect to \( \tilde{h} \) is given by:

\[
\frac{\partial \mathcal{N}_c^n}{\partial \tilde{h}} = 0.5 AR c^2 \Delta x C'(k) \pi \rho \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}}) \\
(\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{\text{wash}} - 2(\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{\text{wash}})/(2 + AR) + AR(F'(k)(\tilde{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}}))/U) + c G'(k)(0.75 c \dot{\theta} + \dot{\theta} U' - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}})))/(2 k U')/(2 + AR) (-(0.5 c \dot{\theta} + U' (\tilde{\theta}_a + \tilde{\theta}_{\text{wash}} - 2(\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{\text{wash}})/(2 + AR) + AR(F'(k)(\tilde{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}}))/U) + c G'(k)(0.75 c \dot{\theta} + \dot{\theta} U' - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}})))/(2 k U')/(2 + AR)))/((2 + AR) k ((U' \cos(\tilde{\theta}_a + \dot{\theta} + \tilde{\theta}_{\text{wash}}) - h \sin(\tilde{\theta} + \tilde{\theta}_{\text{wash}})) h \sin(\tilde{\theta} + \tilde{\theta}_{\text{wash}}))^2 + (0.5 c \dot{\theta} + U' (\tilde{\theta}_a + \tilde{\theta}_{\text{wash}} - 2(\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{\text{wash}})/(2 + AR) + AR(F'(k)(\tilde{\theta} + (0.75 c \dot{\theta} - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}}))/U) + c G'(k)(0.75 c \dot{\theta} + \dot{\theta} U' - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}})))/(2 k U')/(2 + AR)))/(2 + AR))^{1/2} /((2 + AR) k U')) \quad (C.6)
\]

C.1.2 \( \mathcal{N}_a \)

Attached flow apparent mass normal force at time step \( n \) is given by:

\[
\mathcal{N}_a^n = 0.25 c^2 \Delta x \pi \rho (0.5 c \dot{\theta} + \dot{\theta} U' - \dot{\theta} y_{ea} + h \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}}) - h \dot{\theta} \sin(\tilde{\theta} + \tilde{\theta}_{\text{wash}})) \quad (C.7)
\]

Partial derivative of \( \mathcal{N}_a^n \) with respect to \( \dot{\theta} \) is given by:

\[
\frac{\partial \mathcal{N}_a^n}{\partial \dot{\theta}} = 0.25 c^2 \Delta x \pi \rho (-(h \dot{\theta} \cos(\tilde{\theta} + \tilde{\theta}_{\text{wash}})) - h \sin(\tilde{\theta} + \tilde{\theta}_{\text{wash}})) \quad (C.8)
\]

Partial derivative of \( \mathcal{N}_a^n \) with respect to \( \tilde{\theta} \) is given by:

\[
\frac{\partial \mathcal{N}_a^n}{\partial \tilde{\theta}} = 0.25 c^2 \Delta x \pi (U - h \sin(\tilde{\theta} + \tilde{\theta}_{\text{wash}})) \quad (C.9)
\]
partial derivative of $N^a_n$ with respect to $\dot{\theta}$ is given by:

$$\frac{\partial N^n_a}{\partial \theta} = 0.25 c^2 \Delta x \pi \rho (0.5c - y_{ea})$$

(C.10)

partial derivative of $N^a_n$ with respect to $\dot{h}$ is given by:

$$\frac{\partial N^n_a}{\partial \dot{h}} = -0.25 c^2 \Delta x \pi \rho \dot{\theta} \sin(\theta + \bar{\theta}_{wash})$$

(C.11)

partial derivative of $N^a_n$ with respect to $\ddot{h}$ is given by:

$$\frac{\partial N^n_a}{\partial \ddot{h}} = 0.25 c^2 \Delta x \pi \rho \cos(\theta + \bar{\theta}_{wash})$$

(C.12)

C.1.3 $M_{ac}$

Moment about the aerodynamic center at time step $n$ is given by:

$$M^n_{ac} = 0.5C_{mac} c^2 \Delta x \rho U \left( (U \cos(\bar{\theta}_a + \theta + \bar{\theta}_{wash}) - \right.$$

$$\dot{h} \sin(\theta + \bar{\theta}_{wash}))^2 + (-0.5c \dot{\theta} + U (\bar{\theta}_a + \bar{\theta}_{wash} -$$

$$2(\alpha_0 + \bar{\theta}_a + \bar{\theta}_{wash})/(2 + AR) + AR(F'(k)(\theta +$$

$$0.75c \dot{\theta} - \dot{\theta} y_{ea} + \dot{h} \cos(\theta + \bar{\theta}_{wash}))/U) + cG'(k)$$

$$0.75c \dot{\theta} + \dot{h} U - \bar{\theta} y_{ea} + \ddot{h} \cos(\theta + \bar{\theta}_{wash}) - \dot{h} \dot{\theta} \sin(\theta + \bar{\theta}_{wash}))/$$

$$\left( (2k U^2)/(2 + AR) \right)^{\frac{1}{2}}$$

(C.13)
partial derivative of $M_{ac}^n$ with respect to $\tilde{\theta}$ is given by:

\[
\frac{\partial M_{ac}^n}{\partial \tilde{\theta}} = 0.25 C_{mac} c^2 \Delta x \rho L' (2 AR U' \frac{\partial M_{ac}^n}{\partial \tilde{\theta}})
\]

\[
(c G'(k) \left( -(\dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash})) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}) + \frac{F'(k)}{(2 k U^2)} + F'(k) (1 - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash})/U'))
\]

\[
(-0.5 c \dot{\tilde{\theta}} + U' (\tilde{\theta}_a + \tilde{\theta}_{wash}) - 2 (\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{wash})/(2 + AR) + \]

\[
AR (F'(k) (\tilde{\theta} + (0.75 c \dot{\tilde{\theta}} - \dot{\tilde{\theta}} yea + \dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash})/U')
\]

\[
+ c G'(k) (0.75 c \dot{\tilde{\theta}} + \dot{\tilde{\theta}} U' - \dot{\tilde{\theta}} yea + \]

\[
\tilde{\theta} \cos(\tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash})/(2 k U^2)/(2 + AR))/((2 + AR) + \]

\[
2(U' \cos(\tilde{\theta}_a + \tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))/((2 + AR) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash})))/\]

\[
\left( (\dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash})) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))^2 + (0.5 \dot{\tilde{\theta}} + U' (\tilde{\theta}_a + \tilde{\theta}_{wash} - \]

\[
2 (\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{wash}))/((2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 c \dot{\tilde{\theta}} - \dot{\tilde{\theta}} yea + \dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash})/U'))
\]

\[
(0.75 c \dot{\tilde{\theta}} + \dot{\tilde{\theta}} U' - \dot{\tilde{\theta}} yea + \dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))/\]

\[
(2 k U^2)/(2 + AR))/2) \right) (C.14)
\]

partial derivative of $M_{ac}^n$ with respect to $\dot{\tilde{\theta}}$ is given by:

\[
\frac{\partial M_{ac}^n}{\partial \dot{\tilde{\theta}}} = 0.5 C_{mac} c^2 \Delta x \rho L' (0.75 c - yea)/U' + \]

\[
(c G'(k) (L' - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))/((2 + AR) + c G'(k) (0.75 c \dot{\tilde{\theta}} + \dot{\tilde{\theta}} U' - \dot{\tilde{\theta}} yea + \]

\[
\tilde{\theta} \cos(\tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))/((2 + AR))/((2 + AR) + \]

\[
2(\alpha_0 + \tilde{\theta}_a + \tilde{\theta}_{wash})/(2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 c \dot{\tilde{\theta}} - \dot{\tilde{\theta}} yea + \dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash})/U'))
\]

\[
(0.75 c \dot{\tilde{\theta}} + \dot{\tilde{\theta}} U' - \dot{\tilde{\theta}} yea + \dot{\tilde{\theta}} \cos(\tilde{\theta} + \tilde{\theta}_{wash}) - \dot{\tilde{\theta}} \sin(\tilde{\theta} + \tilde{\theta}_{wash}))/\]

\[
(2 k U^2)/(2 + AR))/2) \right) (C.15)
partial derivative of $M_{ac}^n$ with respect to $\tilde{\theta}$ is given by:

$$\frac{\partial M_{ac}^n}{\partial \tilde{\theta}} = 0.25 \ AR \ C_{mac} \ c^3 \ \Delta x \ G'(k) \ \rho \ (0.75 \ c \ - \ y_{ea}) \ (-0.5 \ c \ \dot{\tilde{\theta}} + U' (\tilde{a} + \tilde{\vartheta}_{wash} - 2 (\alpha_0 + \tilde{a} + \tilde{\vartheta}_{wash} )/(2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 \ c \ \dot{\tilde{\theta}} - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/(2 k U^2))/(2 + AR) ))) \ /

((2 + AR) k ((U' \ cos(\tilde{a} + \tilde{\theta} + \tilde{\vartheta}_{wash} ) - \ h \ sin(\tilde{\theta} + \tilde{\vartheta}_{wash} ))^2 + (-0.5 \ c \ \dot{\tilde{\theta}} + U' (\tilde{a} + \tilde{\vartheta}_{wash} - 2 (\alpha_0 + \tilde{a} + \tilde{\vartheta}_{wash} )/(2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 \ c \ \dot{\tilde{\theta}} - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/(2 k U^2))/(2 + AR) )))^{1/2}) (C.16)

partial derivative of $M_{ac}^n$ with respect to $\dot{h}$ is given by:

$$\frac{\partial M_{ac}^n}{\partial \dot{h}} = 0.25 \ C_{mac} \ c^3 \ \Delta x \ \rho \ U' (-2 \ sin(\tilde{\theta} + \tilde{\vartheta}_{wash} )) \

(U' \ cos(\tilde{a} + \tilde{\theta} + \tilde{\vartheta}_{wash} ) - \ h \ sin(\tilde{\theta} + \tilde{\vartheta}_{wash} )) + 2 \ AR \ U' (F'(k) \ cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/U' - c \ G'(k) \ \dot{\tilde{\theta}} \ \sin(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/(2 k U^2)) (-0.5 \ c \ \dot{\tilde{\theta}} + U' (\tilde{a} + \tilde{\vartheta}_{wash} - 2 (\alpha_0 + \tilde{a} + \tilde{\vartheta}_{wash} )/(2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 \ c \ \dot{\tilde{\theta}} - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/(U') + c \ G'(k) ) (0.75 \ c \ \dot{\tilde{\theta}} + \dot{\theta} U' - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ) - \ h \ \dot{\tilde{\theta}} \ \sin(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/ ((2 k U^2))/(2 + AR) ))/(2 + AR)) (((U' \ cos(\tilde{a} + \tilde{\theta} + \tilde{\vartheta}_{wash} ) - \ h \ sin(\tilde{\theta} + \tilde{\vartheta}_{wash} ))^2 + (-0.5 \ c \ \dot{\tilde{\theta}} + U' (\tilde{a} + \tilde{\vartheta}_{wash} - 2 (\alpha_0 + \tilde{a} + \tilde{\vartheta}_{wash} )/(2 + AR) + AR (F'(k) (\tilde{\theta} + (0.75 \ c \ \dot{\tilde{\theta}} - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} )))/(U') + c \ G'(k) ) (0.75 \ c \ \dot{\tilde{\theta}} + \dot{\theta} U' - \dot{\vartheta}_{ea} + \ h \cos(\tilde{\theta} + \tilde{\vartheta}_{wash} ))/ ((2 k U^2))/(2 + AR) ))^{1/2}) (C.17)
partial derivative of $M_{ac}^n$ with respect to $\dot{h}$ is given by:

\[
\frac{\partial M_{ac}^n}{\partial \dot{h}} = 0.25 A R C_{mac} c^3 \Delta x G'(k) \rho \cos(\theta + \theta_{wash}) \\
(-0.5 c \dot{\theta} + U' (\theta_a + \theta_{wash}) - 2(\alpha_0 + \theta_a + \theta_{wash})/(2 + A R) + \\
AR (F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \theta_{yea} + \dot{h} \cos(\theta + \theta_{wash})))/U') + \\
c G'(k)(0.75 c \dot{\theta} + \dot{\theta} U' - \theta_{yea} + \dot{h} \cos(\theta + \theta_{wash}) - \dot{h} \dot{\theta} \sin(\theta + \theta_{wash}))/ \\
(2k U'^2)/(2 + A R))/((2 + A R) k ((U \cos(\theta_a + \theta + \theta_{wash}) - \dot{h} \sin(\theta + \theta_{wash})) \\
2 + (-0.5 c \dot{\theta} + U' (\theta_a + \theta_{wash}) - 2(\alpha_0 + \theta_a + \theta_{wash})/(2 + A R) + \\
AR (F'(k)(\dot{\theta} + (0.75 c \dot{\theta} - \theta_{yea} + \dot{h} \cos(\theta + \theta_{wash})))/U') \\
+ c G'(k)(0.75 c \dot{\theta} + \dot{\theta} U' - \theta_{yea} + \\
\dot{h} \cos(\theta + \theta_{wash}) - \dot{h} \dot{\theta} \sin(\theta + \theta_{wash}))/((2k U'^2)/(2 + A R)))^{1/2}) \quad (C.18)
\]

### C.2 Stalled Flow

Two terms that are needed for Stalled flow are the circulatory normal $(\mathcal{V}_c)_{sep}$ and the apparent mass normal $(\mathcal{V}_a)_{sep}$.

#### C.2.1 $(\mathcal{V}_c)_{sep}$

Stalled flow circulatory normal force at time step $n$ is given by:

\[
(\mathcal{V}_c)^n_{sep} = 0.5 (C_d)_{cf} c \Delta x \rho (\dot{\theta}(0.5 c - y_{ea}) + \dot{h} \cos(\theta + \theta_{wash}) + \\
U' \sin(\theta_a + \theta + \theta_{wash}))((U \cos(\theta_a + \theta + \theta_{wash}) - \dot{h} \sin(\theta + \theta_{wash}))^2 + \\
(0.5 c \dot{\theta} - \theta_{yea} + \dot{h} \cos(\theta + \theta_{wash}) + U \sin(\theta_a + \theta + \theta_{wash}))^2)^{1/2} \quad (C.19)
\]
partial derivative of \((N_c)_{sep}^n\) with respect to \(\hat{\vartheta}\) is given by:

\[
\frac{\partial (N_c)_{sep}^n}{\partial \hat{\vartheta}} = 0.5 (C_d)_{cf} c \Delta x \rho (0.5 c - y_{ea}) \\
(\hat{\vartheta} (0.5 c - y_{ea}) + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + (U') \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash})) \\
(0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + (U') \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash})) \\
((U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + (0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + \\
(U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} + 0.5 (C_d)_{cf} c \Delta x \rho (0.5 c - y_{ea}) \\
((U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + (0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + \\
(U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} - (C.20)
\]

partial derivative of \((N_c)_{sep}^n\) with respect to \(\dot{h}\) is given by:

\[
\frac{\partial (N_c)_{sep}^n}{\partial \dot{h}} = 0.25 (C_d)_{cf} c \Delta x \rho (\hat{\vartheta} (0.5 c - y_{ea}) + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + \\
(U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))( -2 \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}) (U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \\
\dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash})) + 2 \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) (0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \\
\dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))(U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + \\
(0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + (U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} + \\
0.5 (C_d)_{cf} c \Delta x \rho \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) ((U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + \\
(0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + (U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} - (C.21)
\]

partial derivative of \((N_c)_{sep}^n\) with respect to \(\hat{\vartheta}\) is given by:

\[
\frac{\partial (N_c)_{sep}^n}{\partial \hat{\vartheta}} = 0.25 (C_d)_{cf} c \Delta x \rho (\hat{\vartheta} (0.5 c - y_{ea}) + \dot{\vartheta} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + \\
(U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))(2 (U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash})) \\
(-\dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash})) - U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash})) + 2 (U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash})) \\
(0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + (U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} + \\
((U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + (0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + \\
U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} + 0.5 (C_d)_{cf} c \Delta x \rho (U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \\
\dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))(U' \cos(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}) - \dot{h} \sin(\hat{\vartheta} + \bar{\vartheta}_{wash}))^2 + \\
(0.5 c \hat{\vartheta} - \dot{\vartheta} y_{ea} + \dot{h} \cos(\hat{\vartheta} + \bar{\vartheta}_{wash}) + U' \sin(\bar{\vartheta}_a + \hat{\vartheta} + \bar{\vartheta}_{wash}))^2)^{1/2} - (C.22)
\]
C.2.2 \((N_a)_{sep}\)

Stalled flow apparent mass normal force at time step n is given by:

\[
(N_a)_{sep}^n = 0.125 c^2 \Delta x \pi \rho (0.5 c \tilde{\theta} + \dot{\theta} U' - \tilde{\theta} y_{ca} + \tilde{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) - \dot{\theta} \tilde{h} \sin(\dot{\theta} + \bar{\theta}_{wash}))
\]  

(C.23)

partial derivative of \((N_a)_{sep}^n\) with respect to \(\dot{\theta}\) is given by:

\[
\frac{\partial(N_a)_{sep}^n}{\partial \dot{\theta}} = 0.125 c^2 \Delta x \pi \rho (\tilde{h} \cos(\dot{\theta} + \bar{\theta}_{wash}) - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}))
\]  

(C.24)

partial derivative of \((N_a)_{sep}^n\) with respect to \(\dot{\theta}\) is given by:

\[
\frac{\partial(N_a)_{sep}^n}{\partial \tilde{\theta}} = 0.125 c^2 \Delta x \pi \rho (U' - \dot{h} \sin(\dot{\theta} + \bar{\theta}_{wash}))
\]  

(C.25)

partial derivative of \((N_a)_{sep}^n\) with respect to \(\tilde{\theta}\) is given by:

\[
\frac{\partial(N_a)_{sep}^n}{\partial \tilde{\theta}} = 0.125 c^2 \Delta x \pi \rho (0.5 c - y_{ca})
\]  

(C.26)

partial derivative of \((N_a)_{sep}^n\) with respect to \(\dot{\theta}\) is given by:

\[
\frac{\partial(N_a)_{sep}^n}{\partial \dot{h}} = -0.125 c^2 \Delta x \pi \rho \tilde{h} \sin(\dot{\theta} + \bar{\theta}_{wash})
\]  

(C.27)

partial derivative of \((N_a)_{sep}^n\) with respect to \(\tilde{\theta}\) is given by:

\[
\frac{\partial(N_a)_{sep}^n}{\partial \tilde{h}} = 0.125 c^2 \Delta x \pi \rho \cos(\dot{\theta} + \bar{\theta}_{wash})
\]  

(C.28)
Appendix D

1997 Runs and Newmark Predictions

This appendix presents the steady state portion of runs in 1997. Taxi tests were conducted in 1997 from 7 August to 16 September. Data was not collected for all the test in 1997. Of the test done in 1997 the only taxi tests which included "steady-state" portions that can be compared to the Newmark code were conducted on September 1, September 15 and September 16. Fortunately both bending and twisting moments were collected during these dates.
Figure D.1: 1 September 1997 Run 2. Twisting Moment Data
Figure D.2: 1 September 1997 Run 2. Bending Moment Data
Figure D.3: 1 September 1997 Run 4. Twisting Moment Data
Figure D.4: 1 September 1997 Run 4. Bending Moment Data
Figure D.5: 15 September 1997 Run 1. Twisting Moment Data
15 September 1997, Run 1
Newmark Inputs: $\bar{\theta}_a = -3$ deg. $U = 44$ ft/sec $\omega = 0.96$ Hz

![Graph showing Bending Moment Data for 15 September 1997, Run 1, with plots for Bending Moment (Experimental), Bending Moment (Newmark), Air Speed, Angle of Attack, and Throttle Position.](image)
Figure D.7: 15 September 1997 Run 2. Twisting Moment Data
Figure D.8: 15 September 1997 Run 2. Bending Moment Data
16 September 1997, Run 2
Newmark Inputs: $\bar{\theta}_a = -4$ deg. $U = 47$ ft/sec $\omega = 1.32$ Hz

Right Wing Twisting Moment at #3 (ft-lb)

Time from Start of Test (sec)
16 September 1997, Run 2
Newmark Inputs: $\bar{\theta}_a=4$ deg. $U=47$ ft/sec $\omega_o=1.32$ Hz

Figure D.10: 16 September 1997 Run 2. Bending Moment Data
Appendix E

1998 Runs and Newmark Predictions

This appendix presents the steady state portion of runs in 1998. Taxi tests were conducted from 19 September to 8 November 1998. Data was collected for a majority of the tests in 1998. Of the 29 taxi tests conducted in 1998 15 included steady-state portions where the results could be compared to the Newmark code. These are presented in this Appendix. Unfortunately the only remaining torsional strain gage had failed by this point and only bending moment data is available.
19 September 1998, Run 4
Newmark Inputs: $\bar{\theta}_a=0$ deg. $U=35$ ft/sec $\omega=0.80$ Hz

- Bending Moment (Experimental)
- Bending Moment (Newmark)
- Air Speed
- Angle of Attack
- Throttle Position
Figure E.2: 24 September 1998 Run 1. Bending Moment Data
Figure E.3: 24 September 1998 Run 2. Bending Moment Data
24 September 1998, Run 3
Newmark Inputs: $\bar{\theta}_a = -4\,\text{deg.}$, $U = 49\,\text{ft/sec}$, $\omega = 0.97\,\text{Hz}$

Figure E.4: 24 September 1998 Run 3, Bending Moment Data
Figure E.5: 24 September 1998 Run 4. Bending Moment Data
24 September 1998, Run 5

Newmark Inputs: $\theta_a = -6\,\text{deg}$, $U = 58\,\text{ft/sec}$, $\omega = 0.95\,\text{Hz}$

Figure E.6: Bending Moment Data
29 September 1998, Run 1

Newmark Inputs: $\theta_a = -3$ deg, $U = 43$ ft/sec, $\omega = 0.95$ Hz

Figure E.7: 29 September 1998 Run 1. Bending Moment Data
Figure E.8: 16 October 1998 Run 1. Bending Moment Data
Figure E.9: 16 October 1998 Run 2. Bending Moment Data
Figure E.10: 16 October 1998 Run 3. Bending Moment Data
16 October 1998, Run 4

Newmark Inputs: $\bar{\theta}_a = 5$ deg, $U = 63$ ft/sec, $\omega = 1.11$ Hz

Figure E.11: 16 October 1998 Run 4. Bending Moment Data
Figure E.12: 30 October 1998 Run 1. Bending Moment Data
Figure E.13: 30 October 1998 Run 2. Bending Moment Data
Figure E.14: 8 November 1998 Run 1. Bending Moment Data
8 November 1998, Run 2
Newmark Inputs: $\bar{\theta}_a = -6$ deg, $U = 74$ ft/sec $\omega = 1.22$ Hz

Figure E.15: 8 November 1998 Run 2. Bending Moment Data