Topics in Ramsey Theory on Sets of Real Numbers

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
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Abstract

The purpose of this thesis is to explore three topics in Ramsey theory of sets on the real numbers. The first topic of discussion is a program initiated by M. Souslin which focuses on investigating the relationships of chain conditions in topology. I will demonstrate that under the assumption $\aleph_1 < \text{add}_*(\mathcal{L}_1) = c$ there is a c.c.c. nonseparable compactum which does not map onto $[0, 1]^{\omega_1}$. Other aspects of Souslin's program are also investigated. The second topic could be termed "continuous Ramsey theory." In this portion of the thesis we will study colorings $c : [X]^2 \to \omega$ which are continuous and irreducible ($c$ realizes every color on every large square). I will show that such colorings exist which are connected with unboundedness in $(\mathcal{N}, \subseteq)$ and $(\mathbb{P}, <^*)$. The last topic is an investigation of the effects of random forcing on Ramsey theoretic statements about $\omega_1$. We will investigate some of the applications to combinatorics and general topology.
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Dedicated to Stephanie Rihm Moore
and her thesis in progress

Gaudeamus
## Contents

**Introduction** x

1 **Souslinean Spaces** 1  
  1.1 Local and Global Chain Conditions ................. 2  
  1.2 Souslin's Program ................................ 4  
  1.3 Compacta, Boolean Algebras, Partial Orders, and Partitions . 5  
  1.4 Forcing Axioms ..................................... 8  

2 **Global Chain Conditions** 10  
  2.1 Gaps and Souslinean Spaces ....................... 12  
  2.2 Being Separable ................................... 16  
  2.3 Souslin's Condition .............................. 17  
  2.4 Being Linearly Fibered ........................... 19  
  2.5 Helly's Condition ................................ 20  
  2.6 An Example From $\aleph_1 < \text{add}_*(\ell_1) = \mathfrak{c}$ .......... 22  
  2.7 An Example From $b > \aleph_2$ .................... 27

3 **Local Chain Conditions** 30
3.1 Some Consequences $\mathcal{K}_2$ ................................................. 32
3.2 $\mathcal{K}_3$ and Additivity of Measure ................................. 35

4 Continuous Irreducible Colorings ........................................ 39
4.1 A Coloring Associated With Measure ................................. 41
4.2 The Alternation Map ...................................................... 48
4.3 Continuous Codings for Real Numbers ................................. 52

5 Measure Algebras .............................................................. 56
5.1 Measure Algebras ............................................................ 57
5.2 Martin's Axiom and Random Graphs .................................. 59
5.3 A Variation of Maharam's Theorem .................................... 62

6 Combinatorics ................................................................. 65
6.1 Partition Calculus .......................................................... 66
6.2 Uniformizing Ladder Systems .......................................... 73
6.3 Wage's Lemma ............................................................. 77
6.4 Fragments of Martin's Axiom ............................................. 80

7 Topology ........................................................................... 82
7.1 Cometrizable Spaces ....................................................... 85
7.2 Extremally Disconnected Spaces ....................................... 87
7.3 Compact $S$ Spaces ........................................................ 89
7.4 Small Compactifications of $L$ Spaces ................................. 92

A Glossary ............................................................................. 94
Introduction

In 1930, Ramsey published his celebrated theorem [48] today known as Ramsey’s theorem. It states that if the $n$ element subsets of $\mathbb{N}$ are split into finitely many pieces, then for some infinite set $A \subseteq \mathbb{N}$, every $n$ element subset of $A$ is contained in one of the pieces. While it was ignored and lay dormant for some time, it was to eventually influence a wide cross section of mathematics. Over time, various generalizations and variations of Ramsey’s theorem have been studied. In this thesis, we will be particularly interested in those variations which have a close connection to the reals.

The first result of this kind was due to Sierpiński. He proved the following result, showing that an obvious generalization of Ramsey’s theorem to the continuum is simply false.

**Theorem.** [52] There is a partition $[\mathbb{R}]^2 = K_0 \cup K_1$ of the unordered pairs of real numbers into two pieces such that whenever $H$ is a subset of $\mathbb{R}$ and $[H]^2$ is contained in either $K_0$ or $K_1$ then $H$ is countable.

While this might initially seem to render any study of Ramsey’s theorem at the level of the continuum fruitless, this turns out to be far from the case.
It is frequently the case in general topology that we are presented with a partition $[X]^2 = K_0 \cup K_1$ of the pairs of an uncountable set $X$ into two pieces $K_0$ and $K_1$ in such a way that it is desirable to be able to find a large set $H \subseteq X$, all of whose pairs are in one of the pieces. If this was all we knew, then the partition mentioned above would indicate that indeed we can’t hope for much. Often, however, a particular partition has some additional properties which arise from the situation at hand.

Suppose, for instance, that we know that $X$ is a collection of intervals in a linear order and that a pair of intervals $(I, J)$ is in $K_0$ if $I$ and $J$ are disjoint and in $K_1$ otherwise. Whether or not such partitions have uncountable homogeneous sets turns out to be equivalent to a famous problem of Souslin [58] and independent of the usual axioms of set theory (unlike Sierpiński’s example above).

The first three chapters of this thesis will provide some of the history of Souslin’s problem and the program it generated and some of the current research being done in this area. In Chapter 2 I will focus on constructing compact spaces which behave like Souslin lines but which require only mild set theoretic assumptions.

**Theorem.** $(\aleph_1 < \text{add}(\mathcal{L}_1) = \aleph) \text{ There is a compact space satisfying Souslin's condition which is nonseparable and does not map onto } [0, 1]^{\aleph_1}.$

$\mathcal{K}_n$ $(n < \omega)$ is a system of axioms which arose though a careful analysis of Martin’s Axiom its relation to the identification of chain conditions. A program has been launched to study which consequences of $\text{MA}_{\aleph_1}$ can be
derived from the various axioms $\mathcal{K}_n$. I will make the following additions to the known results.

**Theorem.** $(K_2)$ Every set of reals of size $\aleph_1$ has measure 0.

**Theorem.** $(K_3)$ Every union of $\aleph_1$ measure 0 sets has measure 0.

Sierpiński’s partition is defined directly from some well-ordering of the continuum. The partition itself turns out to be a very wild subset of $[\mathbb{R}]^2$ from the point of view of measure and category. It is therefore natural to ask when there are partitions $[X]^2 = K_0 \cup K_1$ on a set of reals $X$ which have properties similar to those of Sierpiński’s but such that $K_0$ and $K_1$ are, say, open subsets of $[X]^2$. It turns out that this question has a very different flavor to it and that there existence of such colorings depend largely on the size of $X$. This will be the topic of our study in Chapter 4. I will demonstrate the following.

**Theorem.** If $X \subseteq \mathbb{N}^\mathbb{N}$ is $2^\mathbb{N}$ thin for some monotonic unbounded $f$ then there is a continuous map $c : [X]^2 \to \mathbb{N}$ such that if $Y \subseteq X$ and $c''[Y]^2 \neq \mathbb{N}$ then $Y$ is $f$ thin.

**Theorem.** There is a $\sigma$-ideal of measure 0 sets $\mathcal{N}_*$, a set $X \subseteq \mathbb{N}^\mathbb{N}$ of size $\text{add}(\mathcal{N}_*)$, and a continuous coloring $c : [X]^2 \to \mathbb{N}$ such that $c''[Y]^2 = \mathbb{N}$ whenever $Y \subseteq X$ has the same cardinality as $X$.

**Theorem.** There is a continuous map $\text{alt} : [\mathbb{P}]^2 \to \mathbb{N}$ such that if $X$ is $\sigma$-directed and unbounded in $(\mathbb{P}, \prec^*)$ then $\text{alt}''[X]^2 = \mathbb{N}$.
The results of this chapter make some progress towards answering a problem of Zapletal asking whether $\text{OCA}$ implies $\text{add}(\mathcal{N}) > \aleph_1$ and a problem of Todorčević asking whether $\text{OCA}$ implies $\mathfrak{c}$ is $\aleph_2$.

Sometimes, when considering applications, it is unnecessary to demand a homogeneous “square” for a partition. Hereditary separability (HS) and the hereditary Lindelöf (HL) property are two topological properties which are equivalent in the class of metric spaces. Whether these two properties can be equivalent in the class of all regular topological spaces is a well known open problem in general topology. It turns out that this question is very closely related to the following question which clearly has a Ramsey theoretic nature.

**Question.** [25] If $[\omega_1]^2 = K_0 \cup K_1$ is a partition of the pairs of the first uncountable cardinal, must there exist uncountable sets $A$ and $B$ such that for some $i = 0, 1$ the pair $\{\alpha, \beta\}$ is in $K_i$ whenever $\alpha$ is in $A$, $\beta$ is in $B$ and $\alpha < \beta$?

While the relationship between HS and HL is an interesting topic of study in its own right, it turns out to have numerous applications in general topology and analysis. Many problems which seem unrelated to the relationship between HS and HL boil down to the above “rectangle problem” and hence do share a deep connection to this Ramsey theoretic consideration.

In some cases it is very important to know how identifying HS and HL in a certain class of topological spaces affects the status of some other combinatorial statement. In Chapters 5-7 we will consider the behavior of HS and HL in the presence of forcing with a measure algebra. A group of problems
in general topology require knowledge of how close the properties HS and HL are to one another in the presence of the inequality $2^{\aleph_0} < 2^{\aleph_1}$. It turns out that one such problem is the following question of Katětov.

**Question.** [35] If $X$ is a compact topological space and $X^2$ is hereditarily normal, must $X$ be metrizable?

Not only does the study of measure theoretic forcing seem as though it will shed light on this area, but this approach is somewhat unique in that it requires an analysis which is often more mathematical than metamathematical. Among the theorems I will prove are the following.

**Theorem.** ($\text{MA}_\theta$) $W(\theta)$ holds after forcing with any measure algebra.

Here $W(\theta)$ is a combinatorial statement studied by M. Wage because of its impact on the relationship between HS and HL.

**Theorem.** ($\text{MA}_{\aleph_1}$) After forcing with any measure algebra

$$\omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2$$

for all $\alpha < \omega_1$.

**Theorem.** ($\text{MA}_{\aleph_1}$) After forcing with a separable measure algebra, HS and HL are equivalent in the class of cometrizable spaces.

**Theorem.** After forcing with a homogeneous nonseparable measure algebra there is a Ostaszewski space. In particular HS does not imply HL in the class of compact spaces.
This last result answers a question of Todorčević and contrasts his result that $\text{MA}_{\aleph_1}$ implies "HL implies HS in the class of compact spaces after forcing with any measure algebra."

Some of the material in this thesis either has been published or will be published in the coming year. Chapter 2 is essentially a reproduction of [46]. The latter section of Chapter 3 and all but the final section of Chapter 4 are reproduced from [45] with permission from Elsevier Science.
Chapter 1

Souslinean Spaces

A well known theorem of Cantor and Dedekind [11] (see [12] for a translation) states that \((\mathbb{R}, \leq)\) is the only linear order which is

1. dense with no first or last element,

2. order complete, and

3. separable

In 1920 Souslin [58] asked whether separability could be relaxed to the following condition:

\textbf{Souslin’s Condition}: Every family of pairwise disjoint intervals is countable.

It was known to Souslin that this problem is really a question of whether Souslin’s condition and separability are equivalent in the class of linear orders. While this question turned out to be undecidable in ZFC, this question
has led to a number of developments in set theory and topology, both at mathematical level and at a metamathematical level and continues to fuel research to this day.

Some of the first progress on this problem was made by Kurepa in 1935. In [38] Kurepa demonstrated that the theory of trees is quite close to the theory of linear orders (see also [43], [53]). Here a tree \((T, \prec)\) is a partially ordered set in which all predecessors of an element of \(T\) are well ordered by \(\prec\). Kurepa showed that every linear order has a \(\pi\)-base \(B\) of intervals such that \(B\) is a tree when ordered by \(\supset\). He established that Souslin's problem is equivalent to the question of whether every tree in which all chains and antichains are countable is itself countable. This formulation of Souslin's problem is perhaps the one which is most commonly quoted today. Notice that this statement of Souslin's problem is purely Ramsey theoretic in nature.

### 1.1 Local and Global Chain Conditions

Further progress was made on Souslin's problem by Knaster and Szpilrajn in the Scottish Book (see Problem 192 in [44]). Knaster introduced the following condition in the context of compact topological spaces.

**Knaster's Condition:** Every uncountable family of open sets contains an uncountable linked family.

Here a family is *linked* if it has the pairwise intersection property. Notice that Souslin's condition naturally generalizes to topological spaces as well and that Knaster's condition clearly implies it.
Knaster demonstrated that Souslin's problem is equivalent to whether Souslin's condition implies Knaster's condition in the class of linear orders (see [44]). Knaster, a geometer, was motivated by Helly's theorem [32] which states that a collection of intervals is linked if and only if it is centered. Knaster and Szpilrajn asked whether there is a topological space which satisfies Souslin's condition but not Knaster's condition.

Another consideration investigated by Knaster and Szpilrajn in the Scottish Book was whether Souslin's condition is preserved by products. They remarked that if Souslin's condition is preserved in the product of two spaces, then it is preserved by arbitrary products. They also showed that Knaster's condition is preserved by arbitrary products. In [54] Shanin, motivated by this result, introduced his own chain condition, which I will denote Shanin's Condition, and showed that it is also preserved by arbitrary products.

**Shanin's Condition:** Every uncountable family contains an uncountable centered family.

Chain conditions such as those of Souslin, Knaster, and Shanin are sometimes referred to as local chain conditions. They assert that any large enough family contains a large subcollection with some additional intersection properties.

Global chain conditions, such as separability, were studied by Horn and Tarski in [33] in the context of von Neumann's problem (see problem 163 in [44]). These chain conditions assert that the open sets of a topological space can be decomposed into countably many families, each of which has some specified intersection properties. In compact spaces separability asserts, for
instance, that the open sets of a topological space can be decomposed into countably many families each with the finite intersection property.

1.2 Souslin's Program

The problem of when (and whether) topological spaces identify chain conditions has become a prominent theme in set theory and topology. In the 1960's and 1970's, Jech, Solovay, and Tennenbaum completely resolved Souslin's problem by showing that it could neither be proven nor refuted using the usual axioms of set theory (see [34], [57], and [63]). Solovay and Tennenbaum's proof in [57] that Souslin's problem can consistently have a positive answer led to the consideration of an important new axiom known today as Martin's Axiom. While Souslin's problem has been resolved, his program remains — the study of chain conditions and their relationships. The focus of the next two chapters will be to explore two modern themes in this program.

The first general program can be considered as an analysis of the interaction between Souslin's condition and the global chain conditions. For example, if a topological space distinguishes between Souslin's condition and separability, what else can be said about the space? Must it map onto \([0, 1]^{ω_1}\)? Typically the analysis of such a question is carried out under a strong Baire category assumption such as \(\text{PFA}\) or \(\text{MA}_{ω_1}\). This is because it is well known to be consistent that objects such as Souslin lines do exist. In Chapter 2 we will investigate some of the limitations of this program by demonstrating how to build nonseparable compacta which satisfy Souslin's condition but which
are in some sense "small."

The second program is to analyze the relationships of the local chain conditions. This program has its roots in Knaster's original consideration of his chain condition, as well as in results of Kunen, Rowbottom, and Solovay concerning the identification of Souslin's condition and Shanin's condition under $\text{MA}_{\aleph_1}$ (an important variation of Martin's Axiom). Current motivation for this program, however, comes out of the result of Todorčević and Veličković in [83] which states that $\text{MA}_{\aleph_1}$ is that same as equating Souslin’s condition and Shanin’s condition in the class of compact topological spaces. What remains is the question of whether Shanin’s condition can be relaxed to some other local chain condition in this characterization (see problem BY of [24]). Chapter 2 will focus on some of the consequences of $\text{MA}_{\aleph_1}$ which can be proven by identifying Souslin's condition with a local chain condition such as Knaster's condition.

1.3 Compacta, Boolean Algebras, Partial Orders, and Partitions

The last two sections of this chapter are intended to provide a little background for some of chapters to come. It turns out that discussions of topics related to Souslin’s program can often be carried out in a number of different but essentially equivalent contexts. The context which is most appropriate generally tends to depend on history, established convention, and occasion-
ally convenience.

**Definition 1.3.1.** A *Boolean algebra* is a ring \((B, +, \cdot)\) with 1 such that \(a^2 = a\) for every element of \(B\).

If \((B, +, \cdot)\) is a Boolean algebra then one can define the operations

\[
\begin{align*}
a \land b &= a \cdot b \\
a \lor b &= a + a \cdot b + b \\
a^c &= 1 + a.
\end{align*}
\]

It is routine to verify that \(\land, \lor,\) and \(\cdot^c\) behave just as \(\cap, \cup,\) and complementation would on a field of sets. This was explained by the following representation theorem of Stone.

**Theorem 1.3.2.** [59] For every Boolean algebra \(B\) there is a 0-dimensional compact space \(X\) such that \(B\) is isomorphic to the collection of clopen sets \(C\) of \(X\) equip with the operations union and intersection (representing \(\lor\) and \(\land\) respectively).

The compact space in the previous theorem is called the *Stone space* of \(B\) and denoted \(st(B)\). It consists of the collection of all maximal centered families (called *ultrafilters*) in \(B\). It is easy to see that the chain conditions we defined earlier have natural translations in the class of Boolean algebra. Thus, in addition to compact spaces, Boolean algebras are another context in which one can discuss Souslin’s program.

If \(B\) is a Boolean algebra then by defining \(a \leq b\) iff \(a \land c = 0\) whenever \(b \land c = 0\) we can equip \(B\) with a natural partial order (which becomes
containment in $\text{st}(B))$. Thus a Boolean algebra is a specific instance of a partial order $(\mathcal{P}, \leq)$. With the Boolean algebraic example in mind, one can define $G \subseteq \mathcal{P}$ to be linked (centered) if every pair (finite subset) of element of $G$ has a lower bound in $\mathcal{P}$. If $G$ is centered and the lower bound can be found in $G$ then $G$ is called a filter. Again, all of the chain conditions mentioned above transfer to the class of partial orders as well. Moreover, for every partial order, there is a Boolean algebra (and hence a compact space) which is equivalent to it as far as the study of chain conditions is concerned (see [37]).

Another special instance of a partial order arises from Ramsey theory. If $S$ is a set and $K$ is a subset of $[S]^n$ then we can consider the collection $Q$ of all finite subsets $H$ of $S$ such that $[H]^n \subseteq K$ ordered by $\supseteq$. One can then define a Boolean algebra from $Q$. Notice that the collection of all $H \subseteq S$ such that $[H]^n \subseteq K$ then this is a compact space when given the subspace topology from $\mathcal{P}(S)$ (see [80]). The translation of Souslin's condition in the class of partitions is "whenever there is an uncountable family $\mathcal{F}$ of disjoint finite sets $F$ such that $[F]^n \subseteq K$, then there is a pair $E \neq F$ in $\mathcal{F}$ such that $[E \cup F]^n \subseteq K."$

Generally the terms used to represent the chain conditions remain the same in these classes. There are some exceptions. Separability in the class of compacta is called $\sigma$-centered in the class of Boolean algebras and partial orders. Souslin's condition, even in the class of compacta, is a somewhat dated term. I have chosen to use it for the discussion of Souslin's program for aesthetic and historical reasons. It is better known today as the countable
chain condition or c.c.c.. We will use the latter terminology when we are discussing partial orders in the later chapters of this thesis.

1.4 Forcing Axioms

In the course of solving the Continuum Problem, Cohen developed a general method known as "forcing" which has become widely used in producing independence results ever since its introduction in the 1960's. Jech and Tennenbaum used it to show that Souslin's problem can have a negative answer. Solovay and Tennenbaum later used it to establish the consistency of a positive answer to Souslin's problem.

Martin noticed that Solovay and Tennenbaum's techniques could be used to prove the consistency of a more general fact about Baire category:

**Martin's Axiom:** Any compactum which satisfies Souslin's condition can not be covered by fewer than continuum many nowhere dense sets.

Martin's Axiom is always true if the Continuum Hypothesis holds. Solovay and Tennenbaum demonstrated that it is consistent with the negation of the Continuum Hypothesis as well. An important variation on this axiom which is actually more relevant to Souslin's program (for $\theta = \aleph_1$) is the following.

**$\text{MA}_\theta$:** Any compactum which satisfies Souslin's condition can not be covered by $\theta$ nowhere dense sets.

Martin's Axiom and $\text{MA}_\theta$ also have formulations in terms of partial orders and Boolean algebras. The former is probably the most useful.
MA$_\theta$: If $(P, \leq)$ is a c.c.c. partial order and $D_\alpha$ ($\alpha < \theta$) is a sequence of dense subsets of $P$ then there is a filter $G \subseteq P$ which intersects each $D_\alpha$.

Here $D \subseteq P$ is dense if for every $p$ in $P$ there is a $d$ in $D$ such that $d \leq p$.

Martin's Axiom is what is sometimes known as a forcing axiom. If one replaces the countable chain condition with some other condition in the definition of MA$_\theta$ then the result is a different forcing axiom. Probably the best known of these forcing axioms is the Proper Forcing Axiom or PFA. As even a brief introduction to PFA would be beyond the scope of this work, we leave this topic to the interested reader. For us it will be enough to know that PFA is a strengthening of MA$_\aleph_1$ which has various additional consequences which shall be mentioned at the appropriate junctures.
Chapter 2

Analysis of Global Chain Conditions

One direction which can be taken from the solution of Souslin’s problem is to ask how strong of a positive answer to Souslin’s problem one can expect to prove is consistent. Since linear compacta are rather simple examples as far as compact topological spaces go, it is reasonable to ask how complex compacta must be if they distinguish between Souslin’s condition and one of the global chain conditions such as separability. Perhaps the simplest examples of compacta which make such a distinction are the large Tychonoff cubes. By Knaster and Szpilrajn’s result, \([0, 1]^I\) always satisfies Knaster’s condition and hence Souslin’s condition. On the other hand, if \(I\) is larger than \(\mathbb{R}\), then \([0, 1]^I\) is never separable.

Treybig and Ward have shown (see [84]) that if \(L\) is a linear compactum and \(L\) maps continuously onto \(X \times Y\) where \(X\) and \(Y\) are infinite compacta
then \( X \) and \( Y \) are metrizable. Thus linear compacta are rather far from mapping onto \([0, 1]^{\omega_1}\) and hence it is natural to ask whether it is consistent that all compacta which distinguish between Souslin’s condition and one of the global chain conditions must map onto \([0, 1]^{\omega_1}\).

The following topological property is useful in understanding why certain compacta don’t map onto \([0, 1]^{\omega_1}\).

**Definition 2.0.1.** A compact space \( K \) is said to be *linearly fibered* if there is a continuous map \( \phi : K \to [0, 1] \) such that the inverse images of points are linearly compacta.

**Fact 2.0.2.** If \( K \) is a compact linearly fibered space then \( K \) does not map onto \([0, 1]^{\omega_1}\).

*Proof.* Clearly being a linearly fibered compactum is inherited to all closed subspaces. Consequently every closed subset \( E \) of a linearly fibered compact space \( X \) contains a linearly orderable subspace which is a \( G_\delta \) set. Since every linear compactum contains a point of countable \( \pi \)-character, \( E \) must contain a point of countable \( \pi \)-character and therefore \( X \) cannot map onto \([0, 1]^{\omega_1}\) by a well known result of Shapirovskii[55].

The purpose of this chapter is to provide a general method for constructing a compactum from a set theoretic object known as a gap. The construction will be carried out in such a way that the combinatorial properties of the gap translate into various topological properties of the resulting compactum. In particular there are restrictions on the gap which ensure that the
resulting space will be nonseparable, satisfy Souslin's condition, be linearly fibered, and so on. Moreover, the gaps required to yield relevant compacta require only mild set theoretic assumptions (e.g. compatible with $\text{MA}_\kappa$).

The bulk of this chapter will be devoted to the method which converts a gap into a topological space and a discussion of how the properties of the gap used in the construction are reflected in the resulting compactum. It will frequently be more convenient for us to work with Boolean algebras and then consider their Stone spaces. Consequently, most of the arguments and language which follows is algebraic in nature.

At the end of this chapter we will consider two applications of this method. One of them will be to produce what is essentially a ZFC example of a compactum which is linearly fibered but which still distinguishes between Souslin's condition and separability. The second application uses the assumption $b > \aleph_2$ to build another linearly fibered compactum, this time differentiating between Souslin's condition and being $\sigma$-linked.

## 2.1 Gaps and Souslinean Spaces

It is frequently the case in general topology that set theoretic objects prove to be rather useful in the construction of topological spaces. One example of such an object is a gap in the structure $([\mathbb{N}]^\omega, \subseteq^*)$. Classically speaking, a gap is a pair of sequences $\{a_\xi : \xi < \kappa\}$ and $\{b_\eta : \eta < \lambda\}$ of infinite subsets of $\mathbb{N}$ such that

1. $a_\xi \subseteq^* a_\eta$, $b_{\xi'} \subseteq^* b_{\eta'}$ for all $\xi < \eta < \kappa$ and $\xi' < \eta' < \lambda$, 
2. \( a_\xi \cap b_\eta \) is finite for all \( \xi < \kappa \) and \( \eta < \lambda \), and

3. there does not exist a \( c \subseteq \mathbb{N} \) with \( a_\xi \subseteq^* c \) and \( c \cap b_\eta \) finite for all \( \xi < \kappa \) and \( \eta < \lambda \).

Gaps always exist by a standard application of Zorn’s Lemma, but the existence of a gap with some additional properties quickly becomes something which cannot be decided within ZFC (for further reading on gaps, see e.g. [79] or [74]).

It turns out that gaps can be quite useful in defining topological spaces which bear some resemblance to Souslin lines (see, e.g. [9] and [66]). In this chapter we will present a generalization of a construction from an early version of [66]. Before proceeding it is necessary to define some notation as well as provide a rough framework for our Boolean algebra. The following gives us a more general formulation of the notion of a gap.

**Definition 2.1.1.** An ideal \( \mathcal{I} \) on \( \mathbb{N} \) is a subset of \( \mathcal{P}(\mathbb{N}) \) which is closed under taking finite unions and subsets. In addition \( \mathcal{I} \) is said to be a \( P \)-ideal if \((\mathcal{I}, \subseteq^*)\) is \( \sigma \)-directed (every countable set has an upper bound in \( \mathcal{I} \)). An ideal on \( \mathbb{N} \) is dense if every infinite set contains an infinite subset in the ideal.

**Definition 2.1.2.** If \( a \) and \( b \) are subsets of \( \mathbb{N} \) and \( \mathcal{I} \) is an ideal on \( \mathbb{N} \) then \( a \subseteq_\mathcal{I} b \) abbreviates \( a \setminus b \in \mathcal{I} \) and \( a \perp_\mathcal{I} b \) abbreviates \( a \cap b \in \mathcal{I} \). A pair \((A, B)\) of subsets of \( \mathcal{P}(\mathbb{N}) \) is said to be orthogonal modulo an ideal \( \mathcal{I} \) on \( \mathbb{N} \) (or orthogonal in \( \mathcal{P}(\mathbb{N})/\mathcal{I} \)) if \( a \perp_\mathcal{I} b \) whenever \( a \) is in \( A \) and \( b \) is in \( B \).
Definition 2.1.3. A subset $c$ of $\mathbb{N}$ is a said to split a set $\Gamma \subseteq A \times B$ modulo an ideal $\mathcal{I}$ if $a \subseteq c$ and $b \perp c$ whenever $(a, b)$ is in $\Gamma$. If there does not exist a $c$ which splits a subset $\Gamma$ of $A \times B$ and $A$ is orthogonal to $B$ then $\Gamma$ is said to be a gap modulo $\mathcal{I}$ (or a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$). If $\Gamma = A \times B$ then I will simply say $(A, B)$ is a gap modulo $\mathcal{I}$.

The advantage of this more general definition is that is allows us to examine gaps which are definable — something which will be of use to us later. The typical definability restriction on ideals is that they are analytic, i.e. the continuous image of a Polish space. It is also that case that it is necessary to consider gaps in algebras other than $\mathcal{P}(\mathbb{N})/\text{fin}$ in order to get results in ZFC (see the remark in [66] following Theorem 8.4). For more information on definable gaps the reader is referred to [64] and [71].

From this point on $(A, B)$ will be a gap modulo an $F_\sigma$ P-ideal $\mathcal{I}$. The ideal $\mathcal{I}$ will moreover be generated by the collection of all finite changes of some compact set $\mathcal{K} \subseteq \mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ (it is easy to verify that in fact all $F_\sigma$ P-ideals are of this form). I will also assume that all finite sets are in $\mathcal{I}$ and that $\mathcal{K}$ is closed under subsets.

The following definitions are modeled after those of an earlier version of [66] which dealt with the special case $\mathcal{I} = \text{fin}$ and $\mathcal{K} = \{\emptyset\}$. Let

$$T = \{(t, n) : t \subseteq [0, n - 1]\}$$

and define $(s, m) < (t, n)$ to be end extension — that is $m < n$ and $t \cap [0, m - 1] = s$. For $n \in \mathbb{N}$, set $\mathcal{K}_n = \{K \cap [0, n - 1] : K \in \mathcal{K}\}$. Instead of considering an arbitrary member of $A \times B$, it will be necessary to restrict our attention
Note that for every \( a \in A, b \in B \) there is an \( n \) such that
\[
(a \setminus [0, n - 1], b \setminus [0, n - 1])
\]
is in \( A \otimes B \) and hence \( A \otimes B \) is also a gap modulo \( \mathcal{I} \).

In addition to \( A, B, \mathcal{I}, \) and \( \mathcal{K} \), the parameters of our Boolean algebra will also include a subset \( \Gamma \) of \( A \otimes B \) which also forms a gap modulo \( \mathcal{I} \). If \((a, b) \in \Gamma \) and \((t, n) \in T\) then define

1. \( T_{(a,b)} = \{(s, m) \in T : a \setminus s \cap [0, m - 1] \in \mathcal{K}_m \) and \( b \cap s \in \mathcal{K}_m \} \).
2. \( T_{(t,n)} = \{(s, m) \in T : ((s, m) < (t, n)) \) or \((t, n) < (s, m)) \}\).

From this define the Boolean algebras

1. \( B = \langle T_{(a,b)}, T_{(t,n)} : (a, b) \in \Gamma, (t, n) \in T \rangle / \text{fin} \).
2. \( A = \langle T_{(t,n)} : (t, n) \in T \rangle / \text{fin} \).

It is now useful to make some observations. First note that \( (T, <) \) is a finitely branching tree. The following two facts are useful in dealing with elements of \( B \).

**Fact 2.1.4.** \( a \) is in \( \mathcal{K} \) if and only if \( a \cap [0, n - 1] \) is in \( \mathcal{K}_n \) for all \( n \).

**Proof.** This follows from the compactness of \( \mathcal{K} \). \( \square \)

**Fact 2.1.5.** If \( F \) is a positive element of \( B \) then there is a finite collection of generators whose meet is positive and contained in \( F \).
Proof. Observe that if \( x \) is the complement of a generator and \((t, n)\) is in \( x \), then

\[
T_{(t,n)} \subseteq^* \{(s, m) \in T : (t, n) < (s, m)\} \subseteq x
\]

(for type 1 generators this is a consequence of the previous fact). By considering the disjunctive normal form of \( F \) and applying this observation the fact follows immediately.

\[\square\]

2.2 Being Separable

An ultrafilter \( v \) in \( A \) records a unique subset \( c \) of \( \mathbb{N} \) which splits a portion of the gap (for every \( n \) there is a unique \( t_n \subseteq n \) such that \( T_{(t_n, n)} \in v \) — let \( c = \bigcup_{n=1}^{\infty} t_n \)). If this ultrafilter is extended to a filter in \( B \) which contains type 1 generators, then the pairs \((a, b)\) corresponding to these generators must be split modulo \( K \) by the set \( c \) (note that even though \( K \) is not an ideal, this notion still makes sense). The following proposition gives us our first indication of how properties of the gap \( \Gamma \) translate into chain conditions in the Boolean algebra \( B \).

Proposition 2.2.1. If both \( A \) and \( B \) are \( \sigma \)-directed under \( \subseteq^* \) then \( B \) is not \( \sigma \)-centered (and hence \( \text{st}(B) \) is not separable).

Proof. Suppose \( B \) is the union of countably many ultrafilters \( \{v_n\}_{n=1}^{\infty} \). Then it is possible to find a countable sequence \( c_n \) of subsets of \( \mathbb{N} \) which correspond to the unique infinite branch each \( v_n \) determines in \((T, <)\). For each \( n \) pick a pair \((a_n, b_n) \in A \otimes B\) which is not split by \( c_n \) modulo \( I \) (i.e. either \( a_n \nsubseteq I c_n \)).
or \( b_n \not\in \mathcal{X} c_n \)\). Since both sides of the gap are \( \sigma \)-directed, find a pair \((a, b)\) such that \( a_n \subseteq^* a \in A \) and \( b_n \subseteq^* b \in B \) for all \( n \). Notice that we may assume \((a, b)\) is in \( A \otimes B \). Pick a \( m \) such that \( T_{(a, b)} \) is in \( \nu_m \). The set \( c_m \) splits \((a, b)\) modulo \( \mathcal{K} \) and hence splits \((a_m, b_m)\) modulo \( \mathcal{I} \), a contradiction. \qed

\section{2.3 Souslin’s Condition}

Element of the Boolean algebra \( \mathcal{B} \) can be thought of as a collection of splitters for some portion of the gap \( \Gamma \) modulo the compact set \( \mathcal{K} \). If we are given that in \( A \)

1. every uncountable \( C \subseteq A \) contains an uncountable \( C_0 \) which is \( \subseteq^* \) bounded

then there is a standard approach to showing that \( \mathcal{B} \) satisfies Souslin’s condition. The general idea is as follows. If \( \mathcal{F} \subseteq \mathcal{B} \) is uncountable, then consider the members of \( A \) used in the definitions of the members of \( \mathcal{F} \). Using 1, refine \( \mathcal{F} \) to an uncountable subfamily \( \mathcal{F}_0 \) for which there is a \( c_0 \) in \( A \) which bounds any \( a' \) such that \( T_{(a', a')} \) is mentioned in \( \mathcal{F}_0 \). By making a finite modification to \( c_0 \), it is possible to produce a \( c \subseteq \mathbb{N} \) which splits many members of \( \mathcal{F}_0 \).

The real trick turns out to be how to make this finite modification. Loosely speaking, we are given a collection \( \mathcal{F} \) of finite pieces of the gap, where each \( F \in \mathcal{F} \) is split by some “local” splitter \( c_F \) modulo \( \mathcal{K} \). We are also given some “global” splitter \( c \) which works for all of the pieces in \( \mathcal{F} \), but only modulo the larger object \( \mathcal{I} \). The goal is to repair \( c \) by altering some finite portion of it so that it also splits many members of \( \mathcal{F} \) modulo \( \mathcal{K} \).
CHAPTER 2. GLOBAL CHAIN CONDITIONS

The sufficient condition which I will use is that \((A,B)\) is actually orthogonal modulo a smaller ideal \(\mathcal{J}\) which satisfies the following “exchange” property:

2. For every \(J\) in \(\mathcal{J}\) there are infinitely many \(n\) such that for every \(K \in \mathcal{K}\) the set \((J \setminus [0, n - 1]) \cup (K \cap [0, n - 1])\) is in \(\mathcal{K}\).

We are now ready to prove the following proposition about \(B\).

**Proposition 2.3.1.** If \(A, B, \mathcal{I}\) and \(\mathcal{J}\) satisfy 1 and 2 then \(B\) satisfies Shanin’s condition and in particular satisfies Souslin’s Condition.

**Proof.** Pick an uncountable family \(\mathcal{F}\) of positive elements of \(B\). Applying Fact 2.1.5 it may be assumed without loss of generality that the members of \(\mathcal{F}\) are the meet of finitely many generators. For each \(F \in \mathcal{F}\) pick a finite set \(S_F \subseteq \Gamma\) and a \((t_F, n_F)\) in \(T\) such that

\[ F = T(t_F, n_F) \cap \bigcap_{(a,b) \in S_F} T(a,b). \]

Applying 1 it is possible to find an uncountable \(\mathcal{F}_0 \subseteq \mathcal{F}\) and a \(c_0 \in A\) such that \(a \subseteq^* c_0\) whenever \(a \in \pi_A(S_F)\) and \(F \in \mathcal{F}_0\). If \(F \in \mathcal{F}_0\), let

\[ J_F = \bigcup_{(a,b) \in S_F} [(a \setminus c_0) \cup (b \cap c_0)] \]

and choose a \(c_F\) such that \((c_F \cap n, n) \in F\) for all \(n\).

Applying 2 find a \(N_F > n_F\) such that

\[ (J_F \setminus [0, N_F - 1]) \cup (K \cap [0, N_F - 1]) \]
CHAPTER 2. GLOBAL CHAIN CONDITIONS

is in \( \mathcal{K} \) whenever \( K \) is in \( \mathcal{K} \). Now pick an uncountable subset \( \mathcal{F}_1 \) of \( \mathcal{F}_0 \) such that \( N_F = N \) and \( c_F \cap [0, N - 1] = t \) for some fixed \( N \in \mathbb{N} \) and \( t \subseteq [0, N - 1] \) whenever \( F \in \mathcal{F}_1 \). Let \( c = (c_0 \setminus [0, N - 1]) \cup t \).

I will now show that \((c \cap n, n) \in F\) for all \( F \in \mathcal{F}_1 \) and \( n \in \mathbb{N} \). Let \( F \in \mathcal{F}_1 \) and \((A, B) \in S_F \). Since \( F \subseteq T_{(t, N)} \cap T_{(a, b)} \) is nonempty,

\[
K_a = a \cap [0, N - 1] \setminus t \quad \text{and} \quad K_b = b \cap [0, N - 1] \cap t
\]

are both in \( \mathcal{K}_N \subseteq \mathcal{K} \). It follows from the choice of \( N \) that

\[
K_a \cup (J_F \setminus [0, N - 1]) \quad \text{and} \quad K_b \cup (J_F \setminus [0, N - 1]) \in \mathcal{K}.
\]

Thus

\[
a \setminus c \subseteq K_a \cup (J_F \setminus [0, N - 1]) \quad \text{and} \quad b \cap c \subseteq K_b \cup (J_F \setminus [0, N - 1])
\]

are both in \( \mathcal{K} \) and therefore \((c \cap n, n) \in T_{(a, b)} \) for all \( n \).

2.4 Being Linearly Fibered

The property of being linearly fibered is, not surprisingly, closely connected to whether the gap is linear.

**Proposition 2.4.1.** If \( \Gamma \) is well ordered by \( \subseteq^* \times \subseteq^* \) then the map \( \phi : \text{st}(B) \to \text{st}(A) \) defined by \( \phi(u) = u \upharpoonright A \) has fibers which are linear compacta.

**Notation.** In this situation we will let \( \Gamma = \{(a_\xi, b_\xi) : \xi < \kappa \} \) be an increasing enumeration of \( \Gamma \). For simplicity we will write \( T_\xi \) instead of \( T_{(a_\xi, b_\xi)} \).
CHAPTER 2. GLOBAL CHAIN CONDITIONS

Proof. Let $v$ be an ultrafilter on $A$ and define $\Gamma_v$ to be the collection of all $\xi < \lambda$ for which $\{T_\xi\} \cup v$ is a filter. It now suffices to show that if $\eta < \xi \in \Gamma_v$ then $T_\eta \upharpoonright v \supseteq T_\xi \upharpoonright v$.

Pick a $m \in \mathbb{N}$ such that

$$a_\eta \setminus [0, m - 1] \subseteq a_\xi \setminus [0, m - 1],$$
$$b_\eta \setminus [0, m - 1] \subseteq b_\xi \setminus [0, m - 1]$$

and let $s \subseteq [0, m - 1]$ be the unique set such that $T_{(s,m)} \in v$. If $(s, m) < (t, n)$ is in $T_\xi$ then

$$a_\xi \cap [m, n - 1] \setminus t \in \{K \cap [m, n - 1] : K \in \mathcal{K}\},$$
$$b_\xi \cap [m, n - 1] \cap t \in \{K \cap [m, n - 1] : K \in \mathcal{K}\}.$$

Since

$$a_\eta \cap [m, n - 1] \subseteq a_\xi \cap [m, n - 1],$$
$$b_\eta \cap [m, n - 1] \subseteq b_\xi \cap [m, n - 1],$$

we also have that

$$(a_\eta \cap [m, n - 1]) \setminus t \in \{K \cap [m, n - 1] : K \in \mathcal{K}\},$$
$$(b_\eta \cap [m, n - 1]) \cap t \in \{K \cap [m, n - 1] : K \in \mathcal{K}\}.$$

Because $T_\eta \cap T_{(s,m)} \neq \emptyset$, $(a_\eta \cap [0, m - 1]) \setminus t \in \mathcal{K}_m$. Therefore $(a_\eta \cap [0, n - 1]) \setminus t \in \mathcal{K}_n$ and $b_\eta \cap [0, n - 1] \cap t \in \mathcal{K}_n$. Thus $(t, n) \in T_\eta$ and $T_\eta \upharpoonright v \supseteq T_\xi \upharpoonright v$. □

2.5 Helly's Condition

Interval algebras have a very interesting intersection property discovered by Helly in [32]: a collection of intervals is linked iff it is centered. Motivated by this I will define
**Helly’s Condition:** There is a base of sets $S$ such that every linked sub-collection of $S$ is centered.

Clearly if a $\sigma$-linked Boolean algebra satisfies Helly’s condition then it is actually $\sigma$-centered. It turns out that Helly’s condition corresponds to a natural consideration in our method of construction as well.

**Proposition 2.5.1.** If $\mathcal{K}$ is an ideal then $B$ satisfies Helly’s condition.

**Remark 2.5.2.** Compact ideals are always trivial — they are of the form $\mathcal{P}(A)$ for some subset $A$ of $\mathbb{N}$.

**Proof.** The base in question is the set of generators. First notice that any linked collection of generators of the form $T(t,n)$ is centered. With a slight amount of argument it is easy to verify that it is enough to consider only linked collections of the generators $T(a,b)$. Suppose that $F \subseteq \Gamma$ is finite and $\{T(a,b) : (a,b) \in F\}$ is linked. This means that for every pair $(a,b), (a',b')$ in $F$ there is a $c \subseteq \mathbb{N}$ such that $a, a' \subseteq \mathcal{K} c$ and $b, b' \perp_{\mathcal{K}} b$ are in $\mathcal{K}$. Since $\mathcal{K}$ is closed under finite unions, this implies that $(a \cap b') \cup (a' \cap b)$ is in $\mathcal{K}$. From this it follows that

$$c' = \bigcup_{(a,b) \in F} a$$

has the property that $c' \cap b$ is in $\mathcal{K}$ whenever $(a,b)$ is in $\mathcal{F}$. Hence $c \cap [0,n-1]$ is in $T(a,b)$ for every $n$ whenever $(a,b)$ is in $F$. $\square$
2.6 An Example From $\aleph_1 < \text{add}_*(\ell_1) = c$

For the first application of the above construction, we will build a linearly fibered compactum which satisfies Souslin's condition (even Shanin's condition) but which is nonseparable. The set theoretic assumption which we will use in the construction is minor in the sense that it actually follows from PFA, an axiom which should identify the chain conditions in as many compacta as possible. Our assumption was removed in a related construction due to Todorčević of a space with the same properties as ours (see Theorem 8.4 in [66] and the remark which follows).

The assumption which we will use is $\aleph_1 < \text{add}_*(\ell_1) = c$. Working in ZFC we will build a pair of analytic $\mathcal{P}$-ideals $A$ and $B$ which form a gap both over their intersection $\mathcal{J}$ and also over a larger $\mathcal{I}$ which is an $F_\sigma$ $\mathcal{P}$-ideal. The assumption $\aleph_1 < \text{add}_*(\ell_1) = c$ will allow us to find cofinal $\subseteq^*$ chains $A_0 \subseteq A$ and $B_0 \subseteq B$, each of which is $\aleph_1$-directed. $\Gamma$ is then chosen appropriately inside of $A_0 \otimes B_0$.

It has recently been shown by Todorčević (see [69]) that if $A$ is an analytic $\mathcal{P}$-ideal then $(\ell_1, \leq)$ can be mapped monotonically and cofinally into $(A, \subseteq)$. Here $\ell_1$ is the collection $\{x \in \mathbb{R}^\mathbb{N} : \sum_{n=1}^{\infty} |x(n)| < \infty\}$ of absolutely convergent series and the order $\leq$ is the coordinatewise order. If $\text{add}_*(A)$ and $\text{add}_*(\ell_1)$ are defined to be the sizes of the smallest unbounded families in $(A, \subseteq^*)$ and $(\ell_1, \leq^*)$ respectively, then it follows that $\text{add}_*(\ell_1) \leq \text{add}_*(A)$. Thus $A$ and $B$ are $\aleph_1$-directed and contain a cofinal $\subseteq^*$-chain under our assumptions.

I will now construct an analytic gap with the properties required for our
compact space to be nonseparable, satisfy Souslin's condition, and be linearly fibered:

1. \( A \) and \( B \) are analytic \( P \)-ideals.

2. \( \mathcal{J} \) and \( \mathcal{K} \) satisfy condition 2 mentioned in Section 2.3.

3. \((A, B)\) is a gap modulo both \( \mathcal{I} \) and \( \mathcal{J} \).

It should be noted that the analytic gap which we will construct below is modeled after a gap of Farah [21] (see Theorem 5.10.2).

First it is necessary to make some preliminary definitions. For \( a \subseteq \mathbb{N} \) define \( \mu(a) = \sum_{n \in A} 1/n \). Let

\[
\varepsilon_n = \ln 2 - \max\{\mu(a) : a \subseteq n \text{ and } \mu(a) < \ln 2\}.
\]

Then for all \( n \) it is true that \( 1 > \varepsilon_n \geq \varepsilon_{n+1} > 0 \). Also it is clear that \( \lim_n \varepsilon_n = 0 \). Define \( h : \mathbb{N} \to \mathbb{N} \) by setting \( h(k) \) to be the least integer \( n \) such that \( 1/n \) is less than \( \varepsilon_k/2^{k+1} \). Define \( g : \mathbb{N} \to \mathbb{N} \) recursively so that \( g(1) = h(1) \) and \( g(n + 1) = h(g(n)) \). For convenience I will also define \( h_k(n) = h(kn) \). Let \( E : [\mathbb{N}]^\omega \leftrightarrow \mathcal{P} \) denote the canonical bijection which identifies subsets of \( \mathbb{N} \) with their increasing enumeration. It will be useful to think of \( E \) as being defined on the on the finite sets as well: if a set \( a \) has no \( n \)th element then set \( E(a)(n) = \infty \). I will use \([m, n]\) denote the interval of integers between (and including) \( m \) and \( n \).

Rather than working inside of \( \mathcal{P}(\mathbb{N}) \) it will be easier to carry out the construction inside of \( \mathcal{P}(R) \) where \( R \) is defined as follows:
1. \( u_n = [g(n) + 1, g(n + 1)] = [g(n) + 1, h(g(n))] \),

2. \( R = \bigcup_{n=1}^{\infty} u_n \times u_n \), and

3. \( R_n = \bigcup_{i=1}^{n} u_i \times u_i \).

Here \( R_n \) takes the place of \([0, n - 1]\) in the construction. Note that \( h(n) \) is at least \( 2n \) and therefore

\[
\mu(u_n) \geq \sum_{i=g(n)+1}^{2g(n)} 1/n \geq g(n) \frac{1}{2g(n)} = \frac{1}{2}.
\]

Define the following:

4. \( L_0 \) is the collection of all \( L \subseteq \mathbb{N} \) such that \( \mu(\{n \in \mathbb{N} : u_n \cap L \neq \emptyset\}) < \infty \) and \( h_k <^* E(L) \) for all \( k \).

5. \( L_1 = \{L \subseteq \mathbb{N} : \mu(L) < \infty\} \)

6. \( A = \{a \subseteq R : \pi_1(a) \in L_0\} \)

7. \( B = \{b \subseteq R : \pi_2(b) \in L_0\} \)

8. \( I = \{I \subseteq R : \pi_1(I) \cup \pi_2(I) \in L_1\} \)

9. \( J = A \cap B = \{J \subseteq R : \pi_1(J) \in L_0 \text{ and } \pi_2(J) \in L_0\} \)

10. \( K = \{K \subseteq R : \mu(\pi_1(K)) \leq \ln 2 \text{ and } \mu(\pi_2(K)) \leq \ln 2\} \)

Remark. Notice that \( L_0 \subseteq L_1 \) since whenever \( 2^n <^* E(L) \) it always follows that \( \mu(L) < \infty \). From this it is immediate that \( J \subseteq I \). Since \( J = A \cap B \), it follows automatically that \( A \perp_{\mathcal{J}} B \) and \( A \perp_{\mathcal{I}} B \).
Lemma 2.6.1. All of the collections mentioned in 4-10 are dense analytic $P$-ideals and $K$ is compact.

Proof. The compactness of $K$ follows from the fact that for any set $L \subseteq \mathbb{N}$, $\mu(L) > \ln 2$ iff $\mu(F) > \ln 2$ for some finite subset $F$ of $L$. It is a routine exercise in descriptive set theory to verify that all the remaining objects are analytic. It is easily seen that $L_1$ is a dense $P$-ideal and since $\pi_1, \pi_2$ are finite-to-one maps, it suffices to show that

$$L = \{L \subseteq \mathbb{N} : \forall k (h_k <^* E(L))\}$$

is a dense $P$-ideal. Let $\{L_k\}_{k=1}^{\infty}$ be a sequence of elements of $L$. For each $k$, pick a $n_k > k$ such that $h_{k(k+1)}(n) < E(L_i)(n)$ whenever $i \leq k$ and $n > n_k$.

Now let

$$L = \bigcup_{k=1}^{\infty} L_k \setminus [1, n_k].$$

To see that $L$ is in $L$, let $k \in \mathbb{N}$ be given and $q > n_k$ and $r < k$. Notice that

$$E(L)(qk + r) \geq E(L)(qk) \geq \min \{E(L_i)(q) : i \leq \max \{j : n_j \leq qk + r\}\}.$$

Furthermore the right hand side is at least

$$h_{k(k+1)}(q) = h_k(q(k + 1)) \geq h_k(qk + r)$$

by our choice of $q$ (note that $r < k \leq n_k < q$).

The density of $L$ follows from the fact that for any $f \in \mathbb{N}^\mathbb{N}$ and any infinite set $L \subseteq \mathbb{N}$, there is an infinite set $L_0$ such that $f <^* E(L_0)$.  \qed
Lemma 2.6.2. If $J$ is in $\mathcal{J}$, there are infinitely many $n$ such that

$$\mu(\pi_i(J \setminus R_n)) < \varepsilon_{g(n+1)},$$

for $i = 1, 2$ and hence $J$ and $K$ satisfy condition 2 of Section 2.3.

Proof. Pick a $N_0$ such that $h(n) < E(\pi_i(J))(n)$ for all $n > N_0$, $i = 1, 2$ and let $N = \max_i E(\pi_i(J)(N_0 + 1))$. Notice that for all $n$, $i = 1, 2$

$$E(E^{-1}(h) \setminus [1, g(N + 1)])(n) < E(\pi_i(J) \setminus [1, g(N + 1)])(n)$$

Now for infinitely many $n > N$, $u_n \times u_n \cap J = \emptyset$. Thus for such $n$

$$\mu(\pi_i(J \setminus R_n)) = \mu(\pi_i(J \setminus R_{n+1})) \leq \sum_{k=g(n+1)}^{\infty} 1/h(k)$$

$$\leq \sum_{k=g(n+1)}^{\infty} \varepsilon_k/2^{k+1}$$

$$\leq \varepsilon_{g(n+1)} \sum_{k=g(n+1)}^{\infty} 1/2^{k+1}$$

$$< \varepsilon_{g(n+1)}.$$  

The proceeding inequality follows from this dominance and the fact that the least element in $\pi_i(J \setminus R_n)$ is at least $g(n + 2) = h(g(n + 1))$, for $i = 1, 2$. $\square$

Finally I will use a Fubini style argument to show that $(A, B)$ is indeed a gap as I have promised all along.

Lemma 2.6.3. $(A, B)$ is a gap in $\mathcal{P}(R)/\mathcal{I}$.

Proof. Suppose that $c \subseteq R$. If $a \subseteq R$, define

$$\mu^2(a) = \sum_{(n,m) \in A} 1/mn$$
and note that this is just the product measure when restricted to the finite rectangles $u_n \times u_n$ (since $\mu$ is determined by its value on the singletons). Let

$$r_n = \mu^2(\bigcup_{k \in \mathbb{N}} u_k \times u_k \cap C) / \mu^2(\bigcup_{k \in \mathbb{N}} u_k \times u_k).$$

I now will consider two overlapping cases.

**Case 1.** $L = \{k \in \mathbb{N} : \tau_k \geq 1/2\}$ is infinite. Let $L_1$ be an infinite subset of $L$ such that $\mu(L_1)$ is finite. Then for each $k$ in $L_1$, let $n_k$ be an element of $u_k$ such that

$$\mu(\{m : (m, n_k) \in C\}) \geq (1/2)\mu(u_k).$$

This choice is possible by Fubini's theorem. Now let

$$b = c \cap \bigcup_{k \in L_1} u_k \times \{n_k\}.$$  

By choice of $n_k$,

$$\mu(c_1) \geq \sum_{k \in L_1} (1/2)\mu(u_k) = \infty.$$  

On the other hand, $\pi_2(b) \cap u_k$ contains at most one element and thus $h_k <^* g \leq ^* E(\pi_2(b))$ for all $k$. Furthermore $\{n \in \mathbb{N} : \pi_2(b) \cap u_n \neq \emptyset\} = L_1$ and therefore $b \in B \setminus I$.

**Case 2.** $\mathbb{N} \setminus L$ is infinite. It is now possible to apply a symmetric argument to find an $a \in A \setminus I$ such that $a \cap c = \emptyset$. 

2.7 **An Example From $b > \aleph_2$**

It should be clear that the previous example fails to satisfy Helly's condition. As the example of Todorčević in Theorem 8.4 [66] is clearly $\sigma$-linked (and
hence fails to satisfy Helly's condition as well), it is natural to ask whether such spaces can be improved to satisfy Helly’s condition.

It turns out that, to some extent, this is not so. The compatibility relation between the generators in these algebras is clearly closed (when viewed as a subset of \([\Gamma]^2\)) and hence is subject to the consequences of \textbf{OCA}.

\textbf{OCA} : If \(X\) is a separable metric space and \(G \subseteq [X]^2\) is open (with \([X]^2\) viewed as an appropriate subset of \(X^2\)) then either \(G\) is countably chromatic or else \(G\) contains an uncountable complete subgraph\(^1\).

This axiom implies that all such Boolean algebras which satisfy Souslin’s condition are \(\sigma\)-linked. Thus, if there is an example of a linearly fibered compactum which satisfies Souslin’s condition but does not have a \(\sigma\)-linked base, it either must require a set theoretic assumption which is incompatible with \textbf{OCA} or else must differ considerably from the examples mentioned in this chapter.

Concerning this, I will reproduce (in the presence of the machinery of this chapter) a construction of Todorčević which appeared in an early draft of [66]. The assumption \(b > \aleph_2\) implies that there is a gap \((A, B)\) with the following properties (see Lemma 3.10 of [74]):

1. \(A\) is \(\aleph_1\) directed.

\(^1\)This is the formulation of \textbf{OCA} currently quoted today. Earlier versions of this axiom for sets of size \(\aleph_1\) can be found in [1]. The only aspect of \textbf{OCA} which we will use in this thesis is its ability to “reflect:” if \(G\) isn’t countably chromatic then it contains a subgraph of size \(\aleph_1\) which is not countably chromatic.
2. $B$ is countably directed.

3. $A$ and $B$ are well ordered by $\subseteq$.

4. $A$ and $B$ are orthogonal modulo finite.

5. $(A, B)$ is a gap modulo finite.

Here $\mathcal{I} = \mathcal{J} = \text{fin}$ and $\mathcal{K}$ consists only of the empty set. It follows that the Boolean algebra $\mathcal{B}$ associated with this gap produces a compactum which is nonseparable, satisfies both Souslin's and Helly's condition, and is linearly fibered. Such an algebra also exists if $\mathfrak{b} = \mathcal{N}_1$ (the example is, moreover, metrizably fibered), though the construction of this example does not involve the use of a gap (see [80] and Section 3 of [66]).
Chapter 3

Analysis of Local Chain Conditions

$\mathbf{MA_{\mathbb{R}_1}}$ was an axiom which came out of the solution of Souslin’s problem — Martin realized that the techniques which Solovay and Tennenbaum used to produce a consistent positive answer to Souslin’s problem in [57] also gave the consistency of the following statement:

$\mathbf{MA_{\mathbb{R}_1}}$: A compact space which satisfies Souslin’s condition can not be covered by $\mathbb{R}_1$ nowhere dense sets.

Even before it appeared in literature in [57], this axiom was already shown to have a number of consequences (see [22]):

1. Souslin’s condition is equivalent to Shanin’s condition.

2. Souslin’s condition is productive.
3. The union of $\aleph_1$ measure 0 sets has measure 0.

4. Every tree of size $\aleph_1$ in which all branches are countable is the union of countably many antichains.

5. There is a non-free group $A$ such that whenever $\pi : B \to A$ is a homomorphism with kernel $\mathbb{Z}$, there is a homomorphism $\rho : A \to B$ such that $\pi \rho$ is the identity (there is a $W$-group which is not free).

While $\text{MA}_{\aleph_1}$ is, on the surface, a statement about Baire category, it was quickly realized that this axiom has a strong influence on the identification of chain conditions in topological spaces (see 1, 2, and 4 above). That the identification of chain conditions in turn has a strong influence on Baire category, however, came as somewhat of a surprise.

**Theorem 3.0.1.** [83] $\text{MA}_{\aleph_1}$ is equivalent to identifying the Souslin's condition and Shanin's condition in the class of compact topological spaces.

In light of this, it became quite natural to ask whether Shanin's condition could be relaxed to some weaker local chain condition. Research in this direction so far has been focused on establishing which consequences of $\text{MA}_{\aleph_1}$ follow from identifying, for example, Souslin's condition and Knaster's condition (see [67], [74], and [78]). To help facilitate the discussion, define the following axioms, for $n \geq 2$:

$\mathcal{K}_n$: Every compact topological space which satisfies Souslin's condition satisfies Knaster's condition $K_n$. 
CHAPTER 3. LOCAL CHAIN CONDITIONS

Here Knaster’s condition \( K_n \) asserts that every uncountable collection contains an uncountable \( n \)-linked subcollection. The focus of this chapter will be to remark on some specific consequences of \( K_2 \) and \( K_3 \). Typically a discussion of this sort is carried out in the language of partitions (see [67] and [74]). We will present the first section in the language of partitions and the latter in the language of Boolean algebras for the sake of contrast.

3.1 Some Consequences \( K_2 \)

In [78] Todorčević demonstrated that \( K_2 \) implies that every tree of size \( \aleph_1 \) without an uncountable branch is the union of countably many antichains. The following is a more general form of this result which was also proven and stated in [78]. I have included a sketch of the proof for completeness and since it involved an interesting new technique.

**Theorem 3.1.1.** [78] (\( K_2 \)) If \( G \subseteq [\omega_1]^2 \) is a graph then either \( G \) is countably chromatic or else there is a pair of sequences \( D_\xi, E_\xi \ (\xi < \omega_1) \) of disjoint \( n \) element sets such that for every \( \xi \neq \eta < \omega_1 \) \( D_\xi \times E_\xi \cap G \) is nonempty.

Before proving this result, let us first examine some of its consequences. If \( T \) is a tree on \( \omega_1 \) without an uncountable branch then the compatibility graph \( G \) on \( T \) can never satisfy the second alternative of the above theorem (see Section 9 of [82]). Also, it should be clear that “\( T \) can be decomposed into countably many antichains” is just a reformulation of “the compatibility graph is countably chromatic.” Thus we have the following.
Theorem 3.1.2. [78](K_2) If \( T \) is a tree of size \( \aleph_1 \) and \( T \) has no uncountable branch then \( T \) can be decomposed into countably many antichains.

Also, \( K_2 \) has an influence on the measure of small sets of reals.

Theorem 3.1.3. (\( K_2 \)) Every set of reals of size \( \aleph_1 \) has measure 0.

The reason for this is as follows. Tall [62] has shown that a set \( X \subseteq \mathbb{R} \) is \( \sigma \)-discrete in the density topology iff it is closed discrete in the density topology iff it has measure 0. Subsets of \( \mathbb{R} \) of size \( \aleph_1 \) are always left separated in the density topology. Thus if \( x_\xi (\xi < \omega_1) \) is a sequence of distinct reals then there is a sequence of closed sets of positive measure \( E_\eta (\xi < \omega_1) \) which contain \( x_\eta \) as a density 1 element but which do not contain any \( x_\xi \) for \( \xi < \eta \). If \( G \) is defined by putting \( \{\xi, \eta\} \) in \( G \) iff \( x_\eta \) is in \( E_\xi \) then \( G \) can not satisfy the latter condition of the above theorem (see the Claim on page 62 of [74]). If \( K_2 \) holds then \( G \) is countably chromatic. Thus \( X = \{x_\xi : \xi < \omega_1\} \) is \( \sigma \)-discrete in the density topology and hence measure 0.

Now we shall sketch the proof of the theorem.

*Proof.* Fix a sequence of 1-1 maps \( e_\alpha : \alpha \to \omega \) for each \( \alpha < \omega_1 \) such that \( e_\alpha =^* e_\beta \upharpoonright \alpha \) whenever \( \alpha < \beta < \omega_1 \) (such a sequence can indeed be found — see [73] or [7, Ch. 4]). Set \( r_\alpha \) to be the range of \( e_\alpha \) and define \( \{\alpha, \beta\} \in K \) iff for all \( n < \Delta (r_\alpha, r_\beta) \) such that \( n \in r_\alpha \) (equivalently \( r_\beta \) \( \{e_\alpha^{-1}(n), e_\beta^{-1}(n)\} \) is in \( G \).

By the remarks made in Section 1.3 \( K_2 \) implies that if \( K \) is c.c.c. then there is an uncountable \( H \subseteq \omega_1 \) such that \([H]^2 \subseteq K \). If such an \( H \) exists,
define $A_{(s,n)}$ for $s \subseteq \{0, \ldots, n\}$ by

$$A_{(s,n)} = \{ \xi < \omega_1 : \exists \alpha \in H(r_\alpha \cap \{0, \ldots, n\} = s \text{ and } e_\alpha(\xi) = n) \}. $$

It is easily checked that the $A_{(s,n)}$'s cover $\omega_1$ and each satisfy $[A_{(s,n)}]^2 \cap G$ is empty. This $G$ is countably chromatic.

Now suppose $K$ is not c.c.c. That is there is a sequence $F_\xi (\xi < \omega_1)$ of disjoint $n$ element $K$ homogeneous sets such that $F_\xi \times F_\eta$ is not $K$ homogeneous for any $\xi \neq \eta$. For each $\xi < \omega_1$ pick a pair of distinct ordinals $\delta_\xi, \epsilon_\xi$ greater than $\delta$ and a integers $m_\xi, n_\xi$ such that:

1. $\#(F_{\delta_\xi}) = \#(F_{\epsilon_\xi})$

2. $e_\alpha \upharpoonright \xi = e_\beta \upharpoonright \xi$ whenever $\alpha$ is the $i^{th}$ element of $F_{\delta_\xi}$ and $\beta$ is the $i^{th}$ element of $F_{\epsilon_\xi}$ ($i^{th}$ in the order on $\omega_1$).

3. $\Delta(r_\alpha, r_\beta) < m_\xi$ whenever $\alpha$ is the $i^{th}$ element of $F_{\delta_\xi}$ and $\beta$ is the $j^{th}$ element of $F_{\epsilon_\xi}$ for $i \neq j$ ($i^{th}$ and $j^{th}$ in the order on $\omega_1$).

4. $m_\xi < \Delta(r_\alpha, r_\alpha') < n_\xi$ whenever $\alpha$ is the $i^{th}$ element of $F_{\delta_\xi}$ and $\alpha'$ is the $i^{th}$ element of $F_{\epsilon_\xi}$.

Define

$$D_\xi = \cup\{e_\alpha^{-1}\{0, \ldots, n_\xi\} : \alpha \in F_{\delta_\xi}\}$$

$$E_\xi = \cup\{e_\alpha^{-1}\{0, \ldots, n_\xi\} : \alpha \in F_{\epsilon_\xi}\}.$$

From item 2 it follows that $D_\xi \cap \xi = E_\xi \cap \xi$. By pressing down and refining to an uncountable set $\Gamma \subseteq \omega_1$ we may assume that:
5. $D_\xi \cap \xi = D_\eta \cap \eta$, $n = n_\xi = n_\eta$ for all $\xi, \eta$ in $\varepsilon$.

6. $\Delta(r_\alpha, r_{\alpha'}) > n$ if $\alpha$ is the $i^{th}$ element of $D_\xi$ (respectively $E_\xi$) and $\alpha'$ is the $i^{th}$ element of $D_\eta$ (respectively $E_\eta$).

7. if $m \leq n$ and $e_\alpha(\zeta) = m$ where $\alpha$ is the $i^{th}$ element of $D_\xi$ for $\xi$ in $\Gamma$ then $e_\beta(\zeta) = m$ where $\beta$ is the $i^{th}$ element of $E_\eta$ for any $\eta$ in $\Gamma$.

8. if $\xi < \eta$ then $\eta$ is greater than the maximum of $D_\xi \cup E_\xi$.

We are now finished once we prove the following claim.

**Claim 3.1.4.** If $(D_\xi \setminus \xi) \times (E_\eta \setminus \eta) \cap G$ is empty then $F_{\xi} \cup F_\eta$ is $K$ homogeneous.

**Proof.** Assume that $(D_\xi \setminus \xi) \times (E_\eta \setminus \eta) \cap G$ is empty. Suppose that $\alpha$ is in $F_{\xi}$ and $\beta$ is in $F_\eta$. By 3, 4, and 6 we know that $\Delta(r_\alpha, r_\beta) < n$. Now suppose that $\zeta_1$ and $\zeta_2$ are such that $e_\alpha(\zeta_1) = e_\beta(\zeta_2) < n$. Notice that if $\zeta_1$ or $\zeta_2$ is less than $\xi$ or $\eta$ respectively then $\zeta_1 = \zeta_2$ by 7. Thus $\zeta_1 > \xi, \zeta_2 > \eta$ and by our assumption $\{\zeta_1, \zeta_2\}$ is not in $G$ and hence $\{\alpha, \beta\}$ is in $K$ and $F_{\xi} \cup F_\eta$ is homogeneous. $\square$

**3.2 $\mathcal{K}_3$ and Additivity of Measure**

The focus of this section will be the following result. The key ideal was already present in the proof of observation (15) in [78]. The general machinery which we will use in our construction was developed in [80].
Theorem 3.2.1. There is a $\sigma$-linked Boolean algebra $B$ such that, if there is a family of $\aleph_1$ measure 0 sets whose union is not measure 0, $B$ does not satisfy Knaster’s condition $K_3$.

Notation. $I$ denotes the collection of all finite unions of rational intervals.

We will need the following standard fact about measure 0 sets (we will prove a variation of this in Section 4.1).

Fact 3.2.2. For every measure 0 set $G \subseteq \mathbb{R}$, there is a sequence $F_G : \mathbb{N} \to I$ such that the measure of $F_G(i)$ is at most $2^{-2i}$ and

$$G \subseteq \bigcap_{k=0}^{\infty} \bigcup_{i=k}^{\infty} F_G(i).$$

For each measure 0 set $G$, we will fix a sequence $F_G$ as above once and for all.

Notation. $T$ denotes all $(t, n)$ such that

1. $n$ is in $\mathbb{N}$
2. $t$ is a subset of $I^n$
3. $t$ (viewed as a finite tree) has no node with more than two immediate successors.

Remark 3.2.3. If $(t, n)$ is in $T$ then condition 3 implies $t$ has at most $2^n$ many elements.

For a measure 0 set $G$, define $T_G$ to be all $(t, n)$ such that the restriction of $F_G$ to $[0, n-1]$ is in $t$. Define $B$ to be the subalgebra of $\mathcal{P}(T)/\text{fin}$ generated by $T_G$ for $G$ measure 0.
CHAPTER 3. LOCAL CHAIN CONDITIONS

Claim 3.2.4. \( \mathcal{B} \) is \( \sigma \)-linked.

Proof. For \( A \) in \( \mathcal{B} \) there is a sequence of measure 0 sets \( G_i \) \((i < l)\) such that \( A \) is equivalent to
\[
\bar{A} = \bigcap_{i=1}^{k} T_{G_i} \cap \bigcap_{i=k+1}^{l} (\mathbb{N} \setminus T_{G_i}).
\]
Define \( n_A \) to be the maximum of all
\[
\Delta (F_{G_i}, F_{G_j}) = \min \{ n : F_{G_i}(n) \neq F_{G_j}(n) \}
\]
for \( 1 \leq i \neq j \leq k \). If \( A \) is positive then there is a \( t_A \) such that \((t_A, n_A)\) in \( \bar{A} \). The collection of all such \( \bar{A} \) with \((t, n) = (t_A, n_A)\) is linked. \( \square \)

Claim 3.2.5. If \( \{ T_G : G \in \mathcal{E} \} \) is 3-linked then the union of \( \mathcal{E} \) has measure 0.

Remark 3.2.6. Notice that this finishes the proof of the theorem. If \( \mathcal{F} \) is a family of measure 0 sets of size \( \aleph_1 \) whose union is not measure 0 then, by taking some unions, there is a collection \( \mathcal{E} \) of measure 0 sets such that \( \cup \mathcal{F} = \cup \mathcal{E}' = \cup \mathcal{E} \) for every uncountable \( \mathcal{E}' \subseteq \mathcal{E} \). In this case \( \{ T_G : G \in \mathcal{E} \} \) will contain no uncountable 3-linked family.

Proof. Define
\[
F_{\epsilon}(i) = \bigcup_{G \in \mathcal{E}} F_G(i).
\]
Note that, for each \( n \), \( \{ F_G \mid [0,n-1] : G \in \mathcal{E} \} \) is in \( T \). Thus \( F_{\epsilon}(i) \) has measure at most \( 2^i \cdot 2^{-2i} = 2^{-i} \). Notice that for all \( k \)
\[
\cup \mathcal{E} \subseteq \bigcup_{i=k}^{\infty} F_{\epsilon}(i)
\]
and
\[ \mu(\cup E) \leq \mu(\bigcup_{i=k}^{\infty} F_{\varepsilon}(i)) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}. \]

Thus the union of $E$ has measure $0$. \qed
Chapter 4

Continuous Irreducible Colorings

In this chapter, we will study strong failures of Ramsey’s theorem at certain cardinals associated with the continuum. A coloring of pairs of a set $X$ is simply a map from the set $[X]^2$ of all unordered pairs of elements of $X$ into some set $Z$ of colors. A coloring $c : [X]^2 \to Z$ is said to be irreducible if for every $Y \subseteq X$ of the same cardinality as $X$, $c'[Y]^2 = Z$.

For arbitrary colorings using 2 colors, Sierpiński (in effect) gave a definitive counterexample at the level of the continuum.

**Example 4.0.7.** [52] There is a coloring $c : [\mathbb{R}]^2 \to \{0, 1\}$ such that $c$ is not constant on any set of pairs of an uncountable set.

The coloring $c$ mentioned above makes crucial use of a well ordering of the continuum — in fact the value of $c$ on a pair is determined by whether or
not a fixed well order on $\mathbb{R}$ and the usual order agree. It is therefore natural to ask for irreducible colorings which are more well behaved in some sense.

One natural restriction to place on the colorings is to ask that they be continuous. Here $X$ and $Z$ are assumed to have a separable metric topology associated with them. In our case $Z$ will typically be countable and discrete. Since $\mathbb{P}$ is the universal Polish space, there is no loss of generality in considering only those $X$ which are subspaces of $\mathbb{P}$. In this context, the continuity of $c$ is equivalent to the property that only a finite number of entries in $x$ and $y$ are needed to compute $c(x, y)$, thus making this a very natural restriction indeed.

The general question is to understand when it is possible to find continuous colorings on sets of reals of a certain size. This is perhaps a little misleading, since in hindsight it is more how a set of reals behaves in a certain structure which is interesting here, rather than its cardinality. For example, we will see that there is always a continuous irreducible coloring on a set of reals of size $\mathfrak{b}$, the cardinality of the smallest unbounded subset of $(\mathbb{P}, <^*)$ where $\mathbb{P}$ is the collection of all increasing functions from $\omega$ to $\omega$. The theorem at hand actually tells us that there is a continuous irreducible coloring on any unbounded chain in $(\mathbb{P}, <^*)$ (it is well known that there is always an unbounded chain of size $\mathfrak{b}$). Thus cardinal invariants are simply a convenient language for discussing results about continuous irreducible colorings.

This chapter has three sections. Section 4.1 investigates a coloring which is closely connected to the ideal of Lebesgue measure 0 sets. Ironically it makes use of a lower bound on a class of finite Ramsey numbers. Ideas from
this section were already put to use in Section 3. Section 4.2 revisits the oscillation map of Todorčević and introduces a variation called the alternation map which is continuous. Section 4.3 puts this mapping to use to produce a new proof that PFA implies $\mathfrak{c}$ is $\aleph_2$. The ideas presented here may be relevant to the problem of whether or not OCA implies that $\mathfrak{c}$ is $\aleph_2$.

4.1 A Coloring Associated With Measure

In this section we will study a subfamily of the Lebesgue measure 0 subsets of $\mathbb{R}$. This family is natural to consider in the sense that it consists of those measure 0 sets which are measure 0 for a concrete reason and in particular encompass all of the measure 0 sets which one is likely to encounter outside of a set theoretic discussion. The family is of interest since the property of being unbounded in the ideal can be connected to continuous irreducible colorings.

To define the family of interest it is first necessary to make a few definitions. Suppose that $G \subseteq 2^\omega$ is a $G_\delta$ set of measure 0 and that $G = \bigcap_{n=0}^{\infty} U_n$ where $U_n$ is an open set of measure less than $2^{-n}$. For each $n$, let $\{[t^n_k]\}$ be a list of disjoint clopen sets whose union is $U_n$. Let $F(i)$ be the collection of all $t^n_k$ in $2^i$. Now notice that $U_n \subseteq \bigcup_{i=n}^{\infty} [F(i)]$ since $U_n$ has measure less than $2^{-n}$. Also,

$$\sum_{i=0}^{\infty} 2^{-i} \cdot \#(F(i)) = \sum_{n=0}^{\infty} U_n \leq 1.$$ 

Thus a set $G \subseteq 2^\omega$ has measure 0 iff there is a sequence $F_G = \langle F_G(i) : i \in \omega \rangle$ such that
CHAPTER 4. CONTINUOUS IRREDUCIBLE COLORINGS

1. \( F_G(i) \subseteq 2^i \),

2. \( G \subseteq \bigcap_{k=0}^{\infty} \bigcup_{i=k}^{\infty} [F_G(i)] \), and

3. \( \sum_{i=0}^{\infty} \#(F_G(i))/2^i < \infty \).

Such a sequence will be called a cover of \( G \). If \( r \in (0,1) \), a cover \( F_G \) of \( G \) is said to be \( r \) nice if

\[
\lim_{i \to \infty} \#(F_i)2^{-r}/2^i = 0.
\]

Thus nice covers are those for which the sum mentioned above converges for a specific reason — the same reason that \( \sum_{n=1}^{\infty} 1/n^p \) converges whenever \( p > 1 \).

Define \( N_r \) to be the collection of all subsets of \( 2^{\omega} \) which have a \( r \) nice cover. Notice that if \( r < s \) then \( N_r \subseteq N_s \subseteq N \). If \( A \subseteq B \) are families of measure 0 sets, define \( \text{add}(A) \) to be the size of the smallest subset of \( A \) which is unbounded in \( (A, \subseteq) \) and \( \text{add}(A, B) \) to be the size of the smallest subset of \( A \) which is unbounded in \( (B, \subseteq) \). It is easy to see that \( \text{add}(N_r, N_s) \leq \text{non}(N_s) \leq \text{non}(N) \). Also, if \( r \leq a < b \leq s \) then \( \text{add}(N_a, N_b) \leq \text{add}(N_r, N_s) \). Thus using a well foundedness argument on the ordinals, it is possible to find a pair \( r < s \) in \( (0,1) \) such that for all \( a < b \) in \( [r,s] \), \( \text{add}(N_r, N_s) = \text{add}(N_a, N_b) \). Define \( N_* = \bigcap_{b>r} N_b \).

**Lemma 4.1.1.** The invariants \( \text{add}(N_*), \text{add}(N_r, N_s) \) are equal. Moreover there are \( a < b \) in the interval \( (r,s) \) and \( A \subseteq N_a \) of size \( \text{add}(N_*) \) which is well ordered by \( \subseteq \) and unbounded in \( N_b \).

**Proof.** It is easy to see that \( N_r \subseteq N_* \subseteq N_s \) and therefore that \( \text{add}(N_r, N_s) \geq \text{add}(N_*) \). Now let \( A \) be an unbounded subset of \( N_* \) of size \( \text{add}(N_*) \). It can
be assumed without loss of generality that $\mathcal{A}$ is well ordered by $\subseteq$. Since $\mathcal{A}$ is unbounded, there is a $b > r$ such that $\bigcup \mathcal{A}$ is not in $\mathcal{N}_b$. Let $a$ be any element of $(r, b)$. Now clearly $\mathcal{A} \subseteq \mathcal{N}_a$ and is unbounded in $\mathcal{N}_b$. Thus $\mathcal{A}$ must have size $\text{add}(\mathcal{N}_a, \mathcal{N}_b) = \text{add}(\mathcal{N}_r, \mathcal{N}_s)$. \hfill $\square$

From this point on $a$ and $b$ will remain fixed and $d$ will be any number such that $a < d < b$.

**Theorem 4.1.2.** There is a subset $X$ of $\omega^\omega$ of size $\text{add}(\mathcal{N}_*)$ and a continuous irreducible coloring $c : [X]^2 \rightarrow \omega$.

The existence of the desired irreducible partition for the cardinal $\text{add}(\mathcal{N}_*)$ is a consequence of the following sequence of results. It is perhaps surprising that this coloring makes crucial use of exponential lower bounds for a certain class of finite Ramsey numbers.

The following fact for $k = 2$ and $l = 1$ is essentially a well known result of Erdős [19] (see also Section 26 of [20]) \footnote{Erdős actually showed $(m)_{2/1}^2 > 2^{m/2}$.} and the methods presented in the proof can readily be adapted to give us the following result.

**Theorem 4.1.3.** If $\binom{k}{l} \leq m$, then $(m)_{k/l}^2 > (k/l)_{(m-1)/2}$. In particular there is a constant $\alpha > 0$ such that $[m]_{k}^2 > 2^{\alpha m/k}$.

**Proof.** First notice that for a fixed integer $n$, the number of colorings $c : [n]^2 \rightarrow k$ is $N = k^{\binom{n}{2}}$. If $S$ is a fixed subset of $n$ of size $m$ and $L \subseteq k$ is a collection of $l$ colors, then there are $l^{\binom{n}{2}} k^{\binom{n}{2} - \binom{m}{2}}$ many colorings $c : [n]^2 \rightarrow k$
which also satisfy $c''[S]^2 \subseteq L$. Since there are $\binom{n}{m}$ ways to choose $S$ and $\binom{k}{l}$ ways to choose $L$, there are at most
\[
N_{m,l} = \binom{n}{m} \binom{k}{l} \binom{m}{2} k \binom{n}{2} - \binom{n}{2}
\]
many colorings $c$ of $[n]^2$ such that there is a subset of $n$ of size $m$ which realizes at most $l$ colors. It now suffices to show that if $n \leq (k/l)^{(m-1)/2}$ then $N_{m,l} < N$. I will use the approximation $\binom{n}{m} < n^m / m!$.

\[
\begin{align*}
n &\leq (k/l)^{(m-1)/2} \\
n^m &\leq (k/l)^{m(m-1)/2} \\
\frac{n^m}{m!} \binom{k}{l} &< (k/l)^{\binom{m}{2} k \binom{n}{2} - \binom{n}{2}} \\
\binom{n}{m} \binom{k}{l} &< (k/l)^{\binom{m}{2}} \\
N_{m,l} &< N
\end{align*}
\]

To see that the constant $\alpha$ exists, note that
\[
1/k \leq \int_{k-1}^{k} \frac{1}{x} dx = \ln\left(\frac{k}{k-1}\right).
\]

Suppose that $T$ is a subset of $\omega^\omega$ and $f \in \omega^\omega$ is a function. I will say that $T$ is $f$ thin if for all but finitely many $n$ in $\omega$ and every $t$ in $\omega^n$ the concatenation $t \cdot i$ is a node of $T$ for at most $f(n)$ many $i$ (i.e. the splitting of the nodes of $T$ is bounded by $f$). Similarly I will call $T$ $f$ splitting if $T$ does not contain a $f$ thin subset of the same size. Define $h(n) = 2\sqrt{n}$ and let $h_k$ be the $k$ fold composition of $h$ with itself.
Remark 4.1.4. Frequently we will be interested in taking the integral part of a real valued function. I will simply write the real valued function and let it be understood that what I really mean is the greatest integer less than this function.

Lemma 4.1.5. Suppose $T$ is any subset of $\omega^\omega$ which is $h(f) = 2^{\sqrt{T}}$ thin for some $f$ with $\lim_n f(n) = \infty$. It follows that there is a continuous coloring $c : [T]^2 \to \omega$ such that $c''[T_0]^2 = \omega$ for any $T_0 \subseteq T$ which is not $f$ thin.

Proof. Notice first that for any $h(f)$ thin subset $T$ of $\omega^\omega$ there is an isometry which embeds $T$ into $\prod_{n=1}^\infty h(f(n))$ (this is in fact an equivalent formulation of “thinness”). Thus I will work in $\prod_{n=1}^\infty h(f(n))$ for convenience. Let $G(n)$ be the greatest integer $k$ such that

$$2^{\alpha f(n)/k} \leq h(f(n)).$$

Note that $G$ is both well defined and satisfies $\lim_n G(n) = \infty$. For each $n$ pick a coloring

$$c_n : [h(f(n))]^2 \to G(n)$$

such that for every subset $S$ of $h(f(n))$ having size $f(n)$, $c''_n[S]^2 = G(n)$ (this is precisely what the definition of $G(n)$ and Theorem 4.1.3 guaranteed). Now define $c : [T]^2 \to \omega$ by

$$c(x, y) = c_n(x(n), y(n))$$

where $n = \Delta(x, y)$. To see that $c$ has the desired properties, let $k \in \omega$ be arbitrary and $T_0 \subseteq T$ be as in the statement of the theorem. Pick a $m$ such
that $G(m) > k$. Now find a $n \geq m$ and a $t$ in $\omega^n$ such that the set

$$S = \{i \in h(f(n)) : \exists x \in S(t^*i \subseteq x)\}$$

has at least $f(n)$ elements in it. Then $c''[S]^2 = G(n)$ contains $k$ and is contained in $c''[T_0]^2$ by definition. \qed

Notice that if $T \subseteq \omega^\omega$ is a $h_k(f)$ thin set which is $f$ splitting then Lemma 4.1.5 tells us that there is a continuous irreducible coloring $c : [T_0]^3 \to \omega$ for some $T_0 \subseteq T$ of the same size. To see this, set $S_k = T$. If $S_k$ is $h_{k-1}(f)$ splitting then this is a consequence of the lemma. If not let $S_{k-1}$ be a $h_{k-1}(f)$ thin subset of $S_k$ having the same cardinality as $S_k$. Now try to apply Lemma 4.1.5 to $S_{k-1}$ and so on. For some $i \geq 1$, $S_i$ has the same size as $T$ and is $h_i(f)$ thin but $h_{i-1}(f)$ splitting (otherwise this would contradict the fact that $T$ is $f = h_0(f)$ splitting).

**Lemma 4.1.6.** Suppose that $T \subseteq \omega^\omega$ is $2^{2n}$ thin but there is not a $T_0 \subseteq T$ of the same size such that $\#(T_0 \upharpoonright n) \leq n^{d-a}$ for all $n$. Then there is a subset $Z$ of $\omega^\omega$ with the same cardinality as $T$ and a continuous irreducible coloring of $[Z]^2$.

**Proof.** For each $x$ in $T$ define $z_x(n)$ to be the finite sequence

$$z_x(n) = (x(n+1), \ldots, x((n+1)^2)).$$

Notice that since $T$ is $2^{2n}$ thin, $Z = \{z_x : x \in T\}$ is $(2^{2(n+1)^2})(n+1)^2$ thin. It is easy to verify that for some $k$, $h_k(n^{d-a}) \geq (2^{2(n+1)^2})(n+1)^2$ for all $n$. It therefore suffices to show that $Z$ is $n^{d-a}$ splitting.
Suppose that $Z_0$ is a $n^{d-a}$ thin subset of $Z$ of the same size. By refining
$Z_0$ if necessary it may be assumed that $n^{d-a}$ controls the splitting at all nodes
of $Z_0$. Let $T_0 = \{x \in T : x \in Z_0\}$ and note that $T_0$ has the same size as
$T$. I will now prove by induction that $\#(T_0 \upharpoonright n) \leq n^{d-a}$ for all $n$. Notice
that, if necessary, it is possible to pass to an appropriate neighborhood of
$T_0$ to ensure that the base case of the induction is satisfied. Now suppose
that $\#(T_0 \upharpoonright m) \leq m^{d-a}$ for all $m$ less than $n$. Fix a $m$ such that $m^2 \leq n <
(m+1)^2$. Then $\#(T_0 \upharpoonright m) \leq m^{d-a}$ and for each $t$ in $T_0 \upharpoonright m$ there are at most
$m^{d-a}$ many $t'$ in $T_0 \upharpoonright (m+1)^2$ which extend $t$. This latter fact follows from
our assumption that $Z_0$ is $n^{d-a}$ thin. Thus $T_0 \upharpoonright n$ can have at most

$$m^{d-a} \cdot m^{d-a} = m^{2(d-a)} \leq n^{d-a}$$

many members, a contradiction. \qed

The following is the last lemma which is needed to show that there is an
irreducible coloring associated with $\text{add}(\mathcal{N}_*)$.

**Lemma 4.1.7.** There is a $2^{2^n}$ thin subset $T$ of $\omega^\omega$ of cardinality $\text{add}(\mathcal{N}_*)$
which has no subset $T_0$ of the same size satisfying $\#(T_0 \upharpoonright n) \leq n^{d-a}$ for all
$n$.

**Proof.** Fix a family $A \subseteq \mathcal{N}_*$ of size $\text{add}(\mathcal{N}_*)$ which is well ordered by $\subseteq$ and
unbounded in $\mathcal{N}_b$. For each $A$ in $A$ choose an $A$ nice cover $F_A$ of $A$ which
also satisfies $F_A(n)n^2/2^n \leq n^a$ for all $n$. Let $T = \{F_A : A \in A\}$ and notice
that $T$ is $2^{2^n}$ thin.
CHAPTER 4. CONTINUOUS IRREDUCIBLE COLORINGS

Suppose for contradiction that there is a $T_0 \subseteq T$ with the same size as $T$ and which satisfies $\#(T_0 \upharpoonright n) \leq n^{d-a}$ for all $n$. Define $F_G(n) = \cup\{F_A(n) : F_A \in T_0\}$ and notice that

$$\#(F_G(n)) \frac{n^2}{2^n} \leq n^{d-a} n^a = n^d.$$  

Thus $\#(F_G(n)) \frac{n^{2-d}}{2^n} \leq 1$ for all $n$. Since $\lim_n n^d/n^b = 0$, $F_G$ is a $b$ nice cover for $\cup\{A \in A : F_A \in T_0\}$. If $T_0$ has the same size as $T$, then

$$\cup A = \cup\{A \in A : F_A \in T_0\} \in \mathcal{N}_b,$$

a contradiction. \qed

4.2 The Alternation Map

In [72] Todorčević introduces and studies the behaviors of the oscillation map (see also Chapter 1 of [74]). If $x$ and $y$ are two elements of $P$, the set of all strictly increasing functions, then $osc(x,y)$ is defined to be the number of times $x$ and $y$ "oscillate." More formally $osc(x,y)$ is the minimum size of a collection of intervals $I$ such that $I$ covers $\omega$ and whenever $I$ is in $I$ and $m, n$ are in $I$ then

$$x(m) \leq y(m) \text{ iff } x(n) \leq y(n).$$

As simple as this definition is, Todorčević shows that it exhibits a rather remarkable behavior.

**Theorem 4.2.1.** [74] Whenever $X$ is an unbounded and $\sigma$-directed subset of $(P,<^*)$ and $n$ is in $\omega$, there is a pair $x, y$ in $X$ such that $osc(x,y) = n$. 
Thus if $X$ is an unbounded chain in $(\mathbb{P}, \leq^*)$ which is well ordered in a regular order type then osc is irreducible when restricted to $[X]^2$. It is clear, however, that the oscillation map is not continuous. The purpose of this section is to produce a continuous variation of the oscillation map of Todorcevic. We will focus on the following definition.

**Definition 4.2.2.** If $S \subseteq \omega$ and $x$ and $y$ are in $\mathbb{P}$ then $x$ and $y$ alternate on $S$ if for every consecutive pair $m, n$ of elements of $S$

$$x(m) < y(m) \text{ iff } y(n) < x(n).$$

This definition also makes sense if $x$ and $y$ are only partial maps, so long as they are both defined on $S$.

Define $S : [\mathbb{P}]^2 \to \mathbb{P}$ is defined recursively as follows. If $x \leq_{\text{lex}} y$ are in $\mathbb{P}$, then define $S(x, y) \subseteq \omega$ recursively as follows. The first element of $S(x, y)$ is $\Delta(x, y)$. Given given the $2n$th element $S(x, y)(2n)$ of $S(x, y)$, define

$$S(x, y)(2n + 1) = y(S(x, y)(2n)),$$
$$S(x, y)(2n + 2) = x(S(x, y)(2n + 1)) = x(y(S(x, y)(2n))).$$

Notice that $S : [\mathbb{P}]^2 \to [\omega]^\omega$ is a continuous map. Also, $S$ can be defined for pairs of finite sequences as well. In this case, the procedure for choosing members of $S(s, t)$ stops when the prospective member of $S$ is no longer in the intersection of the domains of $s$ and $t$.

If $x$ and $y$ are in $\mathbb{P}$ then define $\text{alt}(x, y)$ to be the size of the largest initial segment of $S(x, y)$ on which $x$ and $y$ alternate. Since $S$ is continuous, $\text{alt}$ is continuous as well.
It turns out that the map alt exhibits many of the behaviors of the map osc as the following theorem demonstrates. As it will be needed in the following section, I will prove the next theorem in its full strength (which immediately gives a continuous irreducible coloring in the case \( k = 1 \) for a set \( X \) unbounded and well ordered in \( (\mathbb{P}, <^*) \)). Before stating and proving the theorem it will be useful to first define some notation and terminology.

**Notation.** If \( x \) is in \([X]^k\) where \( X \subseteq \mathbb{P} \) is totally ordered by \( <^* \) then \( x^j \) is the \( j^{th} \) least element in \((x, <^*)\) where \( 0 \leq j < k \). If \( t \) is in \([\omega^n]^k\) and the values elements of \( t \) at \( l - 1 \) are distinct then \( s^j \) is the \( j^{th} \) element of \( t \) ordered by the last coordinate.

**Notation.** If \( x \) and \( y \) are in \([X]^k\) and \([X]^l\) respectively then \( x <^* y \) abbreviates \( x^j <^* y^j \) for all \( j < k \) and \( j' < l \) (similarly one defines \( x <^m y \)).

**Definition 4.2.3.** If \( X \subseteq \mathbb{P} \) then \( F \subseteq [X]^k \) is cofinal if for every \( x \) in \( X \), there is a \( y \) in \( F \) such that \( x <^* y \).

**Definition 4.2.4.** If \( F \subseteq [X]^k \) for some \( X \subseteq \mathbb{P} \) and \( t \in [\omega^n]^k \) then \( t \) is a splitting node of \( F \) if for infinitely many \( i \) there is a \( x \) in \( F \) extending \( t \) such that \( x^j(|t|) > i \) for all \( 0 \leq j < k \).

**Theorem 4.2.5.** If \( X \) is well ordered and unbounded in \((X, <^*)\) and \( F \subseteq [X]^k \) is cofinal then there is an integer \( l \) and a sequence \( x_i \) \((i \in \omega)\) such that for every \( i \) and every \( j \leq k \)

1. \( \Delta(x^j_i, x^j_{i+1}) = l \),
2. \( x_i^j(l) = x_{i+2}^j(l) \) for all \( i \) and \( 0 \leq j < k \), and

3. \( \text{alt}(x_i^j, x_{i+1}^j) = i \).

**Proof.** It may be assumed without loss of generality that there is a \( t_{-1} \) in \([\omega^l]^k\) such that

1. \( t_{-1} \) is an initial segment of every \( x \) in \( F \),

2. \( x^j <^l x^{j'} \) whenever \( x \) is in \( F \) and \( 0 \leq j < j' < k \), and

3. \( t_{-1} \) is a splitting node of \( F \).

Let \( D \subseteq F \) be a countable dense subset and \( a \in X \) be a \(<^*\) bound for \( D \). Pick a \( n_{-1} \) and refine \( F \) to a cofinal set \( E \) such that \( a <^{n_{-1}} x \) for every \( x \) in \( E \). Recursively construct for \( i \) in \( \omega \) integers \( n_i \), elements \( t_i \) of \( \bigcup_{M=0}^{\infty} [\omega^M]^k \), and elements \( x_i \) of \( D \) such that:

1. \( t_{-1} \) is an initial segment of \( t_i \) for \( i \geq 0 \) and \( \Delta(x_i^j, x_{i+2m+1}^j) = |t_{-1}| = l \) for all \( 0 \leq i, m \) and \( 0 \leq j < k \).

2. \( t_i \) is a splitting node of \( F \).

3. \( n_{i-1} < n_i \).

4. \( x_i \) extends \( t_i \).

5. \( x_i <^{n_i} a \).

6. \( t_{i+1} \) extends \( t_{i-1} \).

7. \( s_{i+1}^j(|s_{i-1}^j|) > n_i \).
8. $|t_{i+1}| > s_{i+1}^{k-1}(|t_i|)$.

Carrying out the recursion is routine. Notice that $t_i^j$ and $t_i^j + 1$ alternate on $S(t_i^j, t_i^j + 1) \cap \{0, \ldots, |t_i|\}$. This follows from 7. Furthermore, by 5, 7, and 8, $\text{alt}(x_i^j, s_{i+1}^j) = i$. Since $x_{i+1}$ extends $s_{i+1}$, $\text{alt}(x_{i+1}^j, x_{i+1}^j + 1) = i$. The other two conditions we are interested in follow immediately from 2 and 6.

\[\Box\]

4.3 Continuous Codings for Real Numbers

The focus for this section is to use the map $c$ from the previous section to code real numbers using partitions. The purpose of this is twofold. First, it generates a new proof that \textbf{PFA} implies $c$ is $b \ (\text{and hence } \aleph_2)^2$. Second, it gives an example of a coding device which is continuous and therefore subject to the consequences of \textbf{OCA}. While this still leaves open the question of whether \textbf{OCA} in fact implies that $c$ is $\aleph_2$, it does seem to shed some new light on this problem.

The crucial objects of our study will be the following binary colorings $c_r : [X]^2 \to \{0, 1\}$ defined for each real $r$ in $2^\omega$. Let $\phi : \omega \to 2^{<\omega}$ be a fixed bijection. For $x, y$ in $X$, define

$$c_r(x, y) = \begin{cases} 
1 & \text{if } \phi(\text{alt}(x, y)) = r \upharpoonright y(\Delta(x, y)) \\
0 & \text{otherwise.}
\end{cases}$$

The maps $c_r$ are continuous since $c$ is continuous. Also, no infinite set $H$ can be $1$-homogeneous for both $c_r$ and $c_s$ for two different $r$ and $s$ in $2^\omega$ ($H$ is

\[\text{The original proof is due to Todorčević and can be found sketched in [68] and in more detail in [7] and in [86].}\]
1-homogeneous for \( c_r \) if \( c \) is constantly 1 on \( [H]^2 \). To see this, note that by König's lemma, either \( H \) contains a sequence which converges in \( \mathbb{P} \) or else there is an infinite subset \( H_0 \) of \( H \) and an \( n \) such that \( \Delta(x, y) = n \) for all \( x, y \) in \( H_0 \). Now it is possible to choose a pair \( x <_{\text{lex}} y \) in \( H \) such that \( y(\Delta(x, y)) \) is larger than \( \Delta(r, s) \) which would contradict \( c_r(x, y) = c_s(x, y) = 1 \).

For a \( <^* \)-increasing sequence \( X = \{x_\alpha : \alpha < \kappa\} \subseteq \mathbb{P} \) of regular cardinal length define the following statement:

\((S_X)\) For every \( r \) in \( 2^\omega \) there is a \( \delta_r < \kappa \) of uncountable cofinality and a sequence \( E_{r, n} \) of subsets of \( \delta_r \) such that \( \{x_\alpha : \alpha \in E_{r, n}\} \) is 1-homogeneous for \( c_r \) for all \( n \) and \( \bigcup_{n=0}^{\infty} E_{r, n} \) contains a closed unbounded subset of \( \delta_r \).

Notice that \((S_X)\) implies \( c \) is at most \( \kappa \) and at least \( \aleph_2 \). We will now see that \textbf{PFA} implies \((S_X)\) for all unbounded \( <^* \)-increasing sequences \( X \subseteq (\mathbb{P}, <^*) \) and hence \textbf{PFA} implies \( b = c \).

Let \( X \) be a fixed unbounded \( <^* \)-increasing sequence in \((\mathbb{P}, <^*)\). If \( Y \subseteq X \) then define \( Q(c_r, Y) \) to be the collection of all finite subsets of \( Y \) which are 1-homogeneous for \( c_r \). Observe that if \( Y \) is unbounded and well ordered in type \( \omega_1 \) by \( <^* \) then every finite power of \( Q(c_r, Y) \) satisfies the countable chain condition. This is an immediate consequence of Theorem 4.2.5. Now let \( P \) be the standard partial order for collapsing \( |\kappa| \) to \( \aleph_1 \) without adding reals. In \( V^P \), there is a closed unbounded set \( \dot{E} \) in \( \kappa \) of order type \( \omega_1 \). Since no new reals have been added, \( \dot{Y} = \{x_\alpha : \alpha \in \dot{E}\} \) is unbounded in \( \mathbb{P} \). Applying \textbf{PFA} to \( P \models Q^{<\omega}(c_r, \dot{Y}) \) yields the objects \( \delta_r \) and \( \{E_{r, n}\}_{n=0}^{\infty} \).

For completeness I will now add the proof that \textbf{OCA} (and hence \textbf{PFA})
imply \( b \leq \aleph_2 \) (see Lemma 3.10 and Theorem 8.6 of [74]). If \( b > \aleph_2 \), then let \( a_\alpha (\alpha < \omega_2) \) be any \( \subseteq^* \) chain in \([\omega]^\omega \) (such a chain of length \( b \) exists; just take an initial segment of it). Now, by Zorn's lemma, select another increasing sequence \( b_\alpha (\alpha < \kappa) \) such that \( \kappa \) is a regular cardinal, \( a_\alpha \cap b_\beta \) is finite for all \( \alpha < \omega_2 \) and \( \beta < \kappa \), and there does not exist a \( c \subseteq \omega \) such that \( a_\alpha \subseteq^* c \) for all \( \alpha < \omega_2 \) and \( b_\beta \cap c \) is finite for all \( \beta < \kappa \). A result of Hausdorff [31] and Rothberger [51] implies that \( \kappa \) is uncountable since otherwise there would be an unbounded set of size \( \aleph_2 \) in \((\mathcal{P}, \subseteq^*)\).

Let \( X \) be the collection of all \((a, b)\) such that \( a \) is disjoint from \( b \) and \( a = a_\alpha, b = b_\beta \) for some \( \alpha < \omega_2 \) and \( \beta < \kappa \). Define \( G \subseteq [X]^2 \) by \( \{(a, b), (c, d)\} \in G \) iff
\[
(a \cap d) \cup (b \cap c) \neq \emptyset
\]
Notice that \( G \) is open. Now if \( H \subseteq X \) is such that \([H]^2 \cap G \) is empty, then \( H \) does not form a gap since it is split by
\[
\bigcup_{(a, b) \in H} a.
\]
It follows from the cardinality considerations on \( \kappa \) that \( X \) can not be split into countably many collections, none of which form a gap. Hence \( G \) is not countably chromatic.

On the other hand, if \( X_0 \subseteq X \) has size \( \aleph_1 \), pick an \( \alpha < \omega_2 \) such that \( a \subseteq^* a_\alpha \) whenever \( (a, b) \) is in \( X \). Let \( C \) be the collection of all \( c \) which differ from \( a_\alpha \) on a finite set. Then
\[
H_c = \{(a, b) \in X_0 : a \subseteq c \text{ and } c \cap b = \emptyset\}
\]
satisfies \([H_c]^2 \cap G = \emptyset\) for all \(c\) in \(C\) and hence \(G \upharpoonright X_0\) is countably chromatic. Thus \textbf{OCA} must fail.
Chapter 5

Measure Algebras, Forcing, and Martin’s Axiom

The next three chapters focus on an analysis of random graphs. For us, a random graph is a map $\hat{G} : [S]^2 \rightarrow \mathcal{R}$ where $S$ is some set (often $\omega_1$) and $\mathcal{R}$ is a measure algebra. It should be noted that this definition of “random graph” differs from the one usually considered in combinatorics (see, e.g., [10]) and also should not be confused with the random graph on $\omega$ considered by Rado.

It turns out that these graphs have applications in both topology and in the combinatorics of $\omega_1$. Results concerning random graphs fall into two categories. The first are those which illustrate that some random graphs can exhibit rather unpredictable (or “anti-Ramsey”) behavior. These results are generally results in ZFC and usually take the flavor of what statements always hold concerning graphs on $\omega_1$ after forcing with a measure algebra.

The second class of results are the dichotomies for random graphs. These
require some additional set theoretic assumptions to carry out the analysis and the axiom which has been used for this purpose to date is $\text{MA}_n$. While the results are somewhat varied, the ones in the second category generally share the use of a number of common tools and techniques. The purpose of this chapter is to isolate two of these tools and give an introduction to their role in proving dichotomies.

5.1 Measure Algebras

Before proceeding, let us first define some terms.

**Definition 5.1.1.** A measure algebra is a pair $(\mathcal{R}, \mu)$ where $\mathcal{R}$ is a complete Boolean algebra and $\mu : \mathcal{R} \to [0, 1]$ satisfies

1. $\mu(a) \geq 0$ with equality holding only for $a = 0$.
2. $\mu(1) = 1$.
3. $\mu(\bigvee_{i=0}^{\infty} a_i) = \sum_{i=0}^{\infty} \mu(a_i)$ whenever $a_i \wedge a_j = 0$ for all $i \neq j$.

This is what is usually referred to as a probability algebra ($\mu$ is usually allowed real or even extended real values). The reader is referred to [23] for a more complete account of the different classes of measure algebras.

**Definition 5.1.2.** If $(\mathcal{R}, \mu)$ is a measure algebra, then its character $\tau(\mathcal{R})$ is the cardinality of the smallest subcollection which completely generates it. If the character of $\mathcal{R}$ is countable then $\mathcal{R}$ is said to be separable.
Definition 5.1.3. A factor \( R_a \) of a measure algebra \((R, \mu)\) is the restriction of the measure algebra to one of its positive elements \( a \).

A measure algebra is homogeneous if all of its factors have the same character. Also, I will write the character of \((S, \upsilon)\) is everywhere greater than the character of \((R, \mu)\) if the character of each factor of \( S \) is greater than the character of every factor of \( R \).

The following construction is standard and extremely useful in the study of random graphs.

Definition 5.1.4. If \((R, \mu)\) is a measure algebra, \( I \) is a set, and \( \mathcal{U} \) is an ultrafilter on \( I \), then the ultrapower \((R^I/\mathcal{U}, \mu)\) is defined by setting

\[
\mu(f) = \lim_{i \to \mathcal{U}} \mu(f(i)),
\]

taking the quotient of \( R^I \) modulo the \( \mu \)-measure 0 sets \( \mathcal{N} \), and taking the (unique) completion \( R^I/\mathcal{N} \) with respect to \( \mu \).

Notice that the map which sends \( a \) to the constant map \( \bar{a} \) embeds any measure algebra \((R, \mu)\) into an ultrapower \((R^I/\mathcal{U}, \mu)\) in a measure preserving fashion.

Finally, if \( R \) is a complete subalgebra of a measure algebra \((S, \mu)\) then the projection map \( \pi_R : S \to R \) is defined by \( \pi_R(a) = \wedge \{ b \in R : a \leq b \} \). A useful fact which is easily verified is that if \( S \) is the ultrapower of \( R \) by some ultrafilter \( \mathcal{U} \) then

\[
\pi_R([f]) \leq \bigwedge_{i \in \mathcal{U}} \bigvee_{i \in \mathcal{U}} f(i).
\]
The right hand side is in fact the infimum taken over all representatives $f'$ in $[f]$.

### 5.2 Martin's Axiom and Random Graphs.

The first essential advance towards understanding random graphs was made by Laver in [41]. He proved that, if $\text{MA}_{\aleph_1}$ holds, then in $\mathbb{V}^R$, any tree of size $\aleph_1$ without an uncountable branch can be decomposed into countably many antichains. There were two parts to his argument. The first was to show that $\text{MA}_{\aleph_1}$ implies that a certain dichotomy holds for all random graphs on $\omega_1$. The second part of the proof was an interpretation of this dichotomy in the special case consisting of the incompatibility random graph for a tree in $\mathbb{V}^R$. Much of the latter portion of his argument can be extracted from the next section. The dichotomy result is the following.

**Theorem 5.2.1.** (MA$_{\aleph_1}$) If $\mathcal{G} : [\omega_1]^2 \to \mathbb{R}$ is a random graph over any measure algebra then

1. either there is a sequence $\mathcal{X}_n : \omega_1 \to \mathbb{R}$ such that for all $\alpha < \beta < \omega_1$
   and $n < \omega$
   $$\bigvee_{n=0}^{\infty} \mathcal{X}_n(\alpha) = 1 \quad \text{and} \quad \mathcal{G}(\alpha, \beta) \land \mathcal{X}_n(\alpha) \land X_n(\beta) = 0$$

2. or there is an uncountable sequence $F_\xi (\xi < \omega_1)$ of disjoint $n$ element subsets of $\omega_1$ and a $\delta > 0$ such that for every $\xi \neq \eta$

   $$\mu(\bigvee_{\alpha \in F_\xi, \beta \in F_\eta} \mathcal{G}(\alpha, \beta)) \geq \delta.$$
Remark 5.2.2. Notice that the first conclusion is equivalent to the statement that $\hat{G}$, when viewed as an $\mathcal{R}$-name for a subset of $[\omega_1]^2$, is forced to be countably chromatic. Analyzing the latter conclusion is always where new ideas are required when one tries to use this theorem to attack an unsolved problem in this area.

Remark 5.2.3. It should be remarked here that it is only in hindsight that this dichotomy was isolated as a recurring element of many of these arguments and, as such, did not appear explicitly in either [41] or [75].

Proof. Define a partial order $(Q, \leq)$ from $\hat{G}$ as follows. Let $Q$ be the set of all maps $p$ such that:

1. The domain of $p$ is a finite subset of $\omega_1$.
2. The range of $p$ is contained in $\mathcal{R}$.
3. For all $\gamma$ in the range of $p$, $p(\gamma)$ has measure greater than $1/2$ and is in the span of $\{\hat{G}(\alpha, \beta) : \alpha < \beta \leq \gamma\}$.
4. For all $\alpha < \beta$ in the domain of $p$, $\hat{G}(\alpha, \beta) \land p(\alpha) \land p(\beta) = 0$.

If $p$ and $q$ are two conditions in $Q$ then $q \leq p$ iff the domain of $q$ contains the domain of $p$ and $q(\alpha) \leq p(\alpha)$ whenever $\alpha$ is in the domain of $p$.

Notice that, by considering only those $p$ which map into some appropriate generating set for an algebra containing the events $\hat{G}(\alpha, \beta)$ ($\alpha < \beta < \omega_1$), it can be seen that $Q$ has a dense suborder of size $\aleph_1$. This if $(Q, \leq)$ is c.c.c., then applying $\text{MA}_{\aleph_1}$ to $(Q^{<\omega}, \leq)$ gives the sequence $X_n : \omega_1 \rightarrow \mathcal{R}$ as
specifies in the first conclusion. If \((\mathcal{Q}, \leq)\) isn't c.c.c. then fix an uncountable sequence \(\{p_\xi : \xi < \omega_1\}\) of pairwise incompatible conditions. Let \(F_\xi\) be the domain of \(p_\xi\). Fix an uncountable set \(\Gamma \subseteq \omega_1\) such that:

1. The family \(\{F_\xi : \xi \in \Gamma\}\) forms a \(\Delta\)-system with root \(F\).

2. There exists an \(\varepsilon > 0\) such that for every \(\xi\) in \(\Gamma\) and every \(\alpha\) in \(F_\xi\),
   \[
   \mu(p_\xi(\alpha)) > 1/2 + \varepsilon.
   \]

3. For every \(\xi, \eta\) in \(\Gamma\) and \(\alpha\) in \(F\), the measure of the difference of \(p_\xi(\alpha)\) and \(p_\eta(\alpha)\) is less than \(\varepsilon\).

The last item follows from the fact that the family

\[
\{\hat{G}(\alpha, \beta) : \alpha < \beta \leq \gamma\}
\]

generates a separable measure algebra for every \(\gamma < \omega_1\) and contains

\[
\{p(\alpha) : \alpha \in F\}
\]

whenever \(\gamma\) is greater than \(\max(F)\) and \(p\) is in \(\mathcal{Q}\). Now, for \(p_\xi\) and \(p_\eta\) to be incompatible, it must be the case that, for some \(\alpha\) in \(F_\xi \setminus F\) and \(\beta\) in \(F_\eta \setminus F\), \(\hat{G}(\alpha, \beta)\) has measure at least \(\varepsilon\). Thus, for all \(\xi \neq \eta\) in \(\Gamma\),

\[
\mu(\bigvee_{\alpha \in F_\xi, \beta \in F_\eta} \hat{G}(\alpha, \beta)) \geq \varepsilon
\]

giving the second alternative of the theorem. \(\square\)
5.3 A Variation of Maharam’s Theorem

Perhaps the most famous theorem in the area of measure algebras is the result of Maharam [42] which states that there are remarkably few of them.

**Theorem 5.3.1.** [42] There is a measure preserving isomorphism between any two homogeneous measure algebras of the same character.

If $\kappa$ is a cardinal, then the two point measure on $\{0,1\}$ induces a measure on the product $\{0,1\}^\kappa$. The resulting measure algebra is known as the Haar algebra. It is easy to see that this measure algebra has character $\kappa$ and is homogeneous. Hence Maharam’s theorem states that if $(\mathcal{R}, \mu)$ is homogeneous measure algebra of character $\kappa$ then there is a measure preserving isomorphism between $(\mathcal{R}, \mu)$ and the Haar algebra of character $\kappa$. From this it is immediate that arbitrary measure algebras are just copies of Haar algebras “glued together” with appropriate weights since any measure algebra contains a factor which is homogeneous in character.

This section will focus on the following form of Maharam’s theorem which is useful in the context of analyzing random graphs.

**Theorem 5.3.2.** Suppose $(\mathcal{R}, \mu)$ and $(\mathcal{S}, \nu)$ are measure algebras such that $\mathcal{S}$ has character everywhere greater than $\mathcal{R}$ and $\mathcal{A}$ is a complete subalgebra of $\mathcal{R}$. If $h : \mathcal{A} \to \mathcal{S}$ is a measure preserving homomorphism then $h$ extends to a measure preserving homomorphism $\tilde{h} : \mathcal{R} \to \mathcal{S}$.

In [23] Fremlin presents a full proof of Maharam’s theorem. Although this version of the theorem is not stated in his treatment, it is an immediate
consequence of the lemmas used there to prove the standard version of Maharam's theorem. Todorčević proved a special case of this theorem in [75] to solve problem DV on Fremlin’s list [24]. Theorem 5.3.2 was supplemented in his solution by the following fact.

**Theorem 5.3.3.** Suppose that $h : A \to B$ is a measure preserving homomorphism between two subalgebras of a measure algebra $R$. The map $h$ fixes $A \cap B$ iff $h(a) \leq \pi_{A \cap B}(a)$ for all $a$ in $A$.

**Proof.** Let $h : A \to B$ fix $A \cap B$ and $a$ in $A$ be given. Since $h(a \wedge c) = h(a) \wedge c$ for all $c$ in $A \cap B$,

$$h(a) \leq \pi_{A \cap B}(h(a)) = \pi_{A \cap B}(a).$$

For the other direction, note that $\pi_{A \cap B}(a) = a$ if $a$ is in $A \cap B$ and since $h$ is measure preserving and bounded by $\pi_{A \cap B}$, $h(a)$ must equal $a$ as well.  \[ \Box \]

To demonstrate the usefulness of these two theorems, I shall prove Theorem 4 of [75] using them.

**Theorem 5.3.4.** If $I, \Omega$ are index sets and $(R, \mu)$ is a measure algebra of character everywhere greater than $\#(I) \cdot \#(\Omega)$, $\mathcal{U}$ is an ultrafilter on $I$, and $f_\xi (\xi \in \Omega)$ is a sequence of elements of $(R^I/\mathcal{U}, \mu)$ then there is a sequence $c_\xi (\xi \in \Omega)$ of elements of $R$ such that

1. $\mu(f_\xi) = \mu(c_\xi)$ for all $\xi$ in $\Omega$ and

2. whenever $\Gamma$ is a finite subset of $\Omega$

$$\bigwedge_{U \in \mathcal{U}} \bigvee_{i \in \Gamma} \bigwedge_{\xi \in \Gamma} f_\xi(i) \leq \bigwedge_{\xi \in \Gamma} c_\xi.$$
Proof. Let \( \mathcal{R} \) denote the constant sequences in \((\mathcal{R}^I/\mathcal{U}, \mu)\). Let \( \mathcal{A} \) be a complete subalgebra of \( \mathcal{R} \) of character \( \#(I) \cdot \#(\Omega) \) containing \( f_\xi \) for all \( \xi \) in \( \Omega \) such that \( \pi_\mathcal{R}(a) \) is in \( \mathcal{A} \) whenever \( a \) is in \( \mathcal{A} \). By Theorem 5.3.2 there is a measure preserving homomorphism \( h : \mathcal{A} \rightarrow \mathcal{R} \) such that \( h(\bar{a}) = a \) whenever \( \bar{a} \) is a constant map. By Theorem 5.3.3 \( h(a) \leq \pi_\mathcal{R}(a) \). Since \( \pi_\mathcal{R}(f) \) is bounded by (the map which is constantly)

\[
\bigwedge_{U \in \mathcal{U}} \bigvee_{i \in U} f(i),
\]

the elements \( c_\xi = h(f_\xi) (\xi \in \Omega) \) satisfy the theorem. \( \square \)
Chapter 6

Applications of Random Forcing to Combinatorics

The program of analyzing the effects of forcing with a measure algebra breaks into (at least) two different areas: combinatorics and topology. Among the first results in this line of research was Kunen’s proof that Martin’s Axiom for $\sigma$-linked posets is preserved by adding a random real. Kunen also demonstrated that $\text{MA}_{\aleph_1}$ always fails after adding one random real by modifying a construction of Roitman and showing that Souslin’s condition is not necessarily productive (see [49]).

Laver later showed that forcing with a measure algebra does not produce Souslin trees (assuming $\text{MA}_{\aleph_1}$ in the ground model). Barnett, in [2] and [3], analyzed the effects of random reals on partition relations. In this chapter we will analyze the effect of random reals on a number of combinatorial statements.
6.1 Partition Calculus

Partition calculus is the study of more general forms of Ramsey's theorem. As its discussion is greatly facilitated by notation, it helps to first make a few definitions (see [20] for a more complete treatment).

**Notation.** Suppose that $r, d$ are natural numbers and $\alpha, \gamma_1, \ldots, \gamma_r$ are order types, for $i \leq r$. The expression

$$\alpha \rightarrow (\gamma_1, \ldots, \gamma_r)^d$$

abbreviates "for every coloring $c : [\alpha]^d \rightarrow \{1, \ldots, r\}$, there is an $i \leq r$ and a subset $H$ of $\alpha$ of order type $\gamma_i$ such that $c$ is constantly $i$ on $[H]^d$." If all of the $\gamma_i$'s are the same then the notation becomes $\alpha \rightarrow (\gamma)^d$.

Ramsey's theorem in this language is simply $\omega \rightarrow (\omega)^d_r$ for all $d, r$. There are many variations on this theme. The following will be of interest to us as well.

**Notation.** If $\alpha, \beta, \gamma_0, \gamma_1$ are order types then $\alpha \rightarrow (\beta, (\gamma_0 : \gamma_1))^2$ abbreviates "for every coloring $c : [\alpha]^2 \rightarrow \{0, 1\}$ either

1. there is a subset $B \subseteq \alpha$ of order type $\beta$ such that $c$ is constantly 0 on $[B]^2$ or else

2. there is a pair of sets $C_0, C_1 \subseteq \alpha$ such that the order type of $C_i$ is $\gamma_i$, $C_0 < C_1$, and $c$ is constantly 1 on $C_0 \times C_1$.”

Sierpiński's famous partition [52] shows that $c \not\rightarrow (\omega_1)^2_2$. Hajnal showed in [30] that if the Continuum Hypothesis holds then even $\omega_1 \not\rightarrow (\omega_1, (\omega : 2))^2$
thus implying that the Dushnik-Miller relation $\omega_1 \rightarrow (\omega_1, \omega + 1)^2$ is optimal (at least in ZFC). On the other hand Hajnal and Laver (see [40]) proved $\text{MA}_{\aleph_1}$ implies $\omega_1 \rightarrow (\omega_1, (\alpha : \omega_1))^2$ for all $\alpha < \omega_1$.

Later, Barnett demonstrated in [2] that if $\text{MA}_{\aleph_1}$ holds then $\omega_1 \rightarrow (\omega_1, (\alpha : \omega_1))^2$ still is true after forcing with a separable measure algebra. She notes the result of Todorčević (see [74] Proposition 6.4) which states that if there is a Sierpiński set then $\omega_1 \not\rightarrow (\omega_1, (\omega : \omega_1))^2$. Barnett shows in [3], however, that if $\text{MA}_{\aleph_1}$ holds then $\omega_1 \rightarrow (\omega_1, (\alpha : n))^2$ for all $\alpha < \omega_1$ and $n < \omega$ after forcing with any measure algebra. In this section we will extend this result by replacing $(\alpha : n)$ by $(\alpha : \alpha)$.

Before proceeding, it will be useful to develop some notation and make some definitions. The motivation for the following definitions comes from a class of ultrafilters which were introduced and developed by Hajnal and Laver to show that $\text{MA}_{\aleph_1}$ implies $\omega_1 \rightarrow (\omega_1, (\alpha : \omega_1))^2$ for all $\alpha < \omega_1$.

**Definition 6.1.1.** A partition tree $T$ (on a set of ordinals) is a collection of closed countable sets of ordinals such that the following conditions hold

1. $T$ is a tree under the order of reverse inclusion.

2. If $A$ and $B$ are incompatible elements of $T$ then either $A < B$ (i.e. $\sup A < \inf B$) or $B < A$.

3. An element of $T$ is a terminal node (i.e. has no successors) iff it is a singleton.
4. If $A$ is an element of $T$ which is not a terminal node, then $A$ has infinitely many immediate successors, none of which contain $\max A$.

5. If $A$ is a nonterminal node of $T$ then $\max A$ is the unique limit point of the collection of immediate successors of $A$ (i.e. the minimums of the immediate successors of $A$ forms an $\omega$-sequence converging to $\max A$).

6. If $B, C$ are immediate successors of a node $A$ in $T$ such that $\max B < \min C$ then $\otp B \leq \otp C$.

Remark. This is a slightly different definition than the one usually associated with partition trees on linear orders.

Notation. The set of all $\alpha$ such that $\{\alpha\}$ is a node of $T$ will be denoted by $\pi T$.

Notice that all partition trees are well founded (i.e. have no infinite branch) since the maximum function is a decreasing map into the ordinals along any branch of a partition tree. The definition of a partition tree is made to facilitate the definition and discussion of classes of objects, usually filters, which are associated with them.

Definition 6.1.2. Suppose $T$ is a partition tree. A filter $\mathcal{F}$ on $T$ is a collection of partition subtrees of $T$ such that

1. the empty tree is not in $\mathcal{F}$,

2. if $\mathcal{F}_0 \subseteq \mathcal{F}$ is finite then $\cap \mathcal{F}_0$ is in $\mathcal{F}$, and
3. if $S$ is in $\mathcal{F}$ and $S'$ is a subtree of $T$ which contains $S$, then $S'$ is also in $\mathcal{F}$.

A filter $\mathcal{F}$ on $T$ is *tree-like* if there is an association of a filter $\mathcal{F}_A$ on the successors of $A$ for each nonterminal $A$ in $T$ such that $S$ is in $\mathcal{F}$ iff for every $A$ in $S$, $A$ has $\mathcal{F}_A$-many successors in $T$. A tree-like filter is an *ultrafilter* if the filters $\mathcal{F}_A$ are all ultrafilters. A tree-like filter $\mathcal{F}$ is *uniform* if $\mathcal{F}_A$ contains the Fréchet filter for every $A$ in $T$.

Notice that every filter $\mathcal{F}$ on a partition tree $T$ induces a filter $\pi \mathcal{F}$ on $\pi T$ which is generated by $\{\pi S : S \in \mathcal{F}\}$.

**Definition 6.1.3.** The *order type* $\text{otp}(T)$ of a partition tree $T$ is defined by recursion on the rank of $T$. If $T$ is just a singleton, then $\text{otp}(T) = 1$. If $\{A_n\}_{n=0}^\infty$ is an increasing enumeration of the first level of $T$ above the root of $T$ then $\text{otp}(T) = \sum_{n=0}^\infty \text{otp}(T[A_n])$ (where $T[A] = \{B \in T : B \subseteq A\}$).

It is easily checked that, if $T$ is a partition tree

$$\text{otp}(T) = \min\{\text{otp}(X) : X \in \mathcal{F}\}$$

whenever $\mathcal{F}$ is a uniform tree-like filter on $T$, thus justifying the definition of $\text{otp}(T)$. It is well known and easily verified that for every indecomposable ordinal $\alpha$, there is a partition tree $T$ such that $\pi T \subseteq \alpha$ and $\text{otp}(T) = \alpha$. The following is essentially proven in [40].

**Lemma 6.1.4 (MA$_{\aleph_1}$).** If $\mathcal{F}$ is a uniform tree-like filter on a partition tree $T$ and $\{A_\xi : \xi < \omega_1\}$ is a sequence of elements of $\pi \mathcal{F}$ then there is an uncountable $B \subseteq \omega_1$ such that $\bigcap_{\xi \in B} A_\xi$ has order type at least $\text{otp}(T)$. 
The following standard absoluteness result will be very useful in proving the main theorem (see [6]). The proof is included for completeness.

**Lemma 6.1.5.** Suppose $M \subseteq N$ are models of ZFC which contain the same ordinals. If $\{A_\xi : \xi < \omega_1\} \subseteq M$ is a sequence of subsets of $\omega_1$ and $\alpha < \omega_1$ is such that there are sets $A, B \subseteq \omega_1$ in $N$ with $A \subseteq \bigcap_{\xi \in B} A_\xi$ and $\text{otp}(A) = \text{otp}(B) = \alpha$ then $M$ contains a pair of such sets which have the same properties.

**Proof.** Fix a bijection $e : \omega \to \alpha$. Let $(E, <)$ be the collection of all pairs $(s, t)$ such that

1. $s, t$ are injections from some set $\{0, \ldots, n\}$ into $\omega_1$,
2. $s(i)$ is in $A_{t(j)}$ for every $i, j \leq n$, and
3. for every $i < j \leq n$, $s(i) < s(j)$ iff $t(i) < t(j)$ iff $e(i) < e(j)$

and $<$ orders $E$ by coordinatewise extension. Clearly an infinite branch through $(E, <)$ gives the desired object and $(E, <)$ is well founded in $M$ iff it is in $N$. \qed

We are now ready to prove the main theorem of this section.

**Theorem 6.1.6.** Suppose that $V$ models $\text{MA}_{\aleph_1}$ and $\mathcal{R}$ is a measure algebra. In $V^\mathcal{R}$ the partition relation $\omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2$ holds for all $\alpha < \omega_1$.

**Proof.** Let $\alpha < \omega_1$ be given and assume without loss of generality that $\alpha$ is indecomposable. Suppose that $\dot{G}$ is a name for a graph on $\omega_1$. By Theorem
5.2.1 either \( \hat{G} \) is forced to be countably chromatic or else there is a sequence \( F_\xi (\xi < \omega_1) \) of disjoint \( n \) elements sets and a \( \delta > 0 \) such that for every \( \xi < \eta \) there exist \( i \) and \( j \) such that \( F_\xi(i) \) and \( F_\eta(j) \) are connected in \( \hat{G} \) with probability at least \( \delta \).

Let \( T \) be a partition tree such that \( \pi T \subseteq \alpha \) and \( \text{otp}(T) = \alpha \). Let \( U \) be a uniform tree-like ultrafilter on \( T \). Pick an uncountable set \( B \subseteq \omega_1 \setminus \alpha \) and a pair \( i, j \leq n \) such that whenever \( \eta \) is in \( B \), there are \( \pi U \) many \( \xi \) such that \( F_\xi(i) \) and \( F_\eta(j) \) are connected in \( \hat{G} \) with probability at least \( \delta \). We will now confuse \( B \) with the set \( \{F_\eta(j) : \eta \in B\} \) and \( T \) and \( U \) with the corresponding objects they induce on \( \{F_\xi(i) : \xi \in \alpha\} \).

Fix a measure algebra \( (S, \mu) \) which contains \( R \) and has character everywhere greater then \( \aleph_1 \). Let \( T \) be a tree-like ultrafilter on \( A \) and set

\[
\tilde{f}_\beta(\alpha) = \llbracket \{\alpha, \beta\} \in \hat{G} \rrbracket
\]

for \( \alpha \in A \) and \( \beta \in B \). Applying Theorem 5.3.4 repeatedly, find liftings \( \tilde{f}_\beta : T \to S \) such that

1. \( \tilde{f}_\beta(\{\xi\}) = f_\beta(\xi) \) for all \( \xi \) in \( \pi T \) and \( \beta \) in \( B \),

2. the measure of \( \tilde{f}_\beta(A) \) equals

\[
\lim_{B \to U_A} \mu(\tilde{f}_\beta(B))
\]

where \( A \) is a nonterminal node of \( T \), and

3. for every finite \( \Gamma \subseteq B \) and nonterminal \( A \) in \( T \)

\[
\bigwedge_{\beta \in \Gamma} \tilde{f}_\beta(A) \leq \bigwedge_{U \in U_A} \bigvee_{B \in U} \bigwedge_{\beta \in \Gamma} \tilde{f}_\beta(B)
\]
CHAPTER 6. COMBINATORICS

Define \( \hat{S}_\beta \) by
\[
\lceil A \in \hat{S}_\beta \rceil = \check{f}_\beta(A)
\]
and \( \hat{A} \) by
\[
\lceil \beta \in \hat{A} \rceil = \check{f}_\beta(\text{root}(T)).
\]
It is easy to verify that \( \hat{U} \cup \{ \hat{S}_\beta : \beta \in \Lambda \} \) generates a tree-like filter \( \hat{U} \) on \( T \). Let \( c \in \mathcal{R}^+ \) force that \( \Lambda \) is uncountable. In \( \mathbf{V}^S \) fix a c.c.c. partial order \((\mathcal{P}, \leq)\) which forces \( \text{MA}_{\aleph_1} \). Applying Theorem 6.1.4 in the extension \( \mathbf{V}^{S \ast \mathcal{P}} \) to \( \{ \pi \hat{S}_\beta : \beta \in \hat{\Lambda} \} \), it is possible to find sets \( \hat{A}_0 \subseteq \pi T \) and \( \hat{B}_0 \subseteq \hat{\Lambda} \) such that \( \text{otp}(\hat{A}_0) = \text{otp}(\hat{B}_0) = \alpha \) and \( \hat{A}_0 \times \hat{B}_0 \subseteq \check{G} \) (see [40]). Now applying Theorem 6.1.5, the desired homogeneous set can be pulled back to \( \mathbf{V}^\mathcal{R} \).

It might initially seem that there is no reasonable positive partition relation which lies between \( \omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2 \) and \( \omega_1 \rightarrow (\omega_1, (\alpha : \omega_1))^2 \). This, however, is not the case. Let \((\alpha : \beta : \gamma)\) denote the class of graphs of the form \((A \times B) \cup (B \times C) \cup (A \times C)\) where \( A \prec B \prec C \) and \( \text{otp}(A) = \alpha \), \( \text{otp}(B) = \beta \), and \( \text{otp}(C) = \gamma \). It is clear that \( \omega_1 \rightarrow (\omega_1, (\alpha : \omega_1))^2 \) implies that \( \omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2 \) which in turn implies that \( \omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2 \).

This leads us to the following

**Question 6.1.7.** Does \( \omega_1 \rightarrow (\omega_1, (\alpha : \alpha))^2 \) hold for \( \omega \leq \alpha < \omega_1 \) in an extension of a model of \( \text{MA}_{\aleph_1} \) by an arbitrary number of random reals?

Being even more ambitious, it is also reasonable to ask the following

**Question 6.1.8.** Does \( \omega_1 \rightarrow (\omega_1, \alpha)^2 \) hold for \( \omega + 2 \leq \alpha < \omega_1 \) after forcing with a nontrivial measure algebra over a model of \( \text{PFA} \)?
It should be remarked here that it is unknown whether $\mathbf{MA}_{\aleph_1}$ implies even $\omega_1 \rightarrow (\omega_1, \omega^2 + 2)^2$ (see [22] and [24]). Therefore it is more natural to ask the later question under the assumption of $\mathbf{PFA}$.

### 6.2 Uniformizing Ladder Systems

A ladder system is a sequence of maps $f_\lambda : C_\lambda \rightarrow \omega$ indexed by the limit ordinals $\lambda$ such that $C_\lambda$ is a cofinal subsets of $\lambda$ of order type $\omega$. A ladder system $\{f_\lambda\}$ is uniform if there is a map $g : \omega_1 \rightarrow \omega$ such that for each $\lambda$ $f_\lambda(\alpha) = g(\alpha)$ for all but finitely many $\alpha$ in $C_\lambda$. Devlin and Shelah have shown in [14] that $\mathbf{MA}_{\aleph_1}$ implies all ladder systems are uniform. In this section we will discuss the influence of random reals on the existence of nonuniform ladder systems.

**Theorem 6.2.1.** ($\mathbf{MA}_{\aleph_1}$) After forcing with a separable measure algebra, all ladder systems are uniform.

**Notation.** If $C \subseteq \omega_1$ is a countable sequence of ordinals and $n$ is an integer then let $C/n$ denote the set $C$ with its least $n$ elements removed. Also, let $C \upharpoonright n$ denote the least $n$ elements of $C$.

**Proof.** Let $\{\dot{f}_\lambda : \lambda \in \Lambda\}$ be a given sequence of names corresponding to a ladder system coloring where the domain of $\dot{f}_\lambda$ is $\check{C}_\lambda$. As in Theorem 5.2.1, define a partial order $(Q, \preceq)$ as follows. Elements of $Q$ are functions $p$ such that

1. the domain of $p$ is a finite subset of $\Lambda \times \omega$,
2. the range of \( p \) is contained in \( R \),

3. for all \((\alpha, m)\) and \((\beta, n)\) in the domain of \( p \), \( p(\alpha, m) \wedge p(\beta, n) \) forces that \( \dot{f}_\alpha \) and \( \dot{f}_\beta \) agree on \((\dot{C}_\alpha \setminus m) \cap (\dot{C}_\beta \setminus n)\), and

4. \( p(\alpha, i) \) is disjoint from \( p(\alpha, j) \) whenever \((\alpha, i), (\alpha, j)\) are distinct elements of the domain of \( p \).

The order \( \leq \) is defined by \( q \leq p \) iff

1. the domain of \( q \) contains the domain of \( p \),

2. \( q(\alpha, m) \leq p(\alpha, m) \) whenever \((\alpha, m)\) is in the domain of \( p \), and

3. for all \((\alpha, m)\) in the domain of \( p \), \( \delta(q(\alpha, m)) = \delta(p(\alpha, m)) \).

Here if \( B \) is in \( R \) then \( \delta(B) = 2^{-k} \) where \( k \) is minimal such that \( 2^{-k} < \mu(B) \). Notice that for each \( p \) in \( Q \) there is a \( \dot{g}_p \) which is a name for a function from \( \omega_1 \) into \( \omega \) such that, for every \((\alpha, m)\) in the domain of \( p \), \( p(\alpha, m) \) forces that \( \dot{f}_\alpha \) and \( \dot{g}_p \) agree on \( \dot{C}_\alpha \setminus m \). If \( p \) is in \( Q \), let \( \varepsilon(p) > 0 \) be the least rational (in some fixed enumeration) such that if \((\alpha, m)\) is in the domain of \( p \) then \( \mu(p(\alpha, m)) > \delta(p(\alpha, m)) + \varepsilon(p) \). Note that if \( F \) is a subset of \( \omega_1 \) and \( p, q \) are conditions in \( Q \) such that

1. \( \varepsilon(p) = \varepsilon(q) = \varepsilon \),

2. for every \((\alpha, m)\) in intersection of the domains of \( p \) and \( q \), \( p(\alpha, m) \) and \( q(\alpha, m) \) differ on a set of measure at most \( \varepsilon /3 \),
3. there is a $A$ of measure at least $1 - \varepsilon/3$ which forces $\check{C}_\alpha \cap \check{C}_\beta \subseteq F$ for all $(\alpha, m)$ in the domain of $p$ and $(\beta, n)$ in the domain of $q$, and
4. there is a $B$ of measure at least $1 - \varepsilon/3$ which forces that $\check{g}_p \upharpoonright F = \check{g}_q \upharpoonright F$

then $p$ and $q$ are compatible.

Claim 6.2.2. $\langle Q, \subseteq \rangle$ satisfies the countable chain condition.

Proof. Let $\{p_\xi : \xi < \omega_1\}$ be a given sequence of conditions in $Q$. It suffices to find a pair $\xi$ and $\eta$ and a set $F$ which satisfy conditions 1-4 above. Let $\Gamma \subseteq \omega_1$ be an stationary subset such that the following conditions hold:

1. $\varepsilon(p_\xi) = \varepsilon(p_\eta)$ for all $\xi, \eta$ in $\Gamma$ (condition 1).
2. $\{\text{dom}(p_\xi) : \xi \in \Gamma\}$ forms a $\Delta$-system with root $D$.
3. If $(\alpha, m)$ is in the domain of $p_\xi$ and is not in $D$ then $\alpha > \xi$.
4. If $\xi, \eta$ are in $\Gamma$ then $p_\xi(\alpha, m)$ and $p_\eta(\alpha, m)$ differ on a set of measure less than $\varepsilon/3$ for all $(\alpha, m)$ in $D$ (condition 2).
5. If $\xi < \eta$ are in $\Gamma$ then it is forced that $\check{C}_\alpha$ is a subset of $\eta$ whenever $(\alpha, m)$ is in the domain of $p_\xi$.

By 3 it is possible to find, for each $\xi$ in $\Gamma$, a finite set $F_\xi$ contained in $\xi$ and a condition $A_\xi$ of measure at least $1 - \varepsilon/6$ which forces that $\xi \cap \check{C}_\alpha$ is contained in $F_\xi$ whenever $(\alpha, m)$ is in the domain of $p_\xi$. By pressing down on the $F_\xi$'s, it is possible to find an uncountable $\Gamma' \subseteq \Gamma$ such that $F_\xi = F$ is the same
for all $\xi$ in $\Gamma'$. Notice that for every $\xi \neq \eta$ in $\Gamma'$, $F$ and $A = A_\xi \land A_\eta$ satisfy condition 3 for checking the compatibility of two conditions.

By the separability of $\mathcal{R}$, it is possible to find a $B$ of measure at least $1 - \varepsilon/3$ and $\xi \neq \eta$ in $\Gamma'$ such that $B$ forces $\dot{g}_\xi \upharpoonright F = \dot{g}_\eta \upharpoonright F$. Thus $\xi$ and $\eta$ satisfy condition 4 and, since we have already ensured that they satisfy the other three, $p_\xi$ and $p_\eta$ are compatible. \hfill \square

Define $\mathcal{D}_{\alpha, \varepsilon}$ to be the collection of all $p$ in $\mathcal{Q}$ such that for all $\alpha$ in $\Lambda$ for which $(\alpha, m)$ is in the domain of $p$ for some $m$,

$$\sum_{i=1}^{k} \delta(p(\alpha, m_i)) > 1 - \varepsilon$$

where $m_1, \ldots, m_k$ list the integers $m$ for which $(\alpha, m)$ is in the domain of $p$. It is routine to verify that $\mathcal{D}_{\alpha, \varepsilon}$ is dense for all $\alpha, \varepsilon$.

If $\mathcal{H}$ is a filter intersecting all of the $\mathcal{D}_{\alpha, \varepsilon}$'s then define $P : \Lambda \times \omega \to \mathcal{R}$ by

$$P(\alpha, m) = \land\{p(\alpha, m) : p \in \mathcal{H} \text{ and } (\alpha, m) \in \text{dom}(p)\}.$$ 

In $V^\mathcal{R}$, $P$ defines a function $\dot{P}$ from $\Lambda$ to $\omega$ by $\dot{P}(\alpha) = m$ where $m$ is the unique integer such that $P(\alpha, m)$ is in the generic filter ($\{P(\alpha, m)\}_m$ is a maximal antichain if $\mathcal{H}$ meets $\mathcal{D}_{\alpha, \varepsilon}$ for all $\varepsilon$). Define $\dot{g}(\xi) = \dot{f}_\alpha(\xi)$ if $\xi$ is in $C_\alpha/\dot{P}(\alpha)$ for some $\alpha$ and arbitrarily otherwise. By condition 3 in the definition of $\mathcal{Q}$, $\dot{g}$ is well defined and clearly $\dot{g}$ uniformizes the ladder system. \hfill \square

**Fact 6.2.3.** If $\mathcal{R}$ is a homogeneous nonseparable measure algebra then there is a nonuniform ladder system in $V^\mathcal{R}$. 

CHAPTER 6. COMBINATORICS

Remark 6.2.4. This can be established by some slick arguments using the inequality $2^{\aleph_0} < 2^{\aleph_1}$ and a theorem of Devlin in [14] but I shall give a direct proof.

Proof. Let $\dot{r}: \omega_1 \to \{0, 1\}$ be a random subset of $\omega_1$ and let $\{C_\lambda\}$ be any sequence of ladders in the ground model. Define $f_\lambda$ on $C_\lambda$ to be the constant value $\dot{r}(\lambda)$. Suppose that $\dot{g}: \omega_1 \to \{0, 1\}$ is a name for a potential uniformizing function. By closing off under some appropriate function, it is possible to find a limit ordinal $\delta < \omega_1$ such that $\dot{r}(\delta)$ is undetermined in $V[\dot{g} \upharpoonright \delta]$. Now it is easy to choose a condition which forces $\dot{r}(\delta)$ to have a different value than the value $\dot{g}$ eventually attains on $C_\delta$ (if there is such a value). \qed

6.3 Wage's Lemma

Consider the following combinatorial statement for a regular cardinal $\theta$:

$W(\theta)$ If $A$ is a family of countable subsets of $\theta$ such that every pair has a finite intersection, then there is a cofinal subset of $\theta$ which has finite intersection with every element of $A$.

This statement is known as Wage's Lemma and was considered by Wage in [87] through the course of working on topological problems related to S and L spaces. It also has uses in combinatorics — the interested reader is referred to [87]. In this section we shall interest ourselves in proving that $W(\theta)$ is a consequence of $MA_\theta$ which can hold after forcing with an arbitrary measure algebra.
Theorem 6.3.1 (MA$_{\theta}$). \( W(\theta) \) holds after forcing with any measure algebra.

Proof. Let \( \{ \dot{A}_\xi : \xi < \theta \} \) be a sequence of \( \mathcal{R} \)-names for almost disjoint countable subsets of \( \theta \). For each \( \alpha < \theta \) let \( \mathcal{R}_\alpha \) be the separable complete subalgebra of \( \mathcal{R} \) generated by \( \{ [\xi \in \dot{A}_\alpha] : \xi < \theta \} \) (note that \( [\xi \in \dot{A}_\alpha] \) is \( 0 \) for all but countably many \( \xi \)). Let \( (\mathcal{Q}, \leq) \) be the forcing notion of all pairs \( (X, P) \) which satisfy the following properties:

1. \( X \) is a finite subset of \( \theta \).
2. \( P \) is a finite partial map from \( \theta \) into \( \mathcal{R}^+ \).
3. If \( (\beta, M) \) is in the domain of \( P \) then \( P(\beta, M) \) is in \( \mathcal{R}_\beta \).
4. \( P(\beta, M) \) forces that \( X \cap \dot{A}_\alpha \) has size at most \( M \).

The ordering on elements of \( \mathcal{Q} \) is defined by \( q \leq r \) if the following hold:

1. \( X_q \) contains \( X_r \).
2. The domain of \( P_q \) contains the domain of \( P_r \).
3. For every \( (\beta, M) \) in the domain of \( P_r \), \( \delta(P_r(\beta, M)) = \delta(P_q(\beta, M)) \).

As in the section on random ladder systems, for each condition \( (X, P) \), there is a rational \( \varepsilon = \varepsilon(X, P) > 0 \) such that \( \mu(P(\beta, M)) > \delta(P(\beta, M)) + \varepsilon \).

Claim 6.3.2. \( (\mathcal{Q}, \leq) \) satisfies the countable chain condition.
Proof. Let \( \{(X_\xi, P_\xi) : \xi < \omega_1\} = \{q_\xi : \xi < \omega_1\} \) be a sequence of conditions in \( Q \). Select an uncountable \( \Gamma \subseteq \omega_1 \) and an \( \varepsilon > 0 \) such that the following conditions hold:

1. \( \varepsilon(X_\xi, P_\xi) = \varepsilon. \)

2. The families \( \{X_\xi : \xi < \omega_1\} \) and \( \{\text{dom}(P_\xi) : \xi < \omega_1\} \) form \( \Delta \)-systems with roots \( X \) and \( D \) respectively. Moreover \( X_\xi \cap \xi = X \) and \( \text{dom}(P_\xi) \cap \xi \times \omega = D. \)

3. If \( (\beta, M) \) is in \( D \) and \( \xi, \eta < \omega_1 \) then \( P_\xi(\beta, M) \) and \( P_\eta(\beta, M) \) differ by a set of measure at most \( \varepsilon/2. \)

4. If \( \xi < \eta \) are in \( \Gamma \) then \( X_\xi \subseteq \eta \) and it is forces that \( \hat{A}_\beta \) is contained in \( \eta \) for all \( (\beta, M) \) in \( \text{dom}(P_\xi) \).

Let \( m \) denote the cardinality of \( X_\xi \setminus X \) and \( I \) be the first \( \omega \) elements of \( \Gamma \).

For \( \eta > \sup I \) and \( i \leq m \), define \( f_i^\eta : I \to R \) so that \( f_i^\eta(\xi) \) is the event that the \( i^{\text{th}} \) element of \( X_\xi \setminus X \) is in \( \hat{A}_\alpha \) for some \( (\alpha, M) \) in the domain of \( P_\eta \).

Notice that if \( \eta \neq \zeta \) are in \( \Gamma \setminus I \) then \( \mu(f_i^\eta(\xi) \cdot f_i^\zeta(\xi)) \) vanishes for all \( i \) as \( \xi \to \sup I \). In particular, this means that for some \( \eta \) in \( \Gamma \setminus I \) and some \( \xi \) in \( I \) \( \mu(f_i^\eta(\xi)) < \varepsilon/m \) for all \( i \). It is now easy to verify that \( (X_\xi, P_\xi) \) and \( (X_\eta, P_\eta) \) are compatible.

The proof is now finished with an application of \( \text{MA}_\delta \) to \( (Q, \leq) \) and some suitably chosen dense sets (see the proof of Theorem 6.2.1).
CHAPTER 6. COMBINATORICS

6.4 Fragments of Martin’s Axiom

The very first results concerning random reals and Martin’s Axiom were due to Kunen and concern the effect random reals have on the standard fragments of Martin’s Axiom.

Theorem 6.4.1. (see [49]) The addition of one random real preserves Martin’s Axiom for $\sigma$-linked partial orders.

By making a variation on Roitman’s result for Cohen forcing Kunen showed that $\text{MA}_{\aleph_1}$ always fails after adding a random real:

Theorem 6.4.2. (see [49]) After the addition of a single random real, there is compactum $K$ such that $K$ satisfies Souslin’s condition but $K^2$ doesn’t.

It is still unclear whether this result is optimal.

Question 6.4.3. ($\text{MA}_{\aleph_1}$) After adding one random real, does $\text{MA}_{\aleph_1}$ hold for all partial orders whose countable chain condition is productive?

The situation after adding many random reals (i.e. forcing with a homogeneous nonseparable measure algebra) is much different. In this model, there is always a Sierpiński set and a covering of $\mathbb{R}$ by $\aleph_1$ many nowhere dense sets (hence $\text{MA}_{\aleph_1}$ fails even for countable partial orders). It can also be shown that there is a disparity between most of the local chain conditions (e.g. the example from Section 3.2 shows that there a compactum which has Knaster’s condition $K_2$ but not Knaster’s condition $K_3$). Kunen’s example mentioned above demonstrates that after even one random real there is an
example of a compactum which satisfies Souslin's condition but not Knaster's condition.

What was less clear until recently was whether there was an example which distinguishes between the property of having the c.c.c. productively (i.e. the product with any other c.c.c. partial order is c.c.c.) and Knaster's condition. We will see in Section 7.1 that there is always such an example in the presence of a Sierpiński set. A Sierpiński set with the density topology gives an example of a cometrizable L space (see Section 7.1). The natural partial order for forcing a discrete set through such spaces is always c.c.c. and, since being cometrizable is absolute, productively c.c.c. (see the Claim on page 62 of [74]). The partial order, however, will not satisfy Knaster's condition.

In [66] Todorčević asks whether the nonexistence of a nonseparable perfectly normal compactum (i.e. a compact L space) is equivalent to a "standard fragment" of MA_{\omega_1}. In [75], he has shown that if MA_{\omega_1} holds then forcing with any measure algebra does not introduce a compact L space. At this time, I am unaware of any fragment of MA_{\omega_1} which is considered standard that is not known to fail after forcing with a nonseparable measure algebra (except perhaps SM_{\omega_1}, which was designed for the purpose of killing compact L spaces).
Chapter 7

Applications of Random Forcing to Topology

Two basic properties one encounters in general topology are separability and the Lindelöf property. Here a space is Lindelöf if every open cover has a countable subcover. Thus it is a dual notion to separability. These two properties are hereditary in the class of all metric spaces as they are both equivalent to having a countable base in this context. A natural question to ask is whether being hereditarily separable is equivalent to being hereditarily Lindelöf in the broader class of all regular spaces. Notice that a natural obstruction to being either hereditarily Lindelöf or hereditarily separable is for a space to contain an uncountable discrete subspace. The above question breaks naturally into two halves.

(S) Is every hereditarily separable space hereditarily Lindelöf?

1All spaces considered in this thesis are regular.
(L) Is every hereditarily Lindelöf space hereditarily separable?

Both of these questions are essentially Ramsey theoretic in nature (see [50]). A counterexample to (S) has become known as an S space and a counterexample to (L) has become known as an L space. Given any non Lindelöf space \( X \) it is easy to find a sequence of points \( \{ x_\alpha : \alpha < \omega_1 \} \) in \( X \) such that every final segment of this sequence is relatively closed. Such a sequence is said to be right separated and, as one might expect, nearly every study of an S space immediately passes to such a sequence. Similarly, nonseparable spaces contain such a sequence in which every initial segment is relatively closed (these sequences are said to be left separated).

Eventually a positive answer to (S) was shown by Todorčević [70] to be consistent (and a consequence of PFA). Whether (L) can have a positive answer is still unknown. It is known, however, the a positive answer to (L) does not follows from (S) and that \( \text{MA}_{\aleph_1} \) does not have enough strength to provide a positive answer to either question (see [74]).

Much earlier, a number of counterexamples to both (S) and (L) appeared, following from a variety of axioms such as \( \diamondsuit \), \( \clubsuit \), and \( \text{CH} \). Typically these spaces had additional properties such as compactness or cometrizability (see e.g. [47], [16], and [36]). Even though obtaining consistent positive answers to (S) and (L) turns out to be a somewhat difficult task for which \( \text{MA}_{\aleph_1} \) is not sufficient, \( \text{MA}_{\aleph_1} \) typically gives a positive answer to (S) and (L) in more restrictive classes of topological spaces. For example, in the same article in which Szymański constructed his extremally disconnected S space from \( \clubsuit \),
he also demonstrated that Wage's lemma (a consequence of $\text{MA}_{\aleph_1}$) implies that there are no extremally disconnected S spaces. It was similarly shown by Gruenhage [28] that $\text{MA}_{\aleph_1}$ implies a positive answer to (S) and (L) in the class of cometrizable spaces. Szentmiklóssy [60] has shown that $\text{MA}_{\aleph_1}$ implies a positive answer to (S) and (L) in the class of all compact spaces.

On another front, a program has been initiated to forward our understanding of which consequences of $\text{MA}_{\aleph_1}$ are preserved by forcing with a measure algebra. This began with Laver's result in [41] that there are no Souslin lines in an extension of a model of $\text{MA}_{\aleph_1}$ by random reals. In [75] Todorčević extended this theorem by replacing "Souslin line" with "compact L space."

The reasons for analyzing the effects of random reals on consequences of Martin's Axiom began more or less as a curiosity. It was realized, however, that this scenario has the potential for demonstrating that certain consequences of $\text{MA}_{\aleph_1}$ (particularly those concerning (S) and (L)) are consistent with the inequality $2^{\aleph_0} < 2^{\aleph_1}$. The assumption $2^{\aleph_0} < 2^{\aleph_1}$ can have a strong impact on the normality of certain topological spaces and it is therefore of great interest to general topologists to know when this inequality can hold in conjunction with some particular instance of (S) or (L). In this chapter we will analyze the influence of random reals on (S) and (L) in the classes of cometrizable spaces, compact spaces, and extremally disconnected spaces.
7.1 Cometrizable Spaces

A topological space $X$ is cometrizable if there is a weaker metric topology on $X$ such that every point of $X$ has a neighborhood base of sets which are closed in the metric topology on $X$. In this section we will consider the influence of random reals on (S) and (L) in the class of cometrizable spaces. It turns out that whether these objects are introduced by forcing with a measure algebra depends on the character of the algebra.

**Theorem 7.1.1.** After adding one random real to a model of $MA_{\aleph_1}$ (S) and (L) have a positive answer in the class of cometrizable spaces.

**Proof.** I will only present the proof for (S) as the proof for (L) is symmetric. Suppose that $\mathcal{X} = \{\hat{x}_\xi : \xi < \omega_1\}$ is an enumeration elements of a metric space $(\mathcal{X}, \delta)$ in $V^R$ and $\{\hat{E}_\xi : \xi < \omega_1\}$ is a sequence of names for closed sets such that $\hat{E}_\xi \cap \{\hat{x}_\eta : \eta > \xi\}$ is empty for all $\xi < \omega_1$. Let $\hat{\tau}$ denote the topology on $\mathcal{X}$ defined by declaring the sets $\hat{E}_\xi$ to be clopen. Define $\hat{G} : [\omega_1]^2 \to R_\omega$ by $\hat{G}(\alpha, \beta)$ is the event "$\hat{x}_\alpha$ is in $\hat{E}_\beta$" where $\alpha < \beta < \omega_1$.

By Theorem 5.2.1, either $\hat{G}$ is forced to be countably chromatic (and hence $(\mathcal{X}, \hat{\tau})$ is forced to be $\sigma$-discrete) or else there is an uncountable sequence of disjoint $n$ element sets $F_\xi (\xi < \omega_1)$ and a $\delta > 0$ such that for all $\xi \neq \eta$

$$\mu( \bigvee_{\alpha \in F_\xi} \bigvee_{\beta \in F_\eta} \hat{G}(\alpha, \beta)) \geq \delta.$$ 

We will show that the second alternative can not hold. Suppose that such a sequence $F_\xi (\xi < \omega_1)$ and a $\delta > 0$ are given. For each $\xi$, pick a rational
\( \varepsilon_\xi > 0 \) such that

\[
A_\xi = \left\{ \min_{\alpha \in F_\xi} \min_{\beta \in F_{\xi+1}} d(\dot{x}_\beta, \dot{E}_\alpha) > \varepsilon_\xi \right\}
\]

has measure greater than \( 1 - \delta/2 \). Fix an uncountable \( \Gamma \subseteq \omega_1 \) and \( \varepsilon > 0 \) such that \( \varepsilon_\xi = \varepsilon \) for all \( \xi \) in \( \Gamma \). By the separability of \( R_\omega \) it is possible to find \( \xi < \xi + 1 < \eta \) such that for all \( i = 0, \ldots, n - 1 \)

\[
B_{\xi,\eta} = \left\{ \dot{d}(\dot{x}_{F_{\xi+1}(i)}, \dot{x}_{F_{\eta+1}(i)}) < \varepsilon \right\}
\]

has measure greater than \( 1 - \delta/2 \). It follows that \( A_\eta \cap B_{\xi,\eta} \) has measure greater than \( 1 - \delta \) and forces that \( d(\dot{x}_\alpha, \dot{E}_\beta) > 0 \) whenever \( \alpha \) is in \( F_{\xi+1} \) and \( \beta \) is in \( F_\eta \) and hence is disjoint from

\[
\bigvee_{\alpha \in F_\xi} \bigvee_{\beta \in F_\eta} \dot{G}(\alpha, \beta).
\]

\[\square\]

In [89] White showed that a Sierpiński set with the density topology is an L space (see also [62]). If the Sierpiński set is viewed instead as a subspace of the Stone space of the measure algebra then it is an example of an extremally disconnected L space, a fact that will contrast the result in Section 7.2. A consequence of the Luzin-Menchhoff theorem (see [62]) is that the density topology on the reals has a base of sets which are closed in the real topology. Thus a Sierpiński set with the density topology is a cometrizable L space.

Taking this a step further, suppose that \( \{x_\alpha\}_{\alpha < \omega_1} \subseteq [0, 1] \) is an enumeration of a Sierpiński set which moreover has the property that it is hereditarily Lindelöf in all finite powers in the density topology (an \( \omega_1 \) sequence of random reals is such a set). For each \( \alpha < \omega_1 \) pick a compact set \( E_\alpha \) of positive
measure which misses \( x_\gamma \) for all \( \gamma < \alpha \) but which contains \( x_\alpha \) as a point of density 1. Now the sets \( U_\alpha = \{ \gamma < \omega_1 : x_\alpha \in E_\gamma \} \) are closed in the Vietoris topology on the closed subsets of \([0, 1]\) and, by duality, generate an S space topology on \( \{ E_\alpha : \alpha < \omega_1 \} \). Thus there are always cometrizable S spaces after forcing with a nonseparable measure algebra.

In [74], Todorčević has shown that \( \mathfrak{b} = \aleph_1 \) implies that there is a locally compact cometrizable S space topology on any \( \omega_1 \) sequence of reals.

**Question 7.1.2.** Is it possible to construct a locally compact locally countable cometrizable S space from \( \omega_1 \) random reals?

A closer examination of Todorčević’s construction shows that it is very closely connected to the negative partition relation \( \omega_1 \nRightarrow (\omega_1, (\omega : 2))^2 \). Since, by a result of Barnett [3], \( \omega_1 \rightarrow (\omega_1, (\omega : 2))^2 \) can hold after adding \( \omega_1 \) random reals, if this question has a positive answer, then some new ideas are required.

### 7.2 Extremally Disconnected Spaces

A topological space \( X \) is *extremally disconnected* if the closure of every open set is open. Extremally disconnected S spaces were constructed by Ginsburg [27], Szymański [61], and Wage [88] using a variety of set theoretic assumptions. In the last of the three papers written on extremally disconnected S spaces, Szymański shows that Wage’s Lemma, a known consequence of \( \text{MA}_{\aleph_1} \) (see [87]), implies that (S) has a positive answer in the class of all subspaces
of extremely disconnect spaces. In Section 6.3 we extended Wage's result, showing that if MA$\aleph_1$ holds then Wage's Lemma remains true after forcing with any measure algebra. Combined with the results above, this establishes the consistency of "(S) has a positive answer for all subspaces of extremely disconnected spaces" with statements such as "there is a Sierpiński set" (which yields an extremely disconnected L space) and "there is an Ostaszewski space" (see Section 7.3). The latter is perhaps surprising since historically extremely disconnect S spaces and Ostaszewski spaces have been constructed from similar axioms. I will close this section with the proof of Szymański’s theorem.

**Theorem 7.2.1.** [61] \((W(\aleph_1))\) Extremally disconnected spaces can not contain S spaces.

**Proof.** Suppose that \(X\) is extremally disconnected and that \(\{x_\alpha : \alpha < \omega_1\}\) is a right separated subspace of \(X\). Recursively choose countable sets \(A_\xi\) \((\xi < \omega_1)\) such that

1. \(A_\eta \subseteq \eta\) is almost disjoint from \(A_\xi\) for every \(\xi < \eta\),

2. \(\{x_\xi : \xi \in A_\eta\}\) does not accumulate to any point in \(\{x_\xi : \xi < \omega_1\}\) except for possibly \(x_\eta\), and

3. if possible, \(\{x_\xi : \xi \in A_\eta\}\) accumulates to \(x_\eta\).

Now suppose that \(F \subseteq X\) is almost disjoint from every \(A_\xi\) for \(\xi < \omega_1\). Suppose for contradiction that \(F\) does accumulate to some point \(x_\xi\) and let \(\xi\) be minimal with this property. Let \(W\) be a neighborhood of \(x_\xi\) in \(X\) whose
closure misses the closure of \( \{ x_\eta : \eta > \xi \} \). Since \( A_\xi \cap F \cap W \) is finite and \( A_\xi \) and \( W \cap F \) are countable, it is possible to pick disjoint open sets \( U \) and \( V \) in \( X \) such that \( U \) contains all but finitely many elements of \( A_\xi \) and \( V \) contains all but finitely many elements of \( F \cap W \). Since \( X \) is extremally disconnected, the closures of \( U \) and \( V \) are disjoint. But this is a contradiction since \( x_\xi \) is an accumulation point of both \( A_\xi \) and \( F \cap W \).

\[ \square \]

### 7.3 Compact S Spaces

One of the primary reasons for studying (S) and (L) after forcing with a measure algebra is that this may yield a solution to an old problem of Katětov. In [35] Katětov proved that if \( X \) is a compact space and \( X^3 \) is hereditarily normal then \( X \) must be metrizable. He then asked whether dimension 3 could be lowered to dimension 2. Gruenhage and Nyikos have shown in [29] that under CH and also under MA\(_{\aleph_1}\) this question has a negative answer. Moreover they show that if \( X \) is a counterexample, one of the following must hold:

1. There is a Q set.
2. \( X \) is a compact L space.
3. \( X^2 \) is a compact S space.
4. \( X^2 \) contains both S and L spaces.
After forcing with a measure algebra of character at least $\aleph_2$, there are no Q sets. Also, Todorčević has shown the following.

**Theorem 7.3.1.** [75] $(\text{MA}_{\aleph_1})$ After forcing with any measure algebra every compact space containing an L space also contains an uncountable free sequence.

Thus any counterexample to Katětov's problem in this model would have to be a perfectly normal compactum $X$ whose square is an S space. For some time it was unclear whether or not Todorčević's theorem could be dualized by replacing “S space” by “L space.” The following theorem gives a negative answer to this question.

**Theorem 7.3.2.** In any forcing extension by a homogeneous nonseparable measure algebra there is a perfectly normal countably compact non compact topology on $\omega_1$. In particular, such forcing extensions always contain compact S spaces.

**Remark 7.3.3.** Such a space is often known as an Ostaszewski space (see [47] for Ostaszewski's construction of such a space under $\diamond$). Such a space must be an S space. Eisworth and Roitman [17] have shown that such spaces can not be constructed from $\text{CH}$ alone. This is the only example of an "anti-Ramsey" object on $\omega_1$ that I am aware of which is produced by an $\omega_1$ sequence of random reals but not by $\text{CH}$.

**Proof.** By modifying our ground model if necessary, we may assume that $V^R = V[\check{r}_\alpha : \alpha < \omega_1]$ for some sequence of random reals $\check{r}_\alpha (\alpha < \omega_1)$. It will
be convenient to view \( \hat{r}_\alpha \) as a random real in \( \omega^\omega \) where \( \omega \) is given the atomic measure determined by \( \mu(\{n\}) = 2^{-n-1} \). In \( V \) fix, for each limit ordinal \( \delta \), a strictly increasing sequence \( \delta_n \) cofinal in \( \delta \). Define a sequence of topologies \( \hat{r}_\alpha \) on the limit ordinals \( \alpha \) by recursion so that \( \hat{r}_\alpha \) is locally compact, noncompact topology in \( V[\hat{r}_\xi : \xi < \alpha] \) and \( \hat{r}_\beta \upharpoonright \alpha \) is \( \hat{r}_\alpha \) for \( \alpha < \beta \). \( \hat{r}_\omega \) is the discrete topology. Suppose now that \( \hat{r}_\alpha \) has been defined (limit stages are trivial). Define a topology \( \hat{r}_{\alpha + \omega} \) on \( \alpha + \omega \) as follows. In \( V[\hat{r}_\xi : \xi < \alpha] \), let \( \{U_\alpha(k) : k < \omega\} \) be a partition of \( (\alpha, \hat{r}_\alpha) \) into disjoint compact open pieces. Neighborhoods of \( \alpha + n \) in \( \hat{r}_{\alpha + \omega} \) are of the form \( \{\alpha + n\} \cup \bigcup \hat{V} \) where \( \hat{V} \) is a cofinite subset of

\[ \{U_\alpha(k) : \hat{r}_\alpha(k) = n\}. \]

The space we are interested in is, of course, \( (\omega_1, \hat{r}_{\omega_1}) \). The topology is clearly locally compact and locally countable. It should also be clear that the genericity of the \( \hat{r}_\alpha \)'s ensure that, for a fixed \( \alpha < \omega_1 \), the closure of \( \{\alpha + n : n < \omega\} \) is the set of all \( \beta \geq \alpha \). We will now show that \( (\omega_1, \hat{r}_{\omega_1}) \) has the property that the closure of any set is either compact or cocountable. Suppose that \( \hat{E} \) is a name for an infinite subset of \( \omega_1 \) and assume without loss of generality that \( \hat{E} \) is forced to be either countable or uncountable. Let \( \alpha \) be a limit ordinal such that \( \hat{E} \cap \alpha \) is infinite and is in \( V[\hat{r}_\xi : \xi < \alpha] \). If \( \hat{E} \) is forced to be uncountable, then also arrange that \( \hat{E} \cap \alpha \) is forced to be cofinal in \( \alpha \). Now we will work in \( V[\hat{r}_\xi : \xi < \alpha] \). If \( \hat{E} \cap \alpha \) has compact closure in \( (\alpha, \hat{r}_\alpha) \) then we are done (note that this is impossible if \( \hat{E} \cap \alpha \) is cofinal in \( \alpha \)). If \( \hat{E} \cap \alpha \) does not have compact closure in \( \hat{r}_\alpha \) then there are infinitely many
such that $U_\alpha(k) \cap E$ is nonempty. Moreover, an easy genericity argument
shows that, for each $n$ there are infinitely many $k$ such that $U_\alpha(k) \cap E$ is
non empty and $r_\alpha(k) = n$. It follows that $E \cap \alpha$ accumulates to each of the
$\alpha + n$'s which are in turn dense the set of all $\beta \geq \alpha$.

\section{Small Compactifications of L Spaces}

By a result of Fremlin \[22, 44A\] (see also Section 6 of \[66\]), $\operatorname{MA}_\aleph_1$ implies
that every compact space containing an L space must map onto $[0, 1]^\omega_1$. In
light of Todorčević's Theorem 7.3.1, it is natural to ask whether this result
also holds after forcing with a measure algebra. The following result indicates
that this is not so.

\textbf{Theorem 7.4.1.} If there is a Sierpiński set then there is a compact space $K$
which contains an L space and which is also linearly fibered (and hence does
not map onto $[0, 1]^\omega_1$).

\textit{Proof.} Let $\{x_\alpha : \alpha < \omega_1\}$ be a Sierpiński subset of $2^\omega$ and let $e_\alpha : \alpha \to \omega$
($\alpha < \omega_1$) be a coherent sequence of injections. Suppose further that for all
$\alpha < \beta$ $\Delta(x_\alpha, x_\beta) \leq e_\beta(\alpha)$ (the sequence $e_\alpha$ can always be modified to have
this property — see Chapter 4 and in particular page 96 of \[7\]). Define

$$E_\beta = 2^\omega \setminus \bigcup_{\alpha < \beta} [x_\alpha \setminus e_\beta(\alpha)].$$

Then $E_\beta$ is a compact set of positive measure all of whose elements are of
Lebesgue density 1 and which contains $x_\beta$. Hence if a topology on $2^\omega$ is
generated by the clopen sets and the $E_\beta$'s, $X = \{x_\alpha : \alpha < \omega_1\}$ is an L space when given the subspace topology. Let $\mathcal{B}$ be the Boolean algebra generated by the clopen subsets of $2^\omega$ and the $E_\beta$'s.

**Claim 7.4.2.** The map from the Stone space of $\mathcal{B}$ to $2^\omega$ given by restricting ultrafilters to the algebra of clopen sets has linear fibers.

**Proof.** Suppose that $x$ is in $2^\omega$. Let $\Gamma_x$ be the collection of all $\beta$ such that $x \in E_\beta$. We must show that if $\alpha < \beta$ are in $\Gamma_x$ then, for some neighborhood $U$ of $x$, $E_\beta \cap U \subseteq E_\alpha \cap U$. To this end, it suffices to show that $E_\beta \setminus E_\alpha$ is closed. Let $F \subseteq \alpha$ be a finite set such that $e_\alpha$ and $e_\beta$ agree on $\alpha \setminus F$. Then

\[ E_\beta \setminus E_\alpha = \bigcup_{\gamma \in F} \{x \in E_\beta : \Delta(x, x_\gamma) \leq e_\alpha(\gamma)\} \]

is clearly closed.

\[ \square \]
Appendix A

Glossary

If $A$ and $B$ are subsets of some linear order $(L, <)$ then $A < B$ abbreviates $a < b$ whenever $a$ is in $A$ and $b$ is in $B$.

$b$ The cardinality of the smallest unbounded subset of $(\mathbb{N}^\mathbb{N}, <^*)$.

c The cardinality of $\mathbb{R}$.

c$'$ The image of $S$ under the map $c$.

$\Delta(x, y)$ If $x$ and $y$ are two distinct sequences then $\Delta(x, y)$ is the least $n$ such that $x(n) \neq y(n)$.

$\mathbb{N}$ The set $\{0, 1, 2, \ldots \}$ of natural numbers. Identical to $\omega$.

$[\mathbb{N}]^\omega$ The collection of all infinite subsets of $\mathbb{N}$.

$\mathbb{P}$ The collection of all strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$.

$\mathcal{R}_\omega$ The homogeneous measure algebra of character $\omega$.  

94
#(S) The cardinality of the set $S$.

$|t|$ The length of a finite sequence $t$.

$t^i$ The concatenation of the sequence $t$ with the entry $i$.

$[[\phi]]$ The event "$\phi$ is true" in the context of forcing with a measure algebra.

$x <^* y$ If $x$ and $y$ are in $\mathbb{N}^\mathbb{N}$ then $x <^* y$ means that $x(n) < y(n)$ for all but finitely many $n$.

$x <^m y$ If $x$ and $y$ are in $\mathbb{N}^\mathbb{N}$ then $x <^m y$ means that $x(n) < y(n)$ for all $n \geq m$.

$[X]^k$ The collection of all $k$ element subsets of a set $X$.

**almost disjoint** Two sets $A$ and $B$ are almost disjoint if their intersection is finite.

**analytic** A set of reals is analytic if it is the continuous image of a Polish space.

**Boolean algebra** A Boolean algebra is a ring with 1 $(\mathcal{B}, +, \cdot)$ in which $b^2 = b$ for every $b$ in $\mathcal{B}$. Usually one defines $a \wedge b = a \cdot b$ and $a \vee b = a + ab + b$. Also, $a \leq b$ if $a \wedge c = 0$ whenever $b \wedge c = 0$ and the complement $a^c$ is the unique element of $\mathcal{B}$ such that $a \vee a^c = 1$ and $a \wedge a^c = 0$.

**centered** A collection of sets is centered if it has the finite intersection property. A Boolean algebra is $\sigma$-centered if it can be decomposed into countably many centered families.
character (of a measure algebra) The character of a measure algebra is cardinality of the smallest set which completely generates it (via complementation and arbitrary supremums and infimums).

clique A clique is a synonym for a complete graph — the edge set consists of all unordered pairs of vertices.

cometrizable A topological space $X$ is cometrizable if there is a weaker metric topology on $X$ such that every point in $X$ has a neighborhood base of sets which are closed in the metric topology.

complete (Boolean algebra) A Boolean algebra is complete if every family has a supremum and an infimum in the algebra.

Continuum Hypothesis (CH) The Continuum Hypothesis is the statement that $\mathbb{R}$ can be well ordered in type $\omega_1$.

cover (of a measure 0 set) A cover of a measure 0 set $G \subseteq 2^\omega$ is a sequence $F_n$ ($n < \omega$) such that $F_n \subseteq 2^n$, every element of $G$ is in infinitely many of the clopen sets $[F_n]$, and $\sum_{n=0}^{\infty} \#(F_n) \cdot 2^{-n} < \infty$.

countable chain condition (c.c.c.) A partial order satisfies the countable chain condition if every uncountable subset collection contains two elements with a common lower bound.

countably chromatic A graph $G \subseteq [X]^2$ is countably chromatic if there is a map $f : X \to \omega$ such that $\{x, y\}$ are not an edge of $G$ whenever $x \neq y$ are in $X$ and $f(x) = f(y)$.
**Δ-system** A collection of finite sets $\mathcal{F}$ forms a Δ-system if there is a finite set $R$ (called the root) such that $A \cap B = R$ whenever $A$ and $B$ are distinct elements of $\mathcal{F}$ and all of the members of $\mathcal{F}$ have the same size.

**dense (in a linear order)** A linear order $(L, <)$ is dense if for every $a < b$ in $L$ there is a $c$ in $L$ such that $a < c < b$.

**dense (in a partial order)** A subset $\mathcal{D}$ of a partial order $(\mathcal{P}, \leq)$ is dense if for every $p$ in $\mathcal{P}$, there is a $d$ in $\mathcal{D}$ such that

**dense (in a topological space)** A subset $\mathcal{D} \subseteq X$ of a topological space is dense if $\mathcal{D}$ has nonempty intersection with every nonempty open subset of $X$.

**density topology** The density topology is the topology on the reals in which the neighborhood at a point $x$ is the collection of all Borel sets of positive measure which contain $x$ as a point of Lebesgue density 1. For more information about this topology, see [62].

**discrete** A subset $E$ of a topological space is discrete if for every point $x$ in $E$, there is a neighborhood of $x$ which intersects $E$ only at $\{x\}$.

**event** An event is any element of a measure algebra. The event $\llbracket S \rrbracket$, where $S$ is a sentence concerning objects in a forcing extension by a measure algebra, is the unique element of the measure algebra which forces $S$ is true and for which the complement of $\llbracket S \rrbracket$ forces that $S$ is false.
factor A factor \((R_a, \mu_a)\) of a measure algebra \((R, \mu)\) the restriction of \(R\) to the (positive) element \(a\). Here \(\mu_a(c)\) is defined to be \(\mu(a \cdot c)/\mu(a)\).

filter A collection \(\mathcal{F}\) of subsets of a set \(S\) forms a filter if \(A \cap B\) is in \(\mathcal{F}\) whenever \(A\) and \(B\) are in \(\mathcal{F}\) and \(C \subseteq S\) is \(\mathcal{F}\) whenever \(C\) contains a member of \(\mathcal{F}\). Filters on a Boolean algebra are defined in an analogous way. A subset \(\mathcal{G}\) of a partial order \((\mathcal{P}, \leq)\) is a filter if every pair of elements in \(\mathcal{G}\) has a lower bound in \(\mathcal{G}\) and \(p \in \mathcal{P}\) is in \(\mathcal{G}\) whenever \(p\) is greater than some \(q\) in \(\mathcal{G}\).

free sequence A sequence of points \(\{x_\alpha : \alpha < \delta\}\) in a topological space is a free sequence if for every \(\gamma < \delta\) the closures of \(\{x_\alpha : \alpha < \gamma\}\) and \(\{x_\beta : \beta \geq \gamma\}\) are disjoint. Such a sequence is necessarily discrete.

Haar algebra The Haar algebra of character \(\kappa\) is the measure algebra corresponding to the product measure space \((2^\kappa, \mu)\) where each factor \(2 = \{0, 1\}\) is given the uniform probability measure.

Helly’s condition A topological space (or Boolean algebra) satisfies Helly’s condition if it has a base \(\mathcal{B}\) such that every linked subcollection of \(\mathcal{B}\) is centered.

homogeneous (for a partition \(K\)) If \(K \subseteq [S]^2\) is a partition then \(H\) is \(K\)-homogeneous if \([H]^2\) is contained in \(K\). Similarly, if \(c : [S]^2 \to \mathbb{Z}\) is a coloring, then \(H\) is homogeneous for \(c\) if \(c\) is constant on \([H]^2\). \(H\) is \(i\)-homogeneous if \(c\) is constantly \(i\) on \([H]^2\).
homogeneous (measure algebra) A measure algebra \( (\mathcal{R}, \mu) \) is homogeneous if every factor of \( \mathcal{R} \) has the same character.

ideal A collection \( \mathcal{I} \) of subsets of \( \mathbb{N} \) is an ideal iff it is closed under unions and subsets.

indecomposable An ordinal \( \alpha \) is indecomposable if \( \alpha \) can not be expressed as the sum of two smaller ordinals.

irreducible A coloring \( c : [X]^2 \to Z \) is irreducible if \( c''[Y]^2 = Z \) for every \( Y \subseteq X \) with \( #(Y) = #(X) \).

\( \mathcal{K}_n \) This is the assertion that every uncountably collection of elements of a Boolean algebra must contain an uncountable \( n \)-linked family.

Katetov's problem Katetov asked if \( X \) is metrizable whenever \( X \) is a compact space such that every subset of \( X^2 \) is normal.

Knaster's condition \( K_n \) A topological space satisfies Knaster's condition \( K_n \) if whenever \( \mathcal{F} \) is an uncountable collection of open sets \( \mathcal{F} \) contains an uncountable \( n \)-linked subcollection.

L space An L space is a regular topological space which is hereditarily Lindelöf but not hereditarily separable. Every L space contains a left separated sequence of length \( \omega_1 \).

ladder system A ladder system (on \( \omega_1 \)) is a sequence \( f_\lambda \) of functions indexed by the limit ordinals such that the domain of \( f_\lambda \) is a cofinal
sequence in $\lambda$ of order type $\omega$ and the range of $f_\lambda$ is contained in the natural numbers.

**left separated** A sequence $\{x_\alpha : \alpha < \omega_1\}$ in a topological space is left separated if for every $\gamma < \omega_1$, the closure of $\{x_\alpha : \alpha < \gamma\}$ is disjoint from $\{x_\beta : \beta \geq \gamma\}$.

**Lindelöf** A topological space is Lindelöf if every open cover has a countable subcover.

**linear order** A linear order is a pair $(L, \leq)$ such that $L$ is a set and $\leq$ is a relation which is reflexive, antisymmetric, transitive, and satisfies dichotomy (for every $a, b$ in $L$ either $a \leq b$ or $b \leq a$).

**linear compactum** A linear compactum $L$ is a compact topological space such that for some order $\leq$ on $L$ the open intervals of $(L, \leq)$ form a base for the topology on $L$.

**linearly fibered** A compact topological space $K$ is linearly fibered if there is a continuous map $\phi : K \to [0, 1]$ such that the preimage of every point in $[0, 1]$ is a linear compactum.

**linked** A collection of sets $\mathcal{F}$ is linked (respectively $n$-linked) if every pair (respectively every $n$ element set) of elements has a nonempty intersection.

**Martin's Axiom (MA)** This is a forcing axiom for the class of all c.c.c. partial orders. It asserts that for every c.c.c. partial order and every
sequence of fewer than \( c \) many dense sets, there is a filter meeting each of these dense sets. \( \text{MA}_\theta \), for a cardinal \( \theta \) is the above assertion where \text{“fewer than} \ c \text{ dense sets” is replaced by \text{“at most} \ \theta \text{ dense sets.” Further reading on Martin’s axiom can be found in [7, Ch. 3], [79], [37], and [74].}

**measure algebra** A measure algebra is a pair \((\mathcal{R}, \mu)\) where \(\mathcal{R}\) is a complete Boolean algebra and \(\mu : \mathcal{R} \to [0, 1]\) is a strictly positive map such that
\[
\mu(1) = 1 \quad \text{and} \quad \mu(\bigvee_{i=0}^{\infty} a_i) = \sum_{i=0}^{\infty} \mu(a_i) \quad \text{whenever} \quad a_i \land a_j = 0 \quad \text{for all} \quad i \neq j.
\]

**metrizable fibered** A compact topological space \( K \) is metrizable fibered if there is a continuous map \( \phi : K \to [0, 1] \) such that the preimage of every point in \([0, 1]\) is metrizable.

**\( r \) nice** A cover \( F \) of a measure 0 set \( G \subseteq 2^\omega \) is \( r \) nice if \( \lim_{i \to \infty} \#(F_i)^{2^{r-i}} = 0 \).

**nowhere dense** A set is nowhere dense if it is not dense in any nonempty open set.

**Open Coloring Axiom (OCA)** This is the assertion that whenever \( G \subseteq [X]^2 \) is an open graph on a separable metric space — \( G \) is an open subset of \([X]^2\) — then either \( G \) is countably chromatic or else there \( G \) contains an uncountable clique.

**orthogonal (modulo \( \mathcal{I} \))** Two subsets \( a \) and \( b \) of \( \mathbb{N} \) are orthogonal modulo an ideal \( \mathcal{I} \) if \( a \cap b \) is in \( \mathcal{I} \). Two families \( A \) and \( B \) are orthogonal modulo
If every element of $A$ is orthogonal to every element of $B$.

\textbf{$\pi$-base} A $\pi$-base of a topological space $X$ is a family $B$ of nonempty open sets such that every nonempty open subset of $X$ contains some element of $B$.

\textbf{$\pi$-character} The $\pi$-character of a point is the size of the smallest $\pi$-base at that point.

\textbf{P-ideal} An ideal $\mathcal{I}$ is a P-ideal if every countable sequence of elements of $\mathcal{I}$ is almost contained in a single element of $\mathcal{I}$.

\textbf{Proper Forcing Axiom (PFA)} This is a forcing axiom for the class of all proper partial orders. A full discussion of proper partial orders and PFA is beyond the scope of this thesis. For our purposes it will suffice to know that the proper partial orders encompass both the c.c.c. and the $\sigma$-closed orders and that this class is closed under finite iterations. Interested readers are referred to [7, Ch. 3], [79, Ch. 12], and [74] for a concise sampling of some of the state of the art consequences of PFA in the context of Ramsey theory as well as an introduction to this axiom. [56] provides probably the most complete treatment of PFA and proper partial orders.

\textbf{Polish space} A topological space $X$ is a Polish space if the topology on $X$ is compatible with a complete separable metric.

\textbf{Q set} A set of reals $X$ such that every subset $Y$ of $X$ is a $G_\delta$ set in the subspace topology on $X$. 

random real A generic filter (sometimes construed as a real number) for the Lebesgue measure algebra.

right separated A sequence \( \{x_\alpha : \alpha < \omega_1\} \) in a topological space is right separated if for every \( \gamma < \omega_1 \), \( \{x_\alpha : \alpha < \gamma\} \) is disjoint from the closure of \( \{x_\beta : \beta \geq \gamma\} \).

S space Any space which is hereditarily separable but not hereditarily Lindelöf. Every S space contains a right separated sequence of length \( \omega_1 \).

separable (for measure algebras) A measure algebra is separable if it has countable character. The Lebesgue measure algebra is the unique atomless separable measure algebra.

separable (for topological spaces) A topological space is separable if it has a countable dense subset.

Shanin’s condition A topological space satisfies Shanin’s condition if every uncountable collection of open sets contains an uncountable centered collection.

Souslin’s condition A topological space satisfies Souslin’s condition if every collection of pairwise disjoint open sets is countable. This condition is also known as the countable chain condition.

Souslin’s problem Is \((\mathbb{R}, <)\) the only dense, complete linear order with no first and last elements which satisfies Souslin’s condition?
**f splitting** A set \( X \subseteq \mathbb{N} \) is \( f \) splitting iff \( T \) does not contain an \( f \) thin subset of the same cardinality.

**splitting node** If \( X \subseteq \omega^\omega \) and \( t \) is a finite sequence of natural numbers then \( t \) is a splitting node of \( X \) if there are infinitely many \( i \) such that \( t^i \) is an initial segment of some \( x \) in \( X \).

**subtree** A subtree of a tree \( T \) is a subset of \( T \) which is also downwards closed.

**\( f \) thin** A set \( X \subseteq \omega^\omega \) is \( f \) thin if for all but finitely many \( n \) in \( \omega \) and every \( t \) in \( \omega^n \) there are at most \( f(n) \) many \( i \) for which \( t^i \) is an initial segment of some \( x \) in \( X \).

**tree** A tree \( (T, <) \) is a partially ordered set in which \( \{s \in T : s < t\} \) is well ordered by \( < \) for every \( t \) in \( T \).

**Tychonoff cube** The Tychonoff cubes are the spaces of the form \([0,1]^I\) where \( I \) is any index set.

**ultrafilter** A maximal filter.

**Wage's Lemma** \( W(\theta) \) \( W(\theta) \) is the statement that whenever \( \mathcal{A} \) is a collection of almost disjoint countable subsets of \( \theta \) there is a cofinal subset of \( \theta \) which is almost disjoint from every member of \( \mathcal{A} \).

**well founded** A partial order \( (\mathcal{P}, \leq) \) is well founded it does not contain any infinite descending chains.
Bibliography


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


Index

\((\mathbb{P}, <^*)\), 40
\((m)_{k,t}^2\), 43
\(A < B\), 94
\(A \otimes B\), 14
\(W\)-group, 31
\([X]^k\), 95
\([N]^\omega\), 94
\([m, n]\), 23
\([m]^2_k\), 43
\#(S), 95
\(\kappa_2\), 32, 33
\(\kappa_3\), 35
\(\kappa_n\), 31
\(\text{MA}_{\kappa_1}\), 30, 56
\(\mathcal{N}_*\), 42
\(\mathcal{N}_r\), 42
\(\mathbb{P}\), 94
\(\mathcal{R}_\omega\), 94
\(\text{add}(\mathcal{N}_r, \mathcal{N}_s)\), 42
\(\delta(B)\), 74
\(\text{otp}(T)\), 69
\(\pi T\), 68
\(\pi \mathcal{F}\), 69
\(\sigma\)-centered, 7, 16, 21
\(\sigma\)-linked, 12, 21, 35
\([\phi]\), 95
\(a \perp_T b\), 13
\(a \subseteq_T b\), 13
t\(\sim i\), 95
\(x <^* y\), 95
\(x <^m y\), 95
alternation map, 40, 48, 49
Baire category, 4, 31
Barnett, J. H., 65, 67
Boolean algebra, 6, 12, 35
Cantor, G., 1
cardinal invariant
add($\mathcal{N}$), 35
add($\mathcal{N}_*$), 42, 43
add($\mathcal{N}_r, \mathcal{N}_s$), 42
add$_*(\mathcal{L}_1)$, 22
$b$, 12, 27, 40, 87, 94
c, 94
don($\mathcal{N}$), 32
centered, 6, 21
chain condition
global, 3, 4, 10
local, 3, 5, 30
Cohen, P., 8
coloring, 39
associated with
add($\mathcal{N}_*$), 40, 41, 43
$b$, 40
continuous, 39, 40
irreducible, 39
.. Sierpiński, 39
used in coding, 40
Continuum Hypothesis $\mathbf{CH}$, 8, 66, 90
countable chain condition, 7
cover, 42
$r$ nice, 42
Dedekind, 1
density topology, 33, 86
discrete, 33, 82, 85
Eisworth, T., 90
Erdős, P., 43
Farah, I., 23
filter, 6
free sequence, 90
Fremlin, D. H., 62, 92
gap, 11, 12, 16, 17, 19, 22, 26
analytic, 22
definable, 14
linear, 19
Ginsburg, J., 87
graph, 32
countably chromatic, 32
Gruenhage, G., 83, 89
Hajnal, A., 66, 67
partition under $\mathbf{CH}$, 66
Helly, E., 20
condition, 20, 27, 29
INDEX

theorem, 2, 20
Horn, A., 3
ideal, 13, 21, 24
  analytic, 14, 22, 24
dense, 13, 24
  P-ideal, 13, 14, 22, 24
Jech, T., 4, 8
Katětov, M., 89
Knaster, B., 2, 10
  condition, 2, 3, 5, 31, 80
    in products, 3
  condition \( K_n \), 31, 35
Kunen, K., 5, 65, 80, 83
Kurepa, D., 2
L space, 83, 89
  first, 3, 83, 85, 86
  compact, 81, 83, 89, 90
  compactifications, 92
  extremally disconnected, 86
ladder system, 73
Laver, R., 59, 65–67, 84
Lindelöf, 82
linear compactum, 10
linear order, 1, 2
linearly fibered, 11, 19, 22, 27, 29, 92
linked, 2, 6, 21
Maharam, D., 62
  theorem, 62
Martin’s Axiom, 4, 5, 8, 56, 65
  \( \text{MA}_{\aleph_1} \), 4, 5, 8, 59, 65, 66, 72, 73, 83
Martin’s Axiom \( \text{MA}_{\aleph_1} \), 87
Martin, D. A., 8, 30
measure 0, 30, 33, 35, 40, 41
measure algebra, 56, 57, 62
  character, 57, 62
  factor, 57
  homogeneous, 58, 62
  measure preserving homomorphism, 62
  projection, 58
  separable, 57
  ultrapower, 58
Nyikos, P., 89
INDEX

Open Coloring Axiom **OCA**, 28, 40, 52, 53
orthogonal, 13
oscillation map, 40, 48, 49
Ostaszewski space, 89, 90
partial order, 6
partition calculus, 66
partition tree, 67
\(\pi T\), 68
filter on, 68
order type, 69
tree-like, 69
ultrafilter, 69
uniform, 69
Proper Forcing Axiom **PFA**, 4, 9, 22, 40, 52, 53, 72, 83

**Q** set, 89

Ramsey, F. P., x, 83
numbers, 40, 43
theorem, x, 39, 66
theory, 7
random graph, 56, 59
Roitman, J., 65, 90

Rowbottom, 5
S space, 83, 89
cometrizable, 83, 85–87
compact, 83, 89, 90
extremely disconnected, 83, 87
Scottish Book, 2, 3
separable, 1, 3, 4, 7, 12, 16, 22, 27, 29, 82
Shanin, N. A., 3
condition, 3, 5, 18, 22, 31
in products, 3
Shapirovič, B., 11
Sierpiński, W., x
partition, x, 39, 66
set, 67, 80, 86, 92
Solovay, R., 4, 5, 8, 30
Souslin, M., x, 1
condition, 1–5, 7, 10, 12, 17, 18, 22, 27, 29, 31, 65, 80
in products, 3, 80
line, 5, 13, 84
problem, x, 1, 2, 4, 8, 30
program, 4–6
INDEX

  tree, 65
split, 13
Stone, M. H., 6
  representation theorem, 6
  space, 6, 12, 86
Szentmiklössy, 83
Szpirajn, E., 2, 10
Szymański, A., 87, 88
Tarski, A., 3
tennenbaum, S., 4, 8, 30
Todorčević, 83
Todorčević, S., 5, 22, 27, 31, 32, 48,
  49, 67, 81, 87, 90, 92
tree, 31, 32, 59
Treybig, L. B., 10
Tychonoff cube, 4, 10
Velieković, B., 5, 31
von Neumann’s problem, 3
Wage, M., 77, 87
  \(W(\theta)\), 77, 83, 87
Ward, A. J., 10
White, E. H., 86