DYNAMIC ANALYSIS OF ECONOMIC SYSTEMS

by

Gregory Gagnon

A thesis submitted in conformity with the requirements for the degree of
Doctor of Philosophy
Graduate Department of Economics
University of Toronto

© Copyright by Gregory Gagnon 2000
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author’s permission.

L’auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L’auteur conserve la propriété du droit d’auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-53728-5
Abstract of Chapter 1: This paper examines a monetary economy with financial intermediaries. Several periods of investment are necessary before output is received. To smooth consumption or augment production possibilities, agents can borrow from a competitive banking sector. Loans must be guaranteed by collateral. Thus, the capital market is imperfect. The collateral accepted by banks coincides with one kind of capital used in production. It is shown that agents may overinvest in collateral and increases in statutory reserves can reduce the allocative inefficiency and increase welfare.

Abstract of Chapter 2: This paper examines the issue of whether a small random deviation from a non-random policy process will destabilize the equilibrium exchange rate in a rational expectations economy. The random deviations represent the erratic actions of government which are unobservable by agents until they occur. The degree of randomness is indexed by a parameter $\epsilon \in [0, 1]$, with higher values of $\epsilon$ associated with greater random shocks. As $\epsilon$ changes, the structure of policy changes in a nonlinear way. The random equilibrium exchange rate, $x_t^\epsilon$, converges to the deterministic rate, $z_t$, as $\epsilon \rightarrow 0$. If agents are approximately rational, large deviations between $x_t^\epsilon$ and $z_t$ for small $\epsilon$ are possible.

Abstract of Chapter 3: This paper analyzes the stability of the exchange rate in an economy with noise traders. Noise trading is restricted to agents investing in the domestic stock market, who are less sophisticated than the agents pricing foreign exchange. Monetary policy is affected by the
behaviour of investors in the domestic stock market. We show that small fluctuations in the parameters governing noise trading can have a profound effect on the exchange rate when foreign exchange traders have rational expectations. This shows that instability is the key feature in economies where heterogeneous agents have different levels of sophistication in processing information. Endogenous fluctuations from the stock market spill over into endogenous fluctuations in the exchange rate.
Contents

Introduction 1

Chapter 1: Overlapping Investment With Imperfect Capital Markets

Introduction 4
The Model 7
Characterizing The Optimal Behaviour Of Individuals 13
Equilibrium Without Binding Borrowing Constraints 17
Equilibrium With Binding Borrowing Constraints And No Statutory Reserves 18
Imposing Statutory Reserves And Welfare 21
Conclusion 26
Appendix 27
References 27

Chapter 2: Random Perturbations of Deterministic Equilibria

Introduction 30
Exchange Rates In Infinite Horizon Economies 34
Mathematical Results For Infinite Horizon Economies 40
Convergence Results For Finite Horizon Economies 44
Quasi-Rational Expectations And Instability Of Small Perturbations 48
Exchange Rate Approximation With Target Zones 53
Conclusion 55
Mathematical Results 55
References 65
Chapter 3: Exchange Rate Fluctuations In An Economy With Noise Traders

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>68</td>
</tr>
<tr>
<td>The Stock Market Model</td>
<td>72</td>
</tr>
<tr>
<td>The Exchange Rate Model</td>
<td>77</td>
</tr>
<tr>
<td>Exchange Rate Dynamics</td>
<td>82</td>
</tr>
<tr>
<td>Conclusions</td>
<td>89</td>
</tr>
<tr>
<td>Mathematical Appendix</td>
<td>90</td>
</tr>
<tr>
<td>References</td>
<td>93</td>
</tr>
</tbody>
</table>
1 Introduction

Chapter 1 is entitled Overlapping Investment With Imperfect Capital Markets. This paper examines a monetary economy with financial intermediaries. Several periods of investment are necessary before output is received. To smooth consumption or to augment production agents can borrow from a competitive banking sector. Loans must be guaranteed by collateral. Thus, the capital market is imperfect. The collateral accepted by banks coincides with one kind of capital used in production. Therefore agents have an incentive to overinvest in collateral.

There is a central bank which may impose a minimum level of reserves on the commercial banks. It is widely thought that increases in statutory reserves will cause the return on money to fall and the lending rate to rise. This view has prevailed because of the bias to model reserve requirements as a tax using a partial equilibrium framework. According to this model, increases in statutory reserves lower the return to the depositors who supply money for lending; they also raise the cost of loans from the borrower's view. These conclusions do not hold in the general equilibrium framework of the model. The paper shows that both the return on money and the lending rate may rise as statutory reserves are increased.

It is also shown that imposing reserve requirements may increase welfare. The rationale for this result is that by decreasing the lending base interest rates must rise. This eliminates some of the overaccumulation of collateral and welfare improves.

The question of whether a small change in one aspect of a model will produce a large change in equilibria is becoming a universal issue in science. Although it originated in physics with the discovery of chaos, it is now an issue in finance and economics. It is well known that deterministic economies can exhibit complex fluctuations in prices, see Grandmont (1985) as an example. However, the associated dynamics for random economies remains very much an open question. The last two chapters of my dissertation analyze complex fluctuations in random economies.

My research focuses on the stability of economic models, particularly continuous time asset pricing models. The last two papers study the stability of asset prices when non-fundamentalist traders, noise traders, are market participants. The first model of noise trading was presented in Summers (1986). Since then, the literature has expanded in both theoretical and empirical directions, see Brock and Hommes (1997) and Brock, Lakonishok and LeBaron (1992). Noise traders have the potential to create interesting fluc-
tuations because they introduce a certain amount of erratic behaviour into financial markets.

Chapter 2 is entitled Random Perturbations Of Deterministic Equilibria. This paper is a hybrid, combining economics and an area of probability theory known as Randomly Perturbed Dynamical Systems. In it I integrate this highly evolved area of probability with asset pricing theory.

In probability theory it is known that there are instances when small random shocks can drastically alter the behaviour of a deterministic system. In probability theory, studying the effects of randomness on non-random systems has generated considerable research interest. The mathematical structure of the random and non-random economic models is very different, but no systematic program of comparing random and deterministic equilibria has been undertaken in economics. This is precisely the purpose of chapter 2. I show that a deterministic equilibrium for the foreign exchange market can accurately approximate a random equilibrium provided the random influences are small. The significance is that an unanticipated switch from a random policy to a non-random policy will not produce a large swing in exchange rates if agents have rational expectations.

However, if expectations are not fully rational but are close to being rational, small shocks can induce a potentially large fluctuation. These new expectations, called quasi-rational expectations, are meant to model the behaviour of noise traders. Quasi-rational agents process information intelligently but use a short horizon predictor and a long horizon predictor; rational agents use only a long horizon predictor. Small deviations from a world of perfect certainty can produce large swings in exchange rates if agents are quasi-rational.

Chapter 3 is entitled Exchange Rate Fluctuations In An Economy With Noise traders. This paper analyzes the stability of the exchange rate in an economy with noise traders. It extends the recent papers of Brock and Hommes (1997,1998). Noise trading is restricted to agents investing in the domestic stock market, who are by assumption less sophisticated than the agents pricing foreign exchange. Monetary policy is affected by the behaviour of agents in the domestic stock market. Small fluctuations in the parameters governing noise trading can have a large effect on the exchange rate when foreign exchange traders have rational expectations. Instability emerges as a key feature of the economy when heterogeneous agents have different levels of sophistication in processing information.

Multiple possible exchange rates are a key feature of this model. Given the percentage of noise traders and deviations from fundamental prices in
the stock market there is a unique exchange rate. However, there are many possible combinations of noise trader proportions and price deviations that could exist in the stock market. This is the source of the many possible exchange rates. It is shown that a small change in the amount of trend chasing by noise traders can generate either exchange rate appreciation or depreciation. Similarly, a small change in the propensity to switch between being a noise trader and fundamentalist can generate either exchange rate appreciation or depreciation. If after finitely many periods of noise trading all agents become fundamentalists, both the number and character of the potential exchange rates is unchanged. Thus, even a brief flirtation with noise trading can have lasting impact through multiple possible exchange rates.

Many people have helped in one way or another with this project. First and foremost I wish to thank my principle supervisors in the economics department Miquel Faig and Myrna Wooders, as well as my supervisor in the mathematics department, Luis Seco. Thanks are also due to the other committee members, Angelo Melino and Xiaodong Zhu as well as the external examiner, Andres Erosa. I am indebted intellectually to W. Brock and his research associates, M. Freidlin and A. Wentzell as well as the many other talented mathematicians and probabilists whose work has shaped my thinking. Last, but not least, I wish to thank my parents Linda and Philip Gagnon for their support while writing this dissertation. My mother set my feet on the path of learning and this work represents the time and energy she invested in me.

All of the papers cited here appear as references in the relevant chapters.
Overlapping Investments With Imperfect Capital Markets

Chapter 1 of the dissertation of Gregory Gagnon

University of Toronto, 150 St. George Street, Toronto, Ont., M5S 3G7

Abstract: This paper examines a monetary economy with financial intermediaries. Several periods of investment are necessary before output is received. To smooth consumption or augment production possibilities, agents can borrow from a competitive banking sector. Loans must be guaranteed by collateral. Thus, the capital market is imperfect. The collateral accepted by banks coincides with one kind of capital used in production. It is shown that agents may overinvest in collateral and increases in statutory reserves can reduce the allocative inefficiency and increase welfare.

1 Introduction

Over the last twenty years, there has been a growing movement in the literature to examine economies with financial intermediaries. For examples of this literature see Diamond and Dybvig (1983), Diamond and Rajan (1999), Holmstrom and Tirole (1997) and Stein (1998). A second strand in the literature which appeared at approximately the same time modelled economies with capital market imperfections. These models gave rise to a demand for money that came from the desire of agents to fulfill "liquidity needs", see Bewley (1980) and Woodford (1990). With imperfect capital markets, agents cannot write contracts against every possible contingency and holding money is one way to hedge against adverse circumstances. An alternative source of money demand in these models is the slow or costly liquidation of physical assets. Empirical evidence exists to support the view that agents do hold assets to meet liquidity needs, see Freid (1995) and Bansal and Coleman (1996).

Both the financial intermediaries and imperfect capital markets approaches have tried to reinterpret well known problems while simultaneously throwing light on how these new structural features of the model affect the economy.

1 This chapter is written in collaboration with my advisor Miquel Faig.
This paper synthesizes the two approaches by examining a model of overlapping investment in an economy with financial intermediaries and imperfect capital markets.

By overlapping investment we mean investment projects require several periods to mature and different segments of the population experience a different stage of the investment cycle at a given time. For all agents different forms of investment occur in the two periods before output is received; once output has been produced a new investment cycle begins. Between the periods that output is produced agents have no income. At each time, half of the population receives output when the other half has no income. Thus, while all agents are identical in terms of production technology they begin the investment cycle at different times. These two halves of the population complement each other and this allows mutually profitable trade to occur. Even though output is not available in every period agents enjoy positive consumption at all times by trading goods for money. Specifically, agents with output trade part of their output for money which is supplied by the agents without output.

There is a competitive banking sector which makes loans to the private sector, but agents require collateral to borrow. Money demand arises from the liquidity needs of agents. Money can be used to smooth consumption by allowing agents to purchase consumption when they are not producing output. It can also provide a means of financing second period investment, thereby reducing or eliminating the need to borrow. The collateral against which agents can borrow is one of the kinds of capital which agents use in production. Since this capital is a choice variable agents can in a sense determine their own credit-worthiness. This opens the possibility that agents overaccumulate collateral capital and thus introduce inefficiencies into the economy.

Residential housing accounted for 41.4 per cent of household wealth according to the 1995 United States Survey of Consumer Finances and houses are a form of capital which can be used as collateral for personal loans. It is not inconceivable that agents overinvest in housing relative to other forms of capital as part of a comprehensive plan of lifetime consumption. Similarly, firms use their buildings and other easily observable and tradable forms of capital as collateral. The question we pose in this paper is whether one inefficiency arising from a capital market imperfection could be corrected by increasing another kind of inefficiency.

The tradition in monetary economics has been to view statutory reserves
as a tax on the banking sector because these funds earn no return. Money allocated to reserves could be loaned at a productive rate of interest. However, if agents are overinvesting in collateral relative to other forms of capital it is efficient to reduce the extraneous accumulation. Increasing reserve requirements will decrease the amount banks have to lend and eliminate part of the overinvestment in collateral as interest rates rise. We assert that there is an equilibrium in an optimizing economy where increases in commercial bank reserves do enhance welfare.

Many of the issues arising in this paper have been considered in the literature. Our model is similar in spirit to the model presented in Faig (1999a). One of the major differences with Faig (1999a) is that our model allows for credit markets. It is the investment structure of our model which is very similar to the overlapping investment model described in Faig (1999a). The chief difference between this paper and ours is that Faig (1999a) does not have financial intermediaries. Faig (1999a) discusses the effects of a self-financed central bank on welfare and growth but excludes a competitive banking sector.

Several researchers have analyzed capital market imperfections with the aim of explaining empirical features of asset returns. Aiyagari and Gertler (1991) investigate whether holding liquid assets helps explain several stylized facts of asset returns. Agents hold liquid assets because with imperfect markets they cannot ensure all risk. Their model has a mixed performance in explaining the empirical features of a low real interest rate and large spread between liquid and illiquid assets. Holmstrom and Tirole (1998) also devise a model of imperfect capital markets with the aim of explaining risk premia and the volatility correlations which have been observed in financial data. They introduce a "corporate agency approach to asset pricing". Their model includes financial intermediaries but the supply of assets that could serve as collateral is not endogenous. This represents an important conceptual difference between our model and theirs. Holmstrom and Tirole (1997) model an economy where firms and financial intermediaries are capital constrained. A "collateral squeeze" is harder on the less well capitalized firms than on highly leveraged firms. Bencivenga and Smith (1991) as well as Espinosa and Yip (1995) develop growth models with financial intermediaries. However, neither paper incorporates the requirement that borrowing must be backed by collateral. Gorton and Pennacchi (1990) provide a rationale for deposit insurance in an economy with financial intermediaries. Bank deposits protect uninformed agents when they are risk free. If banks make risky investments
deposit insurance can replicate the allocations that come from riskless deposits.

The paper is organized as follows. Section 2 discusses the model. Section 3 provides an overview of equilibrium conditions. Section 4 discusses the borrowing unconstrained equilibrium. Sections 5 discusses the borrowing constrained equilibrium. Section 6 discusses welfare implications of changing reserve requirements. Section 7 concludes. Section 8 is an appendix containing detailed derivations of some results found in the text.

## 2 The Model

Consider an economy where all agents are identical in terms of preferences, production technology and opportunities. There is a large number of agents so that no one agent affects the behaviour of any other agent. Agents make their production, consumption and portfolio decisions at discrete times \( t = 0, 1, 2 \ldots \). Each agent decides on appropriate capital allocations for production as well as consumption, saving and borrowing. There is one production good which is sold on a competitive market. Since there are many agents, each agent takes the price of the good as given.

Each agent owns a production technology. The most complicated feature of the model is production because it occurs in two distinct stages. Specifically, there is overlapping investment by which we mean that the different kinds of capital are invested at different times. Consequently, the different kinds of capital usually appear with a time subscript. In addition, the production technology is non-transferable and is constant across time.

Although agents are identical in terms of technology they are differentiated by the stage of the investment cycle they experience. At any time \( t \), half the population will undergo stage one of the investment cycle while the other half of the population experiences stage two. The description of the investment cycle which follows applies to half of the population. The other half of the population experiences structurally the same investment process but will be in stage one at odd times rather than even times.

At times \( t = 0, 2, 4, 6 \ldots \) each agent decides on quantities of capital \( k_{0t} \) and \( k_{1t} \) to use in production. The input of another form of capital \( k_{2,t+1} \) occurs at time \( t + 1 \). Final output, \( y_{t+2} \), results at time \( t + 2 \) and a new cycle of investment can begin. Below, the different kinds of capital are given distinct interpretations. Consequently, for expositional purposes we frequently omit
the time subscript and denote the capitals as $k_0$, $k_1$ and $k_2$. It is not possible to invest $k_0$ or $k_1$ at odd times, neither can $k_2$ be invested at even times. This structure implies that investment follows a two stage process over time.

The situation we are modeling is one where investment returns follow with a delay. The investment structure is meant to capture the feature of the real world where investment projects generally require several periods to mature and the output of many productive units varies over time. Allowing three distinct kinds of capital is also a realistic feature of the model since many industrial or agricultural projects use different kinds of capital, each with a different productivity. As an example, consider an agricultural economy. The capitals $k_0t$ and $k_1t$ can represent buildings and plowing investment at time $t$ respectively; $k_{2t+1}$ can represent seeding investment at $t + 1$ and $yt_{t+2}$ the harvest.

Although agents only receive output at even times, they may engage in saving and thus consume at all times. Output is perishable and once received it must be consumed and invested in $k_0$ as well as $k_1$. Then the balance is sold for money to be used in the next period. The money is supplied by the agents who are currently in stage two of the investment cycle i.e. the agents without output. Thus, at time $t$ the agents currently in stage two are able to consume even though they have no output of their own. At time $t + 1$ the agents who were in stage one at time $t$ will be in stage two. They will sell their money holdings for the output of the agents who were in stage two at time $t$. In this way, all agents maintain consumption levels despite the cyclical nature of output.

Money in the form of deposits is the only asset available for saving. Money is only held for the purpose of smoothing consumption between the periods when output accrues or to finance investment in $k_2$. It does not enter the economy artificially through a utility function, neither does the demand for money originate from a cash-in-advance constraint as in Lucas (1987). Instead, money demand arises endogenously as agents determine their optimal cycle of consumption and investment.

The economy has financial intermediaries in the form of a competitive commercial banking sector. The banking sector makes loans to agents. Loans may be used by agents to increase consumption or to augment investment in $k_2$. The credit market is imperfect in the sense that agents require collateral to borrow. We assume that $k_0$ plays the role of collateral. By assumption, $k_0$ will represent an observable form of capital against which agents can borrow. The other forms of capital $k_1$ and $k_2$ are assumed to be either unobservable or
non-transferable and therefore unsuitable as collateral. For example, \( k_0 \) can be the housing stock and \( k_1 \) as well as \( k_2 \) can be different types of circulating capital e.g. materials. There are a large number of banks implying that each bank takes the interest rate on loans as given. There is also a central bank which provides reserves for the commercial banking sector and in some instances requires the banks to hold a minimum amount of reserves. The model allows reserve requirements to be zero as a special case.

This scenario creates an interesting problem. On one hand, agents are constrained in their borrowing because of the amount of collateral they possess. On the other hand, agents effectively choose the amount of collateral they possess via their decision about \( k_0 \). Since \( k_0 \) represents physical, transferable capital such as houses, there is the question of whether or not the economy overinvests in this particular kind of capital to borrow more to finance future expenditures. If overaccumulation of \( k_0 \) exists, is it possible for a small increase in the reserve ratio to improve welfare? Suppose the money supply is constant. If the central bank increased mandatory reserves held by the banking sector there would be a reduction in the funds available for loans. This would engineer an increase in the interest rate and decrease the incentive to accumulate collateral. In the new equilibrium the inefficient accumulation of collateral would be reduced and might even be eliminated. Despite the credit contraction, it is possible that a higher level of welfare could result as resources are allocated more efficiently. This is the central question addressed by the paper.

It is important to emphasize that the problem is typically with overaccumulation of collateral rather than with borrowing for \( k_2 \). However, only by raising interest rates is it possible to counter excessive accumulation of collateral. In a typical equilibrium \( k_2 \) will be too small and \( k_0 \) too large relative to what is optimal. Let us turn toward a formal description of the constraints and choice problem. The next section will discuss the solution concepts and possible equilibria.

Let \( t \) be an even number so that agents are in stage one of the investment cycle. At time \( t \), agents must allocate their wealth \( x_t \) to first stage consumption \( c_{1t} \), first stage investment \( k_{0t}, k_{1t} \) and money holdings \( m_t \). The first budget constraint is

\[
x_t = c_{1t} + k_{0t} + k_{1t} + m_t \quad (2.1)
\]

Let \( R_t \) and \( r_t \) be the gross returns (one plus the net interest rate) on loans and money respectively at time \( t \). As markets are competitive, the agent takes
these rates as given. At time $t + 1$ agents may borrow against the capital $k_{0t}$; denote borrowing by $b_{t+1}$. Borrowing and total savings, $r_{t+1}m_t$, are used to finance second stage consumption and investment, denoted by $c_{2t+1}$ and $k_{2t+1}$ respectively. The resulting budget constraints for $t + 1$ are

$$b_{t+1} + m_t r_{t+1} = c_{2t+1} + k_{2t+1} \quad (2.2)$$
$$b_{t+1} \leq k_{0t} \quad \text{and} \quad m_t, k_{0t}, k_{1t}, k_{2t+1} \geq 0 \quad (2.3)$$

By assumption, the production technology will be Cobb-Douglas. Let $y_{t+2}$ denote the output resulting from the previous two period cycle. Then production is defined by

$$y_{t+2} = A k_{0t}^{\alpha_0} k_{1t}^{\alpha_1} k_{2t+1}^{\alpha_2} \equiv F(k_{0t}, k_{1t}, k_{2t+1}) \quad (2.4)$$

We assume that there is one fixed factor which in turn implies that returns to scale are decreasing i.e. $\alpha_0 + \alpha_1 + \alpha_2 < 1$. Although it is usually conventional to assume constant returns to scale (CRS), we depart from the CRS assumption for reasons of analytic tractability. When returns to scale are decreasing we can find precise equations for the capital stocks, borrowing and consumption.

Agents have an intertemporal utility function with instantaneous utility given by $\log(c_{1t})$ at even times and $\log(c_{2t})$ at odd times $t$. The discount factor $\beta$ is constant over time. Intertemporal preferences are

$$V_0 = \sum_{t=0}^{\infty} \beta^t \log(c_{1t}) \quad (2.5)$$

Standard methods of dynamic programming such as those described in Sargent (1987) allow us to solve the problem recursively. Thus, preferences at an even time $t$ are represented by the value functions $V_t$ and $V_{t+2}$ as well as the one period utilities:

$$V_t = \log(c_{1t}) + \beta \log(c_{2t+1}) + \beta^2 V_{t+2} \quad (2.6)$$

The recursive structure implies that wealth at time $t$ is related to past decisions via the relation $x_t = y_t - b_{t-1}R_t$. At time $t$, the choice variables are $c_{1t}, c_{2t}, k_{0t}, k_{1t}, k_{2t+1}$ and $b_{t+1}$. Clearly, money holdings $m_t$ are determined endogenously.
2.1 Behaviour of the Banking Sector

As mentioned above, the banking sector is competitive. The money held by individuals only exists as deposits at the commercial banks i.e. there is no paper currency. The commercial banks also function as intermediaries transferring goods from borrowers to lenders i.e. they facilitate the transfer of goods for money from individuals in stage one of the investment cycle to individuals in stage two. Deposits are thus shifted from individuals who are currently in stage two of the investment cycle to individuals in stage one. The commercial banks then take these deposits and make loans to the individuals who are currently in stage two of the cycle.

There is a central bank which accepts deposits from the commercial banking sector; these deposits are the reserves which commercial banks decide to maintain in equilibrium. Equivalently, reserves may be held as an unbacked security issued by the central bank which has no intrinsic value. Funds held as reserves by the commercial banks come from deposits and thus cannot be loaned to private agents. The central bank may establish a statutory minimum for reserves. In this case, the commercial banks decide whether or not to hold reserves above the statutory minimum. Thus, in a tangible sense, reserves in the model are always endogenous. Even when there are no statutory reserves the commercial banks may still choose to hold reserves at the central bank; such reserves are referred to as voluntary reserves. The central bank has no taxing power besides requiring commercial banks to hold a minimum level of reserves. For simplicity, we assume that there are no government expenditures.

We will examine several types of equilibria. Each equilibrium can occur without statutory reserves. In one equilibrium the commercial banks do not hold reserves i.e. the reserve ratio is zero. This equilibrium occurs when the borrowing constraint is not binding and when interest rates are positive. In the second equilibrium, the borrowing constraint is binding and the interest rate is positive. Thus, commercial banks do not hold reserves willingly since the money could be profitably loaned. The third equilibrium corresponds to the case when agents are borrowing constrained and where the interest rate is zero. Typically reserves will be positive in this equilibrium because there is not the same incentive to loan all funds on deposit.

It is important to understand that these equilibria are exhaustive in the sense that they characterize all the relevant possibilities for the economy. To see this, observe that in equilibrium reserves must be either positive or
zero and the borrowing constraint must be either binding or non-binding. Hence there are four possibilities: non-binding borrowing with zero reserves, binding borrowing with reserves, binding borrowing without reserves and non-binding borrowing with reserves. The first three possibilities correspond to our equilibria. The fourth possibility is clearly uninteresting since there is no reason to force banks to hold reserves when overaccumulation is not a problem. Neither will banks willingly hold reserves with a nonbinding borrowing constraint because, as we demonstrate, the interest rate is positive when this equilibrium prevails. Voluntary reserves in this instance impose a real cost on the banks since the money could be loaned and earning a positive return. Banks will only be willing to hold reserves voluntarily when the interest rate is zero.

Consider a competitive equilibrium with reserves. Let \( \theta_t \) be the reserve ratio at time \( t \) which consists of both statutory and voluntary reserves; clearly \( \theta_t \in [0,1] \). The total return at time \( t \) paid by the central bank to the commercial banks on their reserves is denoted by \( \rho_t \) i.e. \( \rho_t \) is one plus the rate of interest paid on reserves. The zero profit condition for the commercial banks is

\[
r_t = \theta_t \rho_t + (1 - \theta_t) R_t \tag{2.7}
\]

Equation (2.7) states that the return accruing to the banks in the form of loan repayment and the return on central bank deposits equals the return paid to bank depositors. In the equilibria we analyze \( \rho_t = 1 \) for all times \( t \). i.e. the central bank pays no interest. Thus, our discussion of (2.7) is specialized to this case. Consider the left hand side of the equation. The central bank pays no interest on reserves and \( \theta_t \) percent of each dollar is held in reserve; \( \theta_t \) is therefore the return on reserves held at the central bank. Hence \( 1 - \theta_t \) percent of every dollar is loaned to agents and this commands a return of \( R_t \). If there are no reserve requirements and no voluntary reserves, then \( \theta_t = 0 \) and banks receive a return of \( R_t \) from loans which they pay as interest to the private sector. For \( \theta_t > 0 \) it is clear that \( r_t \leq R_t \) because \( R_t \) is one plus the rate of interest on loans whereas reserves yield no interest. Let us summarize each equilibrium in more detail.

In the first equilibrium, \( r = R = \beta^{-1} \), the borrowing constraint is non-binding and reserves are zero. In this equilibrium there is no inefficient collateral accumulation. Another way of interpreting this equilibrium is that collateral is abundant relative to investment needs. For example, if \( b_{t+1} < k_{0t} \), then \( k_{0t} - b_{t+1} \) is held by agents because it yields a valuable stream of pro-
duction. A positive net worth means that agents value the underlying asset for its productive potential and not its ability to augment consumption as security against a loan.

In the second equilibrium $1 < r = R < \beta^{-1}$, the borrowing constraint is binding i.e. $b_{t+1} = k_{0t}$ and reserves are zero. Agents overaccumulate collateral and investment in the other kinds of capital suffers. In the third equilibrium, the borrowing constraint is binding, interest rates are zero and reserves are typically positive. It is in the third equilibrium that we consider imposing statutory reserves. This causes the economy to shift to a new equilibrium and may increase welfare.

Although agents take $r_t$ and $R_t$ as given, we will analyze equilibria where both returns are endogenous. The rates of return will be determined by market clearing condition

$$ (1 - \theta_t)m_t = b_{t+1} \quad (2.8) $$

The term $(1 - \theta_t)m_t$ is the amount of money which can be loaned after reserve requirements are fulfilled; equation (2.8) is simply the condition of equilibrium in the capital market.

3 Characterizing The Optimal Behaviour Of Individuals

In equilibrium, money holdings $m_t$ as well as both returns $R_t$ and $r_t$ are determined endogenously. However, to solve for individual behaviour in the competitive market, we take both $r_t$ and $R_t$ as given. The structure of the choice problem is the same for both halves of the population because the investment cycle is common to both halves except for the difference in the timing of the stages. Consequently, we may focus on the first order conditions for agents who begin investing at even times. The first order conditions for the other half of the population are structurally identical and will differ only in time subscript. Thus, the conditions derived below characterize individual behaviour for the whole economy. In forming the Lagrangean for the choice problem, we collapse (2.1) and (2.2) into a single constraint by solving for $m_t$ in (2.2) and substituting into (2.1). Thus, the Lagrangean associated with the choice problem for each agent is

$$ L_t(x_t) = \log(c_{1t}) + \beta \log(c_{2t+1}) + \beta^2 V_{t+2}(F(k_{0t}, k_{1t}, k_{2t+1}) - b_{t+1}R_{t+1}) + $$
\[
\lambda_t(x_t - c_{1t} - k_{0t} - k_{1t} - r_{t+1}^{-1}(c_{2t+1} + k_{2t+1} - b_{t+1})) + \lambda'_t(k_{0t} - b_{t+1})
\]

Since the objective function is concave and the constraints convex, standard dynamic programming theory implies that the choice problem has a unique solution. The solution to the individual choice problem is a sequence \(\{(c_{1t}, c_{2t+1}, k_{0t}, k_{1t}, k_{2t+1}, b_{t+1})\}_{t=0}^{\infty}\) which solves the problem defined by (2.1)-(2.6) for given sequences \(\{r_t, t = 0, 1 \ldots\}\) and \(\{R_t, t = 0, 1 \ldots\}\).

We refer to the vectors \((c_{1t}, c_{2t+1}, k_{0t}, k_{1t}, k_{2t+1}, b_{t+1})\) that define a solution as solution vectors. Given each of the variables in the solution sequence, we can determine optimal money holdings, so \(m_t\) does not have to enter the solution vector explicitly.

The first order conditions we derive are the necessary conditions to be satisfied by a solution. For notational convenience, we denote the partial derivative of the production function with respect to \(k_{it}\) by \(F_{it}, i = 0, 1\). The partial derivative of the production function with respect to \(k_{2t+1}\) is denoted by \(F_{2t}\). Using the Kuhn-Tucker theorem the first order conditions for the individual choice problem are:

\[
\begin{align*}
 c_{it}^{-1} &= \beta r_{t+1} c_{2t+1}^{-1} \quad (3.1) \\
 \beta^2 F_{1t} c_{it+2}^{-1} &= c_{it}^{-1} \quad (3.2) \\
 r_{t+1} F_{2t} &= F_{1t} \quad (3.3) \\
 F_{0t} &= F_{1t}(1 - r_{t+1}^{-1}) + R_{t+2} \quad (3.4) \\
 b_{t+1} &= k_{0t} \quad \text{and} \quad F_{0t} \leq F_{1t} \quad (3.5a) \quad \text{or} \\
 b_{t+1} &\leq k_{0t} \quad \text{and} \quad F_{0t} = F_{1t} \quad (3.5b) \\
 x_t &= c_{1t} + k_{0t} + k_{1t} + r_{t+1}^{-1}(c_{2t+1} + k_{2t+1} - b_{t+1}) \quad (3.6) \\
 x_t &= F(k_{0t-2}, k_{1t-2}, k_{2t-1}) - R_t b_{t-1} \quad (3.7)
\end{align*}
\]

Equations (3.1) and (3.2) characterize the intertemporal path of consumption. Equation (3.1) equates marginal rates of utility in consumption. By foregoing a small amount of consumption in period one and buying money, the agent loses utility in period one and earns a return of \(r_{t+1}\) on the incremental money holdings. This amount is available for consumption in period two. Equation (3.1) means that agents have no incentive to shift consumption between periods because the discounted utility in period two of so doing
equals the instantaneous utility of consumption. The term $\beta r_{t+1}c_{2t+1}^{-1}$ is the discounted utility associated with incremental saving in period one and the higher consumption achieved in period two.

Equation (3.2) has a similar interpretation. Agents can shift consumption between periods $t$ and $t + 1$. By lowering consumption at time $t$ and investing more in $k_{1t}$ agents can produce more and enjoy higher consumption at time $t + 2$. In (3.2) agents are indifferent between consuming immediately and consuming more at $t + 2$. The term $\beta^2 F_{1t} c_{t+2}^{-1}$ is the discounted utility of the extra consumption arising from incremental investment in $k_{1t}$ and $c_{1t}^{-1}$ is instantaneous utility of first period consumption.

Equations (3.3) and (3.4) characterize the intertemporal allocation of capital. Equation (3.3) states that agents are indifferent between using $k_{1t}$ and saving in the form of money, then using the return to invest in $k_{2t+1}$. The left hand side of (3.3) represents the incremental production received from shifting from $k_{1t}$ to money and then to $k_{2t+1}$. Equation (3.4) states that investors must be indifferent between using $k_{0t}$ or shifting some investment to one of the other capitals. The expression $F_{1t}(1 - r_{t+1}^{-1}) + R_{t+2}$ is the return that must be received from investing an incremental amount in $k_{1t}$ (which can be increased thanks to the larger collateral) and borrowing to finance investment in $k_{2t+1}$.

Equations (3.5) to (3.7) are constraints associated with the Lagrange multipliers. In (3.5) we constrain borrowing to be less than or equal to collateral. In the case where borrowing equals collateral the agent is overaccumulating. This is referred to as the borrowing constrained case because the constraint is binding and is described in (3.5a). The case (3.5b) is referred to as the borrowing unconstrained case because the constraint is not binding. Condition (3.5) also states the associated conditions on the production function in the borrowing constrained and unconstrained cases. Equations (3.6) and (3.7) are the forward and backward recursions for wealth at time $t$ respectively. As already mentioned, (3.6) comes from collapsing (2.1) and (2.2) into a single constraint. Equation (3.7) expresses $x_t$ as the net worth of the individual after loans have been repaid and the previous production cycle completed.

Problems with time dependent value functions frequently do not have known closed form solutions. To construct an equilibrium, it is convenient to employ the simplifying concept of a steady state. The existence of a steady state equilibrium makes the associated value function time independent. Once $V_t = V \forall t$, the first order conditions become much simpler and
admit closed form solutions for capital and consumption during the decision cycle.

**Definition 3.1:** A solution to the choice problem is a steady state solution if each of the variables defining the solution vector is constant across time and if both returns are also constant over time.

Clearly money holdings and wealth are also constant across time under a steady state solution. Most of the analysis to follow will focus on steady state solutions and their properties. It is left to future research to determine transitional dynamics. In what follows we drop time subscripts because of the stationarity of variables in a steady state.

### 3.1 Steady State Equilibrium

**Definition 3.2:** An equilibrium of the economy defined by (2.1)-(2.8) is a sequence \( \{c_{1t}, c_{2t+1}, k_{0t}, k_{1t}, k_{2t+1}, \theta_t, b_{t+1}, r_t, R_t, \}, t = 0, 1 \ldots \) which solves the individual choice problem for \( \{r_t, t = 0, 1 \ldots \} \) and \( \{R_t, t = 0, 1 \ldots \} \) and which satisfies the capital market clearing condition as well as the zero profitability condition for commercial banks.

The individual vectors of an equilibrium will be called equilibrium vectors. The difference between equilibrium for the economy and a solution for the choice problem is that \( r_t, R_t \) and \( \theta_t \) become endogenous. If statutory reserve requirements are being modelled and they are binding, \( \theta_t \) is equal to the required reserves coefficient. However, it is still endogenous whether the reserve requirements are binding or not.

To find an equilibrium, we must find \( \{r_t, R_t, \theta_t\}, t = 0, 1 \ldots \) such that the associated solution to the choice problem also satisfies (2.7) and (2.8). Generally, this is not analytically tractable and a simplifying technique is necessary. Once again, we invoke the notion of stationarity.

**Definition 3.3:** A steady state equilibrium is an equilibrium of the economy where all the variables in the equilibrium vector are constant across time.

The rest of the paper will focus on steady state equilibria. Each of the three possible equilibria discussed in section 2.1 is a steady state equilibrium and each one will be analyzed in a separate section. In a steady state equilibrium \( \rho = 1 \).

Market clearing in the steady state still requires that the fraction of money not placed on deposit with the central bank equal the desired borrowing. As
discussed above, the differences in timing of the investment cycle across the population implies that the money market always clears. Agents without output at time $t$ always have the incentive to sell money to the agents with output for otherwise they cannot consume. Likewise, agents with output willingly acquire money to finance consumption at time $t+1$ when they have no output. Deposits not held as reserves are available to loan to the agents without output. This yields the market clearing condition

$$m(1 - \theta) = b \quad (3.8)$$

From (2.7) $r$ and $R$, the steady state returns, are related by $r = \theta + (1 - \theta)R$.

4 Equilibrium Without Binding Borrowing Constraints

Consider the case without binding borrowing constraints and without statutory reserves. For convenience we denote the stationary production function $F(k_0, k_1, k_2)$ by $F$ and the derivative with respect to $k_i$ by $F_i$. When $b \leq k_0$ the first order conditions become

$$R = F_2 \quad (4.1)$$

$$F_0 = F_1 = \beta^{-2} \quad (4.2)$$

$$rR = \beta^{-2} \quad (4.3)$$

$$\beta^2 \alpha_1 F = k_1 \quad (4.4)$$

$$\beta^2 r \alpha_2 F = k_2 \quad (4.5)$$

$$\beta^2 \alpha_0 F = k_0 \quad (4.6)$$

This implies that $r = R = \beta^{-1} > 1$. Let $\alpha^* = 1 - \alpha_0 - \alpha_1 - \alpha_2$ and $A^* = (A_0 \alpha_0 \alpha_1 \alpha_2)^{\frac{1}{\alpha^*}}$. Then algebraic manipulation of the first order conditions yields closed form expressions for production, capital and consumption.

$$F = A^* \beta^{(\alpha^*)^{-1}(2(1 - \alpha^*) - \alpha_2)} \quad (4.7)$$

$$k_0 = \alpha_0 \beta^2 F \quad (4.8)$$
Equilibrium in the loan market requires that $m = b$. From (2.2) this means that $\beta(c_2 + k_2 - b) = b$; hence $b = \beta(1 + \beta)^{-1}(c_2 + k_2)$. Finally, for existence of this type of equilibrium we need $b \leq k_0$. Substitution from (4.7) and (4.11) yields

$$
(1 + \beta)^{-1}\left(\frac{1}{2}(\beta^{-1} - (\alpha_0 + \alpha_1 + \alpha_2\beta^{-1})\beta^2) + \alpha_2\right) \leq \alpha_0 \quad (4.12)
$$

In (4.12) we may isolate $\alpha_0$ on one side of the inequality to yield the necessary condition for the equilibrium to exist:

$$
\alpha_0 \geq \frac{\beta^{-1} - \alpha_1\beta + \alpha_2}{2 + 3\beta} = \frac{1 - \alpha_1\beta^2 + \alpha_2\beta}{\beta(3\beta + 2)} \equiv \alpha'_0 \quad (4.13)
$$

5 Equilibrium With Binding Borrowing Constraints And No Statutory Reserves

Consider the case with binding borrowing constraints and when all reserves are voluntary. There are two possibilities: either $r = R > 1$ or $r = R = 1$. In the first instance, banks do not hold reserves because the interest rates are positive. In the second case voluntary reserves can exist and they are determined endogenously.

As in the last section the model yields closed form solutions. Equations (5.1)-(5.7) apply when $r = R > 1$ as well as when $r = R = 1$.

$$
F = A^*\beta^{2(1-\alpha'_1)}r^{\frac{\alpha}{\alpha^2}}(R\beta^2 + 1 - r^{-1})^{-\alpha_0} \quad (5.1)
$$

$$
k_0 = \alpha_0\beta^2(R\beta^2 + 1 - r^{-1})^{-1}F \quad (5.2)
$$

$$
k_1 = \alpha_1\beta^2F \quad (5.3)
$$

$$
k_2 = \alpha_2\beta^2rF \quad (5.4)
$$
Although the order conditions are the same in both cases, the equilibria differ in their money market equilibrium equations. Differences also emerge in the necessary conditions for the equilibria to exist. Let us examine the differences in more detail.

When $r = R > 1$ we must have $r < \beta^{-1}$. To see this observe that if $r = R = \beta^{-1}$ substitution from (3.2) into (3.3) yields $F_2 = \beta^{-1}$. Substitution back into (3.3) yields $F_1 = 1$. Equations (3.4) and (3.5) imply $1 \geq (1 - \beta) + \beta^{-1}$ or that $1 - \beta \leq 1 - \beta^{-1}$. Since $\beta < 1$ this is a contradiction. Hence, the equilibrium interest rate must change between the borrowing constrained and borrowing unconstrained equilibria.

As mentioned above, when $r = R > 1$ there can be no reserves i.e. $\theta = 0$. The money market equilibrium condition implies $m = b = k_0$. To derive a necessary condition for the existence of the equilibrium, use the equation $m = b = k_0$ and substitute it into (5.7). The resulting condition in (5.8) is derived in the appendix.

$$
\beta(1 + \beta)^{-1} + \alpha_2\beta^2(1 + \beta)^{-1} > 2\beta^4\alpha_0(\beta^2 + 1)^{-1} + \beta^3\alpha_1(1 + \beta)^{-1} \tag{5.8}
$$

Now consider the case of $r = R = 1$ and $\theta \geq 0$. When $r = R = 1$ and $\theta \geq 0$ the first order conditions (5.1)-(5.7) can be simplified, but the principle interests are in finding a convenient expression for $\theta$ and a necessary condition for the equilibrium. In this case the interest rate is zero so banks may elect to hold reserves. Reserves are determined endogenously and are given by the money market equilibrium which is $(1 - \theta)m = b = k_0$. Thus, after substitution from the first order conditions we have

$$
1 - \theta = k_0/m = \frac{\alpha_0}{\beta(1 + \beta)^{-1}(1 - \beta^2(\alpha_1 + \alpha_2) - \alpha_0) + \alpha_2\beta^2 - \alpha_0} \tag{5.9}
$$

For the equilibrium to exist, we must have $\theta \in [0, 1]$ and $m > 0$. These restrictions yield two necessary conditions for the equilibrium with $r = R = 1$ to exist; they are presented in (5.10) and (5.11). The derivations again rest on substitution from the first order conditions (5.1)-(5.7).
The first condition arises from the restriction $\theta \in [0, 1]$. Clearly, we must have $\theta \geq 0$, implying $k_0/c_2 + k_2 - k_0 \leq 1$ or after substitution from the first order conditions.

$$\beta(1 + \beta)^{-1}(1 - \beta^2(\alpha_1 + \alpha_2) - \alpha_0) + \alpha_2 \beta^2 \geq 2\alpha_0 \quad (5.10)$$

This can be simplified to

$$\alpha_0 \leq \frac{\beta(1 - \alpha_1 \beta^2 + \alpha_2 \beta)}{3\beta + 2} \equiv \alpha_0^* \quad (5.11)$$

The condition that $m > 0$ is always satisfied. It implies

$$\beta(1 + \beta)^{-1}(1 - \beta^2(\alpha_1 + \alpha_2) - \alpha_0) + \alpha_2 \beta^2 > \alpha_0 \quad (5.12)$$

Clearly (5.12) is satisfied whenever (5.10) is satisfied.

Having discussed the equilibria we now address the question of which equilibria will occur for different parameter values. The answer is provided in Theorem 5.1 the proof of which is deferred to the appendix.

**Theorem 5.1**: Suppose that statutory reserves are zero and $\beta \in (0, 1)$. Then there exist a pair of numbers $\alpha_0^*, \alpha_0'$ determined by (5.11) and (4.13) respectively, with $\alpha_0' > \alpha_0^*$ such that

1) if $\alpha_0 \leq \alpha_0^*$ then $r = R = 1$ and $\theta \geq 0$.
2) if $\alpha_0 \in (\alpha_0^*, \alpha_0')$ then $\beta^{-1} > R = r > 1$ and $\theta = 0$.
3) if $\alpha_0 \geq \alpha_0'$ then $r = R = \beta^{-1}$ and $\theta = 0$.

The theorem states that when $\alpha_0 \geq \alpha_0'$ the borrowing unconstrained equilibrium results. When $\alpha_0 \leq \alpha_0^*$ the borrowing constrained equilibrium with zero interest rates and nonnegative reserves occurs. For $\alpha_0$ between the two bounds the borrowing constrained equilibrium with positive interest rates and zero reserves results.

The occurrence of a particular equilibrium depends on the productivity of collateral. If $\alpha_0$ is large, collateral is very productive and agents have a strong incentive to accumulate it. A high value for $\alpha_0$ will be associated with a high level of collateral. In this instance, the abundance of collateral is not inefficient because the collateral is held for its productive potential. When $\alpha_0$ is small collateral is not very productive. For $\alpha_0$ sufficiently small agents overaccumulate collateral because they want to increase borrowing. In general, $k_0$ will be higher than in the equilibrium which would occur without
the capital market imperfection. For an intermediate range of $\alpha_0$ there will also be overaccumulation but the interest rate will be positive.

6 Imposing Statutory Reserves and Welfare

This section addresses the critical question of whether increases in the reserve ratio can improve welfare when the borrowing constraint is binding. The analysis focuses on the imposition of a slightly binding reserve requirement around the equilibrium with $r = R = 1$. Once again, the analysis concentrates on steady state solutions and the welfare comparisons we make are between steady state equilibria. In other words, we examine how welfare changes when the imposition of statutory reserves shifts the economy from a steady state without statutory reserves to another steady state with positive statutory reserves.

Let us review some salient features of the model. Before turning to this discussion, recall the zero profit and money market clearing conditions.

\[
\begin{align*}
    r - \theta - (1 - \theta)R &= 0 \quad (6.1) \\
    (1 - \theta)m - b &= 0 \quad (6.2)
\end{align*}
\]

We break the set of equations determining equilibrium into two groups. The first group consists of equations (5.1)-(5.7). These equations determine $(c_1, c_2, k_0, k_1, k_2, m, b, F)$. All of these variables determined by (5.1)-(5.7) are implicit functions of $(r, R)$. The second group consists of equations (6.1)-(6.2). When reserve requirements are not binding it determines $r = R$ and $\theta$. When reserve requirements are binding it determines $r$ and $R$.

As mentioned above, all the variables defined by (5.1)-(5.7) can be treated as functions of $(r, R)$, at least in a small neighbourhood of a given equilibrium. In turn, both the $r$ and $R$ which occur in equilibrium are implicit functions of statutory reserves. Denote by $m_r, m_R$ the partial derivatives of $m$ with respect to $r$ and $R$ and $b_r, b_R$ the partial derivatives of borrowing with respect to $r$ and $R$.

The reserve ratio $\theta$ is defined by (5.9). We will focus on an increase in statutory reserves from zero to a small positive quantity about the equilibrium when $R = r = 1$. When statutory reserves are imposed, the banks still make a choice over what voluntary reserves to hold above the statutory minimum. The imposition of statutory reserves has the effect of raising the equilibrium reserve ratio because we assume that if banks voluntarily hold
θ₀ the government forces them to hold θ' > θ₀. The reserves observed in the new equilibrium, θ* must satisfy θ* > θ'.

By applying the implicit function theorem to the system (6.1)-(6.2) we solve for variations in the equilibrium returns as statutory reserves change. From the preceding discussion, this follows because imposing statutory reserves really amounts to an increase in the equilibrium θ. It is important to emphasize that the implicit function theorem is being applied in the neighbourhood of a particular equilibrium. As θ varies we assume that a smooth change in equilibria takes place, at least in a small neighbourhood of a given equilibrium. This means that if the equilibrium reserve ratio is θ₀ for r = R = 1, small variations in reserves around θ₀ still produce borrowing constrained equilibria. Moreover, the equilibrium returns are assumed to be a smooth function of θ in this neighbourhood. In such an instance it is possible to use the implicit function theorem to solve for the change in the equilibrium returns r and R.

It will become clear that changes in welfare depend on the parameters, linear combinations of m_r, m_R, b_r, b_R and the derivatives of the returns with respect to reserves. Thus, we must begin by finding tractable expressions for these derivatives, particularly the derivatives with respect to reserves. To this end we will now apply the implicit function theorem to the system arising from zero profitability by banks and money market clearing.

**Theorem 6.1:** Let \( \delta = (1 - \theta)m_R - b_R + (1 - \theta) [(1 - \theta)m_r - b_r] \). Then implicit differentiation yields

\[
\frac{dR}{d\theta} = \delta^{-1} [m(1 - \theta) - (R - 1)((1 - \theta)m_R - b_R)] \quad (6.3)
\]

\[
\frac{dr}{d\theta} = \delta^{-1} [m + (R - 1)((1 - \theta)m_r - b_r)] \quad (6.4)
\]

**Theorem 6.2:** At \( r = R = 1 \), \( \frac{dr}{d\theta} = (1 - \theta) \frac{dR}{d\theta} \); moreover \( \frac{dR}{d\theta} > 0 \). Thus, provided they are continuous, both derivatives \( \frac{dr}{d\theta} \) and \( \frac{dR}{d\theta} \) must have the same sign in a neighbourhood of the borrowing constrained equilibrium.

**Proof of Theorems 6.1 and 6.2:** Clearly, by proving 6.1 we prove 6.2; thus let us establish 6.1.

An application of Cramer's rule yields the derivatives (6.3) and (6.4). The sign of the derivatives at \( r = R = 1 \) clearly depends on the sign of \( \delta \). We demonstrate the procedure for signing \( (1 - \theta)m_r - b_r \) because the procedure
for $(1 - \theta)m_R - b_R$ is similar. These arguments will imply that $\delta > 0$ and consequently $\frac{d}{d\theta} = (1 - \theta)\frac{dR}{d\theta} > 0$ at $r = R = 1$.

To sign $(1 - \theta)m_r - b_r$ first observe that (5.7) implies $m_r = \frac{d}{dr}r^{-1}(c_2 + k_2 - k_0) = \frac{d}{dr}(\beta c_1 + \alpha_2/\alpha_1 k_1 - r^{-1}k_0)$. After algebraic manipulation this becomes

$$m_r = (m/F)F_r - \alpha_0 \beta^2 (1 + \beta)^{-1}F\frac{d}{dr} R\beta^2 + 1 - r^{-1}$$

Likewise, (5.2) yields $b_r = (k_0/F)F_r + \alpha_0 \beta^2 F\frac{d}{dr} R\beta^2 + 1 - r^{-1}$, implying

$$(1 - \theta)m_r - b_r = \frac{(1 - \theta)m - k_0}{F}F_r - \alpha_0 \beta^2 F\frac{d}{dr} \frac{(1 + \beta)^{-1}(\beta(R + 1) + r^{-1}) + 1}{R\beta^2 + 1 - r^{-1}}$$

Market equilibrium eliminates the first term on the right hand side because $(1 - \theta)m = k_0$. The derivative on the right hand side is negative. Hence, $(1 - \theta)m_r - b_r > 0$.

Similar reasoning shows that at $r = R = 1$, $(1 - \theta)m_R - b_R > 0$. To see this differentiate $m$ and $b$ with respect to $R$. After algebraic manipulation we have

$$(1 - \theta)m_R - b_R = \frac{m(1 - \theta) - k_0}{F}F_r + \frac{\alpha_0 \beta^2 F}{R\beta^2 + 1 - r^{-1}}(\frac{(1 - r^{-1})(1 - \beta^2)}{1 + \beta} + \beta^2)$$

The market clearing condition $(1 - \theta)m = k_0$ again eliminates one term on the right hand side. At $R = r = 1$ the remaining term becomes $\alpha_0 F \beta^2 > 0$. q.e.d.

Before turning to the welfare effects of changes in $\theta$, consider the effects of changes in $\theta$ on the consumption ratio and the ratios of the different kinds of capital. This list of positive changes might be considered of interest in its own right. However, the principle motivation for examining positive changes is to provide supporting analysis for the changes in welfare. As before, the first order conditions have been used to find simplified expressions for the variables under study.

$$\frac{d}{d\theta} \frac{k_2}{k_1} = \frac{d}{d\theta} r > 0 \quad (6.5)$$

The reason that the derivative is positive is that agents hold money in part to finance investment in $k_2$. When the interest rate rises it becomes easier for agents to finance investment in $k_2$ for any level of money holding. Thus,
interest rate increases make \( k_2 \) relatively less expensive and this relative price change induces higher holdings of \( k_2 \).

It is also true that \( k_2/k_1 \) is low by comparison with the efficient equilibrium. Of course the reason for this is that agents have overallocated resources to investing in \( k_0 \). Once investment in \( k_0 \) falls, relatively more investment flows into \( k_2 \) than before. The typical problem in the model is that \( k_2 \) is small relative to its efficient quantity and that \( k_0 \) is large relative to its efficient quantity. Thus, (6.5) suggests that \( k_2 \) moves in the right direction for an increase in welfare to occur.

\[
\frac{d}{d\theta} k_0 = \frac{d}{d\theta} (R\beta^2 + 1 - r^{-1})^{-1} < 0 \quad (6.6)
\]

The effect is clearly negative because as interest rates increase there is less incentive to borrow in period two. Lower borrowing will be reflected in reduced collateral. Again, the effect on \( k_0 \) is what we would expect if welfare increased as interest rates rose because \( k_0 \) must fall relative to \( k_1 \) as we move toward a more efficient equilibrium. As the overinvestment is corrected agents invest relatively more in \( k_1 \). Finally,

\[
\frac{d}{d\theta} c_2 = \frac{d}{d\theta} \beta r > 0 \quad (6.7)
\]

The effect is again positive because higher interest rates increase the ability of agents to finance \( c_2 \) for any level of money holding. Higher interest rates make \( c_2 \) relatively less expensive and \( c_2/c_1 \) increases.

The evidence on adjustment of capital and consumption suggests that increases in \( \theta \) shift the economy to a more desirable state. The suggestion is indeed strong that welfare should improve as increases in \( \theta \) eliminate wasteful borrowing. The next step is to formalize the problem. Since all variables are stationary, the value function \( V_{t+2} \) can be dropped from (2.6) and the measure of welfare is determined by \( c_1 \) and \( c_2 \). Thus, welfare is proportional to

\[
\log(c_1) + \beta \log(c_2) = \log(c_1) + \beta \log(\beta c_1) \quad (6.8)
\]

Since, \( \frac{d}{d\theta} r > 0 \) welfare will be unambiguously improved if increases in \( \theta \) increase \( c_1 \) and that is clearly the case when \( \frac{d}{d\theta} c_1 > 0 \). Thus, the key to analyzing welfare changes is to calculate \( \frac{d}{d\theta} c_1 \). Denote by \( F_\theta \) the derivative of \( F \) with respect to \( \theta \). Denote \( \frac{d}{d\theta} r \) by \( r_\theta \) and \( \frac{d}{d\theta} R \) by \( R_\theta \).

\[
\frac{d}{d\theta} c_1 = \frac{c_1}{F} F_\theta - (1 + \beta)^{-1} F \beta^2 \alpha_0 \frac{dr}{d\theta} R + 1 - r^{-1} + \frac{d}{d\theta} R \frac{\beta^2 (1 + r^{-1})}{R \beta^2 + 1 - r^{-1}} + \frac{d}{d\theta} R_\theta
\]
Then at \( r = R = 1 \), (6.9) simplifies to

\[
\frac{dR}{d\theta} = \frac{d}{dR} \left( \frac{R + 1 - r^{-1}}{R + 1 - r^{-1}} \right) = \frac{c_1}{F} F_{\theta} - \frac{F \beta^2 \alpha_0}{1 + \beta} \times \frac{r_{\theta} r^{-2} (\beta^2 - 1) + R_{\theta} (1 - r^{-1}) (1 - \beta^2)}{R \beta^2 + 1 - r^{-1}} \tag{6.9}
\]

Clearly \( \frac{d}{d\theta} c_1 > 0 \) if \( F_{\theta} \geq 0 \). The monotone increasing property of \( \log(x) \) implies that if \( \log(F(\theta)) \) is increasing in \( \theta \), then \( F(\theta) \) must also be increasing in \( \theta \). This will guarantee that \( F_{\theta} \geq 0 \). The condition \( F_{\theta} \geq 0 \) in no way contradicts the necessary conditions for existence of the borrowing constrained equilibrium with reserves.

\[
\frac{d}{d\theta} \log(F)|_{r=R=1} = R_{\theta} / \alpha^* \left[ (\alpha_2 - \alpha_0 \beta^{-2}) (1 - \theta) - \alpha_0 \right] \tag{6.11}
\]

Thus, all that is necessary for \( F_{\theta} \geq 0 \) is \( (\alpha_2 - \alpha_0 \beta^2)(1 - \theta) - \alpha_0 \geq 0 \). Since \( 1 - \theta \) is determined by (5.9), this condition simplifies to

\[
\alpha_2 - \alpha_0 \beta^2 \geq \beta (1 + \beta)^{-1} \left[ 1 - \beta^2 (\alpha_2 + \alpha_1) - \alpha_0 \right] + \alpha_2 \beta^2 - \alpha_0 \tag{6.12}
\]

Hence, when (6.12) is satisfied \( c_1 F_{\theta} / F \geq 0 \) in (6.10) and welfare increases with \( \theta \). Since this is strictly positive, welfare unambiguously improves. Thus, the conventional assertion that reserve requirements are a tax with negative consequences is not necessarily correct. Here, we mention the special case of \( F_{\theta} = 0 \). When \( F_{\theta} = 0 \) the economy shifts from one steady state to another as \( \theta \) varies. If \( F_{\theta} = 0 \) the change in welfare is determined entirely by the sign of \( r_{\theta} \). Clearly, when \( F_{\theta} = 0 \) there is no new production arising from imposing statutory reserves and all welfare gains come from correcting the allocative inefficiencies associated with a given level of production.

Inspection of (6.10) and (6.12) reveals that welfare could improve even if the economy does not shift between steady states. Suppose for instance that a borrowing constrained equilibrium existed for \( r = R = 1 \) and some parameter values with an endogenously determined \( \theta \). From (6.12) it is possible that the choice of parameters and the resulting \( \theta \) could make \( F_{\theta} \geq 0 \). A numerical example for which \( F_{\theta} > 0 \) in the absence of required reserves when \( r = R = 1 \)
is $\beta = 0.95$, $\alpha_0 = .166494, \alpha_1 = 0.4$ and $\alpha_2 = 0.4$. Using (5.9) this yields $\theta = .34324$.

Perhaps of greater value is the result on the equilibrium interest rates. Partial equilibrium analysis suggests that when statutory reserves are imposed the rate of return on money should fall. This follows because the cost of maintaining the reserves is similar to a tax on real balances. Conventional wisdom suggests that lending rates should rise because a smaller fraction of deposits is available for lending. Clearly, the results from our general equilibrium model are counter-intuitive because around one equilibrium both rates rise. The derivations of these results found above rested heavily on the binding nature of the borrowing constraint. This result suggests that we must be careful in applying partial equilibrium results to the real economy.

7 Conclusion

The equilibria considered in the model have investigated the cases where borrowing is both constrained and unconstrained by collateral. The unconstrained equilibrium is clearly efficient because agents invest in collateral for its productive capacity, not to increase future consumption. When borrowing is constrained agents are accumulating too much collateral. Our work suggests that parameter values exist which guarantee that overinvestment in collateral capital will be an equilibrium.

Constrained equilibria generate two interesting cases. In one case there are no voluntary reserves. In the other equilibrium there are voluntary reserves. Both the borrowing rate and the return on money rose in response to the imposition of statutory reserves. It is in this equilibrium that imposing a small reserve requirement may increase welfare. By reducing the funds available for loans interest rates rise and decrease the excessive accumulation of collateral. It may still be the case that welfare will improve as interest rates rise when both interest rates are positive but small, however the equations involved are more complex and this case is left for future research.

The ultimate source of all these results is the assumption that agents require collateral to borrow. This is a realistic feature of the model and our results suggest that many conventional views about reserve requirements and interest rate dynamics may require closer examination when capital markets are imperfect. There are undoubtedly other conventional views which might be questioned in the context of our model. Future research might address
the impact of government spending and money creation in the model to
determine whether statutory reserve increases can still be welfare enhancing.

8 Appendix

Derivation of (5.8): \( m = r^{-1}(c_2 + k_2 - k_0 = k_0) \Rightarrow c_2 + k_2 = (1 + r)k_0 \).
Substitution and dividing by \( r \) yields

\[
eta(1 + \beta)^{-1} \left[ 1 - \beta^2(\alpha_1 + \alpha_2) - \beta^2 \alpha_0 \frac{r + 1 - r^{-1}}{r\beta^2 + 1 - r^{-1}} \right] + \alpha_2 \beta^2 = \frac{\alpha_0 \beta^2 (1 + r)}{r^2 \beta^2 + r - 1}
\]

Now \( r > 1 \Rightarrow r^2 \beta^2 + r - 1 < r^2 \beta^2 + r^2 \). So the right hand side is greater than
\( 2\alpha_0 \beta^2 / r^2 (\beta^2 + 1) > 2\beta^4 \alpha_0 / (\beta^2 + 1) \) because \( r^{-2} > \beta^2 \). The left hand side is
less that \( \beta(\beta + 1)^{-1} [1 - \beta^2(\alpha_2 + \alpha_1)] + \alpha_2 \beta^2 \). Expand this to yield (5.8).

Proof of Theorem 5.1: There exists \( r \in (1, \beta^{-1}) \) such that

\[
\beta(1 + \beta)^{-1} \left[ 1 - \beta^2(\alpha_1 + \alpha_2) - \beta^2 \alpha_0 \frac{r + 1 - r^{-1}}{r\beta^2 + 1 - r^{-1}} \right] + \beta^2 \alpha_2 = \frac{\alpha_0 \beta^2 (1 + r)}{r^2 \beta^2 + r - 1}
\]

Hence \( \alpha_0 \) satisfies

\[
\alpha_0 = \beta^{-1}(1 - \alpha_1 \beta^2 + \alpha_2 \beta) \frac{r^2 \beta^2 + r - 1}{r^2 \beta + 2r \beta + 1 + r}
\]

This equation determines \( r \) implicitly. As long as \( \beta \geq 0.5 \) we have \( \frac{dr}{d\alpha_0} > 0 \). Thus when \( \alpha_0 \geq \alpha_0' \) we have \( r = \beta^{-1} \) by the analysis in section 4. When
\( \alpha_0 \leq \alpha_0^* \) we must have \( r = 1 \) and such an \( \alpha_0^* \) must exist by the analysis in
section 5. Thus, for \( \alpha_0^* < \alpha_0 < \alpha_0' \) we must have \( 1 < r < \beta^{-1} \). q.e.d.

9 References

with uninsured individual risk. Journal of Monetary Economics 27, 311-331.

Bansal, R. and Coleman, J. (1996). A monetary explanation of the equi-
ity premium, term premium and risk-free rate puzzles. Journal of Political
Economy 104, 1135-1171.

Bewley, T. (1980). The optimal quantity of money. in Models of Mon-
etary Economies, J. Kareken and N. Wallace (eds). Minneapolis, Federal
Reserve Bank of Minneapolis.

27


Random Perturbations of Deterministic Equilibria

by Gregory Gagnon
University of Toronto, 150 St. George Street, Toronto, Ont. Canada
M5S 3G7

Abstract

This paper examines the issue of whether a small random deviation from a non-random policy process will destabilize the equilibrium exchange rate in a rational expectations economy. The random deviations represent the erratic actions of government which are unobservable by agents until they occur. The degree of randomness is indexed by a parameter $\epsilon \in [0, 1]$, with higher values of $\epsilon$ associated with greater random shocks. As $\epsilon$ changes, the structure of policy changes in a nonlinear way. The random equilibrium exchange rate, $x^\epsilon$, converges to the deterministic rate, $z_t$, as $\epsilon \rightarrow 0$. If agents are approximately rational, large deviations between $x^\epsilon$ and $z_t$ for small $\epsilon$ are possible.

1 Introduction

An issue of central importance in economics is whether or not the variables under study are deterministic or random. We would not think of classifying schools of economic thought into a camp which developed deterministic models and another which used stochastic analysis. However, researchers do frequently favour deterministic over random structures or vice-versa. Thus, while we would not think of a deterministic versus random dichotomy in economics, one does in fact exist. For examples of these two approaches see Matsuyama (1991) and Detemple and Zapareto (1991). The assumption of whether or not the environment in which agents operate is random does affect the equilibrium. It is of interest to compare random equilibria that

---

1 Tel. (905) 629-0895 e-mail: phil.gagnon@dpcdsb.org
JEL Classifications: C60, C62, G12, F31
Key Words: Randomly Perturbed Dynamical System, Stability

30
arise as perturbations of a deterministic equilibrium to determine if the two approaches are compatible.

In this paper, the exchange rates and the policy equations that determine them, are highly nonlinear in a parameter that governs the intensity of the random shock. It is known that random systems with nonlinearities are capable of producing large fluctuations e.g. Lim and Martin (1996). Arnold, Bleckert and Schenk-Hoppé (1999) demonstrate that substantial differences can exist between stochastic versions of deterministic systems and the deterministic systems themselves. Kwiecińska (1999) shows that an unstable deterministic system can be made stable when subjected to random shocks. Complex random shocks can have a wide range of effects. Thus, when modeling economic phenomena with complex random shocks, we cannot be certain whether small random perturbations cause radical changes in equilibria or whether there is a measure of central tendency. However, this question has received virtually no attention in economics.

Based on the work of Flood and Garber (1983), this paper examines the issue of whether random equilibrium exchange rates can be approximated by a deterministic exchange rate. Policy rules are modeled as non-random functions and lead to a non-random exchange rate; randomness comes from erratic action and yields a random exchange rate. The degree of randomness is indexed by a parameter $\epsilon \in [0, 1]$, with higher values of $\epsilon$ associated with greater random shocks. We show that in an infinite horizon economy the random equilibrium exchange rate $x_t^\epsilon$ converges almost everywhere to the deterministic exchange rate $z_t$, as $\epsilon \to 0$ for all $t$. So for small random shocks we may approximate the random equilibrium by the deterministic equilibrium at a given instant. For finite horizon economies, we establish that $\lim_{\epsilon \to 0} P(\sup_{t\in[0,T]} |x_t^\epsilon - z_t| > \delta) = 0 \ \forall \delta > 0$ and determine precise bounds on the rate of convergence. We also establish the considerably more difficult result that $\lim_{\epsilon \to 0} \max_{t\in[0,T]} |x_t^\epsilon - z_t| = 0$ almost everywhere.

The equilibrium exchange rate is a forward-looking expression typical of rational expectations models. For general references on rational expectations equilibria see Sargent (1987) and Pentecost (1993). The exchange rate is an integral of discounted expected future policies. Exactly the same expression would arise if forward-looking agents were pricing a stock and they were required to form expectations about future dividends. Thus, the stability results presented here have applicability to more general asset pricing models. Freidlin and Wentzell (1984) proved that small stochastic perturbations of an ordinary differential equation were close approximations to the original
deterministic equation. Our theorems extend the Freidlin-Wentzell results to rational expectation asset pricing equations. This exercise requires methods different from theirs and extends their original results.

All the convergence results rely on the assumption that agents have rational expectations i.e. they forecast using the conditional expectation operator of probability theory. Rational expectations can be interpreted as a forecast of the fundamentals. Evidence exists to support the hypothesis that short term traders called noise traders also participate in the market and that their actions have significant effects, see Allen and Taylor (1990) and Summers (1986). To capture the tendency to focus on short term horizons we define a new form of expectation, quasi-rational expectation, which is nevertheless similar to rational expectation. Quasi-rational agents use a weighted average of short term and long term predictors to forecast policy. If quasi-rational agents are present in the random environment convergence does not necessarily hold: small random shocks could drive a significant wedge between the deterministic and random worlds. The source of discontinuity in our model is a change in forecasting mechanism; different expectations determine whether small fluxes in the perturbation parameter create large price fluxes.

We briefly consider literature related to the Flood-Garber model both as background information and to illustrate that our stability analysis is a new issue. The monetary model of the exchange rate which we extend was introduced by Flood and Garber in a completely different context. They were interested in the equilibrium exchange rate when agents expect a policy change at some random time. In particular, Flood and Garber analyzed the case where the exchange rate would be pegged once the domestic and foreign money supplies reached a threshold level. They failed to find a closed form solution for the equilibrium rate. The problem was taken up and solved independently by Froot and Obstfeld (1991) as well as Smith (1991). The Flood and Garber exchange rate model has been the object of enduring interest. Ikeda and Shibata (1995) analyze the model with rational speculative bubbles. Sutherland (1995) extends the work on policy switches to the case where the change is dependent on income. So the structure introduced by Flood and Garber has been quite versatile in the literature.

This brings us to another issue of central importance, namely how we build random structures into our models. In this paper, we introduce randomness into the policy variables determined by government. Randomness arises as a result of erratic government action. Governments which follow rules have policies described by ordinary differential equations. The random
erratic actions of government result in random perturbations from this deterministic system; the policy process becomes a stochastic differential equation. In this framework, comparing deterministic and random equilibria is possible. It is frequently harder to analyze the sample path properties of random processes than the trajectories of deterministic systems. This is the reason that we model rules by deterministic equations. The deterministic structure should make the environment more predictable for agents.

Although it has been acknowledged by researchers such as Brock, Hsieh and LeBaron (1991) that we must distinguish between random and deterministic dynamics, there appears to be no formal treatment of the subject in the theoretical literature. The BDS test statistic discussed in Brock, Hsieh and LeBaron (1991), is designed to distinguish between random and chaotic motion in data sets. However, there does not appear to be any systematic theoretical investigation into comparing random and deterministic equilibria.

While this is a highly mathematical paper, the motivation is eminently practical. Suppose that foreign exchange traders believed that the structure of the equilibrium price from Flood and Garber's model was correct. But suppose they did not know how much randomness affected policy. There would be a distribution of values for $\epsilon$, the degree of randomness, in the population of traders. If Flood and Garber's model was unstable, then small differences in views over the true value of $\epsilon$ could lead to large bid-ask spreads in the market and a lot of illiquidity as different traders used the same structure of equation, but different policy processes, to price the currency. However, because the model is stable, we know that small differences in beliefs will not create large bid-ask spreads. From a practical standpoint this is important because we could computer simulate the exchange rate from Flood and Garber's model. It is essential that real world traders know the properties of the equations they use. Agents who act as though the world was deterministic when it was actually random could go bankrupt. An argument similar to this one has been made by researchers modelling financial volatility. Engle and Ng (1993) discuss the importance of accurate estimates of the conditional variance of stock prices. Estimates of the conditional variance enter the pricing formula for options; inaccurate estimates lead to mispricing and the potential for enormous losses.

The paper is organized as follows. Section 2 discusses the model. Sections 3 and 4 present the mathematical results for the infinite and finite horizon economies. Section 5 introduces quasi-rationality. Section 6 analyzes exchange rate approximation when the government maintains a target zone.
Section 7 concludes. The appendix presents longer proofs.

2 Exchange Rates In Infinite Horizon Economies

In this section, we adapt the model of Flood and Garber (1983). The Flood-Garber model is a monetary model of the exchange rate, meaning that the monetary conditions rather than trade flows determine the exchange rate. All variables evolve in continuous time. We will compare the cases when the variables are random and deterministic. In our model, random policies lead to random exchange rates and deterministic policies lead to deterministic exchange rates.

The equilibrium exchange rate is the domestic price of foreign currency and its logarithm is denoted by \( x_t \). \( E(x_t^t|\mathcal{F}_t) \) is the expected percentage change in the exchange rate conditional on the time \( t \) information set \( \mathcal{F}_t \). Logarithms of domestic income, prices and the money supply are denoted by \( y_t, p_t \) and \( m_t \); \( v_t^1 \) is a domestic money market shock. Foreign counterparts of these variables are denoted by \( y_t^f, p_t^f, m_t^f \); \( v_t^2 \) is a foreign money market shock. Except in the discussion of the economic structure of the model, we will not distinguish between any one of the variables and its logarithm e.g. we refer to \( x_t \) as the exchange rate.

It is possible that \( v_t^1, v_t^2 \) will be equal to 0 \( \forall t \). In such a case, as the following equations suggest, deterministic evolutions for domestic and foreign prices, interest rates and incomes, will result in a deterministic exchange rate. However, \( v_t^1, v_t^2 \) may be non-zero implying that the economy and exchange rate are random. The economic variables are related by the following equations:

\[
\begin{align*}
    m_t - p_t &= \alpha_0 + \alpha_1 y_t - \alpha_2 i_t + v_t^1 \\
    m_t^f - p_t^f &= \beta_0 + \beta_1 y_t^f - \beta_2 i_t^f + v_t^2 \\
    p_t &= p_t^f + x_t \\
    i_t &= i_t^f + E(x_t^t|\mathcal{F}_t)
\end{align*}
\]

We assume that \( \alpha_1, \beta_1, \alpha_2, \beta_2 > 0 \) and that \( \alpha_2 = \beta_2 \) for simplicity.
Equations (1) and (2) are equilibrium conditions in the domestic and foreign money markets. They equate the supply of real balances to demand. To see how this arises, we treat the domestic and foreign money markets as being structurally identical. In other words, the two economies have the same structure of money demand function although the parameters between the economies may differ. Thus, we need only to derive equation (1). In the derivation we let $M_t, P_t, Y_t, I_t$ be the money supply, price level, real income and nominal interest rate at time $t$. That is, $m_t = \log(M_t)$, $p_t = \log(P_t)$ etc. We assume the domestic money demand function is given by $\delta_0 Y_t^{\alpha_1} I_t^{\alpha_2} \exp(v_t^1)$ where $\delta_0 > 0$. In equilibrium, money demand must equal the real money supply:

$$\frac{M_t}{P_t} = \delta_0 Y_t^{\alpha_1} I_t^{\alpha_2} \exp(v_t^1)$$

Equation (1) follows by taking logarithms and setting $\alpha_0 = \log(\delta_0)$.

Equation (3) is a statement of purchasing power parity. The assumption of purchasing power parity is valid when markets are well integrated. Markets must be integrated enough to allow the free flow of goods and services across international boundaries.

Equation (4) is the interest parity condition. It holds when capital is perfectly mobile with risk neutral agents who know that domestic and foreign assets are perfect substitutes. Interest parity means that the exchange rate change expected by the market equals the difference in the domestic and foreign interest rates. Consider the case where domestic and foreign assets are perfect substitutes and where investors in both countries are risk neutral. By definition, investors from either country must be indifferent to holding assets in one country or another given equal rates of return. However, if each asset pays a stream of returns in the currency of the country in which it is issued, returns must fluctuate with the exchange rate. Therefore, interest parity is the equilibrium condition that balances returns once exchange rate changes are considered.

**Assumption 2.1:** We take as given a probability space $(\Omega, \mathcal{F}, P)$ and a collection of $\sigma$-algebras $\mathcal{C} = \{\mathcal{F}_t, t \in R_+ \}$. Each $\mathcal{F}_t$ is a $\sigma$-algebra and $\forall s < t$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. Whenever a variable in the model is random we assume that it is defined on $\Omega$ and is $\mathcal{F}$-measurable.

**Assumption 2.2:** The expected change in the exchange rate $E(\pi'_t | \mathcal{F}_t)$ is the conditional expectation of $\pi'_t$ given $\mathcal{F}_t$ i.e. agents have rational expectations.
The collection \( C \) is a mathematical representation of the flow of information over time. Under rational expectations, the \( \sigma \)-algebra \( \mathcal{F}_t \) will be the time \( t \) information set. Representing the information available for decision-making at time \( t \) by a \( \sigma \)-algebra is quite natural. For example, \( \mathcal{F}_t \) might contain past realizations of the interest rate i.e. sets of the form \( i_s^{-1}(A) \equiv \{ \omega : i_s(\omega) \in A \} \ s < t \). Also, whenever a condition holds almost everywhere we mean that it is true except possibly on a set of \( P \) measure 0.

To solve the model, rearrange equations 1 to 4 to yield

\[
m_t - m_t^f - x_t = \alpha_0 - \beta_0 + \alpha_1 y_t - \beta_1 y_t^f - \alpha_2 E(x_t'|\mathcal{F}_t) + v_t^1 - v_t^2
\]

Define the policy process \( (k_t) \) by

\[
k_t = m_t - m_t^f + \beta_0 - \alpha_0 + \beta_1 y_t^f - \alpha_1 y_t + v_t^2 - v_t^1
\]

Then the exchange rate satisfies

\[
x_t = k_t + \alpha_2 E(x_t'|\mathcal{F}_t). \quad \text{Although domestic and foreign income both appear in the equation for the government policy process } (k_t), \text{ their behaviour over time is not determined entirely by governments. However, governments may influence them via monetary and fiscal decisions. Thus, we persist in interpreting } (k_t) \text{ as a policy process which includes the policies of both the domestic and foreign governments.}

The equilibrium exchange rate is derived from standard techniques for solving rational expectations models, although as Froot and Obstfeld (1991 p. 242) point out, the exchange rate expression holds for any structure of expectations.

\[
x_t = \alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) E(k_s|\mathcal{F}_t) \ ds \quad (2.1)
\]

Now, we consider the policy process. We denote our policy process by \( (k^\epsilon_t) \). By assumption \( (k^\epsilon_t) \) will satisfy the following stochastic equation,

\[
k^\epsilon_t = 1 + \int_0^t b(s,k^\epsilon_s) \ ds + \epsilon \int_0^t \sigma(s,k^\epsilon_s) \ dW_s \quad (2.2)
\]

where \( \epsilon \in [0,1] \) and 1 is the initial condition. For \( \epsilon > 0 \) we denote the equilibrium exchange rate at time \( t \) by \( x^\epsilon_t \).

**Definition 2.1:** With \( \epsilon > 0 \) we refer to the continuous, adapted process \( (k^\epsilon_t) \) satisfying (2.2) as the solution of a randomly perturbed dynamical

\[\text{---}\]

\[\text{---}\]

\[\text{---}\]
system (RPDS). Equivalently, we refer to \((k_t^\epsilon)\) as the randomly perturbed dynamical system.\(^3\)

For \(\epsilon = 0\) we denote the policy process by \((k_t)\). In this case the policy process is simply the solution to the ordinary differential equation \(dk_t = b(t, k_t)dt\) with initial condition \(k_0 = 1\) and it is deterministic. The term \(\epsilon\) measures the degree of perturbation, i.e., the strength of random forces operating on an otherwise deterministic system. As we have seen, the solution for \(\epsilon = 0\) is a deterministic dynamical system. The reason we call \((k_t^\epsilon)\) a randomly perturbed system is that it is a random deviation from \((k_t)\). As \(\epsilon\) changes so too will \((k_t^\epsilon)\) and probably in a nonlinear way on a path-by-path basis. RPDS make policies and exchange rates sensitive to different levels of erratic government behaviour. Agents cannot observe \(k_t^\epsilon\) until it has occurred; however they estimate it with all available information via \(E(k_s^\epsilon|\mathcal{F}_s), s > t\).

The economic structure of the Flood-Garber model is linear. Nonlinearity is introduced by the assumption of nonlinear process \(k_t^\epsilon\). Although Ito's integral is linear on a space of functions the sample paths are nonlinear in time. Moreover, changes in \(\epsilon\) will create nonlinear fluctuations in the sample trajectories of \(k_t^\epsilon\). With this in mind let us examine the equilibrium rate in more detail.

The formula for the equilibrium exchange rate is valid for any policy process. In particular, it must hold if \((k_s)\) is deterministic. Such a situation arises when there is perfect certainty. For \(\epsilon = 0\) we define the equilibrium exchange rate at time \(t\) as \(z_t\). The deterministic function \((z_t, t \in R_+)\) must satisfy

\[
z_t = v_t(1 + \int_0^t b(s, k_s)\,ds) + \alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right)(\int_s^t b(c, k_c)\,dc)\,ds \quad (2.3)
\]

where \(v_t = \alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) ds = 1\). We make this calculation by substituting the solution for the ordinary differential equation \(k_t = 1 + \int_0^t b(s, k_s)\,ds\) into the formula for the equilibrium exchange rate.

Under \((k_t^\epsilon)\), \(\epsilon > 0\) the exchange rate becomes

\[
x_t^\epsilon = 1 + M_t^\epsilon + \int_0^t b(u, k_u^\epsilon)\,du + \alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) E(\int_s^t b(u, k_u^\epsilon)\,du|\mathcal{F}_t)\,ds \quad (2.4)
\]

where \(M_t^\epsilon = \epsilon \int_0^t \sigma(u, k_u^\epsilon)\,dW_u\).

\(^3\)RPDS have been studied extensively in the probability literature. See Freidlin and Wentzell (1984).
Equation (2.4) would also arise if $k^*_t$ was the process for dividends and rational agents were pricing a stock, see Summers (1986). This follows because for any dividend process the stock will be priced according to (2.1). The equations that arise in rational expectations equilibria have not been analyzed in the probability literature. On the other hand, although economists have widely employed rational expectations, they have not systematically studied the properties of the equations that define rational expectations equilibria.

2.1 Relations To The Probability Literature

This section discusses a variety of technical issues related to the convergence results. Throughout, we assume that $b$ and $\sigma$ satisfy $|b(t, u) - b(t, y)|^2 + |\sigma(t, u) - \sigma(t, y)|^2 \leq K|u - y|^2$ and $|b(t, u)|^2 + |\sigma(t, u)|^2 \leq K|u|^2$.

Simple inspection cannot give bounds on quantities such as $\sup_{t \in [0, T]} |x^*_t - z_t|$. Neither is simple inspection sufficient to determine whether the two quantities in (2.4) differ significantly. This is because the conditional expectations operator enters the rightmost integral as well as the presence of $M^\epsilon_t$ and $\langle k^*_t \rangle$ in the leftmost integral. Standard probability theory shows that if $f_n, n = 1, 2, \ldots$ is a sequence of functions with $f_n \to f$ and $|f_n| \leq g$ where $g$ is integrable then $E(f_n|\mathcal{G}) \to E(f|\mathcal{G})$ a.e. The set of probability one on which convergence holds depends on the sequence $f_n$. It is not generally true that if $f_\epsilon \to f$ as $\epsilon \to 0$ and $|f_\epsilon| \leq g$ then $E(f_\epsilon|\mathcal{G}) \to E(f|\mathcal{G})$ a.e. Using this fact in (2.4), even if $k^*_s \to k_s$ a.e. we could not guarantee that $E(k^*_s|\mathcal{F}_t) \to k_t$ a.e. as $\epsilon \to 0$.

Freidlin and Wentzell (1984) show that $k^*_t$ converges in probability to $k_t$ as $\epsilon \to 0$ i.e. $\lim_{\epsilon \to 0} P(|k^*_t - k_t| > \delta) = 0 \ \forall \delta > 0$. But this does not imply that $\lim_{\epsilon \to 0} k^*_t = k_t$ a.e. Thus, the Freidlin and Wentzell (1984) results on the stability of $k^*_t$ do not give us a starting point for proving stability of $x^*_t$. We need martingale theory, Kolmogorov's theorem on continuous processes, (see Révuz and Yor (1991)), and the concentration on countable sets assumption introduced below, to derive our convergence theorems.

In classical probability, see Ash (1972), if $A_n, n = 1, 2, \ldots$ is a sequence of sets such that $A_n \in \mathcal{F} \ \forall n$ then $\cap_{n=1}^\infty A_n \in \mathcal{F}$. Moreover, if $P(A_n) = 1 \ \forall n$ then $P(\cap_{n=1}^\infty A_n) = 1$. To generalize classical theorems about the convergence of conditional expectation which will establish stability of $x^*_t$, we need to extend this kind of reasoning if we have sets $A_s, s \in [t, \infty)$.

Definition of Concentration on Countable Sets: A probability measure on $\mathcal{F}$ is concentrated on countable sets (CCS) if for any interval $I \subset R$
and any class of sets \( \{A_i, i \in I\} \subset \mathcal{F} \) such that \( P(A_i) = 1 \) \( \forall i \) implies \( \cap_{i \in I} A_i \in \mathcal{F} \) and that \( P(\cap_{i \in I} A_i) = 1 \).

**Assumption 2.3:** The probability space \((\Omega, \mathcal{F}, P)\) satisfies the CCS condition.

It should be noted that previous researchers such as Flood and Garber (1983), Froot and Obstfeld (1991), and Smith (1991), in calculating the exchange rate from a Brownian Motion with drift implicitly assumed that an uncountable intersection of events, each of which has probability one, is a set of probability one. However, standard measure theory only deals with countable intersections, so their implicit assumption should have entered as an explicit assumption.

For technical reasons mainly related to the proof of Theorem 4.1, we assume that \( \mathcal{F} \) is the power set of \( \Omega \) i.e. the set of all subsets of \( \Omega \). This implies that for sets \( A_s, s \in (t, \infty) \cap \mathcal{F}(t, \infty) A_s, \cup_{s \in (t, \infty)} A_s \) are automatically measurable; other intervals may be substituted for \((t, \infty)\).

Since our analysis involves the sensitivity of equilibrium asset prices to small changes in a parameter, we review a selection of articles from the probability literature which deal with the sample path properties of SDE that depend on a parameter. First, the probability literature deals with the SDE sample paths directly, not the complicated function of the sample paths arising from our rational expectations model. Blagovescenskii and Freidlin (1961) establish that for fixed \( t \) the function \( (k_t^e(\omega), e \in [0, 1]) \) is differentiable in \( e \) for almost all \( \omega \), provided that \( b(t, x) \) and \( \sigma(t, x) \) are \( C^2 \) functions with bounded second order derivatives. Métivier (1981) extends the Blagovescenskii-Freidlin results to a case where the stochastic process driving the stochastic integral is more general than a Brownian Motion. Again, he requires twice differentiability of the integrands as well as boundedness of all second order derivatives. However, since continuity in \( e \) only holds for a fixed \( t \), both sets of results are insufficient to guarantee that \( \lim_{\epsilon \to 0} \sup_{t \in [0, T]} |k_t^\epsilon - k_t^0| = 0 \) a.e. Neither can they ensure that \( \lim_{\epsilon \to 0} \sup_{t \in [0, T]} |x_t^\epsilon - z_t| = 0 \) a.e.

Our model yields uniform convergence in \( t \) as \( \epsilon \to 0 \) because we assume concentration on countable sets. The probability literature does not address uniform convergence in \( t \) as \( \epsilon \to 0 \) of \( (k_t^\epsilon) \) or of complex functions of its sample paths. The CCS assumption lets us abandon the differentiability assumption on \( b(\cdot) \) and \( \sigma(\cdot) \) as the driving force behind continuity at 0 in the parameter \( \epsilon \). The CCS assumption allows us to show that the conditions for Kolmogorov's Theorem hold. Kolmogorov's Theorem yields the uniform convergence result.
For a statement of Kolmogorov’s Theorem, see the appendix or Révuz and Yor (1991).

Abandoning the differentiability assumption is desirable since differentiability of the SDE integrands does not have an obvious economic interpretation. Although our results are unrelated to the differentiability of SDE solutions, the appendix shows that an application of Kolmogorov’s Theorem yields continuity of \((k_t^\varepsilon, \varepsilon \in (0, 1])\) and \((x_t^\varepsilon, \varepsilon \in (0, 1])\), for fixed \(t\).

## 3 Mathematical Results for an Infinite Horizon Economy

Stochastic differential equations (SDE) have been widely used to model stock price fluctuations, see Karatzas and Shreve (1991). However SDE arising as perturbed versions of deterministic systems have not entered the literature. By examining the continuity properties of the perturbed system (2.4) we analyze whether small departures from the case of perfect certainty create large jumps in asset prices. When \(\varepsilon\) changes, both integrands change because they depend on previous realizations of the process. Thus, \(k_t^\varepsilon\) and \(x_t^\varepsilon\) exhibit nonlinear dependence on \(\varepsilon\). If (2.4) did exhibit a discontinuity it would imply that prices in an uncertain world could jump markedly from the perfect certainty case, even when agents accounted for all available information.

For the policy process of Flood and Garber (1983), the convergence follows trivially because \(k_t = bt + \varepsilon W_t\) \(\quad x_t = (\alpha + t)b + \varepsilon W_t\). Generalizing this result to more complex integrands is a non-trivial technical exercise. Even though the result for a simple random shock is preserved for a more complex shock, it remains possible that there exist random structures which would create a discontinuity. We begin with a few facts from Real Analysis and Probability to be used in the proof.

**Dominated Convergence Theorem:** Let \(f_n, f, g\) be measurable functions for the measure space \((\Omega, \mathcal{A}, \mu)\), \(g \in L^1\) \(\mid f_n \mid \leq g\) a.e. with \(f_n \rightarrow f\) a.e. Then \(f_n, f \in L^1\) and \(\int_{\Omega} f_n d\mu \rightarrow \int_{\Omega} f d\mu\).

For \((k_t)\) the process \((E(k_s|\mathcal{F}_t), s \geq t)\) represents the predictions at time \(t\) of how the fundamentals will evolve over time. Conditional expectation has many desirable properties which we exploit, see Ash (1972).

**Lemma 3.1:**
1) Let \(f \leq g\) a.e. then \(E(f|\mathcal{G}) \leq E(g|\mathcal{G})\) a.e.
2) $|E(f|\mathcal{G})| \leq E(|f| |\mathcal{G})$ a.e.

3) Dominated Convergence Theorem for Conditional Expectation: Let $|f_n| \leq Z \in L^1; \ f_n \rightarrow f$ a.e. Then $E(f_n|\mathcal{G}) \rightarrow E(f|\mathcal{G})$ a.e.

4) $f \leq g \Rightarrow \int f \ d\mu \leq \int g \ d\mu$

5) $\int f \ d\mu \leq \int |f| \ d\mu$

6) if $f, g \in L^1; \ a, b \in \mathbb{R}$ then $E(af + bg|\mathcal{G}) = aE(f|\mathcal{G}) + bE(g|\mathcal{G})$ a.e.

7) if $f \in \mathcal{G}$ then $E(f|\mathcal{G}) = f$ a.e.

It is necessary to extend the Dominated Convergence Theorem for Conditional Expectation to the following:

**Extended Dominated Convergence Theorem:** Let $\{f_\epsilon, \epsilon > 0\}$ be a continuous process with $\lim_{\epsilon \to 0} f_\epsilon = f$, $|f_\epsilon| \leq g, \ g \in L^1$. Let $\mathcal{G}$ contain all sets of measure 0 and let $P$ satisfy the CCS condition. Then $\lim_{\epsilon \to 0} E(f_\epsilon|\mathcal{G}) = E(f|\mathcal{G})$ almost everywhere.

We cannot use the dominated convergence theorem for sequences to prove the extended dominated convergence theorem, because the set on which convergence holds depends on the sequence that we are analyzing. An extension of the traditional methods of Real Analysis is required. Lemmas 3.2 and 3.3 are proved in the appendix. It is the CCS assumption that permits an extension of the conventional theorems of real analysis which are necessary for all our convergence results.

**Lemma 3.2:** Let $P$ be CCS, then

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} |k_\epsilon^t - k_t| = 0 \ a.e., \ \forall T < \infty$$

**Lemma 3.3:** Let $P$ be CCS, then $\lim_{\epsilon \to 0} \sup_{t \in [0,T]} |M_\epsilon^t| = 0 \ a.e., \ \forall T < \infty$ where $M_\epsilon^t = \epsilon \int_0^t \sigma(s, k_\epsilon^s) \ dW_s$.

The next theorem allows us to approximate a random exchange rate by a deterministic one at a given time $t$. The principle limitation of this result is that the set on which convergence holds will generally vary with $t$. It would be desirable to show that even when agents consider an infinite horizon, we may take any bounded interval and have uniform convergence over the interval almost everywhere as $\epsilon \to 0$. Unfortunately, we cannot guarantee this result because we cannot establish uniform bounds on the convergence of the integrals over $(t, \infty)$. However, the result can suggest that agents who
intend only to conduct finitely many exchange rate transactions at times \( \{t_1, \ldots, t_m\} \) may do so at prices close to \( \{z_1, \ldots, z_m\} \).

**Theorem 3.1:** Let \( b \in C^\infty(R_+ \times R) \) the space of bounded continuous functions on \( R_+ \times R \), and let \( P \) be concentrated on countable sets. Denote by \( v_t = \alpha^{-1} \int_t^\infty \exp(\frac{t-s}{\alpha}) \, ds. \) Let \( (k_t) \), \( (k_t^\epsilon) \) be the solutions of the deterministic and perturbed systems for fundamentals respectively; let \( (x_t^\epsilon) \) be the equilibrium exchange rate under \( (k_t^\epsilon) \). Define

\[
z_t = v_t (1 + \int_0^t b(s, k_s) \, ds) + \alpha^{-1} \int_t^\infty \exp(\frac{t-s}{\alpha})(\int_t^s b(c, k_c) \, dc) \, ds \quad (3.1)
\]

Then

\[
\lim_{\epsilon \to 0} x_t^\epsilon \to z_t \quad a.e.
\]

**Proof:**

\[
k_t^\epsilon = 1 + \int_0^t b(u, k_u^\epsilon) \, ds + \epsilon \int_0^t \sigma(u, k_u^\epsilon) \, dW_u
\]

Let

\[
M_t^\epsilon = \epsilon \int_0^t \sigma(u, k_u^\epsilon) \, dW_u \quad (3.2)
\]

Hence, \((M_s, \mathcal{F}_s)\) is a martingale, so routine calculation using lemma 3.1 properties 6 and 7 yields an expression for \( E(k_s^\epsilon|\mathcal{F}_t) \)

\[
E(k_s^\epsilon|\mathcal{F}_t) = 1 + M_t^\epsilon + \int_0^t b(u, k_u^\epsilon) \, du + E(\int_t^s b(u, k_u^\epsilon) \, du|\mathcal{F}_t) \quad a.e. \quad (3.3)
\]

Recall that the equilibrium exchange rate satisfies

\[
x_t^\epsilon = \alpha^{-1} \int_t^\infty \exp(\frac{t-s}{\alpha})E(k_s^\epsilon|\mathcal{F}_t) \, ds \quad (3.4)
\]

Hence we have (3.5)

\[
x_t^\epsilon = v_t (1 + M_t^\epsilon + \int_0^t b(u, k_u^\epsilon) \, du) + \alpha^{-1} \int_t^\infty \exp(\frac{t-s}{\alpha})E(\int_t^s b(u, k_u^\epsilon) \, du|\mathcal{F}_t) \, ds
\]

By continuity of \( b \) on \( R_+ \times R \), \( b(u, k_u^\epsilon) \to b(u, k_u) \) \( \forall u \) a.e. as \( \epsilon \to 0 \) because \( \lim_{\epsilon \to 0} k_u^\epsilon = k_u \) a.e. by Lemma 3.2. By the dominated convergence theorem,

\[
\int_t^s b(u, k_u^\epsilon) \, du \to \int_t^s b(u, k_u) \, du \quad a.e. \quad (3.6)
\]
By Lemma 3.1 property 7 and the extended dominated convergence theorem,

$$E(\int_t^s b(u, k_u^s)\,du|\mathcal{F}_t) \rightarrow E(\int_t^s b(u, k_u)\,du) = \int_t^s b(u, k_u)\,du \quad \text{a.e.} \quad (3.7)$$

because if $g(z) = c \quad \forall z$ then $g \in \mathcal{G} \quad \forall \mathcal{G}$. Note that the set on which convergence holds in (3.6) and (3.7) depends on $s$. Using variants of arguments in appendix 2, we can establish continuity in $\epsilon$ of the process $(\int_t^s b(u, k_u^s)\,du, \epsilon \in (0, 1))$, so we can apply the extended dominated convergence theorem to yield (3.7).

Now, $|b(x, y)| \leq L$, implying that

$$|\int_t^s b(u, k_u^s)\,du| \leq \int_t^s |b(u, k_u^s)|\,du \leq \int_t^s L\,du = L(s - t) \quad (3.8)$$

by lemma 3.1 properties 4 and 5. Using properties 1 and 6 we have

$$E(\int_t^s |b(u, k_u^s)|\,du|\mathcal{F}_t) \leq E(L(s - t)|\mathcal{F}_t) = L(s - t) \quad (3.9)$$

Consequently,

$$\exp\left(\frac{t - s}{\alpha}\right)E(\int_t^s b(u, k_u^s)\,du|\mathcal{F}_t) \leq \exp\left(\frac{t - s}{\alpha}\right)E(|\int_t^s b(u, k_u^s)\,du|)|\mathcal{F}_t)$$

$$\leq \exp\left(\frac{t - s}{\alpha}\right)E(\int_t^s |b(u, k_u^s)|\,du|\mathcal{F}_t) \leq L \exp\left(\frac{t - s}{\alpha}\right)(s - t) \equiv h(s) \quad (3.10)$$

The function $h$ is integrable on $(t, \infty)$. Let $C_s$ be the set on which

$$E(\int_t^s b(u, k_u^s)\,du|\mathcal{F}_t) \rightarrow \int_t^s b(u, k_u)\,du \quad (3.11)$$

so that $P(\Omega - C_s) = 0$. Take $\mathcal{V} \equiv \{\Omega - C_s \quad s \in (t, \infty)\}$, then by the assumption of concentration on countable subsets

$$P\{\bigcup_{s \in (t, \infty)} (\Omega - C_s)\} = 0$$

Hence on $\cap_{s \in (t, \infty)} C_s$ we have

$$E(\int_t^s b(u, k_u^s)\,du|\mathcal{F}_t) \rightarrow \int_t^s b(u, k_u)\,du \quad \forall s \in (t, \infty) \quad (3.12)$$

43
The concentration on countable subsets assumption is necessary to assure that almost everywhere, we can guarantee convergence \( \forall s \in (t, \infty) \). Taking our measure space as \( (\{t, \infty\}, B(t, \infty), ds) \) where \( ds \) is Lebesgue measure, we see that the hypothesis of the dominated convergence theorem is satisfied almost everywhere. Consequently, we may apply the dominated convergence theorem to obtain that almost everywhere with respect to \( P \)

\[
\alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) E\left( \int_t^s b(u, k_u^\epsilon) \, du \big| \mathcal{F}_t \right) \, ds \to \\
\alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) \int_t^s b(u, k_u) \, du \, ds \quad \text{as} \; \epsilon \to 0 \quad (3.13)
\]

From Lemma 3.3 \( \lim_{\epsilon \to 0} M_t^\epsilon = 0 \) a.e. q.e.d.

4 Convergence Results For a Finite Horizon Economy

The results for finite horizon economies are much stronger than for an infinite horizon. The form of the equilibrium exchange rates is the same; the only difference is that we now integrate over \( (t, T] \) rather than \( (t, \infty) \). So the equilibrium exchange rate is \( x_t^\epsilon = \alpha^{-1} \int_t^T \exp\left(\frac{t-s}{\alpha}\right) E(k_u^\epsilon | \mathcal{F}_t) \, ds ; \; T < \infty \) is terminal time. Likewise, only the interval of integration changes for \( v_t \).

Define \( v_t = \alpha^{-1} \int_t^T \exp\left(\frac{t-s}{\alpha}\right) \, ds \).

**Theorem 4.1:** Let \( P \) be concentrated on countable sets. Then,

\[
\lim_{\epsilon \to 0} P\left( \sup_{t \in [0,T]} |x_t^\epsilon - z_t| \geq \delta \right) = 0 \quad \forall \delta > 0
\]

Consequently,

\[
\lim_{\epsilon \to 0} P\left( \sup_{t \in [0,T]} |x_t^\epsilon - z_t| < \delta \right) = 1 \quad \forall \delta > 0
\]

The significance of this result is that for a given bound on the difference \( \delta \) between random and deterministic exchange rates, with high probability, \( x_t^\epsilon \) and \( z_t \) will differ by no more than \( \delta \) over the entire interval \([0, T]\), provided that \( \epsilon \) is small enough.

Proof:

\[
x_t^\epsilon - z_t = v_t M_t^\epsilon + v_t \left( \int_0^t b(u, k_u^\epsilon) - b(u, k_u) \, du \right)
\]
\[
\alpha^{-1} \int_t^T \exp\left(\frac{t-s}{\alpha}\right) E\left( \int_s^t (b(u, k_u^\epsilon) - b(u, k_u)) \, du \mid \mathcal{F}_t \right) \, ds \quad (4.1)
\]

Obviously, \( \{\nu_t\} \) is bounded and

\[
\left| \int_0^t b(s, k_s^\epsilon) - b(s, k_s) \, ds \right| \leq \int_0^t |b(s, k_s^\epsilon) - b(s, k_s)| \, ds \leq KT \sup_{s \in [0, T]} |k_s^\epsilon - k_s| \quad (4.2)
\]

Furthermore,

\[
|E\left( \int_t^s b(u, k_u^\epsilon) - b(u, k_u) \, du \mid \mathcal{F}_t \right)| \leq E\left( \int_t^s |b(u, k_u^\epsilon) - b(u, k_u)| \, du \mid \mathcal{F}_t \right) \leq KTE\left( \sup_{u \in [0, T]} |k_u^\epsilon - k_u| \right) \quad (4.3)
\]

Let \( Y^\epsilon = \sup_{s \in [0, T]} |k_s^\epsilon - k_s| \). Then almost everywhere for some constant \( C \) which is independent of \( \epsilon \),

\[
\sup_{s \in [0, T]} |x_s^\epsilon - z_s| \leq C\left( \sup_{s \in [0, T]} |M_s^\epsilon| + \sup_{s \in [0, T]} |k_s^\epsilon - k_s| + \sup_{s \in [0, T]} E(Y^\epsilon \mid \mathcal{F}_s) \right) \quad (4.4)
\]

We need to show that the right hand side converges to 0 in probability. Since it is almost everywhere a sum of non-negative random variables, it suffices to show that each random variable converges to 0 in probability. Freidlin has already established that \( \lim_{\epsilon \to 0} P(\sup_{s \in [0, T]} |k_s^\epsilon - k_s| \geq \delta) = 0 \quad \forall \delta > 0 \). So it remains to show that

\[
\lim_{\epsilon \to 0} P\left( \sup_{s \in [0, T]} |M_s^\epsilon| \geq \delta \right) = \lim_{\epsilon \to 0} P\left( \sup_{s \in [0, T]} E(Y^\epsilon \mid \mathcal{F}_s) \geq \delta \right) = 0 \quad \forall \delta > 0
\]

We break this into two parts, proving first that \( \sup_{t \in [0, T]} |M_t^\epsilon| \to^P 0 \).

If \( \{A_s, \epsilon \in (0, 1)\} \) is a class of sets define \( \lim_{\epsilon \to 0} A_s \) by \( \cap_{t \in (0, 1)} A_s \). Obviously, \( \cup_{s < \epsilon} A_s \downarrow \lim_{\epsilon \to 0} A_s \) as \( \epsilon \downarrow 0 \), so that \( \lim_{\epsilon \to 0} P(\cup_{s < \epsilon} A_s) \downarrow P(\lim \sup_{\epsilon \to 0} A_s) \). Define \( A_{s, \delta} = \{\sup_{t \in [0, T]} |M_t^\epsilon| > \delta\} \) and \( B(\delta) = \lim_{\epsilon \to 0} A_{s, \delta} \). Then the set on which \( \sup_{t \in [0, T]} |M_t^\epsilon| \) does not converge to 0 contains \( B(\delta) \) implying \( P(B(\delta)) = 0 \). Hence, \( P(\cup_{s < \epsilon} A_{s, \delta}) \) is 0 as \( \epsilon \downarrow 0 \), implying convergence in probability.

In the appendix, we establish that \( E(\sup_{t \in [0, T]} |k_t^\epsilon - k_t|^2) \leq \epsilon^2 \alpha(T) \) where \( \alpha(T) \) is a constant depending on \( T \). This establishes integrability of \( \sup_{t \in [0, T]} |k_t^\epsilon - k_t| \). We need a few facts from martingale theory to complete the proof:

1) if \( Y \in L^1 \) and \( (\mathcal{F}_t) \) is a filtration, then \( (E(Y \mid \mathcal{F}_t), \mathcal{F}_t) \) is a martingale.
2) If \((\mathcal{F}_t)\) is right continuous and \((M_t, \mathcal{F}_t)\) is a martingale then \((M_t)\) possesses a right continuous modification \((X_t)\): \(P(X_t = M_t) = 1\). We always assume that \(\mathcal{F}_0\) contains all \(P\) null sets, so that \((X_t, \mathcal{F}_t)\) is a right continuous martingale.

3) Doob’s Maximal Quadratic Inequality: Let \((M_t, \mathcal{F}_t, t \in [0, T])\) be a right continuous \(L^2\) martingale (ie: \(M_t \in L^2 \ \forall t\)). Then

\[
E\left( \sup_{s \in [0,t]} |M_s|^2 \right) \leq 4E(M_t^2) \ \forall t \leq T \hspace{1cm} (4.5)
\]

4) Jensen’s Inequality: Let \(g : R \rightarrow R\) be convex, \(X \in L^1\) and \(g(X) \in L^1\), then \(g(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(g(X)|\mathcal{G})\) a.e.

Returning to our central argument, we see that \((E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s||\mathcal{F}_t), \mathcal{F}_t)\) is a martingale. Consequently, it must have a right continuous modification. But by the CCS assumption \((E(Y^\epsilon|\mathcal{F}_t))\) must be continuous. Hence \(\sup_{s \in [0,T]} E(Y^\epsilon|\mathcal{F}_s)\) must be measurable. Jensen’s inequality and the fact that \(E(X) = E(E(X|\mathcal{F}))\) imply that the process is an \(L^2\) martingale. Let \(h(x) = x^2\) if \(x \geq 0\) and \(h(x) = 0\) if \(x < 0\); \(h\) is convex.

\[
(E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s||\mathcal{F}_t)^2 \leq E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s|^2|\mathcal{F}_t) \hspace{1cm} (4.6)
\]

Hence (4.7):

\[
E((E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s||\mathcal{F}_t)^2) \leq E(E((\sup_{s \in [0,T]} |k^\epsilon_s - k_s||\mathcal{F}_t)^2)) = E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s|^2) < \infty
\]

So we may apply Doob’s inequality to yield (4.8):

\[
E(\sup_{s \in [0,T]} (E(Y^\epsilon|\mathcal{F}_s))^2) \leq 4E(E(Y^\epsilon|\mathcal{F}_T)^2) \leq 4E(\sup_{s \in [0,T]} |k^\epsilon_s - k_s|^2) \rightarrow 0 \ \text{as} \ \epsilon \rightarrow 0
\]

Hence, \(\sup_{s \in [0,T]} E(Y^\epsilon|\mathcal{F}_s) \rightarrow L^2 0\) and convergence in \(L^2\) implies convergence in probability. q.e.d.

Using results in this proof and the appendix, we may also prove uniform convergence almost everywhere on \([0, T]\) as \(\epsilon\) converges to 0. Theorem 4.2 is proven in the appendix. As we note in the proof of Theorem 4.1 when discussing \(\sup_{s \in [0,T]} |M^\epsilon_s|\), the existence of a continuous limit a.e. implies convergence in probability. We prove the weaker result in the text because the proof of uniform convergence almost everywhere draws on some facts presented in Theorem 4.1. Its proof also requires the bounds in the \(L^2\) norm.
which we use in Theorem 4.1. The arguments necessary to derive these bounds are complex and the proof of Theorem 4.2 draws on some other involved facts, so its proof is deferred to the appendix.

**Theorem 4.2**: Let \( P \) be CCS. Then \( \lim_{t \to 0} \sup_{s \in [0,T]} |x'_s - z_s| = 0 \) almost everywhere.

To say that \( \lim_{t \to 0} P(\sup_{t \in [0,T]} |x'_t - z_t| \geq \delta) = 0 \ \forall \delta > 0 \) means that we can find \( \gamma = \gamma(\delta, \eta, T) \) such that \( P(\sup_{t \in [0,T]} |x'_t - z_t| \geq \delta) < \eta \ \forall \epsilon < \gamma \). This implies that \( P(\sup_{t \in [0,T]} |x'_t - z_t| < \delta) > 1 - \eta \ \forall \epsilon < \gamma \). So, the time interval in part determines how small \( \epsilon \) must be to obtain a given accuracy with high probability.

**Theorem 4.3**: There exists a bound \( \gamma(\delta, \eta, T) \) which is decreasing in \( T \) for fixed \((\delta, \eta)\). It satisfies \( \gamma(\delta, \eta, T) = \inf \{s \in [0,1] : s^2 \rho(T)C^2(T)/\delta^2 \geq \eta\} \) if the set is nonempty, and 1 otherwise, for some increasing functions \( \rho, C \).

**Proof**: Theorem 4.1 establishes

\[
\sup_{t \in [0,T]} |x'_t - z_t| \leq C(T) (\sup_{t \in [0,T]} |M'_t| + \sup_{t \in [0,T]} |k'_t - k_t| + \sup_{t \in [0,T]} E(Y'|F_t)) \tag{4.9}
\]

The constant \( C(T) \) depends on \( \alpha^{-1} \int_0^T \exp(\frac{t-s}{\alpha}) ds \equiv h(T) \) which must increase in \( T \). This follows because if \( T_1 < T_2 \leq \exp(T_2 - \frac{s}{\alpha}) \leq \exp(T_2 - \frac{s}{\alpha}) \ \forall s \in [0,T_1] \). Hence, by non-negativity of the functions, \( \alpha^{-1} \int_0^{T_1} \exp(\frac{T_2 - s}{\alpha}) ds \leq \alpha^{-1} \int_0^{T_2} \exp(\frac{T_2 - s}{\alpha}) ds \).

\[
P(\sup_{t \in [0,T]} |x'_t - z_t| \geq \delta) \leq P(\sup_{t \in [0,T]} |M'_t| \geq \frac{\delta}{3C(T)}) + \]

\[
P(\sup_{t \in [0,T]} |k'_t - k_t| \geq \frac{\delta}{3C(T)}) + P(\sup_{t \in [0,T]} E(Y'|F_t) \geq \frac{\delta}{3C(T)}) \tag{4.10}
\]

In the appendix, we prove that each of the random variables appearing in the sum which bounds \( \sup_{t \in [0,T]} |x'_t - z_t| \) is bounded in its second moment by a term such as \( \epsilon^2 \beta(T) \) where \( \beta(T) \) increases with \( T \). Recall that \( P(|X| \geq c) \leq c^{-p} E|X|^p \ \forall p > 0 \). Applying this to each term in the sum involving the probabilities yields \( P(\sup_{t \in [0,T]} |x'_t - z_t| \geq \delta) \leq \epsilon^2 \beta(T)^2 \) where \( \beta(T) \) increases with \( T \). Let \( \gamma(\delta, \eta, T) = \inf \{s \in [0,1] : s^2 \rho(T)C^2(T)/\delta^2 \geq \eta\} \) if the set is non-empty and 1 otherwise. So if \( \epsilon < \gamma(\delta, \eta, T) \) we have \( \epsilon^2 \zeta(\delta, T) < \eta \) where \( \zeta(\delta, T) = \rho(T)C^2(T)/\delta^2 \). If \( T < H \), \( \{s \in [0,1] : s^2 \zeta(\delta, T) \geq \eta\} \subset \{s \in [0,1] : s^2 \zeta(\delta, H) \geq \eta\} \Rightarrow \gamma(\delta, \eta, T) \geq \gamma(\delta, \eta, H) \). Here we have assumed
the set related to $T$ is non-empty. If it is empty, then $\gamma(\delta, \eta, H) \leq 1$ by construction. q.e.d.

5 Quasi-Rational Expectations and Instability of Small Perturbations

LeBaron (1996) establishes that simple technical trading rules can improve returns in the foreign exchange market. However, once periods of Federal Reserve intervention are deleted from the data, all excess returns disappear. Summers (1986) suggests that certain investors called noise traders downplay fundamentals and extrapolate trends. Allen and Taylor (1990) discuss a survey of foreign exchange traders; 60 per cent thought that charts were as important in the short term as fundamentals and only 30 per cent relied on purely fundamental analysis. Their survey indicates that noise traders are a viable part of the market and that their effects on prices must be modelled. When there is some measure of noise trading, agents cannot be using rational expectations in a strict sense because rational expectations corresponds to forecasting the fundamentals. Allen and Taylor (1990) suggests traders use a weighted average of fundamental and technical predictors. Consequently, we need to model this kind of mixed expectation and analyze its effects in our exchange rate model.

By definition, agents who are noise traders cannot use the conditional expectations operator exclusively to predict future policies. The task is to model the expectations of agents who deviate from rational expectations because they are noise traders. The expectations introduced here are called "quasi-rational" expectations. They are meant to capture behaviour which is not fully rational in the sense that agents try to simulate the random path of government policy. At the same time quasi-rational expectations are a close approximation to rational expectations in the sense that agents use an intelligent procedure to estimate future policies. This leads to a qualification in the use of the term "noise trader".

Noise traders are frequently associated with the word "irrational". An obvious implication is that noise traders create totally unfounded deviations from fundamental prices. It is possible that noise traders are not so irrational. They may believe that short term trends as well as long term fundamentals affect prices. Charts and technical trading rules are two ways of trying to exploit short term trends. Some methods of discerning trends may be quite
sophisticated, others may be quite crude, but it is possible that noise traders may be trying to make intelligent short term forecasts based on important information. The sophisticated short term forecasting of noise traders is what we want to capture in quasi-rational expectations.

**Definition 5.1**: Agents have quasi-rational expectations if there exists a weight \( w \in [0, 1] \) such that at each time \( t \), the expectation of policy at \( s > t \) is \( wE(k^s_t|\mathcal{F}_t) + (1-w)f^s_{s} \). The process \( (f^s_{s}, t \leq s) \) is a short term forecast of policy and is defined by \( f^s_{s} = 0, s > t + m \) and

\[
f^s_{s} = k^s_t + \int_t^s b(u, k_u) \, du + \epsilon \int_t^s \sigma(u, k_u) \, dW_u \quad s \in [t, t + m] \tag{5.1}
\]

The process \( f^s_{s} \) is defined for \( \epsilon \in [0, 1] \) where we denote \( f^0_{s} = f_{s} \). We set \( f^s_{s} = 0 \), \( \forall s > t + m \) because \( f^s_{s} \) is meant to be a short term forecast. The number \( m > 0 \) is the time horizon of the forecast in the sense that agents are trying to predict policy over the interval \([t, t + m]\). The weight \( w \) reflects the strength of rational expectations in the quasi-rational world; it also reflects the strength attached to the long term forecast.

The process \( f^s_{s} \) is the initial approximation to the policy process. Agents observe \( k^s_t \) at time \( t \) and \( k^s_t \) is the initial condition of the SDE \( (k^s_{s}, s \geq t) \). Recall that \( k^s_{s} = k^s_t + \int_t^s b(u, k_u) \, du + \epsilon \int_t^s \sigma(u, k_u) \, dW_u, s > t \). In (5.1), the agent is trying to simulate the path of \( k^s_{s}, s \geq t \) by replacing \( k^s_u, u \in [t, s] \) in the integrands by \( k_u, u \in [t, s] \). The motivation for doing this, as shown by lemma 5.1, is that for small \( \epsilon \), \( k_u \) and \( k^s_u \) will be close. However, the approximation might not end there. Consider the alternative short term estimator \( h^s_{s} \) defined by

\[
h^s_{s} = k^s_t + \int_t^s b(u, f^s_u) \, du + \epsilon \int_t^s \sigma(u, f^s_u) \, dW_u \quad s \in [t, t + m] \tag{5.2}
\]

and \( h^0_{s} = f^s_{s} = 0, s > t + m \).

Equation (5.2) is closer in spirit to the structure of the SDE governing policy because it allows both integrands to be stochastic. Equation (5.1) only admits deterministic integrands because \( k_u \) is deterministic. Agents trying to capture the structure of the SDE for \( k^s_{s} \) might prefer stochastic integrands for the simulation. Lemma 5.1 demonstrates that \( f^s_{s} \) is an accurate approximation of \( k^s_{s} \) and hence would be a suitable random function to use in the simulation. Both \( f^s_{s} \) and \( h^s_{s} \) are valid estimators of policy \( k^s_{s} \) as the next result shows; the proof is in the appendix. Thus, we could replace \( f^s_{s} \) in the definition of quasi-rational expectations by \( h^s_{s} \).
Lemma 5.1:

\[
\lim_{\epsilon \to 0} E \sup_{s \in [t,t+m]} |f^\epsilon_s - k^\epsilon_s|^2 = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} E \sup_{s \in [t,t+m]} |h^\epsilon_s - k^\epsilon_s|^2 = 0
\]

Quasi-rational agents believe that fundamentals are important but do not believe that any one predictor can capture the path of policy exclusively. Consequently, they balance the strictly rational predictor with a short term forecast of policy. There are valid motivations for believing that agents use both short and long term forecasts. Policy might exhibit short run dynamics that would be difficult to anticipate based solely on fundamentals but which would be easier to detect using a short term predictor. This could happen because a fundamental forecast usually emphasizes long run structural economic relations, it does not extrapolate short term disturbances. This is similar to a noise trader believing that a speculative phase will be short-lived, with complex dynamics that for a time upstage the fundamentals. The noise trader does not lose sight of the fundamentals completely, but modifies his view of the world to account for short term phenomena. This is why quasi-rationality can model the expectations of noise traders.

Quasi-rational expectations are not a radical deviation from rational expectations. Although quasi-rational agents do not use rational expectations exclusively, their simulations of the policy over \([t, t + m]\) are intelligent predictors of government policy. Any agent who can predict policy well will know the value of the currency. Quasi-rational agents try to simulate random policies via \(f^\epsilon_s\) or \(h^\epsilon_s\) because for small \(\epsilon\) the similarity of random and deterministic policies implies that \(f^\epsilon_s\) and \(h^\epsilon_s\) are close to \(k^\epsilon_s\). Quasi-rational agents extract an important subset of information and use it intelligently to forecast future policies.

At the same time, it is clear that quasi-rational forecasts of policy for \(s \in [t, t + m]\) are different from rational forecasts. Equation (3.3) is the rational expectations forecast of policy. Equation (3.3) differs from both (5.1) and (5.2) in that it uses all information to predict the future. Equations (5.1) and (5.2) try to predict the future by using the structure of the SDE and knowledge of \(k_s\) to simulate the path of policy. In elementary probability theory the conditional expectation of \(k^\epsilon_t\) given \(k^\epsilon_t = q\) is calculated by evaluating \(\int_{-\infty}^{\infty} sp^\epsilon(s|q) \ ds\), where \(p^\epsilon(\cdot|q)\) is the conditional density of \(k^\epsilon_t\) given \(k^\epsilon_t = q\). Even if \(p^\epsilon(\cdot|q)\) was known, calculating the conditional expectation would be a far cry from performing either simulation (5.1) or (5.2). Thus,
an agent using all available information via a conditional density would be doing more than simulating the policy path.

Let the quasi-rational expectation forecast use estimator $f_s^\epsilon$. Setting $\alpha = 1$ for simplicity, the equilibrium exchange rate $y_t^\epsilon$ satisfies

$$y_t^\epsilon = \int_t^\infty \exp(t - s)(wE(k_s^\epsilon | F_t) + (1 - w)f_s^\epsilon) \, ds$$  \hspace{1cm} \text{(5.3)}$$

The integral in (5.3) is the discounted forecast of policy over $[t, \infty)$ which is a weighted average of fundamental expectations and the simulation of policy $f_s^\epsilon$. Obviously, the structure of the expression is unchanged if we substitute $h_s^\epsilon$ for $f_s^\epsilon$.

**Theorem 5.1:** Let expectations be quasi-rational with estimator $f_s^\epsilon$ and let $b$ be bounded, then almost everywhere as $\epsilon \to 0$ $y_t^\epsilon$ converges to

$$wz_t + (1 - w) \int_t^{t+m} \exp(t - s) \left[ k_t + \int_t^s b(u, k_u) \, du \right] \, ds \neq z_t$$

The proof is deferred to the appendix. If (5.2) was used in the exchange rate equation (5.3) instead of (5.1), $y_t^\epsilon$ would still not converge to $z_t$. Let $h_s^\epsilon$ replace $f_s^\epsilon$ in (5.3) then $y_t^\epsilon \to wz_t + (1 - w) f_t^{t+m} \exp(t - s)(k_t + \int_t^s b(u, f_u) \, du) \, ds \neq z_t$. Since $f_u = k_u$ the exchange rate limit is independent of the estimator used. Thus, quasi-rational expectations possess a stability property in the sense that the exchange rate when $h_s^\epsilon$ is the estimator will be close to the exchange rate when $f_s^\epsilon$ is the estimator. This makes the divergence with rational expectations more interesting because in using quasi-rational expectations we are not replacing rational agents with agents whose expectations are sensitive to a choice in the underlying estimator $h_s^\epsilon$ or $f_s^\epsilon$. Our analysis suggests that different kinds of intelligent predictions could yield sharp price differences with small departures from the world of perfect certainty.

It is clear that the convergence results of sections 3 and 4 are very sensitive to the assumption that foreign exchange traders behave strictly like fundamentalists. If traders use a weighted average of predictors then $y_t^\epsilon$ will be close to a number potentially very different from $z_t$ for small $\epsilon$. One of the principle determinants of how large the deviation from $z_t$ will be is the weight assigned to rational expectations. If $w$ is close to 1 the deviations will be small. However, Allen and Taylor (1990) suggests that in the short term traders think technical signals are as important as fundamentals. This
empirical evidence suggests sharp deviations from rational values may occur since \( w \) should not be close to 1.

This instability result suggests that economists must exercise considerably more care when choosing between random or deterministic methods in their models. Even a small amount of randomness may have important effects on equilibria. Quasi-rational expectations do not appear to have the potential to create large jumps at first glance. The close relation between quasi-rationality and complete rationality makes the result even more surprising. Only small departures from complete rationality are necessary for small random shocks to matter.

Quasi-rational expectations would probably be much easier to calculate than rational expectations. Even sophisticated traders might find it difficult or extremely costly to use rational expectations because of the amount of information contained in \( \mathcal{F}_t \) or the difficulty in actually finding the conditional density. Economists have been very interested in rational expectations equilibria and have devoted much energy toward their study. Quasi-rational expectations might provide a more realistic model for expectations in the sense that agents simulate the important variable based on a key subset of information. The key subset in this model consisted of the deterministic process \( k_s \), the structure of the diffusion, and the observed shock \( k^*_t \), but the crucial data could vary from model to model.

Most of the motivation for quasi-rationality has come from the noise trader problem. If quasi-rationality was more closely tied to the problem of imperfect information, we might adjust the model so that agents also needed to estimate either \( b \) or \( \sigma \), i.e. they would be less certain of making correct calculations. Clearly, variants on the problem of quasi-rational expectations are possible.

The discontinuity in \( \epsilon \) under quasi-rational expectations appeared because agents used both short and long term forecasts of policy. Assume that agents only used short term forecasts because they believed that short term movements overwhelmed long run structural relations. The description of noise traders in Summers (1986) might account for exclusively focusing on short term predictors. Let

\[
j_t \equiv \lim_{\epsilon \to 0} y_t^\epsilon = (1 - w) \int_t^{t+m} \exp(t - s)(k_t + \int_u^t b(u, k_u) \, du \, ds) + wz_t.
\]

For any \( w, |j_t - z_t| = (1 - w) \int_t^{t+m} \exp(t - s)k_s \, ds \), which is clearly decreasing in \( w \). Hence, quasi-rational agents who use only short term predictors \( (w = 0) \) achieve the maximum deviation from \( z_t \). This demonstrates that it is rational expectations more than the Lipschitz continuity of \( b \) and \( \sigma \) which generates stability of prices. The magnitude of
the jump depends on \( k_s \) and how fast \( \exp(-s)k_s \) decays; it could be a considerable difference. Clearly, it is the long term view provided by rational expectations which generates stability in the model.

6 Exchange Rate Approximation With Target Zones

Many early papers were concerned with finding an easily calculable expression for the exchange rate when it was public knowledge that the government would eventually peg the exchange rate. All earlier results were derived under the Brownian motion with drift assumption for policy. We will now examine how our stability results can be used to approximate the exchange rate when it will be pegged at some random time. Both the rational and quasi-rational cases will be considered.

There are several policy feedback rules that are consistent with maintaining a target zone that have been analyzed in the literature. In Flood and Garber (1983) and Froot and Obstfeld (1991) it was assumed that the exchange rate would be fixed once the policy process hit a point or exited an interval. Flood and Garber (1991) generalized this to pegging the exchange rate once the exchange rate exited an interval. They calculated a closed-form solution provided a one-to-one relation existed between exit times of the policy process and exchange rate. Both Froot and Obstfeld (1991) and Flood and Garber (1991) derived easily calculable expressions for the exchange rate. However, their derivation depended on using the Brownian motion with drift policy process and their method, which depends on second order linear constant coefficient differential equations, does not extend to the more general policy process considered here. Is it possible to find an approximate, easily calculated exchange rate expression when the exchange rate will be pegged at some random time?

Let \( \tau^\epsilon = \inf \{ t : x_t^\epsilon \not\in [a - \delta, a + \delta] \} \) be the first exit time from the target zone. If \( t \geq \tau^\epsilon \) the government sets \( k_t^\epsilon = c \) for some constant \( c \); for \( t < \tau^\epsilon \) \( k_t^\epsilon \) is defined by (2.2). Thus, the exchange rate for \( t < \tau^\epsilon \) satisfies

\[
x_t^\epsilon = \int_t^{\tau^\epsilon} \exp(t - s) E(k_s^\epsilon | F_t) \, ds + c \int_{\tau^\epsilon}^{\infty} \exp(t - s) \, ds \quad (6.1)
\]

Theorem 3.1 proved that \( \int_t^{\infty} \exp(t - s) E(k_s^\epsilon | F_t) \, ds \to z_t \) as \( \epsilon \to 0 \). Even though (6.1) may not be easy to calculate, it can be approximated for small \( \epsilon \) by \( z_t \).
Theorem 6.1: Suppose that $z_t \in (a - \gamma, a + \gamma)$ where $\gamma < \delta$ and that $b$ is deterministic, then $\lim_{t \to 0} \tau^t = \infty$ and $\lim_{t \to 0} x^t = z_t$.

Proof: If there exist $\epsilon_n$ such that $\tau^{\epsilon_n} \leq T$ for some $T > 0$ there must exist a convergent subsequence which for simplicity we also denote by $\tau^{\epsilon_n}$. Let $v^t = \int_t^\infty \exp(t - s) E(k_s^t | \mathcal{F}_i) \, ds = z_t + \epsilon I^t$, $I^t = \int_0^t \sigma(s, k_s^t) \, dW_s$. Clearly, $\tau^t$ is the first time $v^t$ exits $[a - \delta, a + \delta]$ for if $v^t$ never exited the interval it would be the equilibrium exchange rate for all times. Hence, $|\epsilon_n I^{\epsilon_n}| > \delta - \gamma$ but the appendix applies Kolmogorov's Theorem to determine that $\sup_{t \in [0, T]} |\epsilon \int_0^t \sigma(s, k_s^t) \, dW_s| \to 0$ as $\epsilon \to 0$ a.e., a contradiction.

Thus $\int_t^\infty \exp(t - s) \, ds \to 0$ as $\epsilon \to 0$ because $\exp(t - s) \in L^1((t, \infty), ds)$. Theorem 3.1 implies that $v^t \to z_t$ so we need to show that

$$v^t - \int_t^{\tau^t} \exp(t - s) E(k_s^t | \mathcal{F}_i) \, ds = \int_{\tau^t}^\infty \exp(t - s) E(k_s^t | \mathcal{F}_i) \, ds \to 0 \quad (6.2)$$

From the proof of Theorem 3.1 $E(k_s^t | \mathcal{F}_i) = 1 + M^t_s + \int_0^t b(u) \, du$ where $M^t_s = \epsilon I^t_s$: the rightmost integral becomes $\int_t^{\infty} \exp(t - s) k_s \, ds + M^t_s \int_t^{\infty} \exp(t - s) \, ds$. The appendix shows that $M^t_s \to 0$. Integrability of $\exp(t - s) k_s$ and $\exp(t - s)$ on $[t, \infty)$ lets us apply the dominated convergence theorem to prove that both integrals converge to 0. q.e.d.

One of the implications of theorem 6.1 is that small perturbations will delay the switch to a fixed exchange rate. There could be a long period of moderate random shocks before fixing policy and the exchange rate at $c$. Also, the expression for the equilibrium rate under the policy feedback rule would be difficult to calculate directly because of difficulty in calculating the stopping time. Calculating $z_t$ either directly or via numerical simulation would probably be easier. Flood and Garber (1991) derived their results under a restrictive implicit assumption; our approximation result does not depend on linking the first exit time of the exchange rate with the first exit time of the policy process. The Froot and Obstfeld (1991) results are also derived under restrictive assumptions. For instance, they assume $E(k_s^t | \mathcal{F}_i) = G(k_i)$ for some $C^2$ function $G$ where $k_i = bt + \sigma W_t$. There is no reason to suspect that the conditional expectation will be a $C^2$ function in policy on probabilistic or economic grounds. From this vantage point it is more important to determine if the exchange rate can be easily approximated.

It is immediate from the proof that if the policy rule was more complex $z_t$ would still be an accurate approximation to the exchange rate because of the property that the first exit time becomes arbitrarily large as $\epsilon \to 0$. One
example of a more complex policy is to restart at the initial condition every
time the exchange rate hits the boundary of the target zone. This is more
realistic than assuming that the exchange rate is fixed forever.

The proof of Theorem 6.1 shows that if agents have quasi-rational ex-

The results of this paper rely heavily on the structure of random pertur-

8 Appendix: Mathematical Results

Here we take the initial condition of the SDE to be $x$; this bears no relation
to the exchange rate $x_t^e$. The solution to a stochastic differential equation is
found by a method of successive approximation: define $Y_t^{d,e} = x$ and

$$Y_t^{n,e} = x + \int_0^t b(s, Y_s^{n-1,e}) \, ds + \int_0^t \sigma(s, Y_s^{n-1,e}) \, dW_s \quad \forall n \geq 1$$

where the stochastic integral is an $L^2$ martingale.
Theorem 8.1: There exists a constant $C$ independent of $\epsilon \in (0, 1]$ such that

$$E(\sup_{s \in [0,t]} |Y_s^{n,\epsilon}|^2) \leq \sum_{k=0}^{n} \frac{C^{k+1} t^k}{k!} \quad \forall n \geq 1$$

Proof: For notational purposes, we drop the superscript $\epsilon$. Then

$$Y_t^1 = x + \int_0^t b(s, x) \, ds + \int_0^t \epsilon \sigma(s, x) \, dW_s$$

Hence,

$$\sup_{s \in [0,t]} |Y_s^1|^2 \leq 2(|x|^2 + (\int_0^t |b(s, x)| \, ds)^2 + 2(\sup_{s \in [0,t]} |\epsilon \sigma(u, x) \, dW_u|^2))$$

However, $|\int_0^t |b(s, x)| \, ds|^2 \leq t \int_0^t |b(s, x)|^2 \, ds$; this and Doob’s inequality imply

$$E(\sup_{s \in [0,t]} |Y_s^1|^2) \leq 2(|x|^2 + tE \int_0^t |b(s, x)|^2 \, ds + 8E \int_0^t |\epsilon \sigma(s, x)|^2 \, ds)$$

The Lipschitz conditions and the fact that $\epsilon \in (0, 1]$ imply

$$E \int_0^t |b(s, x)|^2 \, ds + E \int_0^t |\epsilon \sigma(s, x)|^2 \, ds \leq Kt|x|^2$$

Accounting for the constants appearing in the inequality, we may find some $C > 1, C = C(T)$ such that

$$E(\sup_{s \in [0,t]} |Y_s^1|^2) \leq C + C^2 t$$

Generally, we have

$$E(\sup_{s \in [0,t]} |Y_s^n|^2) \leq H(|x|^2 + E \int_0^t |b(s, Y_s^{n-1})|^2 \, ds + E \int_0^t \sigma^2(s, Y_s^{n-1}) \, ds)) \leq$$

$$M(|x|^2 + E \sup_{u \in [0,s]} |Y_u^{n-1}|^2 \, ds) = M(|x|^2 + \int_0^t (E \sup_{u \in [0,s]} |Y_u^{n-1}|^2) \, ds)$$

where $H$ is another constant and $M < C$ is also constant. By choice of $C$ we have $M|x|^2 < C$.

So if the result holds for $n \geq 1$, we have

$$E \sup_{s \in [0,t]} |Y_s^{n+1}|^2 \leq M(|x|^2) + C \int_0^t \sum_{k=0}^{n} \frac{C^{k+1} s^k}{k!} \, ds = M|x|^2 +$$
\[
\sum_{k=0}^{n} C^{k+2} t^{k+1} / (k+1)! \leq \sum_{k=0}^{n+1} C^{k+1} t^k / k!
\]
The result follows by induction. q.e.d.

Construction of a solution to a stochastic differential equation as well as all the proofs we present here, depend on this method for bounding the second moments of the iterates \( \{ \sup_{s \in [0,t]} |Y^n_s|, n \geq 1 \} \). We also adopt the convention of using \( C \) to denote a constant, generally depending on \( T \), which may assume different values in different situations. This is valid since we will be dealing with only finitely many constants which are independent of \( \epsilon \).

**Theorem 8.2:** Let \( \epsilon, \delta \in (0, 1] \) then

1) \( E \sup_{s \in [0,t]} |k^\epsilon_s - k^\delta_s|^2 \leq (\epsilon - \delta)^2 \alpha(t) \)
2) \( E \sup_{s \in [0,t]} |k^\epsilon_s - k^\delta_s|^2 \leq \epsilon^2 \beta(t) \)

where \( \alpha(t), \beta(t) \) are constants depending on \( t \) and where \( (k^\epsilon_s) \) satisfies the ordinary differential equation with initial condition \( x \).

**Proof:** Let \( \{(Y^n_s, s \in [0,T]), n \geq 0\} \) and \( \{(Z^n_s, s \in [0,T]), n \geq 0\} \) be the sequence of Picard iterates for \( (k^\epsilon_s) \) and \( (k^\delta_s) \) respectively. Using Doob's inequality and the method in the previous proof, we have \( \forall n \geq 1 \)

\[
E \sup_{s \in [0,t]} |Y^n_s - Z^n_s|^2 \leq H(E \int_0^t |b(u, Y^{n-1}_u) - b(u, Z^{n-1}_u)|^2 ds) +

H(E \sup_{s \in [0,t]} |\epsilon \int_0^s \sigma(u, Y^{n-1}_u) - \delta \int_0^s \sigma(u, Z^{n-1}_u) dW_u|^2)
\]

The expectation of the stochastic integral satisfies

\[
E \sup_{s \in [0,t]} |\int_0^s \epsilon \sigma(u, Y^{n-1}_u) - \delta \sigma(u, Z^{n-1}_u) dW_u|^2 \leq 4E \int_0^t |\epsilon \sigma(s, Y^{n-1}_s) - \delta \sigma(s, Z^{n-1}_s)|^2 ds
\]

Hence, combining these inequalities and invoking the Lipschitz conditions again yields

\[
E \sup_{s \in [0,t]} |Y^n_s - Z^n_s|^2 \leq C(E \int_0^t \sup_{u \in [0,s]} |Y^{n-1}_u - Z^{n-1}_u|^2 ds +

E \int_0^t |\epsilon \sigma(u, Y^{n-1}_u) - \delta \sigma(u, Z^{n-1}_u)|^2 ds)
\]
Observe that,

\[ |\epsilon\sigma(u, Y_{u}^{n-1}) - \delta\sigma(u, Z_{u}^{n-1})| \leq |\epsilon\sigma(u, Y_{u}^{n-1}) - \delta\sigma(u, Y_{u}^{n-1})| + \\
\delta|\sigma(u, Y_{u}^{n-1}) - \sigma(u, Z_{u}^{n-1})| \leq K|\epsilon - \delta||Y_{u}^{n-1}| + K\delta|Y_{u}^{n-1} - Z_{u}^{n-1}| \]

So that

\[ |\epsilon\sigma(s, Y_{s}^{n-1}) - \delta\sigma(s, Z_{s}^{n-1})|^2 \leq 2K((\epsilon - \delta)^2 \sup_{u \in [0,s]} |Y_{u}^{n-1}|^2 + \delta^2 \sup_{u \in [0,s]} |Y_{u}^{n-1} - Z_{u}^{n-1}|^2) \]

Then it follows that

\[ E \sup_{s \in [0,t]} |Y_{s}^{n} - Z_{s}^{n}|^2 \leq C \int_{0}^{t} E \sup_{u \in [0,s]} |Y_{u}^{n-1} - Z_{u}^{n-1}|^2 \, ds + \\
2C(\epsilon - \delta)^2 \int_{0}^{t} E \sup_{u \in [0,s]} |Y_{u}^{n-1}|^2 \, ds + 2C\delta^2 \int_{0}^{t} E \sup_{u \in [0,s]} |Y_{u}^{n-1} - Z_{u}^{n-1}|^2 \, ds \]

This is the central inequality we need to prove assertion 1. If \( n = 1 \)

\[ Y_{t}^{1} - Z_{t}^{1} = (\epsilon - \delta) \int_{0}^{t} \sigma(s, x) \, dW_{s} \Rightarrow E \sup_{s \in [0,t]} |Y_{s}^{1} - Z_{s}^{1}|^2 \leq \\
4K|x|^2(\epsilon - \delta)^2 t \leq C(\epsilon - \delta)^2 t \]

We prove by induction that \( E \sup_{s \in [0,t]} |Y_{s}^{n} - Z_{s}^{n}|^2 \leq (\epsilon - \delta)^2 \sum_{k=1}^{n} \frac{5^{k}C^{k+1}t^{k}}{k!} \). Notice that if this is true, we have proved assertion 1, for

\[ E \sup_{s \in [0,t]} |k_{s}^{\epsilon} - k_{s}^{\delta}|^2 \leq 8(E \sup_{s \in [0,t]} |k_{s}^{\epsilon} - Y_{s}^{n}|^2 + E \sup_{s \in [0,t]} |Y_{s}^{n} - Z_{s}^{n}|^2 + E \sup_{s \in [0,t]} |Z_{s}^{n} - k_{s}^{\delta}|^2) \]

Kallianpur (1984) proves that

\[ \lim_{n \to \infty} E \sup_{s \in [0,t]} |k_{s}^{\epsilon} - Y_{s}^{n}|^2 = \lim_{n \to \infty} E \sup_{s \in [0,t]} |k_{s}^{\delta} - Z_{s}^{n}|^2 = 0 \]

The second integral is uniformly bounded:

\[ E \sup_{s \in [0,t]} |Y_{s}^{n} - Z_{s}^{n}|^2 \leq (\epsilon - \delta)^2 \sum_{k=1}^{n} 5^{k}C^{k+1}t^{k}/k! \leq (\epsilon - \delta)^2 \sum_{k=1}^{\infty} 5^{k}C^{k+1}t^{k}/k! \]

The series converges by Cauchy's ratio test.
Having proven the result for \( n = 1 \), we consider the case for \( n + 1 \) when the result holds for \( n \).

\[
E \sup_{s \in [0,t]} |Y_{s}^{n+1} - Z_{s}^{n+1}|^2 \leq (\epsilon - \delta)^2 \sum_{k=1}^{n} 5^k C^{k+2} t^{k+1} / (k + 1)!
\]

\[+ 2(\epsilon - \delta)^2 \sum_{k=1}^{n} 5^k C^{k+2} t^{k+1} / (k + 1)! + 2C (\epsilon - \delta)^2 \sum_{k=0}^{n} C^{k+1} t^{k+1} / (k + 1)!
\]

This step involves evaluating a Lebesque integral as a Riemann integral. This simplifies to

\[
(\epsilon - \delta)^2 \left( 3 \sum_{k=1}^{n} 5^k C^{k+2} t^{k+1} / (k + 1)! + 2 \sum_{k=0}^{n} C^{k+2} t^{k+1} / (k + 1)! \right)
\]

The bracketed sum becomes

\[
3 \sum_{k=1}^{n} 5^k C^{k+2} t^{k+1} / (k + 1)! + 2 \sum_{k=1}^{n} C^{k+2} t^{k+1} / (k + 1)! + 2C^2 t \leq
\]

\[
5 \sum_{k=1}^{n} 5^k C^{k+2} t^{k+1} / (k + 1)! + 5C^2 t = \sum_{k=1}^{n+1} 5^k C^{k+1} t^k / k!
\]

Now, we prove the second assertion. The function \( b \) satisfies the Lipschitz conditions, hence there exists a unique solution to the ordinary differential equation \( \frac{dx}{dt} = b(t, z(t)) \). We may construct Picard iterates \( z_t^n = x + \int_0^t b(s, z_s^{n-1}) \, ds \quad z_0^0 = x \quad \forall s \) which converge uniformly to \( (z_s) \) on closed bounded intervals. Hence \( \lim_{n \to \infty} E \sup_{s \in [0,t]} |z_s^n - z_s| = 0 \). We will prove the result by establishing inductively that

\[
E \sup_{s \in [0,t]} |Y_s^n - z_s^n|^2 \leq \epsilon^2 \sum_{k=1}^{n} 2^{k-1} C^{k+1} t^k / k!
\]

Cauchy's ratio test implies that the series is summable, so we may apply the same arguments that we used to establish assertion 1 of the lemma. For \( n = 1 \) we have

\[
Y_t^1 - z_t^1 = \epsilon \int_0^t \sigma(s, x) \, dW_s \Rightarrow E \sup_{s \in [0,t]} |Y_s^1 - z_s^1|^2 \leq \epsilon^2 C t
\]

Generally,

\[
E \sup_{s \in [0,t]} |Y_s^n - z_s^n|^2 \leq C \left( \int_0^t E \sup_{u \in [0,s]} |Y_u^{n-1} - z_u^{n-1}|^2 \, ds + \epsilon^2 \int_0^t E \sup_{u \in [0,s]} |Y_u^{n-1}|^2 \, ds \right)
\]
Assuming that the hypothesis holds for \( n \), consider

\[
E \sup_{s \in [0,t]} |Y_s^{n+1} - z_s^{n+1}|^2 \leq e^2 \left( \sum_{k=1}^{n} 2^{k-1} C^{k+2} t^{k+1} / (k+1)! + \sum_{k=0}^{n} C^{k+2} t^{k+1} / (k+1)! \right)
\]

Consider the bracketed term; it simplifies to

\[
\sum_{k=1}^{n} 2^{k-1} C^{k+2} t^{k+1} / (k+1)! + \sum_{k=1}^{n} 2^{k-1} C^{k+2} t^{k+1} / (k+1)! + C^2 t =
\]

\[
\sum_{k=1}^{n} 2^{k} C^{k+2} t^{k+1} / (k+1)! + C^2 t = \sum_{k=1}^{n+1} 2^{k-1} C^{k+1} t^{k} / k! \quad \text{q.e.d.}
\]

Theorem (Kolmogorov): Let \((X_t, t \in [0,1])\) be a class of random variables such that there exist \( \beta, \gamma, C > 0 \) satisfying \( E(|X_t - X_s|^\gamma) \leq C|t-s|^{1+\beta} \). Then there exists a continuous modification \((h_t, t \in [0,1])\) satisfying \( E(\sup_{s \neq t} |h_t - h_s|/|t-s|^\alpha) < \infty \quad \forall \alpha < \frac{\beta}{\gamma} \).

Lemma 3.2: \( \lim_{\epsilon \to 0} \sup_{t \in [0,T]} |k_t^\epsilon - k_t| = 0 \quad \text{a.e. where } (k_s) \text{ satisfies the deterministic equation.} \)

Proof: Define \( X_\epsilon = \sup_{t \in [0,T]} |k_t^\epsilon - k_t| \) if \( \epsilon \in (0,1] \) and \( X_0 = 0 \). If \( \epsilon, \delta > 0 \) then \( |X_\epsilon - X_\delta| \leq \sup_{t \in [0,T]} |k_t^\epsilon - k_t^\delta| \). Hence, by Lemma 8.2

\[
E |X_\epsilon - X_\delta|^2 \leq E(\sup_{t \in [0,T]} |k_t^\epsilon - k_t^\delta|)^2 = E \sup_{t \in [0,T]} |k_t^\epsilon - k_t^\delta|^2 \leq \alpha(T)(\epsilon - \delta)^2
\]

Also, \( E |X_0 - X_\epsilon|^2 = E |X_\epsilon|^2 \leq e^2 \beta(T) \). So the criteria for Kolmogorov’s Theorem is satisfied for \( \gamma = 2, \beta = 1 \).

Fix \( \alpha < \frac{1}{2} \Rightarrow \sup_{t \neq s} |h_s - h_t|/|t-s|^\alpha < \infty \quad \text{a.e.} \) By the CCS assumption, we must have that \( X_\epsilon = h_\epsilon \) almost everywhere for all \( \epsilon \) simultaneously. Let \( s = 0 \) and \( \epsilon > 0 \), then almost everywhere \( \sup_{t \in [0,T]} |k_t^\epsilon - k_t| \leq M \epsilon^\alpha \). Choose \( \epsilon < (\delta/M)^{1/\alpha} \) to assure that \( \sup_{t \in [0,T]} |k_t^\epsilon - k_t| < \delta \quad \text{q.e.d.} \)

The same technique will also establish Lemma 3.3 once we have proven the following:

Theorem 8.3: Let \( M_t^\epsilon = \epsilon \int_0^t \sigma(s, k_s^\epsilon) \, dW_s \). Then if \( \epsilon, \delta > 0 \) we have

\[
E \sup_{s \in [0,t]} |M_s^\epsilon - M_s^\delta|^2 \leq (\epsilon - \delta)^2 \mu(t)
\]
and \( E \sup_{s \in [0,t]} |M_s^n|^2 \leq \epsilon^2 \nu(t) \) where \( \mu(t), \nu(t) \) are constants depending on \( t \) but not on \( \epsilon \).

Proof: Define \( M^n_t = \epsilon \int_0^t \sigma(s, Y^n_s) \, dW_s \) and \( N^n_t = \delta \int_0^t \sigma(s, Z^n_s) \, dW_s \) where \( \{(Y^n_s, s \in [0,T]) \mid n \geq 0\} \) and \( \{(Z^n_s, s \in [0,T]) \mid n \geq 0\} \) are the processes of Picard iterates converging to \((k_t^\epsilon)\) and \((k_t^\delta)\) respectively.

Since

\[
E \sup_{s \in [0,t]} |M_s^n - M_s^\epsilon|^2 \leq 4E \int_0^t |\sigma(s, k_s^\epsilon) - \sigma(s, Y^n_s)|^2 \, ds \leq \frac{8K}{\epsilon^2} \int_0^t |k_s^\epsilon - Y^n_s|^2 \, ds \leq 4tE \sup_{s \in [0,t]} |k_s^\epsilon - Y^n_s|^2 \to 0 \quad \text{as} \quad n \to \infty
\]

we need only show that \( E \sup_{s \in [0,t]} |M_s^n - N_s^n|^2 \leq (\epsilon - \delta)^2 \alpha(t) \). Generally, for all \( n \geq 2 \),

\[
E \sup_{s \in [0,t]} |M_s^n - N_s^n|^2 \leq 4E \int_0^t |\epsilon \sigma(s, Y_s^{n-1}) - \delta \sigma(s, Z_s^{n-1})|^2 \, ds \leq 8K \int_0^t (\epsilon - \delta)^2 E \sup_{u \in [0,s]} |Y_u^{n-1}|^2 + 8K \delta^2 \int_0^t E \sup_{u \in [0,s]} |Y_u^{n-1} - Z_u^{n-1}|^2 \, ds \leq C((\epsilon - \delta)^2 \int_0^t \sum_{k=0}^{n-1} C_{k+1} s^k / k! + (\epsilon - \delta)^2 \sum_{k=1}^{n-1} 5^k C_{k+1} s^k / k! \, ds) \leq C(\epsilon - \delta)^2 (\sum_{k=0}^{n-1} C_{k+1} t^k / (k+1)! + \sum_{k=1}^{n-1} 5^k C_{k+1} t^k / (k+1)!) = (\epsilon - \delta)^2 (\sum_{k=0}^{n-1} C_{k+2} t^k / (k+1)! + \sum_{k=1}^{n-1} 5^k C_{k+2} t^k / (k+1)!) \]

Both series in this expression converge, and the terms are nonnegative; hence, we have the desired bound. Also,

\[
E \sup_{s \in [0,t]} |M_s^n|^2 \leq \epsilon^2 E \int_0^t \sigma^2(s, Y_s^{n-1}) \, ds \leq C\epsilon^2 \int_0^t E \sup_{u \in [0,s]} |Y_u^{n-2}|^2 \, ds \leq \epsilon^2 \sum_{k=0}^{n-1} C_{k+2} t^k / k! \quad \text{q.e.d.}
\]

Theorem 8.4: Let \( x_t^\epsilon, z_t \) be the perturbed and deterministic equilibria at time \( t \). Then
1) $E \sup_{s \in [0,t]} |x^e_s - x^\delta_s|^2 \leq (\epsilon - \delta)^2 \zeta(T)$
2) $E \sup_{s \in [0,t]} |x^e_s - z_t|^2 \leq \epsilon^2 \phi(T)$

where $\zeta(T), \phi(T)$ are constants depending on $T$, the terminal time of the economy.

Proof: From the the proof of Theorem 4.1,

$$\sup_{s \in [0,t]} |x^e_s - z_t|^2 \leq C(\sup_{s \in [0,t]} |M^e_s|^2 + \sup_{s \in [0,t]} |k^e_s - k^\delta_s| + \sup_{s \in [0,t]} E(Y^e|\mathcal{F}_s)^2)$$

Applying our results on the moments of the first two terms contained in Lemmas 2 and 3 of this section as well as Doob's inequality to the last term yields $E \sup_{s \in [0,t]} |x^e_s - z_t|^2 \leq \epsilon^2(5C\beta(t) + CV(t))$. Now,

$$x^e_t - x^\delta_t = v_t(M^e_t - M^\delta_t) + v_t(\int_0^t b(s, k^e_s) - b(s, k^\delta_s) \, ds) +$$

$$\frac{\alpha^{-1}}{\alpha} \int_t^T \exp(\frac{t - s}{\alpha}) E(\int_t^s b(u, k^e_u) - b(u, k^\delta_u) \, du|\mathcal{F}_t) \, ds$$

We have applied the CCS assumption to guarantee that this equality holds almost everywhere for each $t$ simultaneously. Conditional expectation is linear a.e. So, for fixed $s$

$$E(\int_{\nu}^s b(u, k^e_u) \, du|\mathcal{F}_t) - E(\int_{\nu}^s b(s, k^\delta_u) \, du|\mathcal{F}_t) = E(\int_{\nu}^s b(u, k^e_u) - b(u, k^\delta_u) \, du|\mathcal{F}_t)$$

on the set $C(t, s)$. $P(C(t, s)) = 1$. Since we are integrating over $[t, T]$ for a fixed $\omega$, we want expressions like this to hold almost everywhere for all $s \in [t, T]$ simultaneously. The CCS assumption guarantees that this is so: let $C_t = \cap_{s \in [t, T]} C(t, s)$ then $P(C_t) = 1$. Again by CCS, $P(\cap_{t \in [0, T]} C_t) = 1$, so the equality above holds almost everywhere for all $t$; this is necessary for us to bound $\sup_{t \in [0, T]} |x^e_t - x^\delta_t|$ almost everywhere. Hence, accounting for the fact that $g(t, s) = (s - t) \exp(\frac{t - s}{\alpha})$ is bounded on $[0, T] \times [0, T]$, there exists a constant $J$ such that

$$\sup_{s \in [0,T]} |x^e_s - x^\delta_s|^2 \leq J(\sup_{s \in [0,T]} |M^e_s - M^\delta_s|^2 + \sup_{s \in [0,T]} |k^e_s - k^\delta_s|^2 + \sup_{s \in [0,T]} E(Y^{e,\delta}|\mathcal{F}_s)^2)$$

where $Y^{e,\delta} = \sup_{s \in [0, T]} |k^e_s - k^\delta_s|$. To see why this holds, we really only need to examine the integral over the sample paths of conditional expectation.

$$|\int_t^T \exp(\frac{t - s}{\alpha}) E(\int_t^s b(u, k^e_u) - b(u, k^\delta_u) \, du|\mathcal{F}_t) \, ds| \leq$$
\[ \int_t^T M |E(\int_t^s b(u, k_u^\epsilon) - b(u, k_u^\delta) \, du|F_t)| \, ds \]

Fix \( s \in [t, T] \), then
\[ |\int_t^s b(u, k_u^\epsilon) - b(u, k_u^\delta) \, du| \leq \int_t^s \sup_{j \in [0, T]} |k_j^\epsilon - k_j^\delta| \, du = (s - t) \sup_{j \in [0, T]} |k_j^\epsilon - k_j^\delta| \]

Then on \( C(t, s) \), \( P(C(t, s)) = 1 \) we have,
\[ |E(\int_t^s b(u, k_u^\epsilon) - b(u, k_u^\delta) \, du|F_t)| \leq E(\int_t^s |b(u, k_u^\epsilon) - b(u, k_u^\delta)| \, du|F_t) \leq (s - t)E(\sup_{j \in [0, T]} |k_j^\epsilon - k_j^\delta|)|F_t) \leq (T - t)E(Y^{\epsilon, \delta}|F_t) \]

Hence on \( C_t = \cap_{s \in [t, T]} C(t, s) \) \( P(C_t) = 1 \ \forall t \in [0, T] \) we have
\[ \int_t^T |E \int_t^s b(u, k_u^\epsilon) - b(u, k_u^\delta) \, du|F_t) \, ds \leq \int_t^T (T - t)E(Y^{\epsilon, \delta}|F_t) \, ds = \]
\[ 2^{-1}(T - t)^2E(\sup_{j \in [0, T]} |k_j^\epsilon - k_j^\delta|)|F_t) \leq T^2 \sup_{j \in [0, T]} E(Y^{\epsilon, \delta}|F_j) \]

Repeating the argument used in Theorem 4.1, we have \( E \sup_{j \in [0, T]} E(Y^{\epsilon, \delta}|F_j)^2 \leq 4E \sup_{s \in [0, T]} |k_s^\epsilon - k_s^\delta|^2 \) q.e.d.

Proof of Theorem 4.2: Define random variables by \( X_0 = 0 \) and \( X_\epsilon = \sup_{s \in [0, T]} |x_s^\epsilon - z_s| \). Then the proof is exactly the same as the proof of Lemma 3.2.

Theorem 8.5: Let \( P \) be CCS and \( G \) be a sub \( \sigma \)-algebra containing all sets of measure 0. Let \( \{f_\epsilon, \epsilon \in (0, 1] \} \) be continuous with \( |f_\epsilon| \leq Z, \ Z \in L^1 \) and \( \lim_{\epsilon \to 0} f_\epsilon = f \) almost everywhere, then \( E(f_\epsilon|G) \to E(f|G) \) almost everywhere as \( \epsilon \to 0 \).

Proof: Define \( h(t) = \sup_{s \in [0, t]} |f_s - f|, \ \forall t \in (0, 1] \). Then \( h(t) \downarrow 0 \) a.e. and \( |E(f_\epsilon|G) - E(f|G)| = |E(f_\epsilon - f)|G) \leq E(h(t)|G) \) a.e. by the CCS assumption. So the result follows if \( \lim_{\epsilon \to 0} E(h(t)|G) = 0 \) a.e. Consequently, we prove that if \( f_\epsilon \downarrow f \) a.e. as \( t \downarrow 0 \) \( f_\epsilon \leq h, \ E(h) < \infty \) then \( E(f_\epsilon|G) \downarrow E(f|G) \) a.e. For \( h(t) \leq 2Z \), so application of this fact yields the result. If \( s < t \) we have \( E(f_s|G) \leq E(f_t|G) \) on \( C(t, s) \) where \( P(C(t, s)) = 1 \). Let \( C_t = \cap_{s < t} C(t, s) \) and \( C = \cap_{t \in (0, 1]} C_t \). Then \( P(C) = 1 \) and \( s < t \Rightarrow E(f_s|G) \leq E(f_t|G) \) on \( C \). Hence, \( E(f_t|G) \downarrow g \), a \( G \) measurable function. Measurability relative to \( G \) follows because \( C \in G \).

63
Let $A \in \mathcal{G}$, then by extending the monotone convergence theorem to deal with a continuous limit,

$$
\int_A E(f|\mathcal{G}) \, dP = \int_A f \, dP = \lim_{t \to 0} \int_A f_t \, dP = \lim_{t \to 0} \int_A E(f_t|\mathcal{G}) \, dP = \int_A g \, dP
$$

Implies $g = E(f|\mathcal{G})$ a.e. q.e.d.

Proof of Lemma 5.1:

$$
E \sup_{s \in [t,t+m]} |f_s^\epsilon - k_s^\epsilon|^2 \leq C(E \int_t^{t+m} |b(u,k_u) - b(u,k_u^\epsilon)|^2 \, du +
$$

$$
\epsilon^2 E \int_t^{t+m} (\sigma(u,k_u) - \sigma(u,k_u^\epsilon))^2 \, du \leq CK(t + m) E \sup_{s \in [0,t+m]} |k_s^\epsilon - k_s|^2 \to 0
$$

Analogous reasoning holds for $h_s^\epsilon$. q.e.d.

Proof of Theorem 5.1: Observe that $E \int_t^{t+m} (f_s^\epsilon)^2 \, ds \leq M < \infty$ implying that $h_s^\epsilon$ is well defined. Also, $E(f_s^\epsilon)^2 \leq N < \infty \ \forall s \in [t,t+m]$ and

$$
y_t^\epsilon = wx_t^\epsilon + (1-w) \int_t^{t+m} \exp(t-s)h_s^\epsilon \, ds
$$

The proof works by showing that

$$
\sup_{s \in [t,t+m]} |f_s^\epsilon - f_s| \to 0, \sup_{s \in [t,t+m]} |\epsilon \int_t^{t+m} \sigma(s,f_s^\epsilon) \, dW_s| \to 0
$$

a.e. and then exploiting the well-known convergence criteria for Riemann integrals. We have already shown that $\sup_{s \in [t,t+m]} |k_s^\epsilon - k_s| \to 0$ so we can focus on proving the second result. Now,

$$
\sup_{s \in [t,t+m]} |\epsilon \int_t^s \sigma(u,f_u^\epsilon) \, dW_u| \leq \sup_{s \in [t,t+m]} |\epsilon \int_t^s \sigma(u,f_u^\delta) \, dW_u| + \delta \int_t^s \sigma(u,f_u^\delta) \, dW_u| \quad \Rightarrow \quad \sup_{s \in [t,t+m]} |\epsilon \int_t^s \sigma(u,f_u^\epsilon) \, dW_u| - \sup_{s \in [t,t+m]} |\delta \int_t^s \sigma(u,f_u^\delta) \, dW_u| + \delta \int_t^s \sigma(u,f_u^\delta) \, dW_u| \leq |\epsilon - \delta| \sup_{s \in [t,t+m]} |\int_t^s \sigma(u,f_u^\epsilon) \, dW_u| + \delta \sup_{s \in [t,t+m]} |\int_t^s \sigma(u,f_u^\epsilon) - \sigma(u,f_u^\delta) \, dW_u|.
$$

Now,

$$
E \sup_{s \in [t,t+m]} |f_s^\epsilon - f_s^\delta|^2 \leq C(E \sup_{s \in [t,t+m]} |k_s^\epsilon - k_s|^2 + (\epsilon - \delta)^2 E \sup_{s \in [t,t+m]} |\int_t^s \sigma(u,k_u) \, dW_u|^2)
$$
\[ \leq \phi(t + m)(\epsilon - \delta)^2 \quad \text{where} \quad \phi(t + m) \quad \text{is a constant} \]

Let \( v^\epsilon = \sup_{s \in [t, t+m]} |\epsilon \int_t^s \sigma(u, f_u^\epsilon) \, dW_u| \). Then we have shown that

\[ E(v^\epsilon - v^\delta)^2 \leq 2KNm(\epsilon - \delta)^2 + 2Km\phi(t + m)(\epsilon - \delta)^2 \]

Clearly, \( E(v^\epsilon)^2 \leq \epsilon^2 mKN \) and hence defining \( X_\epsilon = v^\epsilon, X_0 = 0 \) we may apply Kolmogorov's Theorem with \( \gamma = 2 \) and \( \beta = 1 \). Hence,

\[ \sup_{s \in [t, t+m]} |\epsilon \int_t^s \sigma(u, f_u^\epsilon) \, dW_u| \to 0 \quad \text{a.e. and} \quad \sup_{s \in [t, t+m]} |f_s^\epsilon - f_s| \to 0 \quad \text{a.e.} \Rightarrow \]

\[ \sup_{s \in [t, t+m]} |h_s^\epsilon - k_t - \int_t^s b(u, f_u) \, du| \to 0 \quad \text{a.e. as} \quad |b(u, f_u^\epsilon) - b(u, f_u)| \leq K|f_u^\epsilon - f_u| \]

Hence \[ \int_t^{t+m} \exp(t - s)h_s^\epsilon \, ds \to \int_t^{t+m} \exp(t - s)(k_t + \int_t^s b(u, f_u) \, du) \, ds \]

The proof of theorem 4.1 shows that

\[ \int_t^\infty \exp(t - s)E(k_s^\epsilon | F_t) \, ds \to \int_t^{t+m} \exp(t - s)k_s \, ds \quad \text{q.e.d.} \]

**References**


Exchange Rate Fluctuations In An Economy With Noise Traders

Chapter 3 of the doctoral dissertation of Gregory Gagnon

Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ont. Canada, M5S 3G7.

Abstract: This paper analyzes the stability of the exchange rate in an economy with noise traders. Noise trading is restricted to agents investing in the domestic stock market, who are less sophisticated than the agents pricing foreign exchange. Monetary policy is affected by the behaviour of investors in the domestic stock market. We show that small fluctuations in the parameters governing noise trading can have a profound effect on the exchange rate when foreign exchange traders have rational expectations. This shows that instability is the key feature in economies where heterogeneous agents have different levels of sophistication in processing information. Endogenous fluctuations from the stock market spill over into endogenous fluctuations in the exchange rate.

1 Introduction

There is a growing literature in financial economics on price dynamics in markets with heterogeneous agents. A typical heterogeneity found in this literature is the distinction between noise traders who use technical trading rules and fundamentalists who are concerned with discounted expected future dividends. Noise traders are meant to be irrational traders who extrapolate trends and who downplay fundamentals, see Summers (1986). Their existence has been a matter of bitter contention in the literature, see Brock and Hommes (1997). However, the literature dealing with their effects on price dynamics is growing. There is evidence that noise traders do exist. Brock,
Lakonishok and LeBaron (1992) find that simple technical trading rules can improve stock returns. LeBaron (1996) suggests that trading rules in the foreign exchange market are profitable because they uncover correlations caused by Federal Reserve intervention. Schliefer and Summers (1990) also present case studies where price dynamics cannot be explained by the fundamentals alone. Empirical phenomena of this nature suggest that markets are somewhat inefficient and that noise trading could be one source of the inefficiency.

Part of the problem in modelling noise traders lies in finding a way to represent these irrational agents. For a long time, it was not clear that the behaviour of irrational agents could be captured in any kind of optimization problem. The fact that these agents are concerned with trends rather than discounted expected profits suggests that their behaviour is less structured than the behaviour consistent with solutions to optimization problems. For this reason, the seminal work in the area of noise trading focused on the effects noise traders have on prices rather than their behaviour. However, a behavioural approach to the noise trader problem now exists. This area is called "evolutionary finance" and it constructs dynamic models of learning in financial markets. Investors may purchase information at a constant cost or trade on the basis of free publically available information. In this approach, the behaviour of noise traders is modelled explicitly and the consequences for prices are derived. The evolutionary finance framework of noise trading plays a major role in our paper.

In this paper, we are concerned with noise trading in the stock market and a mechanism whereby it spills over to the foreign exchange market. The market is segmented in the sense that the agents who may select to be noise traders are present only in the domestic stock market. Exchange rates are determined by sophisticated investors who hold rational expectations and who do not invest in domestic stocks. Domestic investors are meant to be less sophisticated than foreign exchange traders and are consequently more prone to be influenced by market trends in setting prices. Although there is no empirical evidence that foreign exchange traders are more savvy than stock market investors, such an assertion is not beyond the realm of possibility. Despite the preponderance of mutual funds, there are still many small investors in the stock market who trade infrequently and who are relatively uninformed. Stocks are a widely used vehicle for saving whereas foreign exchange is not. Even when investing in stocks priced in a foreign currency, the main motivation surely comes from profitability of the company, not potential exchange rate changes. By contrast, foreign exchange traders are
professionals who need a fairly sound understanding of macroeconomics and international economics to execute intelligent trades.

Summers (1986), Schliefer and Summers (1990) and De Long, Schliefer, Summers and Waldman (1990), analyze the problem of noise trading. Noise trading creates excess speculation and excess volatility in the market. The papers cited above model directly with the price effects of noise traders; they do not construct a behavioural model of noise traders. Brock and Hommes (1997,1998) present a framework for modelling noise trader behaviour. The Brock-Hommes framework allows us to model whether a stock market investor will be a fundamentalist or noise trader. Part of our contribution is to show how noise trading in the stock market can affect the rational expectations price of the currency. This involves building bridges between Brock and Hommes (1997,1998) and a well-known model of the exchange rate presented in Flood and Garber (1983).

Brock and Hommes (1997,1998) present models of noise trading where agents choose between a naive and sophisticated predictor of the price based on past profitability of predictors. In other words, agents may choose a naive predictor based on its past profitability. Their model presents steady states where noise traders are present at all times. Following Brock and Hommes (1997,1998), the sophisticated predictor in our stock market will be the conditional expectations operator. The less sophisticated predictor will attempt to extrapolate market trends. However, to reflect differences in sophistication, stock investors trade only at discrete times \( t_i \) while the exchange rate evolves in continuous time. At times \( t_i > 0 \) data is available from all past trades and stock investors choose their trading type according to Brock and Hommes (1997,1998). Chavas (2000) presents empirical evidence for the presence of heterogeneous expectations. His findings support the feature of Brock and Hommes (1997,1998) that uninformed trading may persist over time for a large number of agents.

Since foreign exchange traders and stock investors are distinct agents some mechanism is needed to create a spillover from the naive traders to the sophisticated traders. The source of the spillover is government. We assume that the monetary authority, the government, is concerned about deviations from fundamental prices in the stock market. Consequently it considers the composition of traders in the stock market in setting the money supply. Since foreign exchange traders have rational expectations, the exchange rate at time \( t \) is an integral of discounted expected future money supplies for times \( s > t \). When the money supply is set according to considerations from the
domestic stock market, rational agents are affected by irrational behaviour even though they have no direct dealings in the stock market. The result is that the equilibrium exchange rate depends on the equilibrium in the stock market. Small changes in the parameters governing the propensity of noise traders to chase trends or switch predictors can create potentially large fluctuations in the exchange rate even when there are no direct links between stock prices and the currency.

There is a long standing debate in macroeconomics over whether observed fluctuations are caused by optimizing behaviour or external random shocks, see Grandmont (1985) and Matsuyama (1991). Most modellers tend to use either the endogenous approach which usually relies on complex deterministic dynamics or the exogenous approach which frequently exploits the machinery of stochastic calculus. In this paper, we couple both approaches by allowing endogenous fluctuations in the stock market to affect the diffusion governing money supply growth. Shocks to the money supply become partially endogenous in that they depend on stock market steady states via the drift function, but they also have a random component independent of the stock market. In so doing, we attempt to build a fluctuation theory in a random rather than a non-random (deterministic) economy. Our results suggest that endogenous fluctuations are important to exchange rate stability even when randomness is present.

Now, we provide a brief survey of related literature. Brock and de-Fontnouvelle (2000) construct an overlapping generations monetary economy where the number of predictors of the future price level approaches infinity. Predictors are again based on past data and predictor selection by agents is determined by the forecast errors of the predictor. The frequency to switch predictors, called intensity of choice, once again affects dynamics tremendously. For low intensities the monetary state is stable. At higher intensities a Hopf bifurcation occurs and prices may have irregular paths. Chiarella and Flaschel (2000) present an open economy growth model with sluggish prices and quantities. They model both real and nominal sectors in an eight dimensional system of deterministic differential equations. The system is stable for slow price-quantity adjustment rates, then exhibits a Hopf bifurcation as the rates of adjustment increase and eventually explodes. They show that external nonlinearities eliminate the chaotic dynamics. Duffy (1994) shows that when agents use adaptive learning rules the economy can converge to a continuum of nonstationary equilibria.

The paper is organized as follows. Section 2 presents the stock market
2 The Stock Market Model

There are two markets in our model, a stock market and a money market. It is assumed that agents allocate their wealth to non-monetary assets which are purchased in the stock market. Money is only held to meet transactions demands and has no effect on the price of stocks. The stock market model is taken from Brock and Hommes (1997,1998). Our analysis focusses on exchange rate dynamics when speculation in the stock market affects the money supply. To preserve a self contained treatment, this section provides a brief overview of Brock and Hommes (1997,1998); for an in depth treatment the interested reader is directed to their paper.

The stock price evolves in discrete time. In the stock market, the stock price depends on the percentages of agents who are fundamentalists and noise traders. All agents in the market are mean-variance maximizers. The distinguishing feature between noise traders and fundamentalists is the expectation of the stock price in the next period. For expositional convenience, fundamentalists and noise traders will be referred to as type one and type two respectively. The expectations of type \( j \) at time \( n \) are denoted by \( E_{jn} \).

The number of noise traders and fundamentalists varies over time and can generate complex dynamics. At each time \( n \), each agent chooses whether to be a noise trader or fundamentalist based on the profitability of each trading type in the previous period. The past profitability of a given type is frequently referred to as a fitness measure. Thus, the distribution of trading types is endogenous. It is the ability of agents to switch types that leads to the possibility of complex dynamics.

Agents investing in stocks use past data from the stock market in forming their expectations and choosing their trading type. Correctly, they do not believe that the money supply has any effect on stock prices. By assumption, the government knows the distribution of trading types in the stock market as well as deviations from the fundamental price of the stock due to noise trading. For reasons not modelled explicitly, it considers both these variables in setting the money supply even though the money supply has no effect on the stock market. This could represent a mistaken belief by government that the money supply affects stock prices but the major motivation is one
of tractability. Our goal is to present a simple theory of fluctuations in a random economy which is similar to the deterministic theory.

The agents pricing foreign exchange are by assumption different than the agents pricing the stock. Foreign exchange traders have rational expectations and price the currency according to the discounted expected value of future money supplies. In this way, even though noise trading is restricted to the domestic stock market it can have an effect on the foreign exchange market with perfectly rational agents. Our scenario attempts to model an economy where savvy professionals price foreign exchange and where the competitive stock market is composed of less savvy agents. We now turn to the summary of Brock and Hommes (1997,1998).

There is one risky asset in the stock market and a risk free bond which has a constant interest rate $r$ and infinitely elastic supply. The gross return is $R = (1 + r)$. Let the dividend process be $(d_i)$. For convenience let the trading times correspond to the nonnegative integers i.e. $t_i = i, i \geq 0$. At each time $i$ the agent will have non-monetary wealth at time $i + 1$ given by

$$W_{i+1} = RW_i + (p_{i+1} + d_{i+1} - Rp_i)h_i \quad (2.1)$$

where $W_i$ is wealth from stock and bond holdings and $h_i$ is the number of shares purchased at time $i$.

Although money is included in total wealth, agents are not choosing money balances. Money is held for transactions purposes and we assume that the money market is always in equilibrium; details are in the next section. Foreign nationals could hold part of the money stock, but this is not necessary for what follows. We will construct a connection from the stock market to the exchange rate and for the derivation to hold it is important that money holdings not enter the agent’s choice problem.

**Assumption 2.1:** The conditional variance of excess returns at time $n$ of type $j$, $V_{jn}$, satisfies $V_{jn}(p_{i+1} + d_{i+1} - Rp_i) = \gamma^2$ for $j = 1, 2$.

Following Brock and Hommes, we assume that each trading type believes that the conditional variance of excess returns per share, $p_{i+1} + d_{i+1} - Rp_i$, will be constant over time and that each type has identical beliefs about the equilibrium value of conditional variance. Brock and Hommes use this assumption to simplify the problem of solving for the dynamics of the investor choice problem which will be examined imminently. There is little loss of generality in making this assumption. Gaunersdorfer (2000) extends Brock

**Assumption 2.2:** Stock market investors do not use any information other than past stock prices to forecast future stock prices. In particular, they do not forecast the future money supplies. Also, the money supply does not affect the stock market.

Thus, the stock market is completely insulated from the money market. Assumption 2.2 is a statement that stock market investors ignore complex correlations of the money supply with the stock price. Although large brokerage houses try to anticipate policy changes by the Federal Reserve, the investors in the Brock-Hommes model are small. It is unlikely that small investors in a competitive market would have the same resources to undertake analysis of monetary policy with the same depth as brokerages.

Brock and Hommes (1997,1998) assume that individual traders cannot determine the fraction of each type in the market. Even if traders knew that the money supply depended on the composition of traders as well as price bubbles it is unlikely that they could forecast future price bubbles and market compositions accurately from the money supply process we introduce in the next section. The reason for this is that agents would have to estimate both the diffusion and drift coefficients of a stochastic differential equation. Although an extensive literature exists on this subject, see for instance Dohnal (1987), it would require agents to use a very sophisticated predictor of the money supply. Such complexity is incompatible with the view that stock market investors are relatively unsophisticated. Thus, there is no loss of generality in assuming that the money supply is not forecasted by stock market traders and that forecasts of stock prices come from past stock prices. Brock and Hommes (1997,1998) use expectations which employ past prices to forecast future ones. The main implication of assumption 2.2 is that there is no need to consider expectations different from the Brock-Hommes trading type expectations. This preserves the resulting Brock-Hommes dynamics.

At each time $n$ traders are mean-variance maximizers. Consequently, each trader of type $j$ solves the following problem for $h_{jn}$, the number of shares to be held.

$$\max \{E_{jn}W_{n+1} - s/2V_{jn}(W_{n+1})\} \quad (2.2)$$
where $s$ is the risk aversion assumed to be constant across traders. The solution to the problem is $h_{jn} = E_{jn}(p_{n+1} + d_{n+1} - RP_n)/s\gamma^2$. This follows directly from calculus because we are able to treat all variables occurring at time $n$ as constants since they are known.

**Assumption 2.3:** The fundamental predictor is available at constant cost $C \geq 0$. This represents the cost of learning in markets.

Agents are assumed to change their trading type based on the fitness measure $\pi_{jn}$, the weighted sum of realized profits of choosing type $j$ up to time $n$. Brock and Hommes (1998) shows that this structure yields transition probabilities for types of traders $q_{nj}$, defined by $q_{jn} = \exp(\beta\pi_{j,n-1})/Z_n$ where $Z_n = \sum_{j=1,2} \exp(\beta\pi_{j,n-1})$ is a normalization. The parameter $\beta$ measures the intensity to switch predictors. This property continues to hold in our model because agents are forced to use past stock prices in their expectations. Again, this is one result of assumption 2.2. When agents cannot forecast the money supply, the Brock and Hommes (1998) dynamics are preserved because there is no source of information external to the stock market that could be used to predict the stock price.

Brock and Hommes make an important assumption. Let $p_n^\ast$ denote fundamental prices. They assume that all expectations have the form

$$E_{jn}(p_{n+1} + d_{n+1})|F_n) = E_{jn}(p_{n+1}^\ast + d_{n+1}|F_n) + g_j(p_n - p_n^\ast) \quad (2.3)$$

where $g_j$ is a parameter called "the trend" of trader type $j$. Clearly, for fundamentalists $g_1 = 0$; for noise traders we denote $g_2$ by $g$. The parameter $g$ determines the strength agents attach to deviations from the fundamental in the sense that a high value of $g$ means that agents expect a strong deviation from fundamentals to persist. Since $g \neq 0$ noise traders are trend chasers because persistent deviations from fundamentals are persistently extrapolated with intensity $g$. If $g < 0$ the agent is called a contrarian because positive deviations from fundamentals lead to an expected negative deviation in the next period.

**Definition 2.1:** An equilibrium is a sequence $((w_n, m_n))_n$ where $w_n = p_n - p_n^\ast$ and $m_n = q_{1n} - q_{2n}$ where $p_n, q_{jn} \quad j = 1, 2$ come from the choice problem described above.

The following theorem is the key result of Brock and Hommes (1997,1998). It is a precise description of stock market dynamics and will form the cornerstone of the endogenous fluctuations in the exchange rate discussed in section 4.
Theorem 2.1:

1) Let $C \geq 0$ and let $0 < g < R$, then the fundamental state for deviations from fundamentals and trader types $E_1 = (0, \tanh(-\beta C/2)) \equiv (0, m^*)$ is globally stable.

2) If $g > 2R$ and $C \geq 0$ there are two non-fundamental steady states, $E_2$ and $E_3$. $E_1$ is a steady state but is unstable for $g > 2R$. The non-fundamental steady states are defined by $E_2 = (a^*, m^*)$ and $E_3 = (-a^*, m^*)$ where $a^*$ is the positive solution of $\tanh(\beta/2(1/s\gamma^2(g - 1)(R - 1)(a^*)^2 - C)) = m^*$ and $m^* = 1 - 2R/g$.

3) For $R < g < 2R$ and $C > 0$, there exists $\beta^*$ satisfying $0 < \beta < \beta^*$ such that for $\beta < \beta^*$, $E_1$ is stable. For $\beta = \beta^*$ a pitchfork bifurcation occurs in which two more non-fundamental steady states are created. These states are $E_2$ and $E_3$ appearing above. If $\beta^* < \beta < \beta^*$ the fundamental steady state is unstable and the non-fundamental states are stable. For $\beta > \beta^*$ all steady states are unstable.

4) For $R < g < 2R$ and $m^* > m^e = \tanh(-\beta C/2)$, $E_1$, $E_2$ and $E_3$ are all steady states and $E_1$ is unstable. If $m^* < m^e$ then $E_1$ is the unique stable steady state.

5) If $g < -2R$, $E_1$ is unstable and there exists a two period cycle defined by $(a^*, m^*)$, $(-a^*, m^*)$ where $m^* = 1 + 2R/g$ and $a^*$ is the positive solution of $\tanh(\beta/2(-1/s\gamma^2 g(R + 1)(a^*)^2 - C)) = m^*$.

6) For $-2R < g < -R$ and $m^* < m^*$ then $E_1$ is the unique stable steady state. If $m^* > m^e$ then $E_1$ is unstable and there exists a two period cycle defined by $((a^*, m^*), (-a^*, m^*))$.

7) Let $\nu$ be the two period cycle defined above and suppose that $-2R < g < -R, C > 0$. Then $\nu$ is stable for $\beta \in (\beta^*, \beta^*)$ and unstable for $\beta > \beta^*$. At $\beta = \beta^*$ the model has an attractor consisting of two invariant circles around $E_2$ and $E_3$. Solutions are periodic or quasi-periodic with values jumping between the two circles.

The significance of theorem 2.1 is that small changes in the propensity to chase trends or switch predictors can have a major impact on the steady states generated by the model. Moreover, some of these steady states involve positive populations of noise traders. For instance, when $C \geq 0$ and $g > 2R$ but is close to $2R$ a small change in $g$ to $g'$ such that $R < g' < 2R$ and $m^* < m^e$ implies that $E_2$ and $E_3$ are no longer steady states. When $C > 0$ part 3 of theorem 2.1 implies that small changes in $\beta$ may alter the stability of $E_1$. From a practical standpoint this means that it may no longer be
possible to start solutions of the Brock-Hommes system close to $E_1$ and guarantee that they are similar to $E_1$ for all times if the propensity to switch predictors changes marginally. Also, if $C \geq 0$ and $g < R$ then $E_1$ is the only relevant steady state. If $m^* > m^*$ a small increase trend chasing to $g'$ such that $2R > g' > R$ will create two more steady states, $E_2$ and $E_3$. Thus, the economy could settle at more than one steady state and two of these involve positive populations of noise traders. Likewise, when agents are contrarians a small change in $g$ from above $-2R$ to $g' < -2R$ can introduce a periodic solution in addition to the fundamental steady state $E_1$. For very high propensities to change predictors part 7 implies that the model generates highly irregular paths.

Part two of the theorem implies that when the tendency to chase trends is strong, fundamentalists will not drive out noise traders even in the long run. One of the early critiques of noise trading was that fundamentalists would eventually drive out noise traders because noise traders would assume extra risk that would inevitably lead to their financial ruin. However, the Brock-Hommes results suggest that when trend chasing is a strong behavioural impulse noise trading can persist. Brock and Hommes (1997,1998) formalize the notion presented in Schliefer and Summers (1990) that noise traders will remain a potent force in the market.

The next step is to endogenize the money supply by linking it with the decisions of stock market traders. We will show that the Brock-Hommes results in Theorem 2.1 imply that small changes in the stock market can affect the exchange rate. In the exchange rate analysis to follow, we focus mainly on the exchange rate that prevails under a steady state or the periodic solution without the complex attractor. Whether the stock market is in a steady state or is described by an equilibrium which fluctuates over time there will be a unique exchange rate. However, changes in the process prevailing in the stock market will induce potentially sharp changes in the corresponding exchange rates.

3 The Exchange Rate Model

In this section, we adapt the model of Flood and Garber (1983). The Flood-Garber model is a monetary model of the exchange rate, meaning that the monetary conditions rather than trade flows determine the exchange rate. All variables evolve in continuous time.
The equilibrium exchange rate is the domestic price of foreign currency and its logarithm is denoted by \( x_t \); \( x_t' \) is the percentage change in the exchange rate and \( E(x_t' | \mathcal{F}_t) \) is the expected percentage change in the exchange rate. Logarithms of domestic income, the domestic price level of goods and the money supply are denoted by \( y_t, p_t^g \) and \( m_t \); \( v_t^1 \) is a domestic money market shock. Foreign counterparts of these variables are denoted by \( y_t^f, p_t^{fg}, m_t^f \); \( v_t^2 \) is a foreign money market shock. Except in the discussion of the economic structure of the model, we will not distinguish between any one of the variables and its logarithm e.g. we refer to \( x_t \) as the exchange rate. The domestic interest rate \( i_t \) is the difference between the bond return \( r \) and the interest rate paid on cash balances. The foreign interest rate \( i_t^f \) is defined analogously. The economic variables are related by the following equations:

\[
\begin{align*}
    m_t - p_t^g &= \alpha_0 + \alpha_1 y_t - \alpha_2 i_t + v_t^1 \\
    m_t^f - p_t^{fg} &= \beta_0 + \beta_1 y_t^f - \beta_2 i_t^f + v_t^2 \\
    p_t^g &= p_t^{fg} + x_t \\
    i_t &= i_t^f + E(x_t' | \mathcal{F}_t)
\end{align*}
\]

We assume that \( \alpha_1, \beta_1, \alpha_2, \beta_2 > 0 \) and that \( \alpha_2 = \beta_2 \) for simplicity.

Equations (1) and (2) are equilibrium conditions in the domestic and foreign money markets. They equate the supply of real balances to demand. To see how this arises, we treat the domestic and foreign money markets as being structurally identical. In other words, the two economies have the same structure of money demand function although the parameters between the economies may differ. Thus, we need only to derive equation (1). In the derivation we let \( M_t, P_t^g, Y_t, I_t \) be the money supply, price level, real income and nominal interest rate at time \( t \). That is, \( m_t = \log(M_t), p_t^g = \log(P_t^g) \) etc. We assume the domestic money demand function is given by \( \delta_0 Y_t^{\alpha_1} I_t^{-\alpha_2} \exp(v_t^1) \) where \( \delta_0 > 0 \). In equilibrium, money demand must equal the real money supply:

\[
\frac{M_t}{P_t^g} = \delta_0 Y_t^{\alpha_1} I_t^{-\alpha_2} \exp(v_t^1)
\]

Equation (1) follows by taking logarithms and setting \( \alpha_0 = \log(\delta_0) \).

---

2 The prices of goods are in nominal terms; stock prices are in real terms.
Equation (3) is a statement of purchasing power parity. The assumption of purchasing power parity is valid when markets are well integrated. Markets must be integrated enough to allow the free flow of goods and services across international boundaries.

Equation (4) is the interest parity condition. It holds when agents are risk neutral and when domestic and foreign assets are perfect substitutes. Interest parity means that the exchange rate change expected by the market equals the difference in the domestic and foreign interest rates. Consider the case where domestic and foreign assets are perfect substitutes and where investors in both countries are risk neutral. By definition, investors from either country must be indifferent to holding assets in one country or another given equal rates of return. However, if each asset pays a stream of returns in the currency of the country in which it is issued, returns must fluctuate with the exchange rate.

**Assumption 3.1:** We take as given a probability space \((\Omega, \mathcal{F}, P)\) and a collection of \(\sigma\)-algebras \(\mathcal{C} \equiv \{\mathcal{F}_t, t \in \mathbb{R}_+\}\). Each \(\mathcal{F}_t\) is a \(\sigma\)-algebra and \(\forall s < t, \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}\). Whenever a variable in the model is random we assume that it is defined on \(\Omega\) and is \(\mathcal{F}\)-measurable.

**Assumption 3.2:** The expected change in the exchange rate \(E(x'_t|\mathcal{F}_t)\) is the conditional expectation of \(x'_t\) given \(\mathcal{F}_t\) i.e. agents have rational expectations.

The collection \(\mathcal{C}\) is a mathematical representation of the flow of information over time. Under rational expectations, the \(\sigma\)-algebra \(\mathcal{F}_t\) will be the time \(t\) information set. Representing the information available for decision-making at time \(t\) by a \(\sigma\)-algebra is quite natural. For example, \(\mathcal{F}_t\) might contain past realizations of the interest rate i.e. sets of the form \(i^{-1}_s(A) = \{\omega : i_s(\omega) \in A\}\) \(s < t\). Also, whenever a condition holds almost everywhere we mean that it is true except possibly on a set of \(P\) measure 0.

To solve the model, rearrange equations 1 to 4 to yield

\[
m_t - m'_t - x_t = \alpha_0 - \beta_0 + \alpha_1 y_t - \beta_1 y'_t - \alpha_2 E(x'_t|\mathcal{F}_t) + v'_t - v_t^2
\]

Define the policy process \((k_t)\) by \(k_t = m_t - m'_t + \beta_0 - \alpha_0 + \beta_1 y'_t - \alpha_1 y_t + v_t^2 - v'_t\). Then the exchange rate satisfies \(x_t = k_t + \alpha_2 E(x'_t|\mathcal{F}_t)\). Although domestic and foreign income both appear in the equation for the government policy process \((k_t)\), their behaviour over time is not determined entirely by governments. However, governments may influence them via monetary and fiscal decisions.
Thus, we persist in interpreting \((k_t)\) as a policy process which includes the policies of both the domestic and foreign governments.

The equilibrium exchange rate is derived from standard techniques for solving rational expectations models, although as Froot and Obstfeld (1991 p. 242) point out, the exchange rate expression holds for any structure of expectations.

\[
x_t = a^{-1} \int_t^\infty \exp\left(\frac{t-s}{\lambda}\right) E(k_s|\mathcal{F}_t) \, ds \quad (3.1)
\]

Now, we consider the policy process denoted by \((k_t^\mu)\).³

**Assumption 3.3:** Government knows the composition of stock market traders and deviations from fundamentals. Its policy, \((k_t^\mu)\), will satisfy the following diffusion process,

\[
dk_t^\mu = b(t, \mu_t, k_t^\mu) \, dt + \sigma(t, \mu_t, k_t^\mu) \, dW_t \quad \text{or equivalently}
\]

\[
k_t^\mu = y_{\mu_0} + \int_0^t b(s, \mu_s, k_s^\mu) \, ds + \int_0^t \sigma(s, \mu_s, k_s^\mu) \, dW_s \quad (3.2)
\]

where \(\mu \equiv (\mu_s)\) is a set of tuples describing evolution in the stock market defined by \(\mu_s = (w_s, m_s)\) where \(w_s = w_n, m_s = m_n\) for \(n \leq s < n + 1\) and \(y_{\mu_0}\) is the initial condition which depends on \(\mu_0\).

The class of equations described by (3.2) is a new class of stochastic differential equations. The integrands depend on measures that arise from agents' equilibrium choices. Conditions for the existence and uniqueness of this class of equations is discussed in the section 6; here we simply concentrate on the motivation for these equations. Researchers have often assumed that key economic variables follow a diffusion process, see Duffie (1992). However, these processes are usually given exogenously without much behavioural foundation for their introduction. The function \(b\) is the government's response function; \(b(s, \mu_s, x)\) is the instantaneous change in the money supply at time \(s\) for conditions \(\mu_s\) prevailing in the stock market when \(x\) is the money supply. Equation (3.2) says that the government examines the distribution of noise traders and fundamentalists as well as deviations from fundamentals in setting the money supply.

³Flood and Garber originally argued that before pegging, the freely floating exchange rate could be represented by a Brownian Motion with drift: \(k_t = \eta t + \sigma W_t\).
The assumption that government knows the composition of traders so that it can set market driven policies is highly stylized. However, we would expect government to know more about the state of the market than individual traders. In our case the government is perfectly informed. In the real world, governments do not make announcements about the composition of the market, thus we assume that the government does not publicize its information.

Likewise, the assumption that the government considers stock market equilibria in setting the money supply is stylized, particularly when we assume that the money supply has no effect on stock prices. In the real economy it is more plausible that money supply changes affect real investment more than stock prices and perhaps monetary policy is governed more by real than financial considerations. Nevertheless, the stock market crash of 1987 did prompt central banks around the world to cut interest rates. Thus, in the real world, stock market fluctuations may induce policy changes even when the link between policy and the market is indirect. Even though the government cannot affect the stock market in our model, we still assume that stock market considerations affect the money supply for reasons of tractability. This is consistent with observed policy at some times.

It may even be the case that government overestimates the effect it will have on the market via its money supply. If government makes policy under the mistaken belief that the money supply affects the stock market, our results apply over the time frame before it realizes its error and changes course. During the time of erroneous belief, the government might suspect that with a high percentage of noise traders any increase in the money supply will result in more speculation than with a high proportion of fundamentalists. The government might want to dampen speculation. Consequently, the rate of money growth will be lower than with only fundamentalists in the market. Alternatively, the government might want to maintain a certain level of stock prices because a depressed stock market might affect consumption expenditure. In this case, the government will fear underevaluation, not overevaluation, and it will adjust the money supply to support prices if they fall below the fundamental price.

By allowing $\sigma$ to depend on stock market conditions, we obtain a very general equation. However, we will shortly assume that $\sigma$ will depend only on $t$. It is not possible to prove precise fluctuation results when the noise trading affects $\sigma$ for technical reasons, see section 6 for details. However, completely endogenous noise and its effects could form a new research question; this is
why we have discussed the more general integrand.

When the equilibrium policy process is \( k^t \) the exchange rate is denoted by \( x_t \). The next section discusses the model in more detail and explores the resulting price dynamics.

4 Exchange Rate Dynamics

The reason that (3.2) is a new class of stochastic equations is that the initial condition depends on the process \( \mu_0 \). In addition, the integrands \( b \) and \( \sigma \) depend on \( \mu \). As the appendix shows, proving that (3.2) possesses well-known properties of stochastic differential equations (SDE) requires some variation in the usual assumptions and proofs. From now on, we will refine the model by assuming that \( \sigma \) depends only on \( t \). Under this assumption the government sets the instantaneous change in the money supply via the function \( b \). Randomness comes from the stochastic integral \( \int_0^t \sigma(s) dW_s \) and the presence of \( k^\mu_s \) in the integrand \( b \). All proofs in this section are in the appendix.

Assumption 4.1: Let \( \mu_s \neq \nu_s \) \( \forall \)s for stock market equilibria \( \mu, \nu \). Then there is a constant \( K \) such that \( \forall \)s, \( f, g \)

\[
|b(s, \mu_s, f) - b(s, \nu_s, g)|^2 \leq K|\mu_s - \nu_s|^2 |f - g|^2 \quad (4.1)
\]

The implication of assumption 4.1 is that differences in the instantaneous growth of the money supply are constrained by differences in the stock market equilibria. Assumption 4.1 is similar to the Lipschitz growth conditions needed to prove existence and uniqueness of SDE. The condition presented here is somewhat stronger in that given \( f \) and \( g \) small differences between \( \mu_s \) and \( \nu_s \) will tend to really restrict the difference in money growth. If the right hand side of (4.1) was \( |\mu_s - \nu_s|^2 + |f - g|^2 \), small differences in stock market equilibria would have less effect on money growth differences. Letting differences in stock market equilibria affect money growth differentials as in (4.1) means that the government does not alter monetary policy radically for small changes in the stock market. This is particularly interesting in our model since small stock market changes will have a potentially greater effect on the exchange rate. Assumption 4.1 ensures that this effect will come from the endogenous behaviour of agents, not the erratic actions of government.
Assumption 4.2: Let $\mu = ((w^1_s, m^1_s)), \nu = ((w^2_s, m^2_s))$ be stock market equilibria such that $\mu_0 \neq \nu_0$. Then $y_{\mu_0} \neq y_{\nu_0}$. In particular, if $w^1_s < w^2_s \ \forall s, \ y_{\mu_0} > y_{\nu_0}$.

The meaning of assumption 4.2 is that the government distinguishes between different equilibria in the stock market when it sets the initial condition. In this sense, the government responds to different conditions in the stock market by creating different monetary conditions. Assumptions 4.1 and 4.2 imply Theorem 4.1.

Theorem 4.1: Let $\mu_s \neq \nu_s \ \forall s$ where $\mu, \nu$ are equilibria from the stock market. Then almost everywhere for all $t$ simultaneously $k^\mu_t \neq k^\nu_t$.

Thus, different equilibria in the stock market generate different money supplies at all times. This is the reason that the money supply is endogenous in our model. By allowing the instantaneous rate of money growth to depend on asset price equilibria, the money supply responds to changes in the optimal choices of agents.

Previous researchers have addressed a variety of questions in the monetary model proposed by Flood and Garber (1983). For examples see Ikeda and Shibata (1995) and Froot and Obstfeld (1991). Virtually all these papers assumed that the money supply was a Brownian motion with drift when the exchange rate was not fixed by government. Endogenizing the money supply in Flood and Garber’s model is interesting because it will provide a tractable way of forming a theory fluctuations for exchange rates in random economies along the lines of deterministic bifurcation theory. Other stochastic bifurcation theories are possible, but they are much more complex, see Arnold (1998) for a comprehensive discussion.

The papers cited above solve complex mathematical problems related to the exchange rate, but they make minimal use of choice theory. At the other extreme lie models where the exchange rate is determined by the supply and demand of traded goods. The exchange rate is related to the Euler conditions governing agents’ choice problem. However, it is frequently the case that the exchange rate cannot be determined from the first order conditions. Researchers have to rely on computer simulations to derive the behaviour of the exchange rate, see Arifovic (1996) for an example.

It might be argued that we have only partially endogenized the money supply because the government does not solve an explicit choice problem and because markets are segmented. Another deficiency is that there is no
feedback from the exchange rate to the money supply. With these considerations, our paper is only a step toward a fully endogenous money supply and exchange rate. However, given the difficulty the trade flow based approach has encountered in finding an explicit exchange rate equation, our model is a step in the right direction. The issue of endogeneity will be briefly discussed again after assumption 4.3.

**Assumption 4.3:** Let \( \mu = ((w^1_s, m^1_s)) \) and \( \nu = ((w^2_s, m^2_s)) \) be stock market equilibria such that \( w^1_s < w^2_s \) \( \forall s \). Then for any \( v, c \) \( b(s, \mu, v) \geq b(s, \nu, c) \).

Assumption 4.3 really is a policy rule for government. It states that the government's objective is to respond negatively to overvaluation as well as positively to undervaluation because the function \( b \) provides for lower money creation in the former instance and higher creation in the latter instance. To see this, take \( \nu \) as the equilibrium with a positive speculative bubble, the condition on \( b \) implies that the rate of money growth will be lower than with equilibrium \( \mu \). Of course, with \( \mu \) there is a smaller deviation from the fundamental provided that \( w^1_s \geq 0 \). Analogous reasoning applies for a negative bubble. Even though the objective of discouraging excess speculation may not come from an optimization problem, it is a reasonable objective for government.

**Theorem 4.3:** Suppose that assumption 4.3 holds. Let \( \mu = ((w^1_s, m^1_s)) \) and \( \nu = ((w^2_s, m^2_s)) \) be stock market equilibria with \( w^1_s < w^2_s \) \( \forall s \). Then almost everywhere for any \( t, x^\mu_t > x^\nu_t \).

**Remark:** Brock and Hommes (1997,1998) establish the existence of several possible steady states in their model. For certain configurations of parameters more than one steady state is possible, but they present no method of determining which steady state will actually be observed. There may be other solutions which fluctuate over time for which no exact expressions are known. Some parameter values generate a periodic solution for which an exact expression is known. Whether the stock market is in a steady state or is described by an equilibrium which fluctuates over time there is a unique exchange rate given by (3.2). Theorem 4.3 illustrates differences in these equilibria for specific variations in the stock market processes. Clearly, under assumptions 4.1-4.3, theorem 4.3 is also valid if \( \mu \) and \( \nu \) are steady states. Thus, the three steady states appearing in Brock and Hommes (1997,1998) correspond to three exchange rates that may be compared via theorem 4.3.
Theorem 4.3 is a fundamental theorem of the paper. When coupled with the Brock-Hommes results, it establishes that instability spills over from the stock market to the foreign exchange market. While theorem 4.3 establishes that prices will be distinct, it does not guarantee that the fluctuations will be large. However, it does predict inequalities for the exchange rate. The theorem does prove that the following scenario arises. Suppose that we make a small change in the trend parameter \( g \). If \( g < R \) but is close to \( R \), then \( E_1 \) is a steady state. Let the small change result in \( g' > R \) and assume that \( m^* > m^\epsilon \). Then \( E_2 \) and \( E_3 \) are also steady states. Clearly, \( E_2 \neq E_1 \) and \( E_3 \neq E_1 \). Let \( \mu_1^0 = (0, m^\epsilon), \mu_2^0 = (a^*, m^\epsilon) \) and \( \mu_3^0 = (-a^*, m^\epsilon) \) \( \forall s \). Then theorem 4.3 implies that \( x_t^{\mu_1^0} > x_t^{\mu_2^0}, x_t^{\mu_2^0} > x_t^{\mu_3^0} \). Conceivably, the fluctuation could be quite large, but we cannot be certain. Other assumptions are necessary to establish that the change is large. However, this discussion contains the essential reasoning behind the stability theorem linking the stock and foreign exchange markets presented below. Before turning to this result there are other issues to be addressed.

**Definition 4.1:** Let \( \mu = ((w^1_s, m^1_s)) \) be a stock market equilibrium. The exchange rate process \( (x_t^\mu) \) is stochastically stable at \( t \) or synonymously, \( S \)-stable at \( t \), if for every \( \epsilon > 0 \) there exists a \( \delta = \delta(\epsilon, t) \) such that if \( \nu \) is any stock market equilibrium with \( |\nu - \mu| < \delta \) \( \forall s \), then \( |Ex_t^\nu - Ex_t^\mu| < \epsilon \). If the exchange rate process is \( S \)-stable for all \( t \) we say that the exchange rate is \( S \)-stable.

The definition of an \( S \)-stable exchange rate is motivated by the definition for a stable deterministic dynamical system. Recall that a deterministic dynamical system \( (y_t) \) solving the problem \( z_t = f(z_t) \) is stable if for any \( \epsilon \) there exists a \( \delta = \delta(\epsilon) \) such that \( |y_0 - v_0| < \delta \) implies \( |y_t - v_t| < \epsilon \) \( \forall t > 0 \) for any other solution \( (v_t) \). Definition 4.1 is similar in that an exchange rate is \( S \)-stable if small changes in stock market equilibria result in small changes in the expected value of the exchange rate. Unlike the deterministic definition, the random definition requires that \( \delta \) also depend on \( t \). The reason for this is that from a practical standpoint \( x_t^\mu \) inherits the diffusion properties of \( k_t^\mu \). Consequently \( x_t^\mu \) exhibits tremendous fluctuation at large times and it becomes impossible to prove that one bound \( \delta(\epsilon) \) would suffice for all \( t \). Clearly, we may also consider \( S \)-stability of \( \mu \) if \( \mu \) is a steady state.

The concept of \( S \)-stability is useful for describing the properties of exchange rate equilibria in the following sense. It is known that the stock market exhibits different steady states for certain parameter values; each of
these steady states generates an exchange rate. If $\nu$ is a stock market equilibrium different from all the steady states it is desirable to compare $x_t^\nu$ with the exchange rates arising from the steady states. Theorem 4.3 allows us to establish inequalities but it is interesting to know if a small deviation from a steady state leads to a small deviation in the exchange rate. The reason is that we want a model where exchange rate fluctuations are caused by endogenous changes in the stock market, not the volatility of the Brownian motion driving the money supply. As we have argued above, small changes in stock market parameters can produce significant fluctuations in the exchange rates corresponding to the steady states. If $x_t^\mu$ is stable then only changes in the stock market parameters can create exchange rate instability.

Different parameter values affect the number of possible steady states. Although there is a unique exchange rate path for each steady state, theorem 4.3 implies that these paths are different. When a small change in trend chasing affects the number of possible steady states, it also affects the number of corresponding paths for the exchange rate. This is a form of sensitivity to initial conditions. This is the substance of the theorem 4.4, which is a version of theorem 2.1 for exchange rates. For convenience, an exchange rate generated by a steady state will be called a steady state exchange rate. For notation in the next theorem the reader should refer to theorem 2.1.

**Theorem 4.4:** Let $\mu^i$, $i = 1, 2, 3$ be the stock market steady states defined by $\mu_1^s = (0, m^e), \mu_2^s = (a^*, m^*), \mu_3^s = (-a^*, m^*)$ $\forall s$. Suppose that the initial condition $y_{\mu_\theta}$ is continuous in $\mu_0$. Assume the stock market is in a steady state. Then the following hold:

1) For $0 < g < R$, $x_t^\mu$ is an S-stable exchange rate.

2) For $g > 2R$ the corresponding exchange rates $x_t^\mu$, $x_t^2$, $x_t^3$ satisfy $x_t^\mu > x_t^2$ and $x_t^\mu < x_t^3$.

3) For $R < g < 2R$ and $m^* > m^e, x_t^\mu, x_t^2, x_t^3$ satisfy the inequalities presented above. If $m^* < m^e$ only $x_t^\mu$ can occur because $\mu_1$ is the only steady state and it is S-stable.

4) For $C > 0$ and $\beta < \beta'$, $x_t^\mu$ is an S-stable exchange rate. For $\beta' < \beta < \beta^*, x_t^2$ and $x_t^3$ are S-stable.

For convenience denote $x_t^\mu$ by $x_t^\nu$. The model exhibits instability in the sense that small changes in either $g$ or $\beta$ can affect the number of exchange rates corresponding to steady states. For $g < R$ there is only one exchange rate corresponding to a steady state, $x_t^\nu$. If we move slightly above $R$ and
For $g > 2R$ there are three rates generated by steady states $x_i^i, i = 1, 2, 3$. If $g$ falls slightly below $2R$ and $m^* < m^*$ there is only one such rate, $x_i^1$. Since $x_i^2 < x_i^1 < x_i^3$ it is evident that small changes in trend chasing may cause either depreciation or appreciation relative to the other exchange rates generated by steady states. Similarly, when $C > 0$ a small movement in $\beta$ above $\beta'$ creates two other possible steady state exchange rates, $x_i^2$ and $x_i^3$. Small changes in the propensity to switch predictors can also induce either exchange rate appreciation of depreciation. If the deviations in $x_i^i$ are substantial, small changes in trend chasing or the propensity to switch predictors would create large exchange rate jumps.

The fact that exchange rate appreciation or depreciation could follow changes in the propensity to chase trends or alter predictors is interesting. It follows because increased trend chasing will perpetuate positive and negative bubbles; in turn this has effects on the money supply growth and the exchange rate. As the propensity to change predictors increases there may be greater tendency to act as noise traders and to deviate from fundamental prices: again this feeds through to the money supply and exchange rate. Similar analysis applies for the case $g < 0$ as the next result shows.

**Corollary 4.1:** Suppose that assumptions 4.2-4.3 are modified so that if a stock market solution exhibits overvaluation or undervaluation in all periods, increases in the money supply are lower than with the fundamental solution $\mu_1$. Then

1) Let $\nu$ be the stock market equilibrium defined by the two period cycle and let $g < -2R$. The two corresponding exchange rates $x_i^{\mu_1}$ and $x_i^{\nu}$ satisfy $x_i^{\mu_1} > x_i^{\nu}$.

2) For $-2R < g < -R$ and $m^* < m^*$, $x_i^1$ is S-stable. If $m^* > m^*$ there are two corresponding rates, $x_i^1$ and $x_i^2$ satisfying the inequality above.

3) Let $\gamma$ be a periodic or quasi-periodic solution which alternates between the circles described in theorem 2.1 part 7. Suppose assumptions 4.2-4.3 are modified so that the money supply with $\gamma$ grows faster than with $\mu_2$ because of its negative bubbles but lower than with $\mu_3$ because of its positive bubbles. Then $x_i^3 > x_i^2 > x_i^1$.

If there are other solutions $\phi$ and $\psi$ for the stock market, the stability of the various steady states allows us to compare $x_i^{\psi}$ and $x_i^{\phi}$. For $g < R$ consider $\phi = ((w_i^1, m_i^1))$ with initial condition sufficiently close to $\mu_0^1$. We have $|\mu_i^1 - \phi_i| < \epsilon \forall t$. Consider a small deviation in trend chasing to $g'$
where \( R < g' < 2R \) and for a solution \( \psi = ((w_i^2, m_i^2)) \) with initial condition sufficiently close to \( \mu_0^2 \) we have \( |\mu_0^2 - \psi_1| < \epsilon \). Suppose that \( \epsilon < \delta < a^* - \epsilon \), then \( w_1^1 < \delta < w_2^2 \) and clearly \( x_i^\psi < x_i^\phi \). Since \( 0 < a^* \) such \( \delta \) and \( \epsilon \) must exist. Thus, theorem 4.4 provides a vehicle for predicting inequalities in the exchange rates corresponding to stock market solutions which are not steady states. This does not guarantee a large fluctuation in \( x_i^\phi \) and \( x_i^\psi \). However, the notion of S-stability may be used to quantify fluctuations in \( E x_i^\phi \) and \( E x_i^\psi \).

Let \( x_i^1 - x_i^2 > \theta \). Then for some \( \epsilon, x_i^1 - x_i^2 - 2\epsilon > \theta \). If \( x_i^2 + \epsilon + \theta < x_i^1 - \epsilon \) then \( E x_i^2 + \epsilon + \theta < E x_i^1 - \epsilon \). Suppose that \( |\phi_t - \mu_1^1| < \delta, |\psi_t - \mu_2^2| < \delta \) imply that \( |E x_i^\psi - E x_i^\phi| < \epsilon, |E x_i^\phi - E x_i^\psi| < \epsilon \). It follows that \( E x_i^\psi + \theta < E x_i^2 + \theta + \epsilon < E x_i^1 - \epsilon < E x_i^\phi \). Consequently, \( E x_i^\phi - E x_i^\psi > \theta \). A sharp fluctuation in \( x_i^1 \) and \( x_i^2 \) will cause a sharp fluctuation in \( E x_i^\phi \) and \( E x_i^\psi \). Now, we present a simple condition on the money growth function which can guarantee sharp deviations in exchange rates.

**Theorem 4.5:** Let \( b(s, \mu_s, u) = b(s, \mu_s) \) and suppose that for stock market equilibria \( \mu = ((w_1^1, m_1^1)), \nu = ((w_2^2, m_2^2)) \) satisfying \( w_2^2 - w_1^1 > \delta \) we have \( b(s, w_1^1) - b(s, w_2^2) > \delta \). Then \( x_i^\mu - x_i^\nu > \delta \).

Under the conditions of theorem 4.5, differences in \( p_t - p_i^* \) have a direct effect on the exchange rate. In the case of the steady states theorem 4.5 also applies and the magnitude of \( a^* \) will play a critical role in deviations of \( x_i^, i = 1, 2, 3 \).

Consider a small variation of the model where agents can only make finitely many choices over their trading types. Assume that agents are allowed \( n \) choices before their type is fixed for all future trading periods. Assume that for times \( k > n \) all agents are rational. The Brock-Hommes model yields a finite system of equations for the proportions of traders and deviations from fundamentals for the first \( n \) periods. These periods characterize the equilibrium because all agents are rational after time \( n \) implying that there are no deviations from fundamentals. However, since \( E_i, i = 1, 2, 3 \) were fixed points of the discrete time dynamical system defined by denumerably many trades, they must be admissible solutions for the finite system. Suppose that assumption 4.1 is modified as follows. When \( \mu_s \neq \nu_s, |b(s, \mu_s, f) - b(s, \mu_s, g)| \leq K|\mu_s - \nu_s||f - g| \), but if \( \mu_s = \nu_s \) \( \forall s \geq n \) then \( |b(s, \mu_s, f) - b(s, \nu_s, g)| \leq K|f - g| \). Then the proof of theorem 4.3 also yields the following result.

**Theorem 4.6:** Let \( \mu = ((w_1^1, m_1^1)), \nu = ((w_2^2, m_2^2)) \) with \( w_i^1 < w_i^2 \) for
Hence, theorem 4.4 is still applicable when all agents convert to rationality in finite time. Thus, the exchange rate level remains as sensitive as before to the initial equilibrium in the stock market. The switch to rationality could occur when \( n = 1 \), i.e. when agents make only one choice. Since integer trading times were only chosen for convenience, time \( t_1 \) when all agents make the conversion could be arbitrarily close to 0, the time when agents can elect to be noise traders. Thus, trend chasing has a permanent effect on the exchange rate level.

It must be emphasized that theorem 4.6 follows both from an economic assumption as well as the theorems of stochastic calculus. On the economics side when \( \mu \) has finitely many components less than \( \nu \) the money supply grows at a faster rate and begins at a higher level i.e. \( b(s,\mu,x) \geq b(s,\nu,y) \) and \( y_{\mu 0} > y_{\nu 0} \). This can allow for equal growth rates once \( \mu \) and \( \nu \) have common components. Adaptation of the usual proofs of stochastic calculus implies \( k^\mu_t > k^\nu_t \forall t \). The appendix shows that when this is true \( x^\mu_t > x^\nu_t \forall t \).

5 Conclusions

An important part of the literature in economic dynamics shows that prices exhibit chaos or bifurcation phenomena. These instability results appear in non-random economies, see Grandmont (1985) or Matsuyama (1991) for examples. However, the instability results presented here occur in a stochastic economy. Brock and Hommes (1997) observe that the numerical patterns generated by chaotic models do not resemble actual economic data. However, instability appears to be a trait of the real world economy. It is thus important to develop a stability theory for stochastic economies and this paper is a move in that direction. The blending of random and deterministic structures shows that endogenous fluctuations can exert an important influence on stability despite the effects of randomness.

The model presented here demonstrates that small changes in parameters linked to noise trading in the stock market generate potentially large fluxes in exchange rates when agents have rational expectations. Clearly, randomness is not the driving force behind the exchange rate instability discussed here. The exchange rate instability arises as a carryover from the instability generated by deterministic dynamics in Brock and Hommes (1997,1998). Our paper suggests that noise traders can have an important impact on the economy via indirect channels. Future research in this area could address
equilibria which result when agents have other learning devices, such as the genetic algorithm considered in Arifovic and Gencay (2000) or the neural networks considered in Heinemann (2000).

6 Appendix: Mathematical Results

Let $\mathcal{M} \equiv \mathcal{M}(R^2)$ denote the space of stock market equilibria; $\mathcal{B}(\mathcal{M})$ will denote the $\sigma$-field for $\mathcal{M}$ inherited from countably many products of $\mathcal{B}(R^2)$. We assume that each sequence in $\mathcal{M}$ has bounded components. We establish some general results for stochastic differential equations which depend on elements of $\mathcal{M}$ before turning to the special one dimensional case of (3.1).

**Definition 6.1** Let $b: R^+ \times R^d \times \mathcal{M}(R^2) \rightarrow R^d$ and $\sigma$ a $d \times d$ matrix with components $\sigma_{i,j} : R^+ \times R^d \times \mathcal{M}(R^2) \rightarrow R$. The components of $b$ and $\sigma$ will be measurable relative to $\mathcal{B}(R^+) \times \mathcal{B}(R^d) \times \mathcal{B}(\mathcal{M})$ and $\mathcal{B}(R)$. Let $|\sigma(s, x, \mu)|^2 = \sum_{i,j} \sigma_{i,j}^2(s, x, \mu)$. Then a solution to the stochastic differential equation $dX^\mu_t = b(t, X^\mu_t, \mu_t) \, dt + \sigma(t, X^\mu_t, \mu_t) \, dW_t$ is a continuous, $(\mathcal{F}_t)$ adapted process such that:

1) $E \int_0^T |b(s, X^\mu_s, \mu_s)|^2 \, ds + E \int_0^T |\sigma(s, X^\mu_s, \mu_s)|^2 \, ds < \infty$

2) $X^\mu_t = \eta_\mu_0 + \int_0^t b(s, X^\mu_s, \mu_s) + \int_0^s \sigma(s, X^\mu_s, \mu_s) \, dW_s$ almost everywhere $\forall t$. This means $X^\mu_t = \eta_{\mu_0,t} + \int_0^t b_i(s, X^\mu_s, \mu_s) + \sum_{j=1}^d \int_0^t \sigma_{i,j}(s, X^\mu_s, \mu_s) \, dW^j_s$

The term $\eta_{\mu_0}$ is the initial condition; we assume $\eta_{\mu_0,i} \in L^2(\Omega, \mathcal{A}, P)$ $\forall i$. As usual $(W_t)$ is a $d$-dimensional Brownian Motion. Theorem 6.1 follows from the well known method of Picard iteration. In fact, Theorems 6.1-6.2 follow because standard methods of proof are applicable to this class of SDE.

**Theorem 6.1:** Let $b, \sigma$ satisfy the measurability requirements above as well as the following Lipschitz conditions

1) $|b(s, x, \mu_s) - b(s, y, \mu_s)|^2 + |\sigma(s, x, \mu_s) - \sigma(s, y, \mu_s)|^2 \leq K(|x - y|^2)$

2) $|b(s, x, \mu_s)|^2 + |\sigma(s, x, \mu_s)|^2 \leq K(|x|^2 + |\mu_s|^2)$

Then, there exists a unique solution to the SDE.

**Definition 6.2:** The function $\eta$ is perfect if $\mu_0 \neq \nu_0 \Rightarrow \eta_{\mu_0} \neq \eta_{\nu_0}$ a.e.

Definition 6.2 generalizes assumption 4.2 to random initial conditions. Theorems 4.1-4.4 also holds under the modification to assumption 4.1 discussed in the text with slight modifications of the proofs to be presented here.
**Definition 6.3:** The flows of the SDE are the functions \( (X_t^u(\omega), t \geq 0) \) indexed by \( \omega \) and \( u \). The flows are weakly injective if \( \forall \mu, \nu \) such that \( \mu_s \neq \nu_s \) \( P(\omega : \exists t \text{ such that } X_t^u(\omega) = X_t^\nu(\omega)) = 0 \).

We cannot guarantee weak injectivity of the flows for an arbitrary matrix \( \sigma \). However we can prove weak injectivity for \( \sigma \) depends only on \( t \). In this case, the components of \( X_t^u \) are

\[
X_t^{\mu,i} = \eta_{\mu_0} + \int_0^t b_i(X_s^\mu, \mu) \, ds + \sum_{j=1}^n \int_0^t \sigma_{i,j}(t) W_s^j
\]

The following result is a variation of an argument found in Protter (1990).

**Proof of Theorem 4.1:** We show the stochastic flows are weakly injective for perfect \( \eta \). Define \( U_t = X_t^\mu - X_t^\nu \). By construction a given solution is a continuous semi-martingale, implying that \( (U_t) \) is a continuous semi-martingale relative to \( (\mathcal{F}_t) \). Under the assumption of right-continuity, \( \mathcal{F}_0 \) containing all null sets and continuity of \( (U_t) \), the hitting time of \( B \in \mathcal{B}(R^d) \) \( \tau_B = \inf \{t \geq 0 : U_t \in B\} \) is a stopping time. If the set is empty, we take \( \tau_B(\omega) = \infty \). In terms of the economic model, we now deal with an infinite horizon. With the Lipschitz conditions we introduced, solutions for the exist on \([0, \infty)\).

Define \( \tau = \inf \{ t \geq 0 : U_t = 0 \} \) and \( \tau_n = \inf \{ t \geq 0 : |U_t| \leq \frac{1}{n} \} \wedge n \) where \( x \wedge y = \min(x, y) \). Now, \( X_0^\mu = \eta_{\mu_0}, X_0^\nu = \eta_{\nu_0} \). By perfection of \( \eta, \tau > 0 \) a.e. This follows because \( U_\tau = 0 \) and \( U_0 = \eta_{\mu_0} - \eta_{\nu_0} \). Now, \( \tau_n < \tau \) \( \forall n \). Clearly this is true when \( \tau = \infty \). Obviously, \( \tau < \infty \iff \{ t : U_t = 0 \} \neq \emptyset \). In this case \( \{ t : U_t = 0 \} \subset \{ t : |U_t| \leq \frac{1}{n} \} \Rightarrow \tau \geq \tau_n \). We must have strict inequality almost everywhere on this set (particularly when \( \tau > 0 \)), for otherwise \( U_{\tau_n} = 0 \) \( \tau > 0 \). Let \( Z_n < \tau = \tau_n, z_n \uparrow \tau_n \). It must be the case that \( |U_{z_n}| > \frac{1}{n} \), but continuity requires \( \lim_{n \to \infty} |U_{z_n}| = 0 \). Hence \( \tau_n < \tau \) almost everywhere.

In fact, \( \tau_n \uparrow \tau \) almost everywhere. Clearly, \( \tau_n \uparrow H = \sup \tau_n, n \geq 1 \leq \tau \).

If \( H < \tau \) then \( \tau < \infty \), for otherwise \( U_t \neq 0 \) \( \forall t \) but \( U_H = 0 \) if \( H < \infty \). So, we have \( H < \tau < \infty \Rightarrow U_H = 0 \), implying a contradiction.

Define a continuous semi-martingale by \( U_t^n = U_t \wedge \tau_n \). Since \( \tau_n < \tau \), we may apply Ito's lemma with our \( f \in C^2(R^d - \{0\}) \) equal to \( \log(|x|) \). Hence,

\[
\log(|U_t^n|) - \log(|\eta_{\mu_0} - \eta_{\nu_0}|) = \sum_{i=1}^d \int_0^t \frac{U_t^{n,i}}{|U_t^n|^2} (b_i(X_s^\mu \wedge \tau_n, \mu_s) - b_i(X_s^\nu \wedge \tau_n, \nu_s)) \, ds
\]
Using the Lipschitz conditions, when $\tau < \infty$, the sum of integrals is bounded by $C(t \wedge \tau) \leq C\tau$. To see this, observe that

$$\left| \frac{U_s^{n,i}}{U_s^n} \right| |b_i(X_s^{\mu \wedge \tau_s, \mu_s}) - b_i(X_s^{\nu \wedge \tau_s, \nu_s})| \leq K\left| \frac{U_s^{n,i}}{U_s^n} \right| |\mu_s - \nu_s| \leq K|\mu_s - \nu_s|$$

because $|U_s^{n,i}| \leq |U_s^n|$. Hence,

$$\sum_{i=1}^{d} \int_0^{t \wedge \tau_s} \left| \frac{U_s^n}{U_s^n} \right| b_i(X_s^{\mu \wedge \tau_s, \mu_s}) - b_i(X_s^{\nu \wedge \tau_s, \nu_s}) \, ds \leq$$

$$\sum_{i=1}^{d} \int_0^{t \wedge \tau_s} K|\mu_s - \nu_s| \, ds \leq C(t \wedge \tau_s)$$

This inequality holds almost everywhere $\forall n$ simultaneously, say on $A$, where $P(AC) = 0$. If $\tau(\omega) < \infty$ for some $\omega \in A$, then $C(t \wedge \tau_s(\omega)) \leq C\tau(\omega)$. Let $t, n \to \infty$, then \( \log(|U_t^n|) - \log(|\eta_{\nu_0} - \eta_{\mu_0}|) \to -\infty \), implying a contradiction. Hence, $B = \{ \omega : \text{such that } X_t^\mu(\omega) = X_t^\nu(\omega) \} \subset AC$ for $\tau(\omega) = \infty \Rightarrow X_t^\mu(\omega) \neq X_t^\nu(\omega) \ \forall t$. Completeness of $P$ on $A$ $\Rightarrow B \in A$ and that $P(B) = 0 \ \text{ q.e.d.}$

**Proof of Theorems 4.3 and 4.4**: We have $y_{\mu_0} > y_{\nu_0}$ and $b(s, \mu_s, u) \geq b(s, \nu_s, u)$. The proof of the comparison theorem found in Protter (1990) holds so that $k_\mu^s \geq k_\nu^s \ a.e. \ \forall s$. But theorem 4.1 implies $k_\mu^s \neq k_\nu^s \Rightarrow k_\mu^s > k_\nu^s$. This implies that $E(k_\mu^s | F_t) > E(k_\nu^s | F_t) \ a.e.$, for let $A_s = \{ E(k_\mu^s | F_t) > E(k_\nu^s | F_t) \} \in F_t$ then $\int_{A_s} k_\mu^s \, dP = \int_{A_s} k_\nu^s \, dP$. Hence $\int_{A_s} k_\mu^s - k_\nu^s \, dP = 0 \Rightarrow (k_\mu^s - k_\nu^s)1_{A_s} = 0 \ a.e$. Since $A_s = \{ (k_\mu^s - k_\nu^s)1_{A_s} \neq 0 \}$ we must have $P(A_s) = 0$ implying $E(k_\mu^s | F_t) > E(k_\nu^s | F_t) \ a.e$. Setting $\alpha = 1$ for convenience,

$$x_t^\mu = \int_t^\infty \exp(t - s)E(k_\mu^s | F_t) \, ds > \int_t^\infty \exp(t - s)E(k_\nu^s | F_t) \, ds = x_t^\nu$$

Otherwise $\int_t^\infty \exp(t - s) [E(k_\mu^s | F_t) - E(k_\nu^s | F_t)] \, ds = 0$ implying for almost all $s$ we have $E(k_\mu^s | F_t) = E(k_\nu^s | F_t)$ q.e.d.

**Proof of Theorem 4.5**: All results are immediate except S-stability.

$$|E(x_t^\mu - x_t^\nu)| = \left| \int_t^\infty \exp(t - s)E(k_\mu^s - k_\nu^s) \, ds \right| \leq \int_t^\infty \exp(t - s)E|k_\mu^s - k_\nu^s| \, ds$$

$$E|k_\mu^s - k_\nu^s| \leq |y_{\mu_0} - y_{\nu_0}| + E \int_0^s |b(u, \mu_u, k_\mu^u) - b(u, \nu_u, k_\nu^u)| \, du \leq \varepsilon +$$

92
provided \( \mu_s \neq \nu_s \) \( \forall s \) but \( |\mu_s - \nu_s| \) small. We may assume that \( \epsilon K < 1 \). Gronwall's inequality implies

\[
E|k_s^\mu - k_s^\nu| \leq \epsilon + \epsilon \int_0^s \exp(\epsilon K(s-u)) \, du \leq \epsilon + \epsilon \exp(\epsilon K s) s \Rightarrow \exp(t-s)E|k_s^\mu - k_s^\nu| \\
\leq \epsilon \exp(t-s) + \epsilon \exp(t) \exp((\epsilon K - 1)s)s \Rightarrow \int_t^\infty \exp(t-s)E|k_s^\mu - k_s^\nu| \, ds \leq \epsilon + \\
\epsilon \exp(t) \int_t^\infty \exp((\epsilon K - 1)s) \, ds = \epsilon + \frac{(1+t)\epsilon}{1-\epsilon K} \exp(t) \exp((\epsilon K - 1)t) = \\
\epsilon + \frac{(1+t)\epsilon}{1-\epsilon K} \exp(\epsilon K t) \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{q.e.d.}
\]

\textbf{Proof of Theorem 4.6}: For \( b(s, \mu_s) - b(s, \nu_s) > 0, \int_0^s b(u, \mu_u) - b(u, \nu_u) \, du > \delta s \). Thus, \( x_t^\mu - x_t^\nu = \int_t^\infty \exp(t-s) [\int_t^s b(u, \mu_u) - b(u, \nu_u) \, du] \, ds \geq \int_t^\infty \delta \exp(t-s) \, ds = (1+t)\delta > \delta \) q.e.d.

\textbf{References}


