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UMI
Accuracy Criteria
and Finite Element Study of a
Highly Saturated Magnetic Device with a Large Air-Gap

by

Masoud Sharifi

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

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0-612-49878-6
To Mehri, Mohammad-Hassan, Mohsen and Mehti:

my roots

To Mansureh:

my ground

To Yasamin:

my ski
Accuracy Criteria
and Finite Element Study of a
Highly Saturated Magnetic Device with a Large Air-Gap

Ph.D. 2000
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Abstract

The Finite Element Method has been frequently used to obtain a reliable virtual prototype of a device. The accuracy of the device model gains considerable importance when measured data are not available for comparison. This thesis proposes to use Ampere’s circuital law to obtain a global accuracy criterion, Ampere’s law error, which measures the accuracy of the FE model and solutions.

The C-shaped magnet of an open-concept Magnetic Resonance Imaging system is used as the focus for this thesis. Being three-dimensional, highly saturated, and having a large air-gap, the open-concept MRI magnet represents an interesting modeling challenge. The objective of the thesis is thus to use this device as a vehicle to explore accuracy issues when modeling saturated, 3D, non-symmetric magnetostatic devices.

A stable 3D modeling approach is developed that encloses the MRI magnet in a spherical FE volume, the exterior surface of which can represent the true infinite boundary condition. A FE model of the MRI magnet is thus developed and studied. It is concluded that the average and standard deviation of Ampere’s law errors provide a valid global accuracy measure for this class of FE solutions.
A selection study is also necessary to obtain the most accurate FE model of a device. This thesis describes (a) a selection strategy and (b) selection criteria. A selection strategy incorporates two methods that simplify the search for better FE models of the device. Both methods have proven to be practical and constructive. The selection criteria include Ampere's law error, the energy content of the model, the energy content in specific regions of the model, and the average of the magnetic field data at specific regions of the magnet. These selection criteria have proven to be feasible for selecting the best FE model of the device.

An alternative design of the MRI magnet is introduced. The performance of the alternative design is shown to be superior to the original design of the MRI magnet.

In addition, the thesis introduces two two-dimensional modeling approaches to simplify the FE study of unsymmetrical 3D devices. One of the 2D modeling approaches is shown to be applicable for calculating the magnet data.
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Chapter 1

Introduction

Typically, most engineering applications require an intensive performance evaluation of a device or a system during the design and development stages. Often, it is not possible to build an actual full-scale prototype of the device or system. Thus, one develops as realistic a simulation model as is possible with Finite Elements, Boundary Elements, Finite Differences, Integral Equations, or other such methods.

Mathematical models of physical devices commonly are posed in terms of boundary value problems. The Finite Element Method simply employs a numerical approach in achieving an approximate solution to such problems. Although the FEM has been applied to and has been calibrated for many problems, each application is unique and each requires its own validation. The output of any FE model is only as good as the model itself. Calibration problems arise when measured data are not available, as will be the case whenever the device in question is large and the cost of its manufacture is high. Therefore, a central modeling issue is how best to validate the FE model under such circumstances.

The accuracy of the FE approximate solutions to highly saturated, nonlinear, magnetostatic field problems will provide a central focus for this thesis. Just as the FEM employs the laws of physics to mathematically represent a physical device, the accuracy of the approximate solutions for that device will be verified in accordance with those laws.

A large air-gap C-core electromagnet serves as a focus for the FE modeling issues that are discussed in this thesis. This particular device is also of considerable practical importance since it is the key component in a recently developed open-concept Magnetic Resonance Imaging (MRI) system. The C-core magnet for this application represents a challenging
problem from a modeling viewpoint. First, the magnetic material of the device, in certain regions, is driven well into saturation. Therefore, there is a considerable leakage magnetic field. Additionally, the device is not entirely symmetrical. It has two symmetry planes at $Y=0$ and $Z=0$ planes of a Cartesian coordinate system; however, such symmetry does not permit a 2D study of the device either in 2D planar or in 2D axisymmetric fashions. Therefore, The MRI magnet problem must be modeled in three-dimensions (3D). The thesis therefore considers the development of a virtual prototype for this device as a means of exploring accuracy issues when modeling highly saturated magnetostatic devices.

1.1 Finite Element Method

When applying the Finite Element Method, there are several issues that have not received adequate attention in the past. First, there is the question of accuracy, particularly in terms of quantities that are derived from the basic solutions. Second, there is the question of identification and selection, which means the achievement of the best possible models and solutions for a given problem. The accuracy and the selection issues are discussed in the following subsections.

1.1.1 Accuracy Criteria

The goal of most FE studies is to obtain the most accurate model possible for a given physical device. Having defined a problem, and then having modeled and solved it using a proper FE formulation, one must assess the quality of the results thus obtained. This leads to an evaluation of model and/or solution error(s) and an assessment of model accuracy. Finally, different models can be compared in order to identify the best possible model for the given problem.

An assessment of the solution quality can be considered in terms of either local or global criteria. Local accuracy criteria have often been discussed in the literature within the context of adaptive meshing [1-3]. It should be noted that a model that complies with local accuracy criteria is not necessarily the best possible model in terms of overall solution quality.
Global accuracy criteria measure the validity and reliability of a model in terms of its solution as a whole. The integral forms of Maxwell's equations, for example, are essentially global criteria and can be used to confirm and evaluate FE solutions. It is surprising that global accuracy criteria have not been widely discussed in the literature. Recently, however, Bossavit has questioned the approximate nature of the FE solutions [4]. He concludes that the FE solution of magnetostatic problems often "fails to have a general validity." The reason is that the FE approximate solution satisfies only one of Maxwell's equations - in their integral forms.

In another recent study, Ampere's circuital law is used to evaluate a posteriori error in the 3D FE solutions of a rectangular coil that surrounds a linear magnetic core of 200×200×100 mm volume size [5]. Two FE models of the problem are introduced: the coarse mesh model (with 8640 tetrahedral elements,) and the fine mesh model (with 43560 tetrahedral elements.) Five integral paths are chosen in the air region of the problem, three of them linking the current source. It is shown that the fine mesh model yields a lower value of the posteriori error. It is also claimed that the posteriori errors of the fine mesh model "show no dependency with path location," - within a 4% discrepancy in errors along five paths.

In this thesis, it is proposed that Ampere's circuital law provides a convenient and reliable quality measure when judging FE solutions to nonlinear, 3D magnetostatic problems. It will be shown that this measure can indicate when a particular model is producing accurate results. However, it should be noted that such a measure does not necessarily show when the most accurate FE results are obtained. Thus, a second aspect of the problem is to propose means, conditions and restrictions whereby the most (possible) accurate FE model can be developed.

1.1.2 A Selection Approach

Powerful software packages implement the FEM and provide the necessary numerical tools to obtain required data such as magnetic field distributions, stored magnetic energy, magnetic flux, etc. The analyst must then examine the solutions, first, to verify the accuracy of the solutions, and second, to find ways to improve the quality of the model and its results. The
accuracy of the solutions should be determined from the accuracy criteria. Therefore, the heart of the selection issue is how to improve the quality of the solutions, how to measure the improvement of the solutions' quality, and how to distinguish the best model for a given problem.

This thesis proposes:

a) to develop convenient methods to obtain an initial model for a given problem; i.e. a model that yields a solution of acceptable accuracy such that the selection process can be initiated,

b) to define criteria to assess the quality of the solutions yielded by the initial as well as the subsequent models,

c) to identify simple methods that improve the efficiency of the selection process, and
d) to define criteria to find the most accurate model for a problem.

1.2 Magnetic Resonance Imaging

Nuclear Magnetic Resonance Imaging (MRI) was discovered in the 1950's and has been used for imaging human body parts since 1970 [6-7]. The major advantages of MRI are the non-invasive operation of the device, the high resolution of its images, and its capability to show both soft and hard tissues. Better and faster computing facilities have substantially decreased the MR-imaging time. Recently, high-speed MRI units have been used to achieve real time monitoring of human tissue. A new development in MR-imaging has been the introduction of small, open-concept MRI units that are particularly attractive for use with surgical procedures that avoid major cuts or openings of the human body. The tissue or body part in question can be imaged in real time and the image then used to guide surgical procedures. In order to achieve these ends, the open concept MRI units represent a major departure from conventional MRI designs.

In a typical air-core MRI unit [8], the human body or the imaged tissue is located in the central volume of a large air-core magnet. Superconducting coils are used to generate a uniform, static magnetic field within the volume containing the tissue to be imaged. Typical levels of such a magnetic field may be up to 3 T. Coils and devices that are
necessary to operate the MRI system surround the imaging volume. Consequently, the inner volume of the magnet, and thus the imaged tissue, are not accessible to the extent that would be required in a surgical procedure.

A recent project at the University of Toronto has focused on the development and testing of an open concept system that has been termed Image-Guided Minimally Invasive Therapy (IGMIT). The main component in this process is an open-concept C-shaped magnet. The C-shaped MRI magnet that is proposed for this system is shown in Figure 1.1, while further details are provided in Appendix A. Open-concept MRI units are economically very promising due to the reduced size, cost, and power consumption of the MRI unit. Thus, there is the possibility that specially designed units for particular parts of the human body can be designed.

Figure 1.1: The half symmetry of the MRI magnet where the coil is around the core.

The main magnetic component of the open-concept MRI unit is its C-shaped magnet. For the purpose of the prototype IGMT unit, the superconducting magnet consisted of a
cylindrical core, two cylindrical poles, and two slab yokes connecting the core and poles. The imaged tissue was to be placed within the large air-gap opening of the magnet; i.e. the free-space volume between the poles. Thus, three sides of the imaged tissue would be accessible for medical treatments. The magnet is energized with a superconducting coil using 120 kA-t (120×10^3 Ampere-turns) of direct current. With this design, the operational magnetic flux density in the air-gap of the MRI magnet is reported at 0.27 T [Appendix A].

Throughout this thesis, the term "MRI magnet" will be used to refer to such a magnet. The magnet material used for core, yokes, and poles is C1006 steel. The B-H characteristic for this material can be found in Appendix B, Figure B.1.

The MRI magnet used in the IGMIT system provides a focus for the research that is reported in this thesis. The primary objective of the thesis is to examine the factors that govern the accuracy of Finite Element models that are used to represent 3D, highly saturated magnetostatic devices. The key issues are the accuracy of a given model and the modification of the model (i.e. the solution grid) such that the most accurate solution possible can be identified. Once an optimum model of the core and the surrounding air has been obtained, alternative coil configurations can be examined.

In the following subsections, the principal features of the C-core MRI magnet will be examined.

### 1.2.1 Air-gap and Magnetic Material of the MRI Magnet

In most magnetic devices, the air-gap constitutes a small fraction of the magnet volume. For example, power electronic transformers and inductors typically have a small air-gap in order to linearise the magnetic circuit. Similarly, electric machines also employ a small air-gap to facilitate the rotational function of the machine. For devices such as these, the magnetic field distribution within the core is of interest during the design stage.

When the air-gap volume of a magnetic device is increased, more current must be supplied to the circuit of the device in order to keep the same level of magnetic field in the air-gap. Consequently, the magnetic material starts to saturate and stores more magnetic
energy. This reduces the linear effect of the air-gap to a point where the nonlinear magnetic material must be included in any model.

The air-gap volume of the MRI magnet accounts for 9% of the magnet volume. However, functionally, it must be large enough to hold the imaged tissue and any accessory equipment. A very high level of magnetic flux density is technically preferred at the air-gap volume. This, in turn, requires a high excitation current in the coil. Consequently, the magnetic flux density in the core is increased to high saturation levels and the core stores a considerable amount of energy. In addition, magnetic leakage flux increases to such levels that the free-space volume around the magnet also stores a significant amount of energy and cannot be neglected. For these reasons, together with the inherently 3D nature of the geometry, the large air-gap MRI magnet represents a challenging modeling problem.

1.2.2 Coil of the MRI Magnet

As is shown in Figure 1.1, the C-shaped MRI magnetic used in the IGMIT system incorporates a long, thin superconducting coil to establish the required magnetic field. This coil encompasses, and is closely coupled to, the core section of the magnet (in the upper left-hand portion of the figure.) It is isolated from the magnet by a cryo-cooler shell. As noted, the coil excitation level is very high in order to maximize the magnetic flux density in the magnet air-gap. Consequently, the magnetic field level in the vicinity of the coil is very high and changes rapidly. Modeling the coil within the large magnet is thus difficult and plays an important role in the accuracy of the simulation results. Finding a convenient method to represent the effect of the long, thin coil would reduce the modeling difficulties. The search for such a method is a consideration in this thesis.

1.2.3 Size of the MRI Magnet

The magnet is exceptionally large. Its frame size is 1.30×1.23×0.50 m and its volume is 0.615 m³. The free-space region that surrounds the magnet stores a considerable amount of energy due to the leakage of the magnetic field. Thus, the modeling of the infinite space
exterior region must be done carefully and will play an important role in determining the accuracy of the results. Due to the exterior region, the overall size of the resulting FE model will be very large. This in turn imposes a constraint on the model since the number of available nodes and elements will be limited by the FE simulation package being used. The trade-off between the limited number of nodes and elements, on the one hand, and optimal accuracy on the other hand, is a central issue in any modeling exercise.

1.3 Thesis Objective

Two aspects of the Finite Element Method provide the focus for this thesis: the global accuracy of the solutions obtained, and selection strategies toward a final and best solution of a given problem. These will be applied and examined in relation to the C-shaped magnet of the open-concept MRI magnet. The thesis objectives are divided among the following topics:

1. Finite Element Modeling:
   - To model the MRI magnet with different gridding strategies, and
   - To develop a stable modeling practice suitable for the MRI magnet.

2. Global accuracy criteria:
   - To develop a global accuracy criterion using Ampere’s circuital law as a basis, and
   - To obtain the limits and conditions under which it can be applied.

3. Selection approach:
   - To develop convenient methods to obtain an initial model of a problem, where the initial model is defined as yielding reasonable and acceptable accuracy,
   - To identify a simple method to guide the improvement of the model such that overall accuracy is increased,
   - To verify that the defined accuracy criteria can be used to assess the quality of the solutions,
   - To define selection criteria that can be used to select the most accurate model of the problem, and
   - To study selection criterion based on the energy content of the FE model.
4. The MRI magnet study:
   To analyze the MRI magnet performance under different operating conditions, and
   To introduce and analyze an alternative design for the MRI magnet.

5. Equivalent 2D models:
   To develop suitable 2D approaches to simplify the study of the MRI magnet and
   similar devices, and
   To validate the use of the 2D equivalent approaches and examin their limitations.

1.4 Thesis Outline

This thesis is composed of five chapters. The first chapter provides the introduction, the
objectives and the outline of the thesis.

Chapter 2 discusses the key issues that arise when the Finite Element Method is applied
to 3D problems. These include the accuracy of the FE model, and the notion that a selection
strategy is required to guide the modification of the model. First, the FE model of the focus
problem used in this thesis, namely the MRI magnet, is completely defined and described.
This description includes: (a) the specifications of the MRI magnet, (b) an outline of the
Finite Element modeling basic steps, (c) the different approaches that are possible when
using the FEM to model a problem, and (d) the domain definition and discretization of the
MRI magnet problem. Second, the applications of Maxwell's equations are discussed in
general terms, and Ampere's circuital law is used to define a global accuracy criterion. Third,
the selection strategy that is used in this thesis is completely introduced. Fourth and last, the
modifications of the Finite Element modeling steps are discussed. Such modifications are
necessary either to achieve an acceptable solution or to improve the accuracy of the solution.

Chapter 3 examines the thesis proposals: the global accuracy criterion and the selection
approach. This study focuses on the MRI magnet problem where the magnetic flux density in
the center of the air-gap is measured and known. First, an initial model of the problem is
obtained and verified against the known solutions in the air-gap of the magnet. Second, a
novel and simple equivalent method that improves the efficiency of the selection process is
introduced and employed. Third, the validity of Ampere's law accuracy criterion is studied
and it is shown that such a measure is not completely flawless and should be used carefully.

Fourth, selection criteria based on the energy content and the magnetic field data in a region of the magnet are investigated for the MRI magnet problem. Fifth, the most accurate FE model of the MRI magnet is selected. Its solution is given graphically and numerically. The magnet is also analyzed for different operational conditions. Sixth, an alternative design of the MRI magnet is introduced and its FE model is studied. Finally, the original and alternative designs of the MRI magnet are compared and conclusions are drawn.

Chapter 4 studies different equivalent two-dimensional (2D) models of the MRI magnet with the caveat that the simplified models should not simplify the basic 3D problem out of existence. Two approximate approaches are introduced. In the first approach, a magnetically isolated region of the magnet is separately modeled and studied. The core and the pole of the magnet are modeled and studied by this approach. In the second approach, the geometry of the magnet is simplified while preserving its critical features. The resultant composite problem has a cylindrical symmetry and is modeled in 2D. The FE solutions in the regions of interest are then sought. The original and alternative designs of the MRI magnet are studied by this approach and the solutions are shown satisfactory.

Chapter 5 summarizes the conclusions and contributions of the thesis. Following this are the References and Appendices.

Appendix A describes the Image-Guided Minimally Invasive Therapy (IGMIT) project. Appendix B shows the B-H and \( v-B^2 \) characteristic curves of the material used in the MRI magnet. Appendix C details the 3D linear and non-linear magnetostatic Finite Element Method based on the magnetic vector potentials. The boundary value problems and the FE approach for solving them are described in a detailed series of steps that lead to the solution of the problem. Working effectively with the FEM - which is the focus of the thesis objectives - requires the complete understanding of the method. Appendix C tries to cover such a background comprehensively. Appendix D summarizes the necessary steps in the application of the axisymmetric Finite Element Method.
Chapter 2

Accuracy and Selection of a Finite Element Model

Over the past 30 years, the Finite Element Method has been widely used to solve a broad class of problems in most fields of engineering. Today, the FEM is the solution method of choice in many of the application areas. This has been accompanied by the wide availability of commercial software packages that implement the method.

Whenever any simulation software is used to model a physical device or process, one should never lose sight of the fact that the results obtained are only as good as the model that is used. This is particularly true when the device being modeled is non-linear and 3-dimensional. The central issues involved in any FE modeling exercise include the following:

a) Generally, the FE software packages have an upper bound on the number of elements or nodes that can be used in the model. Thus, there is an important trade-off between the accuracy desired and the resources available.

b) Calibration and measured data are often not available. This is particularly true when the device being modeled is large or expensive. Indeed, the device may not even exist in prototype form until a significant amount of modeling has been completed. The question is then one of judging the quality of the results that have been predicted by a given model. This can be termed the quality problem; the issue is that of choosing one or more criteria by which the general quality or accuracy of a particular solution can be verified.

c) A related issue is how to deploy the resources available within a given FEM package in order to obtain the most accurate solution to a given problem. This can be termed the
selection problem; the FE model must be modified in order to minimize one or more error measures as defined by the chosen quality factor(s).

The quality and selection problems are discussed in this chapter and, in particular, measures that can effectively be used to judge the quality of a FE solution are described. In order to focus the discussion throughout the thesis, the particular problem of developing an accurate model for an open-concept MRI magnet is considered. ANSYS, a widely used and commercially available FE software package, provides the modeling and solution platform for the thesis.

The FE model of the MRI magnet is detailed in Section 2.1. The MRI magnet specifications are given first. Next, a brief introduction to the Finite Element Method is given which provides the necessary background to the FEM application. The boundary value problem, and the problem domain of the MRI magnet are then defined. Different modeling practices, and the final discretized FE model of the magnet are finally discussed and presented.

The quality of the FE solutions is discussed in Section 2.2. The Finite Element Method employs Maxwell's equations to mathematically model a device. The integral forms of these equations are used to assess the FE approximate solutions and to estimate the general accuracy and quality of the FE results. Different criteria have been defined to assess and verify the accuracy of the results. These criteria are discussed generally and also in regard to the formulation of the FEM that is used in this thesis.

The selection strategy that is used in this thesis is defined in Section 2.3. The selection strategy defines the means and approaches for measuring and improving the accuracy of the solutions. The criteria for the most accurate model are also defined in this section.

Different modifications of the FE model are discussed in Section 2.4. These modifications are introduced as a result of the model selection process; i.e. the modifications of the model improve the accuracy of the solution.
2.1 The FE Model of the MRI Magnet

The main objectives of this section are (a) to discuss the approach used in this thesis to model the MRI magnet, and (b) to introduce the FE model of the magnet. Therefore, the MRI magnet specifications and some general terms are given first in Subsection 2.1.1. The Finite Element Method and the related modeling issues are then discussed in Subsection 2.1.2, leading to Subsection 2.1.3 where the boundary value problem for the MRI magnet is described. The problem domain is defined within Subsection 2.1.4. Different discretization methods are discussed next, and finally, the best modeling practice for the MRI magnet is presented.

2.1.1 The MRI Magnet Specification

The MRI magnet includes one cylindrical core, two cylindrical poles, two yokes, and one coil, all of which were shown in Figure 1.1. The central volume of the air-gap is the main region of interest for MR imaging purposes. There are two symmetry planes in the magnet. The first is the \( z = 0 \) plane which cuts the core at its horizontal mid-plane; the tangential component of the magnetic flux density must vanish across this plane. The second plane of symmetry is the \( y = 0 \) plane. This plane divides the magnet in half vertically and is subject to the boundary condition that the normal components of the magnetic flux density must vanish across it. By exploiting these symmetry planes, a quarter symmetry model of the magnet is used, as is illustrated in Figure 2.1, page 41 - at the end of this chapter. The coil is shown as a white cylindrical shell in the figure.

The \( y = 0 \) symmetry plane is an important cross-section since it shows all the basic features and components that comprise the magnet. This cross-section is shown in Figure 2.2 together with the various paths that will be used to calculate Ampere’s law error.

The MRI magnet specifications are given in Table 2.1. The magnetic material used in the magnet is C1006 steel and its B-H curve characteristic is shown in Figure B.1, Appendix B.
Table 2.1: The MRI Magnet specifications that are used in this thesis.

<table>
<thead>
<tr>
<th>Component:</th>
<th>Material</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>Magnetic Material</td>
<td>Radius = 250 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Height = 300 mm</td>
</tr>
<tr>
<td>Yoke</td>
<td>Magnetic Material</td>
<td>C.L distance = 800 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Height = 315 mm</td>
</tr>
<tr>
<td>Pole</td>
<td>Magnetic Material</td>
<td>Radius = 250 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Height = 150 mm</td>
</tr>
<tr>
<td>Air-gap</td>
<td>Free-space Material</td>
<td>Radius = 250 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Height = 150 mm</td>
</tr>
<tr>
<td>Coil</td>
<td>Super-conductor</td>
<td>Radius = 275 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Thickness = 25 mm</td>
</tr>
<tr>
<td>Coil</td>
<td>Super-conductor</td>
<td>Length = 265 mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{2}$ Current = 60 kA-t.</td>
</tr>
</tbody>
</table>

2.1.2 The Finite Element Method

The Finite Element Method is a numerical technique for obtaining approximate solutions to boundary value problems. The 3-D linear and non-linear magnetostatic Finite Element Method – which are used in this thesis - are detailed in Appendix C. ANSYS, a widely used and commercially available FE software package, provides the modeling and solution platform for the thesis.

The Finite Element analysis of a boundary value problem incorporates the following main steps:

a) Definition of the boundary value problem.

b) Formulation of the problem in terms of the Ritz (variational) or Galerkin approaches.

c) Definition of the problem domain and its boundaries.

d) Discretization of the problem domain by elements and nodes.

e) Selection of the interpolation functions for each category of element.

f) Formulation of the system of the equations in linear or non-linear terms.

g) Solution of the system of equations.

h) Evaluation of the final results.

i) Verification of the final results for their accuracy.

j) Modifications of the steps in the FE approach, if necessary, in order to improve the solution accuracy.
Information as to how these steps affect each other, even at the stages very close to the final solution, is not widely available. The entire model, as well as all the steps, may be modified at any stage due to a failure of the FEM to achieve an acceptable solution.

Whenever a FEM software is used to model a device, one has control over some steps of the FEM application. These modeling selections and controls are:

- The definition of the boundary value problem in step (a),
- The definition of the problem domain in step (c), and
- The discretization of the problem domain in steps (d) and (e).

These modeling controls are discussed in the following subsections while the software implementation of the other steps – steps (b), (f), (g), and (h) – are detailed in Appendix C. Verification of the results, step (i), is discussed in Section 2.2, and modification of the FEM approach, step (j), is given in Section 2.4.

2.1.3 Boundary Value Problem Definition

Mathematical models of distributed parameter physical systems are typically formulated in terms of boundary value problems. Any boundary value problem can be defined in terms of:

a) The conditions that must be satisfied on the boundary $\Gamma$ that enclose the problem domain $D$, and

b) A governing differential equation in the domain $D$. This equation can be written as:

$$ Lu = f, $$

where $L$ is a differential operator, $u$ is the unknown quantity, and $f$ is the excitation or forcing function.

**Maxwell’s Equations:**

In electromagnetics, the differential operator $L$, the unknown quantity $u$, and the forcing function $f$ are all obtained from Maxwell’s equations. In the case of magnetostatic field problems, the differential forms of Maxwell’s equations are

$$ \nabla \times \mathbf{H} = \mathbf{J} \quad \text{(Ampere’s law),} $$

(2.2)
\[ \nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss's law),} \quad (2.3) \]

where \( \mathbf{H} \) is the magnetic field intensity vector, \( \mathbf{B} \) is the magnetic flux density vector, and \( \mathbf{J} \) is the electric current density vector. Using \( \mu \) for the permeability and \( \nu \) for the reluctivity of the medium, the constitutive relation is written as

\[ \mathbf{B} = \mu \mathbf{H} \quad \text{or} \quad \mathbf{H} = \nu \mathbf{B}. \quad (2.4) \]

**Magnetic Vector Potential:**

A FE model of a static magnetic field is traditionally defined in terms of magnetic potentials. The problem can be defined either in terms of the magnetic scalar potential [9-10], or by using the magnetic vector potential [11-13], or by a combination of the scalar and vector magnetic potential [14]. The magnetic vector potential approach is used in this thesis and is introduced in the following subsections. The reasons for not using one of the other two approaches (a scalar potential; a combination of scalar and vector potentials) are given in Section 2.4.

Maxwell's Equations govern magnetostatic field problems. Since the magnetic flux density \( \mathbf{B} \) is a divergence free field, it can be represented in terms of a magnetic *vector* potential \( \mathbf{A} \) as

\[ \mathbf{B} = \nabla \times \mathbf{A}. \quad (2.5) \]

By substituting (2.5) in (2.2), it follows with the aid of (2.4) that a second order differential equation

\[ \nabla \times (\nu \nabla \times \mathbf{A}) = \mathbf{J}, \quad (2.6) \]

is obtained for the magnetic vector potential \( \mathbf{A} \). This equation only partially represents the magnetostatic boundary value problem over the domain \( D \) since any vector field can only be completely defined by imposing conditions on its curl *and* its divergence. A gauge condition must therefore be imposed on the divergence of \( \mathbf{A} \) in order that \( \mathbf{A} \) can be uniquely defined. The commonly used gauge condition for magnetostatic problems is the Coulomb gauge, which is defined as

\[ \nabla \cdot \mathbf{A} = 0 \quad \text{(Coulomb gauge).} \quad (2.7) \]
Problem Description:

In this thesis, the boundary value problem is described by using a vector potential \( A \) that satisfies the following vector differential equation

\[
\nabla \times (\nu \nabla \times A) - \nabla (\nu \nabla \cdot A) = J,
\]

which is obtained by enforcing the Coulomb gauge on to (2.6) and comprehensively explained by Biro and Preis in [13]. The boundary conditions at the outer boundary \( \Gamma \) of the problem domain \( D \) are of two kinds

\[
\begin{align*}
\hat{n} \times A &= \mathbf{P} & \text{Dirichlet condition on } \Gamma_1, \\
\hat{n} \times (\nabla A) &= 0 & \text{Neumann condition on } \Gamma_2.
\end{align*}
\]

These conditions are applicable at planes of symmetry and on \( \Gamma = \Gamma_1 \cup \Gamma_2 \). The continuity conditions applied at the interface between two media of different reluctivity are

\[
\begin{align*}
\nu^+ \hat{n} \times \nabla A^+ &= \nu^- \hat{n} \times \nabla A^- & \text{Continuity of tangential } \mathbf{H}, \\
\hat{n} \cdot \nabla A^+ &= \hat{n} \cdot \nabla A^- & \text{Continuity of normal } \mathbf{B}, \\
\hat{n} \times A^+ &= \hat{n} \times A^- & \text{Continuity of } \mathbf{A},
\end{align*}
\]

where superscripts \( + \) or \( - \) refer to different sides of the discontinuity interface \( S_d \).

2.1.4 Domain Definition

The solution domain in most electromagnetic problems contains the main physical device and the surrounding free-space region. The surrounding region should be defined properly, but this definition will depend on the type and nature of the problem and the objective of the analysis. It is impossible to directly model the whole infinite space around the device, and therefore, the exterior domain should be restricted and should consist of a reasonable volume surrounding of the device. Several methods exist to compensate for those portions of the infinite space region that have been truncated.

The so-called far field or infinite elements are often employed to take the infinite exterior free-space region into account. By employing them, the entire problem domain is divided into an interior region, an exterior region, and a transitional region between the interior and exterior regions. Within the interior and transitional regions, regular elements of finite size
are used. For the exterior region, elements that effectively have an infinite volume are employed. The proper geometry and location of these elements can greatly affect the accuracy of the results. It is known that the infinite elements perform best when the exterior region is symmetrical with respect to the geometric origin of the domain. This symmetry leads to the use of a circular exterior boundary in 2-D problems, and a cylindrical or spherical exterior boundary in 3-D problems.

The interior region contains the device, its components, the regions of interest and the immediate surroundings. A free-space cylindrical or spherical volume envelops this interior region and provides the transitional region between the interior region and the exterior region, whose surface is the infinite side of the infinite element.

The volume of the enveloping cylinder or sphere should be large enough to permit the proper accommodation and performance of the normal and infinite elements. On the other hand, it should be small enough that the model, as a whole, employs a reasonable and limited number of nodes or degrees of freedom.

When a cylindrical volume is used, the infinite elements can be used either on the curved surface or on the flat surface(s) of the cylinder (which can be seen in Figure 2.3.5.) The 3-D infinite elements can have only one face toward infinity. Unfortunately, the corner region of the cylinder has two faces toward infinity and cannot be modeled with infinite elements. Therefore, the boundary conditions on one surface of the cylinder should be set properly - while the infinite elements are employed on the other surface of the cylinder. On the other hand, when a spherical envelope is used, these problems are not encountered (as can be seen in Figure 2.4.1.) However, the spherical envelope presents greater difficulties at the meshing stage, as compared to the cylindrical approach. The unsatisfactory results of the cylindrical envelope approach will be given in Subsection 2.4.2. In this thesis, the spherical envelope approach is primarily used to create the transitional and external regions of the problem domain.

**Domain Definition of the MRI Magnet:**

In the study of the MRI magnet, the problem domain is divided into interior, transitional, and exterior regions. The interior region contains the device, the free-space between different
parts of the device, and the region of interest. In the case of the MRI magnet, the region of the interest is primarily the air-gap between two pole pieces. The device consists of the excitation coil, the core, the yokes, and the pole pieces.

A free-space spherical volume envelops the interior region and provides a transitional region between the interior region and the exterior region. Then, the exterior of such a cylinder or sphere will face infinity and accommodate the infinite side of the infinite element.

During the modeling of the MRI magnet, different radii and element sizes were used to define the envelope spherical volume, with the interior region being kept constant. After several successful solutions, a set of parameters was established for the envelope volume. These parameters are kept as long as a change in them does not improve the accuracy of the result. The final parameters are presented in the following section.

2.1.5 Domain Discretization

The discretization of the problem domain is the most important step when developing a FE model for a given device. The manner in which the domain is discretized will affect the total element and node numbers, and the computer storage requirements. It also defines the computation time, the software requirements, the convergence of the analysis to a solution, and the accuracy of the numerical results. The efficiency or even propriety of the discretization cannot be completely judged till the results are obtained and verified.

The problem domain is typically discretized using one or more of the following techniques:

Semi-computer-aided discretization or extrusion in which the analyst decides on how and where to use a software meshing capabilities. One of the most effective methods of making a 3-D FE model is by extruding a proper 2-D model along a predefined path. First, the analyst makes a 2-D projection model, which includes areas meshed with 2-D elements. Second, software extrudes the 2-D model (or a specific part) along a path to create the 3-D volume. As a result, a circular domain, a triangular element or a quadrilateral element can be extruded to a cylindrical volume, a prism element, or a brick element, respectively. The path should provide enough information to correctly build the model. The extrusion divisions define the
fineness of the elements in the direction of the extrusion path. The extrusion method is mainly used in this thesis.

Computer-aided discretization or free-meshing in which the analyst defines the geometry, sets some control and meshing parameters, and the software creates the elements and the corresponding nodes. To obtain a given quality of discretization, more parameters should be set, and the meshing control should be adjusted from the software to the analyst. Software programs, or at least ANSYS 5.3, fail to discretize a slightly complicated 3-D domain containing many volumes that have curved surfaces - such as the MRI magnet problem domain which include the thin cylindrical coil, the cylindrical core connected to the yoke slab, etc. This failure is mainly from the loss of the node connectivity between adjacent elements of different volumes. The free-meshing method is used selectively in this thesis.

Adaptive discretization in which the analyst only defines the problem domain. A FEM software is then employed to discretize and solve the problem. Different adaptive discretization schemes have been developed and used for 2-D problems. This method is not as widely used in 3-D problems; however, new developments have been reported [15-16]. Tetrahedral elements are used exclusively in adaptive discretization. Such elements are avoided in this thesis due to reasons that will be discussed in Subsection 2.4.4. The adaptive meshing is not used in this thesis.

Non-adaptive discretization and modification in which the meshing of a domain is performed according to the geometry of the domain and some meshing guidelines. These controlling guidelines are mainly the analyst’s specified division number of the geometrical entities, the aspect ratio between different sides of an element, the span angle between adjacent lines or areas of the element, and the visual quality of the element. The discretized problem is solved by the FEM and the accuracy of the result is then verified. If the quality of the results is not satisfactory the meshing parameters are adjusted, and the process starts from the beginning. The principles of this approach is used in this thesis.

**Domain Discretization of the MRI Magnet:**

A convenient, overall problem domain for the MRI magnet is an overall spherical volume which, by definition, includes the previously noted interior, transitional, and exterior regions.
The interior region is cylindrical in shape and contains the entire magnet together with a portion of the surrounding free-space (shown in Figure 2.3.5.) The transitional region is the first layer around the cylindrical interior region and accommodates normal elements (shown in Figure 2.4.1.) The exterior region envelops the transitional region and provides a volume toward the infinitely extending free-space exterior, which accommodate the infinite elements (shown in Figure 2.4.4.) A controlled combination of the extrusion and free-meshing techniques are used to non-adaptively discretize the domain of the MRI magnet problem.

The problem domain data includes two types or sets of parameters. The first set of parameters includes the MRI magnet specifications that were given in Table 2.1 of Subsection 2.1.1. These are, in essence, the dimensions of the magnet and the material parameters. The second set of parameters includes the geometrical information of the surrounding free-space. These data will be given below in order to complete the definition of the problem domain.

The discretization parameters of the most accurate model of the MRI magnet include the element size setting, the number of nodes per line, and other parameters. The discretization method used in this thesis has many steps, and each step includes many discretization parameters. A detailed report of these discretization parameters is not particularly useful. Therefore, only the important discretization parameters of the final FE model are provided in this subsection.

Discretization guidelines have been developed during the modeling of the MRI magnet. These guidelines, with careful modification, are also applicable to other similar problems. These guidelines and the meshing parameters of the MRI magnet are summarized in the following four steps.

1. **2-D projection model:**

   The starting point for the model development is the ½ circular 2-D projection area shown in Figure 2.3.1. This area contains all the necessary details that will be required in the final 3-D model of the MRI magnet. It will be used, together with an extrusion approach, in developing that model. The radius of this circular area is $R_i = 900$ mm and it contains a total of 85 areas. Controlled free-meshing was used to discretize the areas with 1620 elements and 1611 nodes.
Quadrilateral elements are mainly used in this 2-D cross-section. This choice anticipates preferred hexahedral elements that will result during the extrusion stage. (Finer details of this 2-D grid are shown in Figure 2.5.5.)

2. **Interior region:**

The 2-D projection area of Figure 2.3.1 is then extruded along the z-axis - with proper setting for the element height - to create the magnet model. Figure 2.3.2 represents a partial model of the core region, as obtained during the extrusion stage. It shows a portion of the central core, the free-space volume between the core and coil, and the coil.

Figure 2.3.3 shows the continuation of the extrusion that yields the core (the dark cylindrical region), a portion of the yoke (the gray quarter cylindrical region attached to the core), and the free-space above the yoke. The coil volume and the free-space between the core and coil are also shown. Continuing with the development of the model, Figure 2.3.4 shows the model after completion of the core, the pole and the air-gap (both on the right hand side of the model.) Note the creation of the yoke and the free-space beneath the yoke by the extrusion and setting different materials to the extruded volumes.

The complete extrusion of the 2-D projection model yields the interior region of the MRI magnet model. This cylindrical volume is shown in Figure 2.3.5. The height of this cylinder is $H_I=700$ mm. Hexahedral elements are mainly used and prism elements are only employed to accommodate the geometry where necessary. The extrusion is done in seven steps and yields $85 \times 7 = 595$ volumes, $1620 \times 19 = 30780$ elements, and $1611 \times 20 = 32220$ nodes.

3. **Transitional Region:**

In the final model, infinite elements will be used on a spherical bounding surface to properly represent the far field behavior of the magnetic vector potential, and thus the magnetic flux density. It is therefore necessary to properly represent the transition from the previously described (cylindrical) interior region to the outer (spherical) exterior bounding surface.

A $\frac{1}{4}$ sphere with a radius of $R_{trans} = 1600$ mm envelops the inner cylindrical volume. The inner cylindrical volume is then excluded from the sphere to create the transitional volume. Controlled meshing was found to be unsuitable for use with this complementary volume.
Therefor, it is divided into the three sub-volumes that are shown in Figure 2.4.1 – one face of each volume is marked with +X, -X, and +Z volumes. Note the cutting angles of the complementary sphere, which are in the direction of the future radial extrusion.

The two volumes on either side of the y-z plane are discretized using a proper extrusion that yields \(2 \times (4 \times 18 \times 19) = 2736\) hexahedral elements. These volumes are shown in Figure 2.4.2.

In order to discretize the remaining sub-volume (the +Z volume that is absent from Figure 2.4.2), its surface is first meshed with quadrilateral elements. Then, free meshing is used to mesh the volume with 26340 tetrahedral elements. The final form of the discretized transitional region is shown in Figure 2.4.3.

4. Exterior Region:

In order to complete the transition to the spherical far field surface, a \(\frac{1}{4}\) sphere shell having an outer radius of \(R_{\text{ext}} = 1600 + 1000\) mm is used to envelop the transitional volume. This shell is termed the exterior volume and is divided into four sub-volumes, which are shown in Figure 2.4.4. These volumes are discretized with radial extrusion, which yields 1332 hexahedral infinite elements. The final model is shown in Figure 2.4.5, where a cut-out section has been included to show the transition to the exterior volumes.

For the most accurate model of the MRI magnet, the problem domain is a \(\frac{1}{4}\) sphere having a radius of \(R_{\text{ext}} = 2600\) mm. A total 61188 elements and 41193 nodes were used to discretize the problem domain, resulting in a total of 117,345 degrees of freedom. Figure 2.4.6 shows details of the interior region of the FE model, two symmetry planes of the model, and details of the exterior grid.

2.2 Verification of the Results

The Finite Element Method provides an approximate solution to a given boundary value problem. Since the method itself is an approximation and the solution is numerical, there are inherent errors not only in the numerical solution for the potentials, but also in data derived
from that solution. The main objective of this section is to introduce criteria that can be used to evaluate the accuracy of the FE solution.

Different criteria are used to assess the quality and acceptability of the FE results. These criteria can be divided into (a) local quality criteria, and (b) global quality criteria. For the purpose of local verification, a specific location or a region within the problem domain is investigated. On the other hand, global criteria may involve the entire problem domain, or a large number of sub-regions. Either differential or integral forms of Maxwell’s equations can be used as a basis for establishing local or global accuracy criteria.

In Subsection 2.2.1, the errors in the derived magnetic field data are examined. It is argued that the error measures based on the magnetic field discrepancy cannot provide global accuracy criteria. Subsection 2.2.2 and 2.2.3 investigate the agreement between the FE approximate solutions and Maxwell’s equations. It is shown that the FE approximate solutions always satisfy Gauss’s law when magnetic vector potentials are used to define the FE problem. A global accuracy measure is then defined based on Ampere’s law. Subsection 2.2.4 introduces the stored magnetic energy as a selection criterion. In each section, the arguments are first discussed in general terms and then are presented more specifically in the context of the vector potential FE formulation used in this thesis.

2.2.1 Magnetic Field

The most important derived-data of the FE solutions are the magnetic fields B and H. A convenient location to observe and estimate the post-processing error is at the planes of symmetry, on which the field is either tangential or normal, and where one or two component(s) of the magnetic field must vanish. However, the FE solution shows that the out-of-normal or out-of-tangential magnetic fields are not equal to zero. These magnetic field data include the differentiation or post-processing errors. However, such error cannot be used as either a global or a local accuracy criterion.

The magnetic field error can be observed by the following approaches:

a) Using the measured data: The measured magnetic field data can be used to assess the quality of the results if they are available. This is the strongest accuracy check on the FE
results. However, this method should be applied with caution where the results are verified at a specific location. It is always possible to select or modify the discretized location, node or element so as to minimize the difference between the approximate and the measured data. In this thesis, the average of the magnetic flux density in the air-gap center of the MRI magnet is calculated from the FE solution and compared with the measured value. Proximity of two values indicates that the solution of a particular FE model is generally correct. The most accurate model of the problem should then be selected from different correct FE models of the problem. This is the subject of the selection process that is given in the next section.

b) Using contour plots: Contour plots of the magnetic field are frequently used to investigate visually the results. Contour plots can provide a designer with invaluable information, but cannot be used as a local or global accuracy measure for the FE solution.

c) Using a discontinuity error criterion:
The magnetic field data are not continuous at the common node between adjacent elements due to the formulation of the nodal FEM based on the magnetic vector potentials. The element nodal field data are usually averaged to obtain more acceptable and continuous looking results. The discontinuity error $e_{B,j}$ of the $B$ for $i^{th}$ element is then defined as

$$e_{B,j} = \frac{1}{n} \sum_{j=1}^{n} |B_j - B_{ij}|,$$  \hspace{1cm} (2.14)

where $B_j$ is the nodal (averaged) $B$ at node $j$, $B_{ij}$ is the $B$ of the element $i$ at node $j$, and $n$ is the number of nodes in element $i$.

The application of the discontinuity error to 2-D and 3-D problems has produced some guidelines such as avoiding narrow, fat, thin elements, and keeping the aspect ratio of the elements in a reasonable range. However, the local smoothness of the results does not exclusively guarantee the accuracy of the FE solutions. Therefore, a global accuracy criterion cannot be defined based on discontinuity error (2.14). The continuity of the magnetic field, which is the result of Maxwell's equations, is discussed separately in the following.
2.2.2 Gauss’s Law

The differential form of Gauss’s law (2.3) states that the divergence of the magnetic flux density $\mathbf{B}$ is always zero. However, when the magnetic vector potential is used to formulate the nodal FEM, the divergence of the FE solutions of $\mathbf{B}$ is obtained as

$$\nabla \cdot \mathbf{B}_{FEM} = \nabla \cdot (\nabla \times \mathbf{A}_{FEM}) = 0. \quad (2.15)$$

Therefore, by merit of vector identities, (a) the MVP based FE solutions always satisfy magnetic Gauss’s law, (b) the $\mathbf{B}$ solutions are always solenoidal, and (c) the total magnetic flux is always preserved.

The continuity of the normal magnetic flux density between different media is also concluded from Gauss’s law. From the relation $\mathbf{B} = \nabla \times \mathbf{A}$ together with the continuity of the vector magnetic potentials $\mathbf{A}$, it is concluded that the normal $\mathbf{B}$ of the FE solutions is always continuous between different media.

In conclusion, when magnetic vector potentials are used to formulate the FEM, the approximate FE solutions always satisfy the magnetic form of Gauss’s law, and thus, a global accuracy criterion cannot be defined based on Gauss’s law.

2.2.3 Ampere’s Law

The continuity of tangential $\mathbf{H}$ between different media is the direct result of Ampere’s law where the current is zero at the interface between two media or elements, and the magnetostatic field is studied. The complete satisfaction of this condition, in addition to the natural satisfaction of Gauss’s law – continuity of the normal $\mathbf{B}$ and tangential $\mathbf{H}$ - yields the exact solution of a boundary value problem. The exact solution to even a simple problem is not practically achievable. Within an approximate FE solution, the tangential $\mathbf{H}$ is not continuous between different elements and media. However, the local discrepancy of the tangential magnetic field may be used in refining of the elements.

Compliance of the FE solution with Maxwell’s equations is related to the weak or partial satisfaction of Maxwell’s equations by the FE approximate solutions. Regardless of the FE
formulation used. The FE solutions satisfy one of Maxwell’s equations completely and satisfy the other one approximately. In the magnetic scalar potential formulation of the FEM, Ampere’s law is completely satisfied while Gauss’s law is partially satisfied. In the magnetic vector potential formulation of the FEM, Gauss’s law is completely satisfied while Ampere’s law is not completely satisfied. It is therefore proposed that the partial satisfaction of Ampere’s law can effectively be used to define a global accuracy measure for the FE solutions that are considered in this thesis.

The magnetomotive force drop along a contour line is used to define a global accuracy criterion for the approximate solutions of the FEM. Ampere’s circuital law states that the total \( \text{mmf} \) along a closed path must equal the total current passing through the surface that the path bounds. Thus, a convenient error measure is

\[
\text{error}_{\text{mmf}, \ell} = \oint \mathbf{H} \cdot d\ell - I_{\text{enclosed}}.
\]  

(2.16)

This error includes the errors originating from FEM application, from the differentiation in post-processing of potentials \( A \) to obtain \( B \), from material curve fitting or \( B-H \) curve interpolation to obtain \( B \), and from numerical integration as in (2.16).

It should be noted that the \( \text{error}_{\text{mmf}, \ell} \) is path dependent, and that the path may have been selected as a way to increase or decrease this error. As a result, a single path calculation should be avoided. To avoid this misjudgment, different paths, which enclose the same current, should be chosen, and the relevant errors should be studied statistically. This data can then be used to compare different models of the same boundary value problem. Figure 2.2, page 41, as was noted previously, shows the different paths that were used in this thesis for the study of the MRI magnet.

2.2.4 Stored Energy

The FE formulation is generally based on the minimization of an energy related functional. This means that the FE solution for each problem has the minimum energy of the respective discretized model. Moreover, different FE models of the same boundary value problem would introduce different levels of stored energy. Physics states that any natural system is
stable only at the minimum level of its potential energy content. This axiom can be used to compare different models and solutions of a boundary value problem. The FE model which has the lowest energy content between different models of a problem, is expected to be the best approximate model of that problem. This approach is used as one of the selection criteria in this thesis.

2.3 Selection Approach

The Finite Element Method provides an approximate solution to a boundary value problem, and as any numerical method, the results are as good as the model. Accuracy criteria are therefore used to assess the quality of the results. The modeling approach may then be modified to obtain a better (more accurate) model, in which the available resources (i.e. elements) are more efficiently employed.

The basic issues in modeling a complex non-linear problem are:
1) How can one efficiently and conveniently get a correct starting initial model,
2) How can one refine and modify the initial model toward a more accurate model,
3) How much more accuracy improvement is achievable, and
4) What are the selection criteria to measure the accuracy and to select the most accurate model?

The following subsections discuss the approach used in this thesis (a) to obtain the initial model, and (b) to select the most accurate model of the problem.

2.3.1 Initial Model

The initial model of a problem is a correct model: the problem is defined, discretized and solved properly, and thus, the accuracy of the FE solution is acceptable. The initial model is the starting model in the search for the most accurate model of the problem. Achieving the initial model and solution of a problem:

a) is the most time consuming part of solving a boundary value problem,
b) may require many modifications in the FE application, e.g. definition of the problem, definition of the problem domain, and meshing of the problem domain, and
c) is the first and biggest milestone of the FEM toward the final solution of the problem.
This thesis employs the following three steps to simplify achieving the initial FE model of the MRI magnet.

**Free-Space Material Method**

In the Free-Space Material Method, the magnetic material of the discretized model is replaced by free-space, and the total excitation is applied to the model. The MRI magnet problem thus changes to another problem: a coil located in free-space. The FE solution is used to calculate the total stored magnetic energy, the inductance of the excitation coil, and Ampere's law error (2.16). These are compared with the confirmed and published solutions of the new problem. The FE modeling continues till a close agreement is reached between the FE solutions and published solution. Agreement of two results confirms, to some extent, that

a) The boundary value problem is defined and formulated properly, and
b) The current source (coil) and boundary conditions are applied correctly.

This new model of the problem is ready for more examination by the following method.

**Reduced Excitation Method**

In this approach, the coil of the MRI magnet is energized with substantially lower current. Therefore, the magnetic material is not over-saturated, and a mild non-linearity exists in the problem and system of equations. The $e_{\text{emf}}$ (2.16) error and the field at material discontinuity surfaces are studied for future grid refinement. The solutions confirm, to some extent, that

a) The current source (coil) and boundary conditions are applied correctly,
b) The proper interpolation functions or element types are selected, and
c) The material characteristic of the non-linear material is properly defined.

The problem model is modified till a successful solution is achieved. This model is used to obtain the initial model of the given problem.
Chapter 2. Accuracy and Selection of a Finite Element Model

The Initial Model

In continuation of the above method, the current excitation of the FE model is increased in a step-by-step manner. At each level of the excitation, the FE model is modified to maintain a reasonable Ampere's law error $e_{mnf}$ within the solution. The final modified model is then fully excited with the non-linear magnetic material in place. If this model is proved as a correct model with an acceptable accuracy, then it is the initial model of the problem. The correctness of this FE model (or any given problem) is discussed under three categories:

(a) The measured data of the problem are available. Agreements of the FE solutions and measured data confirm the correctness of the model. This is the strongest verification argument. The MRI magnet measured data are available in the center of the air-gap. The FE solutions of the FE model are calculated and compared with such data. A close agreement of two data verifies the validity of the model, and thus, such model is introduced as the initial FE model of the problem. For the purpose of this thesis, the initial model is then used to investigate the application of the proposed general accuracy criteria.

(b) The measured data are available for a similar problem of equal or greater difficulty. The FE approach, which is used for the FE model being investigated, can be applied to a similar problem. Agreement of the FE solutions and the measured data would confirm the validity of the FE model. The similarity of the two problems should have been established beforehand. The thesis will introduce an alternative design to the MRI magnet where the measured data are not available. The FE model of such a design would be acceptable because its application to a similar and more difficult problem - the original MRI magnet - yields solutions close to the measured data of the similar problem.

(c) The measured data of the problem are not available. A global accuracy criterion should be used to assess the accuracy and correctness of the FE model. Such an accuracy criterion measures the approximation of Maxwell's equations in the FE model of the problem. The thesis tries to prove the applicability of Ampere's law as a global accuracy criterion.

The correct initial model serves as the foundation in the search for the most accurate FE model (optimum model) of the problem.
2.3.2 Selection Criteria

Achieving the initial model confirms that the basic approach to the problem is correct, and that the final result would be obtained by fine-tuning of the Finite Element Method steps. The main possible modification fields of the method are (a) the modification of the problem domain definition, and (b) the modification of the problem domain discretization. Beginning with the initial model, a model parameter is changed, and the sensitivity of the solution to such a change is assessed. Improvements in the quality of the results yield a better model, which may be introduced as the new solution. The final FE model of the problem is obtained where the model modifications do not change
(a) The average of Ampere's law error (2.16) along different paths, and
(b) The stored magnetic energy in the problem domain.
The applicability of these measures will be proved in the next chapter.

In conclusion, the most accurate model of a boundary value problem, solved with the Finite Element Method, should:

a) Comply with the known magnetic field data in regions of interest, if such are available,
b) Conform to the integral form of Ampere's law as close as possible, and
c) Contain the minimum amount of stored magnetic energy compared to other models.

2.4 Modification of the Finite Element Model

A key issue in any Finite Element modeling exercise is to ensure that the model yields the best possible results. It is therefore necessary to subject the model to a sequence of modifications in an effort to improve the predicted results. It should be noted, however, that it is not possible to produce an exact diagnostic map to guide this process. Nevertheless, the modification may involve one or some of the following steps:

1) The definition of the boundary value problem.
2) The definition of the problem domain.
3) The discretization of the problem domain.
4) The use of element type.
5) The definition of the material characteristic.
6) The application of the current excitation or load.

These steps are first discussed in general terms and then for particular case of the MRI magnet model.

### 2.4.1 Boundary Value Problem Definition

A boundary value problem is a mathematical model, of the form $Lu=f$, that represents a given physical system. The selection of the unknown quantity $u$, together with the appropriate boundary conditions, defines the problem. Therefore, the variable $u$ should be selected carefully because some available choices will not yield solutions of acceptable accuracy. In magnetostatic problems, for example, the selection of the unknown quantity is mainly affected by the level of magnetic saturation, the connection of the magnetic circuit parts, the geometry and topology of the problem, and the excitation method.

The magnetostatic nodal FEM can be defined either in terms of (a) the Magnetic Vector Potential (MVP), or (b) a Magnetic Scalar Potential (MSP), or finally (c) a combination of the MVP and a MSP. The MVP based FEM is used in this thesis. Other formulations have been tried unsuccessfully. The following subsections describe the MSP and the MSP+MVP approaches and their shortcoming for the MRI magnet problem.

#### Magnetic Scalar Potential:

In a magnetic scalar potential (MSP) formulation of the FEM, the total magnetic field intensity $\mathbf{H}$ in the problem domain $D$ can be decomposed into two terms [10]

$$\mathbf{H} = \mathbf{H}_s - \nabla \phi_s.$$  (2.17)

Then, the analytical or numerical solution of $\mathbf{H}$ should satisfy Gauss’s law in domain $D$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mu (\mathbf{H}_s - \nabla \phi_s) = 0.$$  (2.18)

The magnetic field intensity $\mathbf{H}_s$ is a generalized or a guess magnetic field, which satisfies Ampere’s law and should be defined according to the problem type. Then, the remaining part of the $\mathbf{H}$, can be derived as the gradient of the generalized scalar magnetic potential $\phi_s$. This
Chapter 2. Accuracy and Selection of a Finite Element Model

A general definition then allows for three particular choices of scalar potential: the Reduced Scalar Potential (RSP), the Difference Scalar Potential (DSP), and the Generalized Scalar Potential (GSP). The choice of which scalar potential to use depends only on the problem type. The main advantage of the MSP formulation of the FEM is the reduction of the degrees of freedoms to be solved.

In order to test the accuracy that can be obtained when a scalar potential is used as a basis of formulating a model of the MRI magnet, the DSP-based FEM was used on the ANSYS platform. For a model with less than 16000 elements, the DSP approach yielded the average B at the center plane of the air-gap of 0.384 T, which is 42% more than the measured value of 0.27 T. This test illustrates that the improper choice of potential may yield results that are significantly in error. Meanwhile, the MVP based FEM cannot converge to a solution when the same model is used.

The DSP-based FEM was also tried for the final optimum model of the MRI magnet (this model was obtained by the MVP-based FEM and presented in Subsection 2.1.5.) The results are summarized in Table 2.2. These results clearly show that for a well-developed and optimal model of the MRI magnet, the DSP based formulation of the FEM fails to achieve an acceptable solution. The insufficiency of the DSP-FEM may well be due to (a) the complicated geometry of the MRI magnet, (b) the over saturation of the magnet material, and (c) the weakness of the DSP based elements due to reduced degrees of freedom.

<table>
<thead>
<tr>
<th></th>
<th>B\text{gap along} T</th>
<th>B\text{gap across} T</th>
<th>B\text{core across} T</th>
<th>E\text{air-gap} J</th>
<th>E\text{core} J</th>
<th>E\text{total} J</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSP FEM</td>
<td>0.3148</td>
<td>0.2889</td>
<td>2.1873</td>
<td>593</td>
<td>377</td>
<td>4565</td>
</tr>
<tr>
<td>MVP(A) FEM</td>
<td>0.2903</td>
<td>0.2658</td>
<td>2.1913</td>
<td>512</td>
<td>375</td>
<td>4464</td>
</tr>
<tr>
<td>Discrepancy %</td>
<td>8.4%</td>
<td>8.6%</td>
<td>-0.2%</td>
<td>16%</td>
<td>0.1%</td>
<td>2.2%</td>
</tr>
</tbody>
</table>

**Combined Magnetic Potential:**

A combined vector and scalar potential formulation has been proposed [14] to overcome the shortcoming of the scalar potential formulation. In the combined potential approach, the problem domain is divided into two regions. One region contains the current source and the
surrounding free-space and here, the reduced scalar potential (RSP) formulation is used. In the second region, containing the magnetic material together with the space in between, the magnetic vector potential (MVP) is used.

The combined RSP+MVP based FEM, on the ANSYS platform, has also been used to model the MRI magnet for testing purposes. The MVP based elements were used to model the magnetic material of the magnet, whereas the RSP based elements were used to model the coil of the magnet together with the surrounding free-space region. Interface elements were then employed to connect two segments of the models. For this test, the non-linear iterative solver of the FEM could not converge to a solution for a current excitation higher than 30 kA-t, which is only half of the rated operational current level of the magnet. The results for the 30 kA-t test problem are shown in Table 2.3. The close agreement between the two sets of results seems to confirm that the combined potential formulation of the FEM can be used with confidence when the magnetic material is not heavily saturated. However, as noted above, the combined potential formulation fails to converge when the material is pushed deeply into saturation.

Table 2.3: The results from RSP+MVP and MVP based FE models of the MRI magnet.

<table>
<thead>
<tr>
<th>Current = 30 kA-t</th>
<th>$B_{\text{gap across}}$ T</th>
<th>$B_{\text{core across}}$ T</th>
<th>Ampere’s law error</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSP+A FEM</td>
<td>0.202</td>
<td>1.56</td>
<td>10%</td>
</tr>
<tr>
<td>MVP(A) FEM</td>
<td>0.203</td>
<td>1.67</td>
<td>4%</td>
</tr>
</tbody>
</table>

2.4.2 Definition of the Problem Domain

The domain of a problem contains the main physical device together with the surrounding regions. The first step in defining the problem domain is to decide how to best represent the dimensionality of the device being studied. In practice, there are only three choices: a planar 2-D approximation, an axisymmetric 2-D approximation, or a full 3-D representation of the device. The analyst must decide which approach represents the problem more accurately. This choice must then be validated.
In the case of the MRI magnet, the choice of a planar 2-D FE approximation, using a magnetic vector potential formulation, certainly yielded a solution. The results satisfied both Ampere's law and Gauss's law, and the contour plots of the magnetic field appeared to be quite satisfactory. However, for an excitation current of 60 kA-t, the average magnetic flux densities at the center plane of the air-gap and core were computed to be 0.495 T and 1.35 T, respectively. When a 3-D FE model (whose results will be presented in the next chapter) was used, the corresponding values were found to be 0.268 T and 2.19 T, respectively. The corresponding errors of 85% and 38% are due to an incorrect representation of the problem domain. In other words, a 2-D planar geometry simply cannot represent the magnetic flux density in a core that has a finite cross section, particularly when there is a significant leakage flux due to saturation effects.

The second step in defining the problem domain is to decide how to best represent the free-space region surrounding the device. The surrounding region should be defined properly while the exterior surfaces of this region make most of the FE model boundary. Having a good representation of the surrounding region becomes particularly important when that region stores a significant proportion of the total magnetic energy in the system.

In the case of the MRI magnet, the solution domain can be divided into interior and exterior regions. As noted previously, the interior region contains the magnet core and the coil whereas the exterior region represents free-space and, ultimately, the far fields. The exterior surrounding region can be a cylindrical or spherical volume, either of which will envelop the interior region. Table 2.4 shows the results from two models of the MRI magnet where both cylindrical and spherical exterior envelopes were used. Two designs of the MRI magnet were used, one where the coil was conventionally positioned around the core, and the other where the coil was positioned around each pole piece.

Solving the cylindrical model of the magnet where the coil is around the core, the FE Newton-Raphson method failed to converge to a solution. Initially, 15 iterations were used. Additionally, the convergence could not be achieved with increasing the number of iterations, where a total of 30 iterations were used. Where a cylindrical surrounding region was used for the exterior region of the magnet where the coil is around the pole, the solution
accuracy was not acceptable. The spherical model, which was described previously in Subsection 2.1.5, was chosen as being the best representation of the exterior region.

On the basis of these tests, it was therefore concluded that the solution accuracy was severely compromised when a cylindrical exterior region was used to represent the effect of the far fields. This accuracy problem seems to be a function of the magnet operating characteristics: the device is highly saturated, the air-gap is large, and a considerable amount of magnetic energy is stored in the surrounding region. Consequently, the spherical surrounding region was used for the models that were described in this and next chapter. The main parameters of the spherical surrounding region are its inner and outer radii. Different radii were used to define different models of the problem, and the final radius – which was given in Subsection 2.1.5 - was obtained from the most accurate FE model.

Table 2.4: The results from different models of the MRI magnet.

<table>
<thead>
<tr>
<th>Coil around Core</th>
<th>$B_{\text{across core}}$ T</th>
<th>$B_{\text{across air-gap}}$ T</th>
<th>$e_{\text{mnf}}$</th>
<th>Newton-Raphson</th>
<th>Iteration no.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical model</td>
<td>2.46</td>
<td>0.277</td>
<td>14%</td>
<td>Not converged</td>
<td>30</td>
</tr>
<tr>
<td>Spherical model</td>
<td>2.19</td>
<td>0.268</td>
<td>4%</td>
<td>Converged</td>
<td>16</td>
</tr>
<tr>
<td>Discrepancy %</td>
<td>+12%</td>
<td>+3.4%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coil around Pole</th>
<th>$B_{\text{across core}}$ T</th>
<th>$B_{\text{across air-gap}}$ T</th>
<th>$e_{\text{mnf}}$</th>
<th>Newton-Raphson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical model</td>
<td>0.901</td>
<td>0.397</td>
<td>6.5%</td>
<td>Converged</td>
</tr>
<tr>
<td>Spherical model</td>
<td>0.835</td>
<td>0.419</td>
<td>2.5%</td>
<td>Converged</td>
</tr>
<tr>
<td>Discrepancy %</td>
<td>+8%</td>
<td>-5%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.4.3 Domain Discretization

One of the purposes of this thesis is to study the effects that modifications of the solution grid have on the solution accuracy. For the purpose of this thesis, a controlled combination of the extrusion and free-meshing techniques was used to develop various grids for the problem. The modification of each solution grid involves many parameters, and because each parameter is intimately related to several other parameters, it is very difficult to separate the effect that each individual parameter has on the overall solution accuracy.
model. The problem domain was then discretized and solved with the FEM. The solution for a given grid was examined and a small subset of the discretization parameters was changed. The solution results for the new model were then studied to understand the effect of the parameter changes. This step-by-step approach yielded a sequence of different models and solutions, which can be judged by the accuracy criteria. The knowledge thus acquired was then used to define a new set of parameters and to create the next version or generation of the magnet model. The process was continued until a model obtained that yielded a solution of acceptable accuracy.

Six different models from sequential versions of the MRI magnet model were selected in order to briefly describe the process of grid refinement. These models are shown in Figures 2.5.1 to 2.5.6, inclusive. In order to simplify matters, a 2-D projection of each model is used. As noted in the figures, the exterior region of each model is a spherical envelope, and infinite elements are used to represent far field effects. In each case, the current excitation is 60 kA-t. The non-linear FEM was found to converge for all models. A limited number of parameters and their effects are discussed in the following:

(1) **Figure 2.5.1** shows a rectangular 2-D projection model where 27018 elements have been used. Ampere's law error $e_{\text{nnf}}$, as specified by (2.16), was found to be 13% for this model. The mid-path of the magnet, as defined in Figure 2.2, was used in order to evaluate $e_{\text{nnf}}$. This model has an unacceptably large Ampere's law error and thus provides a starting point for the grid refinement process.

(2) **Figure 2.5.2** shows a circular 2-D projection model where 47833 elements have been used, where a rectangular model was used previously. Note also the extra layer around the coil and the triangular elements in the outer area. The outer $\frac{1}{4}$ circular line is divided into $n_{c-l}=10$ divisions. The error $e_{\text{nnf}}$ of this model was calculated to be 11%.

(3) **Figure 2.5.3** shows a 2-D projection model where 31892 elements (half of the previous model) have been used. In this case, quadrilateral elements have been used in the outer area, and the core region has been sub-divided into different areas. The outer $\frac{1}{4}$ circular line divisions have been increased to $n_{c-l}=12$. As a consequence of these modifications, the error has been reduced to $e_{\text{nnf}}=8\%$. 
(4) Figure 2.5.4 shows a 2-D projection model where 54941 elements have been used. In this model, triangular elements have been used in the outer area, and there is a better representation of the yoke projection area. Finally, the outer ¼ circular line divisions have been further increased to \( n_{c-l} = 14 \). As a consequence of these modifications, the error has been further reduced to \( e_{mnf} = 6\% \).

(5) Figure 2.5.5 shows a 2-D projection model where 61188 elements have been used. The 3-D model for this particular case was previously shown in Figures 2.3 and 2.4. Note the quadrilateral elements in the outer area, together with the division of the core area and its free meshing. The outer ¼ circular line divisions have been increased again, this time to \( n_{c-l} = 18 \), and the error has continued to decrease, now to the level \( e_{mnf} = 3.8\% \).

6) Figure 2.5.6 shows a 2-D projection model where 63650 elements have been used. The discretization of the core area has been refined and the outer ¼ circular line divisions has been kept constant at \( n_{c-l} = 18 \). In this final case, Ampere's law error along the mid-path of the magnet was found to be \( e_{mnf} = 1.5\% \).

It is interesting that the better FE models of the MRI magnet include the better representation and discretization of the surrounding free-space. The complete advantage of hexahedral elements (made from quadrilateral 2-D elements) over the prism and tetrahedral elements (made from triangular 2-D elements) can also be concluded.

### 2.4.4 Element Type

The accuracy of the FE results is affected by the order and type of the elements used in the discretization of the problem domain. Increasing the order of the element type - either uniformly over the problem domain or selectively at specific regions - is not used in this thesis. The type of element (tetrahedral or hexahedral) that is used in developing the FE model has a significant influence on the accuracy of the magnetic field data. After the FE problem is solved, the magnetic field data can be derived using the differential relation between the magnetic vector potential and the magnetic flux density.

The first order tetrahedral element has a potential shape function of the form ‘\( ax+by+cz+d \)’. Thus, the extracted components of the magnetic field - e.g. the curl of the
magnetic vector potential - are constant over each element. Consequently, when developing a FE grid using tetrahedral elements, each element must be sufficiently small to justify the implicit assumption that the magnetic field is constant across that element. This is the main drawback of using the tetrahedral elements, especially where the domain is small and the field is changing rapidly. In the modeling of the MRI magnet, the tetrahedral elements are not employed in the interior region of the FE model. They are only used in one of the free-space transitional volumes to accommodate the geometry – which was shown Figure 2.4.3.

In contrast, the first order hexahedral or brick element has a potential shape function of form ‘axyz+bxy+cyz+dxz+exy+fyz+gzh’ - eight nodes and thus eight coefficients. Thus the resulting magnetic field will have a linear variation within each element. Hexahedral elements are also easy to visualize and convenient to create by extruding methods. They are primarily used in this thesis. The prism elements are a special form of the brick elements and are generally avoided in this thesis. They are only employed to accommodate the geometry.

2.4.5 Material Characteristic

A FE nonlinear solver program may come very close to the solution of a problem, iterate around that solution, and ultimately fail to converge. Further investigation of the non-converged solution may even show partly satisfactory results. This may be due to the least suspicious factor of the entire Finite Element Method: a poor definition of the non-linear B-H characteristic. The basic issue here is that the local reluctivity of the material is required in order to calculate stiffness matrix [K] for each element. In addition to the local reluctivity, the slope of the material reluctivity with respect to the square of magnetic flux density is also required, in this case to calculate each element’s Hessian matrix [H]. Therefore, to insure the uniqueness of the FE result, the reluctivity should be (a) single valued, and (b) monotonic.

The visual checking of either the B-H or the v-B^2 characteristic curve may be satisfactory to confirm the single-valued relation, but it is not enough to confirm the monotonic change of the reluctivity. The v-B^2 characteristic is usually drawn and used to confirm the monotonic change of the reluctivity v with respect to B^2.
For the purpose of this study, C1006 steel is used for the MRI magnet core. The $v-B^2$ characteristic for this material is shown in Figure B-2 of Appendix B.

2.4.6 Current Excitation

In the case of the MRI magnet application, the FEM solver may approach but not actually converge to a final solution. Another factor (other than material characteristic) that has a strong impact on this process is the method that is used to apply the current or load to the relevant portions of the problem domain. Usually, the total excitation current is not applied in one step when nonlinear materials exist in the problem domain. Rather, the current or load should be applied in a step-by-step manner, with the solver being permitted to run for several iterations before applying extra loads. The immediate advantage of such a loading approach is that the elemental non-linear material data can be obtained and updated in each step. This in turn provides good initial material data for further loading of the problem domain.

In a highly saturated problem, such as the MRI magnet, an insufficient number of loading steps will result in the loss of material data from the no-load to the full-load state. As a result, during the solver iterations in the full-load state, the convergence criteria cannot be satisfied. Although this problem is not easily detectable, it is easily resolved by increasing the number of steps from no-load conditions to the full-load excitation of the FE model.

2.5 Summary

Chapter 2 has detailed the application of the Finite Element Method, in general terms, and to the MRI magnet problem. The discussion has focussed on the question of how best to evaluate the accuracy of the solutions, and on the question of how to modify the FE grid such that the best solution can be obtained. The application of the FE modeling to the MRI magnet problem was used to focus the discussion.

Chapter 3 will employ approaches discussed in this chapter for study of the MRI magnet problem. It will verify Ampere’s law error as a reliable global accuracy measure, and furthermore, it will examine the selection criteria that were proposed in this chapter.
Figure 2.1: The $\frac{1}{4}$ model of the MRI magnet where the coil is around the core.

Figure 2.2: The $y = 0$ slice of the MRI magnet showing different paths which are used to calculate Ampere's law error.
Figure 2.3: The steps in defining and discretizing the MRI magnet domain:
1: 2-D projection model, $R_i=0.900$ m.  
2: Extrusion of the core, free-space, coil.
3: Extrusion of the core to yoke, air.  
4: Extrusion of air-gap and yoke.
5: Interior cylindrical region or volume,  
   $H_f=0.700$ m.  
6: The MRI magnet inside the cylindrical volume.
Figure 2.4: The steps in defining and discretizing the MRI magnet domain:
1: Transitional envelope region $R_{\text{trans}}=1.6$ m.
2: Hexahedral elements for side volumes.
3: Tetrahedral elements for front volume.
4: Exterior envelope volume(s), $R_{\text{ext}}=2.6$ m.
5: Hexahedral elements for all volumes.
6: The final model of the magnet within two symmetry plane limits.
Figure 2.5: Six 2-D projection models from different generations of the MRI magnet model:

1: Rectangular interior region, $e_{mmf}=13\%$
2: $n_{c,l}=10, e_{mmf}=11\%$.
3: $n_{c,l}=12, e_{mmf}=8\%$.
4: $n_{c,l}=14, e_{mmf}=6\%$.
5: $n_{c,l}=18, e_{mmf}=3.8\%$.
6: $n_{c,l}=18, e_{mmf}=1.5\%$. 
Chapter 3

Study of the MRI Magnet with 3D FEM

The Finite Element Method and various aspects of its application to the MRI magnet problem were detailed in Chapter 2. It was proposed that the integral form of Ampere’s law could provide a useful quantitative measure when assessing the quality of an individual FE solution for the MRI magnet problem. This accuracy measure could thus guide the process of identifying the best possible solution to the problem.

The present chapter undertakes a systematic study in which the use of Ampere’s law as an error measure is examined for a specific magnet geometry. Its validity as a global error measure is examined and the conditions for its successful application are identified. It is shown that the use of Ampere’s circuital law as an error measure allows one to reliably identify a Finite Element model, among several, that offers optimal accuracy. This can be termed the model selection process. Similarly, the strategy used in this process can be termed the model selection strategy. The applicability of the selection strategy is verified in the search for the most accurate model of the MRI magnet. In addition to using Ampere’s circuital law as an error measure, it is also demonstrated that the magnetic field data and the energy content of the FE model can be reliably used as selection criteria.

In order to start the process of selecting an optimal model, it is first necessary to identify a starting or initial model of the magnet. The initial model is simply a model that provides reasonable results. The grid for this model is modified in the search to improve the quality of the solution. Means of efficiently identifying the initial model are discussed in Section 3.1.

The process of selecting the best FE model of the MRI magnet, from among many such models, is detailed in Section 3.2. A basic issue that is faced in this process is the need to
potentially formulate and solve many nonlinear models of the magnet. This can be prohibitively expensive, certainly in terms of manpower. To avoid this difficulty, an equivalent linear model of the magnet is used that substantially improves the efficiency of the selection process. It is shown that the accuracy of any given nonlinear FE model is very closely matched by an equivalent linear model that uses an appropriately chosen constant permeability. In a second study, the path dependency of Ampere's law error is investigated. It is concluded that the average and standard deviation of Ampere's law errors along different paths provide a reliable global accuracy criterion. Thirdly, in this section, the use of magnetic field data and energy content are examined as alternative selection criteria. The agreement between two selection criteria, (a) minimum of Ampere's law error, and (b) minimum of energy stored in the model, is then confirmed.

Having selected a model that provided the most accurate solution for MRI magnet in Section 3.2, the model is then used in Section 3.3 to examine the magnet performance in detail. The contour plots of the magnetic flux density are shown for different regions of the magnet. The performance of the magnet is also analyzed for different levels of the coil current. The effects of the coil model are then studied. It is shown that an equivalent current sheet coil can effectively simulate the coil of the MRI magnet.

In Section 3.3, it is shown that the existing design of the MRI magnet requires an extraordinarily high excitation in order to produce a relatively modest air-gap magnetic flux density. Therefore, an alternative to the original design is introduced and analyzed in Section 3.4. The magnet geometry is preserved but the coil is split and then positioned around each pole piece, as opposed to being around the core in the original design. In the case of this modified design, no experimental data was available for model calibration purposes. Therefore, the previously described selection process was repeated in order to identify a final model that is believed to yield accurate results. This model was then used to undertake a detailed analysis of the proposed design. The original and the modified design are compared, particularly in terms of magnetic flux density magnitude and the energy content.
3.1 Identification of an Initial Model

The identification of a good starting, or initial, model for the problem at hand is one of the most important milestones in the FEM process. The starting, or initial, model is simply a model that yields a solution having acceptable, but not necessarily optimal, accuracy. This model confirms that (a) the problem and the problem domain are correctly defined, and (b) the problem domain is properly discretized. The initial model thus serves as the foundation in the selection search of the final and most accurate model of the problem. The development of the initial model for the MRI magnet can be divided into two steps, namely, to initially model the coil alone, and then to include the nonlinear core, but with a reduced coil excitation. These approaches were initially discussed in Subsections 2.2.1 to 2.2.3. Their applications to the MRI magnet problem, together with the results thus obtained, are given in the following subsections.

3.1.1 Free-Space Material Method

Using good engineering judgment, a discrete model of the MRI magnet problem, including the coil, the core and the surrounding free-space region is developed. However, the magnetic material of the discretized problem domain is replaced by free-space, but the coil excitation and other features of the FE model remain unchanged. Thus, the problem is that of a coil located in free-space, and the FE solution is used to obtain the total energy of the system. This can be compared to values that are found in the literature for an empty solenoidal coil [17]. If good agreement is obtained between the published and the FE results, the model will be further tested by resetting the property parameters of the core region to those of a nonlinear magnetic material.

As was noted above, the energy stored in an empty solenoidal coil can be easily computed using known expressions. First, the inductance of a cylindrical current sheet of \( N \) turns is given by

\[
L = N \times \mu_0 \int_{\text{coil}} N/A_{\text{coil}} \times K ,
\]
where the correction factor $K$ is Nagaoka's constant [17]. This constant is tabulated in the literature [17] and accounts for the coil end effects. The stored energy of the coil carrying a current $I$ is then given by

$$E = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 A_{coil} \times (NI)^2 \times K . \quad (3.2)$$

Substituting $NI$ with the total excitation of 120 kA-t and the area $A_{coil}$ with $\pi R_{coil}^2$ yields the energy

$$E_{Graver} = 0.2 \pi^2 \times 120^2 \times R_{coil}^2 / l_{coil} \times K \quad [J]. \quad (3.3)$$

Several different models of the MRI magnet have been solved with this approach of replacing the core region by free-space. The results for five such models are summarized in Table 3.1. This table compare the energy predicted by the FE models to the values given by (3.3). The table also includes values for $\epsilon_{mmf. \mu}$, which represents the mid-path Ampere's law error, as was defined by (2.3). Ampere's law errors were obtained by performing a full nonlinear solution for the model in question and have been included for the purpose of comparison. It should be noted that Ampere's law error for the linear models (i.e. the models with the material properties of the core region set to those of free-space) are less than 0.5% and thus are not mentioned.

<table>
<thead>
<tr>
<th>Model</th>
<th>$E_{Graver}$ [J]</th>
<th>$E_{FEM}$ [J]</th>
<th>$\Delta E$ %</th>
<th>$\epsilon_{mmf. \mu}$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2935</td>
<td>2768</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>2674</td>
<td>2730</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2558</td>
<td>2684</td>
<td>0.5%</td>
<td>8</td>
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<tr>
<td>4</td>
<td>2674</td>
<td>2693</td>
<td>0.7%</td>
<td>6</td>
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<tr>
<td>5</td>
<td>2759</td>
<td>2742</td>
<td>0.6%</td>
<td>4</td>
</tr>
</tbody>
</table>

The first and second models present an acceptable level of $\epsilon_{mmf. \mu}$ from corresponding nonlinear analyses (a single digit error); however, the energy differences between $E_{Graver}$ and $E_{FEM}$ are not acceptable, and the models are thus rejected. Model #1 has a grid structure similar to the one that was shown in Figure 2.5.1. The rectangular 2D projection model is the main weakness of this model. A circular 2D projection model and more elements were used
in the grid of Model #2. Model #3 was obtained by (a) better representation of the exterior free-space and (b) by refining the elements in the magnet core where the magnetic flux density is between 1 and 1.7 T – the knee point of the magnetic material B-H curve.

A model with energy discrepancy of less than 1% is considered acceptable. Therefore, the third model is selected as the candidate model for further testing. The 2D projection models of the 3rd, 4th, and 5th models were given in Figures 2.5.3, 2.5.4 and 2.5.6, respectively. The fourth and fifth models are only given here to show the validity of the free-space approach, and Section 3.2 will further inspect their performance.

3.1.2 Reduced Excitation Method: Loading Steps

The next step in the process of determining a good starting, or initial, model is to reset the material properties in the core region to those of a nonlinear magnetic material. Since the problem is now nonlinear, it is necessary to ensure that the Newton-Raphson iteration for a given problem (i.e. model) converges. It is known that the Newton-Raphson method converges easily to a solution when the initial values of the degrees of freedom are close to the final solutions. However, when the full magnet excitation current is applied in one step, the initial and final values of the solution variables can be sufficiently separated that the Newton-Raphson cannot converge to a solution. In order to avoid this problem, the current excitation is applied to the nonlinear model of the MRI magnet in relatively fine steps. The current increments are kept small enough to ensure the convergence of the Newton-Raphson within the first iteration of each step. Should the method fail to converge in the second iteration at the full load condition, the current loading scheme has to be modified to provide smaller current increment. For the purpose of this study, a total of 15 loading steps were used in order to establish the full load excitation of 60 kA-t when energizing the nonlinear magnet model.
3.1.3 The Starting, or Initial, Model

Of the three models that were examined in Table 3.1, Model #3 was chosen as providing a good basis for further refinement. Several factors were considered when making this choice. In particular:

(a) In the first stage of the development, when the core region material properties were set to those of free-space, Model #3 provided an acceptable estimate of the stored magnetic energy.

(b) When the magnetic core properties were reintroduced, the solution of Model #3 had a reasonable Ampere’s law error.

(c) There was good agreement with measured data. The magnetic flux density at the perimeter of the air-gap region of interest is calculated as \( B_{\text{FEM air-gap}} (z = 0, r = R_{\text{air-gap} + 2}) \approx 0.273 \, \text{T} \). The average magnetic flux density at the air-gap center plane \( (z = 0, r \leq R_{\text{air-gap}}) \) is computed as \( B_{\text{FEM air-gap}} = 0.26 \, \text{T} \). The measured value of the same quantity is reported [Appendix A] as \( B_{\text{Measured air-gap}} = 0.27 \, \text{T} \) in the air-gap region of interest \( (z = 0, r \leq R_{\text{air-gap} + 2}) \). The shimming coils and shaping of the pole faces are conventionally used to homogenize the magnetic field within the required limits in the volume of interest – which are out of the scope of this thesis. Therefore, it is not possible to numerically achieve the exact homogeneous measured magnetic field. However, the numerical result should be close to the measured data in order to suggest the possibility of the shimming and calibration procedure.

It is possible to define the volume of the interest in the FE model of the problem in order to simplify the comparison of the solutions with the published data. The initial models of the MRI magnet had such geometric identification in the air-gap of the magnet. However, such extra modeling requirements have restricted the model discretization and reduced the efficiency of the FEM. Therefore, the volume of interest has not been identified in presented models of the MRI magnet.

Having systematically chosen a starting, or initial, model, the next step in the process is to refine that model in order to improve the quality of the solution. This refinement process yields several models of the problem. The process of selecting the most accurate model of the magnet, among many models, will be discussed in the next section.
3.2 Selection of an Optimal Model

Once an initial FE model has been identified, modifications are introduced in order to obtain the most accurate model of the problem. The two main areas where modifications can be introduced are (a) the problem domain definition, and (b) the problem domain discretization. The modification process includes (1) changing a parameter or a set of parameters, (2) solving the model, (3) investigating the quality of the results, and (4) using the results to guide the introduction of further modifications. The search for the best model obviously ends when no further improvement is achieved by model modification.

The two key issues in this process are the accuracy criteria by which individual solutions are judged, and the selection criteria by which the best solution among a group of solutions can be identified. In addition to specifically identifying the best possible solution, this section also studies, applies, and verifies a global accuracy criterion: Ampere's law error, and a global selection criterion: the energy content of the model. As opposed to local or semi-global criteria, a global selection criterion indicates a better model of a problem with a unique, unambiguous and predetermined measure; e.g. the most accurate model has the minimum of the total energy stored in the system. This section also investigates the use of semi-global selection criteria; e.g. the regional average of magnetic field data, and the magnetic energy stored in a region of the model.

Subsection 3.2.1 introduces a novel method, based on an equivalent linear model, that can conveniently and efficiently be used to obtain improved models for a given problem. A set of models is thus generated using this approach. A final subset of models is then selected for further study. In Subsections 3.2.2 to 3.2.5, this subset is used to illustrate the application of different accuracy criteria and to select the best model representing the problem.

The use of Ampere's law as an error measure is the subject of Subsection 3.2.2. The path dependency of this quality measure is examined in detail and it is shown that an accurate model can be identified.

The possibility of using the magnetic field data as a model selection criterion is discussed in Subsection 3.2.3. Different locations of the MRI magnet are selected, and the average
magnetic flux densities for different models are examined. It is shown that the application of the average of magnetic field data as selection criteria are limited and should be used in combination with other selection criteria.

Global and semi-global quality measures based on the energy content of different parts of the model also provide a means of comparing the quality of a solution. Their applications are compared to Ampere's law error approach in Subsection 3.2.4. The selection of the optimum model of the MRI magnet is then concluded.

3.2.1 An Equivalent Linear Model

The process of identifying a good, accurate FE model for a nonlinear 3D problem, such as the one being considered in this thesis, is very time consuming in terms of both human and computer resources\(^1\). The difficulty arises from two characteristics of the MRI magnet problem:

a) the three dimensionality of the model, which cannot be simplified, and

b) the non-linearity and high saturation of the magnetic material in the magnet core.

It is therefore important to identify efficient means of developing and testing nonlinear FE models. The objective in this subsection is to introduce and develop one such method.

A scheme is proposed to approximate the nonlinear material of the magnet using a properly selected linear material. It is postulated that the solutions of the linear and nonlinear models may follow the same quality patterns and may yield similar error measures. Therefore, the accuracy of a given nonlinear model can be estimated by first solving the equivalent linear problem, naturally on the same grid. This scheme to accelerate the selection study of the MRI magnet is described in the paragraphs that follow.

The first problem that must be resolved is to identify a suitable equivalent permeability. In order to do so, four different models (i.e. solution grids) of the MRI magnet were chosen. Full nonlinear solutions were available for each of the solution grids. The four models were

\(^1\) The CPU time on an Ultra Sun Microsystems computer was 8 hours for solving a nonlinear model of the MRI magnet where (a) 61188 elements, (b) 117345 degrees of freedom, and (c) 16 iterations were used. The data are from the most accurate model of the MRI magnet that has been introduced in Section 2.1 and Section 3.3.
chosen such that the solutions covered a reasonable range of Ampere's law error estimates. More specifically, the error estimates ranged from a high of 8% to a low of 4%.

The next step in the process was to re-solve each of the four models for constant relative permeability values that ranged from 50 to 2000. Ampere's law error estimates for all such solutions are shown in Figure 3.1 (page 59 at the end of this section) as a function of relative permeability. Clearly, the trend shown in Figure 3.1 is for each model to converge to a constant Ampere's law error as the relative permeability increases.

A more interesting feature becomes apparent when the error for each of the nonlinear models is plotted on the same set of curves. The four horizontal lines that are shown in the figure indicate these error levels. It will be noted that linear models having a relative permeability in the range 100 to 175 yield error estimates that are comparable to the fully nonlinear cases. It is therefore postulated that linear models of the MRI magnet, with relative permeability values in this range, can be used for the purpose of model refinement when seeking to improve overall solution accuracy. The advantage of this approach is clearly that the effect of a particular refinement to the model is being assessed using a one-step linear solution. For the purposes of this thesis, a constant relative permeability of 100 was chosen for use in the equivalent linear models.

In order to illustrate that the equivalent linear models do, in fact, provide a very good estimate of Ampere's law error, and thus provide a simple means of reducing the computational effort when refining a solution grid, a study was undertaken. A sequence of equivalent linear models, each representing a particular refinement of the solution grid, were solved and Ampere's law errors were computed. The full nonlinear solutions were also computed for a subset of these models and, again, Ampere's law errors were determined. In all cases, Ampere's law error was computed along the mid-path of the magnet.

Figure 3.2 shows the resulting values of the error estimate for each of the 43 models in the sequence. The results for the linear models are shown as white bars, while the black bars are the error estimates from the full nonlinear solutions. There is a very close correspondence between the linear and nonlinear models. Thus, it can be concluded that the equivalent linear approach can indeed be used with advantage when refining the solution grid for problems such as the MRI magnet.
3.2.2 Path Dependency of Ampere’s Law Error

The integral form of Ampere’s law:

\[ e_{\text{mmf},t} = \int \mathbf{H} \cdot d\mathbf{l} - I_{\text{enclosed}} \]  \hspace{1cm} (3.4)

has been proposed as a global accuracy criterion. When this criterion was used in Section 3.1, it was evaluated along the mid-path of the magnet. Note that the mid-path was previously defined in Figure 2.2. In using this definition, it was presumed that the mid-path error provided a good estimate of the global accuracy of the model. However, this assumption was not verified in detail. Therefore, the purpose of the present subsection is to examine the path dependency of Ampere’s law error and to determine whether the mid-path assumption is valid.

From all the models that were developed for the MRI magnet study, a subset of 20 models was selected for discussion in this and the following subsections. The first model of this set is the initial model of the magnet that was identified in Subsection 3.1.3 of this thesis. For each model, Ampere’s law error was calculated along different paths that were shown in Figure 2.2. The models were then sorted with respect to the average of the errors. The mid-path error, the maximum error considering all paths, the minimum error also considering all paths, and the average error were plotted for each model, and the results are shown in Figure 3.3-a. In this figure, the mid-path, maximum and minimum errors are denoted by symbols while the average error is denoted by a bar. The standard deviations of the model errors were also determined and the results are shown in Figure 3.3-b. Note that the 2D projection models of Models 1, 10, 15, 20 were previously given in Figures 2.5.3, 2.5.4, 2.5.5, and 2.5.6, respectively.

On the basis of the results shown in Figure 3.3, it can be concluded that:

a) The mid-path Ampere’s law error is a good accuracy measure and can be used to judge and compare different models of the magnet. In all but two cases (Models 13 and 20), the mid-path error is very close to the average of errors taken over all paths. However, the results for Models 13 and 20 confirm that model modification can minimize the mid-path
Ampere's law error. Therefore, the mid-path error should be used judiciously and in conjunction with other accuracy measures.

b) The average of Ampere's law errors along different paths is a better global accuracy measure. However, the results for Models 13 and 20 show that a model with a low average error may have a wide range of error. Therefore, this measure does not provide the best criterion for selection of the most accurate model, and should not be used as the sole quality measure. Models 10 to 20 have the same average error of 4%.

c) The standard deviation of Ampere's law errors along different paths can be used to overcome the uncertainty of the best model selection. Figure 3.3-b shows the lowest standard deviation of σ = 0.5 for Model #15, which has an average error of 4%.

On the basis of the foregoing discussion, it can be concluded that, Model #15 is the first candidate for the most accurate model.

3.2.3 Magnetic Field as Selection Criteria

The magnetic field solution was used in Subsection 3.1.3 as an accuracy criterion in order to verify the correctness of the initial model. The application of the magnetic field solution as a criterion in selecting the best model among a set of models is examined in this subsection.

The main problem with using the magnetic field as a selection criterion is that it is a local quality of the model. Using local quality measures to globally compare different models of a problem is unproductive and even misleading. Rather than use point values of the magnetic field, it is suggested that the average of the magnetic field be taken over a region in order to obtain a semi-global measure of solution quality.

The first magnetic field selection criterion is the average magnetic flux density along the air-gap centerline. Its normalized value for the selected set of 20 models is shown in Figure 3.4-a. This line integral selection criterion changes less than 2% from the first to the 10th model; however, it oscillates within a 0.5% range afterward. Therefore, the line integral or average of the magnetic field solution is not very useful as a selection criterion.

The second selection criterion to be examined is the average magnetic flux density at the core center plane at z = 0, a surface integral quality measure. It provides the magnet
saturation information and it is shown in Figure 3.4-b. The model modification from the initial model to the 10th model increases such a measure by less than 1%. Models 10 to 20 introduce an almost identical average magnetic flux density of 2.19 T across the core.

The third magnetic field selection criterion is the average magnetic flux density across the air-gap center plane. The data for this criterion are shown in Figure 3.4-c. This surface integral selection criterion changes around 2% from the first to the 10th model. Models 10 to 20 introduce an almost identical average magnetic flux density of 0.266 T across the air-gap. This is very close to the measured value of 0.27 T in the air-gap.

These tests indicate that the averages of magnetic flux density across the core and air-gap are consistent between Models 10 to 20. Therefore, (a) the optimum model is one of these models, and (b) the average magnetic field can be beneficially used as a selection criterion in conjunction with other global accuracy measures.

The magnetic flux density across the air-gap is reported as 0.27 T and computed as 0.266 T from models 10 to 20. Comparison of Figure 3.4-c (the average B across the air-gap; a semi-global selection criterion) and Figure 3.4-d (the average of Ampere’s law error in different paths; a global accuracy criterion) confirms that Ampere’s law error provides a more reliable predictor of the solution’s global accuracy. Thus, Model #15 is still the best candidate for the most accurate model.

3.2.4 Magnetic Energy as Selection Criteria

The total magnetic energy stored in a model is one common global selection criterion that has been used for electromagnetic problems. The selection direction is also known toward the minimum energy; the model with the lowest stored energy is the best model among many models. However, it should be noted that the stored energy does not provide a direct measure of the solution accuracy. The stored energy in a group of elements can also be used as a semi-global selection criterion, but the selection direction is not clear; the model with the lower energy in a group of elements is not necessarily the better model for the problem as a whole. This subsection investigates the use of magnetic energy as selection criteria (both global and semi-global) on the MRI magnet problem.
Figure 3.5 shows the energy content of the MRI magnet, in addition to the average of Ampere's law error and the magnetic field at the air-gap. The same set of 20 models is chosen for this study. The energy values are from ¼ model of the MRI magnet. The values are normalized with respect to the corresponding values of the initial model.

Figure 3.5-a shows the magnetic energy stored in the magnetic components of the magnet, namely the core, the yoke, and the pole. This energy increases by 10% from the first model to the 10th model. From Model 10 to 19, the magnetic material energy stays at an almost constant value of 375 J. Figure 3.5-b shows the magnetic energy stored in the air-gap of the MRI magnet. It increases by 2% from the first model to the 10th model. From Model 10 to 19, the air-gap energy remains at 512 J level.

These cases indicate that better models of the magnet do not necessarily have a lower regional magnetic energy. In other words, minimizing a regional magnetic energy does not produce the better and best model of a problem. It is concluded that the direction of a semi-global selection criterion (magnetic energy stored in a region of the FE model) is not predetermined (toward the minimum or maximum) and better models may have lower or higher values of a criterion. The direction of these criteria should be determined based on other global accuracy or global selection criteria for a given problem.

Figure 3.5-c shows the total magnetic energy stored in the MRI magnet model. This includes the energy stored in the magnetic material, the air-gap, and the surrounding free-space volume. The total energy decreases continuously from 4621 J for the first model to 4477 J for the 10th model. It stays at a minimum value of 4463 J afterwards. The minimum of the total stored energy in a model is associated with the most accurate model of a problem. Thus, any of Models 10 to 20 can be the optimum model of the MRI magnet.

Note the irregular solutions of Model #20. The total energy is similar to the other models, but the energy in the air-gap and magnetic components behave erratically. The Ampere's law errors along different paths were also found to be erratic. Thus, the selection process focuses on Models 10 to 19. This also confirms that more than one criterion should be used to select the most accurate model of a problem.

The pattern and behavior of the energy content of the model (Figure 3.5-c), the average Ampere's law error (Figure 3.5-d), and the average magnetic flux density across the air-gap
(Figure 3.5-e), are very similar. This confirms the validity of using the average of Ampere's law error as a *global accuracy criterion*, and the energy content of the model as a *global selection criterion* for the FE solutions of the problem. Moreover, the magnetic energy stored in a region of the FE model is also concluded as a useful and practical *selection criterion*. All of these measures indicate that any of Models 10 to 19 can be the optimum model of the magnet. Thus, the final selection would be based on the statistical measures of Ampere's law error.

### 3.2.5 The Selected Optimum Model

This section started with an *initial* correct, but not optimum, FE model of the MRI magnet. Better models were efficiently sought using a novel equivalent linear modeling approach. This study then introduced 20 different FE models of the problem from which the most accurate or the optimum model was to be selected. A considerable improvement was shown from Model #1 to Model #10. In the cases of Models 10 to 19, however, modifications to the magnet model did not enhance the solutions. Different quantities of interest - such as the average magnetic field across the core and air-gap of the magnet, and the stored magnetic energy in the magnetic part, air-gap and whole model of the magnet - were found to be very similar for this set of models. Even the average of the Ampere's law errors along different paths was almost stable at a 4% level for these models. It is thus concluded that further model modification cannot significantly improve the quality of the solutions either meaningfully or within a reasonable resource costs. Therefore, the optimum model of the MRI magnet is any of Models 10 to 19.

While all the global and semi-global selection criteria and global accuracy criteria point to the same range of Models 10 to 19, the final model selection is based on the standard deviation of Ampere's law errors along different paths for each model. This is a statistical measure of a global accuracy criterion and thus a solid selection criterion. It can differentiate between models that are very similar in terms of other accuracy and selection criteria. Model #15 was thus selected as the most accurate or the optimum FE model of the MRI magnet.
Chapter 3. Study of the MRI Magnet with 3D FEM

Figure 3.1: Ampere’s law error for four models of the MRI magnet. The linear analyses utilize different values of \( \mu_r \) for the magnet material.

Figure 3.2: Ampere’s law error for different models of the MRI magnet. The linear analyses utilize \( \mu_r = 100 \) for the magnet material.
Figure 3.3: Ampere's law error analysis for 3D models of the MRI magnet:
  a) Error along different paths is shown for each model.
  b) The standard deviations of Ampere's law errors along different paths.
Figure 3.4: The average magnetic flux density at different locations of the MRI magnet
b) Along the air-gap centerline (at $y=0$) normalized w.r.t. $0.28877$ T.
b) Across the core center plane (at $z=0$) normalized w.r.t. $2.1699$ T,
c) Across the air-gap center plane (at $z=0$) normalized w.r.t. $0.2602$ T,
d) The average Ampere's law error for different models (repeat from Figure 3.3).

The results are shown for different models and are normalized with respect to the initial model value. The relevant data for Model #15 are:

$[B_{\text{across core, } z=0} = 2.1913 \text{ T}, \quad B_{\text{across air-gap, } z=0} = 0.2658 \text{ T}, \quad B_{\text{along air-gap, } y=0} = 0.2903 \text{ T}].$
Figure 3.5: The energy contents in different parts of the MRI magnet:

- The magnetic part energy Normalized w.r.t. 338 J.
- The air-gap energy Normalized w.r.t. 500 J.
- The whole model energy Normalized w.r.t. 4621 J.
- The average Ampere's law error for different models (repeat from Figure 3.3).
- The average magnetic flux density across the air-gap (repeat from Figure 3.4).

The results are shown for different models and are normalized with respect to the initial model value. The relevant data for Model #15 are:

\[ E_{\text{magnetic part}} = 375 \text{ J}, \quad E_{\text{air-gap}} = 512 \text{ J}, \quad E_{\text{total}} = 4464 \text{ J}. \]


3.3 The Optimum Model of the MRI Magnet

The *selection* study concluded that Model #15 was the most accurate of the various models that were developed for the MRI magnet. The magnet specifications were given in Section 2.1, and the basic FE model was shown in Figures 2.1 to 2.4. The numerical quantities of interest were summarized in the previous section. Next, the performance analysis of the MRI magnet is presented in this section. The graphical results obtained using Model #15 are shown in Subsection 3.3.1, and then, two methods to model the magnet coil are studied in Subsection 3.3.2. It is shown that an equivalent current sheet coil can simulate the coil closely. The magnet is also analyzed for different values of the coil current and the results are given in Subsection 3.3.3.

3.3.1 Magnetic Field Distribution

Results have been presented in previous sections for (a) the average magnetic flux density at different locations of the MRI magnet, and (b) the energy content of different parts of the MRI magnet. Selected magnetic field distributions for the optimum FE model of the MRI magnet are presented in this subsection. The real challenge in presenting these results graphically is how to best capture the 3D nature of the model. For example, 3D arrow plots of the magnetic field are too packed and hide rather than show information in a useful sense. Therefore, this type of plot has not been used.

More usefully, contour plots of the magnetic flux density magnitude can be used to graphically show distributions. However, the connectivity and clarity of the results are reduced when many 2D slices of the model are used. On the other hand, complete isotropic 3D plots do not show the data properly and should not be used. Therefore, the approach used in this subsection is to graph the data on two important slices of the magnet parts (the core, the yoke, the pole and the air-gap) but within a 3D frame. These slices are selected at the \( y=0 \) and \( z=\) constant planes of the magnet parts. As a result, the magnetic field on the external
surface of these parts is not presented. To compensate for such a shortcoming, Figure 3.6 shows the magnetic flux density, \( B \), on the external surface of the magnet.

**Magnet:**

Figure 3.6 - page 71 at the end of this section - sets the coordinate and the view direction for the subsequent figures. It shows the magnetic field distribution on the external surface of the magnet. The core region (in the upper left-hand portion of the figure) is highly saturated while the pole (in the upper right-hand) remains unsaturated. The yoke experiences a mild saturation close to the core, but is not magnetically saturated close to the pole. The rotational symmetry of the field at the core and pole exteriors, together with the significant changes of the magnetic flux density in different parts of the magnet, are noteworthy features of the distribution.

It should also be noted that the results are not the most exact ones at the external surface of the magnet. Errors are caused by the material discontinuity, the lack of boundary condition enforcement and the resultant discontinuity of the tangential \( H \) between different media, and the post-processing of the magnetic vector potential data.

**Core:**

The results at two important 2D slices of the core are shown in Figure 3.7. The semicircle wire frame is the connection area of the core to the yoke. The \( z = 0 \) plane of the core (the upper right hand semicircular area) has a normal magnetic field boundary condition while the \( y = 0 \) plane has a tangential magnetic field boundary condition. The three degrees of freedom at each node are thus reduced to two degrees and one degree, correspondingly. Therefore, the results at these slices are considered to be the most accurate ones. The core is highly saturated and the magnetic field is almost uniform at a \( z = c \) plane. Both are due to the presence of the coil around, along, and close to the core. The magnetic field has rotational symmetry with respect to the core axis at \( (x = -400 \text{ mm}, y = 0) \).
Yoke:

The magnetic field distribution for the lower plane (connection regions to the core and the pole at $z = 300$ mm) and symmetry plane of the yoke is shown in Figure 3.8. The wire frame (at $z = 615$ mm) shows the airside of the yoke. The symmetry plane $y = 0$ has a tangential boundary condition and thus shows less disruptive or erroneous data. The worst elements of the magnet model are located at (a) the inner connection region of the yoke and the core and (b) the inner connection region of the yoke and the pole. The rapid changes of the field at these locations are shown in Figure 3.8. Refining the size of these elements has not improved the quality of the solutions. The comparison of Figure 3.6 and Figure 3.8 reveals that the yoke is a leaky magnetic flux pipeline from the highly saturated core to the unsaturated pole.

Pole:

Figure 3.9 shows the magnetic flux density distribution at the $y = 0$ plane and the $z = 150$ mm plane of the pole - which is the upper right hand semicircular area. Note that the $y = 0$ plane (the rectangular area of the figure) has an imposed tangential boundary condition. This figure, together with Figure 3.6, shows that the exterior of the pole cylinder is moderately saturated and has a semi-rotational symmetry. The $y = 0$ symmetry plane shows the rapid reduction of the magnetic flux density from the yoke side to the air-gap side of the pole. The pole lower plane at $z = 150$ mm has a low magnetic flux density level which shows a semi-rotational symmetry with respect to the pole axis. This is due to the presence of the free-space on the air-gap side of the pole face.

Air-Gap:

The magnetic flux density of the air-gap is shown in Figure 3.10. The $z = 0$ plane represents the mid-plane of the air-gap and therefore is a symmetry plane with a normal magnetic field boundary condition. This plane is shown as the vertical semicircle in the figure. Similarly, the horizontal plane in the figure is taken at $y = 0$ and is also a symmetry plane, but with a tangential magnetic field boundary condition. The $z$ and $y = 0$ planes in this figure are also two slices of the MRI magnet’s volume of interest within which there should be a high
degree of field uniformity. Recall that this region of field uniformity was specified to be a concentric cylinder within the air-gap but with half of its radius and length. Its borderlines are shown in Figure 3.10. It will be noted in the figure that the magnetic flux density has a general rotational symmetry with respect to the air-gap axial axis. This symmetry is more pronounced in the volume of interest and off-centered outside this volume. The numerical results at these two regions, the air-gap and the region of interest, are:

a) The magnetic flux density is 0.270 T along the perimeter of the volume of interest and reaches a maximum of 0.288 T in the volume center.

b) The magnetic flux density is between 0.225 T and 0.245 T at the periphery of the air-gap volume. The average magnetic flux density across the air-gap, at \( z = 0 \) plane, is 0.2658.

It should be noted that the above-noted figures for field uniformity relate only to the FE model. A very uniform magnetic field is practically produced, once the equipment has been installed, by introducing shimming coils and other devices.

### 3.3.2 Coil Modeling Effect

The coil representation that is used in the FE model of the MRI magnet should closely approximate the actual MRI coil. The model used for the coil affects the accuracy of the solutions, especially a global accuracy measure such as Ampere's law error. The FE modeling and representation of the coil is the focus of this subsection. There are two methods for modeling the coil: (a) to model its volumetric geometry, or (b) to simulate the coil by an equivalent current sheet. It is shown that a current sheet effectively simulates the coil of the MRI magnet. This is very beneficial where the exact data of the coil are not available (e.g. in initial stages of designing a device) and an equivalent coil can be used to simulate a real coil and energize the magnet model.

In the first method, the coil geometry is modeled accurately. The main difficulty is the thinness of the coil with respect to the coil length and the magnet dimensions. The modeling of such a coil is very difficult and time consuming. However, the current loading is easy where the current density is calculated once and applied to the coil elements. The
interpolation functions of each element are then used to obtain the proper current segment value of each node. This method is called the elemental loading of the FE model.

In the second approach, where the coil is narrow enough, a current sheet simulates the coil. Such a coil is very easy to model. The properly placed nodes - at the mid-radius of the coil and along the coil - are used to model the coil. The values of the current segments are calculated and assigned to these nodes directly. In this respect, the coil representation requires more human resources before the solution stage. This method is called the nodal loading of the FE model.

All the magnet models used in this thesis accommodate both types of coil representation. To confirm the accuracy and performance of the magnet models regardless of the coil model, four models of the magnet were selected. These models are - from the previous set of 20 models - Model #1 as the initial model, Model #10 as the entrance model to the optimum model neighborhood, Model #15 as the optimum model, and Model #20 as the erratic model of the set.

These models were solved with the volumetric coil and with the current sheet coil. Different quantities of interest are shown in Figure 3.11 (page 76) for both approaches. The results for the volumetric coil models are shown as white bars, while black bars are used to show the results of the current sheet coil models. The results are also grouped for each quality of interest; namely (a) the average of magnetic flux density across the core (group #1), across the air-gap (group #2), along the air-gap (group #3), (b) the stored magnetic energy in the air-gap (group #4), in the magnetic part (group #5), in the whole model (group #6), and (c) the average of Ampere's law error along different paths (group #7). The results are also normalized with respect to Model #15 results.

The quality measures of both coil models agree closely except for the air-gap energy of the erratic Model #20. Therefore, it is concluded that acceptable solutions can be obtained with a proper equivalent current sheet. This conclusion will also be used when an alternative to the MRI magnet design is examined for the case where the coil data are not available.
3.3.3 Coil Current Level Analysis

The main objective of this subsection is to study the magnet performance for different values of the coil current. The results provide engineering insights for simplifying the magnet model in later studies and for the design of similar devices. This study also explains many questions regarding the modeling of the magnet. The current load of the magnet coil is varied from 10 kA-t to 60 kA-t and quantities of interest are obtained. These quantities are discussed in the following subsections.

**Magnetic Field:**

The values of magnetic flux density, averaged across the core as well as across and along the air-gap, are shown in Figure 3.12 as a function of the coil excitation. The results are all normalized to 100% for the 60 kA-t excitation. It will be noted that the three graphs are almost identical.

It can be concluded that for this device, one well-defined normalized quality measure can effectively represent other quality measures, regardless of the coil excitation level. The effect of saturation is clear for excitation levels higher than 30 kA-t. The averaged field is increased by only 10% after raising the current level from 40 to 60 kA-t. This brings up the question of the magnet design and its operating efficiency.

**Magnetic Energy:**

Figure 3.13 shows three normalized magnetic energy in the magnetic parts (including the core, the yoke and the pole), in the air-gap, and in the total volume of the model. The results are normalized to 100% for the 60 kA-t excitation.

For excitation levels less than 30 kA-t, the magnet is not saturated and most of the coil energy is stored in the air-gap magnetic field. At higher excitation levels, more energy is stored in the magnetic portion of the device in almost a linear fashion. The slope of the total energy graphs changes around 30 kA-t, which indicates that saturation has begun to set in.
Energy Ratio:

To clarify the energy content and its distribution in the magnet, an energy ratio is defined for a volume of interest as follows:

\[ E.R._V = \frac{\text{Volume Energy}}{\text{Total Energy}} \]

Figure 3.14 shows this ratio for the magnetic part of the magnet, for the air-gap of the magnet, and for the total volume of the model excluding the magnetic part and the air-gap. The results are normalized to 100% for total energy of the model at each current level.

As the coil current is increased, the magnetic part becomes more saturated and stores more of the added energy. Meanwhile, the air-gap share of the energy is reduced, but not substantially. Thus, the increased energy share of the magnetic part comes from the energy stored in the external volume. For current levels higher than 30 kA-t, the E.R._V changes almost linearly versus the current excitation.

It is noteworthy that around 80% of the magnetic energy is stored in the free-space region surrounding the magnet and the air-gap, regardless of the coil current level. This justifies the precise and careful approach chosen in this thesis to model the external free-space region. This also explains why the FE model and its solutions were more accurate when a better modeling of the exterior free-space was employed.

Magnetomotive Force:

The magnetomotive force drops along the magnetic part and the air-gap of the magnet are shown in Figure 3.15. These results show the contributions of the magnetic portion of the magnet, as well as the contribution of the air-gap, to Ampere’s law error. At the full excitation, 38% and 58% of the 60 kA-t are dropped along the magnetic part and air-gap of the magnet, respectively. Thus 4% of the excitation is not accounted for and is reported as the error. The graphical similarity of the energy ratio in Figure 3.14 and the magnetomotive force in Figure 3.15 are noteworthy.
3.3.4 Performance Summary of the Optimum Model of the MRI Magnet

For the original model of the MRI magnet, where the coil is placed around the core, and at the operational current excitation level of 60 kA-t, the device performance summary and the general qualities of interest are given as follows:

- The average of Ampere's law error along different paths is 4%.
- The standard deviation of Ampere's law error along different paths is 0.5.
- The magnetic energy stored in the model as 4464 J.
- The magnetic energy stored in the air-gap of the model as 512 J.
- The magnetic energy stored in the magnetic part of the model as 375 J.
- The average magnetic flux density across the core as 2.1912 T.
- The average magnetic flux density across the air-gap as 0.2658 T.

It should be noted that:

a) Only a small amount, 11.5%, of the total magnetic energy is stored in the air-gap volume.

b) Only a small amount, 8.4%, of the total magnetic energy is stored in the magnetic part of the magnet.

c) Most of the total magnetic energy, 80.1%, is stored in the volume exterior to the magnet.

d) Only 12% of the magnetic flux at the core center plane reaches the air-gap center plane.

Therefore, the original design of the MRI magnet, where the coil is placed around the core, is not satisfactory. A better alternative design of the MRI magnet is given in the next section.
Figure 3.6: The contour plots of the magnetic flux density in the MRI magnet.

Portion of the figure that is unsaturated.
Portion of the figure that is highly saturated. The pole is shown in the upper right-hand
where the coil is around the core. The core is the cylindrical volume in the upper left-hand.
Figure 3.7: The contour plots of the magnetic flux density magnitude in the MRI magnet core at \( z = 0 \) plane (across the core cylinder and in the upper right hand portion of the figure) and at \( y = 0 \) symmetry plane (along the core cylinder.) The coil is around the core. The semicircle wire frame (in the lower left hand portion of the figure) is the connection area of the core to the yoke.
Figure 3.8: The contour plot of the magnetic flux density magnitude in the MRI magnet shows the inside of the yoke.
Figure 3.9: The contour plots of the magnetic flux density magnitude in the MRI magnet pole at $z = 150$ mm plane (across the pole and adjacent to the air-gap) and at $y = 0$ symmetry plane (the rectangular area along the pole cylinder.) The coil is around the core. The semicircular borderline or wire frame is the connection area of the pole to the yoke.
Figure 3.10: The contour plots of the magnetic flux density magnitude in the MRI magnet air-gap at $z = 0$ plane (across the air-gap) and at $y = 0$ symmetry plane (along the air-gap.) The coil is around the core. The borderlines of the region of interest, that is a concentric cylinder within the air-gap but with half of its radius and length, are shown inside the air-gap. The semicircular borderline or wire frame (in bottom left hand portion of the figure) is the connection area of the air-gap to the pole.
Figure 3.11: The average magnetic flux density across the core Group #1, across the air-gap Group #2, along the air-gap Group #3.

The stored magnetic energy in the air-gap Group #4, in the magnetic part Group #5, in the whole model Group #6.

The average of Ampere's law error Along different paths Group #7.

The nodal (equivalent current sheet coil) and elemental (volumetric coil) current excitations are used. The data are normalized with respect to the Model #15 results.
Figure 3.12: The average magnetic flux density at different locations of the magnet versus the coil excitation levels of 10 to 60 kA-t. The results are normalized with respect to the final results of the 60 kA-t excitation. The normalization values are given on each graph.

Figure 3.13: The stored magnetic energy in different parts of the magnet versus the coil excitation levels of 10 to 60 kA-t. The results are normalized with respect to the final results of the 60 kA-t excitation. The normalization values are given on each graph.
Figure 3.14: The energy ratio of the magnetic part of the magnet, the air-gap of the magnet, and the remaining volume out of the magnet versus the coil excitation levels of 10 to 60 kA-t. The results are normalized with respect to the total energy of the model at each load step.

Figure 3.15: The average of the magnetomotive force drop along the magnetic part, the average of the magnetomotive force drop along the air-gap, and Ampere’s law error versus the coil excitation level of 10 to 60 kA-t. The results are normalized with respect to the current level at each load step.
3.4 Design Alternative: Coil around the Pole

Amongst different components and parameters of the original MRI (O-MRI) magnet, the location of the coil offers one attractive possibility to improve its performance. The performance of the magnet, as originally designed, was summarized in Subsection 3.3.4. The study of an alternative design of the MRI (A-MRI) magnet and its performance is the main focus of this section. The FE model and solution of this magnet is given in Subsection 3.4.1. The contour plots of the magnetic flux density are presented in Subsection 3.4.2. The A-MRI magnet is also analyzed for different levels of the coil current and the results are given in Subsection 3.4.3.

3.4.1 FE Analysis of the Alternative MRI Magnet

The analysis of the alternative design of the MRI (A-MRI) magnet is a typical engineering problem: the analysis of a physical device before its manufacture. Questions like how best to define the boundary value problem, should it be defined in terms of MVP or MSP? What is the domain definition and discretization? Are the solutions of the FE model correct, and by what measure? Is the developed FE model, an optimum model? Can its accuracy be enhanced, and if so by how much, and at what cost? In other words, all the steps taken for the study of the O-MRI magnet can/should be repeated.

Previous studies of this thesis have provided the advantage of having the answers to most of these questions. The current A-MRI magnet is a similar problem to the previous O-MRI magnet, except it is less difficult due to a low level of magnetic saturation in the magnet. Therefore, the FE study of the A-MRI magnet would be brief. It focuses only on some critical issues such as accuracy and the development and identification of an optimum model. The steps that were taken to study this device are summarized in this subsection.
A-MRI Specifications and Model:

The geometry of the A-MRI magnet is similar to the O-MRI magnet, the latter having been given in Section 2.1. The same magnetic material, core, yokes, and poles are used for the alternate design. The only modification is that the coil is split and positioned around the poles, as opposed to being around the core. The total current of the two coils in the alternate design is equal to the O-MRI magnet current of 120 kA-t. Due to symmetry, only one quarter of the magnet, and thus only one of the coils, needs to be modeled. The current excitation of each coil is 60 kA-t. The coil height is 115 mm, while the pole height is 150 mm. The air-gap side of the coil is flush with the pole face at z=150 mm.

As before, the boundary value problem was defined in terms of the magnetic vector potential. The problem domain was then defined and discretized. Next, the Newton-Raphson method was used to solve the nonlinear system of equations. The magnetic field data were then obtained and investigated.

It should be noted that all the FE models of the MRI magnet were developed with the provision of accommodating a volumetric coil or a current sheet coil being placed either around the core or around the pole. To differentiate between models of two magnets, the alternate MRI magnet models are designated with a suffix "A." For example, Model #15A is the FE model of the A-MRI magnet whose problem definition and formulation, domain dimension, domain grid, and material all are the same as Model #15 (shown in Figures 2.1 to 2.5), but with a current sheet coil being around the pole, and of course, different solutions.

Initial Model:

Model #1A is the first candidate for the initial model. Its problem domain and discretization are identical to the model that was detailed in Section 2.1. Its 2D projection model is also identical to the one shown in Figure 2.5.3. Ampere's law errors were calculated along the same paths of the magnet that were shown in Figure 2.2. The average of errors was obtained as 4.7%, which is within a reasonable limit. Therefore, Model #1A is introduced as the initial model of the A-MRI magnet.
Selection Study:

The initial model was modified systematically in order to identify the most accurate model of the A-MRI magnet. The chosen selection criteria were (a) the magnetic flux density, (b) the energy contents, and most importantly (c) the Ampere’s law errors of the models.

From the set of 20 models, presented in Section 3.2, only four models were selected:

1) Model #1A, as the initial model of the O-MRI model,
2) Model #10A, as the entrance model to the optimum model neighborhood,
3) Model #15A, as the optimum model candidate, and
4) Model #20A, as the erratic model of the MRI magnet.

These FE models were solved and the results are shown in Figure 3.16 - page 88 at the end of this section. The quantities of interest are normalized with respect to the solutions of Model #15A of the A-MRI magnet, which are also given on the graphs.

The Optimum Model:

Among different models, Model #15A is the optimum model of the A-MRI magnet because it provides:

a) The minimum average of Ampere’s law errors along different paths at 2.5%,
b) The minimum standard deviation of Ampere’s law errors along different paths at 0.27,
c) The minimum magnetic energy in the whole model of the magnet at 2990 J,
d) The maximum magnetic flux density across the air-gap center plane at 0.419 T,
e) The maximum magnetic flux density across the core center plane at 0.835 T,
f) The maximum magnetic energy stored in the air-gap of the magnet at 1477 J, and
g) The maximum magnetic energy stored in the magnetic part of the magnet at 22 J.

This of course is not surprising since Model #15 has closely been identified as being the optimum model for the original MRI magnet. All that has changed in Model #15A is the location of a current sheet approximation of the coil.
3.4.2 Magnetic Field Graphical Results

Contour plots of the magnetic flux density are used to show the results. The sequence of graphs, as before, starts with the general external overview of the magnet, as shown in Figure 3.17. The magnetic field at selected slices of the magnet core, yoke, pole, and air-gap are then presented.

**Magnet:**

The distribution of the magnetic flux density magnitude over the external surface of the core, yoke and pole is shown in Figure 3.17. This figure also defines the coordinates and the view direction of the subsequent figures. As can be seen in the figure, the pole is only saturated in a relatively small region at the pole face edge. The magnetic field over the surface of the yoke is lower than the saturation level of 1.5 T, and its value diminishes at the corner sections of the yoke. The magnetic flux density at the inner side of the core (toward the air-gap) is around 1 T, while its value is reduced to 0.5 T at the outer side of the core (far from the air-gap). The following observations are noteworthy:

- The magnetic field at the exteriors of the pole is rotationally symmetric,
- The magnetic flux density changes smoothly between different parts of the magnet, and
- The magnetic flux density level is between 1 to 1.2 T in most of the magnet volume.

**Core:**

The results at two important 2D slices of the core are shown in Figure 3.18. The normal and tangential magnetic field boundary conditions are set at the \( z = 0 \) and \( y = 0 \) planes of symmetry, respectively. The results at these slices are thus the most accurate ones. The core magnetic flux density is slowly changing from 1 T to 0.7 T, and its average across the core at the \( z = 0 \) plane is 0.8350 T. The core magnetic flux density lacks a rotational symmetry.

It is concluded that the core only closes the magnetic circuit of the magnet and that it is not saturated. This can be used to enhance the future designs of the magnet in two ways: first, the radius of the core can be reduced in order to decrease the core volume, and second, a less expensive magnetic material can be used in order to reduce the cost of the core.
Yoke:

The results for the yoke lower plane at $z = 300$ mm and the yoke symmetry plane at $y = 0$ are shown in Figure 3.19. In the major volume of the yoke - from the centerline of the core to the centerline of the pole - the magnetic flux density inside the yoke is approximately 1.2 T. On the other hand, in the top corners of the yoke, the magnetic flux density is between 0 and 0.4 T. A rapid change of the magnetic field exists only at the inner connecting regions of the yoke with the core and the pole. The following observation can be concluded:

- The magnetic flux density in the active volume of the yoke is around 1.2 T,
- The low magnetic field regions are in the outer sides and corners of the yoke, and
- The yoke acts as a good magnetic flux pipeline between the core and the pole.

Pole:

The results for the pole face at $z = 150$ mm and the pole symmetry plane at $y = 0$ are shown in Figure 3.20. The magnetic flux density in most of the pole volume is between 1 T and 1.4 T. It is between 0.55 T and 2 T at the pole face. However, it changes only between 0.55 T and 0.77 T in the center region of pole face. The average magnetic flux density at the pole face is 0.62 T.

The rapid change of the magnetic field is due to the proximity of the coil and the air-gap to the pole face. The magnetic flux density at the pole face has a clear rotational symmetry, which is mainly due to the location of the coil around the pole.

Air-Gap:

The magnetic flux density in the air-gap is shown in Figure 3.21. The $z = 0$ and the $y = 0$ planes are the normal and tangential magnetic field planes. The $z$, $y = 0$ planes include two slices of the MRI magnet volume of interest; i.e. the volume within which maximum uniformity is desired for the flux density distribution. These are also shown in Figure 3.21 by the borderlines of the inner cylinder. The results at the air-gap regions are:
a) The magnetic flux density is 0.45 T around the volume of interest and reaches a maximum of 0.48 T in the volume center. It is suggested that the 0.03 T deviation could be reduced by introducing some measure of pole shaping.

b) The magnetic flux density's approximate value is 0.33 T at the periphery of the air-gap volume. The average magnetic flux density across the air-gap at \( z = 0 \) plane is 0.4193 T.

The magnetic flux density has a pronounced rotational symmetry with respect to the air-gap axial axis, especially in the air-gap volume of interest.

### 3.4.3 Coil Current Level Analysis

The main purpose of this subsection is to study the alternative model of the MRI magnet for different levels of the current excitation. The current level of the coil is changed from 20 kA-t to 120 kA-t and the quantities of interest are calculated. The operational current level of the coil is 60 kA-t. The quantities of interest are discussed in the following paragraphs.

**Magnetic Field:**

The magnetic field within the core was found to be lower than the saturation level, even for the high excitation of 120 kA-t. The pole and yoke connection area was thus selected to show the highest magnetic flux densities in the magnet. Figure 3.22 shows the normalized average magnetic flux densities at the pole and yoke connection area, across the air-gap, and along the air-gap. The magnetic flux density across the yoke-pole area changes linearly with respect to the current levels that are lower than 80 kA-t. However, for excitation levels higher than 80 kA-t, the magnetic flux density in this region exceeds the saturation level of 1.5 T. The magnetic field at the air-gap of the magnet changes almost linearly with respect to the current excitation.

**Magnetic Energy:**

The normalized stored magnetic energy in different parts of the magnet is shown in Figure 3.23. The results are normalized to 100\% for the 120 kA-t excitation. The graph of the total energy shows that a mild saturation dominates the magnet at current levels higher than 80
kA-t. The same holds true for the air-gap energy content because it holds half of the magnet energy. The magnetic part of the magnet begins to store energy at the outset of the saturation approximately at 80 kA-t.

Energy Ratio:
The energy ratios of the magnetic part, the air-gap, and the free-space exterior to the magnet are shown in Figure 3.24. The magnetic part of the magnet stores only 1% of the total energy while the air-gap stores around 50% of the total energy - regardless of the current level. At the operational current level of 60 kA-t, 0.7%, 49.3%, and 49.9% of the total 2990 J is stored in the magnetic part, the air-gap, and the exterior free-space region, respectively.

Magnetomotive Force:
The magnetomotive force drops and the mmf error are shown in Figure 3.25. Most of the mmf drop occurs along the air-gap. For current levels higher than 80 kA-t, the saturation increases the mmf drop along the magnetic part of the magnet, which consequently decreases Ampere's law error.

The summary of the O-MRI magnet performance data will be given in the next section.

3.5 Comparison of MRI Magnet Designs

Although the focus of the thesis has been to develop global accuracy criteria and selection criteria, the comparative study of two MRI magnet designs offer engineering insights that lead toward a better design of the magnet. Table 3.2 compares the results from two designs of the MRI magnet. The last column is the ratio of the alternative (A-MRI) design results to the original (O-MRI) model results. The current excitation level for both models is 60 kA-t for the ¼ symmetry model, or equivalently 120 kA-t for the whole magnet.

The stored magnetic energy in the A-MRI magnet is 67% of the O-MRI magnet stored energy. Such a reduction in total energy is accompanied (a) by an increase in the air-gap energy by 188%, and (b) by a substantial decrease of the energy stored in the magnetic part. The energy ratio of the A-MRI air-gap is three times more than the original design's value.
Therefore, the A-MRI magnet is far better than original design in terms of the magnet energy. Most importantly, the better energy distribution and efficiency of the alternative design is accompanied by a significant 58% increase in the magnetic flux density across the air-gap, and moreover, such increase is achieved without significantly sacrificing the field uniformity. It is concluded that the alternative design performance is far better than the original model of the MRI magnet.

<table>
<thead>
<tr>
<th>Quantities of interest</th>
<th>A-MRI</th>
<th>O-MRI</th>
<th>A/O</th>
</tr>
</thead>
<tbody>
<tr>
<td>The magnetic energy stored in the whole model:</td>
<td>2990 J</td>
<td>4464 J</td>
<td>67%</td>
</tr>
<tr>
<td>The magnetic energy stored in the air-gap:</td>
<td>1477 J</td>
<td>512 J</td>
<td>288%</td>
</tr>
<tr>
<td>The magnetic energy stored in the magnetic part:</td>
<td>22 J</td>
<td>375 J</td>
<td>6%</td>
</tr>
<tr>
<td>The magnetic energy stored out exterior to the magnet:</td>
<td>1499 J</td>
<td>3577 J</td>
<td>42%</td>
</tr>
<tr>
<td>The energy ratio, E.R., of the air-gap:</td>
<td>49.3%</td>
<td>11.5%</td>
<td>429%</td>
</tr>
<tr>
<td>The energy ratio of the region exterior to the magnet:</td>
<td>49.7%</td>
<td>80.1%</td>
<td>62%</td>
</tr>
<tr>
<td>The energy ratio of the magnetic part:</td>
<td>0.7%</td>
<td>8.4%</td>
<td>8%</td>
</tr>
<tr>
<td>The average magnetic flux density, B, across the air-gap:</td>
<td>0.419 T</td>
<td>0.266 T</td>
<td>158%</td>
</tr>
<tr>
<td>The average magnetic flux density across the core:</td>
<td>0.835 T</td>
<td>2.19 T</td>
<td>38%</td>
</tr>
<tr>
<td>The magnetic flux of the air-gap + the core magnetic flux:</td>
<td>50.2%</td>
<td>12%</td>
<td></td>
</tr>
<tr>
<td>The average of Ampere's law errors, $e_{mef}$:</td>
<td>2.5%</td>
<td>4%</td>
<td></td>
</tr>
<tr>
<td>The standard deviation of Ampere's law errors:</td>
<td>0.27</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>

3.6 Conclusion

The accuracy and selection issues of the basic FE model were generally discussed in Chapter 2. The MRI magnet problem was then solved in this chapter. An alternative design for the MRI magnet was also suggested. The accuracy of the A-MRI model and the optimum model were studied. The performances of both designs of the MRI magnet were then compared, and the A-MRI design was concluded to be superior to the original design of the MRI magnet.

The main conclusions of this chapter are the following:
a) Ampere’s law error measure is a valid global accuracy measure that should be used carefully. The path dependency of such a measure causes a few limitations in its application as a global accuracy criterion.

b) Statistical indicators of Ampere’s law error along different paths, its average and standard deviation, are reliable global accuracy measures.

c) The stored magnetic energy in the FE model is a valid selection criterion. The magnetic energy in the FE model parts, such as the magnetic part, and the air-gap of the MRI magnet, can also be used as selection criteria.

d) The average of the magnetic flux density at different locations of the magnet is a good selection criterion. When such an average is taken across the core or air-gap, it shows a trend similar to the average of Ampere’s law error measure.

e) The free-space material method and the reduced excitation method were very effective in obtaining the initial model of the MRI magnet.

f) The equivalent linear modeling method - introduced in Subsection 3.2.1 - substantially improved the efficiency of the selection search for better models of the MRI magnet. It was shown that the accuracy of the nonlinear model could be estimated from a linear model of the problem.
Figure 3.16: The average magnetic flux density across the core Group #1, across the air-gap Group #2, along the air-gap Group #3. 

The stored magnetic energy in the air-gap Group #4, in the magnetic part Group #5, in the whole model Group #6. 

The average of Ampere's law error Along different paths Group #7. 

Models 1A, 10A, 15A, 20A are used. The nodal current excitation of 60 kA-t is used. The data are normalized with respect to the Model #15A results:

\[ B_{\text{cross core}} = 0.8350 \text{ T} \quad B_{\text{across air-gap}} = 0.4193 \text{ T} \quad B_{\text{along air-gap}} = 0.4840 \text{ T} \]
\[ \text{Energy}_{\text{air-gap}} = 1477 \text{ J} \quad \text{Energy}_{\text{magnetic part}} = 1477 \text{ J} \quad \text{Energy}_{\text{total}} = 2990 \text{ J}. \]
Figure 3.17: The contour plots of the magnetic flux density magnitude in the A-MRI magnet, where the coil is around the pole.
Figure 3.18: The contour plots of the magnetic flux density magnitude in the A-MRI magnet core at $z = 0$ plane (across the core cylinder) and at $y = 0$ symmetry plane (along the core cylinder.) The coil is around the pole.
Figure 3.19: The contour plots of the magnetic flux density magnitude in the A-MRI magnet yoke at $z = 300$ mm plane (lower plane of the yoke slab) and at $y = 0$ symmetry plane. The coil is around the pole.
Figure 3.20: The contour plots of the magnetic flux density magnitude in the A-MRI magnet pole at $z = 150$ mm plane (across the pole and adjacent to the air-gap) and at $y = 0$ symmetry plane (along the pole cylinder.) The coil is around the pole.
Figure 3.21: The contour plots of the magnetic flux density magnitude in the A-MRI magnet air-gap at $z = 0$ plane (across the air-gap) and at $y = 0$ symmetry plane (along the air-gap.) The coil is around the pole.
Figure 3.22: The average magnetic flux density at different locations of the magnet versus the excitation level of 20 to 120 kA-t. The results are normalized with respect to the final results of 120 kA-t excitation. The normalization values are given on each graph.

Figure 3.23: The stored magnetic energy in different parts of the magnet versus the excitation level of 20 to 120 kA-t. The results are normalized with respect to the final results of 120 kA-t excitation. The normalization values are given on each graph.
Figure 3.24: The energy ratio of the magnetic part of the magnet, the air-gap of the magnet, and the remaining volume out of the magnet versus the excitation level of 20 to 120 kA-t. The results are normalized with respect to the total energy of the model at each load step.

Figure 3.25: The average of the magnetomotive force drop along the magnetic part, the average of the magneto-motive force drop along the air-gap, and Ampere's law error versus the excitation level of 20 to 120 kA-t. The results are normalized with respect to the current level at each load step.
Chapter 4
Study of the MRI Magnet with 2D FEM

4.1 Introduction

The 3D FE study of the MRI magnet was given in Chapter 3. However, the development of a 3D model of even such a simple structure as the MRI magnet is very time consuming, both in terms of human effort as well as computational resources. An attractive alternative is to use a suitably designed 2D model with the caveat that it does not simplify the basic problem out of existence. Defined correctly, 2D models can be used cost effectively to explore design alternatives prior to undertaking a full 3D modeling effort. The purpose of this chapter is to compare 2D (approximate) and 3D (exact) models of the MRI magnet system with a view of developing a method that simplifies the analysis without sacrificing the integrity of the results thus obtained.

It is always preferable to undertake the analysis of an electromagnetic device using 2D models, provided suitable planes of symmetry can be identified. The main objective of this chapter is to show that even in the case of the highly saturated MRI magnet, engineering and physical insight aid in identifying suitable 2D models. By direct comparison with the results of measurement and of an optimum 3D FE analysis, it is shown that the 2D approximate models provide surprisingly good results in selected regions of the magnet system.

A physical system with either longitudinal or rotational symmetry can be entirely modeled in 2D. In reality, most physical systems have no such symmetry and cannot be
modeled entirely in a 2D space. However, there are two generic types of approximations to reduce the dimensionality of a physical system and simplify its study:

a) If the portion of the physical system that is of interest can be isolated from the system as a whole and studied conveniently in 2D, one obtains a partial 2D model. This approach is used and discussed in Sections 4.2 and 4.3.

b) If the geometry of the physical system can be modified such that some critical features are preserved in either 2D translational or rotational symmetry, one obtains a composite 2D model. This method is detailed in Section 4.4.

Both approaches are examined within the context of the 3D MRI magnet system. The main objective is to determine whether the 2D approximate models can yield useful and accurate results.

The remainder of this section discusses the axisymmetrical 2D approaches for study of the MRI magnet. It also clarifies the objectives and structure of this chapter in more detail.

4.1.1 Axisymmetric Approximation of the MRI Magnet

In an axisymmetrical problem, a rotational or cylindrical symmetry exists in the problem that simplifies its representation. This method can also be used to study some specific regions and aspects of the MRI magnet problem. The steps in the axisymmetric FE modeling of the MRI magnet are summarized in Appendix D.

The focus of the thesis, the MRI magnet, has been studied in 2D by employing three approximate models:

a) A partial 2D model where the coil is located around the core. The super-conductive coil and the enclosed core magnetic material were removed from the MRI magnet and studied using an axisymmetric 2D FEM. This study is given in Section 4.2. It is shown that this approach yields reasonable results when the core is highly saturated.

b) A partial 2D model where the coils are placed around the poles. The combination of the pole piece, the coil, and the air-gap were separated from the 3D problem domain and analyzed with the axisymmetric 2D FEM. This study is detailed in Section 4.3. It is
confirmed that the air-gap magnetic field solutions can be conveniently approximated by this approach.

c) A composite 2D model where the coil is located around either the core or the poles. The effect of the yoke on the core and air-gap magnetic field is assumed to be negligible. Thus the 3D problem domain was modified, first, by preserving the core, poles, and coil cylindrical configuration, and second, by attributing a cylindrical shape to the yoke. The whole composite magnet was then modeled and studied with the axisymmetric 2D FEM. The study and the relevant results are given in Section 4.4. This modeling practice has proven to be the most practical one to estimate the magnetic field in the core and air-gap of the magnet. The solutions are shown to be surprisingly close to the 3D solutions regardless of the coil location and excitation level.

Objectives:

The main objectives in the study of the approximate 2D FE models are (a) to determine whether they can yield useful and accurate results, and (b) to obtain the extent of their applicability. These issues are explored in this chapter. The quantities of interest are limited to the magnetic flux density in the core, the pole and the air-gap. The averages of the magnetic flux density from 2D and 3D analyses are compared.

4.2 A Partial 2D Model of the Coil and the Core

When the coil is concentric with the core of the MRI magnet, the partial modeling and study of the core can be justified by two arguments:

a) The high level of the coil excitation causes a high magnetic saturation in the core. It is assumed that the region of the magnet that is completely inside the coil is not affected by other components of the magnet such as the yoke, pole and air-gap. In other words, it is assumed that the core is magnetically isolated from the remainder of the magnet.

b) Figures 3.6 and 3.7 (from 3D FE solutions of the MRI magnet) displayed the magnetic flux density contour lines at the core. The core magnetic field was rotationally symmetric with respect to the core central axis, and it was almost uniform at any $z = h$ plane – where
\( h < L_{\text{coil}} \). This suggests that the core magnetic field can be presented in an axisymmetrical fashion. Therefore, it is suggested that the coil and the enclosed core magnetic material be separated from the rest of the magnet and be modeled individually.

The new problem of the coil and the enclosed magnetic material has a cylindrical symmetry, and therefore, can be modeled with axisymmetric 2D FEM. The problem domain of this partial 2D model is shown in Figure 4.1 - page 103, at the end of this section. The quantity of interest is the magnetic flux density at the core center plane \((z=0)\). By comparing with the full 3D model, it is possible to determine the limits and conditions under which this approximation is justified. The approximate model can be beneficially used to estimate \( B \) at the core central plane as a function of the core geometry.

The unclear part of this model is the definition of the proper boundary conditions. However, the field data are mainly sought in the center region of the core, which is surrounded by the coil. Therefore, the boundary condition values should not affect the results greatly. The normal, tangential and far-field boundary conditions are used and are shown in Figure 4.1.

The 2D partial model of the core is studied in the next subsection. The accuracy of this approach is then investigated in Subsection 4.2.2.

4.2.1 Problem Definition and Results

In this subsection, the domain of the partial 2D model is first defined, and then its dimensions are obtained. The general guidelines for the specification of the domain parameters are also developed and hold for the rest of this chapter. The solution of the partial 2D model of the core is finally given.

**Domain Definition:**

The problem domain is a rectangular area in the \( rz \) plane, which includes the core, the coil, the air in between, and the surrounding free-space. Figure 4.1 shows the simplified domain, the domain parameters, and the boundary conditions. It should be noted that the length of the
core and the height of the problem domain are equal. The domain width \(R_{\text{domain}}\) and domain height \(Z_{\text{domain}}\) will be obtained by using methods that will be described next. The defining parameters of the problem domain are given in Table 4.1. The \(B-H\) curve of the core magnetic material is given in Appendix B. The dimensions of the domain width and height are obtained in the following paragraphs.

Table 4.1: The problem domain parameters for partial model of the core and the coil.

<table>
<thead>
<tr>
<th>Domain width, (R_{\text{domain}}) (to be obtained)</th>
<th>Domain height, (Z_{\text{domain}}) (to be obtained)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core radius, (R_{\text{core}}) 250 mm</td>
<td>Core height, (Z_{\text{domain}}) (to be obtained)</td>
</tr>
<tr>
<td>Coil width, (W_{\text{coil}}) 25 mm</td>
<td>Coil length, (L_{\text{coil}}) 265 mm</td>
</tr>
<tr>
<td>Core - coil separation, (S_{cc}) 25 mm</td>
<td>Coil excitation, (I_{t_{\text{coil}}}) 60 kA-t.</td>
</tr>
</tbody>
</table>

**FE Study and Domain Dimensions:**

This study incorporates two issues: first, determining the domain of the problem, and second, obtaining the proper discretization of the problem domain. These studies are related and cannot be completely separated. The approach here is (a) to employ a very fine mesh and focus to obtain the proper dimensions of the problem domain, and (b) to use such problem domain and obtain the optimum discretization of the domain. This sequence yields the proper domain dimensions and discretization, which in turn yield the optimum model of the problem in hand. The selection criterion during this study is the average of the magnetic flux density at the core center plane at \(z = 0\).

To obtain the dimensions of the problem domain, a numerical value was assigned to the domain height, \(Z_{\text{domain}}\), and the domain width was obtained from the following relation

\[
R_{\text{domain}} = \alpha Z_{\text{domain}}
\]

where the coefficient \(\alpha\) should be determined. The coil and core parameters were kept constant. Different models were then solved by axisymmetric FEM, and the average magnetic flux density at the core center plane were computed. These field data are shown in Figure 4.2 as a function of \(Z_{\text{domain}}\) and for different values of coefficient \(\alpha\). It is concluded that in order to approximate consistent solutions, the domain parameters should satisfy the following criteria:
\[ Z_{\text{domain}} \geq 2 \times L_{\text{coil}} = 530 \text{ mm} \]
\[ R_{\text{domain}} \geq 4 \times Z_{\text{domain}} = 2120 \text{ mm}. \]  \hspace{1cm} (4. 2)

Different discretization schemes were applied to the problem domain. The optimum element sizes for different areas were found to be:

- \( 1/3 \text{rd} \) of the coil width, \(( W_{\text{coil}} + 3 )\) for small areas such as the coil area, and
- coil width \(( W_{\text{coil}} )\) for areas such as core and air, and with a proper spacing factor.

Further refinement of the elements did not change the average magnetic flux density at the core center plane.

**The Optimum Model and Results:**

The problem domain of the optimum model was thus defined by:

\[
\text{Domain width ( } R_{\text{domain}} \text{) = 2120 mm} \quad \text{Domain height ( } Z_{\text{domain}} \text{) = 530 mm.}
\]

A total number of 4400 elements was used to discretize the problem domain. Figure 4.3 shows the contour plot of the magnetic flux density magnitude at the core of the magnet.

The following observations are concluded from the results:

a) A high level of magnetic saturation exists in the core.

b) The average magnetic flux density at the core center plane is computed as 2.32 T versus 2.19 T that was computed from 3D analysis. This translates to a discrepancy of 6%.

c) Ampere's law error \( ( e_{\text{mmf}}, r ) \) along different paths is less than 5%.

d) The magnetomotive force drop \( ( \text{mmf} ) \) at the core and along the line from \((0,0)\) to \( ( r=0, z=L_{\text{coil}} ) \), is around 37 kA-t. Thus the mmf drop along the air-gap can be at most \((60-37=23) \text{ kA-t, which would suggest a magnetic flux density of around 0.20 T along the air-gap.}\)

The measured corresponding value is 0.27 T, and thus a 26% discrepancy would be derived. It shows that (1) the magnetic flux density of this model is only acceptable at the \( z=0 \) plane of the core, and (2) even such a simple partial model yields better results than the 2D planar model of Subsection 4.1.1.
4.2.2 Limiting Effect of the Coil Excitation Level

The objective here is to obtain the limits and the conditions to which the 2D approximation is justified and the solutions of the partial 2D model of the core are accurate. Thus the magnetic field of the core for different levels of the coil excitation is examined. The optimum FE model was chosen and the coil excitation level was varied in the range of 10 to 60 kA-m. The averages of magnetic flux density at the core center plane were calculated. These data are shown in Figure 4.4 as a function of the coil current along with the results of 3D analyses. The error is also given in the same figure.

For the nominal excitation of the 60 kA-m, the results disagree within an acceptable margin of 6%. As the excitation is reduced to 50 kA-m, the error is still acceptable and lower than 10%. However, for lower levels of the coil excitation, 2D and 3D studies diverge considerably from each other. The error reaches 200% for the case of 10 kA-m excitation level. The results of the 3D analyses of the MRI magnet have been proven to be accurate. Therefore, it can be concluded that the axisymmetric model fails to correctly evaluate the core magnetic field for low levels of the coil excitation.

The solutions of the 3D model of the MRI magnet, for a low current level of 10 kA-m, yielded a magnetic flux density of 0.7 T and 0.5 T at the inner side and the outer side of the core, respectively. The results clearly did not have a rotational symmetry and differed in value by more than 30%. For the same excitation level, the axisymmetric analysis computes a smooth magnetic flux density of 1.7 T, which is more than double the actual amount of 0.58 T.

In conclusion, the 2D partial model of the coil and the core yields acceptable result only when the magnet core is highly saturated, and thus, the application of this method is limited.
Figure 4.1: Problem domain of the partial 2D (axisymmetrical) model of the core and the coil.

Figure 4.2: The relation between the average magnetic flux density ($B$) across the core center plane, and the dimensions of the problem domain for the 2D partial model of the core and the coil. The length of the coil is $L_{coil} = 265$ mm.
Figure 4.3: The contour plots of the magnetic flux density magnitude at the core of the O-MRI magnet. The 2D partial model of the coil and the core (Figure 4.1) was used.

Figure 4.4: The average magnetic flux density ($B$) in the center plane of the core, for different levels of the coil excitation. The 2D partial model of the coil and the core (Figure 4.1) was used.
4.3 An Acceptable Partial 2D Model of the Coil and the Pole

The partial modeling of the MRI magnet, where the coils are concentric with the poles, is the subject of this section. The results of 3D analysis of the A-MRI magnet (Section 3.4) confirmed that regardless of the location of the coil and the excitation level of the coil

a) the magnetic field is normal at the air-gap $z=0$ plane,
b) the poles are not saturated, and
c) the air-gap magnetic field has a cylindrical symmetry with respect to the air-gap axis.

These arguments suggest that the pole and air-gap can be separately studied. A new problem is defined that includes only the pole, the air-gap, and a coil that is placed around the pole. In this section, this simplified version of the magnet is studied in 2D and the results are compared with 3D results. The objectives of this study are:

1) to show the applicability of the 2D partial model, in providing useful solutions, and
2) to obtain the extent and the conditions to which the 2D approximation is justified.

The partial 2D model can be used to estimate the air-gap magnetic flux density as a function of the air-gap geometry or the coil current in initial stages of the magnet design.

4.3.1 Problem Domain and Results

The problem domain includes the pole, the air-gap, and the coil. Due to the symmetry of the problem, only one half of the domain is modeled in a cylindrical coordinate system. The problem domain is a rectangular area in the $roz$ plane. Figure 4.5 shows the simplified domain and the boundary conditions. The domain parameters are given in Table 4.2. The material and parameters were selected to simulate the same magnet as in the 3D study given in Section 3.4. The domain height, $Z_{\text{domain}}$, is a problem parameter. The domain width, $R_{\text{domain}}$, was selected according to the optimizing relation of $R_{\text{domain}} \geq 4 \times Z_{\text{domain}}$, given in (4.2).

A total number of 7500 elements was used to discretize the problem domain of this model. Increasing the domain width and refining the domain grid did not change the approximate solutions. Therefore, this model is the optimum partial 2D model of the problem.
Table 4.2: The problem domain parameters for partial model of the pole and the coil.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain width, $R_{domain}$</td>
<td>1.2 m</td>
</tr>
<tr>
<td>Pole radius, $R_{pole}$</td>
<td>250 mm</td>
</tr>
<tr>
<td>Coil width, $W_{coil}$</td>
<td>25 mm</td>
</tr>
<tr>
<td>Pole - Coil separation, $S_{pc}$</td>
<td>25 mm</td>
</tr>
<tr>
<td>Domain height, $Z_{domain}$</td>
<td>265 mm</td>
</tr>
<tr>
<td>Air-Gap length, $L_{air-gap}$</td>
<td>150 mm</td>
</tr>
<tr>
<td>Coil length, $L_{coil}$</td>
<td>115 mm</td>
</tr>
<tr>
<td>Coil excitation, $I_{coil}$</td>
<td>60 kA-t</td>
</tr>
</tbody>
</table>

Results:

Figure 4.6 shows the contour plot of the magnetic flux density magnitude at the pole and the air-gap. The following conclusions can be obtained from the results:

a) The pole is not magnetically saturated, and an almost uniform magnetic field exists in the center region of the air-gap.

b) The discontinuity of the magnetic field at the pole and air-gap interface is due to the discontinuity of the tangential component of the magnetic flux density, $B_r$.

c) The normal component of the magnetic flux density, $B_z$, is continuous across the interface. It is the principal part of the magnetic field in the air-gap.

d) Ampere’s law error ( $e_{mmf}$, $I$ ) along different paths is less than 3%.

e) The magnetic flux density indicators at different locations are given in Table 4.3.

Table 4.3: The solutions from 3D and the partial 2D model of the pole and the coil.

<table>
<thead>
<tr>
<th>$B$ at a location</th>
<th>T</th>
<th>2D</th>
<th>3D</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$ along air-gap</td>
<td></td>
<td>0.46187</td>
<td>0.48396</td>
<td>4.5%</td>
</tr>
<tr>
<td>$B$ across air-gap</td>
<td></td>
<td>0.39476</td>
<td>0.41925</td>
<td>6%</td>
</tr>
<tr>
<td>$B$ across pole, air-gap</td>
<td></td>
<td>0.64539</td>
<td>0.62279</td>
<td>3.5%</td>
</tr>
</tbody>
</table>

Therefore, the partial 2D model of the pole and air-gap provides acceptable magnetic field data for the nominal current excitation of 60 kA-t.

4.3.2 The Effect of the Coil Excitation Level

The objective of this subsection is to determine the extent and the conditions to which the 2D approximation is justified. Thus, the magnetic field of the air-gap is studied for different
levels of the coil excitation. The current was changed from a low level of 20 kA-t to a high level of 120 kA-t. The average magnetic flux density across the air-gap, along the air-gap, and across the pole and air-gap were computed. These are shown in Figure 4.7, along with those of the 3D analyses.

It is confirmed that the maximum discrepancy between 2D and 3D results are 6%, 5% and 8% for the averages of magnetic flux density across the air-gap, along the air-gap, and across the air-gap and pole interface area, respectively. Moreover, a higher excitation level reduces the error in the air-gap magnetic field.

**Conclusion:**

This study confirms that the partial 2D study of the pole and air-gap can be justified. This is mainly due to the moderate magnetic field level in the pole, the proximity of the air-gap region, and the cylindrical shape of the pole. The 2D axisymmetric study conveniently and efficiently yielded satisfactory results in the air-gap. Such solutions can provide useful engineering information about the device in its initial stages of the design and study.

Figure 4.5: Problem domain of the partial 2D (axisymmetrical) model of the pole, the air-gap, and the coil.
Figure 4.6: The contour plots of the magnetic flux density magnitude at the pole and air-gap (top) of the MRI magnet. The 2D partial model of the coil and the pole was used.

Figure 4.7: The average of the magnetic flux density at the pole and air-gap. The 2D partial model of the coil and the pole (Figure 4.5) was used.
4.4 A Novel Composite 2D Modeling Method

The 2D partial model of the core, in Section 4.2, was shown to be satisfactory only when the core was highly saturated. The partial 2D model of the pole and air-gap was examined in Section 4.3, and the results were shown to be acceptable. In those studies, the yoke was excluded from the models. The main objective of this section is the study of a composite 2D axisymmetric model that includes all the components of the MRI magnet. It is shown that such a novel modeling approach yields acceptable solutions across the core and air-gap of the MRI magnet.

Development of the composite model is based on the following:

1) The full 3D analysis result shows that the yoke magnetic field is always between the highly saturated levels of the core, and the unsaturated levels of the pole.
2) The cross-sectional area of the yoke is 80% of the core cross-section.
3) The yoke is assumed as a magnetic transmission media that does not greatly affect the magnetic field at the regions of interest.
4) The yoke provides a magnetic path whereby magnetic flux closes on itself. The simplified composite model should provide an alternate geometry for the same function.
5) Cylinders can approximate and simulate the actual yokes. The cylindrical core, the simulated yoke, the pole, and the air-gap all have a common axis in the composite model.
6) The material and radius of the simulated cylindrical yoke are similar to the core and pole. The 25% increase of the yoke cross-sectional area translates to a lower magnetic field in the simulated yoke for the same transmitted magnetic flux.
7) The length of the simulated yoke is assumed to be equal to the mean magnetic path of the yoke in the 3D magnet.
8) At this point, all the components of the composite problem have a cylindrical symmetry and can be conveniently modeled by a 2D axisymmetric approach.

Therefore, a new problem is defined that includes the core, the simulated yoke, the pole, and the air-gap. The problem domain of this composite MRI magnet is defined in the next subsection. Then, the O-MRI magnet (where the coil is around the core) is modeled and
solved. The results and conclusions are given in Subsection 4.4.2. Subsection 4.4.3 studies the 2D composite model for the A-MRI magnet (where the coil is around the pole.) The objectives of each study are:

- To determine the extent and conditions to which the 2D composite modeling approach provides useful, accurate and acceptable solution, and
- To calculate three quantities of interest as the average of the magnetic flux density (1) at the core center plane, at \( z = 0 \) of the 2D model, (2) at the air-gap center plane, at \( z = Z_{\text{domain}} \) of the 2D model, and (3) along the air-gap centerline, at \( r = 0 \) of the 2D model.

These magnetic flux indicators are compared against the 3D counterparts in order to evaluate the accuracy of the 2D composite modeling approach.

### 4.4.1 Problem Domain

The 2D problem domain is shown in Figure 4.8. Due to the symmetry, only half of the domain is modeled in a cylindrical coordinate system. The material and the domain parameters, given in Table 4.4, were selected to simulate the same magnets as in the 3D studies of O-MRI and A-MRI magnets. The domain height, \( Z_{\text{domain}} \), is the sum of the core, the simulated yoke, the pole, and the air-gap lengths. The domain width, \( R_{\text{domain}} \), was selected according to the optimizing relation of \( R_{\text{domain}} \geq 4 \times Z_{\text{domain}} \) in (4.2). A total number of 15500 elements was used to discretize the problem domain. Similar solutions were obtained when the domain grid was refined and/or the domain width was increased. Therefore, this model is the optimum composite model of the MRI magnet.

| Domain width, \( R_{\text{domain}} \) | 7000 mm | Domain height, \( Z_{\text{domain}} \) | 1700 mm |
| Core/Yoke/Pole radius, \( R_{\text{core}} \) | 250 mm | Core length, \( L_{\text{core}} \) | 300 mm |
| Coil length at core, \( L_{\text{coil}} \) | 265 mm | Yoke length, \( L_{\text{yoke}} \) | 1100 mm |
| Coil length at pole, \( L_{\text{coil}} \) | 115 mm | Pole length, \( L_{\text{pole}} \) | 150 mm |
| Coil excitation, \( I_{\text{coil}} \) | 60 kA-t | Air-Gap length, \( L_{\text{air-gap}} \) | 150 mm |
| Core - Coil separation, \( S_{\text{cc}} \) | 25 mm | Coil width, \( W_{\text{coil}} \) | 25 mm. |
4.4.2 A Composite 2D Model of the O-MRI Magnet

The first objective of this study is to verify the accuracy of the composite modeling approach. The magnetic flux density indicators were computed from the FE solution of the composite 2D model of the O-MRI magnet, where the coil was placed around the core and its excitation level was 60 kA-t. For this nominal current, the results agree within 2%. Therefore, the accuracy of the composite 2D model is acceptable. The magnetic flux density indicators are summarized in Table 4.5.

Table 4.5: The results from 3D and composite 2D models of the O-MRI magnet.

<table>
<thead>
<tr>
<th>Location</th>
<th>2D</th>
<th>3D</th>
<th>Error</th>
<th>Max. Error for current of 10 to 60 kA-t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Along air-gap</td>
<td>0.2886</td>
<td>0.2909</td>
<td>1%</td>
<td>1.5%</td>
</tr>
<tr>
<td>Across air-gap</td>
<td>0.2605</td>
<td>0.2658</td>
<td>2%</td>
<td>2.5%</td>
</tr>
<tr>
<td>Across core</td>
<td>2.2307</td>
<td>2.1913</td>
<td>2%</td>
<td>4%</td>
</tr>
</tbody>
</table>

Figure 4.9 shows the contour lines of the magnetic flux density magnitude at the core of the model and from the 2D analysis. A high level of magnetic saturation exists in the core, and the field slowly and smoothly changes in the z direction. Figure 4.10 shows similar data from the 3D analysis of the O-MRI magnet. The magnitude and the changing pattern of the magnetic field in both graphs are very similar. It is concluded that the 2D composite model efficiently represents the core of the O-MRI magnet.

The contour lines of the magnetic flux density magnitude at the pole and the air-gap of the 2D model are shown in Figure 4.11. The regions and view directions were shown in Figure 4.8. The magnetic field is uniform in the center region of the air-gap. The pole is not magnetically saturated. Figure 4.12 shows similar data from the 3D study of the MRI magnet. It is concluded that the 2D composite model can efficiently compute the magnitude and pattern of the magnetic flux density in the air-gap of the magnet. The 2D composite model also yields acceptable magnetic field data in the air-gap side of the pole.

The next objective of the study is to find the extent and conditions under which the composite 2D modeling is valid. The 2D model was solved for the coil excitation levels of 10 to 60 kA-t. The magnetic flux indicators were calculated and compared with the results from
the 3D analyses of the O-MRI magnet. Figure 4.13 shows the average magnetic flux density across the core. The error does not exceed 4%. Figure 4.15 shows the average magnetic flux density across and along the air-gap. The discrepancy between 2D and 3D results is less than 2.5%.

**Conclusion:**
The composite 2D modeling approach provided acceptable solutions when used to study the O-MRI magnet. The magnetic flux density at the air-gap and core were conveniently obtained from the composite 2D model. The model also yielded reasonable results for different excitation levels of the coil. Therefore, the 2D composite modeling is concluded to be an effective approach for approximating the O-MRI magnet and similar devices.

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**Figure 4.8:** Domain definitions for composite 2D (axisymmetric) models of the MRI magnet. The coil is placed either around the core (O-MRI) or around the pole (A-MRI).
Figure 4.9: Magnetic flux density contour line at the core of the O-MRI magnet using a composite 2D model. The coil is around the core and its excitation level is 60 kA·t.

Figure 4.10: Magnetic flux density contour line at the core of the O-MRI magnet using a 3D model. The coil is around the core and its excitation level is 60 kA·t.
Figure 4.11: Magnetic flux density contour line at the pole and the air-gap (top) of the O-MRI magnet using a composite 2D model. The coil excitation level is 60 kA-t.

Figure 4.12: Magnetic flux density contour line at the pole and the air-gap (top) of the O-MRI magnet using a 3D model. The coil excitation level is 60 kA-t.
Figure 4.13: The average of the magnetic flux density across the core of the O-MRI magnet using 2D and 3D FE models. The coil is around the core.

Figure 4.14: The average of the magnetic flux density across and along the air-gap of the O-MRI magnet using 2D and 3D FE models. The coil is around the core.
4.4.3 A Composite 2D Model of the A-MRI Magnet

The composite 2D modeling approach was applied to the O-MRI magnet, and the solutions were acceptable. The alternative design of the MRI magnet (A-MRI), which was introduced in Section 3.4, is a similar but less difficult device than the O-MRI magnet. Therefore, it is suggested to use the composite 2D modeling approach and to study the A-MRI magnet. As before, the main objective of this study is to determine whether the composite 2D model can yield useful and accurate results. The other objectives are, first, to obtain the magnetic flux density at the core and at the air-gap, and second, to find the extent and conditions to which the 2D approximation is justified.

The composite 2D model of the A-MRI magnet and the boundary conditions are shown in Figure 4.8, page 112 – where the coil is located around the pole. The magnetic flux density indicators of the FE solutions, at the coil excitation level of 60 kA-t, are summarized in Table 4.6. The solution of the composite model is found similar to the solution of the 3D analysis.

<table>
<thead>
<tr>
<th>B at a location</th>
<th>T</th>
<th>2D</th>
<th>3D</th>
<th>Error</th>
<th>Max. Error for current of 20 to 120 kA·t</th>
</tr>
</thead>
<tbody>
<tr>
<td>B along air-gap</td>
<td>T</td>
<td>0.4596</td>
<td>0.4840</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>B across air-gap</td>
<td>T</td>
<td>0.3947</td>
<td>0.4193</td>
<td>6%</td>
<td>6%</td>
</tr>
<tr>
<td>B across core</td>
<td>T</td>
<td>0.8243</td>
<td>0.8350</td>
<td>1.5%</td>
<td>8%</td>
</tr>
</tbody>
</table>

To find the extent and conditions where the 2D composite model is valid, the coil excitation level was varied from 20 kA-t to 120 kA-t. The study indicators were then calculated and compared with results from the 3D analyses of the A-MRI magnet. Figure 4.15 shows the average magnetic flux density across the core, across the air-gap, and along the air-gap. The discrepancy of the results does not exceed (a) 6% for the nominal current excitation of 60 kA-t, and (b) 8% for the full range of the current excitation.

Figure 4.16 shows the contour lines of the magnetic flux density magnitude at the core of the model. This result from the 2D analysis shows that the core is not saturated. Figure 4.17 shows similar data from the 3D analysis of the A-MRI magnet. Although the profile of the
magnetic field from 2D model is different from the 3D model, the averages of the magnetic flux density from the two models are very close.

The contour lines of the magnetic flux density magnitude at the pole and the air-gap of the model are shown in Figure 4.18. The magnetic field is uniform in the center region of the air-gap. The pole is not magnetically saturated, except at the corners. Figure 4.19 shows similar data from the 3D study of the A-MRI magnet. The magnitude and the changing pattern of the magnetic field in both graphs are similar. It can be concluded that the 2D model efficiently computes the magnitude and pattern of the magnetic flux density in the air-gap and the pole of magnet.

Conclusion:

The magnetic flux density for the pole and the air-gap of the model were acceptable for different excitation levels of the coil. The core magnetic field results were also acceptable for coil excitation levels of less than 80 kA-t. It is therefore concluded that the composite 2D modeling approach provides satisfactory solutions for the A-MRI magnet design.

Figure 4.15: The average magnetic flux density across the core, along the air-gap and across the air-gap. The A-MRI magnet is solved with 3D and 2D FEM, where the coil is located around the pole.
Figure 4.16: Magnetic flux density contour line at the core of the A-MRI magnet using a composite 2D model. The coil is around the pole and its excitation level is 60 kA-t.

Figure 4.17: Magnetic flux density contour line at the core of the A-MRI magnet using a 3D model. The coil is around the pole and its excitation level is 60 kA-t.
Figure 4.18: Magnetic flux density contour line at the pole and the air-gap (top) of the A-MRI magnet using a composite 2D model. The coil excitation level is 60 kA-t.

Figure 4.19: Magnetic flux density contour line at the pole and the air-gap (top) of the A-MRI magnet using a 3D model. The coil excitation level is 60 kA-t.
4.5 Conclusions

The primary objective of this chapter was to introduce and employ simplifying methods, which yield acceptable 2D models for magnetic devices such as the MRI magnet. The partial and composite 2D modeling approaches were then introduced.

The 2D partial model of the core only yielded an acceptable result when the core was highly saturated. However, the 2D partial model of the pole and air-gap yielded reasonable results in the air-gap. This was due to the inclusion of the air-gap in the model and also to the moderate magnetic field level in the magnetic poles.

Finally, the composite 2D model was shown to be the most practical 2D modeling approach, giving acceptable results in the air-gap and the core of the magnet regardless of the coil location and excitation level. The power of this model is that it can be used cost effectively to explore design alternatives prior to undertaking a full 3D modeling effort.

The composite 2D modeling approach can be used for the study of other 3D physical systems that do not have a complete symmetry and cannot be modeled in 2D. The non-symmetrical part of the system can be properly substituted and the resultant simulated system can be studied.
Chapter 5

Conclusions

This thesis proposed to use Ampere's circuital law to obtain a *global accuracy measure* for assessing the correctness of a FE model and solution. The open-concept MRI magnet was used as a basis for the study. This device includes a large air-gap, and a large and saturated magnetic structure. A *selection* strategy and *selection* criteria were introduced to aid in the study of the MRI magnet. A precise 3D modeling method, partly based on the use of a constant permeability model of the saturated magnet, was developed and then used to obtain the most accurate model of the MRI magnet. This model was analyzed for different operational conditions. An alternative design of the MRI magnet was then introduced and studied. The thesis also presented 2D modeling approaches to simplify the FE study of magnetic devices that do not have a complete symmetry and cannot be modeled in 2D.

5.1 Contributions

The main contributions of the thesis are as follows:

a) The application of the Ampere's law error to evaluate the accuracy of a 3D FE model has been extensively investigated. The conditions under which this *global accuracy criterion* is reliable have been obtained.

b) The applications of different *selection* criteria have been examined. Both *global selection criterion*, e.g. the total energy stored in the model, and the *semi-global selection criterion*, e.g. the energy stored in a region of the model or the integral of the magnetic
Chapter 5: Conclusions

flux density over a region of the model, were shown to be practical in identifying the optimum FE model of a problem.

c) A novel equivalent linear 3D modeling approach has been introduced. Its application in (a) evaluation of the model global accuracy, and (b) enhancing the efficiency of the selection search and study, was shown. It was also shown that the global accuracy of the nonlinear 3D model (Ampere's law error) of a problem could be estimated from the equivalent linear 3D model of the problem.

d) A stable 3D modeling approach has been developed that expands a 3D volume of a device to a spherical volume and exterior with a limited number of elements. The main advantage of this modeling approach is that it provides a representation of the infinite boundary condition when infinite elements are used.

e) A 2D composite modeling approach has been introduced that simplifies the study of magnetic devices that cannot be completely modeled by conventional 2D methods. The 2D composite model of the MRI magnet was shown to be a practical and reliable model, giving acceptable solutions in the air-gap and the core of the MRI magnet.

f) Open concept MRI magnets, of the type that was considered in this thesis, conventionally have the excitation coil located far from the air-gap. By simply moving the excitation coil(s) to the pole region of the magnet, it was shown that the available air-gap flux density, per unit of excitation, could be increased by almost 60% without significantly compromising flux density uniformity.

5.2 Conclusions

The following conclusions have been drawn from the studies conducted in this thesis:

1) Ampere's law error has been shown to be an acceptable global accuracy measure, provided that its path dependency has been addressed. It was found that the error along a specific path could be minimized without a general improvement in the quality of results. This limitation called for a statistical examination of the error along different paths.

The average of Ampere's law errors along different paths has been shown to be a better global accuracy measure than the error along a single path. However, such a
measure could not properly address the wide range of errors along different paths. The combination of the standard deviation and the average of Ampere’s law errors along different paths have been shown to be an applicable global accuracy measure for evaluating the FE solution.

2) The total magnetic energy stored in the FE model of the device was shown to be a trustworthy global selection criterion. The selection direction is predetermined and is toward the minimum energy content of the model.

The magnetic energy stored in specific regions of the FE model was shown to be a valid semi-global selection criterion. The selection direction of such a measure depends on the chosen region and should be obtained for each problem and region.

3) The average of the magnetic field at a specific region of the FE model was found to be a valid semi-global selection criterion. When the region was selected within one material neighborhood, the selection direction could be uniquely determined.

4) The equivalent linear 3D model of the MRI magnet - where the nonlinear material of the magnet was replaced with an equivalent linear material - was concluded to be efficient in the search for better models of the device.

5) The original design of the MRI magnet, where the coil is around the core, was found to be inefficient. Less than 20% of the total energy was stored in the magnet and air-gap, and only 12% of the core magnetic flux reached the air-gap.

The alternative design of the MRI magnet, where the coils are placed around the poles, was concluded to be more efficient than the original design. It provided 58% more magnetic flux across the air-gap while using 33% less magnetic energy.

6) It was shown that a current sheet coil could effectively simulate the thin and long superconducting coil of the MRI magnet.

7) The 2D composite modeling approach was found to be a useful and practical approach for simplifying the study of 3D magnetic devices that cannot be modeled by conventional 2D methods.
5.3 Future Work

Further research studies, which can be addressed by application of methods and proposals discussed in this thesis, are suggested. These are:

1- The MRI magnet can be studied with different materials for the core, yokes and poles. These materials can be a permanent magnet, powder material, or magnetic materials of different characteristics.

2- The design of the MRI magnet can be modified in more details. The geometry of the air-gap, poles, yokes and core can be redesigned, and the device performance can be studied.

3- The 2D partial and composite modeling approaches can be used to study smaller magnets, and magnets with linear material.

4- The effect of the yoke length in the composite 2D model of the MRI magnet can be studied.

5- The modeling approach used in this thesis, together with its principles, can be used for the ac analysis of magnetic devices, such as inductors with large air-gaps that are used in the power systems of moderate to large size industrial networks.
References


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Appendix B

C1006 Characteristic Curves

Figure B.1: The B-H characteristic curve of C1006 used in the MRI magnet.

Figure B.2: The v-B² characteristic curve of C1006 used in the MRI magnet
Appendix C

Magnetostatic Finite Element Method in 3D

The Finite Element Method is a numerical technique for obtaining approximate solutions to boundary value problems in mathematical physics. The principle of the method is to replace the entire continuous domain of the problem by a number of sub-domains in which the unknown function is approximated using interpolation functions with unknown coefficients. Thus the solution of the whole system is approximated by a finite number of unknown coefficients.

The theory of the Finite Element Method has been studied and reported in the literature. Software packages are also available that effectively implement the method. ANSYS is one of the leading FEM software programs and is used exclusively in this thesis. The main objective of this chapter is to introduce and detail the 3D linear and nonlinear magnetostatic Finite Element Method comprehensively.

There are two reasons for inclusion of this appendix. First, to use a method to solve a problem correctly and effectively, the method itself has to be understood. The Finite Element Method incorporates many steps that one has to understand in order to effectively implement and solve a problem. The use of a FEM package to solve a problem seems very trivial at first; however, as soon as a difficulty appears in any stage approaching the final and correct solution of the problem, the reasons for the failure should be sought. A clear and comprehensive knowledge of the method and its steps are necessary to trace and to understand the source of the method failure. This knowledge is more critical when dealing with difficult problems that require the utmost capabilities of the FEM application. The MRI magnet has been shown to be a difficult problem, which is caused by nonlinearity and the
high saturation level of the magnet, the huge size of the magnet, a very large air-gap, and a high fringing of the magnetic field to the surrounding free-space regions. Thus all the FEM steps are detailed in this appendix.

The second reason for the inclusion of this appendix relates to the 3D nonlinear FEM implementation steps. They have been discussed in the literature and in the software manuals; however, some steps are discussed in detail while other steps in the modeling process are often omitted. References are usually made to multiple sources in the literatures, and the FEM user does not obtain a complete picture of the method without an exhaustive search of old and new literature in different disciplines and with different terminology. Thus the implementation steps, numerical sub-steps, and iterative routines are clearly detailed in order to provide enough information and guidelines for a person to write a program and to solve a problem.

The FEM analysis of a boundary value problem incorporates the following main steps:

1. Definition of the boundary value problem.
2. Formulation of the problem in terms of Ritz's (variational) or Galerkin's approach.
3. Definition of the problem domain and its boundaries.
4. Discretization or subdivision of the domain by elements and nodes.
5. Selection of the interpolation functions for each category of the used elements.
6. Formulation of the system of the equations in linear or nonlinear terms.
7. Solution of the system of equations.
8. Evaluation of the final results.

The proper implementation of each step is necessary, although there is not a clear-cut division between the steps. Information as to how these steps affect each other, even at the stages very close to the final solution, is not widely available. The whole model and all the steps may be modified at any stage due to a failure of the FEM to achieve an acceptable solution.

This appendix focus is on the whole formulation and numerical sequence of the FEM, and thus, each of the FEM steps is discussed in one separate section. It should be noted that all of the FEM steps are implemented by software, however, some steps are under the control.
of the FEM user. These steps (step 1,3,4,5) were detailed in Chapter 2 and are briefly discussed here.

The formulations are customized for the purpose of this thesis: the magnetostatic analysis of the MRI magnet. This thesis uses both linear and nonlinear analyses of the MRI magnet, and therefore, the formulation and method is generally given and discussed for both kinds. The solution to a nonlinear problem is given separately by the Newton-Raphson method in the system solution step.

C.1 Boundary Value Problem Definition

Mathematical modeling of physical systems typically yields boundary value problems. Any boundary value problem can be defined in terms of:

1. the boundary conditions on the boundary \( \Gamma \) that enclose the problem domain \( D \), and

2. a governing differential equation in the domain \( D \) as

\[
Lu = f, \tag{C.1}
\]

where \( L \) is a differential operator, \( u \) is the unknown quantity, and \( f \) is the excitation or forcing function. The form of the differential equation and unknown are problem dependent. Analytical solutions to boundary value problems are available for only a few cases. Therefore, approximation methods have been developed to numerically obtain the unknown quantity. The Rayleigh-Ritz and the Galerkin’s methods are usually used to approximate the solution. It should be noted that both methods yield a similar system of equations for a problem. The Rayleigh-Ritz method that is preferred in this thesis is discussed in Section C.2.

There are mainly two methods for defining a FE boundary value problem in electromagnetics. The difference is in the selection of the unknown quantity \( u \) when posing the boundary value problem. In one method, components of the magnetic field are used to define the problem. This method uses edge or face based elements. It uses a vector basis or vector elements and assigns degrees of freedom to the edges rather than to the nodes of the
elements [18–20]. The edge elements have not been used and discussed in this thesis, and the
detailed discussions are given in the literature [21].

In the second method, magnetic potentials are used to define the boundary value problem.
Selecting the unknown quantity \( u \) as the magnetic potential has an advantage because it can
be chosen to be continuous across interfaces between different media. Therefore, difficulties
that may arise due to the discontinuity of the magnetic field or the magnetic flux density
across those surfaces can be avoided. This method, which assigns the degrees of freedom to
the nodes rather than to the edges of the element, is referred to as a node based Finite
Element Method. The thesis uses the nodal FEM based on the Magnetic Vector Potential
(MVP) [13]. This is discussed in the following subsections after a review of Maxwell’s
equations. The FEM based on the Magnetic Scalar Potential (MSP) were discussed briefly in
Section 2.4.1.

C.1.1 Maxwell’s Equations

In electromagnetics, the differential operator \( L \), the unknown quantity \( u \), and the forcing
function \( f \) are all obtained from Maxwell’s equations, and the selection of \( u \) defines the final
form of the differential operator \( L \). The differential forms of Maxwell’s equations describing
the magnetostatic field are

\[
\nabla \times \mathbf{H} = \mathbf{J} \quad \text{(Ampere’s law),} \tag{C.2}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss’s law),} \tag{C.3}
\]

where \( \mathbf{H} \) is the magnetic field intensity vector, \( \mathbf{B} \) is the magnetic flux density vector, and \( \mathbf{J} \)
is the current density vector. Using \( \mu \) for the permeability and \( \nu \) for the reluctivity of the
medium, the constitutive relation is written as

\[
\mathbf{B} = \mu \mathbf{H} \quad \text{or} \quad \mathbf{H} = \nu \mathbf{B}. \tag{C.4}
\]

The boundary conditions at the outer boundary \( \Gamma \) are

\[
\mathbf{H} \times \hat{n} = 0 \quad \text{(normal field on } \Gamma_{H}), \tag{C.5}
\]

\[
\mathbf{B} \cdot \hat{n} = 0 \quad \text{(parallel flux on } \Gamma_{B}). \tag{C.6}
\]
where $\hat{n}$ is the unit vector normal to the boundary, and $\Gamma = \Gamma_H \cup \Gamma_B$. The field continuity conditions at the interface between the two media are

\begin{align*}
\hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) &= 0 \quad \text{(tangential field continuity),} \\
\hat{n} \cdot (\mathbf{B}^+ - \mathbf{B}^-) &= 0 \quad \text{(normal flux continuity),}
\end{align*}

where superscripts + or - refer to different sides of the discontinuity interface.

### C.1.2 Magnetic Vector Potential

The FE modeling of the static magnetic field is traditionally defined in terms of magnetic potentials. The problem can be defined in terms of the magnetic scalar potential [9-10] or the magnetic vector potential [11-13], or the combination of the scalar and vector magnetic potential [14]. The magnetostatic FEM based on the magnetic vector potential is used in this thesis and detailed in the following. The reasons for not using the scalar and combined potential formulation FEM are given in Subsection 2.4.2.

Maxwell’s Equations, given in (C.2) and (C.3), govern magnetostatic field problems. Since the magnetic flux density $\mathbf{B}$ is a divergence free (or solenoidal) field, it can be represented in terms of a magnetic vector potential $\mathbf{A}$ as

$$
\mathbf{B} = \nabla \times \mathbf{A}.
$$

Substitution of (C.9) in (C.2) with the aid of (C.4) yields the second order differential equation

$$
\nabla \times (\nabla \times \mathbf{A}) = \mathbf{J},
$$

which defines the magnetostatic boundary value problem on its domain $D$. Equation (C.10) does not determine a unique solution of $\mathbf{A}$. If $\mathbf{A}$ is a solution to (C.10), then any function that can be written as $\mathbf{A}' = \mathbf{A} + \nabla f$ is also a solution of (C.10). That is due to the vector identity of $\nabla \times (\nabla f) = 0$. A gauge condition can be imposed on the divergence of $\mathbf{A}$ in order that (C.10) determines $\mathbf{A}$ uniquely. One of the most used gauges is the Coulomb gauge defined as

$$
\nabla \cdot \mathbf{A} = 0 \quad \text{(Coulomb Gauge).}
$$

The necessity of enforcing the Coulomb gauge has been a very controversial subject [22]. However, if the ultimate goal of the problem study is to compute the magnetic flux density $\mathbf{B}$
or the energy associated with the problem domain, the uniqueness of \( A \) does not affect the uniqueness of the magnetic field or energy [23]. There is not a final resolution on the dilemma of the gauge problem. The general definition and formulation of the problem is conventionally done with the use of the Coulomb gauge [13, 23] and is outlined below.

**C.1.3 Problem Description**

The boundary value problem can be described by using vector potential \( A \) that satisfies the vector differential equation of

\[
\nabla \times (\nabla \times A) - \nabla (\nabla \cdot A) = J, \tag{C.12}
\]

which is obtained by enforcing the Coulomb gauge on to (C.10). The boundary conditions at the outer boundary \( \Gamma \) of the problem domain \( D \) are of two kinds

\[
\hat{n} \times A = P \quad \text{Dirichlet condition on } \Gamma_1, \tag{C.13}
\]
\[
\hat{n} \times (\nabla \times A) = 0 \quad \text{Neumann condition on } \Gamma_2, \tag{C.14}
\]

which are applicable at planes of symmetry and \( \Gamma = \Gamma_1 \cup \Gamma_2 \). The continuity conditions applied at the interface between two media of different permeability or reluctivity are

\[
\nu^+ \hat{n} \times \nabla \times A^+ = \nu^- \hat{n} \times \nabla \times A^- \quad \text{Continuity of tangential } H, \tag{C.15}
\]
\[
\hat{n} \cdot \nabla \times A^+ = \hat{n} \cdot \nabla \times A^- \quad \text{Continuity of normal } B, \tag{C.16}
\]
\[
\hat{n} \times A^+ = \hat{n} \times A^- \quad \text{Continuity of } A, \tag{C.17}
\]

where superscripts + or - refer to different sides of the discontinuity interface \( S_d \).

**C.2 Variational Formulation**

Two of the most common approximation methods for solving a boundary value problem - which was defined by (C.12) to (C.17) - are Galerkin's method and the Ritz method. Galerkin's method belongs to the family of weighted residual methods, which seeks a solution by weighting the residual of the differential equation (C.1). The best approximate is the solution that reduces the residual to the least value over the problem domain [23].
The Ritz method is a variational method in which the boundary value problem is formulated in terms of a variational expression or functional. The minimum of the functional corresponds to the governing differential equation under the given boundary conditions. The approximate solution is obtained by minimizing the functional with respect to its variables. This method is included here to complete the theory of the magnetostatic FEM.

According to the variational principles, the solution to (C.12)-(C.17) can be obtained by extremizing the modified energy functional [11,23]

\[ F(A) = \frac{1}{2} \int_V \nabla \cdot \left( \nabla \times A \right) \cdot \left( \nabla \times A \right) dV - \int_V J \cdot A dV + \frac{1}{2} \int_V \nabla \cdot \nabla \cdot A \ dV \ , \quad (C.18) \]

under the condition given by (C.13) and potential continuity (C.17). The first and second terms of (C.18) are standard energy functions, and their integrals yield the stored energy in the volume. The last term of (C.18) is the enforcement of the Coulomb gauge. The first variation of the functional is used to find

\[ \delta F(A) = \frac{1}{2} \int_V \nabla \cdot \left( \nabla \times A \right) \cdot \left( \nabla \times \delta A \right) dV - \int_V J \cdot \delta A dV + \int_V \nabla \cdot \left( \nabla \cdot A \right) \delta A dV \ . \quad (C.19) \]

Using the vector identities

\[ \mathbf{X} \cdot \nabla \times \mathbf{Y} = \nabla \cdot \left( \mathbf{Y} \times \mathbf{X} \right) + \mathbf{Y} \cdot \nabla \times \mathbf{X}, \quad (C.20) \]
\[ a \nabla \cdot \mathbf{X} = \nabla \cdot (a \mathbf{X}) - \mathbf{X} \cdot \nabla a, \quad (C.21) \]

for the first and third terms of (C.19), respectively, yields

\[ \delta F(A) = \int_V \nabla \cdot (\delta \mathbf{A} \times \nabla \times A) \ dV + \int_V \delta \mathbf{A} \cdot \nabla \times (\nabla \times A) dV - \int_V J \cdot \delta \mathbf{A} dV \]
\[ + \int_V \nabla \cdot (\nabla \cdot \mathbf{A}) \delta \mathbf{A} \ dV - \int_V \delta \mathbf{A} \cdot \nabla \left( \nabla \cdot \mathbf{A} \right) dV \ . \quad (C.22) \]

Applying the divergence theorem,

\[ \int_V \nabla \cdot \mathbf{f} dV = \oiint S \hat{n} \cdot \mathbf{f} dS \ , \quad (C.23) \]

to the first and fourth terms, and then regrouping the second, third and fifth terms of (C.22)

\[ \delta F(A) = \int_V \left[ \nabla \times (\nabla \times A) - \nabla \left( \nabla \cdot \mathbf{A} \right) - J \right] \cdot \delta \mathbf{A} dV \]
\[ + \oiint_S \hat{n} \cdot (\delta \mathbf{A} \times (\nabla \times A)) dS + \oiint_S \hat{n} \cdot (\nabla \cdot \mathbf{A} \delta \mathbf{A}) dS \ . \quad (C.24) \]

Using the vector identity of \( \mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z}) = -\mathbf{Y} \cdot (\mathbf{X} \times \mathbf{Z}) \) and \( \mathbf{X} \cdot a \mathbf{Y} = a \mathbf{X} \cdot \mathbf{Y} \) yields
The surface integrals, which result from application of the divergence theorem (C.23), should be taken carefully over the surface of each media in the domain. Assuming two volumes of different reluctivity in the domain, where their interface discontinuity surface is $S_d$, the integral domain $S$ can be arranged as

$$S_t = S_{Total \, v1} + S_{Total \, v2} = S_{External \, v1} + S_{discontinuity \, v1} + S_{External \, v2} + S_{discontinuity \, v2}$$

Since Dirichlet's conditions (C.13) are prescribed over $\Gamma_1$, $\hat{n} \times \delta A$ vanishes on $\Gamma_1$ and (C.25) changes to

$$\delta F(A) = \int_S \left[ \nabla \times (\nabla \times A) - \nabla (\nabla \cdot A) - J \right] \cdot \delta A \, dV$$

$$- \int_{S_d} \left[ \nabla^+ (\hat{n} \times \nabla \times A^+ \ n^- (\hat{n} \times \nabla \times A^-) \right] \cdot \delta A \, dS$$

$\int_{\Gamma_2} \nabla (\hat{n} \times \nabla \cdot A) \cdot \delta A \, dS$

$$+ \int_{S_d} (\nabla^+ \cdot A^+ - \nabla^- \cdot A^-) \hat{n} \cdot \delta A \, dS$$

Then the solution is obtained when the stationary requirement $\delta F(A) = 0$ is imposed for any arbitrary variation of $\delta A$. It is then evident that $A$ must satisfy:

The boundary value problem as (C.12)

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) = J$$

The continuity of the tangential $H$ (C.15) on $S_d$: $\nabla^+ \hat{n} \times \nabla \times A^+ = \nabla^- \hat{n} \times \nabla \times A^-$

The Neumann boundary condition (C.14) on $\Gamma_2$: $\hat{n} \times (\nabla \times A) = 0$

The Coulomb gauge on $S_d$: $\nabla^+ \cdot A^+ = \nabla^- \cdot A^-$

The Coulomb gauge (C.11) on $\Gamma$: $\nabla \cdot A = 0$

The continuity of the normal $B$ on $S_d$ (C.16) is already satisfied from the continuity of $A$ in (C.17). The Dirichlet condition on $\Gamma_1$ (C.13) should be forced on the solution. Therefore, the approximate solution of the boundary value problem - which is defined in Section C.1.3 - can be obtained by minimizing the variational functional (C.18).
C.3 Domain Definition

The domain in most electromagnetic studies contains the device and the surrounding free-space. The surrounding space should be defined properly, depending on the type and nature of the problem and the objective of the analysis. It is impossible to model an infinite outer surrounding space, and therefore, the domain should be restricted within a reasonable space around the device. A method to compensate for the lost infinite space should be considered beforehand. The so-called far field or infinite elements are usually employed to take the infinite space into account. Failure of the FEM to converge to either a solution or an accurate solution necessitates the modification of both the domain definition and the domain discretization. A complete discussion of the domain definition was given in Subsection 2.1.4.

C.4 Domain Discretization

The discretization of the domain is the most important and the most time-consuming step in any FE analysis. The manner in which the domain is discretized will affect the total element and node numbers, and the computer storage requirements. It also defines the computation time, the software requirements, the convergence of the analysis to a solution, and the accuracy of the numerical results. The efficiency or even propriety of the discretization cannot be completely judged till the results are obtained and verified.

The problem domain is typically discretized using one or more of the following techniques:

1. Computer-aided discretization or free-meshing,
2. Semi-computer-aided discretization or extrusion,
3. Adaptive discretization, and

This thesis uses a combination of the extrusion and free-meshing techniques in order to non-adaptively discretized the MRI magnet problem. These methods were discussed and demonstrated in Subsection 2.1.5, Subsection 2.4.4.
C.5 Interpolation Functions Selection

The boundary value problem is defined in terms of the magnetic vector potential $A$ in Section C.1, and then the proper variational formulation is introduced in Section C.2. The problem domain, which was defined in Section C.3, is subsequently discretized into small elements of hexahedral, prism and tetrahedral elements. The next steps of the FE analysis are, first, to select interpolation functions that provide an approximation of a scalar function within each element, and second, to choose a proper elemental coordinate system.

The interpolation or shape functions are usually selected to be polynomial of the first order (linear), the second order (quadratic), or a higher order. The linear interpolation function is used in this thesis due to the ease and availability of its formulation, representation, and application. A typical interpolation function for an element $e$ having $n$ nodes is

$$f^e = \sum_{i=1}^{n} N_i^e f_i^e = (N^e)^T \{f^e\} = \{f^e\}^T \{N^e\}, \quad \text{(C.27)}$$

where the continuous function $f$ is approximated - at any point inside the element - by the values of $f$ at the nodes of the element $f_i$ and the continuous interpolation functions $N_i$. Therefore, the function $f$ can be differentiated or integrated with respect to global 3D coordinates, as it is required in functional formulation (C.18). Thus a local coordinate system should be defined to facilitate the differentiation and integration within the element. The intrinsic coordinate system is the most commonly used system in the FEM.

A reference normalized element in the local intrinsic 3D coordinate system is defined by three coordinates $(r,s,t)$ which are normalized in the range of $[-1,1]$. The interpolation functions are then defined in terms of the intrinsic coordinates as $N_i(r,s,t)$. Then, differentiation w.r.t. global coordinates can be carried out in terms of intrinsic coordinates. Two kinds of elements were used in the analysis of the MRI magnet: the hexahedral elements for the interior region, and the infinite elements for the exterior region. These elements are detailed in the following discussion in order to help clarify the terminology that is used in the calculation of the system of equations in the next step of the FE formulation.
C.5.1 Linear Isoparametric Hexahedral Element

Different elements of the model, which are defined in the global coordinate system, should be mapped to the normalized reference element in the intrinsic coordinate system. This mapping is also necessary to express the global coordinates in terms of the intrinsic local coordinates to facilitate the integration of the variational functional (C.18) over the volume of each element. The same family of interpolation functions that is used to approximate the potential functions can also be used to express the element shape and coordinate transformation. Such an element system is called an isoparametric element system [24-26].

An eight-node linear isoparametric hexahedral element is shown in Figure C.1. The global coordinates and local intrinsic coordinates of a typical corner node, \( i \), are \((x_i, y_i, z_i)\) and \((r_i, s_i, t_i)\), respectively. The local interpolation functions or shape functions for this element are

\[
\{N^e\}_{81} = \{(1 + r_i)(1 + s_i)(1 + t_i) + 8\}; \quad \text{for } i = 1: 8,
\]

where \( r_i, s_i, \) and \( t_i \) are \(+1\) or \(-1\) for the node \( i \). Therefore, (C.27) can be written as

\[
f^e(r, s, t) = \sum_{i=1}^{8} N_i^e(r, s, t) f_i^e = (N^e)^T \{f^e\} = [f_1^e N_1 + f_2^e N_2 + \ldots + f_8^e N_8]
\]

\[
= f_1^e (1 - r)(1 - s)(1 - t) + f_2^e (1 - r)(1 + s)(1 - t) + \ldots + f_8^e (1 + r)(1 - s)(1 + t),
\]

where the scalar function \( f \) is any quantity of interest such as the global coordinates \( x, y, z \), or any components of magnetic vector potential \( A_x, A_y, \) or \( A_z \). The isoparametric shape functions (C.28) satisfy the relations...
\[ \sum_{i=1}^{8} N_i(r,s,t) = 1 \]

\[
N_i(r_j,s_j,t_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]  

Along the edges of the element, where \( r = \pm 1 \), \( s = \pm 1 \), or \( t = \pm 1 \), the interpolation functions (C.28) become linear. Then, the function \( f^e \) is defined by the two corresponding nodal values on that edge. Since the function \( f^e \) varies linearly on the edge of the element, the element with interpolation functions (C.28) is called a linear isoparametric hexahedral element, although the interpolation functions are trilinear inside the element.

Although the global coordinates \( x, y, \) and \( z \) of an isoparametric element are defined in terms of local coordinates \( r, s, \) and \( t \) (C.28), a unique inverse transformation defining \( r, s, \) and \( t \) in terms of \( x, y, \) and \( z \) is not usually needed. However, the relation between derivatives in two coordinate systems is necessary. The Jacobian matrix relating two coordinate systems is generally written as

\[
\left[ \frac{\partial}{\partial \text{local}} \right] = \left[ J(r,s,t) \right] \left[ \frac{\partial}{\partial \text{global}} \right] = \begin{bmatrix}
\frac{\partial}{\partial r} & \frac{\partial}{\partial s} & \frac{\partial}{\partial t} \\
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t}
\end{bmatrix}
\]  

Symbolically, the relation between the derivatives of a quantity, such as \( f \), is also written as

\[
\left\{ \frac{\partial}{\partial \text{local}} f(r,s,t) \right\} = \left[ J(r,s,t) \right] \left\{ \frac{\partial}{\partial \text{global}} f(x,y,z) \right\},
\]

\[
\left\{ \frac{\partial}{\partial \text{global}} f(x,y,z) \right\} = \left[ J(r,s,t) \right]^{-1} \left\{ \frac{\partial}{\partial \text{local}} f(r,s,t) \right\}.
\]

Therefore, to evaluate the global and local derivatives, the matrices \( J \) and \( J^{-1} \) should be obtained. In practical applications, these two quantities are evaluated numerically. For an isoparametric element, the global coordinate \( x \), from (C.29), is

\[
x'(r,s,t) = \sum_{i=1}^{8} N_i r s t = [N_1 x'_1 + N_2 x'_2 + \ldots + N_8 x'_8] = (N')^T \{x'\}.
\]

Applying (C.33) on the first row of the Jacobian matrix yields
Generalizing (C.34) yields the Jacobian matrix as
\[
\begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial r} \{N(r, s, t)\}^T \\
\frac{\partial}{\partial s} \{N(r, s, t)\}^T \\
\frac{\partial}{\partial t} \{N(r, s, t)\}^T
\end{bmatrix} = \begin{bmatrix}
\{x^r\} \{y^r\} \{z^r\}_b\delta \cdot (C.35)
\end{bmatrix}
\]

The partial derivatives of the interpolation functions, which are also the mapping functions, with respect to the local coordinate system are obtained in the matrix form of
\[
\begin{bmatrix}
\frac{\partial}{\partial r} \{N(r, s, t)\}^T \\
\frac{\partial}{\partial s} \{N(r, s, t)\}^T \\
\frac{\partial}{\partial t} \{N(r, s, t)\}^T
\end{bmatrix} = \begin{bmatrix}
\{x^r\} \{y^r\} \{z^r\}_b\delta = \begin{bmatrix}
\{x^r(1+s,t)(1+t,t)\}^T \\
\{x^r(1+r,t) s(1+t,t)\}^T \\
\{x^r(1+r,t) s(1+s) t\}^T
\end{bmatrix}
\end{bmatrix} . \quad (C.36)
\]

Thus (C.35) defines the Jacobian matrix \( J \), at a local point inside a typical element in terms of the spatial global coordinates of the element nodes \( \{x^r\}, \{y^r\}, \{z^r\} \), and the local derivatives of the interpolation functions \( \{\Delta_n(r, s, t)\} \) (C.36). Therefore, at any point of interest, such as the numerical integrating points in an element, the matrices \( J, J^{-1} \), and \( |J| \) can be numerically obtained.

### C.5.2 Infinite Element

The mapped infinite brick elements are used to discretize the exterior region of the MRI magnet domain. The basic idea is to map the domain of a regular finite element to an infinite element [27-28].

---

**Figure C.2**: One dimensional infinite element
In the one-dimensional case of Figure C.2, the coordinate transformation yields the mapping function as

\[ x^e(r) = \sum_{i=1}^{2} M_i^e(r) x_i^e = -\frac{2r}{1-r} x_1^e + \frac{1+r}{1-r} x_2^e. \]  
(C.37)

Then, the scalar function \( f^e \) can be approximated with a second order function as

\[ f^e(r) = \sum_{i=1}^{2} N_i^e(r) f_i^e = \frac{1}{2} (r^2 - r) f_1^e + (1 - r^2) f_2^e. \]  
(C.38)

An eight-node linear infinite hexahedral element is shown in Figure C.1 with the local coordinate \( r \) toward infinity. The global coordinates and local coordinates of a typical corner node \( i \) are \((x_i, y_i, z_i)\) and \((r_i, s_i, t_i)\), respectively. By extension of (C.37) for the 3D infinite element, the coordinate transformations or mapping functions are

\[ x^e(r, s, t) = \sum_{i=1}^{8} M_i^e(r, s, t) x_i^e = \{M^e\|^T \{x^e\}_b, \]  
(C.39)

where

\[ M_i^e(r, s, t) = \begin{cases} \frac{-2r}{4(1-r)}(1+s)(1+t) & i=1,2,3,4 \\ \frac{1+r}{4(1-r)}(1+s)(1+t) & i=5,6,7,8 \end{cases}, \]  
(C.40)

and where \( s_i \) and \( t_i \) are \(+1\) or \(-1\) for the \( i^{th} \) term. Substitutions of \( x \) with \( y \) or \( z \) yield the mapping functions of \( y \) or \( z \) global coordinates.

Approximation of the potential \( A \) with second order functions in the direction of infinity, \( r \), yield the interpolation functions of

\[ f^e(r, s, t) = \sum_{i=1}^{8} N_i^e(r, s, t) f_i^e = \{N^e\|^T \{f^e\}_b, \]  
(C.41)

where

\[ N_i^e(r, s, t) = \begin{cases} \frac{1}{2} (r^2 - r)(1+s)(1+t) & i=1,2,3,4 \\ \frac{1}{4} (1-r^2)(1+s)(1+t) & i=5,6,7,8 \end{cases}. \]  
(C.42)

where \( s_i \) and \( t_i \) are \(+1\) or \(-1\) for the \( i^{th} \) term, and \( f \) is any one of the magnetic vector potential components \( A_x, A_y, \) and \( A_z \).

The Jacobian matrix is obtained, similar to the isoparametric elements, as
The partial derivatives of the mapping functions with respect to the local coordinate system are obtained in the matrix form of

\[
[J(r, s, t)]_{bd} = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t}
\end{bmatrix}_{bd}
= \begin{bmatrix}
\frac{\partial M_i(r, s, t)}{\partial r} \\
\frac{\partial M_i(r, s, t)}{\partial s} \\
\frac{\partial M_i(r, s, t)}{\partial t}
\end{bmatrix}^T_{bd} \begin{bmatrix}
[x^e] \\
y^e \\
z^e
\end{bmatrix}_{bd}.
\]  

(C.43)

Thus, (C.43) defines the Jacobian matrix \(J\), at a local point inside a typical element in terms of the spatial global coordinates of the element node \([x^e], [y^e], [z^e]\), and the local derivatives of the mapping functions \(\Delta_M(r, s, t)\) (C.44). Therefore, at any point of interest, such as the numerical integrating points in an element, the matrices \(J\), \(J^{-1}\), and \(|J|\) can be numerically obtained.

For future use, the partial derivatives of the interpolation functions with respect to the local coordinate system are obtained in the matrix form of

\[
[\Delta_M^*(r, s, t)]_{bd} = \begin{bmatrix}
\frac{\partial}{\partial r} \{N^e\}^T \\
\frac{\partial}{\partial s} \{N^e\}^T \\
\frac{\partial}{\partial t} \{N^e\}^T
\end{bmatrix}_{bd}
= \begin{bmatrix}
\frac{\partial N_i(r, s, t)}{\partial r} \\
\frac{\partial N_i(r, s, t)}{\partial s} \\
\frac{\partial N_i(r, s, t)}{\partial t}
\end{bmatrix}^T_{bd}.
\]  

(C.45)
C.6 System Formulation

The domain of the problem is first discretized into $M$ elements, where each element has $n$ nodes. The interpolation functions of elements are then selected. The next step of the FE analysis is to apply the variational method to formulate the system of equation [23,24]. The following discusses the implementation of the variational formulation and the different techniques that are necessary to obtain the final form of the system of equations.

C.6.1 Variational Implementation

The variational principles and formulation are given in Section C.2 and (C.18). Some preliminary steps are necessary to arrive at the final form of the modified energy functional (C.18). The energy functional for each element is written in the form of

$$F'(A^e) = \int_{\Omega^e} W'(A^e) \, dV - \int_{\Gamma} J^e \cdot A^e \, d\Gamma,$$

where $W'(A^e)$ is the energy density associated with the solution $A^e$ given by

$$W'(A^e) = \int_0^g H \cdot dB - \int_0^g V \cdot B \cdot dB. \quad (C.47)$$

The components of the magnetic vector potential $\{A_x, A_y, A_z\}$ and current density $\{J_x, J_y, J_z\}$ in element $e$ are approximated as

$$A^e_p = \sum_{i=1}^n N^e_i A^e_{pi} = \{N^e\}_n^T \{A^e_p\}_n = \{A^e_p\}_n^T \{N^e\}_n, \quad \text{for } p = x, y, z, \quad (C.48)$$

$$J^e_p = \sum_{i=1}^n N^e_i J^e_{pi} = \{N^e\}_n^T \{J^e_p\}_n = \{J^e_p\}_n^T \{N^e\}_n$$

where the nodal values of the potential $A_p$ are $\{A^e_p\}_n$, the number of nodes is $n=8$, and the interpolation functions are either isoparametric elements (C.28) or infinite elements (C.42). The minimization of the functional with respect to the nodal values of the $A$ yields

$$\frac{\partial F'}{\partial A_i} = \int_{\Omega^e} \frac{\partial}{\partial A_i} (W^e - J^e \cdot A^e) \, dV = 0. \quad (C.49)$$

The first part of the integral (C.49) is derived from (C.47) as
Applying the chain rule of differentiation to (C.50) yields
\[
\frac{\partial W^*}{\partial A_i} = \frac{1}{2} \nu(b^2) \frac{\partial}{\partial A_i}(B^2) \ .
\]  
(C.51)

Substitution of (C.51) in (C.49) yields
\[
\frac{\partial F^*}{\partial A_i} = \frac{1}{2} \nu^* B^2 dV - \frac{\partial}{\partial A_i} \int_{\Omega} J^* \cdot A^* dV = 0 .
\]  
(C.52)

By substitution of \(B = \nabla \times A\), (C.52) becomes the FE form of the minimized energy functional (C.18) without the Coulomb gauge enforcing term. The minimization of the functional (C.52) yields the matrix form of
\[
[K']\{A^*\} = \{b^*\},
\]  
(C.53)

where the element degrees of freedom vector \(\{A^*\}\) is
\[
\{A^*\} = \{A^*_x\}^T \quad \{A^*_y\}^T \quad \{A^*_z\}^T, \quad 2 \times 1 .
\]  
(C.54)

The element load vector \(\{b^*\}\) is
\[
\{b^*\} = \{b^*_x\}^T \quad \{b^*_y\}^T \quad \{b^*_z\}^T, \quad 2 \times 1 ,
\]  
(C.55)

and the element stiffness matrix \([K']\) is
\[
[K'] = \int_{\Omega} \nu^* (\nabla \times [N_A]^\top) (\nabla \times [N_A]^\top) dV .
\]  
(C.56)

The application of the curl operator on the shape function matrix \([N_A]\) yields
\[
\nabla \times [N_A]^\top = \begin{bmatrix}
\{0\}^\top \\
\frac{\partial (N^*)^\top}{\partial z} \\
\frac{\partial (N^*)^\top}{\partial y} \\
\frac{\partial (N^*)^\top}{\partial x}
\end{bmatrix} + \begin{bmatrix}
\{0\}^\top \\
\frac{\partial (N^*)^\top}{\partial z} \\
\frac{\partial (N^*)^\top}{\partial y} \\
\frac{\partial (N^*)^\top}{\partial x}
\end{bmatrix},
\]  
(C.57)

where the vector \(\{N\}\) is the shape function vector as defined for each element.
To complete the formulation of the stiffness matrix, the Coulomb gauge condition is included from functional (C.18) as

\[
[K'_s]_{24\times24} = \iiint_{\Omega} \nu^* \left( \nabla \cdot [N_A]^T \right) \left( \nabla \cdot [N_A]^T \right) dV,
\]

where the divergence of the shape function matrix \([N_A]\) yields

\[
\nabla \cdot [N_A]^T = \left[ \frac{\partial (N^*)^T}{\partial x} \quad \frac{\partial (N^*)^T}{\partial y} \quad \frac{\partial (N^*)^T}{\partial z} \right]_{24\times24}.
\]

Finally, the element stiffness matrix becomes

\[
[K^e] = [K'_s] + [K^*_s],
\]

and the system of equations becomes

\[
[K^e]_{24\times24} \{A^e\}_24 = \{b^e\}_24.
\]

The detached representation of the element system of equation (C.61) is

\[
\begin{bmatrix}
[K'_{xx}] & [K'_{xy}] & [K'_{xz}] \\
[K'_{yx}] & [K'_{yy}] & [K'_{yz}] \\
[K'_{zx}] & [K'_{zy}] & [K'_{zz}]
\end{bmatrix}
\begin{bmatrix}
\{A^e_x\} \\
\{A^e_y\} \\
\{A^e_z\}
\end{bmatrix} =
\begin{bmatrix}
\{b^e_x\} \\
\{b^e_y\} \\
\{b^e_z\}
\end{bmatrix},
\]

where the \(\{b^e_p\}_{6\times1}\) load vector is given by (C.55), and the \([K_{pq}^e]\) element sub-matrices are given by

\[
[K^e_{xx}]_{6\times6} = \iiint_{\Omega} \nu^* \left( s \frac{\partial (N^*)}{\partial x} \frac{\partial (N^*)}{\partial x} + \frac{\partial (N^*)}{\partial y} \frac{\partial (N^*)}{\partial y} + \frac{\partial (N^*)}{\partial z} \frac{\partial (N^*)}{\partial z} \right) dV,
\]

\[
[K^e_{yy}]_{6\times6} = \iiint_{\Omega} \nu^* \left( s \frac{\partial (N^*)}{\partial x} \frac{\partial (N^*)}{\partial x} + \frac{\partial (N^*)}{\partial y} \frac{\partial (N^*)}{\partial y} + \frac{\partial (N^*)}{\partial z} \frac{\partial (N^*)}{\partial z} \right) dV,
\]

\[
[K^e_{zz}]_{6\times6} = \iiint_{\Omega} \nu^* \left( s \frac{\partial (N^*)}{\partial x} \frac{\partial (N^*)}{\partial x} + \frac{\partial (N^*)}{\partial y} \frac{\partial (N^*)}{\partial y} + \frac{\partial (N^*)}{\partial z} \frac{\partial (N^*)}{\partial z} \right) dV,
\]

\[
[K^e_{pq}]_{6\times6} = \iiint_{\Omega} \nu^* \left( s \frac{\partial (N^*)}{\partial p} \frac{\partial (N^*)}{\partial q} - \frac{\partial (N^*)}{\partial q} \frac{\partial (N^*)}{\partial p} \right) dV; \quad p, q = x, y, z; \quad p \neq q,
\]

where the term \(s=1\) is included to show the terms which are generated from the divergence part of the functional \(F^e\) (C.18). It is concluded that \([K^e_{xx}]_{6\times6} = [K^e_{yy}]_{6\times6} = [K^e_{zz}]_{6\times6}\), which is the result of the Coulomb gauge enforcement in the functional (C.18).
The element matrices and load vector are calculated numerically. These quantities are obtained from the volume integral of integrands that contain global derivatives of interpolation functions. Therefore, first, a numerical integration method should be selected, and second, the global derivatives of interpolation functions should be obtained.

C.6.2 Gaussian Element Integration

The most accurate numerical method is the Gauss method, in which a typical integral of

\[ I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} h(r, s, t) \, dr \, ds \, dt \]  

(C.67)

is approximated by

\[ I_T = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} W_{Gi} W_{Gj} W_{Gk} h(r_{Gi}, s_{Gj}, t_{Gk}) \]  

(C.68)

where the integration points \((r_{Gi}, s_{Gi}, t_{Gi})\) and weighting factors \(W_{Gi}, W_{Gj}, W_{Gk}\) are given by the Gaussian quadrature formula. ANSYS software uses the Gaussian quadrature formula of the second order, \(n=2\), for hexahedral elemental calculation in which the integration points \((r_{Gi}, s_{Gi}, t_{Gi})\) are selected at \(\pm 0.577350269189626\), and the weighting factors are defined at \(W_{Gi}, W_{Gj}, W_{Gk}=1\). Thus the numerical integration (C.68) is written as

\[
I_T \approx \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} h(r_{Gi}, s_{Gi}, t_{Gi}) = a(-1)^i, s_{Gi} = a(-1)^j, t_{Gi} = a(-1)^k \\
a = 0.577350269189626
\]  

(C.69)

C.6.3 Global Derivative of Interpolation Functions

The integral definitions of the element matrix (C.63) to (C.66) require the global derivatives of interpolation functions \(\{N^r(r, s, t)\}\), which are functions of local coordinates \((r, s, t)\). By using (C.29), the local derivatives of quantities of interest, such as \(A_x\), are related to the nodal values of the quantity \(A_x\) by
The matrix $D$, which relates the global derivatives of the quantity $A_x$ to the quantity's nodal values, can be partitioned as

$$
[D(r,s,t)]_{\text{bd}} = 
\begin{bmatrix}
\{Dx(r,s,t)\}_{\text{bd}}^T & \{Dy(r,s,t)\}_{\text{bd}}^T & \{Dz(r,s,t)\}_{\text{bd}}^T
\end{bmatrix}_x = 
\begin{bmatrix}
\frac{\partial N_1^e(r,s,t)}{\partial x} & \frac{\partial N_2^e(r,s,t)}{\partial x} & \cdots & \frac{\partial N_8^e(r,s,t)}{\partial x} \\
\frac{\partial N_1^e(r,s,t)}{\partial y} & \frac{\partial N_2^e(r,s,t)}{\partial y} & \cdots & \frac{\partial N_8^e(r,s,t)}{\partial y} \\
\frac{\partial N_1^e(r,s,t)}{\partial z} & \frac{\partial N_2^e(r,s,t)}{\partial z} & \cdots & \frac{\partial N_8^e(r,s,t)}{\partial z}
\end{bmatrix},
\tag{C.75}
$$
Therefore, each row of matrix $D$, which is numerically available at any point of interest such as the Gaussian integration points inside the element, represents a derivative of the interpolation functions with respect to one global coordinate direction.

### C.6.4 Element Matrix Calculation

To obtain the element matrix, all the element sub-matrices $[K_{pq}]_{6\times 8}$ are to be calculated. The element $(l, m)$ of matrix $[K_{pq}]_{6\times 8}$ is obtained by evaluation of the matrix elements as

$$K^e_{lm}(l, m) = \iiint_V \varphi \left( \frac{\partial N_i^e \partial N_m^e}{\partial x} + \frac{\partial N_i^e \partial N_m^e}{\partial y} + \frac{\partial N_i^e \partial N_m^e}{\partial z} \right) dx
dy
dz .$$

(C.76)

The Jacobian matrix (C.43) is used to change the integration from global coordinates to local element coordinates. As a result, the volume integral takes the form of

$$K^e_{lm}(l, m) = \iiint_{-1}^{+1} \varphi \left( \frac{\partial N_i^e \partial N_m^e}{\partial x} + \frac{\partial N_i^e \partial N_m^e}{\partial y} + \frac{\partial N_i^e \partial N_m^e}{\partial z} \right) J(r, s, t) |ds
dr
dt .$$

(C.77)

The integrand function can then be written as

$$h_{lm}(r, s, t) = \varphi \left( \frac{\partial N_i^e \partial N_m^e}{\partial x} + \frac{\partial N_i^e \partial N_m^e}{\partial y} + \frac{\partial N_i^e \partial N_m^e}{\partial z} \right) J(r, s, t) .$$

(C.78)

To apply the Gaussian numerical integration (C.69), the integrand function $h_{lm}(r, s, t)$ should be evaluated at the Gaussian points $(r_{gi}, s_{gi}, t_{gk})$. Therefore, for an element with $n=8$ nodes, and for each Gaussian integration point, the following quantities should be calculated:

- **For isoparametric elements:** $[\Delta h(r_{gi}, s_{gi}, t_{gk})]_{3\times 8}$ from (C.36),
  
  $[J(r_{gi}, s_{gi}, t_{gk})]_{3\times 3}$ from (C.35), then $[J]^{-1}$ and $|J|$, 
  
  $[D(r_{gi}, s_{gi}, t_{gk})]_{3\times 8}$ from (C.74).

- **For infinite elements:** $[\Delta h(r_{gi}, s_{gi}, t_{gk})]_{3\times 8}$ from (C.43),
  
  $[J(r_{gi}, s_{gi}, t_{gk})]_{3\times 3}$ from (C.43), then $[J]^{-1}$ and $|J|$, 
  
  $[\Delta h(r_{gi}, s_{gi}, t_{gk})]_{3\times 8}$ from (C.45),
  
  $[D(r_{gi}, s_{gi}, t_{gk})]_{3\times 8}$ from (C.74).

Therefore, vectors $[Dx(r_{gi}, s_{gi}, t_{gk})]_{8\times 1}$, $[Dy(r_{gi}, s_{gi}, t_{gk})]_{8\times 1}$, and $[Dz(r_{gi}, s_{gi}, t_{gk})]_{8\times 1}$ are numerically available at $(r_{gi}, s_{gi}, t_{gk})$. The integrand (C.78) can be obtained as
\[ h_{i,m}(r, s, t) = v^e(Dx_r, Dy_r, Dz_r) |J| \]  
(C.79)

The proper value of \( v^e \), and application of the Gaussian integration (C.69) yield

\[ K^e_{lm}(l, m) = \sum_{j=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} h_{i,m}(r, s, t) \]  
(C.80)

Since vectors \( Dx, Dy, Dz \) are not changing at each integration point within each element, the element matrix can be obtained by calculating

\[ [K^e]_{ba} = \sum_{j=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} v^e [(Dx)^T(Dx) + (Dy)^T(Dy) + (Dz)^T(Dz)] |J| \]  
(C.81)

Element sub-matrix \([K^e]_{ba}\) is numerically obtained by the same approach as

\[ [K^e_{pq}]_{ba} = \sum_{j=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} v^e [(Dp)^T(Dq) - (Dq)^T(Dp)] |J|; \quad p, q = x, y, z; \quad p \neq q. \]  
(C.82)

It should be noted that since the integration points within an element are constant, the same vectors \( Dx, Dy, Dz \) of (C.81) are used in (C.82).

**C.6.5 Load Vector Calculation**

To calculate the load vector for each element with \( n=8 \) nodes, (C.55) is written as

\[ \{f^e\} = \iiint_{v} [N^e]^T J^e \{f^e\} dV = [T^e]_{ba} \{f^e\}; \quad p = x, y, z . \]  
(C.83)

where

\[ [T^e]_{ba} = \iiint_{v} [N^e] dV . \]  
(C.84)

Applying the Jacobian relation between global and local volume differential yields

\[ [T^e]_{ba} = \iiint_{v} [N^e] J^e(r, s, t) |J| dr ds dt . \]  
(C.85)

To apply the Gaussian numerical integration (C.69), the integrand function should be evaluated at the Gaussian points \((r_1, s_1, t_1)\). Therefore, for an isoparametric element with \( n=8 \) nodes, and for each Gaussian integration point, the following quantities should be calculated:

\[ [\Delta^e_{N}(r, s, t)]_{ba} \] from (C.36), and

\[ [J^e(r, s, t)]_{ba} \] from (C.35), and then \(|J|\).
Consequently, the Gaussian integration (C.64) yields the $[T]$ matrix in the form of

$$
[T^e]_{k=0} = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \left[ N^e(r_{el}, s_{el}, t_{el}) \right] \left[ N^e(r_{el}, s_{el}, t_{el}) \right] \left[ J(r_{el}, s_{el}, t_{el}) \right].
$$

(C.86)

Finally, the load vector $\{b_r^e\}$ is obtained from (C.83).

C.6.6 System of Equations Assembly

After calculating all of the element matrices and element loads, the system of equations is then obtained by assembling all the elements, by global numbering the nodes, and imposing the stationary condition $\delta F = 0$. The final form of the equations is

$$
\begin{align*}
\left[ \frac{\partial F}{\partial A_x} \right] &= \sum_{k} \left( [K_{xx}][A_x^r] + [K_{yx}][A_y^r] + [K_{xz}][A_z^r] - \{b_x^r\} \right) = \{0\}, \\
\left[ \frac{\partial F}{\partial A_y} \right] &= \sum_{k} \left( [K_{xy}][A_x^r] + [K_{yy}][A_y^r] + [K_{yz}][A_z^r] - \{b_y^r\} \right) = \{0\}, \\
\left[ \frac{\partial F}{\partial A_z} \right] &= \sum_{k} \left( [K_{xz}][A_x^r] + [K_{yz}][A_y^r] + [K_{zz}][A_z^r] - \{b_z^r\} \right) = \{0\}.
\end{align*}
$$

(C.87)

The Boundary Condition:

The boundary conditions should be applied at this stage of the FE procedure. The total exterior surface of the problem domain can be divided into symmetry planes and the exterior surface that faces the infinite. The boundary condition at the infinite exterior surface is accounted for by using the infinite elements. The symmetry planes of the geometry are usually characterized as either normal or parallel magnetic field planes. These conditions should be enforced on the set of equation (C.87).

For the normal magnetic field plane of symmetry, the normal component of the magnetic vector potential must be set to zero. Thus the degrees of freedom of all the nodes in such a plane of symmetry are reduced by one, and for each node the relative row of the element matrix and the relative element of the load vector are removed from the set of equations (C.87).
For the tangential magnetic field plane of symmetry, the in-plane or tangential components of the magnetic vector potential must be set to zero. Therefore, the degrees of freedom of all the nodes in such a plane of symmetry are reduced to only one. Thus for such a node, the relative rows of the element matrix and relative elements of the load vector are removed from the set of equations (C.87).

The Final Set of Equations:

The system of equations can finally be written as

\[
\begin{bmatrix}
K_{xx} & K_{xy} & K_{xz} \\
K_{yx} & K_{yy} & K_{yz} \\
K_{zx} & K_{zy} & K_{zz}
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix} =
\begin{bmatrix}
b_x \\
b_y \\
b_z
\end{bmatrix}
\]

or symbolically

\[
[K][u] = \{b\}. \tag{C.89}
\]

Matrix \([K]\) can be termed the global coefficient matrix, Dirichlet matrix, or stiffness matrix. Vector \([u]\) is the non-zero global vector of the unknown quantity or degrees of freedom vector, and vector \([b]\) is the global load vector or forcing function vector.

C.7 System Solution

To obtain the solution to the boundary value problem, defined in Section C.1.4, the system of equations is formulated and then assembled into one form of (C.89). If all the materials in the domain of the problem are linear materials, then the solution of (C.89) yields the solution to the problem. The magnetic flux density of the problem domain is then obtained from the appropriate use of (C.9) and (C.57).

Existence of a nonlinear magnetic material in the domain of the problem leads to a nonlinear stiffness matrix \([K]\) of (C.89). The nonlinearity arises from dependency of the magnetic material permeability on the magnetic field intensity. Therefore, the \(B-H\) curve of any nonlinear material in the problem domain should be discussed accordingly.

There are two important considerations for the material property specification. First, to insure the uniqueness and solvability of the FE approach, the permeability of materials
should be single-valued and monotonic [29]. Second, to increase the efficiency of the calculation, the \( B-H \) curve of the material is usually converted to a spline fit curve of \( v \) versus \( B^2 \), from which the \( v^* \) is evaluated for each element.

A simple iterative method can be set up to solve the nonlinear equations of (C.89) [24]. Starting with a zero magnetic vector potential, a guess set of reluctivity values for the elements is assumed. The stiffness matrix is calculated, and then the linear \( KU=F \) is solved for \( U \). The new set of \( U \) is used to obtain a new set of element reluctivity values in which relaxation methods can be used. The iterative calculation of \( K \), \( U \), and \( v \) is terminated after two successive solutions agree within an acceptable tolerance. This approach converges very slowly. The preferred method of solving the nonlinear set of equation (C.89) is the Newton-Raphson method.

### C.7.1 Newton-Raphson Method

The Newton-Raphson method has proven itself to be a very stable and fast converging iterative method [24,30]. Considering the correct solution of (C.89) as vector \( \{A^\text{exact}\} \), and the iteration solution as vector \( \{A^*\} \), then the distance to the solution or displacement vector is

\[
\{A^\text{exact}\} - \{A^*\} = \delta\{A^*\}.
\]  
(C.90)

The multidimensional Taylor's series expansion of the gradient of the \( F(A) \) functional near \( \{A^*\} \) yields

\[
\frac{\partial F^*}{\partial A_i} = \left. \frac{\partial F}{\partial A_i} \right|_{A^*} + \sum_j \left. \frac{\partial^2 F}{\partial A_i \partial A_j} \right|_{A^*} \delta A_j + \cdots. 
\]  
(C.91)

Minimization of the functional (C.52) requires that at the exact solution, \( \{A^*\} = \{A^\text{exact}\} \), all the gradients of the functional must vanish. Approximating the Taylor series with its first two terms, and equating the left-hand-side with zero, yields the matrix form of (C.91) as

\[
\{V^*\} + [H^*][\delta A^*] = 0, 
\]  
(C.92)

where \([H^*]\) is the element Hessian or Jacobian matrix and \( \{V^*\} \) is the element gradient vector.
The element gradient vector or residual vector \( \{ V' \} \) is obtained from the non-zero functional (C.52) and the Coulomb gauge enforcement as
\[
\{ V' \} = [K']\{ A' \} - \{ b' \},
\]
where \([K']\) is given by (C.60).

To evaluate the element Hessian matrix, (C.49) and (C.51) are used to obtain
\[
H_{i,j} = \frac{\partial^2 F}{\partial A_i \partial A_j} = \int_{\nu'} \frac{\partial^2 W'}{\partial A_i \partial A_j} dV = \int_{\nu'} \frac{\partial}{\partial A_i} \left( \frac{\nu}{2} \frac{\partial (B^2)}{\partial A_j} \right) dV = \int_{\nu'} \left( \frac{\nu}{2} \frac{\partial^2 (B^2)}{\partial A_i \partial A_j} + \frac{1}{2} \nu \frac{\partial (B^2)}{\partial A_i} \frac{\partial (B^2)}{\partial A_j} \right) dV.
\]

By application of \( B = \nabla \times A \), and writing \( A \) in terms of nodal values of \( A' \) as (C.54), the element Hessian matrix (C.94) is written as
\[
[H'] = [K'_f] + [K'_h],
\]
where \([K'_f]\) is the element stiffness matrix given in (C.56), and \([K'_h]\) is
\[
[K'_h]_{\nu'\nu'A} = \int_{\nu'} 2 \frac{\partial V'}{\partial (B^2)} \left( [B']^T (\nabla \times [N_A]^T) \right) \left( [B']^T (\nabla \times [N_A]^T) \right)^T dV.
\]

The slope of reluctivity is obtained from the iteration value of the magnetic flux density
\[
\{ B' \} = \{ B_{x}' B_{y}' B_{z}' \}^T_{3\text{D}}.
\]

The curl of the shape function matrix \((\nabla \times [N_A]^T)\) is given in (C.57). The element Hessian matrix is numerically calculated in the same fashion as the stiffness matrix outlined in Section C.6.3.

The global numbering and assembly of all the related matrices and vectors lead to the system stiffness matrix, the Hessian matrix, and the gradient vector.

The iterative Newton-Raphson method is started with the initial guess vector \( \{ A \}_{i=0} \). Then, in each \( i^{th} \) iteration,

1. The elements reluctivity from iteration value of potentials,
2. the stiffness matrix \([K]\) from (C.89),
3. the gradient vector \([V]\) from (C.93).
4. the Hessian matrix $[H]$, from (C.95),
5. the new displacement or correction vector $\{\delta A\}$, from (C.92),
6. the new degrees of freedom vector from $\{A\}_{i+1} = \{A\}_i + \{\delta A\}_i$,

are calculated. The iteration continues till a convergence or an error measure is satisfied.

The convergence criterion uses the norm of the displacement or distance vector $\{\delta A\}$, or, more logically, that of the gradient or residual vector $\{V\}_i$. In the preferred latter case, the limit is expressed as a percentage of the norm of the external load or forcing function vector as

$$\|V_i\| \leq \varepsilon \|b\|,$$  \hspace{1cm} (C.99)

where the norm of a vector is defined as $\|x\| = (x^T x)$, and $\varepsilon$ is selected at 0.1%.

In step five of the Newton-Raphson method iteration (C.98), a set of linear algebraic equations of the $Ax=b$ form has to be solved. The different solution techniques are primarily divided into direct methods and iterative methods. The direct methods commonly use decomposition techniques to arrive at a solution. Their main disadvantage is an increase in both the memory storage and processing time, coupled with an increase in the number of the problem unknowns, in this case the degrees of freedom associated with all nodes. The iterative methods have proven themselves very efficient with respect to the processing time and the memory requirements for the solution. Among various iterative methods, the Jacobi-Conjugate-Gradient, JCG, is widely used. In this thesis, the software ANSYS employs this method as outlined in [31].

C.8 Evaluation of the Final Results

The Finite Element formulation and its immediate results are given in terms of magnetic vector potentials, which are mathematical functions and not physical quantities. Therefore, the primary results of the FEM cannot be used, or compared with measured data, or verified in their original format. Different quantities of interest must be obtained in order to analyze the device, to estimate the error in the solutions, and to verify the results. Quantities of interest include the magnetic flux density $B$, the magnetic field intensity $H$, the magnetic flux
through a closed surface \( \Phi \), the magnetomotive force along a contour line \( \text{mmf} \), and the stored energy in a part of the model. It should be noted that the post-processing of the primary data introduces a new level of error in the solution due to numerical differentiation and integration of the primary data. The derived data are usually calculated after the convergence of the Newton-Raphson method while elemental data are still available. The final solutions - which are described in the following subsections - are calculated by the ANSYS program.

**C.8.1 Magnetic Field**

The magnetic flux density solution to a boundary value problem is sought for the following reasons. First, \( \mathbf{B} \) is generally the primary physical quantity of interest when considering the design of a device and should be obtained within a tolerable accuracy. Second, it is important to compare the approximation solution with the measured data when such data are available for the device. This would provide the best general quality criterion for the solution verification. Third, Ampere's law (C.2) is used as an effective global accuracy criterion. Its calculation requires \( \mathbf{H} \), which is obtained from \( \mathbf{B} \) and the \( \mathbf{B}-\mathbf{H} \) curve.

The magnetic flux density \( \mathbf{B} \) is defined as the curl of the magnetic vector potential \( \mathbf{A} \). The nodal magnetic flux density of each element can then be obtained from numerical differentiation of nodal values of the vector potential \{\( \mathbf{A} \)\}. However, a more accurate approach is to use the element interpolation functions, which yields

\[
\{\mathbf{B}^e\} = (\nabla \times [\mathbf{N}_\mathbf{A}]^T)\{\mathbf{A}^e\},
\]

where the curl of shape functions is given in (C.57). To increase the accuracy of the calculation, the field is computed at the integration points of each element. It should be noted that the curl of the shape functions has already been calculated at each integration point within the element. The integration points' values are then extrapolated to the element nodes to yield the element nodal magnetic flux density \( \mathbf{B}_{\text{nodal}}^e \). The extrapolation is based on the element shape function, or its simplified form as
where the hexahedral element is considered. Because of the elemental calculation, the elemental magnetic flux density is not continuous from each element to the adjacent elements, while the primary nodal magnetic potential is continuous. The averaging of the element nodal field is usually employed to obtain more acceptable and continuous looking results where there is no material discontinuity between elements. The element nodal magnetic field intensity, \( H_{\text{nodal}} \), is then calculated from the \( B-H \) relation of the element material. The magnetic field data are used to assess the quality of the results if the measured data are available in regions of interest.

Contour plots of the field data are usually drawn at places of interest to graphically study the magnetic field data. 3D contour plots can be viewed on color monitors that make them very difficult to reproduce on paper for later analysis. Contour plots are usually drawn for a 2D slice of the problem domain at a region of interest. The first and best location for contour plots is at the planes of symmetry on which the tangential or normal component of the magnetic field vanishes. The out-of-normal, or tangent, field can also be drawn to view the error due to the numerical post-processing of degrees of freedom. The second good location for contour plots is at the planes of material discontinuity, where the boundary conditions can be verified. Continuity and the rate of field change, closeness of the contour lines, and the location of a very high and low field are used to evaluate the study model.

### C.8.2 Magnetic Flux

It is always more desirable to assess the quality of the results from primary data than from derived data. The total magnetic flux passing through a closed surface can be calculated directly from the magnetic vector potentials. By definition, the magnetic flux passing through the closed surface \( S \) is written as

\[
\Phi = \int_{S} \mathbf{B} \cdot d\mathbf{s} = \int_{S} (\nabla \times \mathbf{A}) \cdot d\mathbf{s}.
\]

Applying Stokes's theorem yields
\[ \Phi = \int_{l} A \cdot \hat{t} \, dl , \]  

(C.103)

where \( l \) is the bounding contour line of the surface \( S \), and \( \hat{t} \) is the unit tangent vector to line \( l \). The nodal values of magnetic vector potential \( \{A\} \) are then used to approximate the magnetic flux through a closed surface as

\[ \Phi = \int_{l} \{A\} \cdot \{t\} \, dl , \]  

(C.104)

where the integration is carried out numerically. Flux calculations are used extensively by design engineers to analyze different parts of a magnetic device. Leakage of the magnetic flux from the magnetic material is also used to evaluate the efficiency of the magnetic circuit design.

### C.8.3 Magnetomotive Force

The integral form of Ampere’s law (C.2) is written as

\[ \int_{l} \mathbf{H} \cdot \hat{t} \, dl = \int_{s} \mathbf{J} \cdot ds = I_{S} , \]  

(C.105)

where \( I_{S} \) is the total current passing through the surface \( S \), \( l \) is the bounding contour line of the surface \( S \), and \( \hat{t} \) is the unit tangent vector to contour line \( l \). Then, (C.105) is used to approximate the magnetomotive force drop, \( \text{mmf} \), along a contour line, \( l \), as

\[ \text{mmf}_{l} = \int_{l} \{H\} \cdot \{t\} \, dl , \]  

(C.106)

where \( l \) is the bounding contour line of the surface, and vector \( \{H\} \) contains the nodal values of magnetic field intensity. The magnetomotive force drop along a path provides a valuable global accuracy measure for verification of the FE results with the magnetic vector potential formulation.

### C.8.4 Stored Energy

The total energy stored in the problem domain is often used to assess the quality of the FE results. The stored energy in an element is
where the energy density is given by

\[ E^*_S = \int_{V^*} W^*(B^*) dV \quad \text{(C.107)} \]

The energy density (C.108) is calculated either directly from the $B$-$H$ curve of the element material, or from the spline fit function of the material $B$-$H$ curve. The Gaussian integration (C.69) is then used to evaluate the stored energy as

\[ E^*_S = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} W(B^*(r_{ik},s_{ik},t_{ik})) \left| J(r_{ik},s_{ik},t_{ik}) \right| \quad \text{(C.109)} \]

The stored energy in different parts is then calculated from the elements' stored energy. The total stored energy is used to calculate the inductance of the magnetic circuits. It was used in the selection study of the FE application.

### C.9 Conclusion

This appendix has detailed the Finite Element Method for magnetostatic 3D problems: from definition of the problem, to the solution of the system of equations. Should the Newton-Raphson method (C.98) fail to converge to a solution, the Finite Element approach must be modified properly. Then, the final converged solutions of the FEM should be verified because of the numerical nature of the method. This has been discussed in Chapter 2.
Appendix D

Axisymmetric Finite Element Method

The axisymmetric Finite Element modeling of the MRI magnet is summarized in the following steps:

1. The boundary value problem was defined in terms of the magnetic vector potential $\mathbf{A}$ in a cylindrical coordinate frame $(r, \phi, z)$. The $r$ and $z$ components of $\mathbf{A}$ are zero, and there is only one degree of freedom for each node denoted as $A_\phi(r, z)$.

2. The variational formulation was used [23, 24].

3. The problem domain was defined in the $r\phi z$ coordinate plane. The azimuthal coordinate $\phi$ is the symmetry plane and is not used in the domain definition and representation.

4. Due to the specific symmetry of the MRI magnet configuration, only one quarter of the MRI magnet problem domain was modeled. The exterior lines of the domain divided into the normal field, the tangential field, and the far field lines. The extent of the domain definition should be verified by a selection approach. The $B-H$ characteristic of the magnetic material may be specified in this step.

5. Two element types were used: the linear isoparametric quadrilateral elements, and the Lagrangian isoparametric infinite element. The shape functions and the mapping functions are the simplified and modified notation of the three-dimensional hexahedral and infinite elements given in Appendix C.
6. The domain of the problem was properly discretized. The refinement of the element size and shape is used to obtain the *optimum* model and final discretization of the problem domain.

7. The element matrices were calculated and assembled into the system of equations. The boundary conditions of $A_\phi=0$ were applied to the nodes on the tangential flux exterior line. The normal boundary condition is satisfied by the problem formulation. The proper current density was applied to the coil. Finally, the nonlinear system of equations was obtained.

8. The Newton-Raphson method was used to solve the nonlinear equations.

9. The Magnetic vector potential results were post-processed to obtain the magnetic field data. The results were then studied to obtain the proper conclusion.

10. The total magnetic flux passing a circular surface normal to the roz plane is given as

$$\Phi_{\text{total}} = \int B \cdot ds = \int A \cdot dl = 2\pi RA_\phi(R). \quad (D.1)$$

where $R$ is the radius of such surface. Then the average magnetic flux density at such a circular cross-section was obtained as

$$B_{\text{across}} = \Phi_{\text{total}} + S = 2A_\phi(R) + R. \quad (D.2)$$

Equation (D.2) was used to calculate the average magnetic flux density at the core central plane, the pole and air-gap interface plane, and the air-gap central plane.

11. The average magnetic flux density along the center line of the air-gap was obtained from

$$B_{\text{along}} = \mu_0 H_{\text{along}} = \mu_0 \int H \cdot dl + l. \quad (D.3)$$

Finally, the accuracy of the solutions should be examined.