Supporting Student Efforts to Learn with Understanding:
An Investigation of the Use of JavaSketchpad Sketches in the Secondary
Geometry Classroom

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Department of Curriculum, Teaching and Learning
Ontario Institute for Studies in Education of the
University of Toronto

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Abstract
This qualitative case study sought to understand the benefits and limitations of using
JavaSketchpad sketches in learning tasks aimed at developing reasoning and communication
skills related to geometric proof in the secondary school classroom. In addition to video and
audiotaped observations of student pairs, data was collected via student questionnaires and
teacher interviews.

The systematic analysis of the data led to the building of interrelationships among the
following elements of the learning situation: teacher, student, student partner, labsheet questions,
and sketch. Categories developed by Towers (1999) to describe teacher interventions were
extended to provide a more appropriate framework for analysis of the roles of JavaSketches and
labsheets in the secondary school geometry program.

The study results showed that JavaSketchpad-supported activities offer several benefits.
JavaSketches motivate and engage students through their interactive elements. They support
exploration of geometric relationships through the provision of action buttons, dragging
capabilities, and onscreen measurements. They also provide opportunities for students of
different ability levels to investigate geometric ideas, and to develop further visual, analytic,
and deductive reasoning skills.

When students work in pairs, the activities enable student-student and student-teacher
communication through the following: sketch colour and labels aid referencing; the onscreen
diagram acts as a shared image; and the sketch provides an easily accessible tool for
demonstration during discussion.

Several limitations were also identified. The extent to which JavaSketches are beneficial
is dependent on the skills of the sketch creator. Sketches must be carefully linked to the
accompanying labsheets to be effective in guiding students to extend their geometric understanding. Additional limitations are that details cannot be added, and that geometric terminology is not displayed during use of the pre-constructed sketches.

The results of the study indicate that pre-constructed JavaSketches and the accompanying labsheets provide effective interaction opportunities for student pairs, and intervene in the learning situation in ways that parallel those of a good teacher. To support student efforts they must complement one another in the roles of focusing attention--drawing attention, prompting action--providing affordances, inviting exploration--providing alternate paths, introducing uncertainty--supporting experimentation, checking for understanding--providing a shared image.

When developed with attention to these parameters, JavaSketchpad-centred learning situations can provide valuable opportunities for students to enhance their understanding of geometric problems.
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Chapter One  
Introduction

In all didactical situations, the teacher attempts to tell the students what she wants them to do. Theoretically the transition from the information and the teacher's instructions to the expected answer must require students to bring the target knowledge into play... The teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem.  

(Brousseau, 1997, p. 32)

The dynamic-geometry supported classroom offers a challenge regarding the creation of the didactical situations described by Brousseau. Students in such classes may spend much of their class time interacting with a computer program, rather than communicating with a teacher. We hope that they will actively explore visual images, discuss, analyse, and communicate their findings. In order to ensure such focused and productive activity, we must develop worthwhile tasks that take into account the features of the particular software and our understanding of how these features impact student efforts to learn geometry with understanding.

Purpose of the Research

The purpose of this study is to identify and describe the benefits and limitations of using sketches, pre-constructed with JavaSketchpad, in learning situations designed to help students develop reasoning skills related to geometric proof in secondary school.

The choice of this topic stems from my ongoing interest in using computers in teaching geometry at the secondary level. From the start I was captivated by the dynamic computer images produced by Cabri Géomètre (Baulac, Bellemain, and Laborde, 1992), The Geometer's Sketchpad (Jackiw, 1991), and later, JavaSketchpad (Jackiw, 1998), but as a mathematics teacher I needed to explore how to create tasks with this exciting software that would support student learning. A focus of my teaching became the attempt to design situations to help students use their technological experiences to understand the underlying mathematical concepts.
Clements and Battista (1992), Yerushalmy and Chazan (1993), and de Villiers (1998a) have shown that students benefit from using dynamic software to explore geometric concepts; however, it is insufficient at the secondary level for them to use the software merely to empirically verify results. They must learn to explain clearly why a particular result is true. This implies that students at the secondary level must learn to develop proofs. Since researchers have shown that proof is difficult for most students (Hoffer, 1983; Clements and Battista, 1992; Hoyles, 1997; Hanna, 1998), the challenge is to design situations that are motivating, and that help strengthen reasoning and argumentation skills.

This study describes and analyses student responses to geometry learning tasks, designed around JavaScript images in order to shed light on how such situations might enhance student understanding of geometric proof.

**Definitions**

**Dynamic Geometry**

Dynamic geometry is the exploration of geometric relationships by observing geometric configurations in motion.

**Dragging**

Dynamic geometry software uses the operation of dragging to move constructed objects through a continuous range of possibilities. Dragging involves selecting a movable point in a sketch and pulling it to a new location using the computer mouse. As the point is dragged, the onscreen image displays the resulting configurations that share the constructed properties of the original figure. Features that remain invariant under dragging provide visual evidence for the truth of statements that are to be explained or proven. Under dragging, any onscreen measurements that are affected by the movement automatically update to new values.

**Dynamic Geometry Software: General Purpose**

General-purpose dynamic geometry software allows the user to construct an accurate geometric diagram, and to investigate conjectures about the diagram by dragging.

*The Geometer's Sketchpad, and Cabri Géomètre, are examples of general-purpose dynamic software. Unlike objects in draw programs, figures constructed in the Sketchpad or Cabri environments retain constructed relationships under the operation of dragging.*
example, if a segment AB is constructed to be perpendicular to segment AC, then angle BAC will remain 90° no matter where the point B is moved.

In addition to constructing and dragging, students can also use general-purpose dynamic geometry software to trace loci, animate objects, reflect, rotate and translate figures, create scripts that replay constructions, and carry out some work with equations and graphs.

**Dynamic Geometry Software: Domain-Specific Applications**

Domain-specific is a term that refers to the needs of a certain environment. It is commonly used to describe a category of programming languages that create custom applications for particular markets. A domain-specific software application, in contrast to a general-purpose program, is limited by parameters determined by the customer's needs. For example, Lotus 1-2-3 (1983) is a general-purpose spreadsheet program, whereas The Cruncher (1993), an animated spreadsheet program aimed at helping young children learn arithmetic, could be described as domain-specific.

*The Geometer's Sketchpad* and *Cabri Géomètre* in their blank states are examples of general-purpose dynamic geometry programs. Students can construct, manipulate and explore using the full range of program features. On the other hand, both Sketchpad and Cabri can be used to produce pre-constructed sketches that limit students' options. These sketches are domain-specific.

With *JavaSketchpad* (described in the next section) students can only manipulate and explore diagrams that have been pre-constructed for particular geometry investigations. Since it is a no-tools interface (i.e., the user cannot add or delete details), the user is limited to actions pre-set by the sketch creator. Thus, *JavaSketchpad* sketches are also domain-specific.

*JavaSketchpad*

*JavaSketchpad* produces dynamic geometry sketches. It converts sketches constructed with *The Geometer's Sketchpad* to HTML format, allowing them to be viewed and manipulated through any Java-compatible web browser such as Internet Explorer, or Netscape Navigator.

*JavaSketches* are interactive models. Like sketches pre-constructed with Sketchpad or Cabri, they can be dragged to enable reasoning about invariant properties, and to provide evidence about the validity of conjectures. Pre-set relationships, such as measurements and ratios, change to match the change that has occurred as a consequence of dragging.

*JavaSketchpad* supports action buttons to hide or show additional details, to move and to animate
objects; however, the *JavaSketch* creator's definitions remain embedded in the sketch and elements cannot be deleted.

To study the use of *JavaSketchpad*, it is important to observe how students use it within the mathematics learning environment. Like any teaching tool used in a mathematics classroom, its ability to help students develop an understanding of the concepts can only be evaluated when viewed in relation to specific activities (Meira, 1998). Unlike the geoboard where mathematical objects are always created with rubber bands on a flat studded surface and are static, *JavaSketchpad* can support dynamic experiments. Although choices for action are restricted compared to those available in a general-purpose environment such as Sketchpad, many rich activities are possible.

Design choices may facilitate or impede interpretation of specific sketches. If we know that an expert can interpret the geometric information presented in a particular *JavaSketch*, we can still ask: Is this criterion sufficient, or are there additional details that should be provided to aid student understanding? Colour, line thickness, text, labels, action buttons, included elements and excluded elements, permit infinite variety. It is important that we know, not only the type of mathematical detail that is beneficial, but also the elements of sketch and labsheet that help students focus their attention.

*Rationale*

This study uses dynamic-geometry learning tasks to investigate what features of pre-constructed sketches help students focus on and select appropriate information, notice and explain new relationships, and recognise the absence of potential relationships.

There is a continuing discussion in the educational technology community about whether it is better to give students powerful general-purpose programming and construction tools or to have them interact with pre-constructed, interactive models. Wilensky (1990) strongly supports student constructions because he believes that students develop a deeper understanding of the object by explicitly connecting the parts. His view is echoed by LOGO researchers—among them Clements, Sarama, and Battista (1998), Cohen and Geva (1987), Kieran and Hillel (1990), and Hoyles (1991) who found that the activity of constructing shapes using the turtle pointer increased elementary students' ability to describe geometric relationships. Support for constructing also comes from researchers who have examined the dynamic geometry environment, notably Hoyles and Noss (1994), and Hadas and Hershkowitz (1999) who present quite different reasons for their stance. Hoyles and Noss (1994) believe that when an improperly
constructed figure falls apart under dragging, the student is forced to notice relationships among the geometric objects. Hadas and Hershkowitz (1999) maintain that the experience of not being able to construct an object that seems intuitively possible to make, is a powerful incentive for students to investigate geometric ideas.

From the opposite viewpoint, Whiteley (personal communication, 2000) believes that pre-constructed models are valuable as learning tools because ability to recognise the connections between geometric objects is a necessary stage before students can effectively carry out many constructions. Schumann and Green (1994) give additional strength to this position:

Experience has shown that the weaker student often fails in the construction of a geometric configuration, which is essential for the solution of the underlying geometric problem…. By using Cabri-geometre, we can reduce the demands on the student to a minimum by supplying prepared configurations and simplifying or enhancing menus appropriately. (p. 204)

The views of Whiteley, and Schumann and Green draw attention to an interesting dilemma--students cannot begin to do a question about geometric objects because they can't construct the diagram, and they can't construct the diagram because they don't understand the connections between geometric objects.

In their study of the transition from exploring with Cabri, to conjecturing and proving, Arzarello, Micheletti, Olivero, Robutti, Paola, and Gallino (1998) classified modalities of dragging and identified those that were linked with productive student reasoning. One modality, "dragging test," is associated with constructing. When a figure has been constructed correctly it will pass the dragging test, that is, it "will not be messed up by dragging" (p. 33). The other two forms, "wandering dragging" and "lieu muet" (dummy locus), are associated with student explorations of already constructed models.

Arzarello et al. found that students who produced good conjectures made use of "lieu muet" dragging (p. 33), a purposeful mode which "can be seen as a wandering dragging which has found its path" (p. 37). Although the students used all three forms in their investigations with Cabri, it is interesting to note that the researchers did not draw a link between "dragging test" and good conjectures, giving further support to the position that pre-constructed diagrams can provide sufficient material for developing reasoning skills related to proof.

Since I believe that students must have the opportunity to observe the mathematical relationships in a completed diagram I chose to investigate the role of pre-constructed dynamic sketches in the development of student geometric reasoning skills.
The Geometer's Sketchpad, Cabri, and JavaSketchpad were three possible software choices. If Sketchpad or Cabri were used to display the sketches, students would have a full range of dynamic tools available, and would be able to add or delete details at any time. This posed several problems. Troubleshooting would be required to help students recover deleted details. Students, always curious, might pursue their own exploration of the program and ignore the goal of the tasks, which was to observe and describe pre-selected geometric relationships. If JavaSketchpad were used, since pre-constructed JavaSketches are indestructible and require no knowledge of dynamic-geometry construction techniques, troubleshooting would be kept to a minimum and students would be able to devote their full attention to the tasks.

I chose JavaSketchpad based on these considerations, aware that in creating the sketches I would need to carefully consider all possible paths students might choose to take, since students trying to probe connections that were not pre-set could become frustrated.

Research Questions

In this research I set out to investigate the benefits and limitations of using JavaSketches in a senior mathematics classroom. The focus is on four broad categories within which I developed the research questions: the student, the teacher, the JavaSketches, and the labsheets. Labsheets are the pages that the students use in the computer lab; they are more than instruction sheets because they contain task questions and space for answers.

My research questions within the four categories are:

The Student

1. Are students interested and involved throughout the session?
2. How does the organisation of students affect the learning environment?
3. How do students use the onscreen image?
4. How do student pairs overcome obstacles to understanding?
5. What are student opinions on the use of JavaSketches in geometric learning situations?

The Teacher

1. What is the teacher's role in the JavaSketchpad-supported environment?
2. What is the function of class discussions in the JavaSketchpad-supported classroom?
3. What are teachers' opinions on the use of JavaSketches in geometric learning situations?
The JavaSketches

1. Which affordances of the JavaSketches did students use and what was their impact?
2. What affordances were not noticed, and what affordances did students need that were missing?
3. How did the pre-constructed nature of the sketches affect student efforts?

The Labsheets

1. What are the functions of the labsheet in the learning task?
2. How do students respond to exploration instructions and questions?

Significance of the Study

The results of this study extend our knowledge of computer use in secondary school geometry with regard to understanding what helps students interpret visual information in dynamic-software-produced diagrams, and use their experiences to develop reasoning skills related to geometric proof. This new knowledge about how students use both the static and dynamic forms of these diagrams, could help teachers and researchers design more effective technological geometric learning situations.

Limitations of the Study

Since the research was conducted as a case study of three classes in two different schools in the same school board, the results may not be generalizable to every teacher and every school. The small sample provides an in-depth picture of the participating classes. Nevertheless, it may serve to inform educators whose situation is similar to that of the participants and to provide insight into why students have difficulty with reasoning in similar situations.
Chapter Two

Literature Review

Computers provide ways of doing and experiencing mathematics that we simply did not dream of thirty years ago. The touchstone of these experiences is the experience of direct manipulation of mathematical objects and relations—a new experiential mathematical realism.

(Balacheff and Kaput, 1996, p. 470)

Introduction

For the past thirty years "innovators" and "early adopters" (Rogers, 1995) have experimented with using technology in mathematics education. Some of these pioneers have reported successes (Clements and Battista, 1989; Chazan, 1990; Day, 1993; Alkalay, 1993; Laborde, 1993; Soloway and Norris, 1998), others have reported problems (Becker, 1990; Bellamy, 1997).

In assessing the impact of technology, researchers have suggested a revamped curriculum (Fey, 1989; Keitel and Ruthven, 1991; Kilpatrick and Davis, 1991; Noddings, 1994), warned of the need for teacher training (Means and Olson, 1994; Fullan, Galluzzo, Morris, and Watson, 1998), predicted changes in roles and attitudes of teachers and students (Heid, Sheets, and Matras, 1990; de Villiers, 1998a), and found that students benefit from activities using LOGO (Clements, Battista and Sarama, 1998; Kieran and Hillel, 1990; Hoyles, 1991), the Geometric Supposer (Schwartz, Yerushalmi, and Wilson, 1993), and dynamic geometry software (Chazan, 1990; Hannafin and Scott, 1998; Jackiw, 1997).

Studies on the use of technology in mathematics education are related to extensive research in areas of teaching and learning mathematics such as, (a) how students develop understanding in mathematics, (b) the structure of mathematical learning environments, and (c) the role of the teacher. If we consider the use of dynamic software in geometry education specifically, the research topics include (a) levels of geometric thinking, (b) how to help students
develop reasoning skills related to geometric proof, and (c) visual and spatial understanding. There are also interesting findings related to the use of technology from research on student achievement. And finally, if we intend to apply this research to the classroom we must examine the design of activities, and the specifics of the software.

**Teaching and Learning Mathematics**

The main factors in the process of mathematics education are the student, the environment, the teacher, and the learning materials.

**How Students Develop Understanding in Mathematics**

Knowledge of how students develop mathematical understanding is the basis for any discussion about what works or does not work in the technology-supported mathematics classroom.

**Constructivist theory.**

Constructivist theory is based on the Kantian philosophical idea that objective reality does not exist outside of our perceptions—which are informed by our experiences (Brown, 1999). In light of this view, constructivists assert that the individual learns, not by passively receiving facts, but by constructing his or her own knowledge through interactions with the environment (von Glasersfeld, 1995; Simon, 1995). Simon describes the process as follows:

> When what we experience differs from the expected or intended, disequilibrium results and our adaptive (learning) process is triggered. Reflection on successful adaptive operations (reflective abstraction) leads to new or modified concepts. (p. 115)

Constructivist learning theory does not imply that humans must 'construct' meaning by manipulating concrete objects. For example, under the theory, when we listen to a lecture we learn by constructing our own understanding of the information (Noddings, 1990).

Within constructivism there are two main perspectives. Individual constructivists view learning as a cognitive endeavour and focus on the individual's knowledge constructions. The most vocal supporters of this viewpoint are called radical constructivists (Smith, 1999). The application of this approach in the classroom presents a challenge since the student may construct an understanding that is incomplete or in conflict with the understandings of others. One proposal from the individual constructivist perspective is that the students use discussions to develop taken-as-shared meaning, in which they "achieve a sense that some aspects of
knowledge are shared [although they] have no way of knowing whether the ideas are in fact shared" (Simon, p. 116).

Social constructivists, on the other hand, see learning as a social process and emphasise the role of social interactions and language in the learner's construction of meaning (Smith, 1999). From the perspective of the social constructivist, students engage in social negotiation through whole class or group discussions to share and refine their personally constructed meanings in light of the understandings of others (Savery and Duffy, 1995). These open dialogues allow teachers to listen to students' ideas, and to offer additional perspectives. Students may use this feedback to modify their understandings.

Thus the constructivist teacher's goal is not to transmit information but to design learning situations that will help students develop good constructions—ones that build connections to previously learned concepts, and that provide the impetus for creating new knowledge. Although constructivist learning theory does not define how teachers should proceed, it is often linked to active learning or discovery models, which focus on the development of knowledge through experiences. For example, in a problem-based learning model students learn through interacting with their environment and with their peers. To this social constructivist base is added the cognitive challenge of a problem to be solved.

Many mathematics education researchers believe that a classroom based on a constructivist foundation offers the best environment for mathematics learning (Confrey, 1990; Saxe, 1995; Savery and Duffy, 1995; Kieren, 1997; Crawford and Witte, 1999). In order to apply mathematical concepts effectively students must develop tentative solutions, examine these in the light of their experience and knowledge, and create their own understanding of mathematical relationships. Von Glasersfeld (1995) explains that only after students have built up a "conceptual repertoire" of skills and personally constructed meanings are they able to successfully solve non-routine problems (p. 5).

It is interesting to consider how the computer impacts the application of constructivism in the mathematics classroom. Research from a constructivist perspective by Blais (1988), Confrey (1990), Honebein, Duffy, and Fishman (1993), Ernest (1992), Brooks and Brooks (1999), and others, has influenced studies of technology use in geometry, since both emphasise student participation in the learning process. In the computer-based curriculum an important focus is the idea that mathematics is something students "do", rather than something they are told (Dossey, 1992; Chazan and Ball, 1999; Skemp, 1987; Clark, 1993).
Smith (1999) suggests that in the social constructivist model the individual uses the computer as a mediating tool to act on the mathematics problem. In other words, the computer changes how the mathematics is done. Smith proposes that from the viewpoint of the individual constructivist, the change occurs in the mathematics itself. That is, students construct a solution that is "inseparable from their understanding of the actions which were possible using the available tools" (Smith, p. 422).

**Cognitive apprenticeship.**

Collins, Brown and Newman (1989) have suggested that learning mathematics can be compared to learning a trade. It is an apprenticeship—a "cognitive apprenticeship". An apprentice watches the master, then practices under close supervision, and finally works independently. Paralleling the master's role, the mathematics teacher models strategies, then acts as coach and finally withdraws (fares) as the student works independently.

A key feature of cognitive apprenticeship is its attention to the idea that knowledge is partly a product of the activity, context and culture in which it is developed and used (Collins, Brown and Newman, 1989). This approach to mathematics learning highlights the importance of hands-on learning. Merely listening to explanations is insufficient if the student is intent on becoming an expert. (Blais, 1988).

**Sense making.**

The word 'apprenticeship' may evoke the notion of 'practice', but learning mathematics involves more than practice under supervision. Alan Schoenfeld (1989) refers to doing mathematics as an act of "sense making", and Goldenberg, Cuoco and Mark (1998) note that we must help students develop mathematical "habits of mind". These expressions highlight the idea that learning mathematics is really learning to think mathematically--to approach problems by looking for patterns, making conjectures, and describing relationships. We cannot hope to achieve such a goal by simply telling students what they need to know (Chazan and Ball, 1999).

**Dynamical model for the growth of mathematical understanding.**

Pirie and Kieren (1994) developed a theory for the growth of mathematical understanding, which describes how the student passes through stages from not knowing to knowing. Based on observations of students they note that:

One's understanding is seen to grow out of one's knowing experiences and [one's attempts] to organize them and complementarily one's understanding configures one's knowing actions. (Kieren, 1994, p. 213)
In their model, Pirie and Kieren have identified eight modes of understanding, which they depict as nested rings (see Figure 2.1). Students come to the learning task with certain skills and concepts that form the innermost ring—"primitive knowledge". When students are operating in any of the first three modes—"image making", "image having", and "property noticing"—their understanding is informal and dependent on the local context. The next three rings represent more abstract and less context dependent modes of understanding—"formalizing", "observing", and "structuring". The outermost ring is labelled "inventizing". Students operating in this ring are able to use prior knowledge to create completely new ways of looking at concepts. The rings are not concentric to illustrate that students can move very quickly through these stages if they are close to the adjacent points.

![Diagram of nested rings illustrating eight modes of understanding.]

**Figure 2.1**

**Pirie and Kieren model for the growth of mathematical understanding.**

Rather than progressing through stages in a linear way, the student in Pirie and Kieren's model moves back and forth between the rings. The three dark rings in the diagram illustrate the idea that once students are past each of these particular landmarks they can proceed without the necessity of referencing earlier stages; however, at any time the model provides for students to return to an inner layer—to "fold back". A student folds back to retrieve information, to rethink ideas learned earlier, or to use previous concepts to create new ways of solving the current problem (Martin and Pirie, 1998).
The Structure of Mathematical Learning Environments

Mathematical learning environments range from the traditional, in which the teacher delivers the information via a lecture or didactical lesson, and the students practice the skill that was introduced, to the innovative, in which students posit and investigate their own mathematical theories. Whatever form the learning environment takes, it will impact how and what the students learn.

Social interaction.

Many researchers believe that collaboration and communication have a beneficial impact on learning (Collins, Brown and Duguid, 1989; Confrey, 1990; De Kerckhove, 1996). As an explanation of this phenomenon, De Kerckhove suggests that when people work together there is energy generated by the fact that some people don't know. This 'not knowing' is important for the rest of the group because it helps them begin working towards solving the problem (De Kerckhove, 1996).

In his classroom, Alan Schoenfeld (1994) stresses the social skills of communication and negotiation because he believes that people develop their values and beliefs as a result of their interactions with others. His students become part of the local mathematical community of a classroom that shares and evaluates ideas.

Collins, Brown, and Duguid (1989) recommend group work in mathematics because:

Teachers rarely have the opportunity to hear enough of what students think, to recognize when the information that is offered back by students is only a surface retelling for school purposes that may mask deep misconceptions about the physical world and problem solving strategies. (p. 40)

However, it is important to be cautious about group work. It is a common concern that students (especially at the secondary level) who work in groups to develop and share results may just be sharing ignorance. Cobb recommended in his 1995 study of small group learning in mathematics that teachers must assume an active monitoring role when group work is used, checking students' ability to argue and discuss, and intervening as necessary to guide the learning process.

Brousseau's theory.

Sutherland and Balacheff (1999) have proposed Guy Brousseau's theory of didactical situations (1997) as a model for the mathematics classroom. Brousseau's theory asserts that the teacher must develop classroom situations through which pupils can experiment with and
develop understanding of the intended knowledge (i.e., the set of mathematical ideas that society expects pupils to learn).

Since students' individual constructions will likely differ from the intended knowledge, Brousseau's theory of didactical situations contains a step called the "situation for institutionalization" through which a teacher draws on examples of pupils' work to emphasise those that relate to the intended knowledge. Brousseau's theory does not negate the importance of social interaction in the classroom but as Sutherland and Balacheff remark:

What seems to be characteristically different between the theory of didactical situations and the dominant constructivist research is that in the former, the notion of intentional mathematical knowledge is prioritized and made explicit. In the latter, the emphasis tends to be on supporting diversity in student's methods, because this is viewed as being the driver of mathematical learning. (p. 5)

**Computer-based learning environments.**

The mathematical learning environment is affected when students use technology. In 1986 Yerushalmi and Houde noted that.

The pedagogy we used most closely resembles the teaching ordinarily found in science classes where the primary focus is on the scientific process of collecting data, conjecturing and finding counterexamples or generalizations....Students spent the majority of class time discussing and doing geometry rather than listening to a teacher talk about it. (quoted in Fey, 1989, p. 245)

In 1991 Dreyfus predicted that computer use would not affect the topics in the curriculum but would result in completely revised ideas about how to teach mathematics. In fact, change has been very slow but there is evidence that Dreyfus was correct about the effect of technology on teaching methodology. According to McDougall (1999), teachers new to technology experience an initial loss of control and need to rethink and revise their teaching approaches to gain "confidence in their ability to teach effectively with the new methods" (p. ii). Nevertheless, additional research is required to help us understand the interaction between the teacher and the machine in the learning process (Balacheff, 1998).

Much of the research on computer environments for learning mathematics has focused on the student and the development of student-centred software. Alkalay (1993) reported strongly positive reactions of students toward using the computer for independent exploration in pre-calculus. Day (1993) found that the combination of graphical, numerical and symbolic representations through technology had a profound impact on student learning of algebra. Dugdale (1994) reported that slow learners were able to solve simple algebraic problems by using spreadsheet addresses to represent variables. Hoyles and Noss (1994) found that increased
geometric understanding was fostered by use of geometry software. And in 1998 Soloway and Norris reported that using software to give students a place to display their work was an effective motivator.

Dugdale, Thompson, Harvey, Demana, Waits, Kieren, McConnell, and Christmas, in 1995 raised an important question for researchers. They wondered whether software images are successful at helping us develop intuitive understanding only because we already know the ideas, and thus whether students new to the subject will benefit in the same way. In 1999 Sutherland and Balacheff suggested that visual images displayed on computer screens do allow students to gain access to mathematical knowledge. They reported:

We suggest that computer-based environments provide access to formal mathematical knowledge, through the nature of the 'intermediate' screen objects with which students interact. ... From the perspective of the teacher the computer affords the possibility of rendering more visible the nature of the objects with which a student is engaging. (p. 2)

Their analysis has special import for the teaching of geometry—a field that would seem to benefit greatly from the computer's ability to display visual images, and which has often suffered from students' inability to break through the formal language to the underlying concepts.

The Role of the Teacher

Researchers have described the role of the teacher in a variety of ways. As reported by Gravemeijer (1998), Freudenthal saw the teacher as a guide, who maps out a learning route to give the student the opportunity to progress via a more direct route to reinvent the mathematical concept. Alan Schoenfeld (1989) downplays the role of teacher as authority figure and recommends that the task of judging and certifying solutions be shared with the local community of the class. Ball (1993), Chazan and Ball (1999), Clark (1993), and Collins, Brown and Newman (1989) see the teacher as facilitator of an inquiry process that must include active participation of students.

The common thread among these views is that the teacher is not merely a transmitter of knowledge.

The teacher in Brousseau's model.

In his didactical theory, Brousseau (1997) sets out two major roles for the teacher in a mathematics classroom: devolution and institutionalization. In describing a learning activity, he states:
Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. This situation or problem chosen by the teacher is an essential part of the broader situation in which the teacher seeks to devalue to the student an adidactical situation, which provides her with the most independent and most fruitful interaction possible. (p. 30)

After the devolution, the next phase in Brousseau's model is the institutionalization of the concepts through a teacher-guided sharing of ideas.

Thus, Brousseau's theory gives support to a teacher's natural inclination (and duty) to develop tasks that help students move towards the intended aim. Within the theory a teacher is described as "grounded in the students' experiences" and "purposeful about what he/she wants to teach" (Sutherland and Balacheff, p. 6).

**Towers' intervention categories.**

Towers (1999) researched the role of the teacher vis-à-vis individual students. She examined the nature of teachers' interventions and identified those that contributed to the growth of student understanding, thereby extending the work of Pirie and Kieren. Her work is especially relevant for the technology-supported mathematics classroom because the nature of the lab classroom supports teaching small groups or pairs rather than delivering whole-class lectures.

Towers believes the model for the growth of mathematical understanding is more linear than Pirie and Kieren have proposed. However, in her research she found strong evidence of the "image making", "image having", and "property noticing" stages. She also observed "folding back" in student learning behaviour.

In her research Towers categorized the following intervention strategies: managing, checking, reinforcing, inviting, clue-giving, enculturating, blocking, modelling, anticipating, praising, retreating, and rug-pulling. She also identified three teaching styles: (a) showing and telling, (b) leading, and (c) shepherding. She found that the shepherding style and the strategies of rug-pulling and inviting consistently contributed to the growth of student understanding (Towers, 1999, pp. 249-251). She defines her categories as follows:

**Teaching Styles:**

- **Showing and Telling**
  - An extended stream of interventions involving the giving of new information but usually without checking that the students are following the explanation.

- **Leading**
  - An extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations. Differs from showing and telling by its attempts to involve the students in the explanation through frequent questioning.
Shepherding: An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing and prompting.

(Towers, 1999, pp. 200-201)

Intervention Strategies:

Checking: The teacher is checking for student understanding.
Reinforcing: Giving further emphasis to a significant point (often one already made by a student).
Inviting: Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than clue-giving.
Clue-giving: A deliberate attempt to point the student to the correct answer or preferred route.
Managing: Including discipline, keeping students on task, giving instructions etc.
Enculturating: Inducting students into the language, symbolism and practices of the wider mathematics community.
Blocking: Preventing a student from following a certain path (sometimes preventing a student from folding back to Image Making activities).
Modelling: The teacher explicitly models her own thought processes.
Praising: Praising individual students, groups or the whole class.
Rug-pulling: A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing. Often results in a return to Image Making activities.
Retreating: A deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem.
Anticipating: Preventing students from falling into common pitfalls, trying to prevent mistakes before they happen, protecting students from error, or removing the challenging aspects of a task.

(Towers, 1999, pp. 201-202)

Towers concludes that the growth of understanding does depend on the interventions of the teacher but that the student's actions, abilities and prior knowledge determine the depth and direction of this growth.

Teaching and Learning Geometry

In his lecture at the Canadian Mathematics Education Study Group 1999 annual meeting, geometer Walter Whiteley argued that, "discrete geometry virtually died as an 'important' field of mathematical research through the twenties and thirties and forties" (Whiteley, 1999, p. 8), leading to a decline in the importance of geometry in high school, where it is now often viewed as an optional topic—a skippable collection of dusty, irrelevant theorems. But Whiteley believes
that the research community is experiencing a renewed interest in geometry because spatial
conscepts have taken on new importance. Many disciplines and industries are discovering new
applications for geometric models, and the explosion in visual communication, fuelled by
technology, is driving researchers to investigate how we reason with visual and spatial
information. At the same time, software tools now allow researchers, teachers and students to
experiment with geometric forms and to see those dusty theorems from a new perspective.

*Levels of Thinking in Geometry*

Traditionally secondary school geometry was based on Euclid's formal axiomatic
geometry. In the last century, some mathematics educators, aware of Piaget's groundbreaking
work on how children learn, began to question the wisdom of this approach. Pierre van Hiele and
Dina van Hiele-Geldof carried out in depth studies of students learning geometry, and developed
a model to help teachers provide experiences appropriate to the student's level of thinking (van
Hiele, 1999).

**The van Hiele model.**

The van Hiele model identifies five main levels through which students pass. Pegg and
Davey (1998) describe the van Hiele levels as they relate to two-dimensional figures as follows:

Level 1. Students recognize a figure by its appearance (i.e., its form or shape). Properties of a figure play no explicit role in its identification.

Level 2. Students identify a figure by its properties, which are seen as independent of one another.

Level 3. Students no longer see properties of figures as independent. They recognize that a property precedes or follows from other properties. Students also understand relationships between different figures.

Level 4. Students understand the place of deduction. They use the concept of necessary and sufficient conditions and can develop proofs rather than learn them by rote. They can devise definitions.

Level 5. Students can make comparisons of various deductive systems and explore different geometries based on various systems of postulates.

(Pegg and Davey, 1998, p. 111)

Van Hiele believes that progress through the levels is influenced by the student's process
of exploration, the learning environment, and the student's response to the guidance offered by
the teacher, rather than by maturation. He maintains that the traditional Euclidean geometry
curriculum starts at too high a level—omitting the visual and even the descriptive stages of
development despite the fact that very few students (in his opinion) ever move beyond Level 3 (van Hiele, 1986).

Pegg and Davey note that van Hiele has also used an alternative set of levels: the visual, the descriptive, and the theoretical, in his writings, but does not link these to the five levels in his original model. To increase the confusion, some researchers use different numbering systems when referring to the van Hiele levels (Clements and Battista, 1992). In this thesis I will consider that the van Hiele model begins at level one.

Hoffer categorises the van Hiele levels as 1) recognition, 2) analysis, 3) ordering, 4) deduction and 5) rigor (Hoffer, 1981). De Villiers uses visualisation, analysis, and ordering as the descriptors for the first three levels. In level one, students think and reason visually. If they are asked to prove something, to classify or define, they will try to use an argument based on visual evidence. At level two, learners are able to think and reason using the properties of geometric figures. They will try to explain and prove things in terms of properties; however, their definitions and proofs are uneconomical, that is, they include superfluous information. At level three, students begin to notice logical causal relationships between properties and begin to understand that one property implies another. This allows them to develop economical definitions and simple proofs (de Villiers, personal correspondence, March 18, 2001; de Villiers, 1998b). When learners understand the role of postulates and theorems in proof they are functioning at level four.

**Pegg and Davey proposed model.**

The van Hiele model provides a framework for examining conceptual development in geometry but it has been criticised for the discrete nature of its levels. Clements and Battista (1992), Pegg and Davey (1998) and Lehrer, Jenkins and Osana (1998) contend that the levels do not adequately describe the reality, which appears to be continuous.

In an attempt to interpret the van Hiele model in a more relevant way for present research, Pegg and Davey (1998) have proposed a synthesis of the model with the SOLO taxonomy for intellectual development first proposed by Collis and Biggs in 1982. SOLO stands for Structure of the Observed Learning Outcome, and stems from analyses of student responses. The model postulates that all learning occurs in one of five modes of thinking: sensorimotor, ikonic, concrete symbolic, formal, and postformal. Pegg and Davey describe these modes and approximate ages at which they begin, as follows:
1. **Sensorimotor** (soon after birth). The individual reacts to the physical environment. For the very young child, it is the mode in which motor skills are acquired.

2. **Iconic** (from 2 years). The individual internalizes actions in the form of images. It is in this mode that the young child develops words and images that can stand for objects and events. This mode of functioning later assists in the appreciation of art and music and leads to a form of knowledge referred to as intuitive.

3. **Concrete symbolic** (from 6 to 7 years). The individual is capable of using or learning to use a symbol system such as written language and number systems, which have an empirical referent. This is the most common mode addressed in learning in the upper primary and secondary level.

4. **Formal** (from 15 or 16 years). The individual can consider more abstract concepts and work in terms of "principles" and "theories". The individual is no longer restricted to a concrete referent.

5. **Postformal** (possibly at around 22 years). The individual is able to question or challenge the fundamental structure of theories or disciplines.

(Pegg and Davey, 1998, pp. 116-117)

An individual may operate in different modes, depending on the situation, for example, responding in the formal mode in one context and in the concrete symbolic in another.

Within each of the modes, the individual progresses through five levels. The characteristics of these are as follows:

1. **Prestructural** (Response is below the target mode). The learner is frequently distracted or misled by irrelevant aspects and does not engage the task in the mode involved.

2. **Unistructural** The student focuses on the domain/problem, but uses only one piece of relevant data.

3. **Multistructural** The student uses two or more pieces of data without perceiving any relationships between them. No integration occurs.

4. **Relational** The student uses all data available, with each piece woven into an overall mosaic of relationships. The whole has become a coherent structure with no inconsistency within the known system.

5. **Extended abstract** The student goes beyond the data into a new mode of reasoning and can generalize from new and abstract features.

(Pegg and Davey, 1998, p. 117)

The SOLO model provides for cyclic growth. Students repeatedly pass through the five levels as they move through the modes, and can operate in a target mode while using skills from an earlier (or later) mode. Although Biggs and Collis never attempted to apply SOLO to geometry, Pegg and Davey propose that the taxonomy can be linked (see Table 2.1) to the van Hiele model to help us describe student understanding in geometry. In their synthesis Pegg and
Davey consider only three levels—unistructural, multistructural, and relational, since they argue that the prestructural level can often be associated with the preceding mode and the extended abstract level with the subsequent mode.

Table 2.1.

Synthesis of the van Hiele and SOLO models.

<table>
<thead>
<tr>
<th>Van Hiele theory</th>
<th>Mode</th>
<th>SOLO taxonomy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level</td>
<td></td>
<td>Level</td>
</tr>
<tr>
<td>1</td>
<td>Ikonic</td>
<td>Relational</td>
</tr>
<tr>
<td>2</td>
<td>Concrete symbolic</td>
<td>Unistructural/multistructural</td>
</tr>
<tr>
<td>3</td>
<td>Concrete symbolic</td>
<td>Relational</td>
</tr>
<tr>
<td>4</td>
<td>Formal</td>
<td>Unistructural/multistructural</td>
</tr>
<tr>
<td>5</td>
<td>Postformal</td>
<td>Unistructural/multistructural</td>
</tr>
</tbody>
</table>

(Pegg and Davey, 1998, p. 123)

Table 2.2 illustrates the relationship between the five van Hiele levels and the 25 stages of the SOLO model in a slightly different way. It shows that the van Hiele levels are associated with moderately high to high levels of functioning on the SOLO model.

Table 2.2.

Illustration of the synthesis of the van Hiele and SOLO models.

<table>
<thead>
<tr>
<th>Sensorimotor</th>
<th>Ikonic</th>
<th>Concrete symbolic</th>
<th>Formal</th>
<th>Postformal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prestructural</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unistructural</td>
<td></td>
<td>van Hiele level 2</td>
<td>van Hiele level 4</td>
<td>van Hiele level 5</td>
</tr>
<tr>
<td>Multistructural</td>
<td></td>
<td>van Hiele level 1</td>
<td>van Hiele level 3</td>
<td></td>
</tr>
<tr>
<td>Relational</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extended Abstract</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Pirie and Kieren’s model*

Pirie and Kieren’s dynamical theory for the growth of mathematical understanding (1994) also offers an alternative to the van Hiele model. It is not specific to geometry but incorporates the idea that students move back and forth between levels as they progress in understanding. Under this model, a university student who is dragging a figure to investigate invariant properties has ‘folded back’ to work in a previous stage. A grade school student, exploring the same
dynamic image is moving within the boundaries of the three informal and context dependent stages of image making, image having and property noticing (see Figure 2.1).

**Revisiting the van Hiele model**

Although there are alternatives, de Villiers (1999) finds the general structure of the van Hiele model effective for describing student development. He does not accept that the hierarchy of the van Hiele model is rigid. In particular, he argues that attainment of higher levels in the van Hiele hierarchy does not make the use of the lower level skills obsolete. He notes:

Very often people will use the level most appropriate (or easiest) to the task - we as teachers (or examiners) may think a task is at Level 3 or 4 but I've often seen students at Olympiad level solving such problems through visualisation (Van Hiele 2), which does NOT mean that if the situation demands, they are incapable of higher levels of thinking. (de Villiers, personal correspondence, March 14, 2001)

De Villiers suggests that students carry out mathematical processes in accordance with the essence of the level at which they are operating. For example, at the visualisation stage (level one) all processes are connected to the use of visual evidence. Table 2.3 presents an illustration with regard to defining, proving and classifying.

**Table 2.3.**

**Defining, proving and classifying at van Hiele levels one, two, three.**

<table>
<thead>
<tr>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>visual defining</td>
<td>uneconomical defining</td>
<td>economical defining</td>
</tr>
<tr>
<td>visual proving</td>
<td>analytic proving (using</td>
<td>deductive proving</td>
</tr>
<tr>
<td></td>
<td>properties)</td>
<td></td>
</tr>
<tr>
<td>visual classifying</td>
<td>analytic classifying</td>
<td>logical classifying</td>
</tr>
</tbody>
</table>

(de Villiers, personal correspondence, March 14, 2001)

De Villiers contends that the use of dynamic software makes it easier to fully develop levels one and two, which, in turn, makes it easier for students to develop their mathematical process skills.

**Visual: Mode versus level**

Although the word visual is often associated with level one on the van Hiele model (van Hiele, 1986; de Villiers, 1999), the visual mode of reasoning is used throughout all levels of geometric thinking. De Villiers' diagrammatic representation of the skills used at each van Hiele level, shown in Figure 2.2, is an attempt to illustrate this double meaning (de Villiers, personal
communication, March 18, 2001). It clearly shows that visualisation remains an important element as students progress.

Figure 2.2.

De Villiers' conception of the skills used at each van Hiele level.

For clarity in this thesis, the phrase inspection alone will be used to denote that the student is using visual evidence at van Hiele level one (i.e., the student recognises the figure but does not consider properties).

Proof in Secondary School Geometry

Research has shown that many students in high school have great difficulty with deductive proof (Hoyles, 1997; Hanna, 1998; Lehrer and Chazan, 1998). Perhaps this problem can be partially explained by the claim that most children between 13 and 15 are "concrete generalizers," not "formal thinkers"; that is, a few specific instances satisfy them of the reliability of a rule (Pegg and Davey, 1998, p. 116). This observed behaviour corresponds to level 3 in Table 3.1. Taking into account that the intended knowledge for most secondary geometry curricula is aimed at the skills of level 4, this discrepancy is an important consideration for the design of geometry learning environments. It implies that we must define a realistic path that teachers can follow to lead their students to success.

Development of proof skills.

Gutierrez and Jaime (1998), recommend leading students through the following stages to achieve the equivalent of level 4: (a) recognition of types and families of geometric figures, (b)
identification of components and properties of figures, (c) use and formulation of definitions, (d) classification of geometric figures, and (e) proof of properties or statements (p. 29).

Edwards (1997) has investigated the development of proof skills and stresses the importance of teaching students the thinking skills necessary to establish the mathematical certainty of a result. She defines proof as:

The set of processes involved in translating intuitions or generalizations into assertions of certainty, expressed in language which is unambiguous, precise, and accepted within a community of mathematicians. (Edwards, 1997, p. 190)

This definition includes the importance of a community taking into account the accepted standards of mathematicians, a concept supported by others such as Schoenfeld (1989) and Lampert (1990).

Koedinger (1998) reported student difficulties with stating conjectures and developing arguments. He recommends that students acquire practice in these skills in a discovery-based environment that includes the use of software tools to facilitate conjecturing.

*Conjecture and proof with dynamic software.*

Even the most powerful dynamic software can only show students that something is the case. True understanding demands that the students see why it is the case, and furthermore why it must always be the case. (Hanna, 1998, p. )

Dynamic software allows students to search for counter examples to conjectures they have posed. Failure to find a counter example becomes strong evidence for the truth of the conjecture. Thus, dynamic exploration leads to conviction, which leads to a search for a proof. This process more closely models the actual practice used by mathematicians. Unlike students in a traditional geometry program, who are required to prove a statement, given a set of facts, mathematicians usually satisfy themselves that a theorem is true before attempting to prove it.

When students are engaged in dynamic explorations, they cannot simply follow a memorised set of procedures. Rather, as Chazan (1990) suggests, to successfully explore a geometry problem within a computer microworld students require six inquiry skills: verifying, conjecturing, generalising, communicating, proving and making connections (p. 630). It is necessary to establish the pedagogy for helping students develop these geometric proof skills— not only with regard to the teacher's role but also with respect to the design of activities for particular software.
Constructions.

Secondary school geometry programs require students to carry out constructions (e.g., bisecting an angle using a compass and straight edge). Constructions are usually taught at the beginning of a course before students have had opportunities to observe, recognise, and discuss geometric properties, leaving students unaware of the wider application of construction techniques to solving geometric problems (Koedinger, 1998).

Whiteley believes that constructing--especially diagrams for complex problems--may require students to be at a fairly high van Hiele level (personal correspondence, 2000). In support of this position Schumann and Green (1994) recommend that students do some work with pre-constructed diagrams. They report that the requirement for students to construct everything themselves has the following drawbacks:

- It discourages trial and error;
- It diverts attention away from the aim of the overall construction;
- It may significantly reduce the accuracy of the whole construction;
- It is extremely time-consuming, without producing new insight or understanding, more so because of forgetfulness or fatigue;
- It makes the final drawing appear complex and unclear because of the clutter of all the intermediate lines, which must be drawn to achieve the final construction;
- It may lead to the actual construction process (many small steps) failing to correspond to the mental modular representation of the construction process (a few large steps) (p. 204).

The only one of these drawbacks that cannot apply to the dynamic-geometry environment is the third--by the nature of the software, figures constructed with Sketchpad or Cabri are accurate. The other five may or may not affect student learning. A skilled user of dynamic geometry software can hide construction lines to simplify the final drawing, correct errors by retracing steps, and carry out difficult constructions quickly; however, the actual construction process may draw attention from the overall image and multiple steps may interfere with the mental representation of the construction process.

Visual and Spatial Understanding

The growing importance of visual and spatial understanding in our world presents a challenge for educators. Evidence from physiological research suggests that spatial tasks (e.g., mentally visualizing an object, orienting oneself in a room) are processed by the right side of the brain while the left hemisphere specialises in analytic/logical thought (Battista, 1990). Since the
geometry curriculum emphasises deductive proof. students do not have sufficient opportunity to develop visual/spatial skills (Goldenberg, Cuoco, and Mark, 1998).

In the belief that it is no longer appropriate for the study of geometry to be limited to formal Euclidean proof, Lehrer and Chazan (1998) have called for a reintegration of spatial reasoning into the core mathematics program (p. ix). But what skills must students possess to use spatial reasoning? Goldenberg, Cuoco, and Mark (1998) contend that before students are able to look for invariants in geometric relationships they must first develop "the ability to take apart in the mind, see the individual elements, and make sufficiently good conjectures about their relationships to guide the choice of further experimental and analytic tools" (p. 7).

Although educators have long recommended that young children use concrete objects to gain a sense of shape and form, the secondary student until recently often began the study of geometric properties and proof without the benefit of actual models. The introduction of dynamic geometry software has changed that experience for some students, and in their summary analysis of research into geometry and spatial reasoning, Clements and Battista (1992) concluded that students gain greater understanding of spatial relationships by constructing and exploring a dynamic geometry model.

Students' Use of Visual Information

In their research, Eisenberg and Dreyfus (1991) found that although many teachers and mathematicians extol the benefits of visual images, many students prefer algorithmic over visual thinking. They theorised that a significant reason for this is that visual thinking requires more cognitive effort (p. 25). Despite this they recommended increased emphasis on a visual approach in order to help students make connections between the visual and analytic aspects of the underlying mathematical concepts (p. 27).

Vinner and Kopelman (1998) conducted a study of gifted grade nine students to determine if they would use symmetry considerations to construct a proof in a situation in which these appeared to be the most obvious and natural concepts to use. The students had worked with transformations in an earlier grade, and in grade nine were introduced to Euclidean geometry in the traditional way. The researchers were surprised to find that even these gifted students did not use the visual evidence to construct their proofs but relied on other methods. In addition, when asked to decide which of two given proofs was most convincing—a proof based on symmetry or one based on congruency theorems, they overwhelmingly (13 out of 17) chose the latter (p. 23)
Vinner and Kopelman theorised that although many mathematicians and mathematics teachers enjoy using visual intuition, this skill has developed after years of training in formal reasoning. However, students cannot depend on their visual intuitions because these intuitions do not yet exist (p. 25). In contrast to Eisenberg and Dreyfus, Vinner and Kopelman recommend that teachers not push students to reason visually until they have developed the ability to discern when it is valid to do so (p. 25).

Despite the cautionary note by Vinner and Kopelman, both sets of researchers acknowledge that visual reasoning is an important skill for mathematicians. Thus it is important for us to develop methods for teaching students how to use visual evidence in reliable ways at appropriate levels in order to model the visual reasoning of mathematicians.

**Diagrammatic Reasoning**

Philosophers and mathematicians have debated the use and effectiveness of diagrams in mathematical reasoning for a long time (Brown, 1999). Based on his analysis of the role of pictures in proof, Brown postulates that some diagrams are "instruments... which help the unaided mind's eye" (p. 39). Barwise and Etchemendy (1998) go even further to conclude that diagrams play an integral role in reasoning. They note:

> To solve a reasoning problem, we explore a space of possible situations ... consistent with the initial information we are given. ... Sentences. ... partition this space of possibilities, dividing it up into fiefdoms with a multitude of overlapping claims. The study of traditional logic deals with the relations among these claims. Thus, traditional proof techniques allow us to add to a collection of sentences that characterize a set of possibilities, additional sentences that also hold in that set. ... Diagrams, like sentences, carry information: they carve up the same space of possibilities, though perhaps in very different ways. ... Maps, charts, diagrams, and other nonsentential forms of representation can be, and often are, of equal importance to sentences. (p. 109)

Barwise and Etchemendy (1991) found that use of a diagram can help students visualize steps (p. 16). Since the student can add details or modify the sketch, even a pencil and paper diagram is dynamic. It represents not only the initial information but also the process of reasoning (p. 17). However, if we expect students to develop proofs based on a given diagram, we must ensure that they are able to interpret what is shown.

Goldenberg, Cuoco and Mark, (1998) claim that mathematical pictures and diagrams are difficult to interpret because they contain a great deal of information represented in a concise but "nonsequential" format (p. 11). Wheatley (1998) has found that understanding and interpreting what we see—a process he calls "imaging"—is not simple. It actually involves three activities:
constructing an image, re-presenting the image, and transforming the image. The implication of this is that the meaning we take from an image is not necessarily the meaning that someone else takes, because it depends in part on what we know about what we are looking at (p. 129).

The latest research is concerned with how we take meaning from an image that is moving—a topic which has important implications for the design of dynamic software diagrams. Rensink (2000) in a study of change blindness reports that we are only able to grab four to six visual objects at once and that focused attention is needed to notice change. The challenge will be to create images that help students concentrate on important details.

**Geometry Software**

School geometry includes three main themes: spatial visualisation, deductive reasoning and measurement (Schell, 1998). The fact that the visual aspect of geometry naturally lends itself to computer representation has led to the development of Logo-based Turtle Paths (Clements and Meredith, 1995), The Geometric Supposers (Schwartz and Yerushalmi, 1986), The Geometer’s Sketchpad (Jackiw, 1991), and Cabri Géomètre (Baulac, Bellemain, and Laborde, 1992). Unlike Supposer and Turtle geometry, Sketchpad and Cabri are referred to as dynamic. Dynamic software updates onscreen measurements and relationships as objects are dragged.

Much of the research with geometry software to date has focused on elementary students and the use of Logo to develop concepts of shape and measurement (Papert, 1980; Clements and Battista, 1989; Cohen and Geva, 1987; Clements, Battista, and Sarama, 1998a). The link between dynamic software use and the development of deductive reasoning in secondary school has only recently been the focus of study (Olive, 1998; de Villiers, 1998a; Edwards, 1997; Vinner and Kopelman, 1998).

With regard to primary education results, early research showed strong evidence that the Logo environment had a positive effect on children's geometric conceptualisations (Clements and Battista, 1989, p. 466). Clements, Battista and Sarama (1998b) theorised that this effect was due to the fact that, unlike textbooks which often move to higher levels of geometric thinking, Logo reinforced properties at the descriptive/analytic level of the van Hiele hierarchy. Nevertheless, Logo researchers Hillel, Kieran and Gurtner (1989) reported that the feedback offered by the program encouraged students to use visual clues rather than higher level thinking in solving problems. A new Logo environment called Geo-Logo was designed to provide tasks that required students to use analytic thinking and other mathematical skills (Clements et al., 1998b, p. 48). The researchers found that the new environment gave a context for the
development of geometric ideas, was motivating for students, aided in the construction of mental
collections between symbolic and graphic representations of geometric figures and gave
students feedback so that they could reflect on their thinking (p. 62).

Despite these results, neither Turtle geometry nor Geo-Logo has made an impact at the
secondary level. One reason may be that secondary school teachers viewed using Logo as too
time-consuming, given the demands of the curriculum.

In contrast, generally positive feedback has greeted the incorporation of dynamic-
geometry in secondary school geometry (Keyton, 1997; Boehm, 1997; Chazan and Yerushalmy,
1998; Hadas and Hershkowitz, 1999), and a large proportion of the material written for
Sketchpad and Cabri focuses on topics in the secondary or senior elementary curriculum (de
Villiers, 1998a; Exploring Geometry, 1993; Chazan, 1989).

**Student Achievement**

Despite the necessity of developing a repertoire of inquiry skills, and the challenges
presented by visual images, there are compelling reasons to consider the use of software for
geometry. Hannafin and Scott (1998) conducted an investigation of the effects of students' 
working memory capacity, preference for amount of instruction, spatial problem-solving ability,
and school mathematics grades on success in a dynamic geometry environment. They found that
it isn't always the student with the top mark who excels at the tasks of exploring, conjecturing
and explaining in a dynamic geometry environment. Students with relatively high mathematics
school grades scored higher than students with lower grades on the factual recall test items but
not on the conceptual understanding items (p. 3). Hannafin and Scott comment that: "[the fact]
that low achievers in school mathematics performed relatively better in these non-traditional
mathematics activities is an encouraging finding" (p. 3).

Possible explanations of this result might be due to the following factors:

1. Good students frequently believe they can progress faster if the teacher simply tells
   them what they need to know. They don't want to waste time thinking deeply about a
   problem, but they don't realise that they will learn far more through the hands-on
   experience. Skemp (1987) notes that *instrumental* understanding (i.e., ability to apply
   rules) is easy for some students and leads to immediate rewards. On the other hand, 
   *relational* understanding (i.e., understanding the concepts) takes more effort on the
   part of the student.
2. Geometry software has a creative impact on student work. Sendov and Sendov (1992), and Schwartz (1994) both comment on the fact that students often taught themselves to use the software to create geometric patterns or objects, which in turn extended their geometry knowledge.

3. In the past, knowledge meant facts, usually presented in words, sentences and paragraphs. Today more and more information is presented visually in diagrams, charts, and other displays. As data reaches our brains via this new schema there may need to be a re-evaluation of the concept of intelligence--some linear thinkers may be poor at processing the nonsequential information in diagrams, while some previously considered low achievers may learn more readily from a visual image (Kilpatrick and Davis. 1991).

4. Dynamic geometry software generates excitement among students (Keyton. 1997). This has physiologic significance. Neurobiologist Marian Diamond reports that a stimulating environment causes dendrites to form neural pathways of insight--"magic trees of the mind" (quoted in Fogarty. 1999).

Another explanation may be that this environment enables the teacher to interact more frequently and more effectively with all students, including the low achievers. While moving around the class the teacher is able to see what the student is responding to and (if students are in pairs) to hear what the students say about the mathematics. Thus teachers can readily identify areas of need and intervene to correct misinterpretations, to strengthen concepts, and to encourage extended thinking.

**Development of Activities**

Developing appropriate activities so students will be actively involved is an important and complex process. Based on his study of the structure of learning tasks, Flewelling (2000a) concluded that too many activities are developed so that students can do them easily. He recommends that instead, the emphasis should be on designing tasks that present a challenge and are worth doing. (Flewelling (2000b) has since used these results to develop guidelines for teachers on the creation of rich learning tasks).

Brousseau's didactical theory revolves around developing situations that will lead the student to develop mathematical understanding. The focus of each situation is the *obstacle* that the student must deal with—the essence that makes the task worth doing. Brousseau explains:
The posing of a problem consists of finding a solution with which the student will undertake a sequence of exchanges concerning a question which creates an 'obstacle' for her, and from which she will derive support for her acquisition or construction of a new piece of knowledge. ...Organizing the overcoming of an obstacle will consist of offering a situation which is likely to evolve and to make the student evolve. (pp. 87-88)

Dynamic Geometry Tasks

Hadas and Hershkowitz (1999) have done extensive research on the activities that students undertake using dynamic software. They divide such tasks into two categories: constructing, and conjecturing. They do not believe that conjecturing tasks are as valuable because, "the only motivation driving students to prove is to explain" (p. 57). In contrast, they contend that construction tasks, by allowing for uncertainty, spur students to investigate and learn. In their research Hadas and Hershkowitz have demonstrated that uncertainty is a critical feature that results from students being unable to complete a construction that seems intuitively possible but is actually impossible (p. 57).

Although these researchers favour construction tasks, dynamic-geometry software can be used to create pre-constructed (i.e., conjecturing) tasks that exploit uncertainty. There are endless creative opportunities for the software. As Laborde enthuses:

Dynamic drawings offer stronger visual phenomena than does one static drawing...The software environment may facilitate the appropriation of the explanation or proof task by the child, or in Brousseau's terms, it may facilitate the devolution process of the problem; the pupil acquires ownership of the task whereas in a traditional environment a proving task can be viewed as a school task with no relation to visual phenomena. (Laborde, 1998, p. 117)

Curriculum materials that accompany The Geometer's Sketchpad include activities in which students learn to construct and analyse figures, as well as activities in which they learn to postulate and make conjectures using pre-constructed sketches. These learning situations encourage student investigation and exploration, and support teachers acting as facilitators. Each activity aims to motivate learning by providing a good problem.

Study Tasks

The tasks involved in this research were based on the synthesis of the van Hiele levels and the SOLO taxonomy outlined in Table 2.1. They were aimed at students in level 3, that is, students operating in the concrete symbolic mode at a relational level and were created to
encourage development towards level 4: they also attempted to support students still operating at level 2.

The JavaSketches and the accompanying labsheets were designed to provide a challenging obstacle and the means to overcome it, carefully structured to build a deep understanding of the mathematical concepts.

I used the following basic principles in preparing the dynamic sketches and the accompanying labsheets for the study tasks.

1. The task must present an interesting obstacle that encourages acquisition of new knowledge.
2. The sketch must focus student attention on specific relevant details via affordances such as colour and motion.
3. The labsheet questions, supported by the interactive features of the pre-constructed sketch, must motivate students to explore the geometric relationships and to organise conceptual links to facilitate the transition from exploring, to conjecturing and proving.

**JavaSketchpad**

JavaSketchpad is a new tool in the dynamic geometry software category that is currently under development by the authors of The Geometer's Sketchpad. As a prototype of domain-specific, interactive geometry software it offers interesting options for geometry learning, such as a Web interface.

Since JavaSketchpad uses a restricted tool set, it resembles what Jackiw (1997) calls a "micro-microworld", such as those created with meta-geometric authoring tools to explore non-Euclidean geometry or investigate application-modelling scenarios. Jackiw argues that this format has advantages for students because it protects them from viewing the intimidating array of software features (p. 179). However, there may be disadvantages. For example, a JavaSketch cannot be messed up, a feature that Hoyles and Noss (1994) believe is one of the most important attributes of a dynamic sketch for student learning. They contend that in observing the results of dragging, students come to understand the relationships between objects. If the sketch falls apart (i.e., gets messed up) when being dragged, the relationships in the figure have not been defined properly. Students are forced to re-evaluate their techniques and to focus on the causes for the 'disaster'. (Of course, an intermediate JavaSketch could be designed to fall apart on dragging if the sketch creator deemed that this was important for student learning).
Arzarello et al. (1998) note that extensive studies of Cabri have been undertaken by researchers such as Balacheff, Mariotti, Laborde and Strasser and have led to an understanding of some components of Cabri's "epistemological domain of validity". One such finding is that a geometry problem cannot be solved simply by perceiving the images on the Cabri screen, even if these are animated. The student must bring some explicit mathematical knowledge to the process (p.32). A similar conclusion may be true for images created with JavaSketchpad but there has been no systematic analysis.

Summary

Teaching mathematics must be more than training students to apply various algorithms. It must focus on developing the ability to recognise and describe relationships between ideas and objects (Skemp, 1987). This is especially valid in today's information age when procedures can often be accessed and carried out in seconds.

The research reviewed so far indicates that to help students develop geometry skills required at the secondary level, the learning environment should include: (a) a focus on visual interpretation and spatial reasoning, (b) opportunities for collaboration and communication, (c) an emphasis on exploration, (d) the development of conjecturing skills, and (e) an emphasis on explaining why a particular result is true and why it must always be true.

There is an added dimension that must be considered in the technology-supported classroom—the relationship between student and machine. Sutherland and Balacheff (1999) have suggested that the relationship between teacher, student and machine be examined in light of Brousseau's theory of didactical situations. I believe that Towers' teacher intervention categorisations can be extended to help describe this important connection.

More research is needed to establish guidelines for the tasks in which students are engaged in the dynamic-geometry-supported classroom. There is a consensus that an element that leads to uncertainty or confusion is a motivating factor for students but there is a need for further research. For example, there is still disagreement about whether students can profit from investigating pre-constructed diagrams.

With JavaSketchpad, a domain-specific geometry software program, interactive and animated diagrams can be pre-constructed for use in dynamic geometry tasks. This study investigates the benefits and limitations of this program within the framework of geometric learning tasks in a senior mathematics program.
Chapter Three
Method

Teaching mathematics in secondary school is a complex process. Among the many factors that influence the outcomes are choice of content, student ability and attitudes, teacher knowledge and experience, and the design of the learning situation. In order to understand and fully describe the experiences of the participants in this study it was necessary to observe the interplay of several factors in the learning environment. As Merriam (1998) notes:

In contrast to quantitative research, which takes apart a phenomenon to examine component parts ..., qualitative research can reveal how all the parts work together to form a whole. It is assumed that meaning is embedded in people's experiences and that this meaning is mediated through the investigator's own perceptions. (Merriam, p.6)

From the perspective of a former secondary school mathematics teacher who has used software in the grade 12 geometry program, I have interpreted the interactions between the student and the other elements within the learning situation to enable the reader to form a deeper understanding of the JavaSketchpad-supported environment.

Research Approach

I chose the case study approach from the qualitative research methodologies because as Merriam (1998) notes:

A case study design is employed to gain an in-depth understanding of the situation and meaning for those involved. The interest is in process rather than outcomes, in context rather than a specific variable, in discovery rather than confirmation. Insights gleaned from case studies can directly influence policy, practice and future research. (Merriam, p. 19)

The case study approach afforded a good measure of flexibility in terms of data collection methods. Multiple sources of information were used in this study—observation field notes, videotape, audiotape, a student questionnaire, and interviews with the teachers. The data analysis methods—coding, developing categories, describing relationships, and applying simple statistical tests were applied where appropriate.
Participants

Three classes were chosen as illustrative of the typical grade 12 advanced mathematics class in Ontario. Within the classes, student pairs chosen to observe more closely via audio and videotape were selected by the teacher, as representative of the mix of students in the class with respect to achievement and computer expertise.

The three grade 12 advanced mathematics classes that participated in this study were a purposive sample. Class A from School S1, and Classes B and C from School S2. The students were between seventeen and eighteen years of age. There were 26 students in Class A, 22 in Class B and 21 in Class C for a possible total of 69 students; however due to student absences and an outside school activity at School S2, actual combined attendance totals at the sessions ranged from 50 to 66. The sites and particular classes were chosen by the researcher, in consultation with board and school administration on the basis of course availability, lab and computer capacity, and teacher agreement to participate.

Students worked in pairs. There were 12 pairs in Class A, 11 in Class B and 10 in Class C. In each class, several pairs were studied in more depth by audiotaping or videotaping their activities. The students in the taped pairs were chosen by the teacher to represent the range of achievement levels within the class. The students in any particular pair were not necessarily at the same achievement level.

Preliminary Procedures

I visited the teachers before the first in-class session, outlined the plan for the study, set a time frame for the class observations and ensured that the computer set-up was adequate. The topics for the sessions were chosen from the required Ontario grade 12 advanced mathematics curriculum (Curriculum guideline, 1985). The three teachers and I settled on a section of the geometry curriculum on deductive proof involving triangles and parallel lines for the focus of the investigations. All three teachers were using the textbook: Mathematics: Principles and Process, Book 2 (Ebos, Tuck, and Schofield, 1986) and the selection corresponded roughly to the material in chapter six of the text.

Each teacher spent one lesson reviewing the geometry terms and concepts learned in earlier grades. The original design of the study called for each teacher to give a brief quiz on this material which would be included in the data set. Due to circumstances at school S2, only Class A was able to complete a quiz before the observation sessions. Therefore teachers' anecdotal evidence of student prior knowledge was used for classes B and C.
Six JavaSketches were prepared for students to explore. The JavaSketches were created after consultation with the teachers so as to address particular problems or proof concepts. They included facilities for hide/show, movement, and animation where appropriate. One additional JavaSketch was prepared for use in a class discussion. The sketches were examined by a colleague and by two of my advisors. Details were added and corrections were made in light of their comments.

Labsheets, to accompany the sketches, were developed in line with Geometer's Sketchpad materials such as those of de Villiers (1999) and Schumann and Green (1994). These included directions for opening and manipulating the sketches, a description of the problem to be investigated, instructions for exploration, and questions to focus student attention. Space was provided on the sheets for students to draw and mark diagrams, list given properties, list new information that they noticed or inferred, and compose conjectures and proofs.

**Data Collection**

**Sessions**

The sketches were designed to be used in two sessions of 75 minutes each, including time for a group discussion. However, due to differences in class timetables and lab availability the breakdown of the actual sessions for each class were slightly different as shown in Tables 3.1, 3.2 and 3.3.

Class A had all three sessions in the lab. Class B had sessions one and two in the lab, then completed the questionnaire in a regular classroom the next day. Because Class C was timetabled for only 45 minutes each morning they had four timeslots. They spent the first three sessions in the lab, then completed the questionnaire in a regular classroom the next day.

**Table 3.1.**

**Class A—session breakdown.**

<table>
<thead>
<tr>
<th>Lab Session 1</th>
<th>Lab Session 2</th>
<th>Lab Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brief introduction - 10 min.</td>
<td>Group session - 20 min.</td>
<td>Finish up sketches - 10 min.</td>
</tr>
<tr>
<td>Day 1 sketches - 55 min.</td>
<td>Day 2 sketches - 45 min.</td>
<td>Questionnaire - 15 min.</td>
</tr>
</tbody>
</table>
Table 3.2.
Class B—session breakdown.

<table>
<thead>
<tr>
<th>Lab Session 1</th>
<th>Lab Session 2</th>
<th>Regular class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brief introduction - 10 min.</td>
<td>Group session - 20 min.</td>
<td>Questionnaire - 15 min.</td>
</tr>
<tr>
<td>Day 1 sketches - 60 min.</td>
<td>Day 2 sketches - 50 min.</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3.
Class C—session breakdown.

<table>
<thead>
<tr>
<th>Lab Session 1</th>
<th>Lab Session 2</th>
<th>Lab Session 3</th>
<th>Regular class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brief introduction - 10 min.</td>
<td>Day 1 and Day 2 sketches - 40 min.</td>
<td>Group session - 20 min. Day 2 sketches - 20 min.</td>
<td>Questionnaire - 15 min.</td>
</tr>
<tr>
<td>Day 1 sketches - 30 min.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data Collection Methods

The data collection methods were: videotaping of selected student pairs, audiotaping of selected student pairs and of the group sessions, field notes, three semi-structured interviews with each teacher, and a student questionnaire.

Videotaping.

In each lab session two pairs of students were videotaped using two cameras on tripods. One exception was the third lab session for Class A. Since the work session that day was very short no video camera was set up. As the students worked on the problems outlined in the labsheets (see Appendix D), the cameras recorded the students' actions with the sketches, their gestures and comments during the process and any interventions by me or the teacher.

Audiotaping.

In each lab session one pair of students was audiotaped. The tape recorded the students' comments during the process and the comments of the teacher or myself as we interacted with them.

In addition, the group session was audiotaped. Since some students did not give videotape authorization I was not able to videotape the whole class.

Field notes.

The teachers and I circulated to give assistance throughout the sessions. During and after each session I noted items of interest in hand-written field notes.
Semi-structured interviews.

Before the study and after two lab sessions I conducted audiotaped semi-structured interviews with each teacher (for preliminary questions, see Appendix C).

The goal of the pre-study interview was to understand the teacher's level of knowledge about geometry software, their interpretation of student experiences in geometry and their methods of teaching and remediation. Additionally, it helped to situate me in terms of the students' prior mathematical experiences.

The other interviews focused on the teacher's impressions of the sessions with respect to the flow of the class and the students' responses to the sketches and follow-up discussion. It provided the opportunity for the teacher to suggest revisions for subsequent sessions or future applications, and to comment on benefits and disadvantages of the use of pre-constructed dynamic sketches.

Student questionnaire.

A student questionnaire, (see Appendix B) was administered to each class. It collected information about students' prior experience with computers and mathematics software, in addition to their reactions to the dynamic geometry activities and the features of the JavaSketches themselves.

Quality Concerns

Internal validity.

Internal validity assesses whether the findings accurately reflect the reality of the situation being studied. In order to increase internal validity, I used several data collection methods to achieve triangulation. I also asked the teachers to check my descriptions of activities and discussions, and noted in this report, biases that I may have brought to the study.

External validity.

External validity deals with the extent to which the findings of one study can be applied to other situations. Quantitative results can be generalised by virtue of procedures such as random sampling. Some researchers reject the idea of generalising qualitative data, others try to strengthen external validity by incorporating sampling procedures; however, there are several interpretations that can help the qualitative researcher. One of these is naturalistic generalisation, which tries "to illuminate the general through the particular" (Ernest, 1992, p. 34). In the spirit of this interpretation, each local community of mathematics students that I observed and described
is one particular instance which I have used to illustrate the larger domain of secondary geometry classes.

Another way to look at external validity is to leave decisions about generalising up to the reader (Merriam, 1998). This approach assumes that the reader takes from the study those results that are applicable to his situation. By providing a rich, thick description of the context of this study, I hope that I have enabled the reader to compare his situation with that of the participants.

**Reliability.**

Reliability deals with the extent to which the research findings can be replicated. This issue presents a problem for qualitative researchers who are often collecting data from a small purposeful sample. Merriam (1998) suggests that the most important criterion is that the results be consistent with the data collected. In order to assure the reader of the reliability of the data I have attempted to describe fully the social context of the study, the basis on which participants were selected, the participants themselves and my relationship to the pairs being studied. To enhance reliability I have also used triangulation and have established an audit trail to enable others to follow my reasoning through the data analysis.

**Data Preparation Procedures**

The raw data set consisted of field notes of classroom activities, 7 audiotapes, 12 videotapes (one of which did not record properly), 33 student labsheets, and 56 student questionnaires. The information provided by each of the five data collection methods offered a different view of the learning environment. In order to use this information to develop a thick description of the people and events involved, and to interpret and explain significant aspects of the sessions, I prepared the raw data by transcribing the tapes, creating a questionnaire response matrix, and marking student labsheets.

**Transcripts**

I transcribed all audiotapes and pertinent sections of the videotapes myself. Since full anonymity was assured on the consent forms (see Appendix A), all participants' names and school names were changed to ensure confidentiality. This resulted in transcripts of 8 audiotaped sessions, and of extended relevant segments of 14 videotaped sessions with annotations of student gestures and motions. Table 3.4 identifies the taped pairs for each class, the taping method used and the dates.
Table 3.4.

Taping method and timelines for taped pairs by class.

<table>
<thead>
<tr>
<th>Class</th>
<th>Alias</th>
<th>Video</th>
<th>Audio</th>
<th>Dates</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Sue and Paul</td>
<td>X</td>
<td></td>
<td>March 28.29</td>
</tr>
<tr>
<td>A</td>
<td>Pat and Dave</td>
<td>X</td>
<td></td>
<td>March 28.29</td>
</tr>
<tr>
<td>A</td>
<td>Barb and Clara</td>
<td></td>
<td>X</td>
<td>March 29</td>
</tr>
<tr>
<td>A</td>
<td>Joe and Bob</td>
<td></td>
<td>X</td>
<td>March 28.30</td>
</tr>
<tr>
<td>B</td>
<td>Beth and Kim</td>
<td>X</td>
<td></td>
<td>April 3.5</td>
</tr>
<tr>
<td>B</td>
<td>Ray and Owen</td>
<td></td>
<td>X</td>
<td>April 3.5</td>
</tr>
<tr>
<td>B</td>
<td>Lou and Rob</td>
<td>X</td>
<td></td>
<td>April 5</td>
</tr>
<tr>
<td>B</td>
<td>Doug and Sal</td>
<td></td>
<td>X</td>
<td>April 3</td>
</tr>
<tr>
<td>C</td>
<td>Sarah and Earl</td>
<td>X</td>
<td></td>
<td>April 3,4,5</td>
</tr>
<tr>
<td>C</td>
<td>Lily and Fran</td>
<td></td>
<td>X</td>
<td>April 3.4</td>
</tr>
<tr>
<td>C</td>
<td>Jan and Pam</td>
<td></td>
<td>X</td>
<td>April 5</td>
</tr>
<tr>
<td>C</td>
<td>Katy and Bea</td>
<td></td>
<td>X</td>
<td>April 3,5</td>
</tr>
<tr>
<td>C</td>
<td>Tara and Mary</td>
<td></td>
<td>X</td>
<td>April 4</td>
</tr>
</tbody>
</table>

In addition, the audiotaped teacher interviews were transcribed.

**Questionnaire Response Matrix**

Student responses on the questionnaire were set into a matrix using the questions as headings. Information from this display was used to provide a rich description of student background, attitudes and experiences and to identify task design features that helped students.

**Marking of Labsheets**

Student labsheets were not marked as tests but instead were examined to identify areas of student understanding, to uncover difficulties, and to look for evidence of visual reasoning.

For the non-taped pairs this data formed a general background of information about students' experiences during the sessions. For the taped pairs, the notations were later compared to the transcripts of particular tasks.

**Data Coding and Analysis**

The theoretical links that I identify and describe are the result of a systematic approach to the data. A qualitative analysis computer program, ATLAS.ti (Scientific Software, 1997), was used to codify, annotate, and sort text segments of the transcripts. Using the software I created families of codes, and built visual networks to uncover the relationships between student responses and the stimuli offered by teacher, student partner, labsheet or sketch. As common
themes emerged, however, I found it more efficient to work through the data by hand, highlighting text and making notations on the transcripts, since I had been immersed in all its subtleties from the very beginning.

In the first run-through, the transcripts were codified according to what was happening at the time. Some examples of these first codes were: students on task, pointing to screen, dragging diagram, using motion button, excited, reading, appealing for help, student states theorem, student restates known information, deducting from visual, erroneous conclusion. The results were a rather disconnected series of observations.

On the second run-through each of these annotated activities was examined to uncover the underlying motivation. Some of the codes at this stage were: checking understanding, modelling thinking, reinforcing ideas, posing inviting questions, colour used as reference, students use redo capability.

As the work progressed I realised that many of the codes that described teacher interactions with taped pairs were similar to those described by Towers' (1999) interventions. I replaced codes (where appropriate) with hers and looked for evidence of the other styles and strategies that she had identified. Although Towers had not done her research in a dynamic geometry environment, almost all her intervention categories were represented in my transcript data. Since I was analysing student pairs working together in an informal setting, rather than participating in a teacher-led lesson, I did find that certain interventions such as "shepherding", which is a more subtle form of teaching, were more prominent—just as Towers (1999) found in her thesis research.

While some of the coding was beginning to highlight themes with respect to the teacher's role, most of the transcripts contained very little teacher-student discourse. There was more discussion between student partners. New codes were developed to describe this kind of interaction: students correct one another, working at cross-purposes, struggle.

However, students were reacting not just to their teacher or their partner but to some external stimulus. I started to investigate the relationship between student comments and particular labsheet questions, or sketch features. First, I created an analysis table for each labsheet (see Appendix E). The first column included the instructions, statements, and questions from the original labsheet. The second column set forth the purpose I had intended that statement or question to fulfil. The third column described the role that I had envisioned for the sketch in relation to the statement or question. In choosing entries for the columns I began with Towers' category labels in order to facilitate comparisons to the transcripts, then developed additional
descriptions as required. For example, beside "Open Jsketch 1", was the word "managing" since this instruction was only intended to contribute to the management of the activities. Beside "drag each red point and observe the diagram", the sketch column includes the notation, "draws attention--colour. movement."

I then re-examined the transcripts to determine student responses to these statements. In addition to new labsheet and sketch codes which I inserted at appropriate questions and statements, new codes for student responses emerged, such as: hypotheses, struggles to remember, answers, is confused, concludes. These represented not just the outward action but the inner thinking of the students, explicitly linked to the task components—the labsheets and the sketches.
Chapter Four
The Study Setting

This chapter introduces the study participants and examines the materials used in the sessions.

The Study Participants

As noted earlier, the participants were students between seventeen and eighteen years of age, in three grade 12 advanced mathematics classes from two schools.

The Schools

School S1.

School S1 was recommended by the mathematics co-ordinator for the school board as a possible site for the study because its mathematics department head was open to innovative ideas and interested in using software for mathematics.

The school is a co-educational secondary school with a population of just over 1000 students, located in a residential urban area. Most students live in the surrounding area, which contains a mix of single family houses, multiple-dwelling units, and high rise apartments. Some students are bussed in for special programs from outside the area. The students come from a wide range of backgrounds, both in terms of ethnicity and income.

School S1 runs a semestered program on a four-period day (i.e., students attend the same four courses each day for the entire semester). Each period is 76 minutes long.

The members of the mathematics department were very helpful in setting up the lab and ensuring that the sessions went smoothly. They were also very interested in the project. Several popped in during the sessions to watch the students work with the sketches. It was a very positive experience to work with this very professional group of teachers.
School S2.

School S2 was a replacement for another school whose teacher pulled out of the study at the last minute due to ill health.

I had originally passed over school S2 because I had taught there for three years before leaving to become a vice principal; however, I knew that it would be a good setting. The teachers in the mathematics department are experienced, committed to excellence in education and open to new ideas. At the same time they are cautious about hopping on bandwagons, and insist on evidence that a new method will help their students learn.

I examined and rejected several possible sources of bias.

- I was not in a position of authority at the school with respect to teachers or students.
- Although I had taught some of the students, I had not taught the grade 10 course which contains the geometry unit that is the basis for the grade 12 geometry section in the study.
- I had used dynamic software as a teacher at the school in my grade 12 classes, but none of the present grade 12's had been in those classes.
- I knew the volunteer teachers, but I believed that my relationship with them could only enhance my understanding of their comments regarding the sessions.

After considering these facts I decided that my previous relationship with the school would not interfere with my impartiality, and that my familiarity with the school routines would be an advantage. Nevertheless, throughout the study I remained cautious to ensure that there were no other sources of bias.

When I then approached the department head at school S2 to inquire if a teacher was interested, he informed me that two teachers would like to participate. I had intended to use only one class but I broadened the study to include a second

School S2 is similar in some aspects to school S1. It is a co-educational secondary school in the same urban board, with a population of just over 1000 students. Most students live in the surrounding area, which contains a mix of single family houses, low rise units and high rise apartments. Some students are bussed in for special programs from outside the area. The school is multicultural with a large percentage of the students being of Asian descent. Most students come from low to middle class backgrounds.

School S2 runs a non-semestered program on a four and a half period day. The half period class is taught for 45 minutes Monday through Thursday; the other four classes are taught for 76 minutes on a day 1, day 2 schedule (e.g., if grade 12 advanced mathematics is a 76 minute
class it is taught either on day 1 or on day 2.) For this reason Class B met on April 3, 5 and 7, whereas Class C met on April 3, 4, 5 and 6.

The sessions went very smoothly at the school. The computer site administrator checked that the files and logon procedures were in place, and the teachers ensured that students were organised quickly. The atmosphere was relaxed and friendly, and the students were anxious to learn.

The Teachers

Teacher A.

Teacher A is a mathematics teacher with 23 years experience. She had had no previous experience with dynamic geometry software but was curious and enthusiastic about being part of the study. It was clear from her interview responses and her manner with the students that she cares deeply about helping students develop understanding, and is constantly looking for ways to improve her teaching of mathematics.

Teacher A finds that students often have trouble with the geometry unit—especially with proof. In the initial interview she noted that they find proof difficult and that the marks usually drop when they are asked to prove. She added that some students find it extremely hard to stay on task and motivated during geometry.

Teacher A uses traditional methods—she introduces a new concept, does a few examples and the next day takes up any problems. However, she is aware of the pitfalls of merely showing students one method. She explained during the interview that since there are so many different ways of doing each question she attempts to alter her approach to meet student needs. As she says,

I sort of believe that students are wired differently, and it doesn't make them superior to other students. It's just, that's their way of learning and that's how—how they get the information and are able to work with it. So, hopefully I give them enough options to work with so they're successful. (interview, March 27, 2000)

Teacher A is very attuned to the importance of a visual image. She uses coloured chalk to outline various parts of the diagram as she teaches, often uses the overhead to show rotations or that two shapes are congruent, and encourages the students to use memory diagrams. These are diagrams (such as those one might find in a text) that have angles or sides marked to remind students of a particular theorem. For example the memory diagram for ITT (the isosceles triangle theorem) was an isosceles triangle with two sides marked equal and the two angles opposite these sides marked equal.
**Teacher B.**

Teacher B has had 22 years experience teaching mathematics—6 years in Saudi Arabia, 7 years in Lebanon and 9 years in school S2. She had not used dynamic geometry software with her classes but was interested in learning about *JavaSketchpad*, and hopeful that the tasks would help her students develop a deeper understanding of geometry proof.

Teacher B teaches geometry in a traditional manner. For example she stated:

If we're doing just definitions and you know, concepts—that would be different. But if we're doing proofs then I put on the board, let's say, a question. We work on it together. I try to ask them questions you know, to analyse the problem. So I do one or two and then I ask them to do one that is quite close to what we did let's say. And then I go to a more difficult one. We work together again on the more difficult one and then I give them some time to do it on their own. (interview, April 3, 2000)

She is keenly aware of the needs of her students, especially those who have difficulty with English. She explains:

And then what I did--some questions, they don't have the diagrams. So we started working on those--how to go from a sentence to drawing a diagram, because some of them have even difficulty you know going from the English sentence and transforming it to a diagram. (interview, April 3, 2000)

In addition, although she finds that students have difficulty with proof concepts, she believes that active learning can make a difference. She says:

Using the compass, ruler ... I've brought to class a compass, a ruler et cetera. For example when we were—even in working with the ambiguous case for example I remember this very well too. They understood it very, very well when we did it by construction. (interview, April 3, 2000)

**Teacher C.**

Teacher C is a mathematics teacher with 29 years experience in the same urban centre and 13 of these in school S2. She had experimented with dynamic software briefly but had not used it with her students. She uses the didactic approach for most lessons, although she occasionally develops activities to help students understand new concepts. Her focus is on making sure that students work with understanding. She explained:

Well, usually I would review whatever concepts or what I tried to teach the day before and make sure that they have all those theorems and ideas in mind. And uh, then I'll put up a problem that's trying to take us to the next step or something and then I generalise about it and talk about theory and then—we always do—we do a couple together. I try to establish the theorem so that it makes sense to them as opposed to just telling them what it is. (interview, April 3, 2000)
Teacher C was curious about *JavaSketchpad* and hopeful that it might add a new dimension to the geometry unit. Since she has taught for many years she is familiar with the old program in which Euclidean geometry was taught for an entire year and students were required to memorise lists of propositions. Although she does not want to return to such formality she does feel that the present geometry curriculum lacks a clear focus. She noted:

That's the unit that I dislike teaching the most, because I'm not sure where I'm headed. Right, like I don't have a clear idea of what I'm trying to show the students. You know, so it's not really logic. It's not really properties and--and it just kind of--you know a few silly theorems and--and a few 'show that this is congruent to this' and blah. You know, so I'm not happy with it the way it is at all. So I'd welcome something different. (interview, April 3, 2000)

Table 4.1 presents a summary of the study teachers' years of experience, prior use of dynamic software, and teaching styles.

**Table 4.1.**

**Summary of teacher data.**

<table>
<thead>
<tr>
<th>Years of experience</th>
<th>Teacher A</th>
<th>Teacher B</th>
<th>Teacher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>22 (9 in Canada)</td>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Prior personal use of dynamic software</th>
<th>Teacher A</th>
<th>Teacher B</th>
<th>Teacher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Prior use of dynamic software in program</th>
<th>Teacher A</th>
<th>Teacher B</th>
<th>Teacher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teaching style</th>
<th>Teacher A</th>
<th>Teacher B</th>
<th>Teacher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>Traditional</td>
<td>Traditional</td>
<td>Traditional</td>
</tr>
</tbody>
</table>

**The Classes**

*Geometry background.*

All students in the study were enrolled in the grade twelve advanced mathematics program. This course covers topics in algebra, geometry, analytic geometry and trigonometry, and is a prerequisite for OAC mathematics (i.e., Ontario Academic Credits, which are required for application to university). The geometry section of the course has three main headings: (1) congruence and parallelism, (2) similar figures, and (3) the circle (Curriculum guideline, 1985, p. 60). As noted earlier, the teachers and I chose to focus on the first topic. The students had not done any work on the other two areas in the course thus far.

It is very difficult to determine the extent of the prior geometry instruction of the students in the study. From grades seven to twelve the students would have been subject to the requirements of the Ontario curriculum (1985). An abbreviated outline of the geometry
requirements of that curriculum most pertinent to this study is shown in Table 4.2.

(Measurement, three-dimensional and co-ordinate geometry topics are also taught).

**Table 4.2.**

Outline of selected geometry topics in the Ontario mathematics curriculum (1985), grades 7 to 12.

<table>
<thead>
<tr>
<th>Basic Principles</th>
<th>Symmetry and Transformations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7</strong></td>
<td></td>
</tr>
<tr>
<td>Classify angles, pairs of lines, triangles, quadrilaterals</td>
<td>Classify triangles and quadrilaterals according to symmetry</td>
</tr>
<tr>
<td>Identify congruent figures and properties of congruent figures</td>
<td>Identify line symmetry and rotational symmetry of regular polygons</td>
</tr>
<tr>
<td>Construct geometric figures using properties and a variety of techniques and instruments</td>
<td>Sketch figures under slides, flips, and turns</td>
</tr>
<tr>
<td>Identify and apply geometric principles in designs</td>
<td>Recognize images under slides, flips, turns</td>
</tr>
<tr>
<td>Test for perpendicular and parallel lines</td>
<td>Construct patterns with congruent tiles</td>
</tr>
<tr>
<td>Construct special angles without protractor</td>
<td>Identify slides, flips, turns in tiling patterns</td>
</tr>
<tr>
<td>Construct triangles, quadrilaterals and circles from given information</td>
<td>Make enlargements and reductions using a geoboard, dot paper, grid or tiles.</td>
</tr>
<tr>
<td>Locate center of a circular arc by folding</td>
<td></td>
</tr>
<tr>
<td>Investigate angle relationships for parallel and intersecting lines, and isosceles and equilateral triangles.</td>
<td></td>
</tr>
<tr>
<td>Investigate the sum of angles property of a triangle.</td>
<td></td>
</tr>
<tr>
<td>Review constructions and angle properties</td>
<td>Identify the image of a figure under slides, flips, turns</td>
</tr>
<tr>
<td>Learn SSS, SAS, ASA</td>
<td>Identify the transformation relating two congruent figures</td>
</tr>
<tr>
<td>State equidistant property for perpendicular bisector</td>
<td>Draw enlargements and reductions</td>
</tr>
<tr>
<td>Investigate inscribed angles in a circle, and angle properties of a cyclic quadrilateral</td>
<td>Identify the properties of similar figures (shape, sides, angles, area)</td>
</tr>
<tr>
<td><strong>9</strong></td>
<td></td>
</tr>
<tr>
<td>Distinguish defined and undefined terms</td>
<td>Use grid to identify, and draw the image of a figure under translation, reflection and rotation.</td>
</tr>
<tr>
<td>Illustrate application of conditional statements in deductive thinking</td>
<td>Identify the transformation relating two congruent figures</td>
</tr>
<tr>
<td>Identify properties of parallel lines,</td>
<td>Draw dilatation images</td>
</tr>
<tr>
<td><strong>10</strong></td>
<td>None</td>
</tr>
<tr>
<td>Basic Principles</td>
<td>Symmetry and Transformations</td>
</tr>
<tr>
<td>------------------------------------------------------</td>
<td>------------------------------</td>
</tr>
<tr>
<td>intersecting lines, sum of angles of a triangle, isosceles triangles and congruent triangles as postulates</td>
<td></td>
</tr>
<tr>
<td>Identify the properties of reflection, rotation and translation as postulates</td>
<td></td>
</tr>
<tr>
<td>Analyze and prove statements</td>
<td></td>
</tr>
<tr>
<td>Prove the Pythagorean theorem.</td>
<td></td>
</tr>
</tbody>
</table>

11 All geometry in the grade 11 program is analytic geometry.

12 Identifying undefined and defined terms. Identifying as axioms such properties as: the sum of angles in a triangle, angle properties related to parallel lines, intersecting lines and isosceles triangles, sufficiency conditions for congruent triangles. Analysing and proving simple deductions and theorems. Solving problems involving numerical applications. Identifying the features of indirect proof. Using indirect proof to establish the inequality relations for a triangle.

Identify as axioms the basic properties of reflection, rotation, and translation. Define similar figures. Identify sufficiency conditions for similar triangles. Solve numerical problems and prove simple deductions based on sufficiency conditions for similar triangles. Identify and prove the relationship between the linear dimensions and area of similar figures.

Out of a total of 110 hours per course, the number of hours recommended in the curriculum guideline for the topics above is:

Grade 7: 10 hours (basic notions of geometry)
Grade 8: 10 hours (constructions), 6 hours (properties of plane figures), 6 hours (transformations)
Grade 9: 5 hours (constructions), 6 hours (properties of plane figures), 4 hours (dilatations)
Grade 10: 13 hours (deductive thinking through geometry)
Grade 12: 8 hours (congruence and parallelism), 7 hours (similar figures), 8 hours (the circle)
It is very unlikely that all the students in the study were exposed to all topics. The amount and quality of geometry instruction in grades seven and eight varies from school to school as a result of the use of non-subject specialists for mathematics. In grades nine, ten and twelve, geometry topics are sometimes eliminated or assigned as independent work to allow more time for other areas of the curriculum.

Class A.

Class A was a group of students at school S1 enrolled in the Ontario grade twelve advanced mathematics program in the 1999-2000 school year.

There were 26 students in class A, but three students were absent for all three sessions. An additional three students were absent at least one day.

Prior to the sessions the students completed a quiz on the material from grade 10 that Teacher A had reviewed. This covered definitions of geometric terms such as 'equilateral', as well as theorem statements for: complementary angle theorem (CAT), supplementary angle theorem (SAT), opposite angle theorem (VOAT), parallel line theorems (PLT), angle sum triangle theorem (ASTT), exterior angle theorem (EAT), and isosceles triangle theorem (ITT). Students were asked to match a theorem name with its statement, to match a memory diagram with its theorem, to fill in the reasons for the statements in a proof, and to carry out the following proof: "In the diagram, PQ \parallel RS and PQ = RS. Prove that M is the midpoint of PS and QR."

Teacher A marked the quiz. The class average was 81% with marks ranging from 50% to 94%. Despite the high class average on this quiz, it was apparent during the sessions that many students had difficulty applying these concepts.

It was difficult to base conclusions about student ability on the results of the quiz since the marks, as noted, were very high, and some students missed the quiz; therefore, for an overview of student ability based on teacher assessment see Table 4.3.

Class B.

Class B was a group of students at school S2, enrolled in the Ontario grade 12 advanced mathematics program in the 1999-2000 school year.

Class B was scheduled for a full period every second day. There were 22 students on the attendance list; however, two students were absent for all three sessions. An additional five students were absent at least one day. Some of these absences were due to a mandatory school related activity. Unfortunately I was not made aware of this activity until midway through the
first session and thus, a different pair was audiotaped in the second session than in the first session.

Prior to the sessions the students did one review class on material from grade 10. The topics covered were similar to those covered by Class A: definitions of geometric terms, complementary angle theorem (CAT), supplementary angle theorem (SAT), opposite angle theorem (VOAT), parallel line theorems (PLT), angle sum triangle theorem (ASTT), exterior angle theorem (EAT), and isosceles triangle theorem (ITT). As with Class A, students had difficulty with many of the reviewed concepts during the sessions. Due to circumstances at the school the class did not write a quiz on this review. Teacher B found this class very weak. For an overview of student ability based on teacher assessment see 4.3.

Class C.

Class C was a group of students at school S2, enrolled in the Ontario grade 12 advanced mathematics program in the 1999-2000 school year. School S2 has already been described above in relation to Class B.

This class was scheduled for a half period every day, four days a week. There were 21 students in the class; however, one student was absent for all four sessions. An additional five students were absent at least one day. Some of these absences were again due to the same mandatory school related activity as mentioned in the introduction to Class B. This resulted in having to choose different pairs for audio and videotaping when students were absent. I was concerned about the lack of continuity in the videotaped pair, however, the enlarged set of taped students that resulted, gave me a wider variety of responses to examine.

Prior to the sessions the students did one review class on material from grade 10. The topics covered were those covered by Class B. Due to circumstances at the school, the class did not write a quiz on this review. As with Classes A and B, students had difficulty with many of these concepts during the sessions. For an overview of student ability based on teacher assessment see Table 4.3.

Table 4.3.
Number of students by class and by assigned grade.

<table>
<thead>
<tr>
<th>Assigned Grade</th>
<th>Class A</th>
<th>Class B</th>
<th>Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excellent (A+)</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Very good (A)</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Good (B)</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Average (C)</td>
<td>5</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>Weak (D and below)</td>
<td>9</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
Set-up

School S1.
Since the computer labs are used for regularly scheduled classes, arrangements were made in advance for a class to switch rooms with Class A for the three consecutive sessions. The lab had never been used for mathematics so *The Geometer's Sketchpad* and *JavaSketchpad* were installed prior to the sessions. Although the students would not require *The Geometer's Sketchpad* in the sessions, having it on the network allowed me to troubleshoot if necessary.

Originally I had intended to have the students access the sketches via the Internet; however this was too slow. Putting the files on the school network improved access times somewhat, and eliminated worries about websites crashing during the sessions.

The lab was well lit and spacious. It contained approximately 25 computers located around the perimeter and a U-shaped set of tables for seatwork in the middle of the room. Since the students worked in pairs we needed only 12 computers.

The school data projector was used during the group session to project the *JavaSketch* image on a screen at the front of the lab. This was not entirely satisfactory since details on the image were difficult to see from the back of the room.

School S2.
One computer lab at school S2 is set aside for cross-curricular use. Several weeks before the sessions I arranged to sign up for the lab. The computers are all Pentium 100's or better, but as with school S1, I installed *JavaSketchpad* prior to the sessions to avoid problems of access to an external website. The lab is large and comfortable. It contains approximately 30 computers located around the perimeter and along aisles in the centre of the room. Since the students worked in pairs we needed only 11 computers for Class B and 10 computers for Class C.

School S2 does not own a data projector. There is an overhead monitor in the cross-curricular lab that displays whatever is on the attached computer. This at first, seemed to be an advantage, but the resolution of the monitor proved to be a problem during the introduction and the class session. It was difficult to see details from the back of the classroom.
The Sessions

Class A.

At the beginning of the first session I gave a brief overview of what the sessions would involve, then led the students through the process of opening the first sketch and locating the corresponding labsheet in their booklet.

During the sessions students worked in pairs on the learning tasks while Teacher A and I moved around the classroom, helping with the software as required and answering questions about the tasks. Each of the 12 pairs of students was given a folder containing the labsheets for all sketches. These were collected after the sessions and examined. If a pair answered at least one question on the labsheet it was included in the following totals. Over the three days, 10 pairs investigated Day 1, task 1, 6 pairs explored Day 1, task 2, 11 pairs did Day 2, task 1, and 12 pairs attempted Day 2, task 2. In addition, 6 pairs worked on Jsketch4, and 10 pairs examined Jsketch7.

Since students chose their own partners there were many variations with regard to gender and ability. Six pairs were boy/boy, five were girl/girl, and one was girl/boy. On the first day, the cameras focused on Sue and Paul, and Pat and Dave. Joe and Bob were audiotaped. On the second day the same two pairs were videotaped, but I moved the audio tape recorder to Barb and Clara to get a different perspective. On the second day we also had a group session which was audiotaped. On the third day Joe and Bob were audiotaped again for a brief time until students began working questionnaires individually.

Class B.

The introduction to the sessions in Class B was the same as that in Class A. During the sessions students worked in pairs on the learning tasks while Teacher B and I circulated. When the labsheet folders were collected at the end of the sessions it was found that: all 11 pairs had investigated Day 1, task 1, and Day 1, task 2. Seven pairs did Day 2, task 1, and 9 pairs attempted Day 2, task 2. In addition, 4 pairs worked on Jsketch4, and 1 pair examined Jsketch7.

As with Class A, students chose their own partners. Six pairs were boy/boy, four were girl/girl, and one was girl/boy. On the first day the cameras focused on Beth and Kim, and Ray and Owen. Doug and Sal were audiotaped. On the second day the same two pairs were videotaped, while a new pair – Lou and Rob were audiotaped. On the second day we had a group session which was audiotaped. On the third day, since the lab had no regular desks, assignments and questionnaires were completed in a regular classroom.
Class C

Since this class was offered for 45 minutes each morning, the lab time was split into three sessions rather than two. Because sessions were shorter, students were still working on Day 1 tasks at the second session and Day 2 tasks at the third. Students in this class actually had 20 minutes less time in the lab than the students in the other two classes. This may explain why they completed fewer tasks.

When the labsheets were collected, students had attempted the following: all 10 pairs investigated Day 1, task 1, Day 1, task 2, and Day 2, task 1. In addition, 4 pairs worked on Jsketch4; however, none of the pairs wrote responses to Day 2, task 2 or Jsketch7. On the video, Jan and Pam can be viewed exploring Day 2, task 2; however, they did not write it up, perhaps due to time constraints.

As with the other classes, students chose their own partners. In this class there were two boy/boy pairs, four girl/girl pairs, and four girl/boy pairs. On the first day the cameras focused on Sarah and Earl and Lily and Fran. Katy and Bea were audiotaped. On the second day the same two pairs were videotaped, while Tara and Mary were audiotaped. On the third day Sarah and Earl and Jan and Pam were videotaped and Katy and Bea were audiotaped. On day three we also had a group session which was audiotaped. On the fourth day since the lab had no regular desks, assignments and questionnaires were completed in a regular classroom.

Tables 4.4 and 4.5 summarise information about the makeup of the pairs by gender and by class, and also give an overview of how many pairs in each class attempted particular tasks.

Table 4.4.
Number of pairs by gender and by class.

<table>
<thead>
<tr>
<th></th>
<th>Class A</th>
<th>Class B</th>
<th>Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girl/girl pairs</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Boy/boy pairs</td>
<td>6</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Boy/girl pairs</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 4.5.
Number of pairs by task and class, n = 33.

<table>
<thead>
<tr>
<th>Task</th>
<th>Sketch</th>
<th>Class A</th>
<th>Class B</th>
<th>Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Day 1, task 1: Jsketch2</td>
<td>10 pairs</td>
<td>11 pairs</td>
<td>10 pairs</td>
</tr>
<tr>
<td>2</td>
<td>Day 1, task 2: Jsketch3</td>
<td>6 pairs</td>
<td>11 pairs</td>
<td>10 pairs</td>
</tr>
<tr>
<td>3</td>
<td>Day 2, task 1: Jsketch1</td>
<td>12 pairs</td>
<td>7 pairs</td>
<td>10 pairs</td>
</tr>
<tr>
<td>4</td>
<td>Day 2, task 2: Jsketch6</td>
<td>12 pairs</td>
<td>9 pairs</td>
<td>1 pair</td>
</tr>
<tr>
<td>E1</td>
<td>Extra sketch 1: Jsketch4</td>
<td>6 pairs</td>
<td>4 pairs</td>
<td>4 pairs</td>
</tr>
<tr>
<td>E2</td>
<td>Extra sketch 2: Jsketch7</td>
<td>10 pairs</td>
<td>1 pair</td>
<td>0 pairs</td>
</tr>
</tbody>
</table>
**The Taped Students**

**Class A.**

1. **Barb and Clara**

   Teacher A rated Barb and Clara as above average students. Barb missed the quiz, but Clara achieved 91%. This pair was audiotaped on the second day. They explored Day 2, task 1, and Day 2, task 2 while being taped. These two girls were on task, enthusiastic, and worked well together, but were unaccustomed to investigations, and had some difficulty with geometry terms.

2. **Sue and Paul**

   Sue and Paul were both excellent mathematics students, as reported by their teacher. They had achieved 90% and 94%, respectively, on the geometry review quiz. During the three sessions, they completed the four regular tasks and the two additional ones. Since Paul had a broken arm Sue did all the writing (and typed in URL's), while Paul manipulated the mouse. Thus, Sue is often heard telling Paul to drag or click or show some item, while Paul frequently directs Sue to write a particular statement.

3. **Pat and Dave**

   Pat was an average student and Dave a poor student according to their teacher. On the quiz they had received 77% and 56% respectively. During the first session they worked very slowly, considering only Day 1, task 1. In the second session they explored Day 2, task 1 and Day 2, task 2. They shared the task of manipulating the sketches but Pat recorded all results. On the video, Dave is often seen gazing at the sketch and experimenting with the various options while Pat is more concerned with answering the questions.

4. **Joe and Bob**

   Joe and Bob were average students according to Teacher A. On the quiz they received 73% and 84% respectively. They were audiotaped on the first and third days. At the first session, there was a problem with the microphone. Thus, their investigation of Day 1, task 1 was lost. After the technical problem was fixed, they can be heard discussing Day 1, task 2 briefly, before the class ended. On the third day Joe and Bob explored Day 2, task 2, and the extra problem Jsketch7. They missed Day 2, task 1 altogether.

**Class B.**

1. **Beth and Kim**

   According to Teacher B’s assessment Beth was a very good student (A average) but Kim had some difficulty with mathematics (C average). They were videotaped on the first and second
days. At the first session, they investigated Day 1, task 1 and Day 1, task 2. Although they continued Day 1, task 2 and began Day 2, task 2 at the second session, the videotape of their discussion that day is very unsatisfactory, in part because they speak very little. In addition the microphone was muffled by some obstruction.

2. Ray and Owen

Ray and Owen were both good students (B averages). They were videotaped on the first and second days. At the first session they explored Day 1, task 1, and Day 1, task 2. They briefly also examined Jsketch4. At the second session they worked almost exclusively on Day 2, task 1 and only had a few moments to discuss Day 2, task 2.

3. Doug and Sal

Doug and Sal were audiotaped on the first day. Both were average students. On the tape they discuss Day 1, task 1, Day 1, task 2, and spend a short time on Jsketch4.

4. Lou and Rob

Lou and Rob were audiotaped at the second session. They were an unusual pair because Lou was an A student while Rob's average hovered around 50%, according to teacher B. They worked on Day 2, task 1 and on Day 2, task 2 while being taped.

Class C

1. Sarah and Earl

Sarah and Earl were videotaped at all three sessions. According to Teacher C, Sarah was a good student (B average). She commented that although Earl had a C average, in her estimation he was capable of better work. This pair worked methodically but slowly through the sketches, covering Day 1, task 1, Day 1, task 2, then Day 2, task 1.

2. Lily and Fran

Lily and Fran were videotaped on the first two days. Another pair was videotaped on the third day because Lily was away at the school activity. Fran had an A average in the course thus far, while Lily had a C average. The girls worked on Day 1, task 1 at the first session and tackled Day 1, task 2 at the second.

3. Jan and Pam

Jan and Pam were videotaped on the third day. They did tackle Day 2, task 2 however, they spoke so little and the sound quality is so poor on the videotape that it is impossible to use their discussion as a basis for analysis.
4. Katy and Bea

Katy and Bea were audiotaped at the first and third sessions. During the taping they worked on Day 1, task 1, Day 1, task 2 and Day 2, task 1. In the course they were both carrying a C average.

5. Tara and Mary

Tara and Mary were audiotaped at the second session. Teacher C reported that Tara had an A average while Mary had a C average. They explored Day 1, task 2 while being taped.

A summary of the foregoing information by class is found in Tables 4.6 to 4.8.

Table 4.6.
Taped pairs in class A by ability level, taping method and tasks attempted.

<table>
<thead>
<tr>
<th>Names (aliases)</th>
<th>Ability (based on teacher assessment)</th>
<th>Method</th>
<th>Tasks attempted during taping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sue and Paul</td>
<td>Excellent/Excellent</td>
<td>Video</td>
<td>1,2,3,4, E1, E2</td>
</tr>
<tr>
<td>Pat and Dave</td>
<td>Average/Weak</td>
<td>Video</td>
<td>1,3,4</td>
</tr>
<tr>
<td>Joe and Bob</td>
<td>Average/Average</td>
<td>Audio</td>
<td>2,3,E2</td>
</tr>
<tr>
<td>Barb and Clara</td>
<td>Very good/Very good</td>
<td>Audio</td>
<td>3,4</td>
</tr>
</tbody>
</table>

Table 4.7.
Taped pairs in class B by ability level, taping method and tasks attempted.

<table>
<thead>
<tr>
<th>Names (aliases)</th>
<th>Ability (based on teacher assessment)</th>
<th>Method</th>
<th>Tasks attempted during taping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beth and Kim</td>
<td>Very good/Average</td>
<td>Video</td>
<td>1,2,4</td>
</tr>
<tr>
<td>Ray and Owen</td>
<td>Good/Good</td>
<td>Video</td>
<td>1,2,3</td>
</tr>
<tr>
<td>Doug and Sal</td>
<td>Average/Average</td>
<td>Audio</td>
<td>1,2,E1</td>
</tr>
<tr>
<td>Lou and Rob</td>
<td>Very good/Weak</td>
<td>Audio</td>
<td>3,4</td>
</tr>
</tbody>
</table>

Table 4.8.
Taped pairs in class C by ability level, taping method and tasks attempted.

<table>
<thead>
<tr>
<th>Names (aliases)</th>
<th>Ability (based on teacher assessment)</th>
<th>Method</th>
<th>Tasks attempted during taping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sarah and Earl</td>
<td>Good/Average</td>
<td>Video</td>
<td>1,2,3</td>
</tr>
<tr>
<td>Lily and Fran</td>
<td>Very good/Average</td>
<td>Video</td>
<td>1,2</td>
</tr>
<tr>
<td>Jan and Pam</td>
<td>Very good/Good</td>
<td>Video</td>
<td>3</td>
</tr>
<tr>
<td>Katy and Bea</td>
<td>Average/Average</td>
<td>Audio</td>
<td>1,2,3</td>
</tr>
<tr>
<td>Tara and Mary</td>
<td>Very good/Average</td>
<td>Audio</td>
<td>2</td>
</tr>
</tbody>
</table>
The Sketches and Labsheets

To prepare for the data collection segment of the study I prepared four JavaSketches for student use, and one to be used in a whole class lesson. In addition, I made two extra sketches in case students finished early and wanted to experiment. Each student sketch was accompanied by a labsheet containing instructions, questions, and space for answers.

Guidelines for Preparation

In deciding which problems to use as the basis for the sketches, I used these guidelines:

1. I chose problems that were similar in difficulty to those in the student text. They related to triangles and quadrilaterals, and were based on the theorems the students had studied.

2. I chose problems that could be done in at least two ways. Each of the diagrams included the possibility of arriving at a solution from a transformation perspective as well as from a straightforward application of congruency theorems.

Note: The 1985 Ontario curriculum, of which the students' grade twelve advanced program is a part, includes work on rotations (turns), reflections (flips), and translations (slides) in grade eight. In grade ten, students carry out simple transformations on the Cartesian plane (e.g., reflecting a triangle in the y-axis, rotating a triangle 90° about the origin) (Curriculum guideline, 1985, p. 50).

In designing the sketches and labsheets it was not enough to have chosen a question or problem. I needed to decide which details to focus on, to support students in their efforts to understand the concepts. Thus,

1. Each sketch addressed areas that teachers had identified in pre-session discussions as problems for their students. These areas included being able: to pick out smaller triangles within a larger diagram, to focus on the whole shape, to mentally separate overlapping shapes, to notice relevant details, to understand that rotated or reflected copies of a shape are congruent to the original. In order to accomplish these goals I,
   - used colour to highlight small triangles or pairs of triangles
   - provided capabilities to remove detail to allow students to focus on the whole shape
   - created motion buttons to allow overlapping shapes to separate and join
   - made hide/show buttons to add details as needed, and
   - provided shapes that could be reflected or rotated.
2. Each labsheet addressed other important skills for students, such as being able: to describe what they notice, to explain their thinking, to use the results of one proof in the next, to conjecture. Each question on the labsheets was intended to do at least one of the following:

- to focus student attention on details in the sketch
- to encourage students to explain their thinking in full sentences
- to help students move through an investigation by examining the evidence in the onscreen model, checking hypotheses, then considering other possibilities
- to write a proof.

A full discussion of the elements of each learning task is provided in the next section. Although the sketches were called jsketch1, jsketch2 and so forth, the tasks were named Day 1, task 1, Day 2, task 2, etc. For each task, two views of the sketch are provided in Appendix F. The accompanying labsheet (except for group sketch, jsketcha) is included in Appendix D. The sketches themselves may also be viewed online at:

http://fcis.oise.utoronto.ca/~masinclair/jsketch1.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketch2.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketch3.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketch4.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketch6.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketch7.htm
http://fcis.oise.utoronto.ca/~masinclair/jsketcha.htm

Note: Since JavaSketchpad is still in development there are some incompatibilities with various browsers. The best results will be obtained using Netscape Navigator.

Overview of Tasks

I will usually refer to the tasks by the names used in the sessions because it situates the activity in relation to the sessions. However, in any tables I will use the numbers 1 through 4 as set out in this section. The two extra sketches will be referred to as E1 and E2.

1. Day 1, task 1: Jsketch2.

A. Purpose

This sketch was developed to introduce students to JavaSketchpad, and specifically, to address student difficulties with overlapping figures and selection of triangles. It provided an opportunity to view the problem as an application of a reflection.
B. Design

In the sketch, ΔABC and ΔFCB are reflections of one another in the perpendicular bisector of BC, which can be toggled on or off using the "Show perpendicular through H" and "Hide" buttons. When the sketch is first opened, point A is red, indicating that it can be manipulated. As point A is dragged, point F undergoes opposite motion as a reflection should.

The two triangles ABC and FCB can be separated using an action button with label "Separate ABC and FCB," so students can examine the triangles apart. The button "Show reflection and mirror." reflects triangle ABC in a red mirror line. The button "Match FCB and A'B'C" causes triangle FCB to move on top of the reflection of triangle ABC, demonstrating congruency. A reset button is provided to move triangle FCB to its original position.

In addition, the "Show Given Information" button controls the display of the markings that indicate the equality of AB and FC, and ΔABC and ΔFCB, as well as the measures of these lengths and angles. As the figure is dragged, the measurements update. The measurements were included to help students notice that the two triangles remain congruent even when they change size, and thus to understand that this happens because they were constructed with particular relationships.

PROVE: Triangle ABC congruent to Triangle FCB

![Diagram of triangles and buttons]

Figure 4.1:

Day 1 - Task 1 - Jsketch 2 - View on selecting: "Show Given Information," "Separate ABC and FCB," "Show reflection and mirror," "Show perpendicular through H."
As a method of proof, students could use a straightforward application of SAS (side, angle, side) congruency after noting that BC was a common side. Alternatively, \( \triangle ABC \) is congruent to \( \triangle FCB \) because when the triangle is flipped over FC will lie on AB, CB will lie on BC and \( \angle FCB \) will lie on \( \angle ABC \). Thus, students could use the idea of reflection to explain the congruency.

The labsheet presented the problem: "Prove triangle ABC congruent to triangle FCB" It then led students through instructions to help them learn how to work with a JavaSketch. Students were informed that red points were "draggable", instructed how to access details with hide/show buttons, and told what to do if the sketch became too messed up. (Sometimes if a point is dragged too quickly or too far it is difficult to bring the sketch back to the original configuration. If this happens it is easier to reload the sketch.)

C. Expectations

Students were expected to begin noticing details and exploring the mathematics from the very beginning. They were told to "observe the diagram", to "drag point A", and to "explain the meaning of the tick marks and the angle shading". In addition, they were asked (a) to interpret the meaning of the image: "How can the information provided by these images be used to explain why \( \triangle ABC \) is congruent to \( \triangle FCB \)?" (b) to use deductive reasoning: "What additional information can you deduce about point H from the diagram?" and (c) to extend their experience: "Find another pair of congruent triangles in the figure". In all subsequent labsheets, the instructions and questions focused on encouraging the same student behaviours: explore, notice, interpret, deduce, and extend.

2. Day 1, task 2: Jsketch3.

A. Purpose

The second sketch also presented a problem, which could be viewed as an application of a reflection.

B. Design

The sketch was designed to address student difficulties with overlapping triangles, selection of triangles, and two step proofs. The sketch was constructed to help students:

1. Notice details:
   - the four chosen pairs of congruent triangles were shaded in four different colours;
   - given equal angles were shaded red;
   - information could be toggled off and on to allow details to stand out;
- triangle pairs could be separated;
- measurements for the given angles and lengths were displayed;
- measurements updated as the sketch was dragged.

2. Pick out a shape within the larger diagram:
- overlapping figures could be separated;
- colour was added to emphasise the shapes;
- colour was used to overlay angles and sides within the shape.

This task also used a form of rug-pulling, by including pairs of triangles that could not be proven congruent with the given information. This is not usually done in textbook problems.

In Day1, task 2, students were asked to prove that BA = BC. This could be done in several ways, the most straightforward being the proof that ΔABE is congruent to ΔCBD by ASA (angle, side, angle) congruency. However, in all cases students needed to deduce at least one piece of information from the given information before developing the proof. (i.e. all options involved at least two steps). For example, in the case just given, students needed to deduce that ∠DEF = ∠EDF by the isosceles triangle theorem, and that ∠BED + ∠DEF = ∠BDE + ∠EDF before employing ASA.

PROVE: BA = BC

```
Show Given information
Show Pair #1
Hide

m DF = 1.3 cm
m FE = 1.3 cm
m BED = 65°
m BDE = 65°

Show Copy
Hide copy
```

![Diagram](image)

**Figure 4.2:**

Day 1, task 2 - JSketch3 - View on selecting: "Show Given Information," "Show Copy," and "Show Pair #1."
All chosen pairs of triangles were reflections, and congruency could be established or not established by considering what would happen if one member of the pair was flipped over. For example, with $\triangle BDF$ and $\triangle BEF$, BE would lie on BD (equal sides of an isosceles triangle). EF would lie on DF (given equal) and BF would lie on BF (common sides). However, for pair #2, $\triangle BFA$ and $\triangle BFC$, only the behaviour of BF could be predicted.

C. Expectations

The labsheet for this task asked students to prove that $BA = BC$, but it asked them to consider several different methods of accomplishing this proof. It included a few instructions on using JavaSketches, (i.e. You may click Join at any time to show triangles together) but the aim was to help students become immersed in the mathematics. For example, the questions encouraged them to:

1. Explore: Drag each red point...
2. Notice: …observe the measurements.
3. Interpret: Write two additional facts that you know and explain why they are true.
4. Deduce: If you proved the pair congruent, how would this help you prove $BA= BC$?
5. Extend: What is an alternative explanation for the congruency of triangle ABC and triangle FCB?

3. Day 2, task 1: Jsketch1.

A. Purpose

This sketch was constructed so that students could work on a proof involving a quadrilateral. It allowed students to apply properties of parallel lines and provided an opportunity to solve a problem using a rotation.

B. Design

The triangles to be proven congruent were coloured to attract student attention. When the quadrilateral was dragged, AD and BC appeared to remain equal and parallel, as did AB and DC. When the "Show Given Information" button was used, students could deduce that $ABCD$ was indeed a parallelogram since opposite sides were marked equal.
To prove that $\triangle ABD$ and $\triangle CBN$ were congruent, students could use ASA congruency:

- $AD = BC$ (given),
- $\angle DAM = \angle BCN$ (parallel line law), and
- $\angle MDA = \angle NBC$ ($\angle AMD = \angle CNB = 90^\circ$, so the remaining angles in the triangles are equal).

They could also make use of the extra triangle that was provided. This triangle could be placed on top of $\triangle ABC$ to demonstrate that it was congruent to that triangle. As $\triangle ABC$ changed shape, the extra triangle changed as well to remain congruent. The extra triangle could then be rotated about point O (which lined up with the midpoint of AC), to lie on $\triangle ADC$. This demonstrated that AC split the parallelogram into two congruent triangles. The students could then deduce further that, since a triangle can only have one altitude from a particular vertex to the opposite side, AM must equal BN.

C. Expectations

In this labsheet, students received some instructions on handling the sketch: "It's easier if you have AC in a horizontal position". However, as usual, the labsheet encouraged students to:

1. Explore: Drag...Rotate...
2. Notice: What do you notice about the new triangle?
3. Interpret: What is the relationship between the new triangle and $\triangle ADC$?
4. Deduce: Use your conclusions...to prove that \( \triangle AMD \) is congruent to \( \triangle CNB \).

5. Extend: How can the information provided by these images be used to explain why \( DM = BN \)?


A. Purpose

This sketch was designed to allow students to investigate the question: "When do the diagonals of a parallelogram right bisect one another?"

B. Design

The sketch included parallelogram ABCD, with diagonals AC and BD. The opposite sides were marked with arrows, the traditional markings for parallel lines. Measurements of the sides, diagonals, and semi-diagonals could be toggled on or off using Show/Hide buttons.

Since it can be frustrating to drag an angle until the measurement is precisely 90°, I provided a line, perpendicular to AC, that was activated by the button "Show Perpendicular". This allowed students to drag the diagram until BD was aligned with the perpendicular - a slightly easier task.

Figure 4.4:

Day 2, task 2 - Jsketch4 - View on selecting: "Show Perpendicular," and "Show."
This sketch was undertaken after a short class session on what it means to conjecture. It attempted to give the students, not a statement to prove, but the chance to decide what that statement would be. Since the question involved the term "right bisector", the first section of the labsheet asked questions about the meaning of "bisect", "right bisect", and "right bisect one another". It then asked students to drag the diagram (explore, notice), and to conjecture a response to the original question (interpret). Once students had formed a conjecture the labsheet asked them to develop a proof of the conjecture (deduce) and then to outline an alternate proof (extend).

The group problem: Isketcha.

The group problem sketch consisted of nested isosceles triangles ABC and DBC, and segment AD which extended to meet BC at point E. The Show Given Information button toggled on the measurements of \( \angle ADC, \angle ADB, DC \) and DB, and displayed markings on the diagram for these measurements.

When using the sketch with Class A, the image was projected via a data projector. I was able to walk up to the screen to talk about the sketch and to manipulate the mouse to point to objects. The resolution was good and small details could be seen by students, except at the very back of the classroom. In Class C, the image was shown on a rather fuzzy monitor. It was too high for me to easily point to details and I found it difficult to refer to items because the labels were too small for students to see. Before using the sketch with Class B, I added coloured shading and several Show/Hide buttons. Although the monitor was the same, the discussion with Class B went much more smoothly. I was able to talk about "the blue triangle", the "red lines", and "the pink angle markings".

The initial aim of this problem was to go over triangle selection, and to discuss the various possible methods for proving that \( \triangle ABC \) was isosceles. The second was to demonstrate that shapes with the same givens retain their properties on dragging. I also wanted to ensure that students understood and could explain why line segment AE was perpendicular to BC.
PROVE: Triangle ABC is isosceles

**Figure 4.5:**

*Extra sketches: Jsketch4 and Jsketch7.*

Two additional tasks were provided to enable students who finished early to investigate other problems. They included colour, dragging capabilities and hide/show features as in the other sketches. I mention them here because some student pairs did work on them, albeit for a short time, and I make reference to their experiences in a later section.

The extra problem, Jsketch4, invited students to examine a sketch that looked like two isosceles triangles JCL and ECG. When the sketch was opened J, E, G, and L were collinear.

Students were able, using the Show More button, to examine a situation where these points might not be collinear, and thus to consider why it is important to clarify initial assumptions. If the points are collinear, then the two triangles are isosceles. If the points are not, then the outer triangle is not necessarily isosceles.
Figure 4.6:
Jsketch4 - View on selecting: "Show Given Information," "Show CN," "Show More."

The labsheet for this task included one special element. It provided an attached sheet on which were printed copies of the sketch. Students were asked to shade triangle pairs that could be used in the proof. This was a variation on the previous task in which the JavaSketch itself had shaded pairs of triangles for the students.

The extra problem, Jsketch7, presented a quadrilateral ABCD. When the given information was toggled on, the markings showed that the shape was a kite. In the sketch, diagonal AC was cut by segment EF, such that AE was equal to AF. Students were asked to prove that AC was the right bisector of EF.

This proof could be approached by proving ΔABC and ΔADC congruent by SSS to establish that ∠BAC = ∠DAC; then proving ΔEAP congruent to ΔFAP by SAS. Students could then deduce that EP = FP, and that EF was perpendicular to AC.

PROVE: Triangle CJL is isosceles

▲ Show Given information ▲ Hide

▲ Show CN ▲ Hide

m∠CEG = 59°
m∠CGE = 59°
JE = 2.7 cm
LG = 2.7 cm

▲ Show More ▲ Hide
PROVE: AC bisects EF and AC is perpendicular to EF.

\[ m \overline{BA} = 3.9 \text{ cm} \]
\[ m \overline{AD} = 3.9 \text{ cm} \]
\[ m \overline{CD} = 3.4 \text{ cm} \]
\[ m \overline{BC} = 3.4 \text{ cm} \]

\[ m \overline{FD} = 1.9 \text{ cm} \]
\[ m \overline{EB} = 1.9 \text{ cm} \]

Figure 4.7: Sketch - View on selecting: "Show Given Information #1," "Show Given Information #2," "Show Triangles."

Alternatively, students could develop a proof using reflections, by first noting that triangles ABC and ADC are identical because they have three pairs of equal sides. If AC is used as a mirror, AD falls on AB, so AF falls on AE. Therefore point F will fall on point E. But if E is the mirror image of point F, then the segment joining them must be perpendicular to the mirror and bisected by the mirror. This group of students would have met these facts about mirrors both in science and also in transformation work in the grade eleven mathematics program, although they would not have had experience writing a geometric proof based on symmetry concepts.

Summary

This chapter introduced the study participants and outlined the learning materials and settings. Details of each learning situation were presented—from the aim of the task, to the questions on the labsheet, to the design of the sketch. The schools, teachers and classes were introduced to familiarise the reader with the setting of the study.
Chapter Five

Observations

This chapter describes some of the observations regarding students and their interactions with student partners, teacher, sketches, and task labsheets during the sessions. I introduce the emergent themes and illustrate these using excerpts from the responses of the taped pairs.

In the next chapter this observation data forms the basis for answers to the research questions posed in chapter one.

**Theme 1: Student**

In the examination of the tapes several issues emerged with respect to the individual student and what he or she brought to the sessions.

**Difficulty with Terms**

Bisect means cuts in half you know. Wait. DC--DB bisects AC [pointing--spreads fingers to span each segment] it means it cuts it in half, right? Yeah. So that's one half—that's one half.

(Pat and Dave, March 29, 2000)

This quote is a rare example of a student in the study who had a firm understanding of the word *bisect*. In the following exchange, students thought that bisector always signified "angle bisector". In this case they should have considered GH a bisector of segment BC.

Oh, Ok, angle GBH is equal to angle GCH because this is a right angle triangle. This dissect[sic] this. And these are equal. I'm not sure. If you bisect it—wouldn't it be equal anyways because it's a bisector? Depends on the angle though—on the measurements of the angle.

(Ray and Owen, Day 1, task 1, April 3, 2000)

The transcripts revealed that many students had difficulty with basic geometric terms such as common, half, equidistant, bisect. Here are just two additional examples:
We proved that the parallelogram is equidistant from one another.

(Barb and Clara, Day 2, task 1, March 29, 2000)

DB bisects that—the first line, so that means the two halves of that line are equal, but the other two aren't.

(Student comment during group lesson in class A, March 29, 2000)

In response to, "Click Show Perpendicular through H. What additional information can you deduce about point H from the diagram?" (Day 1, task 1), Pat and Dave's lack of precision in geometric language hampered their progress.

Pat: What additional information?
Dave: G is the midpoint
Pat: No H is the midpoint. G is the line in the middle.
Dave: I'm going to write 'H is the line that crosses the midpoint'.
Pat: Is H the line or the point?
Dave: H is the line that crosses the midpoint.
Pat: H is a point, not a line.

(Pat and Dave, March 28, 2000)

After this interchange Dave appealed for help from the teacher and discovered that Pat was correct (although Pat himself had called G 'a line' earlier in the conversation).

This pair's poor grasp of geometric concepts meant that they progressed very slowly through the task questions. Nevertheless, I contend that Pat and Dave profited from the experience of using dynamic software because it helped them correct misconceptions. I particularly like the following exchange between them. I have included comments at the side to note particular behaviours and attitudes.

Pat: No, it says rotate....
Dave: How do you rotate it? You can't unless it's round. You can only rotate it--
Pat: Oh, it rotates on one point. ...
Dave: Yeah so—so it stays in one point.
Pat: It goes in a circle. It goes around the midpoint.
Dave: Yeah, it goes in a circle.
Pat: And doesn't stretch.
Dave: It goes in a circular motion.
Pat: It goes around the midpoint and doesn't stretch.
Dave: How do you know it's a midpoint? How do you know it's a midpoint? Huh? And why does this keep on going? It goes around on one point, right?

Pat: Yeah.

[They gaze intently at the screen]

(Pat and Dave, March 29, 2000)

**Difficulty Communicating Ideas**

The following interchange reveals how even simple concepts could prove complicated once students were removed from the familiar textbook-driven milieu and asked to explain. It is in response to: "Use your conclusions and the appropriate congruency theorem (SSS, SAS or ASA) to prove that \( \Delta AMD \) is congruent to \( \Delta CNB \)" (Day 2, task 1).

Lou: Can't do angle, angle, side. It has to be angle, side, angle.....If you have a triangle--if you have a triangle, right? This--you can't do--you can't do angle and then this side and then this angle. It has to be angle, side, angle.

Rob: I'll show you what I'm talking about.

Lou: Angle, side, angle, or side, side, side...it has to be--

Rob: It doesn't matter. I'll show you guys what I'm talking about.

Lou: I'm almost positive it does.

Rob: Look at this--this is what we're given. As long as they have three things the same--

Lou: Wait, wait, wait. But what you're saying--if you have--if you have two angles--if you have three things the same in a triangle--in two triangles and three things are the same it doesn't matter which order they come in--where they are.

...Are you sure?

Rob: If you flip it--I'm telling you, I'm telling you.

Lou: I'm pretty sure it's two sides. This is what I'm talking about. See this triangle ABC? And triangle ADC?

Rob: But see, now you've just taken an extra step--and you don't need to do that triangle. You're taking extra steps. You don't need to do the triangle that way. You could do that angle right there or that angle. Or actually that's cutting it in half. That's a--that's an angle right there. See how it goes like this--like a zed? Or like an N? That angle up there? Is--angle A is equal to angle C. I'm telling you. It's just parallel.

Lou: I hear that. I know that.
Rob: That's true, and then you already have your 90's right there and you have your--your third--your side. Angle, side, angle.

Lou: That--No, but see, isn't that angle. angle. side? Like that. I dunno if you can do that.

Rob: I'm telling you. You can do it. It's the same thing as angle, side, angle, isn't it?

Lou: It's not though because I--I dunno. Think about it. What are you saying?

Rob: Because no matter where you put the angle it is like as long as--OK if we prove that this angle's equal to that angle cause it's given and that angle's equal to that angle because of the parallel--the Z theorem--whatever. then--Would you say? Wouldn't that angle have to be equal to that angle?

Lou: If we already proved that and that?

Rob: Yeah

Lou: Yeah

Rob: So then

Lou: Of course.

Rob: We have angle, side, angle.

Lou: Now we have to write it!

(Lou and Rob, April 5, 2000)

This exchange illustrates that students were eager to argue and discuss, but it is just one of many in the transcripts that demonstrates the struggle many students experienced trying to communicate their thinking on relatively simple concepts. In fact, most of the students in the study were clearly operating at a 'superficial' formal level--discussing angle, side, angle and developing congruency proofs, but lacking a basic understanding of geometric relationships.

**Theme 2: Student and Student Partner**

An important theme in the data was the relationship between student and student partner. Students in all three classes were unaccustomed to working in pairs to learn concepts. The teachers, as noted earlier, usually taught the material in a traditional lesson format; nevertheless, the industrious atmosphere in the classes, mentioned by all three teachers in their interviews, showed that students enjoyed working in pairs.

**General**

Students were usually focused on the work but there was some off-topic chatter by all pairs. In transcribing, I ignored these conversations.
The transcripts of the conversations for boy/boy, girl/girl and girl/boy pairs were approximately the same length, with three exceptions. Doug and Sal's transcript was almost twice as long as other pairs. They verbalised almost non-stop, and engaged in very little off-topic chatter. Another exception was a girl/girl pair that was quiet for extended periods of time. Beth and Kim, the girls in this pair, gazed at the screen, or wrote on their labsheet, but communicated very little. In addition, as mentioned earlier, one other pair of girls, Jan and Pam, talked in such low tones that I was unable to use their discussion in the analysis.

**Beneficial Interactions**

In annotating the tapes I observed that students not only worked together, but often intervened with one another as a teacher might. Some students were very patient; others ignored their partner's frustration. One notable trait of student pair, Sue and Paul, was the way they often acted in concert with one another. (I also noticed this behaviour with a pair of very good students in class C.) Sue and Paul can be seen on videotape gazing intently at the sketch as it moves, spending time to take in the detail, then working rapidly—correcting one another, giving feedback, questioning, even occasionally completing one another's sentences as shown in the following excerpt.

Sue: [reading] Use your observations and the appropriate congruency theorem to prove that—
Paul: So we just have to do—Angle F. Separate. It's easier. OK. angle FCB is equal to angle ABC.
Sue: What, what, what? Angle FCB right?
Paul: Yeah. Angle B—line
Sue: Line FC, right?
Paul: And angle BC is common
Sue: Right
Paul: Or not angle BC—line BC
...
Paul: Ok, so we have side, angle, side. Therefore triangle ABC is congruent to
Sue: FCB
Paul: FCB by
Sue: Side, angle, side.

(Paul and Sue, March 28, 2000)

In a later exchange Paul uses a simple illustration to help Sue grasp the form of deductive reasoning.

Sue: What's an If . . . then statement?
Paul: If yadayadayada, then yadayadayada.
Sue: Oh. ok.

(Paul and Sue, March 29, 2000)

In the next excerpt, Barb is confused by the need to mentally separate nested angles—a difficulty that was shared by other student pairs. Clara intervenes as follows:

Barb: Angle A equals angle C.
Barb: Angle A equals angle C.
Clara: You can't do that because this is angle C too.
Barb: Oh yeah. Angle A1 equals angle C1. Would that be?
Clara: Because of alternate parallel line theorem.
Barb: Hold on. Hold on.
Clara: Ok. we'll just say angle A1 equals angle C1.
Barb: So what? So just cause these are equal doesn't—Oh yeah. All right. all right.
Clara: Ok?

(Barb and Clara, March 29, 2000)

In this exchange, Barb shows that she understands the subtle point when she says, "Oh yeah. All right, all right". but it has taken her some time and Clara is clearly ready to explain again if Barb doesn't quite get it.

The teachers and I often tried to help pairs that were confused or stumped, by demonstrating how to go through possibilities systematically, or by restating known information. Students imitated these strategies later to organise their thinking. An example of this is:

Doug: Do we have 3 pieces of information to decide if it's congruent? We have an angle. We have--we have a side. First of all we have an angle and a side. Ok, the D angle--so we have BDF, right? Is congruent to BEF--that's what we're trying to prove.

(Doug and Sal, April 3, 2000)

**Modelling Ways of Seeing**

Most students were willing to help their partners but had trouble with the material. Paul, a student in class A, was an exception. In the exchange below, Paul realises that a previous explanation to help his partner, Sue, hasn't had the desired effect so he tries to model for her his method of seeing. This is quite a wonderful effort on his part but she is so wrapped up in her attempt to make sense of the ideas that she misses its power.
Another example is seen in this excerpt from Lou and Rob. Here Rob has grasped the idea that one can have equal angles even if the triangles aren't the same size. He models it aloud—probably for himself as well as for Lou. This is a nice illustration of Rob's ability to use mental visualisation.

**Difficulties**

Sometimes peer-provided information was accurate; sometimes it was completely wrong. Some peer interactions left students with misunderstandings of subtle points, others caused students to argue in circles until they gave up in frustration or a teacher arrived to help.

The next exchange between Barb and Clara, about Day 2, task 1, results in a correct response but leaves Clara with a misconception about terminology.
Barb: Angle M equals angle N, given.
Clara: Angle, side, AD and BC, given.

(Correct use of 'given'

(Barb and Clara, March 28, 2000)

At one point in this exchange Barb thinks aloud about a known theorem about right triangles—the hypotenuse, side, congruency theorem. She has noted the markings on the sketch to indicate that M and N are right angles and she attempts to think back to remember, but Clara ignores this attempt. Barb does not appear to notice Clara’s slip about M and N being common angles and writes (correctly) that these angles are equal because they are given.

In the next excerpt, Doug has the same problem seeing the whole triangle that Sue did in an earlier example; however, Sue had the advantage of a partner who tried to help her visualize. Sal hesitates, as if he knows that Doug is wrong, but then retreats.

Sal: So we have—what?
Doug: We have FD.
Sal: No, no that's one big triangle.
Sal: This splits the triangle in half, right? So, FD is a common? A common side. They have a common side here, right?
Sal: But F's not—Oh. All right.

(Doug and Sal, Day 1, task 1, April 3, 2000)

Unfortunately, no teacher arrived at this point, so the students based their conclusions on this line of reasoning, unaware of the error.

**Theme 3: Student and Sketch**

A central theme in the data was the visual image—how students reacted to it, and used it.

**Student Reactions to the Visual Image**

When using the *JavaSketches* students often talked in animated fashion and expressed excitement. Here are a few (unconnected) comments from among the 25 annotated on the tapes.

- Ray: Oh, wicked! Wild stuff.
- Ray: Watch this!
- Sarah: Whooo —Hey cool [the triangles separate].
- Clara: Oh look, wow, wow, wow! They're connected!
- Clara: Oh it could be in—Oh, wow it goes to O!
- Clara: Oh wow—roooootate it! That's so cool.
- Clara: This program is fun. I understand it
- Doug: Ooh...it moves
- Lou: It just goes together. Wow! Did you see that? Wow!

The student manipulating the mouse was the most likely to get excited. The writing partner sometimes had to get the pair back on task. To illustrate, Barb says:

All right. Anyway, we're getting kind of carried away. All right, just leave it. All right, what do we see?

(Barb and Clara, March 29, 2000)

The program hooked students at both ends of the spectrum. Teacher A, commented:

"Two girls at the back [who] never do anything--were actually working" (interview, March 29, 2000), and Sue and Paul, two excellent mathematics students were openly enthusiastic about the program and its capabilities. When working on Day 1, task 1, Paul says, "Separate. Hide reflection? Oh that's wicked!"

**Student Use of the Onscreen Image**

All videotaped students can be seen gazing at, poking at, drawing on, pointing to, and playing with the onscreen image. I annotated 16 segments in which students actually traced on the screen—six of them with a pencil!

**Gazing, pointing, tracing.**

All taped pairs demonstrated three common responses to the onscreen image—gazing, pointing and tracing. Here are some annotated excerpts from various pairs to illustrate this behaviour.

<table>
<thead>
<tr>
<th>Then--this equals this--that equals that--and that equals that.</th>
<th>Sue points to sketch. She keeps tracing on the screen talking to herself.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Paul and Sue, March 28, 2000)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>[Silence]</th>
<th>Pat gazes at the sketch for several seconds He points to sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>That's a 90 degrees. Angle M is equal to angle N.</td>
<td>(Pat and Dave, March 29, 2000)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Like--this is the same. This is the same--right? This is shorter. This is the same as this but this is not the same as this.</th>
<th>Katy points to sketch as she talks.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Katy and Bea, April 3, 2000)</td>
<td></td>
</tr>
</tbody>
</table>
Playing.
Another common response was playing with the sketch. I use this term to denote students engaged in some or all of: dragging, using action buttons, restarting. The following excerpts illustrate three different types: play to explore new possibilities, play to investigate a question, and aimless play.

In Day 1, task 1. Sue and Paul jabbed at the screen as they referred to items, using the familiar geometry terms correctly; however, they didn't just leave the sketch static. They continued to play with the sketch and explore new possibilities.

Paul: GBH is congruent to GCH because angle, angle, given, common. common
Sue: So wait a second. Side, side--no side, angle, side
Paul: Angle B and angle C are equal and then H is the midpoint
Paul: It's the right bisector too

Points to the sketch
Focused attention
Models reasoning
Discovers a new idea by playing with the sketch

(Paul and Sue, March 28, 2000)

Other students played to investigate a particular question as seen in this quote from Sarah:
I feel obligated that there must be 3 pieces of information given.
Let's play with it and maybe we'll see.

(Sarah and Earl, April 5, 2000)

Others, as this excerpt from Doug and Sal's tape shows, played just to see what would happen.

[Serious, interested, on task. While one wrote, the other played with the sketch.]
"You can really mess this one up. Look what I did!"

(Doug and Sal, April 3, 2000)

The Sketch as a Shared Image
Students treated the sketch as a shared object. Joe and Bob made this explicit in the following interchange:

Bob: Put 'therefore triangle--this one, is equal to triangle'--
Joe: Triangle this one? Eh?
Bob: You're looking at the screen too
I observed that students quickly adopted this new visual exploration tool. It gave focus to their investigation and support for their use of mathematical language. For example, in response to, "How can the information provided by these images be used to explain why $\Delta$ABC is congruent to $\Delta$FCB?" (Day 1, task 1), Doug and Sal used the image to illustrate their discussion.

Doug: Because the angle and side are equal. Therefore—
Sal: Try this. [unsure] Match FCB with A'B'C'.
Doug: Ooh—it moves.
Sal: There we go.
Doug: [laughing] Do it again.
Sal: Ooh. There we go.
Doug: See--so we separated them and we matched...
Sal: So--[reading] How can this information provided by those images show why they're congruent?
Doug: Why? As I say--you know FC. FC equals A--
Sal: AB
Doug: Right and you have angle ABC congruent to...
Sal: FCB

Uses sketch affordance
Uses redo capability
Instructions followed
Reviews his reasoning
Shows collaboration

(K Doug and Sal. April 3. 2000)

Katy and Bea responded to, "What shape is ABCD? Explain" (Day 2, task 1), by referring to details of the sketch as follows:

Katy: We have to explain this.
Bea: Because--um. because two sides are equal.
Katy: Opposite sides are equal.
Bea: Yep
Katy: Because opposite sides are equal and there's no 90 degree angle..
Bea: Actually there is a 90--uh
Katy: Just in the little triangle
[Katy writes on the labsheet] Because opposite sides are equal and there's no 90 degree angle.

Corrects partner
Poorly formed concept of parallelogram
Notices visual detail

(Katy and Bea. April 5, 2000)

In this excerpt we can see that Katy held a misconception about parallelograms. Because students wrote explanations on the labsheets, the teacher could address this problem later with Katy or with the whole class if necessary.

Most taped pairs became deeply involved in discussions while manipulating the shared image. Here Doug and Sal discuss as they explore together:
Doug: It just means they’re equivalent right?
Sal: Ya, these tick marks mean they’re equivalent.
Doug: No matter how much you change the points they move. [pause] Hey look at this. [He is quiet while experimenting with the sketch]
Doug: Tick marks and shading mean that they are equivalent…
Sal: And the shading, right? Is the angles--those are angles.
Doug: Right.
Sal: We’re just observing--
Doug: I did A, and it is—it is moving

(Doug and Sal, Day 1, task 1, April 3, 2000)

Student Use of Sketch Affordances

The sketches were designed to attract student attention to details through the use of program affordances such as colour, hide/show buttons that displayed measurement information, or action buttons that highlighted and moved shapes or allowed the sketch to be reset. In addition, points could be dragged to allow students to investigate which properties of the sketch were invariant. Students were enthusiastic about these capabilities as shown by over fifty positive comments on the questionnaires (see Table 6.3).

In the following example, the program provided the capability to manipulate the sketch and to toggle the colour off and on. On listening to the tape one can hear the insistence and focus in Clara’s voice when she says:

OK change it so the colour comes back in. Stretch out the sides. Go here. Drag it.
Who cares about that, just go here and drag it. You go here and drag it. What are you doing Barb? Where’s the colour? Oh, show given information.

(Barb and Clara, March 29, 2000)

Provision of onscreen measurements.

Students may have noticed the measurement data provided by the hide/show buttons but it was mentioned explicitly by only two student pairs. However, it can be assumed that when students made comments such as the following, they were basing their conclusions on the measurement data, since in these cases, tick marks were not included.

Well this equals—yeah, they’re pretty close…

Oh the sides are equal [gazing]

(Barb and Clara, March 29, 2000)

(Pat and Dave, March 29, 2000)
**Action buttons.**

Three actions that were very popular with the students were the flipping of the triangles in Day 1, task 1, the separating and highlighting of congruent pairs in Day 1, task 2, and the rotation in Day 2, task 1. All taped pairs repeatedly used these action buttons. Many positive student comments on the questionnaires (see Table 6.3 and 6.4) mentioned that being able to separate triangles, see shapes from a different angle, or highlight triangles helped them to understand.

The following excerpts are just a few of over 50 annotated examples related to these action buttons.

Try this. Match FCB with A'B'C'.

(Doug and Sal, April 3, 2000)

They're the same. We're still looking for a third. OK reset it. Oh, see how they're sharing a side. It has to be equal.

(Katy and Bea)

What pairs? AHHHH Go up a bit. OK go show pair one.

(Paul and Sue, March 28, 2000)

Let me see it separated OK?

(Pat and Dave, March 29, 2000)

[Comment on congruency of two triangles after using rotation.]

Because the triangle fits—the triangle fits both.

(Barb and Clara, March 29, 2000)

That's what I thought we had. So basically what you said, but we didn't see it until we put it together again.

(Katy and Bea, April 3, 2000)

The mirror effect shows that ABC just like FCB—shows that it's the same triangle.

(Pat and Dave, March 28, 2000)

No but listen—the reflection—show the reflection. This angle FCB is a reflection of ABC. Therefore it has to be equal—equal because it's a reflection.

(Katy and Bea, April 3, 2000)

**Redo capabilities.**

Being able to redo, tidy up and start again were frequently used features. Here are some (unconnected) examples from the 26 annotated on the tapes:
Clara: "OK let's get rid of this. Show triangle it says. Click show triangle."
Katy: "That's what I thought we had. So basically what you said, but we didn't see
it until we put it together again."
Owen: "Wait. Bring the tick marks back."
Sarah: "Hide them all. Start again."
Earl: "Put it together again. Separate it again."
Pat: "Let's go step by step. Go back. Go to the first one. Now go to the second one.

**Tendency to quit dragging.**

Although students frequently used the hide/show capabilities, and the separate
commands, it can be seen on the tapes that they usually avoided dragging the figures once they
began to talk about the answers to questions.

This behaviour was especially obvious in Day 1, task 2. The inner pairs of congruent
triangles could be highlighted, and the sets moved apart to view separately. All students
repeatedly used these capabilities. However, the main triangle could also be deformed by
dragging. Despite the fact that students were initially asked to drag the vertices and observe the
changes, once they began discussing the congruency of the sets no students repeated the
dragging process. I deliberately moved triangles for several student pairs to help them see that
certain relationships they may have deduced (e.g., the equality of two segments) were not
invariant.

In analysing the tapes I found the following example, in which abandoning dragging led
to an erroneous conclusion. Doug and Sal are looking at the triangle in Day 1, task 2. Angle BEA
may have been close to a right angle on their diagram, but if they had dragged the sketch they
would have seen it change.

Doug: But this is not according to the sheet. BEA--angle E is 90 degrees..
Sal: There's no thing [referring to the symbol]
Doug: Well, you can't see it.
Sal: That's right
Doug: Well, I'm thinking this is an[sic]--cause it looks like it, right?

(Doug and Sal, April 3, 2000)
Student Desire to Add Details

While some affordances were available to support student actions, others were needed. The program suffered from the inability to add details. In the following excerpt it is clear that Pat and Dave want to be able to mark the diagram as a memory aid, while they talk.

Dave: We need to know. Ok we have angle--
Pat: Angle B equals angle D because--
Dave: You know what they should do? When we find out an angle we should write it in and then they should like--
Pat: By circling it--
Dave: Ya. We should be able to draw it and like erase. Like you would have.

(Pat and Dave, March 29, 2000)

Several times in the sessions, Sue appeared frustrated by her inability to write onscreen, although her partner Paul did not share this attitude. For example, she said: "Wish we could type in our answers," and "I need to draw on something. That's why I prefer paper--so I can draw on it." Paul and Sue were one of only five pairs that did draw some pencil diagrams on their submitted labsheets. In the brief conversation below we see one problem with their shared paper diagram.

Sue: What are you doing?
Paul: Erasing your lines...
Sue: Why are you erasing them?
Paul: Because I don't want them there.

(Paul and Sue, March 28, 2000)

In order to tidy up the pencil sketch, lines or labels had to be erased whereas in the JavaSketch environment the screen could be reset at any time to allow a fresh start.

Paper sketches have some disadvantages but they do provide a record of conclusions. During interactions with pairs I found that most students were intent on analysing the onscreen image and did not make a paper sketch, despite having the materials at hand. Consequently, they sometimes forgot what they had decided only moments earlier. I speculate that this problem could be alleviated if students could record their results using onscreen notations.
Using the Sketch to Reason

It is not enough to notice details. Once students have focused their attention on a particular object they need to do something meaningful. The JavaSketches provided dragging capabilities, onscreen data, accurate representations, and familiar labels and marks to support student investigation of the problem.

The sketches could be used in two ways—as static diagrams, or as dynamic images. In the static form each sketch could depict a general case or a special case. By using the dragging affordance to check invariant properties students could check whether their conclusions about the sketch were valid. In its dynamic form, a sketch allowed students to combine an initial guess about what looks true with a search over cases. As noted earlier, although initially intrigued by the ability to drag points, many students quit dragging and reasoned with the static form augmented by information provided by the action buttons.

Here are some observations on how students used the JavaSketches to reason about the tasks.

Conjecturing.

Students had no experience developing their own conjectures. During a brief session in each class students were introduced to the idea of conjecturing. I used the isosceles triangle theorem, with which students were familiar, to demonstrate how to construct an if...then statement. I presented a conjecture as a guess, which then needed to be proved (i.e., they needed to explain why it was true).

Paul and Sue provide an example of a pair that made a conjecture and discovered their error by using the JavaSketch. They had decided the following:

Sue: Ok, if DB right bisects AC then the parallelogram will become a square.
Paul: Two diagonals bisect each other at right angles then the parallelogram becomes a square.

(Paul and Sue, Day 2, task 2, March 29, 2000)

Sometime later, after dragging the sketch they were rather surprised to find that their conclusion was not true:

Sue: Obviously it's not a square now...it's a parallelogram
Paul: Still a parallelogram--so we were wrong
Sue: So we have to erase all of it
Paul: Let's take a look if the sides are all equal

(Paul and Sue, Day 2, task 2, March 29, 2000)
Some students had difficulty developing a conjecture about the actual problem. One pair provides an example: Day 2, task 2 asked students to conjecture a response to: "When do the diagonals of a parallelogram right bisect one another?" I expected students to write a statement similar to the following: "If all sides of a parallelogram are equal, then the diagonals right bisect one another." In responding to this task, Barb and Clara kept falling back on: "If AC right bisects DB, then AH equals HC," focusing on a part, rather than looking at the entire shape, and thus not conjecturing a result about the parallelogram.

Barb: Miss, we still don't get this.
Me: OK. Now, what I'd like you to do is drag the parallelogram until DB right bisects AC, and then tell me what you think is true about the parallelogram. Conjecture what on earth is true if the parallelogram has—diagonals that right bisect one another. What do you think would be true of the parallelogram if they did?
Clara: The sides would be equal?
Me: Do you think so? OK, why don't you experiment and see.

Barb: If AC right bisects DB, then DH equals BH and AH equals CH. All right, we're right. OK. Miss, is this right?
If AC right bisects DB, then DH equals BH and AH equals CH.
Me: Yes. That's true, but ... I want you to conjecture an answer to this: When do the diagonals of a parallelogram right bisect one another?
Barb: They right bisect one another when.
Me: When, when,
Barb: This equals—Do I just reverse that answer here?
Me: Um, DH equals BH. But, no, I want to know about the parallelogram. What—
Clara: Oh, all the sides are equal.

Me: OK, is that what you found? When it right bisects it, you found that all the sides are equal?
Barb: Well this equals—Yeah, they're pretty close.
Clara: When the diagonals right bisect each other,
Barb: If the diagonals right bisect each other then they're like—Opposite sides are equal.
Clara: All sides are equal

(Barb and Clara, Day 2, task 2, March 29, 2000)
Student difficulties with conjecturing may be explained by the observations that follow, which are related to 1) using "looks like" evidence, and 2) being able to switch between seeing the whole shape and seeing a figure within.

**Using "looks like" evidence.**

Although the appearance of a diagram can act as the impetus for further exploration, students cannot assume the truth of a statement based on inspection without recourse to reason. In the following, Paul demonstrates that he understands the distinction between assumptions and deductions:

Sue: [r] Observe the measurements. Write two additional facts that you know.
Paul: Ok. Segment DF equals that. Obviously DF is equal to FE—it's—
Sue: Given. Angle BED is equal to angle BDE—that's given. Something else that's given—
Paul: No that is not given. Write two additional facts that you know.
Sue: Ok that we know. Well, this has got to be—[pointing]—this looks equal—
Paul: Oh yeah, that's good, let's just make assumptions.

*(Paul and Sue, Day 1, task 2, March 28, 2000)*

Although all students had been exposed to deductive reasoning, there are many instances of students concluding without recourse to deductive logic. In the exchange that follows, Barb and Clara are working on Day 2, task 1. They have just read that they are to drag each of the red points A, B, and C and observe the diagram.

Barb: Drag the red dots.
Clara: What is that—a rectangle?
Barb: Looks like it.
Clara: Yeah.
Barb: It's still a rectangle?
Clara: It's still a rectangle.

Barb: Bring it in.
Clara: Parallelogram?
Clara: Like, what if we can make it a square? You can make a square.
Barb: Not equal
Clara: It always remains.

*(Note: no marked right angle)*
Barb: It's some sort of a quadrilateral.
Clara: It's um
Barb: It's a--It's a--
Clara: Parallelogram
Barb: Parallelogram. hold on. cause it's slanted. But maybe it's just slanted
Because--ok. make it straight.
Barb: Bring it down and drag A and B over there.
Clara: It is a parallelogram.
Barb: Wait. let me see it. Yeah it is. It's a parallelogram.
There it is. It's a parallelogram.

(Barb and Clara, March 29, 2000)

The labsheet instructions had invited Barb and Clara to drag and observe. The reference to red points--and specifically to the labels A, B, and C were meant to focus student attention on the visual detail. In this exchange. Barb and Clara freely dragged the figure. but interestingly did not note specifics. Rather, they looked at the entire figure. categorising it alternately as rectangle. quadrilateral. and parallelogram--not because they had systematically examined measurements or relationships between sides and angles as they dragged, but because the figure conformed to their conception of what a particular shape should look like. This is shown when Barb referred to the shape being a parallelogram because it was "slanted". On the labsheet the girls entered that the shape was a parallelogram but ignored the requirement to explain (which was in boldface type).

The tendency to conclude based solely on "looks like" evidence from the static form was common. I annotated over 30 examples of conclusions that seemed to come out of thin air. Here are some (unconnected) samples in which students explicitly gave this as a reason:

Barb: "I wrote that opposite angles will be equal and so are the triangles because they look the same."

(Barb and Clara, March 29, 2000)

Doug: "Angle E is 90 degrees. Well. I'm thinking this is--cause it looks like it, right?"

(Doug and Sai, April 3, 2000)

Sue: "Ok that we know. Well. this has got to be-- [pointing] this looks equal."

(Paul and Sue, March 28, 2000)

Some students did question one another on the use of "looks like", although they were usually tentative. Here are two examples:

Bea: "Doesn't that DM equal BN? How did you prove that? I mean I can tell looking at it. yeah. but--"

(Katy and Bea, April 3, 2000)
Owen: "How do you know that H is in the middle? Well it looks like it, but..."

(Ray and Owen. April 3. 2000)

However, in the following exchange, Paul demonstrates that he understands the need for evidence in deductive proof.

Sue: It could be side, side, side.
Paul: Where's the side, side, side?
Sue: [pointing] side, side, side.
Paul: Who says that's a bisector?
Sue: It is a bisector
Paul: Does it say it's a bisector?
Sue: You know it's a bisector. If it goes from one point here to one point there.
Paul: All right, but it might be a little bit off.

(Paul and Sue, March 28, 2000)

In contrast to these examples, I was surprised to observe that students sometimes made conclusions that were in direct conflict with the visual evidence. An example is Barb's observation, "Maybe cause it's slanted you can't tell it's a square." I hypothesised that this response might stem from her prior use of textbook diagrams. Geometry teachers frequently warn their students not to make conclusions based on the appearance of these diagrams because they are not necessarily accurate. Teachers in future may need to revise this approach as they move towards greater use of accurate dynamic images.

*Changing focus: Whole versus part.*

But that's an angle within the triangle. We want the whole triangle. What does that little angle prove? It just proves that on the little triangle they're equal but we want the big triangle. You get what I mean?

(Doug and Sal. April 3, 2000)

This excerpt illustrates student difficulty with focusing on the whole triangle. They needed to change their focus from looking at an angle or side that was within a triangle to those that made up the main triangle. In the next excerpt, Paul has correctly determined that Sue's problem is her inability to see the whole picture. She is concentrating on trying to deduce information about items extraneous to the problem. The conversation is very interesting because it is an example of a student attempting to help another learn to change focus—albeit impatiently.

Sue: OK second pair. Do you have 3 pieces of information?
Paul: NO
Sue: UM [she moves the diagram and looks at the screen]. Ok [she uses a pencil to point to the screen] DF

*Uses sketch affordances*  
*Refers to the visual image*
During the sessions I helped many other pairs who also had difficulty focusing on the whole, as shown in an earlier section when Barb and Clara had a similar problem while trying to develop a conjecture.

Using deductive reasoning.

OK no that's true if you think about it. OK if we prove that this angle's equal to that angle cause it's given and that angle's equal to that angle because of the parallel--the Z theorem, whatever, then--would you say? Wouldn't that angle have to be equal to that angle?

Students had been introduced to deductive reasoning through previous work in grade 10 and their brief review before the sessions. Evidence in taped segments, such as the example above, demonstrates that some students grasped the form although most were weak in understanding of background concepts.

The next four excerpts give evidence to show how the affordances provided by the program were used by students to deduce results.

1. In response to, "Use your observations and the appropriate congruency theorem (SSS, SAS or ASA) to prove that ΔABC is congruent to ΔFCB" (Day 1, task 1):

Katy: They're the same. We're still looking for a third. OK reset it. OH see how they're sharing a side? It has to be equal.
Bea: BC?
Katy: BC. Cause they share the side--so it has to be the same length.

(Katy and Bea, April 3, 2000)
2. In response to, "Do you have 3 pieces of information to decide whether the pair is congruent" (fourth pair, Day 1, task 2):

Sarah: Angle B equals angle B since the angle is common.
Earl: How do you know that?
Sarah: We can put them together.  

(Sarah and Earl, April 5, 2000)

3. In several of their responses to prompting questions on the day 2, task 1 labsheet, Pat and Dave show how the image affected their investigation. The following are disconnected snippets of their conversation:

Pat: It's a rectangle--ABCD
Dave: This equals that and this equals that. [He points to each side.]  

Uses the static diagram as a reference

Pat: AMD matches B--
Dave: CNB--the coloured ones, right? [He hits each vertex of each triangle on the screen with his pencil to show the triangles AMD and BNC.]

Uses the colours as reference

Dave: Ya. AD is equal to BC. right? Right?
Pat: AD is equal to BC
Dave: Is that given? I think that's given.
Pat: Let me see it separated, OK?

Looks for evidence

Uses separate button

(Pat and Dave, March 29, 2000)

4. In response to: "Click Show Given Information. Drag the red points again and observe the measurements" (Day 2, task 1), two students noted the markings and decided:

Lou: It's a parallelogram.
Rob: Opposite sides--
Lou: You are given an angle and you are given a side.
Rob: It's a parallelogram, cause opposite sides are equal.

(Lou and Rob, Day 2, task 1, April 5, 2000)

**Theme 4: Student and Labsheets**

In the taped segments students constantly refer to the labsheets. In addition to a first reading of statements and questions there are many instances of rereading, and as many as 14 where students specifically mentioned rereading to clarify what was being asked. In many of
these latter situations students were considering a question that was worded in a slightly different way than those they normally met. For example, "Write two additional facts that you know are true" is very straightforward, but seldom found in a textbook.

I have separated the responses of the taped students to the labsheet material by considering the type of question or statement.

A. Response to Focusing Statements

Labsheet statements such as the following, directed students to use particular affordances of the sketches. They were intended to focus student attention on details.

1. Notice that some points in the sketch are red (Day 1, task 1)
2. Click Show Given Information #1 (Day 2, task 3)
3. Click Show Triangle (Day 2, task 1)

Here is a selection of (unconnected) comments, which demonstrate that students did notice details.

Doug: "Angle BED--hey!...Angle BED is 72.455."
Katy: "Um, uh the angle shadings They're the same angles. Yeah, I would say that. The angle shadings mean that they're congruent angles. So, congruent sides and congruent angles."
Dave: "They match. It matched it with the other one. It shows us that they're congruent."
Ray: "Black means the angles are equal. And the tick marks mean that the sides are equal."
Pat: "The tick marks are showing that AB is equal to FC."

B. Response to Prompting Statements

Some examples of prompting questions were:

1. Explain the meaning of the tick marks and the angle shading.
2. Write two additional facts that you know and explain why they are true.
3. What shape is ABCD? Explain.

All students responded to these straightforward action questions by diligently setting to work, examining the sketch, discussing their answers, and responding on the labsheets.
One question which was intended to prompt students to gather evidence to determine congruency was:

Do you have 3 pieces of information to show whether each pair of triangles is congruent?

(Day 1, task 2)

This particular question generated a great deal of discussion and investigation because, in fact, there were not always three pieces of information. I also discuss this particular question in section D, but it is interesting to note here that in only one case did students discuss the format of the answer. Here, Paul has correctly determined that they are not required to give the three pieces of information (although later that information was requested).

Paul: Yes.
Sue: But don't we have to say what they are?
Paul: Yes, just write yes. No, no, don't write AB. Just write yes. Do you have--just answer the question.

(Paul and Sue, March 28, 2000)

C. Response to Inviting Questions

Questions such as the following, were more open-ended than the prompting questions mentioned above.

1. What additional information can you deduce about point H from the diagram?
   (Day 1, task 1)
2. How can the information provided by these images be used to explain why DM = BN? (Day 2, task 2)

These questions invited students to use their observation and interpreting skills and to look at the problem from a different perspective. To answer, students needed to investigate by exploring alternate paths.

The JavaSketches in this study were created to offer students several options. Any sketch could be dragged into configurations that went well beyond the needs of the particular assignment. Measurements of various sides and angles were provided, even when they were not strictly required. And although students usually carried out traditional congruency proofs, all sketches could be explored via transformation relationships.

Towers' definition of inviting is: "Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than clue-giving" (Towers, 1999, p. 200). This word appropriately
describes the labsheet questions that did not lead students step-by-step, but tried to spark student interest in looking at a problem from a new perspective.

An example of an inviting question from Day 1, task 1 was: "How can the information provided by these images be used to explain why $\triangle ABC$ is congruent to $\triangle FCB$?" Here, the students were asked to prove the two triangles congruent—not in the usual way by "trolling for triangles" (Whiteley, personal communication, 2000), but by applying their observations of images that were controlled by onscreen buttons. In response to this question, Katy shows that she appreciates the significance of reflections:

Katy: No but listen—the reflection—show the reflection…. This triangle $\triangle FCB$ is a reflection of $\triangle ABC$. Therefore it has to be equal—equal because it's a reflection.

(Paty and Bea, April 3, 2000)

Pat and Dave, working on the same problem have the following conversation:

Pat: It breaks it apart. It helps you to see it better.  
Dave: This is the mirror  
Pat: It breaks it apart so it's easy to picture it.  
Dave: They match. It matched it with the other one.  
Pat: It shows us that they're congruent….  
Dave: The mirror effect shows that $\triangle ABC$'s just like $\triangle FCB$.

(Paty and Dave, March 28, 2000)

Note that this conversation draws attention to the idea of 'taking apart'. As noted in chapter two, Goldenberg, Cuoco and Mark (1998) count 'the ability to take apart in the mind' as one of the geometric reasoning skills that must be developed before students can search for invariants.

However, not all inviting questions were successful in engaging students to see a problem from a new perspective. In some cases, the sketches were not sufficiently flexible:

Tara: Cause that means that's the uh—the median? The centre point I mean.  
Mary: Yeah  
Tara: I dunno—is it? Can you turn it around? Probably not. You can't turn it around.

(Tara and Mary, Day 1, task 2, April 4, 2000)

In some cases, students were aware of alternative ideas but unable to express them formally. In this example from Day 1, task 1, Katy and Bea recognised that a reflection could explain the relationship but they were unable to construct a formal transformation proof so they reverted to the traditional congruency proof on their labsheet.
Katy: Can't I just put BG equals CG?
Bea: What?
Katy: Because, when you flip this over--right?
Bea: I know.
Katy: BG's going to equal CG.
Bea: Right!

(Katy and Bea, April 3, 2000)

Barb and Clara's work also illustrates student responses to the availability of alternate approaches and the directive to explain. The last question on Day 2, task 1 was: "How can the information provided by these images be used to explain why DM equals BN?" As noted in chapter 4, the sketch allowed students to use rotation to explain why DM equals BN, instead of deducing the result via a triangle congruency proof. Students intuitively understood that when the triangle was rotated, DM would fall on BN. Clara briefly commented: "Because the triangle fits--the triangle fits both," however, she did not follow up with a step-by-step analysis. I suggest that the result was so obvious that she felt it needed no further explanation. Students had had no practice with formal transformation proofs, although they frequently referred to rotations and reflections. Thus it is questionable whether she could have developed anything other than a very rudimentary proof based on the idea of rotation.

D. Response to Surprise and Uncertainty

As noted earlier, research undertaken by Hadas and Hershkowitz (1999) demonstrated that uncertainty helps students develop understanding of geometric concepts. When students are unable to construct an example using dynamic software, to confirm an apparently reasonable conjecture, uncertainty results.

It is not the case that construction is always a requirement for the creation of uncertainty; pre-constructed diagrams can also introduce the element of surprise. I compare it to Towers' rug-pulling intervention through which teachers deliberately confuse students to draw their attention to subtle relationships. The actual question does not usually introduce uncertainty. In most cases, the student considers that a particular question is easily answered, but upon further investigation the student realises that the result is not straightforward.

In the study tasks, there were several examples of questions that led to uncertainty when the accompanying sketch was explored.
Example 1:

Day 1, task 3 was deliberately designed to surprise. Initially it showed a simple nested set of isosceles triangles. On experimenting, students discovered that the base could be broken and thus, that the isosceles nature of the exterior triangle should not have been assumed. Unfortunately because of time constraints only two taped pairs completed this task. In the following excerpt, Joe and Bob come to a new awareness. In response to, "Is it true that ΔCJ'L' is isosceles?" (Day 1, task 3) they have the following discussion:

Bob: Um. Isosceles?
Joe: Sure. But it--but it might not be--like--like Bob. it could not be an isosceles. Say I went like this…
Bob: Look--L N J.
Joe: Ya, but what do you need that for? You don't need that for anything.
Bob: Oh, you can need it.
Joe: You know what the answer is? No they're not isosceles because if you move one they both have to move. Is it isosceles right now? I don't think so.

Example 2:

The second example of rug-pulling was not intentional. I included pairs of triangles in Day 1, task 2 that did not have three pieces of information available to prove them congruent. This confused students because their experience of geometry problems was limited to situations that could be solved. Students reacted to this surprising situation in several ways. Some spent a great deal of time--certain that they must be missing something. Others made up information!

When working on pair number two in Day 1, task 2, Sarah and Earl decided that there were three pieces of information because of the "bisector theorem". They returned to the problem several times. Sarah can be heard asking, "Is the bisector allowed?" She didn't question their decision to include BF as a bisector--just worried that they might not be allowed to use it. At one point she said, in a rather frustrated voice, "I feel obligated that there must be 3 pieces of information given" (Sarah and Earl, April 4, 2000).

Lily and Fran, working on the same problem, couldn't find anything more than the fact that BF was a common side. They asked the next group for help, but were not told anything. They re-examined the sketch--separating it again and again. Fran attempted to include in the proof, relationships of sides and angles inside the triangle (another instance of whole vs. part confusion) but Lily corrected her, saying: "We're not supposed to be focusing on that". After I
led them through an organised check, they realised that they didn't have three pieces of information, but they weren't confident enough to abandon the search until I arrived (Lily and Fran, April 4, 2000). This episode demonstrates the importance of teachers circulating throughout sessions and also of gathering students together to discuss ideas and conclusions.

**E. Response to Checking Questions**

All labsheets asked students to explain in order to check student understanding. In general, explanations were very sparse if they were included at all. For example, in response to "Find another pair of congruent triangles in the figure. Explain your reasoning" (Day 1, task 1) three students wrote the following:

**Example 1:**

\[ \triangle GBH \text{ and } \triangle GCH \text{ are two new congruent triangles that are formed. The perpendicular line forms a common side.} \]

**Example 2:**

\[ \triangle ABG = \triangle FGC \text{ because} \]

\[ \angle ABC = \angle FCB \text{ given [note: this angle isn't in } \triangle ABG] \]

\[ AB = FC \text{ given} \]

\[ \angle AGB = \angle FGC \text{ VOAT} \]

**Example 3:**

\[ \triangle ABG \text{=} \triangle CGH \]

Proof: \[ GH = GH \text{ common} \]

\[ BH = HC \text{ right bisector} \]

\[ BG = CG \text{ ITT} \]

When inviting type questions asked students to explain, students wrote sentences; however, these were always very general in nature. For example, in response to "How can the information provided by these images be used to explain why \( \triangle ABC \) is congruent to \( \triangle FCB \)?" (Day 1, task 1), three students wrote as follows:
Example 1:
Both triangles have uneven sides (scalene). From the reflection in the mirror they are both symmetrical. When ΔFBC is placed upon ΔABC they cover one another exactly.

Example 2:
The buttons do reverse images and try to place the triangles together. They fit so they are congruent.

Example 3:
It breaks it into two diagrams so it is easy to picture it. The mirror effect shows that it is the same triangle. It shows that it is exactly the same.

This tendency—to write very little, or to substitute traditional congruency proofs for explanations could be explained by time constraints, or by student realisation that the labsheets were not to be marked by their teacher, but the examples above suggest that students need more practice developing explanations of their mathematical reasoning.

Labsheet Wording of Questions and Statements

Two particular concerns arose about wording during the analysis of the data, trivial responses and ambiguous wording.

Trivial responses.

Although directions on labsheets were intended to focus student attention to detail, and to invite them to explore and to explain, some were more effective than others. For example, the question, "What do you notice about the new triangle?" was too vague for some student pairs, resulting in trivial conclusions:

Clara: It gets bigger and smaller.
Barb: It moves according to how you--It changes.

(Barb and Clara. Day 2, task 1, March 29, 2000)

In the next snippet Lou and Rob go a little further than most students in trying to answer this rather vague question.

Lou: It changes.
Rob: It's bigger.
Lou: Why is that?
Rob: Drag point B again...
Lou: It doesn't necessarily increase. When point B goes up...
Rob: Let's put, "It changes in size."

(Lou and Rob, Day 2, task 1, April 5, 2000)

**Ambiguous wording.**

During the study, some communication difficulties between student pairs appear to have been caused by student lack of knowledge, and some by inattention to partners' comments, but ambiguous wording of labsheet questions also caused communication problems.

The next examples are drawn from the transcript of Barb and Clara, working on Day 2, task 2. In this task, students in all classes struggled with the idea of right bisecting. Some students thought the introductory questions, which were intended to help them go over the ideas of bisecting and right bisecting line segments in preparation for later work, were questions about the parallelogram in the sketch. The teachers and I spent some time correcting this impression; however, in this example, the girls are working at cross-purposes because Barb is considering the parallelogram while Clara is thinking only of the line segments DB and AC.

| Barb: [reading] What additional facts will be true if DB and AC right bisect one another? DB and AC right bisect one another? It would be a square, wouldn't it? No. It doesn't have to be. | Considering the shape |
| Clara: If you have two lines how can you have a square? | Looking at the segments |
| Barb: Do you know what I'm saying? They're saying that what if this goes like this, and it's like a 90 degree. | Correcting her partner |

(Barb and Clara, March 29, 2000)

Clara interpreted the question as I intended, however, Barb's interpretation was also logical. This exchange highlights the importance of careful wording of task questions to avoid ambiguity.

**Theme 5: Student and Teacher**

Throughout my analysis of the data I noted examples of the teacher's role in the learning environment. Teachers' interventions were essential to help catch student misunderstandings that interfered with their progress, and to help them consider new ideas. I briefly go through the most evident interactions here.

**Telling, Leading, Shepherding**

Although most tape segments involve students only, there are instances of teacher interventions with each taped pair. As I analyzed these interactions I classified approximately one half of them as leading, one third as shepherding, and one sixth as showing and telling.
In the exchange that follows, Barb and Clara have called me over to ask whether ABCD is a square. I initially moved to clarify concepts by directly telling them that a square is a parallelogram, but then used a shepherding style to subtly nudge them towards further understanding. They folded back to retrieve the necessary information, then applied their knowledge to the sketch and ended by congratulating themselves.

Me: Well. I mean you could make it a square. You could make it into one--because a square is a parallelogram, right?..... But even if it's a parallelogram like this, there's one more thing you know about these two triangles. Clara: That they're equidistant from the midpoint, right? Me: Well, they are but, but ah, that isn't. Barb: Proven. Me: Ya. You don't have that given here. Barb: See! Me: What do you know? There's something else you know. Think about parallels, parallelograms. What do you know? Clara: Opposite sides are parallel. Me: OK, what else do you know? What do you know about parallel lines? Barb: Oh, the Z, the F and the. Clara: Oh, yeh. This angle's the same. Oh yeh. It's the same. Barb: That is the same as that angle, cause of PLT. Clara: This one and this one is the same Barb: Ohhh! Whose got the brains? Clara: Me, Barb. It's not you! [laugh]

(Barb and Clara, March 29, 2000)

As shown in this example, shepherding interventions encouraged students to think back to previously learned material (such as the laws of parallel lines). In this case, students realised how to solve the problem and ended up feeling that they were smart—a real confidence builder.

Teachers did use the telling intervention with some pairs—sometimes because time was short. I noted that this intervention was not effective in helping students develop understanding. For example, in a very lengthy taped segment, teacher B worked with Doug and Sal who were having difficulty finding the three pieces of information in Day 1, task 2. Here is a very brief snippet of this segment.

Teacher: Right there. So this is a common angle. So what did you prove now? This side--OK? This angle--and this angle. Therefore you get what? Doug: Angle, side Teacher: Angle, side, angle and the two triangles are

Questions but doesn't wait long enough

Telling

Barb folds back

Clara connects

Understanding

Celebrating
congruent.

Doug: All right
Teacher: If they are congruent now—what is the result?
What do they want? Which side? BA equals to BC? To prove BA equals to BC—so then BA equals to BC by what?
Corresponding sides of congruent triangles. Right?

(Teacher B working with Doug and Sal, April 3, 2000)

After the teacher left, neither student could remember the information to record it.

Modelling

When students appealed for help teachers sometimes modelled their thinking for students as illustrated by this brief example of teacher A's interaction with Pat and Dave:

It's like a knife. If—if you have a piece of bread and you cut the bread in half—it's the two pieces of bread that are in half. You're not looking at the knife.

(Teacher A, March 29, 2000)

Teachers also frequently helped pairs review what they knew in an organised way, as shown by my interaction with Lily and Fran:

Me: BA equals BC?
Lily: No
Me: BF equals FB?
Lily: Yes
Me: What about AF and CF?
Lily: No
Me: DBF and CBF?
[They try to say they're equal but I remind them that they don't know that BF is a bisector.]

(Me, April 4, 2000)

Checking

It was important to check carefully the exact nature of the student difficulty and address this. I noted that all of us did ask students to explain what they were having trouble with, but we were quite rushed. When I reviewed the tape I discovered that I had not uncovered the source of one pair's difficulty and had spent considerable time talking to them about an unconnected issue, while they politely listened!
**Inviting**

Teachers posed inviting questions as they went from pair to pair. These were usually simple, but designed to get students to look at underlying reasons. Some (unconnected) examples from my interactions with Pat and Dave:

- OK, how can you get that other angle?
- What do you know about a triangle? How many heights can it have?
- But what about a square? Would a square work?

(working with Pat and Dave, March 29, 2000)

**Rug pulling**

Most of the rug-pulling questions that I annotated were in the group sessions— a wonderful opportunity to surprise and to get students thinking about new possibilities. Because most students were unsophisticated in their knowledge of geometry, even basic questions about figure properties caused uncertainty. Some examples are:

Is a square a rhombus?  
(To class A, March 29, 2000)

Watch what happens here. [I drag the point D below to form a kite] Does the diagram still have the same properties as before? What do you think?  
(Group session with class A, March 30, 2000)

Yes--a kite. How do you know something's a kite? [can't hear] That's right. Two sides equal and two sides equal. But what about a parallelogram? It has two sides equal and two sides equal.  
(Group session with class A, March 30, 2000)

Ok but if you had BD and DC equal—say this equals this. And you've got this line that comes down—maybe it's crooked. How do you know [this is] a right angle?  
(Group session with class B, April 3, 2000)

If you look at those, do you have enough information to prove [them congruent]?  
[COMMENT: There wasn't enough information. At the start of the sessions the students assumed there was always enough information and were surprised/confused when we dealt with cases in which there wasn't enough.]  
(Group session with class C, April 5, 2000)

When you have two equal angles you don't always know they're both 90. How come you know they're both 90 here?  
(Group session with class C, April 5, 2000)
As mentioned in the labsheet observations, when pairs worked on the rug-pulling questions on the labsheets it appeared to give them new motivation. The surprise led them to discuss excitedly and to turn to the sketch to experiment further.

Retreating

As we went around the classroom, we tried to give students engaged in active exploration the opportunity to proceed without interruption. We listened for the sounds of frustration and quickly moved to help students but it was important to avoid spending excessive time with a particular pair. I annotated two examples of situations in which I extricated myself from lengthy interactions with student pairs who, I suspected were trying to get me to do their work. In one case, although I gave the pair a specific direction, they floundered and appealed for help within minutes; however, the other pair resumed their investigation and reached a satisfactory conclusion.

Managing, Praising, Reinforcing

The tapes reveal that teachers engaged in all these interventions during the sessions. As in any classroom, they managed as necessary, praised students for their work (and the students were wonderful), and repeated ideas to reinforce them before leaving to help another pair.

Inclusion of Non-Taped Pairs

While I am able to discuss the activities of the non-taped pairs in broad terms based on field notes, and submitted labsheets, I do not have specific evidence with which to analyze their responses to the interrelating factors in the learning situations, (i.e., their partner, the teacher, the labsheet questions and sketch). Nevertheless, it is important to recognise that the inclusion of these non-taped students affected the experiences of the taped pairs. The taped students were exposed to the conditions of an actual secondary class. They worked amidst normal hubbub, were subject to class time constraints, and were forced to compete for the attention of the teacher. This realistic atmosphere added a complexity to the interrelationships under examination, by creating a dependency on partner and learning materials that might otherwise have been absent.

Summary

In this chapter I have examined student activity in the sessions, devoting special attention to the behaviour of the taped pairs, and to the influences of teacher, partner, labsheets and sketches in supporting student efforts to carry out the learning tasks.
Chapter Six
Discussion and Interpretation of Findings

Introduction

In the first part of this chapter I describe the impact of the activities by discussing specific study observations in light of de Villiers' interpretation of the van Hiele model of geometric thinking. I then revisit Towers' interventions and discuss and extend these categories by considering the study observations presented in chapter five.

In the second part, I formulate answers to my research questions in light of this extended theory and the themes that emerged during the data analysis. I then provide a summary of the benefits and limitations of using JavaSketchpad in the secondary classroom and discuss guidelines for the design of JavaSketchpad tasks.

Study Activities and Levels of Geometric Thinking

In chapter two I presented several models of geometric thinking. In particular, I noted de Villiers' perspective on the van Hiele model (see Table 2.3). According to de Villiers, students carry out mathematical processes (e.g., proving, classifying), in accordance with their level of reasoning. He maintains that dynamic software makes it easier to fully develop the first two levels—visualisation and analysis (de Villiers, personal correspondence, March 14, 2001).

During the study, students were engaged in three main activities: exploring and interpreting the sketches, using the sketches to reason, and communicating. By reviewing student difficulties and successes in these areas I will shed light on the impact of using pre-constructed JavaSketches in the secondary geometry program.

Expectations

The grade twelve advanced geometry requirements are focused on deductive proving. The curriculum guideline (1985), states that the "emphasis should be placed on consolidating the principles of deductive proof as initiated in Mathematics, Grade 10, Advanced Level, and on
reviewing some of the basic properties of figures, including those related to transformations" (p. 60). Thus, the curriculum expectations that formed the basis for the study tasks are aimed at developing the reasoning skills associated with van Hiele level four.

The students in this study were all over sixteen years of age. They were enrolled in grade twelve advanced mathematics, which implies that they had shown above average competency in earlier secondary school mathematics. In light of their age (and their ability), we might expect them to be working at the Formal stage of the SOLO model, the stage which can begin at around fifteen years of age and at which, "the individual can consider more abstract concepts and work in terms of 'principles' and 'theories'. The individual is no longer restricted to a concrete referent" (Pegg and Davey, 1998, p. 116). This stage is associated in Pegg and Davey's synthesis with van Hiele level 4, where "students understand the place of deduction. They use the concept of necessary and sufficient conditions and can develop proofs rather than learn them by rote. They can devise definitions" (p. 111).

Thus from a curriculum perspective and from consideration of age and ability factors, van Hiele level four seems an appropriate stage at which to place the students. However, the results noted in chapter five suggest that many students were, in fact, operating at or below level three on the van Hiele model.

**Exploring and Interpreting the Sketches**

Students in the study were motivated and actively involved in exploring the sketches. Since no learning can take place unless students are engaged, this is not a trivial point. *JavaSketchpad* grabbed student interest initially through its colourful and dynamic interface. As noted in the section on student reactions to the visual image, students were openly enthusiastic. Students demonstrated confidence in their ability to work through the activities by their on task behaviour, which was noted by all three teachers.

**Using sketch affordances.**

The software allowed students to interact with the onscreen image by providing the means to drag points, reflect, rotate, and translate objects, hide/show details, watch onscreen measurements update, and restart. Study results show the following:

1. All students gazed at, pointed at, and traced the onscreen image. As they did so they identified objects, and noted details and relationships between them as shown by the following (unconnected) comments from students, quoted in chapter five:
   - It's a rectangle.
1. That's a 90 degree angle.
2. GBH is congruent to GCH because angle angle. given. common. common.
   Students played with the sketches. As noted in the discussion in chapter five, some
students looked for new possibilities, some investigated a particular question and
others played aimlessly. Thus, whether students were at level one or level four on the
van Hiele model, JavaSketchpad supported their actions.
3. Students restarted frequently and used the hide/show capabilities to tidy up their
   sketch. The pre-constructed nature of the JavaSketches made these actions possible,
but the particular details that could be toggled on and off were chosen by the sketch
creator. Thus, the extent to which the use of JavaSketches is beneficial is dependent
on the skills of the sketch creator.

Motion
The introduction of motion to a diagram allows students to visually verify specific
conjectures by observing all cases. Motion also makes it possible for students at an earlier
developmental level to investigate geometric relationships (a level three skill); however, it also
makes the analysis of the diagram more complex. To interpret dynamic diagrams students need
to be able to notice details in moving objects. To address this need, the dragging capability of
dynamic software programs, including JavaSketchpad, gives students control over the speed and
direction of a moving point, and permits students to repeat a motion.

The dragging capability did help students notice details. For example, Sue noted,
"Obviously it's not a square now... it's a parallelogram" (Paul and Sue, Day 2, task 2, March 29,
2000). As noted in the section on difficulties with terms, Pat and Dave, who had trouble with
many of the geometry concepts, cleared up a misunderstanding by rotating an object.

Nevertheless, there were some problems during the study. Hardware problems, especially
in class A caused delayed motion of objects, which sometimes resulted in points moving
erratically when students tried to drag them. My inexperience as a sketch designer also led to
some difficulties. For example, when Tara and Mary tried to turn a figure around in Day 1, task 2
they could not do so.

As we moved around the classroom, the teachers and I demonstrated effective use of
dragging to the students. For example, I deliberately moved triangles for several student pairs to
help them see that certain relationships they may have deduced in the static diagram were not
invariant. JavaSketchpad provided the means for me to instantly demonstrate, but the structure of
the class was also an important factor. Since the students worked in pairs I was able to hear their comments and recognise their need for clarification of concepts.

Study results show that all students participated in the activities. A closer examination of the dragging capability demonstrates how the sketches met the needs of students at different levels.

1. Deliberate dragging to examine all cases is an ordering skill (level three) and thus comparable to analysing relationships between figures. Only Paul and Sue clearly demonstrated this skill. As mentioned in chapter five, they dragged to verify their own conjecture and realised their error.

2. Dragging and pausing to note and list properties reflects level two behaviour. Study results show that most students were at this level.

3. Dragging aimlessly, the wandering dragging of Arzarello et al. (1998), is illustrated by the playing of Doug and Sal—"You can really mess this one up. Look what I did!" (April 3, 2000). This is an example of thinking at or below van Hiele level one.

**Tendency to quit dragging**

The study results show that students usually stopped dragging after a short time and concentrated on interpreting the static figure. When students recognised that undirected dragging wasn't providing the information they needed, it was natural for them to stop dragging and look for details in the static image. This indicates that students were not yet prepared to use directed dragging to explore the relationships among the displayed images—a level three skill.

These results show that the provision of the dragging capability is not enough to help students interpret dynamic figures. In order to make effective use of dynamic diagrams students must be able to plan directed dragging and to interpret the resulting image, that is, they must move on to level three.

*JavaSketchpad* can be used to show students the effects of directed dragging. For example, action buttons in the *JavaSketchpad* activities demonstrated separating, joining, rotating, translating, and reflecting motions for students in the study. As noted earlier, these affordances generated interest and helped students formulate conclusions. The study results also show that viewing these pre-built motions raised student awareness of possible actions. One pair, Tara and Mary, after using various affordances to explore sketches, attempted to rotate a figure but found that the particular sketch did not permit the motion.
Student desire to add details.
The results show that some students were frustrated by the fact that JavaSketchpad does not permit the addition of details to a sketch. For example, Pat and Dave wanted to circle items and Sue explicitly mentioned that she preferred paper because she could draw. In addition, the tapes clearly show students tracing onscreen during the sessions, which demonstrates a need, albeit unstated, to mark up the diagram.

There are no plans at present to add the capability for onscreen writing to JavaSketchpad; however, Cinderella (1999, Richter-Gebert and Kortenkamp), a new geometry software program, addresses this issue. Students can access a teacher-specified set of online tools to add details to a pre-constructed Cinderella diagram.

Provision of an accurate image.
The ability to display an accurate image is commonly assumed to be a benefit of dynamic geometry software—it seems reasonable to conclude that the task of noticing and interpreting relationships between objects is easier if figures are drawn to scale. However, the study results showed that students do not automatically understand that the onscreen image is accurate.

Some students treated the onscreen image as if it were a pencil sketch—as if the diagram represented objects and their relationships, but was not drawn to scale. For example, they made comments such as: "Maybe cause it's slanted you can't tell it's a square" (Barb and Clara, March 29, 2000), and "If this is equilateral these sides would have to be equal" (Paul and Sue, March 28, 2000). [In this latter instance, the triangle was clearly not equilateral]. Students also, as noted, made little use of the onscreen numerical information.

Using the Sketch to Reason
Throughout the sessions the students were involved in reasoning with the sketches. They interpreted details, developed conjectures, used deductive reasoning skills and wrote explanations. Study results show that all students practised their geometric reasoning skills and in some cases displayed a growth in understanding.

Developing conjectures.
This was the first time that students in the study had been asked to develop their own conjectures. The study results show that most students found the process difficult. Only one pair of students, Paul and Sue, formulated a conjecture, tested it, refined it, and then verified their new conjecture. Other students struggled with:
1. Expressing their idea using precise mathematical language. As noted in chapter five, many students had trouble with terms. Since the sketch that was used for conjecturing involved the idea of a right bisector, the term right bisector was especially important. Unfortunately, many students, as noted earlier, had difficulty with the word bisect, thinking that it referred only to bisecting an angle.

2. Separating givens from conclusions. Some students ended up using the statement they were supposed to prove as one of the givens in the argument. As noted in chapter five, this may have been caused by ambiguous wording of the particular questions.

**Changing focus: Whole versus part**

Goldenberg, Cuoco and Mark (1998) consider the ability to "take apart in the mind" a critical skill for geometry understanding. The participant teachers, as noted in chapter four, mentioned several areas of difficulty for their geometry students that are related to this skill. These areas include being able to pick out smaller triangles within a larger diagram, to focus on the whole shape, and to mentally separate overlapping shapes (level three skills).

As described in the section on guidelines for preparation, I designed the JavaSketches to help students overcome these difficulties. I included overlapping triangles, and details that could be toggled off and on, and figures that could be separated and joined. Study results show that these provisions helped some students understand; as one student put it "We didn't see it until we put it together again" (Katy and Bea, April 3, 2000).

However, as noted in chapter five, being able to switch between seeing the whole and seeing the particular was a continuing source of difficulty for some students in the study. The teachers and I addressed this problem throughout the sessions. Only one student, Paul, showed that he was able to mentally dissect a figure, when he advised Sue to "stare at the red parts [and] blur out the black parts" (Paul, March 28, 2000).

**Using "looks like" evidence.**

According to de Villiers, students at the visualisation level engage in visual verification. Students in the study who made "looks like" conclusions without exploring further, illustrate this process. As noted in chapter five, these students used inspection alone to determine 1) the shape of ABCD, 2) that an angle was a right angle, 3) whether two triangles were congruent. Recall that in chapter two inspection alone is used to denote that the student is using visual evidence at van Hiele level one (i.e., the student recognises the figure but does not consider properties).
On the other hand, students at the analytic, ordering, or deductive levels can also use visualisation (de Villiers, personal communication, March 14, 2001). The characteristics of a visual image can provide the impetus for further investigation, and visual reasoning skills can be used to carry out an investigation. Paul, as noted in chapter five, understood the concept of assumptions, which shows that he was at van Hiele level three or four; yet, as noted in chapter five, he used the word *look*: "Let's take a look if the sides are all equal". He first determined by sight that the image was not a square, then decided to examine the list of measurements to see if the sides were nevertheless equal. His investigation of relationships indicates level three skills.

Thus, results show that some students were operating at level one, while others were at a higher van Hiele level employing their visual reasoning as an efficient analysis tool. For both groups the *JavaSketches* offered support. They provided the student at level one additional practice and a base from which to develop the skills of level two. They gave the student at the higher level the opportunity to select an appropriate analysis tool.

*Using deductive reasoning.*

The students used theorems such as the isosceles triangle theorem, the parallel lines theorem, and the congruency theorems to deduce results, as noted in the chapter five section on using deductive reasoning. For example,

OK if we prove that this angle's equal to that angle cause it's given and that angle's equal to that angle because of the parallel--the Z theorem, whatever. then... Wouldn't that angle have to be equal to that angle?

(Lou and Rob, April 5, 2000)

However, students had difficulty developing logical written arguments. As noted in chapter five, some students gave a congruency proof as the only explanation of their reasoning. The use of a congruency proof could indicate the ability to use deductive proving (a level three skill); however, in many cases the proof was simply a list of given properties, illustrating the level two tendency to include superfluous information—to be *uneconomical* (de Villiers, personal correspondence, March 18, 2001).

Lou and Rob's discussion on the angle, side, angle congruency theorem clearly demonstrates how the *JavaSketch* played a part in their growth of geometric understanding by providing an immediately accessible and interactive model; in addition, the structure of the class (i.e., students working in pairs, freely discussing), was an important contributing factor.
Using symmetry.

Each *JavaSketch* in the study was designed so students could arrive at a solution from a consideration of symmetries as well as from a straightforward application of congruency theorems. Students had studied transformations in earlier years as noted in the section on the curriculum, however, students in the study had not been taught to communicate arguments based on transformation relationships.

Though the *JavaSketches* allowed students to reflect, rotate and translate objects, study results show that this was insufficient support for students to develop symmetry-based proofs. Students in the study recognised that certain shapes were related by transformations. They referred to reflections and rotations while working with the sketches. Some even mentioned on labsheets that shapes were reflections or rotations; yet no students applied symmetry concepts in writing a proof.

Thus, if we expect students to develop proofs based on transformation relationships in pre-constructed sketches, we must provide focused instruction on this skill.

Communicating

Many students had difficulty communicating ideas, both orally and in writing, either because of unfamiliarity with terminology or lack of practice at identifying, and describing properties of figures.

In addition, students did not use diagrams to communicate their understanding. As noted in chapter five, only a few students drew even rudimentary sketches on their labsheets. Although *JavaSketches* can be printed, students were not provided with the capability to print. In future, it may be advisable to allow students to print the onscreen image when it displays evidence for or against a conjecture that they have made. The printout could be annotated and would provide support for communicating via diagrams as well as words. It might also address the need, mentioned by several students, to mark details onscreen.

The *JavaSketch* activities supported the development of communication by providing numerous opportunities for student-student and student-teacher discussions about the properties of onscreen figures. Regular classes should provide a channel for students to discuss mathematical concepts but time constraints often mean that only a few get the chance to express their ideas. Since students in the study were in pairs, the sessions provided each student with approximately three hours to talk about mathematics.
**Difficulty with terms.**

Students in the study sometimes struggled to communicate ideas to their partners because of difficulty with terms such as bisector, common, and half. As noted in chapter five this led to working at cross purposes (e.g., each student referring to a different triangle in the sketch), errors in deductions (e.g., "Clara: So angle M and angle N are common"). and extra time spent on ideas that should have been straightforward (e.g., Lou and Rob's discussion of angle, side, angle). Lack of familiarity with basic geometric terms is related to the students' earlier geometric experiences or lack thereof, but it highlights a limitation of working with pre-constructed *JavaSketches*. When students construct their own dynamic diagrams with the Geometer's Sketchpad, they select tools from drop-down menus containing mathematical object names such as midpoint and angle bisector. Pre-constructed *JavaSketches* do not provide a comparable opportunity to explicitly link a term with its geometric representation. This limitation could be addressed by redesigning the software to display a label as the mouse is dragged over an object. Indeed, one student recommended on the questionnaire that onscreen displays of theorems and terms would be a valuable addition to the program. Teachers could also use *JavaSketchpad* itself to address the problem by designing pre-constructed *JavaSketches* to help students develop their own definitions—stressing as de Villiers (1998b) does, the importance of *economical* language in mathematics.

**The sketch as a shared image.**

In other classrooms each student has their own diagram, but each *JavaSketch* in this study belonged to pairs of students. Study results show that students worked together, using the sketch as a shared image. They pointed to the sketch, and used colour, and visual detail as references in their conversations.

**Modelling ways of seeing.**

There were several instances of students modelling their way of seeing to communicate an idea to their partner—Rob helping Lou to imagine a scaled copy of a triangle, and Paul, helping Sue "see" the whole. Teachers, as noted in chapter five, also used this tool. For example, Teacher A talks about a knife while trying to help students understand the difference between bisecting and bisecting one another. These examples show that the environment supported students in using visual reasoning.
Although constructing should remain an important part of the geometry curriculum, study results show that the exploration of ready-made sketches provided valuable learning opportunities. Pre-constructed dynamic images helped students develop visualisation, analysis, and in some cases ordering and deductive skills by supporting exploration, reasoning and communication. Thus, they deserve additional emphasis in future geometry programs, though strategies for teaching with these new tools are yet to be developed.

*Towers' Interventions Revisited*

Towers outlined the important ways in which teachers interact with students, and identified those that occasion the growth of mathematical understanding. In her model, interventions help by creating stepping stones to extend student understanding, by encouraging students to think back to earlier concepts, and by supporting jumps in understanding (i.e., a student is able to move forward more than one level in Pirie and Kieren's model).

In the *JavaSketchpad* environment, the interventions that bring about understanding occur not only between teacher and student, but also between all the elements of the learning task and the student. This is an advantage of using *JavaSketches* (or other pre-constructed diagrams). The teacher is not always available, but each student continues to interact within the learning situation—with his or her partner, with the sketches, and with the material on the labsheets.

To illustrate, I have summarised observations from the study with respect to the elements of the learning situations, under each of Towers' intervention categories.

*Teaching Styles*

**Leading.**

Teacher  Very often used—used in the large group setting, and also with student pairs.
Student Partner  Sometimes used when a student went through a step by step explanation for their partner.
*JavaSketches*  Leading implies a set path. *JavaSketches* can be developed to permit alternate approaches.
Labsheets  Some elements of leading

**Showing and telling.**

Teacher  Used—Information presented in this way was not always remembered by the student pairs.
Student Partner  Frequently used.
JavaSketches Measurements, labels, and markings "tell" given information, although there is no guarantee that students will notice or use this data.

Labsheets Some information was included on the labsheets (e.g., that red points are draggable).

**Shepherding.**

Teacher Used with pairs—led to renewed confidence and to students feeling pleased with themselves.

Student Partner Most students did not use this intervention. There was one notable exception when Paul helped Sue "see" the whole.

JavaSketches Shepherding implies adaptation. The JavaSketches were designed to provide feedback to the students and this feedback was dependent on the action of the student. However, the sketch could not tailor its response to meet students' needs. That is, if a sketch was misinterpreted, it could not morph into a new image to help correct that student's misconception.

Labsheets Shepherding implies adaptation. The labsheets could not change to help students who misinterpreted a statement.

**Teaching Strategies**

**Inviting.**

Teacher Teachers posed new questions and encouraged students to look a little deeper and to experiment with the sketches in different ways.

Student Partner Students sometimes played to explore a new direction—to see 'what if' (i.e., they "invited" themselves).

JavaSketches The dragging capability makes JavaSketches inviting; however, students often stopped dragging and exploring after a few minutes. They used the less open-ended hide/show and motion buttons more frequently.

Labsheets Some questions were designed specifically to be more open-ended (e.g., What is an alternate explanation for the congruency of triangle ABC and triangle FCB?). They invited students to examine the sketch and to move outside the formal congruency proof process of their text. There were some difficulties with this type of question.

**Rug-pulling.**

Teacher Used with some pairs. Also used in the large group session—for example I challenged the students to consider that the segment, which appeared to be perpendicular, might actually not be.

Student Partner In the group sketch, the students explored a diagram with a triangle inside another triangle. They were asked to consider how the proof was affected if the figure was deformed into a kite with the same givens. Rug pulling in the sketches caused students to investigate with renewed interest.

JavaSketches In JsSketch4 the sketch appears to be an isosceles triangle but when the Show More button is pressed, students can explore what happens if the base is not
a single segment. This was designed to help them with the idea of prior assumptions: however, few pairs had time to investigate this sketch.

Labsheets The inclusion of pair#2 in Day 1, task 2 proved to be an example of rug-pulling. Students were faced with a pair of triangles that they could not prove congruent, which frustrated some but led others to organise what they knew. Some pairs assumed no question would be impossible to answer so they made up facts. This demonstrates the need for an active teacher presence.

**Modelling.**

Teacher Mainly used to model going through possibilities.
Student Partner Sometimes used by good students. They went through their thinking processes aloud—sometimes too quickly for their partner to follow. In one case the partner said "Tell me again." When Lou helped Rob understand that Angle, Angle, Side (AAS) is acceptable as a proof of congruency (see theme 1), he modelled his reasoning for his partner.

JavaSketches Some sketches modelled the thinking of an expert. For example—separating overlapping triangles, reflecting and superimposing triangles, and highlighting pairs of triangles.

Labsheets Each labsheet was a model for investigating a problem. In particular, Day 1, task 2 modelled the process of going through possibilities, and noticing optional methods.

**Checking.**

Teacher All teachers used this intervention with the study pairs; however, in one case, failure to pay careful attention to the student response led to an unprofitable interaction.
Student Partner Seldom used by students—thus some students didn't know what their partner was thinking. This resulted in students working at cross-purposes.

JavaSketches JavaSketches could be used to check reasoning but students frequently stopped dragging after initial exploration and did not go back to examine all possibilities.

Labsheets The word "explain" was always included to encourage students to examine their own thinking. Response to this direction was disappointing. In many cases, a formal congruency proof was the only "explanation".

**Enculturating.**

Teacher Teachers always used correct terminology with student pairs and corrected misinterpretations of concepts.
Student Partner Students sometimes corrected one another with respect to terminology although occasionally the correction was wrong.

JavaSketches Sketches employed common format for diagram markings, measurements, and button labels.

Labsheets Questions and sentences were always provided using precise mathematical language.
**Anticipating.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>It was difficult to identify this intervention. In some ways, every time a teacher helped a student pair and prevented a possible error they were using this strategy; however, it was not deliberately used.</td>
<td>N/A</td>
<td>Pre-construction prevented errors that might have been based on incorrect diagram construction.</td>
<td>In Day 2, task 2, the first 3 questions on bisectors may have anticipated student difficulty with that topic. Through those questions they were led to the solution (i.e., They didn't investigate freely).</td>
</tr>
</tbody>
</table>

**Blocking.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used as a time-saver with several pairs.</td>
<td>Students sometimes used blocking to keep the pair focused, especially if the mouse wielding student was getting carried away.</td>
<td>The design of the sketches prevented some free exploration. Students did not have access to details that were not provided.</td>
<td>N/A</td>
</tr>
</tbody>
</table>

**Clue-giving.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>It was difficult to differentiate leading from clue-giving. Since most interventions were lengthy, the categories telling and leading were used in preference.</td>
<td>N/A</td>
<td>Some measurements were provided to allow numerical checking. They were used infrequently.</td>
<td>N/A</td>
</tr>
</tbody>
</table>

**Managing.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teachers engaged in management as necessary.</td>
<td>N/A</td>
<td>N/A</td>
<td>Written procedures allowed students to proceed with a minimum of help.</td>
</tr>
</tbody>
</table>

**Reinforcing.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequently used to sum up after helping a pair</td>
<td>Students often repeated a theorem or finding for shared understanding.</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

**Praising.**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Student Partner</th>
<th>JavaSketches</th>
<th>Labsheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Often used.</td>
<td>Students sometimes praised themselves—after they had worked something out, or after a shepherding encounter.</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>
As seen in the foregoing synopsis, Towers' intervention styles and strategies describe not only the behaviour of teachers in the study but also many of the interactions between student partners. Since the categories were meant to classify the behaviour of human beings, the fact that they are also somewhat useful in analysing the relationships among student, labsheet and sketch is quite remarkable. Perhaps the reason for this fortuitous match is that Towers has captured in the categories the most common behaviours that experienced and caring teachers use to support student efforts to understand. When teachers develop tasks to help their students work independently they create materials that are an extension of their intervention strategies.

Towers' Interventions Extended

As noted earlier, the teacher, the student partner, the labsheets and the sketches themselves all have an impact on the experience of the student. All must act in concert to move students along the path towards understanding. But there is a need to broaden Towers' list to satisfactorily describe all the interventions in the JavaSketchpad environment.

During leading or shepherding encounters Towers suggests that a teacher engages in many behaviours. For example, she identifies nudging and coaxing as two indicators of the shepherding style. Since teachers are capable of intervening in the full range of behaviours within a particular style, Towers did not explicitly identify every strategy used within each style; however, on considering the possibility of labsheet interventions and sketch interventions I found it necessary to subdivide Towers' categories. This is not to imply that teachers do not also use these sub-strategies, but to clarify the idea that the non-human components of the environment engage in a more limited set of interventions.

Labsheets

Each statement or question on the labsheet can be categorized in terms of its function. Some of the directions manage as a teacher would, (e.g., Open jsketch1); some questions check for understanding, (e.g., What do you notice?). The labsheets invite, by asking students to investigate, explain, and explore more. However, we must be careful not to take this analogy too far. A teacher may use a shepherding style, in which he or she modifies the next comment or question to suit the student's response. A labsheet cannot do this, and although the JavaSketch offers feedback as it is dragged, this response does not necessarily address a particular student's needs.
One new category that I have developed to describe labsheet interventions is **prompting**, which suggests guidance that is more open-ended than telling or clue-giving, but which avoids the idea of adaptation that is implicit in the leading and shepherding styles.

A second new category is **focusing student attention**. When the labsheet directs students to notice various aspects of a sketch such as a measurement, a relationship between sides or angles, or behaviour under dragging, it is engaged in an intervention that is not adequately described by any of Towers' categories. Clue-giving applies in cases where the instruction is specific to a particular question, but does not capture the intent of a general and more open instruction to "observe" what happens.

**Sketches**

The sketches can also be examined with regard to their function, and several new categories added to help describe their particular interventions.

In general, *JavaSketchpad*:

1. **Provides a visual image**—Through the affordances of colour and motion the sketches draw student attention to important details and provide accurate mathematical objects to investigate.
2. **Supports exploration**—By giving students the capability to drag, examine, hide/show, and redo, the sketches allow students to experiment with mathematical relationships.

The sketches in this study were designed to exploit these functions. As students worked it became clear that there were two other important ways that sketches intervene in the learning process:

1. **Providing a shared space**—When students work in pairs, the pre-constructed sketch acts as a catalyst to encourage collaboration and mathematical discussion.
2. **Surprising**—Several sketches introduced the element of uncertainty to the learning situation. Students thought they knew what was true and were taken aback by the visual evidence. In several cases this sketch intervention piqued student curiosity and led to profitable investigations.

**Summary**

During these sessions involving *JavaSketchpad* I observed that the teacher intervened in all of the ways described by Towers; that student partners intervened by showing and telling, leading, inviting, checking, reinforcing, enculturating, blocking, modelling, praising and rug-
pulling; and that the labsheets intervened by managing, and rug-pulling. Further, using additional categories that I developed, I observed that the labsheets intervened by prompting, and focusing student attention, and that the sketches intervened by providing a visual image, supporting exploration, providing a shared space, and surprising.

**The Research Questions**

As noted in the introduction, my research questions are grouped into four categories: the student, the teacher, the *JavaSketches*, and the labsheets. In this section I answer each question using the observation results reported in chapter five, the questionnaire data, and the teacher interview comments.

**A. The Student**

**Q1: Are students interested and involved throughout the session?**

The answer to this question is an unqualified "yes". The observations confirmed that most students were involved and interested throughout the sessions. There was some off-topic chatter by all pairs, but all three teachers commented on the fact that students were on task and enthusiastic during the activities.

**Q2: How does the organisation of students affect the learning environment?**

I would like to address this question in terms of the general decision to pair the students, and then comment briefly on the issues of gender, and ability.

There was no shortage of computers in any of the study labs, but in order to stimulate communication (especially for purposes of taping), students were assigned to work in pairs. Some students requested permission to work in groups of three or four but, except for temporary arrangements when students were late, I refused. This decision was based on my own experience working with students in a lab situation. I had observed during sessions with my own classes, that a third student seldom got the opportunity to manipulate the mouse, and often had difficulty getting close to the screen. Occasionally I had noticed that a third student was left out of the conversation and the action. The decision to have students work in pairs during the sessions is one that I do not regret. Students were able to sit comfortably and access their screen easily. They divided responsibilities for dragging and writing without conflict. In addition, the teachers and I were able to interact easily with the number of pairs in the lab.

Students actively engaged with their partners throughout the sessions. In fact, the role of the partner had a far greater impact on the learning environment than I expected. Students
corrected one another, kept each other on task, and argued over interpretations. As noted earlier, they even assumed some of the intervention strategies associated with teachers.

Towers found that a situation in which a teacher interacts with two or three students is most conducive to the shepherding style discussed earlier. Although the tapes reveal some telling and leading interventions, I documented examples of the use of shepherding by all three teachers. As noted in the discussion on student and teacher in chapter five, these resulted in helping students correct misunderstandings and resume their exploration with renewed confidence.

Most pairs in the study were boy/boy or girl/girl but two of the videotaped pairs were boy/girl. There was some off-topic chatting but most pairs actively focused on the work. There were no consistent differences between the amount of discussion between taped pairs based on gender. In the wider group we did need to actively encourage some of the girl/girl pairs to talk to one another about the sketches when we noticed them wordlessly looking at the screen and writing responses. This behaviour is also apparent in one of the cases of videotaped girl/girl pairs.

Neither the teachers, nor I regulated the choice of partners (except to place students who arrived late). In each class there were examples of all possible ability pairings. The pairings that were problematic were student pairs where both were very poor students. Although they were often interested in the sketches, these pairs struggled with the content. Because their background knowledge was so poor, the teachers and I sometimes became frustrated at helping these pairs. This frustration was usually evidenced by the tendency to tell the students the required information. If developing the ability to explain, rests in large part on actively using mathematical language to discuss ideas with peers, then it may be advisable to deliberately avoid pairing two very quiet, or very poor students.

**Q3: How do students use the onscreen image?**

As noted in the discussion in chapter five, the onscreen image was a central feature of the learning situations. Because partners focused on one onscreen image, each sketch acted as a shared space for exploration and an enabler of mathematical communication.

Students pointed to the sketch, traced onscreen figures (sometimes with a pencil!), encouraged one another to drag or click, and gazed intently at the image as if trying to soak up the details.

I expected students to draw pencil and paper diagrams but only five pairs included a diagram with their labsheets—and in all cases these were rudimentary sketches. Students focused on the onscreen diagram. Unlike a paper diagram, it wasn't the sole property of one person.
Students can be seen on videotape gazing intently at the sketches, watching the motion of objects, and gesturing toward the screen while talking to their partner.

The dragging capabilities of the software, coupled with the indestructible nature of the sketches, provided an environment in which students played, and restarted frequently to make sense of the visual image.

**Q4: How do student pairs overcome obstacles to understanding?**

Students had the sketch at their disposal, their texts, the teacher and myself. No one took out their text. Students who didn't initially understand spent considerable time talking the problem out, rereading the questions and directions on the labsheet, gazing at the sketch, clicking all possible buttons, and dragging the shape while observing the details. They enjoyed the investigating and most pairs were loath to ask for help.

Some notable points from the observations in chapter five:

1. In most pairs students tried to help one another to understand concepts. The exceptions were pairs in which both students had serious problems with the content.

2. There were examples of students working at cross-purposes. (e.g., each student in a pair talking about a different triangle in the diagram).

3. Students' use of mathematical language in the tapes suggests that they were unaccustomed to communicating with one another about mathematical objects. They frequently groped for basic words to express an idea. Some breakdowns in communication occurred as the result of students' misuse of basic words such as 'common' and 'half'.

4. Students did appeal for help from the teacher if they couldn't solve the problem. They also called the teacher to settle arguments about who was correct.

**Q5: What are student opinions on the use of JavaSketches in geometric learning situations?**

Students' frequent expressions of excitement during the sessions showed that they enjoyed using JavaSketchpad. For more of their opinions I turned to the questionnaire data which had been collected from 56 students.

All students had used computers at home, but only eight had used a computer for mathematics and none had used dynamic geometry software. On question four, which asked students to check off all the descriptions that matched their experience, the results shown in Table 6.1 were obtained.
Table 6.1.

Student responses to question 4, n = 56.

[Students could give more than one answer.]

<table>
<thead>
<tr>
<th>Experience</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Using the sketches was OK</td>
<td>39</td>
</tr>
<tr>
<td>The sketches really helped me understand the ideas</td>
<td>29</td>
</tr>
<tr>
<td>The sketches were fun to use but I needed to use the textbook to really understand the ideas</td>
<td>16</td>
</tr>
<tr>
<td>I liked using the sketches but they were confusing because-- (see Table 6.2)</td>
<td>11</td>
</tr>
</tbody>
</table>

The reasons given by the 11 students who checked the statement "I liked using the sketches but they were confusing because..." were divided into roughly four categories as shown in Table 6.2.

Table 6.2.

Students' reasons for confusion—question 4, n = 11.

<table>
<thead>
<tr>
<th>Description of category</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sketch deficiencies - small letters, inability to write onscreen.</td>
<td>3</td>
</tr>
<tr>
<td>difficulty dragging (a hardware problem), colours disappearing</td>
<td></td>
</tr>
<tr>
<td>Tasks too difficult</td>
<td>3</td>
</tr>
<tr>
<td>Instructions not clear--needed more information.</td>
<td>3</td>
</tr>
<tr>
<td>Don't like using computers</td>
<td>2</td>
</tr>
</tbody>
</table>

There were also several places on the questionnaire for students to share their opinions. They were largely positive, although some reflected student frustration with the hardware, some showed that students have difficulty adopting new methods, and others highlighted the fact that using dynamic diagrams does not guarantee understanding. Table 6.3 displays the breakdown for the comments and Table 6.4 gives a sample of comments. (Some students wrote more than one).

Table 6.3.

Students' stated comments on questionnaire by category, n = 56.

[Students could give more than one comment.]

<table>
<thead>
<tr>
<th>Category</th>
<th>Number and Type of Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Movement (e.g., dragging, separating)</td>
<td>17 positive</td>
</tr>
<tr>
<td>Visual (e.g., colour, shading, easy to see)</td>
<td>21 positive</td>
</tr>
<tr>
<td>Miscellaneous (e.g., different approaches, easy to understand, could start again)</td>
<td>20 positive</td>
</tr>
<tr>
<td>Hardware related</td>
<td>3 negative</td>
</tr>
<tr>
<td>Miscellaneous (e.g., hard to understand, not enough information,)</td>
<td>7 negative</td>
</tr>
</tbody>
</table>
Table 6.4.

Sample student comments from student questionnaire.

<table>
<thead>
<tr>
<th>Positive</th>
<th>Negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sketches on the computer, overall was more fun and more interesting than using a textbook.</td>
<td>I would prefer to do the work on paper, rather than on screen, it helps one to understand more.</td>
</tr>
<tr>
<td>The sketches highlighted different pairs of triangles and outlined elements in certain problems that I never would have thought of.</td>
<td>I didn't really like it because I constantly wanted to draw on the screen when I proved something.</td>
</tr>
<tr>
<td>[They] helped isolate the congruent triangles.</td>
<td>Too confusing because I'm not used to it.</td>
</tr>
<tr>
<td>It was helpful because each pair of triangles were given in colour and we were able to slip them apart to isolate the pair that we were looking at. I think that it [is] easier on computer rather than on paper.</td>
<td>It was difficult to manipulate the shapes (mainly due to hardware), also for a mediocre math student such as myself, it was difficult for me to learn something new.</td>
</tr>
<tr>
<td>The triangles could be moved around, triangles shaded in for better visuals.</td>
<td>It didn't create difficulties. I just found no difference from regular sketches.</td>
</tr>
<tr>
<td>The questions allowed me to try to use different approaches in each problem.</td>
<td></td>
</tr>
<tr>
<td>They were able to be shaded which allowed us to view what was important.</td>
<td></td>
</tr>
<tr>
<td>They helped show the triangles in a different way.</td>
<td></td>
</tr>
</tbody>
</table>

Twenty one students also made recommendations on their forms. These fell into five main categories as displayed in Table 6.5.

Table 6.5.

Student recommendations by category, n = 21.

[Students could give more than one comment.]

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improve sketches</td>
<td>larger letters, clearer labels, 3D, more colour</td>
<td>6</td>
</tr>
<tr>
<td>Provide a means to write onscreen</td>
<td>to draw on the diagram, to add marks</td>
<td>4</td>
</tr>
<tr>
<td>Give more onscreen information</td>
<td>theorems, notes, asides</td>
<td>4</td>
</tr>
<tr>
<td>Make more interactive</td>
<td>include solution pop-ups</td>
<td>2</td>
</tr>
<tr>
<td>Improve question sheets</td>
<td>more explanations, especially for the computer illiterate</td>
<td>6</td>
</tr>
</tbody>
</table>
B. The Teacher

**Q1: What is the teacher's role in the JavaSketchpad-supported environment?**

In each class, the teacher and I acted as facilitators during the sessions. To understand exactly what this implies, I will examine the nature of our interactions with the students according to Towers' intervention styles and strategies. The following were the most evident:

(a) **Shepherding**

Being unfamiliar with an investigative approach, the students in the three classes were possibly more tentative than others might have been. Whenever one of us was near they appealed for help to ensure they were on the right track. The use of the subtle shepherding approach often resulted in students figuring the problem out, and consequently feeling very pleased with themselves.

(b) **Checking**

All of us used this intervention strategy when approaching students but we occasionally did not take the time to uncover the students' underlying difficulty. When we didn't understand exactly what students were thinking we sometimes provided help on the wrong concept.

(c) **Modelling**

We often tried to help pairs that were confused or stumped, by demonstrating how to go through possibilities systematically, or by restating known information. Students imitated these strategies later to organise their thinking.

(d) **Inviting**

As we interacted to help pairs explore more effectively we often posed inviting questions on the spot - questions that piqued student interest and increased motivation. These were not necessarily difficult, but they were aimed at encouraging students to think about their reasons or to experiment a little longer.

(e) **Rug-pulling**

There were several instances of rug-pulling in the interactions that caused students to rethink their position. The elements of surprise and confusion led to animated conversations between partners.

(f) **Retreating**

Each of us used this technique to encourage student pairs to explore on their own. As noted in the analysis of the observations of the taped pairs, this intervention did lead in one case
to confusion and to students abandoning their efforts, but in most cases to a renewed enthusiasm for exploring.

**Q2: What is the function of class discussions in the JavaSketchpad-supported classroom?**

It was difficult for the teacher to orchestrate class discussions since students were often working on different tasks; however, it was crucial to conduct these group sessions in order to situate the concepts in the larger framework. Students used the ideas from the group sessions in later work and one pair explicitly mentioned that particular points had helped. Unfortunately, in school S2 the overhead monitor presented a poor quality image. This made it difficult for students to see details. Thus, in order to carry out these group discussions a suitable demonstration screen is important.

**Q3: What are teachers' opinions on the use of JavaSketches in geometric learning situations?**

All three teachers expressed satisfaction with the sessions. Teacher A was the most positive. She enthused, "It went well!" and noted that the "students were involved and seemed interested" (interview, March 29, 2000). She commented that the students found the visuals helpful—especially the colour, the fact that they could go back and forth without messing anything up and the fact that they didn't have to draw the diagrams themselves.

She did note, however, that students found the work challenging. In answer to "Do you think the guidelines on the sheets were needed?" she replied "Oh, yes, even with the step-by-step, some of them still had trouble."

Teacher B mentioned that being able to change things onscreen, and move them, were helpful to her students. She also pointed out the following benefits: "They could visualize better....They could play also, play around with the diagrams....They were very happy with the colours ... This helped a lot" (interview, April 5, 2000). However, she did comment on the following problems.

"The kids were able to visualize but they ...had difficulty to express themselves."

She believed this difficulty stemmed from the fact that "they're not used to explorations," and to the following: "In class we, .... solve a problem ... so they have an example in front of them that they can follow. But here they did not have an example they could follow and I guess some of the kids did not know how to express themselves" (interview, April 5, 2000)
Teacher B theorised that the sketches help because the students are able to be more active in the learning process (interview, April 7, 2000). Between sessions two and three, she held a discussion with her class and reported:

It develops some kind of interest in them, you know....Today I asked them this question. I said, 'how about, if I bring a movie here, let's say. You watch that movie—you can visualize too... would it be better to be in front of a computer or to watch the movie? What's the difference?' They said, 'Miss. we interact there and we can change the diagrams' and they are right....that's what they can do. ...I can show them a movie where they have all those colours and everything..but they cannot interact... they cannot change anything (interview, April 7, 2000).

Teacher C agreed that the students were "on task" (interview, April 5, 2000), but she was less enthusiastic than the other two teachers. She commented: " I really liked the separating [of the triangles in Day 1, task 1 and in Day 1, task 2]. But other than that... I don't know. I think it's just about the same. ... Like, it's an alternative method--but you know I don't think, 'well, if we do this students will understand deductive geometry."

She argued that the weak students had experienced difficulty with the sketches and that "for the good students nothing would make any difference because they could learn any which way" (interview, April 5, 2000).

Teacher C did admit however, that the visuals might be helpful to students. She said, " I think yesterday at the beginning when they actually saw it separate!...Maybe separating the triangles would help them to visualize it and even if they just did that once then maybe when they got another one they would be better able to picture it separated" (interview, April 5, 2000)

Although some of her statements were negative, teacher C did note that all her students had been serious about the tasks. She also explained her views on possible advantages of using dynamic diagrams in class as follows:

A change is good, right? So, to do this--whether it's for fifteen twenty minutes in a couple of classes or one whole period—it's good for them because then they remember it. ...Like, this might be the part that they remember, right. That they were in the computer lab and that they were flipping those triangles around...I'm visual, so I have to see it. Somebody can talk all they want but if I see something I remember it (interview, April 5, 2000)

She proposed that the sketches might be even more helpful when the idea of congruency is being introduced. She said: "I think something on computers would be very helpful in grade 9
when you're first introducing the congruent triangles, right, because then you can swing them around. You can flip them" (interview, April 5, 2000).

All three teachers insisted that bringing the class together during sessions was very important. Teacher A expressed this best when she said: "The teacher's role is still really important. It's important to bring them together and get them all at the same place--like those two girls. One said: 'you know when you went over the kite? Then I understood'--Well, that's important—that you bring their ideas out and check that they understand" (interview, March 30, 2000).

Teachers all said that they would consider using pre-constructed sketches in some way--either by using them for periods in a regular class, or as teacher A suggested, using them at home as an optional extra (interview, March 30, 2000).

C. The JavaSketches.

Q1: Which affordances of the JavaSketches did students use and what was their impact?

The ability to interact with the image was very appealing to students. As noted in chapter five there were many instances of students getting excited about what they could do with the sketch. JavaSketchpad supports this interaction through the provision of affordances. When I examined the transcripts, certain affordances of the sketches supported student efforts to investigate, but there were some difficulties that must be reported.

As noted in the questionnaire responses (see Tables 6.1 to 6.5), the students gave many positive comments about the motion, colour, and ability to redo features of the program.

Motion.

The flipping of the triangle in Day 1, task 1 helped students visualize the reflective relationship between the two, as evidenced by the comments of three taped pairs and one questionnaire comment. The separate and join command in Day 1, task 2, helped students understand the concepts of common angle, and common side. This was a favourite of students as evidenced by comments of all taped pairs.

Although all sketches could be dragged, students frequently stopped manipulating an image after some initial exploration. On the tapes students are seen at first using the mouse to drag the figure and experiment. They continued to use the mouse to access buttons such as hide/show, but stopped dragging and exhibited gazing and pointing behaviour as they answered
the questions on the labsheet. This caused problems for taped pairs as noted in chapter five. If the sketch displayed a particular case—e.g. if students dragged a triangle until it was right angled.

**Colour.**

Colour drew students' attention. Students noticed items that were coloured and sometimes missed those that weren't—such as the student who didn't see the little box that marked the perpendicular in Day 1, task 1.

The ability to toggle colours off and on allowed students to view particular figures in isolation and to use colour as a simple and effective means of referencing objects in discussion. Students are heard referring to "the red one" or "the blue one" when communicating to their partner, the teacher, and in the whole class discussion.

There were some problems caused by colour disappearing when figures overlapped. Two students noted this on their questionnaires as a recommendation for improvement.

**Ability to redo, tidy up, and repeat.**

Observations confirmed that students used the hide/show capability frequently to view overlapping figures separately. Many also liked to keep the sketch clean and can be heard using expressions such as, "let's get rid of this", or "reload". This is reminiscent of the tendency to start a new paper diagram when the old one gets crowded and confusing.

**Q2: What affordances were not noticed, and what affordances did students need that were missing?**

**Provision of an accurate image.**

*JavaSketchpad* offers teachers the ability to create precise diagrams. I expected the accuracy of the sketches to be an advantage to students but was quite surprised that many taped students were observed to ignore this feature.

Students were heard to say that an angle was 90°, or lengths were equal, or a segment was half of another segment, although they had no evidence on which to base the deduction. I speculate that in some of these cases they may have dragged the diagram to a stage at which the statement looked like it was true and they then treated the diagram as if the dragged case was the only case. However, in other cases, students hadn't (or couldn't have) dragged it to that stage and they made the deduction even though the naked eye would tell them they were wrong.

I was surprised at these obvious errors and considered that they might be the result of student inattention to detail. However, there may be an alternative explanation. Students are
instructed not to trust the appearance of diagrams in their texts. It is possible that they are extending this attitude to dynamic diagrams that, in fact, are accurate.

**Provision of onscreen measurements.**

As noted in the analysis, only two taped pairs mentioned the numerical information that was displayed onscreen. Perhaps this was because the measurements were in a list, rather than being attached to an object. (It is extremely difficult at this point to design a JavaSketch with a measurement that moves with a segment or angle). Another explanation may be that students have been taught to ignore such information in deductive proof.

**Ability to add details.**

The tracing that all students engaged in implies that students wanted to mark the sketch but several students made this need explicit. During the sessions, Pat and Dave mentioned that they would like to circle objects onscreen. As noted in chapter five, Sue wanted to type answers on the screen. Four students included on their questionnaire that they would like to be able to draw onscreen and add marks to the onscreen diagrams.

In the JavaSketchpad environment affordances available in Sketchpad and Cabri such as those for constructing lines, midpoints and other geometric objects, for measuring angles and segments, and for calculating ratios are not available. On reviewing the tapes I did not find any instance in which students commented that they wanted to add details such as extra line segments, to a sketch or to construct geometric objects. Perhaps an explanation for this is that none of these students had used dynamic software before, so they had no experience of constructing with such software, and were unaware of the possibilities. In addition, when designing the sketches I did attempt to outthink the students, to ensure that they could display any additional information they might require.

**Q3: How did the pre-constructed nature of the sketches affect student efforts?**

Students in the study were unaccustomed to an investigative approach, and found the learning tasks very time-consuming. However, since the JavaSketches were pre-constructed, students were able to spend all the available time exploring and developing their responses. From the teacher's perspective this was an efficient use of class time. In planning the sessions all three teachers had been concerned that learning the program would take precious time from the curriculum, but they were all relieved during the first session to observe how easily the students were able to proceed to consider the mathematics.
D. The Labsheets

Q1: What are the functions of the labsheet in the learning task?

From a practical point of view, the labsheet provided written procedures, exploration instructions and questions, and space to record answers. However, after examining student responses I realised that the exploration instructions and questions were an active part of the didactical situation—working with the sketch to create an environment for exploration.

Practical. Since students were often at different places, having written instructions for each sketch was an aid to the teacher. Students didn't need to wait before starting a new sketch, and the teacher could spend her time helping with the concepts.

The labsheets provided space for the students to record their responses to the task questions. This was an essential element; first, because it indicated to students that they were expected to do more at this level than play with the sketches; second, because the teacher could collect it to check for evidence of student understanding or lack of understanding.

Creating an environment. Based on the observations outlined in chapter five, as noted earlier in this chapter the labsheets "intervened" in the learning process by focusing student attention on particular details, prompting students to take action, inviting them to explore and experiment, introducing the element of uncertainty, and by asking students to explain what they knew. To ensure that these interventions encourage students to explore, interpret, deduce, and explain, labsheet questions and statements must be carefully worded and linked to the sketches.

Q2: How do students respond to exploration instructions and questions?

The instructions on the labsheet were intended to facilitate the exploration process and to focus student attention on specific details. On the questionnaire, five students complained that the instructions were unclear, but most students followed the labsheet directions independently.

As noted in the chapter five discussion on wording of questions, it became clear during the examination of the tapes that some of the instructions did not produce the expected results. For example, the word "notice" was too vague. Taped students were seen checking the diagram for the required item or items, but they proceeded to give trivial responses on their labsheets. If the item was colourful or if it moved, the students usually spent some time watching its behaviour. In contrast, onscreen measurements were only mentioned by two taped pairs.

The most fruitful explorations resulted from student responses to a labsheet problem or question. For example, "What shape is ABCD?" in Day 2, task 1, led to animated discussions between student pairs such as Katy and Bea, and Barb and Clara.
### Benefits and Limitations

Mindful of the answers to the research questions, it is helpful to summarise the discussion in terms of overall benefits and limitations of using *JavaSketches* in the secondary school geometry program.

<table>
<thead>
<tr>
<th>Benefit</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motivates</td>
<td>Teachers noted that students were on task and involved throughout the sessions. Students made positive comments about the activities. They specifically enjoyed: - the chance to actively participate; - the interactive nature of the sketches.</td>
</tr>
<tr>
<td>Supports exploration</td>
<td>Students used action buttons to join and separate figures and commented on the fact that these actions helped them understand. Motion capabilities were used extensively. Students played with the figures--some aimlessly, others to investigate particular relationships. Students made frequent use of the ability to restart. The manipulation capabilities permitted the teacher to demonstrate exploration techniques during class sessions.</td>
</tr>
<tr>
<td>Provides opportunities for a range of students</td>
<td>The ability of the students in the study ranged from very weak to excellent; however, all students were able to use the JavaSketches to investigate geometry concepts at some level. - All students used their visualisation skills. - Many students worked at improving their analysis skills. - Some students worked at the ordering level. - At least one student strengthened deductive level skills. Since sketches were pre-constructed, students were able to explore diagrams even if these were beyond their construction skill level.</td>
</tr>
<tr>
<td>Supports reasoning</td>
<td>Students posed and investigated conjectures. JavaSketchpad supported this process by: - providing accurate images</td>
</tr>
</tbody>
</table>
- providing dragging capabilities that allowed students to visually inspect all cases.

Enables communication
Student-student, and student-teacher communication about the geometric objects was facilitated by the use of the JavaSketch as a shared image.
Students used sketch details, especially colour, as referencing aids in discussions.
Students pointed to the image and traced on the screen to illustrate their reasoning to their partner or to the teacher.
Student difficulties with terminology and reasoning were exposed by the need to discuss.
During the class sessions the sketch acted as a focus for the discussion.

Is web-based
The sketches were accessed through a browser, but the study did not make full use of the web-based capabilities of JavaSketchpad. Students were not permitted to explore JavaSketches at online sites, nor did they use the JavaSketches from home. Nevertheless, two students mentioned on their questionnaires that online JavaSketches posted by their teachers would be helpful to them in learning the geometry unit.

<table>
<thead>
<tr>
<th>Limitation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actions dependent on sketch creator's ideas</td>
<td>Students were not always able to carry out a particular action because I had not included that option in the design. The sketch creator must consider possible actions that students might require in conducting an open-ended exploration.</td>
</tr>
<tr>
<td>Sketches alone are insufficient</td>
<td>The provision of the sketch is insufficient to ensure that students engage in meaningful activity. Students require carefully prepared labsheet instructions and questions to guide their investigation.</td>
</tr>
<tr>
<td>Details cannot be added</td>
<td>Several students wanted to add details to the sketches. At the present time this is not possible. Cinderella (1999, Richter-Gebert</td>
</tr>
</tbody>
</table>
and Kortenkamp). A new geometry software program addresses this issue. Students can access a teacher-specified set of online tools to add details to a pre-constructed Cinderella diagram.

Pre-construction limits exposure to terminology

Students had difficulties with terminology. If they had constructed the sketches themselves, they would have selected items such as "point", and "midpoint of segment", thus reviewing basic terms. Pre-constructed sketches present the student with a collection of unnamed objects. Adding a feature to automatically display the name of an object would address this problem.

Cannot be messed up

The JavaSketches in the study did not allow students to observe the results of dragging an incorrectly constructed sketch. It is possible to create a sketch that can be messed up but students would not be able to repair the damage and would need a correct sketch for comparison.

Student mathematical understanding is enriched by focused, yet open-ended opportunities to explore (Towers, 1999). Despite some limitations it is clear from the benefits noted, that JavaSketchpad offers educators an effective medium for designing creative activities to provide such opportunities.

**Designing Learning Tasks with JavaSketchpad**

The activities for the study were designed to help students notice geometric details, explore relationships, and develop reasoning skills related to geometric proof. Initially, I focused on the sketch as the most important element of an activity, with the labsheets as backup. However, an analysis of the data showed that task question and sketch provision must work together.

Although I have used dynamic geometry software with many classes, some of the questions I posed were successful, others were less so. Some task questions were inviting. They excited students and encouraged exploration, just as an inviting question from a teacher would. In turn the sketch supported the required exploration. Some were rug-pulling questions. They led to folding back in some cases and thence to understanding, but they sometimes caused frustration. Since the teacher was not always available, unless the sketch helped students to quickly overcome their puzzlement this acted as a roadblock.
Colour and motion attracted the students' interest, but this was not always enough to help them interpret visual details. Students needed to be prompted to notice particular features and relationships. Vague prompts like "What do you notice?" resulted in scattered and trivial observations, whereas more specific prompts such as "What shape is ABCD?" led to good discussion and conclusions.

The requirement to "Explain" was also vague and did not guide students to clearly communicate their understanding. Students did not use multiple representations such as diagrams and words to express their ideas—perhaps because they were not directly instructed to do so.

Some questions were ineffective because they did not take into account the levels at which the students were working. The question "What additional information can you deduce about point H from the diagram?" (Day 1-task 1) was appropriate for students developing level three skills, however, students still working at level two were unable to carry out the necessary reasoning.

These results show that labsheets and sketch must be very carefully designed together to exploit the features of the software and provide interventions that closely parallel the beneficial interventions of teachers. The sketch designer must create opportunities for students to investigate by including appropriate affordances—not necessarily just buttons to add a line or measure an angle, but also capabilities for revealing information in a novel way. For example, an animated sketch point could provide a dynamic illustration of a range of possible configurations for students who avoid dragging, or who drag aimlessly, to help them move to level three. The labsheet questions must, in turn, help students to notice, interpret, deduce and extend their thinking.

During the analysis I examined more closely how the interventions of the labsheet questions and sketches worked towards these goals. I divided the task questions and statements into five types: 1) Focusing attention, 2) Prompting action, 3) Inviting exploration, 4) Introducing uncertainty, and 5) Checking for understanding. In creating these five categories I was guided by Towers' labels for interventions, (inviting, and checking), and my extensions of these categories. I then examined the responses of the taped students to see what sketch feature corresponded to a particular question type. From this analysis I have developed the guidelines below.

1. When the question attempts to focus student attention so that they will notice and interpret details, the sketch must provide the visual stimulus. It must **Draw attention**.
2. When the statement prompts action, such as asking students to drag, observe or deduce, the sketch must contain the necessary provisions. It must *provide affordances* so that the student can take the required steps.

3. Prompting statements or questions suggest a path, or a required action, whereas questions that invite exploration are open-ended. In order to allow open-ended exploration into uncharted territory, the student requires a sketch that allows options. Thus, when the question invites exploration, the sketch must *provide alternate paths*. It can also *surprise* - which may lead to further exploration.

4. A statement or question that introduces uncertainty is similar to Towers' tug-pulling intervention; however, the teacher is not necessarily there to correct any misinterpretation. Thus, the sketch must *support experimentation* to unmask the confusion. It must be flexible enough to help students examine cases, yet constrained enough to prevent aimless wandering.

5. Questions that check understanding are important parts of any learning situation. One familiar problem is the tendency of students who work alone to gloss over such questions assuring themselves that they know the answers. In these tasks, the checking involved students looking together for the answer. Thus, the sketch aided this process by *providing a shared image* for students to consider and discuss. A printout of the onscreen image could extend this benefit by providing a place for students to communicate their ideas using diagrams in addition to words.

These links between labsheet questions and sketch provisions should provide other researchers with a basis for designing learning tasks around *JavaSketches*, or other pre-constructed sketches. They are summarised in Table 6.6.

### Table 6.6.

**Labsheet-JavaSketch** intervention links.

<table>
<thead>
<tr>
<th>Labsheet</th>
<th>JavaSketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Focusing attention</td>
<td>Drawing attention</td>
</tr>
<tr>
<td>Prompting action</td>
<td>Providing affordances</td>
</tr>
<tr>
<td>Inviting exploration</td>
<td>Providing alternate paths; Surprising</td>
</tr>
<tr>
<td>Introducing uncertainty</td>
<td>Supporting experimentation</td>
</tr>
<tr>
<td>Checking understanding</td>
<td>Providing a shared image</td>
</tr>
</tbody>
</table>

*Questions for Future Research*

Since the use of dynamic geometry software in secondary classrooms is a very recent innovation not much is known—about how to teach with it, and about what to teach with it. This
study attempted to extend our knowledge of dynamic geometry activities using pre-constructed sketches, but there are many other areas that remain to be explored. I suggest the following questions as possible starting points for future researchers:

1. How can teachers help students consolidate and communicate knowledge attained through explorations with dynamic software?
2. What are the visual skills necessary to reason with dynamic images, and how do we support the development of those skills?
3. What supplementary support do students require to use similar web-based applications from home in the absence of partner and teacher interventions?
4. How can we teach students to use symmetry concepts in reasoning and in proof writing?
5. What are the levels of working with visual images, and how do these relate to the levels of geometric thinking?
6. How can we identify the level of visual reasoning at which a student is working?
7. What tasks or experiences will help move students up in their level of reasoning?
8. When is it more beneficial for students to do their own constructions?

**Conclusion**

In this study I set out to observe secondary school students in a dynamic-geometry-supported classroom. I examined the interrelationships between the student and the elements of the learning task in order to describe the benefits and limitations of *JavaSketchpad* with regard to the development of reasoning skills related to geometric proof.

The study results show that *JavaSketchpad* motivates and engages students. It helps students strengthen their geometric thinking skills—especially at the visualisation and analysis levels, by supporting student exploration, visual reasoning, and communication activities.

In addition, by examining the data I identified interrelationships between sketch and labsheet, and extended Towers' teacher intervention categories to describe the impact of the *JavaSketches* and the accompanying labsheets on student responses. From these results I developed guidelines for the design of *JavaSketchpad* learning tasks.

Through my analysis I hope that I have furthered our understanding of the role that pre-constructed dynamic sketches can play in a geometry learning situation. And I hope that my analysis will help teachers and researchers to meet the challenge set by Flewelling (2000a)—to design tasks that are not only do-able, but also worth doing.
References


Becker, H. J. (1990). Effects of computer use on mathematics achievement: findings from a nationwide field experiment in grade five to eight classes: Rationale, study design, and aggregate effect sizes (Center for Research on Elementary and Middle Schools No. 51). Baltimore, MD: Center for Research on Elementary and Middle Schools.


**Software**


Appendix A - Consent Forms

Principal Agreement Form

I, ____________________________, principal of ____________________, consent
(Print name) (Name of school)
to have Margaret Sinclair conduct research for the study entitled:

Developing understanding in the secondary geometry program: A study of the use of
JavaSketchpad sketches,

with selected mathematics classes at ____________________________
(Name of school)

The nature and general purpose of the research procedure has been explained to me by Margaret
Sinclair, B. Math, M.A. who is conducting the research at the Ontario Institute for Studies in
Education of the University of Toronto under the supervision of Professor Gila Hanna. A copy of
the letter explaining the study is attached.

I understand that there are no anticipated risks or benefits to the subjects from participating in the
study and that I may review and comment on any parts of the dissertation that represents this
research before publication.

The researcher is authorised to proceed on the understanding that I may withdraw my consent at
any time, without reason.

Witness ____________________________ Signed ____________________________

(Principal)

Date ____________________________

Please sign two copies and keep one copy for your records
Teacher Agreement

I, ____________________________, consent to participate in the research study entitled: Developing understanding in the secondary geometry program: A study of the use of *JavaSketchpad* sketches.

The nature and general purpose of the research procedure has been explained to me by Margaret Sinclair, B. Math, M.A. who is conducting the research at the Ontario Institute for Studies in Education of the University of Toronto under the supervision of Professor Gila Hanna. A copy of the letter explaining the study is attached.

The researcher is authorised to proceed on the understanding that I may withdraw from the study at any time, without reason.

I understand that there are no anticipated risks or benefits from participating in the study and that I may review and comment on any parts of the dissertation that represents this research before publication.

Witness ____________________________ Signed ____________________________
(Participant)

Date ____________________________

Audio/Videotaping Authorization and Release

I authorise Margaret Sinclair (the Principal Investigator) to create photographs and video and audio recordings of my likeness and/or voice, for use in research activities related to the project: Developing understanding in the secondary geometry program: A study of the use of *JavaSketchpad* sketches.

I understand that these recordings will not be licensed or otherwise provided to other researchers, and will not be published to the public at large, without my additional consent.

_______________________________ (Signature)

_______________________________ (Date)

Please sign two copies and keep one for your records
Parent/Guardian Permission Form for Student Research Participation

I give permission for ___________________________ to participate in the research study entitled: Developing understanding in the secondary geometry program: A study of the use of JavaSketchpad sketches.

I have read the attached letter explaining the nature, purpose and procedures of the study. I understand that Margaret Sinclair, B. Math, M.A. is conducting the research at the Ontario Institute for Studies in Education of the University of Toronto under the supervision of Professor Gila Hanna.

I understand that there are no anticipated risks or benefits to my child from participating in the study.

The researcher is authorised to proceed on the understanding that I may withdraw my son/daughter/ward from the study at any time, without reason.

Signed ___________________________ Date ___________________________
(Parent/guardian)

Parent/Guardian Audio/Videotaping Authorization and Release

I authorise Margaret Sinclair (the Principal Investigator) to take video and audio recordings of ___________________________’s likeness and/or voice for use in research activities related to the project: Developing understanding in the secondary geometry program: A study of the use of JavaSketchpad sketches.

I understand that these recordings will not be licensed or otherwise provided to other researchers, and will not be published to the public at large, without my additional consent.

Signed ___________________________ Date ___________________________

Please sign two copies and keep one for your records.
Student Agreement

I, ________________________________, consent to participate in the research study entitled: Developing understanding in the secondary geometry program: A study of the use of JavaSketchpad sketches.

I have read the attached letter explaining the nature, purpose and procedures of the study. I understand that Margaret Sinclair, B. Math, M.A. is conducting the research at the Ontario Institute for Studies in Education of the University of Toronto under the supervision of Professor Gila Hanna.

I understand that there are no anticipated risks or benefits from participating in the study.

The researcher is authorised to proceed on the understanding that I may withdraw from the study at any time, without reason.

Signed ___________________________ Date ___________________________ 

Student Audio/Videotaping Authorization and Release

I authorise Margaret Sinclair (the Principal Investigator) to take video and audio recordings of my likeness and/or voice for use in research activities related to the project: Developing understanding in the secondary geometry program: A study of the use of JavaSketchpad sketches.

I understand that these recordings will not be licensed or otherwise provided to other researchers, and will not be published to the public at large, without my additional consent.

Signed ___________________________ 
Date ___________________________

Please sign two copies and keep one for your records.
Appendix B

Student Questionnaire

Please Circle One Response

1. Gender: Male  Female

2. Do you have a computer at home?  YES   NO

3. Have you used a computer for math before this study?  YES   NO  
   If your answer was YES, describe how you used a computer for mathematics.
   ____________________________________________

4. JavaSketches: Check all descriptions that match your experience.
   __ Using the sketches was ok.
   __ The sketches really helped me understand the ideas.
   __ The sketches were fun to use but I needed to use the textbook to really understand the ideas.
   __ I liked using the sketches but they were confusing because ___________
   __ Other (please specify) _______________________

5. Did the activities with the JavaSketches help you to develop explanatory proofs for the assignment questions?

   YES   NO

   If YES, describe how the sketches were helpful.
   ____________________________________________

   If NO, describe your difficulties.
   ____________________________________________

6. Please share any suggestions you have for using dynamic sketches in geometry.
   ____________________________________________
Appendix C - Possible Interview Questions

Example Teacher Interview Questions/Topics

Pre-study:
1. Have you used dynamic geometry software in your program before?
   - If so, what were some of the benefits/difficulties?
   - If not, what were your reasons?

2. From your experience how does student achievement in the geometry unit of the grade 12 program compare with their achievement in the other units of the program?
3. What concepts/skills in the geometry unit do students find most difficult?
4. What methods have you found helpful in assisting students who are finding the geometry unit difficult?
5. How do you usually conduct a geometry class?
6. How do you expect your students to write up a proof in geometry?

Post session 1:
1. What were your impressions of the session?
2. What did you notice that your students:
   a) Found helpful?
   b) Had difficulty with?
3. What effect do you think the visual component of the sketches had on the students' follow-up discussion and assignment work?
4. What changes would you recommend for the second session?

Post session 2:
1. What were your impressions of the session?
2. What specific student difficulties did you notice?
3. What features of pre-constructed sketches do you think:
   a) Help the students in their development of proof?
   b) Cause problems for the student?
4. What do you see as your role in the classroom when using sketches such as those used? How would using such sketches impact your teaching of the geometry unit?
5. Would you consider using pre-constructed sketches on your own? If so, what would you require in terms of support?
Appendix D - Labsheets

Day 1 - Task 1
Jskecht2

1. Open Jskecht2.htm as described in the general instructions.

2. Notice that some points in the sketch are red. They are "draggable."
   * Drag each red point and observe the diagram.

   Tip: If the sketch gets really messed up just click Reload in the tool bar.

3. Click [Show Given Information]. Drag point A again and observe the diagram. Explain the meaning of the tick marks and the angle shading.

4. Click each button on the top right. How can the information provided by these images be used to explain why ΔABC is congruent to ΔFCB?

5. Use your observations and the appropriate congruency theorem (SSS, SAS or ASA) to prove that ΔABC is congruent to ΔFCB.

6. Click [Show Perpendicular through H]. What additional information can you deduce about point H from the diagram?

7. Find another pair of congruent triangles in the figure. Explain your reasoning.
Day 1 - Task 2
Jsketch3

1. Open Jsketch3.htm

2. Drag each red point and observe the diagram.

3. Click [Show Given Information] Drag points A and B again and observe the measurements. Write two additional facts that you know and explain why they are true.

4. There are several possible paths to prove that BA = BC.
   - Click [Show Copy] and [Separate] (Either can be moved so they don't overlap).
   - Tip: You may click [Join] at any time to show the triangles together
   - Show each pair of triangles in turn, and answer the following questions in the space provided:
     a) Do you have 3 pieces of information to decide whether the pair is congruent?
     b) If you proved the pair congruent, how would this help you prove BA=BC?

   First Pair
   
   Second Pair
   
   Third Pair
   
   Fourth Pair
   
5. Consider your work in Task 1. What is an alternate explanation for the congruency of triangle ABC and triangle FCB?
Day 1 - Task 3
Jsketch4

1. Open Jsketch4.htm

2. Drag the red points and observe the sketch.

3. Click [Show Given Information] and drag the points again. Write one additional fact that you know is true.

4. What information do you need to have in order to prove that ΔCJL is isosceles?

5. On the attached sheet, shade pairs of triangles that could be used in the proof. Note in the space underneath whether you have enough information to prove them congruent.

6. Click [Show CN]. Write a proof that uses the information provided by CN.

7. Click [Show More]. Drag the red points to examine the diagram.
   1. Is it true that ΔCJ'L' is isosceles? Explain.

   2. What statement should you have included in the Given information for the proofs above?
Day 2 - Task 1
Jsketch1

1. Open Jsketch1.htm

2. Drag each of the red points A, B, and C and observe the diagram.

3. Click [Show Given Information]. Drag the red points again and observe the measurements. What shape is ABCD? Explain.

4. Use your conclusions and the appropriate congruency theorem (SSS, SAS or ASA) to prove that $\triangle AMD$ is congruent to $\triangle CNB$.

5. Click [Show Triangle].
   - Drag point B. What do you notice about the new triangle?
   - Rotate the new triangle using Handle. What do you notice?

6. Click on Move O->Midpoint. [It's easier if you have AC in a horizontal position.]
   - Line the Handle point up with point C. What is the relationship between the new triangle and $\triangle ABC$?
   - Use the Handle to rotate the triangle about point O. What is the relationship between the new triangle and $\triangle ADC$?

7. How can the information provided by these images be used to explain why $DM = BN$?
Day 2 - Task 2
Jsketch6

1. Open Jsketch6.htm

In this task you are not asked to "prove" a statement, but to investigate the question, "When do the diagonals of a parallelogram right bisect one another?"

*The line through EF, perpendicular to AC, has been provided as a measurement tool.*

2. Answer the following:
   - What will be true about the measurements if DB *bisects* AC?
   - What will be true if DB *right bisects* AC?
   - What additional facts will be true if DB and AC *right bisect one another*?

3. Drag the red points in the diagram and conjecture a response to the question. Remember to state the conjecture as an 'If... then' statement.

4. Use the information in the diagram and the appropriate congruency theorem to develop a proof of your conjecture.

5. Outline one alternative proof.
Day 2 - Task 3
Jsketch7

1. Open Jsketch7.htm

2. Prove that AC is the perpendicular bisector of EF.

3. Drag each of the red points and observe the diagram.

4. Click [Show Given Information #1]. Drag the red points again and observe the measurements. What shape is ABCD? Explain.

5. What two facts must be proved to conclude that AC is the perpendicular bisector of EF?

6. Click [Show Given Information #2] and [Show Triangles]. Use these two triangles and the given information to develop the proof.

7. Give an alternate proof explanation.
<table>
<thead>
<tr>
<th>Task</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day 1 - Task 1: Sketch</strong></td>
<td><strong>Appendix E - Lashaud Question Analysis</strong></td>
</tr>
</tbody>
</table>

**Explanation**: Check for understanding of the figure and the diagram.

**Step 1**: Find another part of the diagram to note.

**Step 2**: Provide shared image.

**Step 3**: Supporting explanation.

**Step 4**: Drawing attention - document.

**Step 5**: Prompting action.

**Step 6**: Check show reference through H & A.

**Step 7**: Check the highlighted ACB.

**Step 8**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 9**: Check the highlighted ACB.

**Step 10**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 11**: Check the highlighted ACB.

**Step 12**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 13**: Check the highlighted ACB.

**Step 14**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 15**: Check the highlighted ACB.

**Step 16**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 17**: Check the highlighted ACB.

**Step 18**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 19**: Check the highlighted ACB.

**Step 20**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 21**: Check the highlighted ACB.

**Step 22**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 23**: Check the highlighted ACB.

**Step 24**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 25**: Check the highlighted ACB.

**Step 26**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 27**: Check the highlighted ACB.

**Step 28**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 29**: Check the highlighted ACB.

**Step 30**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 31**: Check the highlighted ACB.

**Step 32**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 33**: Check the highlighted ACB.

**Step 34**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 35**: Check the highlighted ACB.

**Step 36**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 37**: Check the highlighted ACB.

**Step 38**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 39**: Check the highlighted ACB.

**Step 40**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 41**: Check the highlighted ACB.

**Step 42**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 43**: Check the highlighted ACB.

**Step 44**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 45**: Check the highlighted ACB.

**Step 46**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 47**: Check the highlighted ACB.

**Step 48**: Check the highlighted AS, SAS, or ASY to prove the conclusion.

**Step 49**: Check the highlighted ACB.

**Step 50**: Check the highlighted AS, SAS, or ASY to prove the conclusion.
<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Consider your work in Task 1. What is an alternate explanation for the confusion of the image of ABC and...</td>
</tr>
<tr>
<td>2.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
<tr>
<td>3.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
<tr>
<td>4.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
<tr>
<td>5.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
<tr>
<td>6.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
<tr>
<td>7.</td>
<td>Check for understanding of the image of ABC and...</td>
</tr>
</tbody>
</table>

Task 2: Sketch

---

**Day 1**

- **Task 1**: Sketch
- **Task 2**: Sketch
<table>
<thead>
<tr>
<th>Instructions</th>
<th>Drawing attention - colour, movement</th>
<th>Providing shared image</th>
<th>Supporting explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Open sketchbook</td>
<td>Focusing attention - visual</td>
<td>Check for understanding</td>
<td>Checking for understanding</td>
</tr>
<tr>
<td>2. Draw the red points and observe the sketch</td>
<td>Focusing attention - detail</td>
<td>Providing action</td>
<td>Promoting action</td>
</tr>
<tr>
<td>3. Check Show CN</td>
<td>Focusing attention - detail</td>
<td>Providing shared image</td>
<td>Promoting action</td>
</tr>
<tr>
<td>4. What information do you need to have in order</td>
<td>Focusing attention - visual</td>
<td>Drawing attention - hide/show</td>
<td>Providing action</td>
</tr>
<tr>
<td>5. On the attached sheet, shade parts of triangles</td>
<td>Check Show CN</td>
<td>Supportive explanation</td>
<td>Drawing attention - colour, movement</td>
</tr>
<tr>
<td>6. Check Show CN</td>
<td>Corrects understanding</td>
<td>Checking for understanding</td>
<td>Checking for understanding</td>
</tr>
<tr>
<td>7. Check Show CN</td>
<td>Corrects understanding</td>
<td>Checking for understanding</td>
<td>Checking for understanding</td>
</tr>
</tbody>
</table>

Given information for the tasks above:

- Is it true that ACG is isosceles? Explain.
- Bring the red points to examine the diagram.
- What a proof that uses the information provided by.
- Through information to prove them correct.
- Note in this space underneat whether you have.
- That could be used in the proof.

Day 1 - Task 3 - Sketches
<table>
<thead>
<tr>
<th>Instruction</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 2 - Task 1 - Sketch</td>
<td></td>
</tr>
<tr>
<td><strong>Provide detailed drafts</strong></td>
<td></td>
</tr>
<tr>
<td>Drawn with attention - colour, movement</td>
<td></td>
</tr>
<tr>
<td>Providing shared image</td>
<td></td>
</tr>
<tr>
<td>Supporting explanation</td>
<td></td>
</tr>
<tr>
<td>How can the information provided by these images</td>
<td></td>
</tr>
<tr>
<td>VADID?</td>
<td></td>
</tr>
<tr>
<td>Use the handle to outline the image shown point 0</td>
<td></td>
</tr>
<tr>
<td>Check the relationship between the new handle and VAD</td>
<td></td>
</tr>
<tr>
<td>Line the handle point up with point C which is the</td>
<td></td>
</tr>
<tr>
<td>AV in a horizontal position</td>
<td></td>
</tr>
<tr>
<td>Y in the same position</td>
<td></td>
</tr>
<tr>
<td>K. Check on Move O - Withpos, it's easier if you have</td>
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<tr>
<td><strong>Provide shared image</strong></td>
<td></td>
</tr>
<tr>
<td>Drawn with attention - colour, movement</td>
<td></td>
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<tr>
<td>Supporting explanation</td>
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<tr>
<td>Checking for understanding</td>
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<tr>
<td>What do you notice about the new</td>
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<td>Drawn with attention - colour, movement</td>
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<tr>
<td>What do you notice about the new</td>
<td></td>
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<tr>
<td><strong>Drawn with attention - hidden show</strong></td>
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<tr>
<td>Providing explanation</td>
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<tr>
<td>Checking for understanding</td>
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<tr>
<td>What do you notice about the new</td>
<td></td>
</tr>
<tr>
<td><strong>Drawn with attention - visible</strong></td>
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<tr>
<td>Providing explanation</td>
<td></td>
</tr>
<tr>
<td>Checking for understanding</td>
<td></td>
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<tr>
<td>Providing explanation</td>
<td></td>
</tr>
<tr>
<td>What do you notice about the new</td>
<td></td>
</tr>
<tr>
<td>Sketch</td>
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</tbody>
</table>
# Day 2 - Task 2 -- Jsketch6

<table>
<thead>
<tr>
<th>Instructions</th>
<th>Labsheet</th>
<th>Sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Open Jsketch6.htm</td>
<td>Managing</td>
<td>Supporting experimentation</td>
</tr>
<tr>
<td>In this task you are not asked to &quot;prove&quot; a statement, but to investigate the question, &quot;When do the diagonals of a parallelogram right bisect one another?&quot;</td>
<td>Telling: Inviting explorations</td>
<td></td>
</tr>
<tr>
<td>The line through EF, perpendicular to AC, has been provided as a measurement tool.</td>
<td>Managing</td>
<td>Providing affordances</td>
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<tr>
<td>2. Answer the following:</td>
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<tr>
<td>• What will be true if DB right bisects AC?</td>
<td>Leading: Enculturating - mathematics; Providing shared image; Providing data</td>
<td></td>
</tr>
<tr>
<td>• What will be true if DB bisects AC?</td>
<td>Leading: Enculturating - mathematics; Providing shared image; Providing data</td>
<td></td>
</tr>
<tr>
<td>• What additional facts will be true if DB and AC right bisect one another?</td>
<td>Leading: Enculturating - mathematics; Providing shared image; Providing data</td>
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<tr>
<td>3. Drag the red points in the diagram and conjecture a response to the question. Remember to state the conjecture as an 'if...then' statement.</td>
<td>Focusing attention - colour, movement Prompting action</td>
<td>Drawing attention - colour, movement Proving affordances</td>
</tr>
<tr>
<td></td>
<td>Enculturating - mathematics; Inviting exploration;</td>
<td>Providing alternate paths; Supporting experimentation</td>
</tr>
<tr>
<td>4. Use the information in the diagram and the appropriate congruency theorem to develop a proof of your conjecture.</td>
<td>Prompting action Checking for understanding</td>
<td>Providing shared image</td>
</tr>
<tr>
<td>5. Outline one alternative proof.</td>
<td>Inviting exploration</td>
<td>Providing alternate paths</td>
</tr>
<tr>
<td>Instructions</td>
<td>Prompting actions</td>
<td>Initial explanation</td>
</tr>
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<td>--------------</td>
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</tr>
<tr>
<td>1. Open Sketch.js file</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Prove that AC is the perpendicular bisector of EF</td>
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<tr>
<td>3. Drag each of the red points and observe the change</td>
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<tr>
<td>4. Check Show Given Information #1</td>
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<tr>
<td>5. What shape is ABCD? Explain.</td>
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<tr>
<td>6. Click Show Given Information #2 and Show</td>
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<tr>
<td>Drawing attention - click</td>
<td>Focusing attention - visual</td>
<td></td>
</tr>
<tr>
<td>Drawing shared Image</td>
<td>Checking for understanding</td>
<td></td>
</tr>
<tr>
<td>Drawing attention - click</td>
<td>Focusing attention - visual</td>
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</table>

Day 2 - Task 3 - Sketch.js
Day 1 - Task 1 - Isketch 2

1. View on Opening:

PROVE: Triangle ABC congruent to Triangle FCB

- Show Given Information
- Hide
- Separate ABC and FCB
- Show reflection and mirror
- Hide Reflection
- Match FCB and A'B'C'
- Reset

![Diagram of triangles ABC and FCB with instructions to prove congruence]

- Show perpendicular through H
- Hide
PROVE: $BA = BC$

1. View on Openline:
   1. Task 1 - Fetch 2 - Fetch 3
2. View all hidden - This view shows pair #1 coloured. Pairs 2,3 and 4 can be similarly displayed

PROVE: BA = BC
PROVE: Triangle CJI is isosceles

1. View on opposite

Day 1 - Task 3 - Sketch a
2. View all hidden

PROVE: Triangle CJL is isosceles

- Show Given Information
- Hide
- Show CN
- Hide
- Show More
- Hide

$\angle CEG = 59^\circ$
$\angle CGE = 59^\circ$
$JE = 2.7 \text{ cm}$
$LG = 2.7 \text{ cm}$
PROVE: Triangle AMD is congruent to Triangle CMB

1. View on Opening

Day 2: Task 1 - Sketch 1
PROVE: Triangle AMD is congruent to Triangle CMB

2. View all hidden items

Try 2. Task 1: Sketch 1
When do the diagonals of a parallelogram bisect one another?

1. View on Opposite

Day 2 - Task 2 - Sketch 6
When do the diagonals of a parallelogram bisect one another?

2. View all hidden lines

Day 2 - Task 2 - Sketch 6
PROVE: AC bisects EF and AC is perpendicular to EF.

1. View on opposite

Day 2 - Task 3 - Jketch
Prove: AC bisects EF and AC is perpendicular to EF.
Day 2 - Group - Jsketcha

1. View on opening

PROVE: Triangle ABC is isosceles
PROVE: Triangle ABC is isosceles.

2. View all hidden - Triangles #2 and #3 are not shown in this view.