Establishing the Practical Frontier in Data Envelopment Analysis

By

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Abstract

Data Envelopment Analysis (DEA) assigns a score to each production unit (DMU) considered in the analysis. Such score indicates whether the unit is efficient or not. For inefficient units, it also identifies a hypothetical unit as the target and thus suggests improvements to their efficiency. However, for efficient units no further improvement can be indicated based on a DEA analysis. Nevertheless, it is important for management to indicate targets for their efficient units if the organization is to improve as a whole. If the inputs and outputs of efficient units can be varied within a specified range, then it is possible to find other combinations of inputs and outputs from which new, “artificial”, DMUs can be created. These DMUs are constrained to be more efficient than the DEA efficient unit from which they were created.

This thesis presents a linear programming model and a methodology for improving the efficiency of empirically efficient units by defining a new “Practical Frontier” and utilizing management input. This new frontier allows the analyst to identify adjusted efficiency scores for DMUs which were on the frontier when only real DMUs were considered. The new frontier, formed mostly from the new, artificial DMUs, thus ranks the efficient units which will now have scores less than 1.0. Available bank branch data was used to illustrate the applicability of this theoretical development.

To my dear family

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CHAPTER 1

Introduction

This chapter presents an overview of the thesis. First a background on measuring productivity and efficiency is given which leads to the need for this research. A detailed problem definition and research objectives are presented next. Then the approach used to address the goals is outlined. This chapter concludes by presenting the organization of the rest of the document.

1.1. Background

There has been an increasing emphasis on measuring and comparing the efficiency of organizational units such as bank branches, where there is a relatively similar set of units. The growing competition and globalization provides additional motivation to these efforts. The traditional measure of efficiency, which is the ratio of output to input, is often inadequate due to the existence of multiple inputs and outputs related to the different resources and activities of units. Other concerns in assessing performance are how to improve those who are

not efficient and how to persuade them to accept the results without any “push back”.

Economic production process is a term used by economists to describe a process that transforms inputs into useful outputs. They use *production functions*, which describe input and output relationship, to determine the optimum potential for a production unit. A production function, or *production frontier*, is the frontier of the production possibility set and is used to measure efficiency. However, the true frontier is not known and an empirical frontier is usually constructed based on the production data, which is the observed and achieved levels of inputs and outputs.

There are two main techniques for measuring *productive efficiency*, which describes how well a production process transforms resources into useful outputs. The parametric method, as exemplified by econometric approaches, requires an explicit formation of the production functional form. The non-parametric approach as represented by Data Envelopment Analysis, provides greater flexibility since it does not require a priori assumptions on the functional relationship of inputs and outputs.

Data Envelopment Analysis is a powerful technique for measuring the relative efficiency of organizational units with multiple inputs and outputs. This novel approach was introduced by Charnes, Cooper and Rhodes in 1978, and is gradually becoming a useful management tool. In addition to the efficiency score, DEA indicates targets for inefficient units. These targets, which are shown to the inefficient units as models, are their actual peer units, therefore they are more likely to be accepted by these units. DEA's advantages over other methodologies resulted in widespread application of this technique in over 50 industries.

1.2. Problem Definition

Data Envelopment Analysis is a linear programming technique which gives a single measure for efficiency. The method has the ability to simultaneously handle multiple inputs and outputs without requiring any judgments on their relative importance, so it does not need a parametrically driven input and output production function.

A DEA analysis provides a variety of valuable information. It establishes a best practice frontier among the units based on comparison process. The units on this frontier are efficient units and the rest deemed inefficient. The level of inefficiency is measured by the unit's distance from this frontier. One of the important advantages of DEA is its ability to identify performance targets for inefficient units and indicate what improvements can be made to achieve Pareto-Efficiency. Since, in the real world, all inputs and outputs of inefficient units cannot be adjusted as one might wish to, Kao [Kao94] presented a modified version of DEA in which bounds are imposed on inputs and outputs. The results from his proposed model provide efficiency improvement for management, which is feasible in practice.

DEA limitation is that it does not provide a mechanism for improving the performance of the best practice units that form the frontier. Therefore, for efficient units, no further improvement can be considered based on DEA results alone. Yet, improving the efficiency of DEA efficient units is very important for management.

1.3. Thesis Objectives

The objectives of this research are:

- To define targets for empirically efficient units by establishing a new frontier in DEA based on new mathematical developments and utilizing management's opinion.
- To test the solution approach with real data and to examine if it is a suitable approach.

1.4. Solution Approach

I have shown in my research that to find targets for DEA efficient units a new frontier, which I call "Practical Frontier" can be defined based on new mathematical developments in DEA utilizing management input. This new frontier, which envelops or touches the normal DEA frontier, allows the analyst to identify the adjusted efficiency scores for empirically efficient units and to rank them based on their new scores. Figure 1-1 shows the theoretical, practical and empirical frontiers for the case of one input and one output. In this figure the curve shows the theoretical frontier, which of course is unknown in any analysis where human performance is studied.

FIGURE 1-1: THE THEORETICAL, PRACTICAL AND EMPIRICAL FRONTIERS

1.5. Illustrating the Theory Using Banking Data

The Canadian banking industry, with over 8,000 branches around the country, is one of the fundamental strengths of the Canadian economy. Canadian banks can be divided into two categories, *Schedule I* and *Schedule II* banks. Schedule I or Class A banks are majority owned by Canadians. Their shares are traded on the major stock exchanges and no one party is allowed to own more than 10% of the shares. Schedule II or Class B banks are foreign owned and have the same power as Schedule I banks. Schedule I banks control about 90% of the total Canadian bank assets and operate in an environment similar to each other.

The Canadian banking industry is experiencing spirited competition due to rapid technological and major legislative changes in Canada and abroad.

Therefore, they need to continuously analyze and improve their performance in order to meet these challenges and remain market leaders.

Performance evaluation and efficiency measurement is an important issue for managers since the inherent inefficiencies can be identified and eliminated. Measuring the banks' performance has been widely based on a number of key performance indicators (KPIs); however, each of these indicators gives an incomplete picture of the banks' performance. In order to have a meaningful overall measure of their efficiency a more sophisticated method than the traditional performance measurement techniques is needed.

Data Envelopment Analysis (DEA) is a non-parametric technique which has proven to be useful for the efficiency analysis of service organizations. It measures the relative efficiency of a group of similar units and identifies the best practice frontier. It also indicates targets for inefficient units to improve. Different studies have been done in the financial services industry in different countries using DEA. However, none of them focused on improving the efficiency of a unit which has been considered efficient in the DEA analysis. This provides a strong research opportunity to investigate methods to increase the performance of a DEA efficient unit which helps the banks to improve even their best practice units and gain more competitive advantage. Available bank data will be used to illustrate the applicability of this theoretical development.

1.6. Thesis Organization

The structure of the rest of this thesis is as follows:

- Chapter 2 provides a thorough literature review in the area of performance assessment in service organizations and Data Envelopment Analysis. It identifies different approaches used for

measuring productivity in service organizations and explains the advantages of Data Envelopment Analysis over other methods. The development of this methodology, its application areas, and important advances in DEA are highlighted.

- Chapter 3 is dedicated to the topic of DEA. It explains the mathematical formulation of four basic DEA models (CCR, BCC, Additive and Multiplicative) along with the associated terminology. It presents key extensions of DEA: incorporating non-discretionary variables and categorical inputs and outputs into the basic DEA model. The DEA model characteristics, units and translation invariance, are introduced. The last section of this chapter discusses the complete process of using DEA: choosing the units, the inputs and outputs, correlation analysis, and collecting the data. The short-term and long-term management usages of DEA results are also explained in this chapter.
- Chapter 4 explains the solution approach. The model and methodology to define a new frontier in DEA is explained. The mathematical formulation of the model is presented and the variables are explained. The limitations of the model are also illustrated. The approach is extended to the log-linear frontier and its mathematical formulation is presented.
- Chapter 5 describes the preliminary test. It illustrates how the data was obtained and how the DEA models were constructed. It also introduces the initial analysis of the data. It presents the results of the DEA models and how the management's opinion was incorporated in the model. Finding the new units and establishing the new frontier are described. It also summarizes the management usages of the results.

- Chapter 6 reports on a sensitivity analysis test of the results as these relate to the parameters defined by management in the P-DEA model. It also describes the different models and compares the frontiers constructed from these models.
- Chapter 7 presents the conclusions of the work and provides recommendations for future research. It also summarizes the contributions of this work.
- Appendix A – contains the bank branch data used in the DEA analysis.
- Appendix B – contains the correlation analyses plots
- Appendix C – contains the inputs and outputs of the new units created from changing the parameters in the P-DEA model (sensitivity analysis) and the efficiency scores of all the units for different models.

CHAPTER 2

Literature Review

This chapter provides a thorough review of the literature in the area of performance assessment and Data Envelopment Analysis. Productivity in service organizations, its elements and different techniques available to help manage service organization productivity are described in the first part. Literature relating to the powerful and increasingly popular productivity management technique, Data Envelopment Analysis, is presented in the second part of this chapter.

2.1. Performance Assessment

There has been a great deal of effort spent in the process of counting, measuring and comparing people performance levels in government and business. Once the intention is to measure how well an organization is performing and how much it could improve, an appropriate performance management technique must be chosen.

Performance assessment in the private sector is typically based on ratios. The best-known ones are financial ratios. The popularity of these ratios is mainly due to their simplicity and ease of calculation, however, each ratio gives only a partial picture of a company's health. Sometimes, different ratios can present a very different picture. One of these efforts led economists to develop a more complex indicator known as the Z-score, which is a composite measure comprising of the weighted sum of some of the key financial ratios, often used to measure corporate financial well being.

In the public sector, where profit is not an objective such as in health care, social services and education, a wide range of performance ratios have been used. The most fully developed of these are "Performance Indicators (PIs)" used in U.S. National Health Service. [Norm91]

Performance assessment is the key to progress in any organization. In order to be competitive, improving productivity is an important issue. Two different approaches for improving productivity were discussed in [Kao95] based on the data gathered from 15 machinery firms in Taiwan. One approach, the *efficiency approach*, refers to improving productivity via internal cooperation without consuming extra inputs. Another approach, *effectiveness approach*, is to increase the level of technology and management but this typically requires additional capital investments. It was explained that technology and management are two broad categories of factors, which have major influences on productivity. Raising the level of technology implies hiring more skilled persons and purchasing advanced machines, while introducing new management techniques or carrying out the management tasks in a better way increases the level of management. Both approaches have cost implication, however.

2.1.1. Efficiency Elements

In many papers written on the subject of performance measurement, there has been confusion in the use of the terms: *efficiency*, *effectiveness* and *productivity*. The reason is that these terms are related to each other. Sherman [Sher88] explained that for a manager these terms are so close and indeed efficiency can be viewed as a part of effectiveness. Effectiveness is the ability of an organization to attain its pre-determined goals and objectives; i.e. to do the right job. Efficiency is the ability to attain the outputs with a minimum level of resources; i.e. to do the job right. Productivity is commonly defined as the ratio of outputs to inputs. It is comprised of several components or efficiency elements, which are: price efficiency, allocative efficiency, technical efficiency, and scale efficiency [Sher88]. These elements influence the overall efficiency of the organization.

Price efficiency is the efficiency of the organization to purchase the inputs that meet the quality standard at the lowest price. *Allocative efficiency* gives a measure of whether the organization is using the optimal mix of inputs to produce outputs for example a bank's use of automatic teller machines versus reliance on tellers or customer service representatives. *Technical efficiency* is the efficiency in converting inputs to outputs. Technical inefficiency exists when it is possible to produce more outputs with the inputs used or to produce the present level of outputs with fewer inputs. *Scale efficiency* examines whether an organization is operating at its optimal size. Producing more or fewer goods or services than the optimal level results in added costs only due to the volume and size. A comprehensive productivity management approach requires explicitly recognizing, analyzing and managing all of these components.

2.1.2. Productivity Management Techniques

Different techniques have been used to evaluate and manage productivity, however generally, one approach is not sufficiently comprehensive and adequate to use alone. Moreover, based on differences in organizations, like leadership style, environment, culture and resources available, any one or even a group of techniques may not necessarily be equally useful for all organizations.

2.1.2.1. Standard Cost System

When a good standard cost system is available, then the manager can compare the actual cost of services to this standard and determine whether the organization is producing these services efficiently. This approach has been used in management accounting and manufacturing rather than service organizations, since the effective costs are rarely known for services.

Most of the time the standard cost does not exist even in manufacturing and most systems use historical standards instead. The historical cost is not necessarily an efficient cost. It is the actual cost of service or product in prior periods. Using historical standard cost, managers can realize whether the operation is below or above the efficiency levels of the past. It does not actually indicate whether the operations are actually efficient or not and therefore it is not sufficient for modern productivity management. [Sher88]

2.1.2.2. Comparative Efficiency Analysis

When an efficient standard is not available, comparative efficiency analysis (CEA) is generally used to evaluate productivity. In CEA the performance of an organization is compared to judgments, opinion, past history or other organizations. When considering CEA techniques, one needs to understand their potential limitations which are: inherent flaws in the benchmark may incorrectly

indicate problems in the organization instead of benchmarking problems, the historical standard may be considered as the efficient standard over time and its problems may be forgotten. [Sher88]

2.1.2.3. Ratio Analysis

Ratio analysis is used to compare various aspects of performance among comparable units and within a single unit over time. When an efficient standard is available, the ratio of standard to actual resources used or actual to standard output produced represent a measure of efficiency, otherwise other ratios such as cost per unit of output may be calculated and analyzed.

Different ratios may be used to comprehend the results and capture different types of inefficiency. Although, ratio analysis has been extensively used for productivity measurement, especially in service organizations, it has several limitations, which was alluded to in section 2.1. [Sher88], [Norm91]

2.1.2.4. Profit and Return on Investment Measures

Profitability and return on investment (ROI) are two extensively used ratios in service organizations as well as manufacturing, excluding government operations since they are non-profit organizations where Cost/Benefit ratios are used (see [Herz80] for more discussion on non-profit organizations). Profitability is defined as the ratio of income to revenue and ROI is defined as the ratio of net income to invested capital.

In services, the main investment is in human resources (rather than capital equipment), which is expressed as an ongoing expense, not as an asset for financial accounting purposes. Since training and hiring costs reduce income, ROI

measures in services result in different sets of relationships than in manufacturing. These differences are important to managers who are responsible for evaluating, managing and allocating resources between both service and manufacturing units.

Although ROI is a more comprehensive measure when compared to profitability, which ignores the amount of invested funds used to generate profits, they are both subject to short-term bias. Using these ratios, current performance can appear strong, sacrificing long-term performance, for example delayed training and hiring will increase current profits but will likely reduce benefits from the lack of these activities.

2.1.2.5. Zero-Base Budgeting

In many service organizations where actual performance is compared to a budget, there are no standards to develop a budget that determines the revenue and expenses related to operations. Zero-Base Budgeting (ZBB) is a useful tool for managers to develop budgets where standards are not available. It is most appropriate for service areas where little or no revenue is generated and when determining the efficient and effective amount of resources needed for service objectives is difficult. Therefore, it is applicable to most government activities.

In this approach, managers separate their departments' activities into decision-making units and define the functions, goals, costs and the most feasible and alternative programs to attain the goals of the decision-making unit in a decision package. Based on the costs and benefits of the proposed program, each decision package is evaluated and then ranked in order to select those that are important to the organization. Allocating resources is based on the ranking process and resources are normally assigned to the decision packages with the highest ranking.

ZBB is a useful and widely used technique for managing productivity in the service environment where standards are not available (a survey indicates that many large corporations have found ZBB to be beneficial in managing internal staff functions [Sher88]), however it does not typically lead to increased productivity or reduced costs.

2.1.2.6. Program Budgeting

Program budgeting, referred also as Program Planning and Budgeting Systems, is an approach to assess the adequacy of specific services or group of services identified as programs of an organization. This is done by comparing the resources used by the program to generate revenue. Tax collection, higher education, elderly care and postal services can be some examples of programs.

Program budgeting enhances the result of resource allocation and indirectly improves productivity. Programs with unjustified costs and benefits may be terminated or cut back and resources from these programs may be assigned to other programs in order to increase productivity. Segregating the costs and benefits of one program from other programs in the organization should result in more focus on ways to improve each program and consequently improving productivity. The reader is referred to [McCl71] and [Anth80] for further discussion on program budgeting.

2.1.2.7. Best Practice Analysis

Best practice analysis is a useful approach when the efficient standard is not available and a historical standard is not reliable. By comparing the operating methods, outputs and resources used of individuals, groups or organizational units

that provide similar services, a benchmark for efficient operations can be established.

Developing a standard that is efficient in resource utilization and meets the quality and service objectives of an organization is the result of careful analysis, discussion and negotiation of how the service should be produced. Therefore, sharing information among service providers is desirable but obtaining such information and implementing it across competing business organizations is unlikely. This approach is suitable for governments where sharing data is not a problem.

Best practice standards are more likely achievable since they are based on actual operating experience of a comparable unit, although it may require major operating changes in the organization. In some situations when there are differences in culture, size, geographical location, etc. between the organization and the best practice standard, targeting such standard may not be achievable. The key benefit of this approach is its ability to develop standards for service activities, however, it is notable that the benchmarks provide a best practice which may not actually be an efficient one. [Sher88]

2.1.2.8. Peer Reviews

In this approach, outside professionals or consultants, who have a broad range of knowledge and experience, provide input in quality, effectiveness and efficiency of an organization. One of the outcomes of the Peer Review process is that the input may provide information that leads to improved productivity. Even if the review does not improve productivity, it does assure top management that the issues of productivity have been explicitly considered and the organization has had the opportunity to benefit from the perspective of knowledgeable professionals. [Chur82],[Sher88]

2.1.2.9. Management Reviews

This is more comprehensive than a Peer review and the results are presented to a third party, usually a senior management group, in addition to the audited unit. Typically, individuals from outside the organization, including analytical experts and qualified peers, conduct the review. In order to assess the efficiency and effectiveness of an organization reviewers analyze financial, operational and managerial performance. The audit may provide an analysis of past and present issues and propose solutions to improve productivity within the organization. [Herb79], [Thom86], [Sher88]

2.1.2.10. Activity Analysis

This technique is used to establish the normal, more efficient and less efficient units among similar units by comparing the time employees spent in each unit. First, a profile of all tasks and functions need to be prepared. Then, the employees are asked to estimate the time spent on each activity. Using basic statistical analysis for the responses, an array of time allocations by personnel type to each of the work functions can be developed. Management evaluates these results according to the organization's objectives. This may result in reallocation of tasks to individuals most qualified to perform them and indicate activities where inappropriate time is devoted to certain tasks based on the importance of that task.

The insight provided by the array of time spent on functions can lead to productivity improvements; however, the accuracy of the data is difficult to determine since it relies on the ability of employees to estimate their time spent in each activity. [Schr85]

2.1.2.11. Process Analysis - Functional Cost Analysis

Process analysis evaluates the work methods and procedures of services within a system. This requires a detailed review of each procedure and the development of flow charts. Analysts consider alternative procedures, design of the job function and layout of service facilities to make the process more efficient.

Functional cost analysis is a type of Process analysis, which examines the cost of each function or activity. In order to improve productivity, analysts modify the process of the more costly functions or activities with the goal of cost reduction.

Activity analysis, Process analysis and Functional cost analysis are variations of essentially the same approach. This approach examines the existing system and refines it to improve productivity. Different statistical analysis and simulation techniques may be used to indicate the effects of changes on the system. The results of the analysis rely on the analysts' skills and the scope of the analysis. [Schr85]

2.1.2.12. Staffing Models

This approach indicates the personnel needs for activities and is generally developed from other approaches such as process analysis and activity analysis. It assists managers in staff allocation based on activities' level and determines where excess resources are being utilized. It was primarily developed for services with multiple locations, i.e. branches or outlets.

There are different approaches to manage productivity in service organizations and the important challenge for managers is the selection and use of one or a combination of these techniques. [Sher88]

The next section provides a detailed literature review of Data Envelopment Analysis.

2.2 Data Envelopment Analysis

DEA is a relatively new technique in productivity management. Although the first paper written on DEA was in 1978 [Char78], the practitioner community has been slow in adapting the technology to real life problems, perhaps because it is a more complex method than some of the other approaches. Moreover, DEA research is almost entirely by academics and there are very few who are motivated to transfer the technology to organizations. Another issue is that to prepare the DEA model, the analyst is required to thoroughly understand the strengths and limitations of DEA. It is admittedly more difficult to apply than ratios, regression analysis and many other well used methodologies. This is unfortunate because it is a very powerful technique which withstands well the typical objections by those being measured. DEA establishes a best practice group of units, identifies inefficient units compared to the best practice group and quantifies the amount of potential improvement possible for each inefficient unit. In simple terms DEA indicates the level of resources savings and/or services improvements possible for each inefficient unit if it is to achieve the level of efficiency of the best practice units.

2.2.1. Background

There are two empirical approaches for measuring efficiency. One is the parametric approach, favored by economists. In this approach, the form of the production function is either known or is estimated statistically. In many cases,

however, the functional form of the production function is not known. Farrell's method [Farr57] of computing the facets of the efficient function from a set of observations was the foundation for non-parametric approaches in measuring efficiency and productivity. In the non-parametric approach, no assumptions are made about the form of the production function. Instead, a best practice function is built empirically from observed inputs and outputs. [Norm91]

Charnes, Cooper and Rhodes' research in 1978 [Char78] forms the basis for subsequent developments in non-parametric approaches used to evaluate relative efficiency, based on Farrell's pioneering work. They introduced Data Envelopment Analysis, which is an operational research methodology based on a linear programming technique used to measure the relative efficiency of Decision Making Units (DMUs). DEA is especially useful where the presence of multiple inputs and outputs makes conventional, ratio-based comparisons difficult. It does not require any judgment as to the relative importance of inputs and outputs. It has received significant attention from academia in recent years with over 1,200 publications in existence [Sinu98].

2.2.2. Different DEA Models

In their original DEA model, Charnes, Cooper and Rhodes (CCR) adopted a ratio definition of efficiency. It generalizes the single-output to single-input classical engineering ratio definition to multiple inputs and outputs without requiring preassigned weights.

In the CCR model, it is proposed that the efficiency of any DMU can be obtained as the maximum of a ratio of weighted outputs to weighted inputs subject to the condition that similar ratios for every DMU are less than or equal to one. Using the fractional programming theory, the ratio optimization problem is transformed into an ordinary linear programming problem [Char62]. To obtain the

efficiency of all DMUs, it is necessary to solve a series of linear programs, one for each DMU as the objective function.

DEA identifies the most efficient units and indicates the inefficient units in which real efficiency improvement is possible. The amount of resources saving or services improvement that can be achieved by each inefficient unit to make them efficient is identified and can be used as indications for management action.

Banker, Charnes, and Cooper in 1984 [Bank84b] introduced the BCC model in which the envelopment surface is variable returns to scale. The CCR model is employed to estimate the overall technical and scale efficiency of a DMU. However, the BCC model takes into account the possibility that the most productive scale size may not be attainable for a DMU which is operating at another scale size. It estimates the pure technical efficiency of a DMU at the given scale size of operation.

Charnes et al. in 1982 [Char82] developed a multiplicative model for efficiency analysis. It has a theory similar to that of the CCR model; however, a multiplicative combination instead of an additive combination of outputs and inputs were used to achieve virtual outputs and inputs, and it has a piecewise log-linear envelopment surface.

The additive model was developed in 1985 by Charnes et al. [Char85a]. While it has the same envelopment surface as the BCC model, i.e. variable returns to scale, it projects the inefficient units onto the envelopment surface by decreasing their inputs and increasing their outputs simultaneously.

2.2.3. DEA Advantages

The efficiency of a machine can be determined by comparing its actual output to its engineering specifications. However, when we consider service organizations, we generally do not know what the optimum efficiency is and therefore we cannot determine whether a service unit is absolutely efficient. DEA enables us to compare several service units with each other and determine their relative efficiency.

DEA produces a single score for each unit, which makes the comparison easy. Unlike ratios, it can accommodate multiple inputs and multiple outputs. These inputs and outputs can be in different units of measurement.

In contrast to regression methods, DEA focuses on individual observations and optimizes the performance measure of each DMU. A priori knowledge of weights or prices for inputs and outputs is not required in DEA; however, managerial judgment can be accommodated when desired.

Another DEA advantage that attracts analysts and management is its ability to identify the potential improvement for each inefficient DMU. For units enveloped by the frontier, the inefficient units, DEA compares the unit with a convex combination of DMUs located on the frontier and enables the analyst to indicate the sources and the level of inefficiency for each of its inputs and outputs. [Char97], [Sher88]

2.2.4. Application Areas

Since 1978, numerous papers and books have been published on extending the basic methodology of DEA, and it has been widely applied in different production situations in the public as well as in the private sectors. Its applications

involve a wide range of contexts, such as agriculture, airline industry, armed forces, banking, construction, education, health care, marketing, mining, software production, sports, telecommunication, transportation, etc. [Char97]

2.2.5. Returns to Scale

In DEA literature, the question of scale is addressed in terms of increasing, constant or decreasing returns to scale. An increasing (decreasing) returns to scale is when an increase (decrease) in inputs result in a greater (less) than proportionate increase (decrease) in outputs [Appa99]. The estimation of returns to scale in DEA was first investigated in [Bank84a] and [Bank84b].

Banker [Bank84a] developed an LP-based method of determining the most productive scale size (MPSS) to set targets for scale inefficient DMUs. The targets vary depending on whether efficiency is analyzed in terms of minimizing inputs or maximizing outputs.

Banker et al. [Bank84b] presented a modification to the CCR (constant returns to scale) model by adding a convexity constraint. The presence of the convexity constraint decreases the feasible region from conical (or convex cone) hull in CCR to convex hull of DMUs. In his model, now known as BCC, all DMUs were assumed to be efficient in their current scale so that the efficiency measured was independent of scale considerations (variable returns to scale). Figure 2-1 illustrates the returns to scale in CCR and BCC models.

FIGURE 2-1. RETURNS TO SCALE

Chang and Guh [Chan91] and Ganley and Cubbin [Ganl92] noted that the Returns to Scale (RTS) determination approach is problematic when there are alternative multiple solutions. Banker and Thrall [Bank92] devised a method to deal with such multiple solutions, although they did not discuss when and why the alternative optima occurs. Sueyoshi [Suey99] discussed the concept of RTS in the framework of DEA production and cost analyses, focusing on the occurrence of multiple solutions and how to deal with such a difficulty.

Golany et al. in 1997 addressed the issues of returns to scale in DEA [Gola97]. They proposed a simple technique based on solving two variants (input and output oriented) of the BCC model to estimate the returns to scale for each unit. The advantage of their technique is that it provides sharper results for efficient DMUs.

2.2.6. Scale Efficiency

It is interesting to investigate whether the source of inefficiency in a DMU is caused by the inefficient operation of the DMU itself or by the disadvantageous conditions under which the DMU is operating. For this reason, we can compare CCR and BCC models. The CCR model assumes that the constant returns to scale production model is the only possibility and provides overall efficiency measures on this basis. While, the BCC model assumes a convex combination of the observed DMUs as the production possibility set and provides technical efficiency. If a unit is fully efficient in both the CCR and BCC models, it is operating in the most productive scale size (MPSS) [Bank84a]. If a DMU is BCC efficient but inefficient in the CCR model, then it is locally efficient but not globally and this is due to its scale size. Scale efficiency is defined as the ratio of overall efficiency to technical efficiency [Coop00]. In the two dimensional example, with one input and one output, shown in Figure 2-2, unit A on the CCR frontier is both technically and scale efficient.

$$\text{Technical Efficiency of Unit A} = \text{Scale efficiency of Unit A} = KA / KA = 1.0$$

Unit B is a DMU under evaluation. Equation 2-1 shows its technical, scale and overall efficiencies.

$$\text{Technical and Scale (Overall) Efficiency of Unit B} = MN / MB$$

$$\text{Technical Efficiency of Unit B} = MP / MB \quad (\text{EQ 2-1})$$

$$\text{Scale Efficiency of Unit B} = MN / MP$$

FIGURE 2-2: TECHNICAL AND SCALE EFFICIENCY

Appa et al. [Appa99] discussed the theoretical and practical aspects of setting scale efficient targets in DEA. They provide new models for setting scale efficient targets for overall inefficient DMUs, in which the new targets are the same under input minimization and output maximization criteria.

2.2.7. Restricting the Factor Weights

Apart from being positive, input and output weights in a DEA assessment are restricted such that the efficiency of the DMUs do not exceed the upper limit of 1.0. The total weight flexibility in DEA allows the assessing of the relative efficiency of DMUs to ensure the best possible outcomes. If a unit is assessed to be inefficient, it can not be argued that the weights did not fairly represent the values of that DMU. In some situations; however, it may lead to assessing some DMUs only on a subset of their inputs and outputs, while ignoring the remaining

ones. It has proven beneficial to impose additional restrictions on the multipliers in situations when management has strong preferences about the relative importance of different factors, or the DEA model fails to discriminate between DMUs. Imposing additional restrictions on the multipliers increases the discriminating power and flexibility of DEA and thus yields sharper efficiency scores by incorporating expert information, managerial preference, or other judgmental information into the analysis [Char97]. Proposed techniques for restricting the multipliers include imposing bounds on ratios of multipliers [Thom86], [Bank89], imposing upper and lower bounds on multipliers [Dyso88], [Roll91], and appending multipliers inequality [Wong90].

In order to increase DEA's discrimination, Thompson et al. in 1986 [Thom86] proposed the use of weight restrictions in a DEA analysis to evaluate the advantages of one site versus another from six different feasible sites for locating a high-energy physics laboratory.

Dyson and Thanasoulis in 1988 [Dyso88] discussed the issue of getting zero weights and suggested a method for limiting such flexibility in the CCR model by constraining the output weights for DMUs with a single input.

Roll et al. discussed the difficulties with unbounded DEA, which are:

- Getting zero weights for some inputs and outputs seems strange considering the careful selection of inputs and outputs.
- Getting widely different weights for the same input/ output of different DMUs may be unacceptable.
- Covering up some serious deficiencies of DMUs (low outputs and/or high inputs) by the unbounded DEA model in order to represent them in the best possible light [Roll91], [Dyso88].

It is important to bear in mind that there is no single correct process for determining numerical values of bounds and each case is different. Different techniques were presented in [Roll91] for setting the factor bounds:

- Choose appropriate bounds for input and output weights from the results of unbounded DEA, for example by eliminating the outliers (those with zero or very high weights) or defining numerical values of bounds such that a certain percentage of the results fall within the bounds.
- Specify an acceptable ratio of variation for each weight and set bounds at this ratio within the range of the unbounded DEA.
- Start from some known and feasible set of weights and set the bounds such that a certain percentage of variation around these values is permitted.

One of the problems with absolute bounds on multipliers is that these bounds are dependent on the units of measurement of inputs and outputs; however, virtual input and output is dimensionless. Wong and Beasley [Wong90] have suggested the use of virtual weights to solve the problem with absolute weights.

Podinovski [Podi99] analyzed the effects of incorporating absolute weight bounds in classical DEA models. He indicates that although a DEA model with such restrictions maximizes the absolute efficiency, it may not maximize the relative efficiency of the unit under consideration. His suggested approach is to incorporate the weight bounds in a "maximin model", which is a non-linear model that maximizes the relative efficiency of an assessed DMU.

2.2.8. Ranking the Efficient Units

The DEA approach is successful in discriminating among the inefficient units; their efficiency score provides a form of ranking from best to worst. This is not true for efficient units, because they have the same efficiency score: 1.0. In many applications, the problem of prioritizing frontier resident units is very important. Different studies have been done in this area. Cook et al. [Cook92] developed different models for ranking efficient units based on the assumptions made by management about the factor weights. They recommended an approach for prioritizing the efficient units when there are ordinal relationships on the multipliers or upper and lower bounds on them. A minimum-range discrimination model for prioritizing frontier units for the unbounded case, when all factors are equal in importance and no upper or lower limits are imposed, was also discussed.

Andersen et al. [Ande93] developed a modified version of DEA for ranking efficient DMUs. The basic idea of their model is to compare the unit under consideration with a linear combination of all other units. The model is identical to the BCC-model, except that the unit under evaluation is not included in the reference set. Hence, the efficient DMU may obtain a score above one. The approach provides a rating of the efficient units similar to the rating of inefficient units.

Suld [Suld96] used a BCC output oriented model for prioritizing IS projects. The projects in the reference set were pre-defined by the decision-makers and all other real projects received a score by being compared to the management-defined reference set in the model. In Suld's model only the inputs are known. The role of the model is to determine the priority score, which is the output.

FIGURE 2-3: SULD'S MODEL

Torgersen et al. in 1996 suggested a method for ranking the efficient units based on their importance as benchmarks for the inefficient units [Torg96].

A new method, Discriminant Data Envelopment Analysis of Ratios (DR/DEA), was developed by Sinuany-Stern et al. in 1998 to rank all the units, efficient and inefficient units, on the same scale [Sinu98]. They found the best common weights for all the units by a new non-linear ratio that optimizes the goodness of separation between the two groups of efficient and inefficient units of DEA. Based on the common weights, they constructed a new efficiency score for each unit as the ratio between the composite output and the composite input.

Friedman et al. in 1998 [Frie98] presented a combined ranking method to fully rank the units from the most efficient to the least efficient within the DEA context. The combined ranking is based on three recent ranking methods developed within the DEA framework: Canonical Correlation Analysis (CCA), Discriminant Analysis of Ratios (DR/DEA), and Cross Efficiency (CE/DEA). The advantage of the combined ranking is that it incorporates all the other methods, since each has some advantages. They illustrated the approach by ranking Israeli industrial plants with at least 75 employees.

2.2.9. Stochastic DEA

A meaningful extension to the DEA method is how to incorporate stochastic effects in performance measurement. Gong et al. in 1995 introduced some approaches for measuring efficiency with stochastic inputs and outputs, which extended DEA into the stochastic sphere [Gong95]. One suggested approach to extend the efficiency measure to deal with random inputs and outputs was to compute the expected efficiency score for DMUs. The other proposed method for evaluating the efficiency under uncertainty was by applying the concept of certainty equivalent. Given a random variable z and a utility function $u(\cdot)$, the certainty equivalent $C_u(\cdot)$ of z can be defined as:

$$C_u(z) = u^{-1}[E(u(z))], \quad (\text{EQ 2-2})$$

where $E(\cdot)$ is the expected random variable, and $u^{-1}[\cdot]$ is the inverse function of the utility function u . They extended the latter method to accommodate various risk attitudes of evaluators in 1998 [Gong98].

Sengupta generalized DEA for stochastic variations of input and output data [Seng87]. He considered the case of one output and many inputs and applied the chance constrained programming method, which was introduced by Charnes et al. in 1959 [Char59], to measure the efficiency in constraint variation type.

A model based on chance constrained programming was developed by Olesen et al. in 1995, which allows random disturbances such as measurement errors in data [Oles95]. It uses a piecewise linear envelopment of confidence regions for observed stochastic multiple inputs and multiple outputs.

Li in 1998 developed stochastic DEA models based on a chance constrained programming problem [Li98]. She took random disturbances into account and defined the stochastic efficiency measure of a DMU via joint probabilistic comparisons of inputs and outputs with other DMUs.

Cooper et al. [Coop98] extended DEA to stochastic situations using joint probabilistic comparisons of inputs and outputs. They assumed that the statistical distributions are known and evaluated the stochastic efficiency by solving the chance constrained programming problem.

2.2.10. Sensitivity Analysis in DEA

Since a separate linear program must be run to determine the relative efficiency of each DMU and in real applications the number of units is usually large, it is important to know how sensitive the efficiency scores are to the inputs and outputs. Sensitivity analysis is used to assess by how much the inputs and outputs of DMUs can be changed without serious effects on their efficiency. Different studies have been done on sensitivity analysis of DEA models.

Charnes et al. [Char85b] studied the sensitivity of the CCR model. They focused on ranges of variation in a single output for a particular DMU which do not affect the efficiency score. Since an increase in any output cannot worsen the efficiency score, they restricted their study to reductions of outputs.

Charnes and Neralic [Char90] studied the sensitivity analysis of the additive model in DEA for simultaneous change of all inputs and outputs of an efficient unit.

Zhu in 1996 [Zhu96] used modified versions of the CCR model for sensitivity analysis. Sufficient and necessary conditions for upward variations of inputs and downward variations of outputs of an efficient unit retaining its efficiency at 1.0 were provided. Seiford and Zhu [Seif98a] provided a procedure for the sensitivity analysis of an efficient unit in a CCR model and extended Zhu's approach by allowing simultaneous changes in all inputs and outputs. They developed a new

sensitivity analysis approach for CCR, BBC and additive models in 1998 [Seif98b] and generalized the sensitivity approach by allowing data perturbation simultaneously for all DMUs.

2.2.11. Window Analysis

In real applications when data is available over multiple time periods, it is important to measure efficiency changes over time using a technique known as window analysis [Char97]. This technique was used and well illustrated in [Char85c] for the study of aircraft maintenance operations, where the data were obtained for 14 tactical fighter wings in the U.S. Air Force over 7 monthly periods.

A DEA window analysis works on the principle of moving averages. Each unit in a different year is treated as a different unit in the analysis. When there are n units in a given time period and each window has a width of k periods, then there will be $(n*k)$ units in each window. This feature is important when there are a small number of units with a large number of inputs and outputs in the DEA analysis since it increases the discriminating power of DEA [Agga96].

The width for each window in window analysis is currently determined by trial and error. Too small a width for the window decreases the discriminating power of DEA while a too large width gives misleading results since the changes occur over a longer period. The problem of choosing the width of a window and the sensitivity of DEA results to window width are areas for further research in DEA. In [Thom92], [Tulk95], [Hart96], [Agga96] and [Tall97], window analysis has been used to measure the efficiency changes over time.

2.2.12. Efficiency Studies of Banking Industry Using DEA

There has been a considerable number of studies done on the banking industry in many countries using DEA.

Parkan in 1987 [Park87] studied the efficiency of thirty-five bank branches of a large Canadian bank in Calgary using a CCR model. This is the first application of DEA in Canadian branches. In this study the possibility of returns to scale was not addressed. Besides, in the production model, error correction was one of the outputs. Therefore, the branches with high number of errors may have been classified as efficient.

Aggarwal [Agga96] presented a comprehensive analysis of the performance of Canadian banks during the period from 1981 to 1995. She combined the Malmquist Index technique with window analysis to show productivity changes in Canadian banks. The Malmquist Index is an index of productivity change. It decomposes the change into performance change and technological frontier movement. She developed two banking models for determining the cost efficiency and the organizational efficiency of Canadian banks. The efficiency ratings were obtained using a DEA window analysis and productivity changes were analyzed using a variation of the Malmquist Index technique. She investigated the effects of the changes in the economic climate, the management teams and the nature of the banking operations on the performance of schedule I banks during the 15 year period. She summarized most published banking applications using DEA in terms of sample size, banking model and technique used. The banking model used by researchers can be classified into *Production models* and *Intermediation models*. Banks are considered as producers of different services like loans and deposits using labor, capital and operating expenses in the Production model. While in the Intermediation model, banks are considered as

financial institutions borrowing funds from depositors and lending them to others for profit.

Schaffnit et al. [Scha97] analyzed the performance of branch personnel of the Ontario based branches of a large Canadian bank. They considered five types of staff as the inputs and both transactions and account maintenance as the outputs of their model. Each type of staff was measured by the number of efficient hours converted into the number of people in the branch. Constraints on output multipliers were considered to sharpen the efficiency estimates, they also used constraints on input multipliers to estimate allocative efficiency. Studies published on bank branches using DEA were summarized in their paper.

The reader is referred to [Scha97] and [Agga96] for further explanation and discussion on banking application using DEA.

CHAPTER 3

Data Envelopment Analysis

This chapter discusses DEA theoretical concepts and mathematical models in some detail. DEA has gained considerable interest in the productivity management literature because it has proven to be particularly effective in many service organizations including the government.

Managers are familiar with some of the productivity management techniques such as ratio analysis due to their ease of use and calculation. While DEA uses a linear programming technique, which is a more complex method and requires explanation, its concepts and applications can be presented without the need for mathematical notation; it provides useful information for managers that can be understood and adopted without theoretical understanding. It identifies the efficient and inefficient units in which real efficiency improvements are possible. The amount of resource savings (service improvements) that can be achieved by making inefficient units efficient can also be indicated for management. These results can further be used to transfer system and managerial expertise from

efficient units, which are deemed to be relatively better managed units, to the inefficient units.

The last section of this chapter deals with the management issues of DEA and describes how to set up a performance measurement system using DEA.

3.1. *DEA Models*

In this section, the focus is on describing the basic DEA models and in particular the CCR, BCC, additive and multiplicative models are examined. Primal and dual characterizations for each model are presented. Comparison based on their envelopment surface, returns to scale properties, projections onto the efficient surface are provided as well.

3.1.1. *The CCR Model*

This is one of the most basic DEA models, proposed by Charnes, Cooper and Rhodes in 1978 [Char78] based on Farrell's [Farr57] method to measure efficiency. They introduced the term Decision Making Unit (DMU) to describe the organization under efficiency study, which can for example be a firm, a department store, or a bank branch, with common inputs and outputs. A DMU is an entity, which converts inputs to outputs, and has a certain degree of managerial freedom in decision making.

3.1.1.1. *CCR Input Oriented Model*

Charnes et al. generalized the concept of the classical engineering ratio to multiple inputs and outputs. They proposed that the efficiency of a DMU can be obtained as the maximum of a ratio of weighted outputs to weighted inputs,

subject to the condition that the same ratio for all DMUs must be less than or equal to one.

Suppose there are n DMUs: $DMU_1, DMU_2, \dots, DMU_n$, with m inputs: $X_1, X_2, \dots, X_m$ and s outputs: $Y_1, Y_2, \dots, Y_s$. The following fractional programming model can be solved to obtain the efficiency score, input and output weights:

$$\begin{aligned}
 \max \quad & h_o = \frac{\sum_{r=1}^s u_r \cdot y_{ro}}{\sum_{i=1}^m v_i \cdot x_{io}} \\
 s.t. \quad & \frac{\sum_{r=1}^s u_r \cdot y_{rj}}{\sum_{i=1}^m v_i \cdot x_{ij}} \leq 1 \quad j = 1, \dots, n \\
 & u_r, v_i \geq 0 \quad r = 1, \dots, s \quad i = 1, \dots, m
 \end{aligned} \tag{EQ 3-1}$$

Here x_{ij} and y_{rj} (all non-negative) are the inputs and outputs of the j^{th} DMU, v_i and u_r are the input and output weights (also referred to as multipliers).

The objective is to obtain weights (v_i, u_r) that maximizes the efficiency (ratio) of DMU_o , which is the DMU under evaluation. The constraints mean that the efficiency of none of the DMUs should exceed one, while using the same multipliers.

The above fractional programming model can be transformed to a linear programming problem [Char62]:

$$\max \quad h_o = \sum_{r=1}^s u_r \cdot y_{ro} \quad (\text{EQ 3-2})$$

$$\text{s.t.} \quad \sum_{i=1}^m v_i \cdot x_{io} = 1$$

$$\sum_{r=1}^s u_r \cdot y_{rj} - \sum_{i=1}^m v_i \cdot x_{ij} \leq 0 \quad j = 1, \dots, n$$

$$u_r, v_i \geq 0 \quad r = 1, \dots, s \quad i = 1, \dots, m$$

The fractional program is equivalent to the linear program and they have the same optimal objective value, h_o^* .

When DMU_o has $h_o^* < 1$, then it is CCR-inefficient. Therefore, there must be at least one constraint for which the optimal weights (v_i^* , u_r^*) produces equality between left and right hand sides, otherwise h_o^* could be enlarged. This means that there must be at least one CCR-efficient DMU. The set of CCR-efficient DMUs is called the reference set or the peer group for DMU_o. Actually, the existence of these efficient units forces DMU_o to be inefficient. The set of efficient units form the efficient frontier. Figure 3-1 shows the efficient frontier and production possibility set for the CCR model in two dimensions, the single input and single output case.

FIGURE 3-1: CCR PRODUCTION POSSIBILITY SET AND FRONTIER

The dual problem of (EQ3-2) is expressed as follow:

$$\begin{aligned}
 &\min \theta \\
 &s.t. \quad \theta \cdot x_{io} - \sum_{j=1}^n \lambda_j \cdot x_{ij} \geq 0 \quad i = 1, \dots, m \\
 &\quad \sum_{j=1}^n \lambda_j \cdot y_{rj} \geq y_{ro} \quad r = 1, \dots, s \\
 &\quad \lambda_j \geq 0 \quad j = 1, \dots, n
 \end{aligned}
 \tag{EQ 3-3}$$

In the above formulation, θ and λ_j ($j=1, \dots, n$) are the dual variables of the linear program model (EQ 3-2). The scalar variable θ is the (proportional) reduction which should be applied to all inputs of DMU_o in order to make them

efficient. This reduction is applied to all inputs simultaneously and since the result is in a radial movement toward the envelopment surface, the efficiency is called "radial efficiency".

In order to transform the dual problem into the linear programming standard form, slack variables s^- and s^+ should be added to the model. "Slack variables" is a standard LP terminology for additional variables added to the model in order to convert inequality constraints to equality constraints. This terminology in DEA is also used when additional improvement is possible in specific inputs or outputs. The standard linear program is as follow [Coop00]:

$$\begin{aligned}
 &\min \quad \theta \\
 &s.t. \quad \theta \cdot x_{io} - \sum_{j=1}^n \lambda_j \cdot x_{ij} - s_i^- = 0 \quad i = 1, \dots, m \\
 &\quad \quad \sum_{j=1}^n \lambda_j \cdot y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\
 &\quad \quad \lambda_j, s_r^+, s_i^- \geq 0 \quad j = 1, \dots, n, r = 1, \dots, s, i = 1, \dots, m
 \end{aligned} \tag{EQ 3-4}$$

If θ for a DMU is 1.0, but the slack variables are not zero, it means additional improvements in the efficiency of this DMU is possible by reducing (increasing) specific inputs (outputs). Charnes, Cooper and Rhodes [Char78] removed this ambiguity by amending the objective function to maximize the slack variables, but in a manner which did not impair the minimization of θ . This resulted in the following amended objective function:

$$\begin{aligned}
 \min \quad & \theta - \varepsilon \sum_{i=1}^m s_i^- - \varepsilon \sum_{r=1}^s s_r^+ \\
 \text{s.t.} \quad & \theta x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0 \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\
 & \lambda_j, s_r^+, s_i^- \geq 0 \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
 \end{aligned}
 \tag{EQ 3-5}$$

where ε is a very small constant usually chosen as 10^{-6} [Norm91]. Therefore, the optimization can be achieved in two-steps: first the maximal reduction of inputs is computed (by the optimal θ^*), then movement on the efficient frontier is achieved using slack variables s^+ and s^- .

Note that improper selection of a value for ε can result in serious errors and was indicated by computational testing in [Ali93]. Cooper et al. [Coop00] mentioned that it is not advisable to represent ε by a small number since it can lead to errors, besides it is not even necessary to specify a value for ε explicitly. A two phase procedure was described in [Coop00] which eliminates the difficulty with choosing the ε value. In phase I, the optimal objective value of θ (θ^*) is computed, then in phase II the sum of input excess and output shortfalls will be maximized while setting θ by θ^* . The reader is referred to [Coop00] for more discussion in this area.

DMU_o is efficient if and only if $\theta^*=1$ and all slacks are zero. $\theta^*<1$ and non-zero slacks indicate the sources and amount of inefficiencies. To determine the efficiency of all DMUs, a separate LP must be solved for each.

The linear program (EQ 3-2) also referred to as "multiplier form" and the dual program (EQ 3-3) as "envelopment form", from which the name Data Envelopment Analysis was derived [Coop00]. As shown in Figure 3-1, all the data are inside the frontier and hence they are enveloped by the efficient frontier.

It is advisable to solve the CCR model using the dual (envelopment form) [Coop00]. In DEA, the number of DMUs (n) is considerably larger than the sum of inputs and outputs ($m+s$), therefore it is easier to solve the dual, which has $m+s$ constraints, comparing to primal, which has n constraints. Another reason is that the interpretation of the solutions of the dual is more straightforward than the interpretation of the primal. The results give the possible proportional reduction in inputs and the amount of slacks which indicate the improvement possibilities for an inefficient unit.

Up to this point, we have considered a version of the CCR model in which the objective is to minimize inputs while producing at least the given output levels. This is called the *input-oriented* model. The envelopment surface for the CCR input oriented model and projections of the inefficient units (B, C and D) to this efficient frontier for the case of one input and one output are shown in Figure 3-2.

FIGURE 3-2: ENVELOPMENT SURFACE AND PROJECTIONS IN THE CCR-I MODEL

3.1.1.2. CCR Output Oriented Model

There is another type of CCR model, the *output oriented* model, which aims to maximize outputs while not exceeding the observed input levels. The primal (multiplier) form of CCR output oriented is as follow:

$$\begin{aligned}
 \min \quad & q_o = \sum_{i=1}^m v_i x_{io} \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r y_{ro} = 1 \\
 & \sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s u_r y_{rj} \geq 0 \quad j = 1, \dots, n \\
 & u_i, v_r \geq \varepsilon \quad i = 1, \dots, m, \quad r = 1, \dots, s
 \end{aligned} \tag{EQ 3-6}$$

and the dual for it is formulated as:

$$\begin{aligned}
 \max \quad & z_o = \phi + \varepsilon \cdot \sum_{r=1}^s s_r^+ + \varepsilon \cdot \sum_{i=1}^m s_i^- \\
 \text{s.t.} \quad & \phi \cdot y_{ro} - \sum_{j=1}^n \lambda_j \cdot y_{rj} + s_r^+ = 0 \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j \cdot x_{ij} + s_i^- = x_{io} \quad i = 1, \dots, m \\
 & \lambda_j, s_i^-, s_r^+ \geq 0 \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
 \end{aligned}
 \tag{EQ 3-7}$$

In the dual model, maximum output augmentation is accomplished through the variable ϕ . If $\phi > 1.0$ and/or slacks are not zero, then the unit is inefficient. To improve inefficient units, first a proportional increase of ϕ in all outputs is required, and then additional improvement to the envelopment surface may be necessary based on positive slack variables. As illustrated in Figure 3-3, the envelopment surface in the CCR output oriented model is the same as in the CCR input oriented model. However, the projection of inefficient units to the envelopment surface is different.

FIGURE 3-3: ENVELOPMENT SURFACE AND PROJECTIONS IN THE CCR-O MODEL

A DMU is characterized as efficient in an input oriented CCR model, if and only if it is characterized efficient in the corresponding output oriented CCR model.

3.1.2. The BCC Model

The CCR model evaluates both technical and scale efficiency via the optimal value of the ratio form. The envelopment in CCR is constant returns to scale meaning that a proportional increase in inputs result in a proportionate increase in outputs.

Banker et al. in 1984 [Bank84b] developed a model to estimate the pure technical efficiency of decision making units with reference to the efficient

frontier. It identifies whether a DMU is operating in increasing, decreasing or constant returns to scale.

3.1.2.1. BCC Input Oriented Model

The BCC input oriented model evaluated the efficiency of DMU_o by solving the following linear program:

$$\begin{aligned} \max \quad & h_o = \sum_{r=1}^s u_r \cdot y_{ro} + u_o \\ \text{s.t.} \quad & \sum_{i=1}^m v_i \cdot x_{io} = 1 \\ & \sum_{r=1}^s u_r \cdot y_{rj} - \sum_{i=1}^m v_i \cdot x_{ij} + u_o \leq 0 \quad j = 1, \dots, n \\ & u_r, v_i \geq \varepsilon \quad r = 1, \dots, s \quad i = 1, \dots, m \\ & u_o \quad \text{free} \end{aligned} \tag{EQ 3-8}$$

The dual form of this program is expressed as:

$$\min \quad \theta - \varepsilon \sum_{i=1}^m s_i^- - \varepsilon \sum_{r=1}^s s_r^+$$

$$\begin{aligned}
 s.t. \quad & \theta x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0 \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \quad j = 1, \dots, n \\
 & \lambda_j, s_r^+, s_i^- \geq 0 \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
 \end{aligned}
 \tag{EQ 3-9}$$

A unit is BCC-efficient if and only if $\theta^* = 1$ and all slacks are zero. The envelopment surface in BCC model is variable returns to scale and this is the result of the presence of the convexity constraint ($\sum \lambda_j = 1$) in the dual and, equivalently, the presence of u_o , which is an unconstrained variable, in the primal problem. Figure 3-4 is a two dimensional example and illustrates the envelopment surface and projections to this frontier. Inefficient units are projected to the efficient frontier, first by reducing their input, and then by accommodating the slack variables if any.

FIGURE 3-4: ENVELOPMENT SURFACE AND PROJECTIONS IN THE BCC-I MODEL

Units A, B, C, D, E are the efficient units and form the efficient frontier. Units F and G are inefficient. In order to make unit F efficient, a proportional decrease in its input is needed. For unit G, first a reduction in input level and then an increase in its output is necessary, since its non-zero output slack indicates that additional improvement is possible.

If a unit is characterized as efficient in the CCR model, it will also be characterized as efficient in BCC model, however the converse does not necessarily hold true.

3.1.2.2. BCC Output Oriented Model

While the envelopment surface for the BCC output oriented model is the same as BCC input oriented one, the projection to the envelopment surface in the two

models is different. The objective in BCC-O is to maximize the output production while not exceeding the actual input level. EQ 3-10 gives the primal formulation for the BCC output oriented model.

$$\begin{aligned}
 \min \quad & q_o = \sum_{i=1}^m v_i . x_{io} + v_o \\
 s.t. \quad & \sum_{r=1}^s u_r . y_{ro} = 1 \\
 & \sum_{i=1}^m v_i . x_{ij} - \sum_{r=1}^s u_r . y_{rj} + v_o \geq 0 \quad j = 1, \dots, n \\
 & u_i, v_r \geq \varepsilon \quad i = 1, \dots, m, \quad r = 1, \dots, s \\
 & v_o \quad \text{free}
 \end{aligned} \tag{EQ 3-10}$$

The dual (envelopment) form of the problem is as follow:

$$\begin{aligned}
 \max \quad & z_o = \phi + \varepsilon . \sum_{r=1}^s s_r^+ + \varepsilon . \sum_{i=1}^m s_i^- \\
 s.t. \quad & \phi . y_{ro} - \sum_{j=1}^n \lambda_j . y_{rj} + s_r^+ = 0 \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j . x_{io} + s_i^- = x_{io} \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j, s_i^-, s_r^+ \geq 0 \quad j = 1, \dots, n, \quad i = 1, \dots, m, \quad r = 1, \dots, s
 \end{aligned} \tag{EQ 3-11}$$

For BCC output oriented models, similarly to the CCR output oriented models, maximal output augmentation is accomplished through ϕ . Based on this model, a unit is efficient if and only if $\phi^* = 1$ and all slacks are zero. In order to show graphically the difference between BCC-I and BCC-O models in projecting inefficient units to the efficient envelopment surface, consider the BCC-I example in Figure 3-4, now shown as BCC-O in Figure 3-5:

FIGURE 3-5: ENVELOPMENT SURFACE AND PROJECTIONS IN THE BCC-O MODEL

As is shown in Figure 3-5, while the envelopment surface of BCC-O is identical to the envelopment surface of BCC-I (Figure 3-4), units F and G are projected to significantly different points on the envelopment surface.

3.1.3. The Additive Model

In the preceding models (CCR and BCC), the projection of inefficient units to the envelopment surface is based on the model orientation. An input orientated model focuses on maximal movement toward the frontier through the proportional reduction of its inputs, while an output orientated model does this through proportional augmentation of outputs. Charnes et al. in 1985 [Char85a], introduced the additive model, which combines both orientations in one model. In this model, the projection of the inefficient units onto the envelopment surface is accomplished by decreasing their inputs and increasing their outputs simultaneously.

The primal (multiplier form) problem of the additive model can be expressed as follows:

$$\begin{aligned}
 \max \quad w_o &= \sum_{r=1}^s u_r \cdot y_{ro} - \sum_{i=1}^m v_i \cdot x_{io} + u_o & (\text{EQ 3-12}) \\
 \sum_{r=1}^s u_r \cdot y_{rj} - \sum_{i=1}^m v_i \cdot x_{ij} + u_o &\leq 0 \quad j = 1, \dots, n \\
 u_r, v_i &\geq 1 \quad r = 1, \dots, s \quad i = 1, \dots, m \\
 u_o &\text{ free}
 \end{aligned}$$

The dual (envelopment form) is:

$$\min \quad z_o = -\varepsilon \sum_{i=1}^m s_i^- - \varepsilon \sum_{r=1}^s s_r^+$$

$$\begin{aligned}
 \text{s.t. } & x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0 \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \quad j = 1, \dots, n \\
 & \lambda_j, s_r^+, s_i^- \geq 0 \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
 \end{aligned}
 \tag{EQ 3-13}$$

DMU_o is efficient if and only if $z_o^* = w_o^* = 0$. When any of the slack variables, s_i^- or s_r^+ , is not zero it means DMU_o is inefficient and slack values identify the sources and amounts of inefficiency in the corresponding inputs and outputs. A unit is Additive-efficient if and only if it is BCC-efficient, which is proven in [Coop00] as a theorem.

The envelopment surface in the Additive model is the same as that in the BCC model, which is variable returns to scale. This is due to presence of the convexity constraint in the dual and equivalently u_o in the primal problem. The one input, one output example in Figure 3-6 illustrates the envelopment surface and the way inefficient units are projected onto the frontier in the Additive model.

FIGURE 3-6: ENVELOPMENT SURFACE AND PROJECTIONS IN THE ADDITIVE MODEL

3.1.4. The Multiplicative Model

In the preceding DEA models, efficiency is viewed as the sum of outputs divided by the sum of inputs. This means that adding one more output results in added input without any effect on the other outputs. However, in some processes output levels (or input levels) may be interdependent [Sher88]. Charnes et al. in 1982 [Char82] suggested an alternative formulation of DEA to reflect these interactions. In their model, efficiency is measured as the multiplicative combination of outputs divided by multiplicative combination of inputs. Its theory is similar to the CCR model. The formulation for Multiplicative model can be expressed as follows:

$$\max \quad \frac{\prod_{r=1}^s y_{ro}^{u_r}}{\prod_{i=1}^m x_{io}^{v_i}} \quad (\text{EQ 3-14})$$

$$\begin{aligned} \text{s.t.} \quad & \frac{\prod_{r=1}^s y_{rj}^{u_r}}{\prod_{i=1}^m x_{ij}^{v_i}} \leq 1 \quad j = 1, \dots, n \\ & u_r, v_i \geq 1 \quad r = 1, \dots, s, \quad i = 1, \dots, m \end{aligned}$$

By taking logarithms (to any base), the above formulation can be written as a linear program:

$$\begin{aligned} \max \quad & \sum_{r=1}^s u_r \cdot \log y_{ro} - \sum_{i=1}^m v_i \cdot \log x_{io} \quad (\text{EQ 3-15}) \\ \text{s.t.} \quad & \sum_{r=1}^s u_r \cdot \log y_{rj} - \sum_{i=1}^m v_i \cdot \log x_{ij} \leq 0 \quad j = 1, \dots, n \\ & u_r, v_i \geq 1 \quad r = 1, \dots, s, \quad i = 1, \dots, m \end{aligned}$$

The dual formulation of the linear program is given as:

$$\min \quad - \sum_{i=1}^m s_i^- - \sum_{r=1}^s s_r^+$$

$$s.t. \quad \sum_{j=1}^n \log x_{ij} \lambda_j + s_i^- = \log x_{io} \quad i = 1, \dots, m \quad (EQ\ 3-16)$$

$$\sum_{j=1}^n \log y_{rj} \lambda_j - s_r^+ = \log y_{ro} \quad r = 1, \dots, s$$

$$s_i^-, s_r^+, \lambda \geq 0 \quad i = 1, \dots, m, \quad r = 1, \dots, s, \quad j = 1, \dots, n$$

DMU_o is efficient if and only if all slacks are zero. The envelopment surface in the Multiplicative model is piecewise log-linear instead of piecewise linear, which is the envelopment surface for the other DEA models. Sherman [Sher88] mentioned that the Multiplicative model would be useful in a situation where management's insight indicated that the production process was more represented by multiplicative relationship.

The above model is also called *Variant Multiplicative* model, which has a constant returns to scale envelopment surface. The *Invariant Multiplicative* model has the same formulation for the primal and the dual except that the convexity constraint in the dual and the variable u_o in the primal are added to the model. As a result, the envelopment surface will be variable returns to scale.

3.2. Non-Discretionary Inputs and Outputs

In all the preceding model formulations, it was assumed that all inputs and outputs are *discretionary*, which means they are controlled by management. However, in many real applications, there are some variables that are beyond the control of management [Char97]. These variables are *non-discretionary* or *exogenously fixed*, as Banker and Morey [Bank86a] referred to them. Snowfall, weather, age of store, store location, drive-in capability and soil characteristics are

some instances of these variables from DEA literature. Banker and Morey [Bank86a] analyzed a network of sixty fast-food restaurants and evaluated their performance using DEA. They illustrated the impact of exogenous inputs in their study. The modified CCR model incorporating non-discretionary variables is as follows:

$$\begin{aligned}
 \min \quad & \theta - \varepsilon \sum_{i \in I_D} s_i^- - \varepsilon \sum_{r \in O_D} s_r^+ \\
 \text{s.t.} \quad & \theta x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0 \quad i \in I_D \\
 & x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0 \quad i \notin I_D \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\
 & \lambda_j, s_r^+, s_i^- \geq 0 \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{EQ 3-17}$$

where I_D and O_D refer to the sets of discretionary inputs and outputs. The variable θ is not applied to non-discretionary inputs because it is not possible to vary them under the direction of the management. The slacks in the objective function are for discretionary variables. Non-discretionary variables do not enter directly into the efficiency measure, however they affect the efficiency evaluation by their presence in the constraints.

The treatment for non-discretionary inputs and outputs in the BCC and Additive models are similar to what was explained here for the CCR model. The reader is referred to [Char97] for further discussion and explanation on this

subject and the BCC and Additive model modifications to incorporate non-discretionary variables.

3.3. Categorical Inputs and Outputs

In the previous DEA models, it was assumed that all inputs and outputs were continuous variables. In some real situations, the input and output data reflect the categories of service units instead of a continuous measure of resources used and outputs produced. For instance, some bank branches may have drive-in capability and some others may not; some branches may have ATM machines and some others may not. Incorporating the categorical variables into the basic DEA models was first discussed by Banker and Morey [Bank86b]. They proposed a mixed-integer LP model for categorical variables.

When there is a natural nesting or hierarchy of the categories, another applicable approach for incorporating these variables is to evaluate the efficiency of each DMU with respect to the envelopment surface forming from its category and all "disadvantages categories", i.e. those DMUs operating under the same or worse conditions [Char97]. The advantage of this approach is its capability to extend to multiple categorical variables.

In some cases where categories are not comparable, for example public universities and private universities, a separate analysis should be performed for each category.

3.4. Units and Translation Invariance

Units invariance means that the efficiency scores from the DEA model are independent of the units in which the inputs and outputs are measured. For

example, when we evaluate the efficiency of the same collection of automobiles, inputs and outputs can be measured in miles and gallons respectively or can be measured in kilometers and liters, the obtained efficiency scores will be the same [Coop00].

In many applications it may be necessary to deal with negative data, for instance when it is possible to have losses as well as profits as the output of the DEA model. Therefore, it is necessary to go beyond the assumption of non-negative data in a DEA model. This can be achieved by a property of the BCC and the Additive models known as *translation invariance*. A DEA model is translation invariant if the efficiency scores are invariant to the translation of inputs and/or outputs by a scalar [Coop00]. The BCC input oriented model is translation invariant with respect to outputs (but not inputs). Figure 3-7 gives a graphical interpretation of this property. In this figure the efficiency of unit D is MN/MD and this ratio is invariant if the output value is shifted by changing the origin from O to O' .

FIGURE 3-7: TRANSLATION IN THE BCC INPUT ORIENTED MODEL

The BCC output oriented model is translation invariant to inputs (but not outputs) using the same reasoning.

The Additive model is translation invariant in both inputs and outputs because the efficiency score does not depend on the origin of the coordinate system when this model is used. Figure 3-8 shows this property of the Additive model.

FIGURE 3-8: TRANSLATION IN THE ADDITIVE MODEL

3.5. Using DEA – The Complete Process

Decision and Control are the two key attributes of management and for both of them information plays a vital part. A major amount of information needed to manage a system is related to the performance of people and processes. In order

to provide relevant information for decision makers, sophisticated tools, which involve collection and analysis of data, are needed. In certain situations, where the performance of a group of units is desired, Data Envelopment Analysis can be used. As Norman explained [Norm91] "The value of DEA contribution will depend on how well the analysis is planned and how well the results are integrated with other elements of management information". He explained the complete process of setting up the DEA performance measuring system and its benefits for management. These benefits are that management gains a better understanding of the process within each unit of organization and knows when and where action is needed to improve performance.

The first step in using DEA is to define the units, the role of the units, and the units' objectives.

Choosing the outputs and inputs of the units is the next step before the analysis can be undertaken. Outputs are the outcomes that reflect and support the unit's objectives. The rule is to choose those outputs that cover the whole range of the unit's work. Inputs are factors that aid the production of outputs. In the stage of choosing inputs and outputs, it is important to involve as many people as possible from the organization, because they will help to ensure that no factor is missed. Other reasons are that when people are involved they will be more likely to support and assist the work and they might be more ready to accept the results. At the end of this stage, a list of factors will be produced which can be reviewed and irrelevant or duplicate factors can be eliminated.

Another major step in setting up the DEA performance measuring system is collecting the data. Staff and management are more willing to help in collecting data if they were involved in the earlier stages. One of the main problems in this stage is that there will be a number of chosen inputs and outputs for which no data exists. By checking through the factor list, if there are two or more factors that cover the same aspect, the one for which data is not available can be dropped.

Factors for which collecting data would be too inconvenient or expensive will be eliminated. It is also necessary to drop the factors for which the data is incomplete. If the data for one or more factors relate to a different timescale or period from the other factors, it should be recalculated on a common basis. In this stage management may find out that the information they considered to be important is either not available or not reliable. Therefore, one of the outcomes of a DEA analysis can be the introduction of a reliable and sophisticated system to collect and store information.

When the data have been collected and saved in a file, usually in the form of a computer spreadsheet, initial analysis will be performed to check the consistency and integrity of the data. Correlations between inputs and outputs should be examined. Identifying the correlated factors offers a further opportunity to reduce the number of factors in the DEA model. In dropping the correlated factors, those people who are involved with creating the list of factors should be consulted because they may suggest other factors or if this is not possible at least they are informed of changes.

Once the initial analysis is completed, a DEA model will be developed and the results will be interpreted. The first DEA analysis and results sometimes bring up questions about the model construction and alternative models will be developed to have a comprehensive performance measurement system.

Presenting the results of a DEA model to management is important and should provide insight into the operation of the organization. A list of units sorted in descending order based on their efficiency score is the most common way [Norm91]. For efficient units, which have an efficiency score of 1.0, the number of times they appear in the reference set of inefficient units can be calculated and added to the list. Norman has explained the main four groups of units in the efficiency list:

- The *robustly efficient* units are those which appear in many reference sets. Units in this group effectively manage their resources and are examples of good practice.
- The *marginally efficient* units are units which appear in only one or two reference sets and a small change in the value of an input or output may make them inefficient.
- Units with an efficiency score of more than 0.9 but less than 1.0 are *marginally inefficient* units.
- The *distinctly inefficient* units are those units with efficiency score of less than 0.9, say 0.7, and will have difficulty to make them efficient in short term. Units in this group are not succeeding and questions must be asked about the management of the unit.

One of the important pieces of information from a DEA analysis is the set of target factor values for inefficient units. A short term management action is to set targets for inefficient units to improve their efficiency based on the analysis results and if it is possible, achieve the efficiency score of 1.0, because for most of the inefficient units reaching the targets might be impractical.

After setting the targets for inefficient units, examining the results periodically to check what progress has been made allows the setting of new targets, hence underlying the important long-term usages of DEA. It is necessary to ensure that the DEA results truly reflect the organization; therefore if changes occur over time, the DEA model should be revisited.

CHAPTER 4

Solution Approach

This chapter presents the solution to the problem of believable efficiency goals for already DEA efficient units. First the model and then the methodology to incorporate this model are discussed. Next, the limitation of the model is explained and at the end, the solution is extended for log-linear piecewise frontier.

4.1. Model - Linear Program: Practical DEA (P-DEA)

I have examined the potential usefulness of different approaches for defining the Practical Frontier. Consequently, I developed a novel and eminently suitable mathematical programming model, which I will explain in this section. To begin, first consider the BCC ratio model:

$$\begin{aligned}
 \text{Max } h_o &= \frac{\sum_{r=1}^s u_r y_{ro} + u_o}{\sum_{i=1}^m v_i x_{io}} \\
 \text{s.t. } &\frac{\sum_{r=1}^s u_r y_{rj} + u_o}{\sum_{i=1}^m v_i x_{ij}} \leq 1, \quad \forall j, \\
 &u_r \geq \varepsilon \quad \forall r, \\
 &v_i \geq \varepsilon \quad \forall i, \\
 &u_o \text{ free.}
 \end{aligned} \tag{EQ 4-1}$$

In the above model x_{ij} and y_{rj} are the inputs and outputs of the j^{th} DMU; and u_r and v_i are the output and input weights, respectively. The objective is to obtain those weights that maximizes the efficiency of the unit under evaluation, DMU_o, while the efficiency of all DMUs must not exceed 1.0. The efficiency score and input output weights are the variables of the BCC model. The inputs and outputs of DMU_o are known. If DMU_o is efficient then $h_o = 1.0$. The first objective of this thesis is to define a practical target, which can be achievable in reality, for each DEA efficient unit, hence extending DEA theory and enlarging its application area. Specifying targets for efficient units is of interest to operations analysts, management and industrial engineers.

In the real world, some of the factors (inputs and outputs) are fixed, and it is not possible to vary their values, e.g. store area. However, changes in other factors are permitted within certain ranges, i.e., $L_{xio} \leq x_{io} \leq U_{xio}$ and $L_{yro} \leq y_{ro} \leq U_{yro}$. Furthermore, some factors may have a specific relationship with some other factors. This information about inputs and outputs can be obtained from management.

Suppose that there are upper and lower bounds for some or all inputs and outputs. Our goal is to look for the inputs and outputs of a new DMU within the specified range, but one that has an efficiency score greater than that of DMU_o, which is, at present, 1.0. In effect, we are attempting to create new DMUs by adjusting the already efficient DMUs' input and output variables according to limits determined by management. This should produce DMUs which could be used as models for the efficient DMUs from which they were derived. The Practical DEA (P-DEA) model then becomes:

$$\begin{aligned}
 \text{Max. } h_o &= \frac{\sum_{r=1}^s u_r \tilde{y}_{ro} + u_o}{\sum_{i=1}^m v_i \tilde{x}_{io}} \\
 \text{s.t. } \frac{\sum_{r=1}^s u_r y_{rj} + u_o}{\sum_{i=1}^m v_i x_{ij}} &\leq 1, \quad \forall j, \\
 1 &\leq \frac{\sum_{r=1}^s u_r \tilde{y}_{ro} + u_o}{\sum_{i=1}^m v_i \tilde{x}_{io}} \leq 1 + \delta \\
 L_{xio} &\leq \tilde{x}_{io} \leq U_{xio}, \quad \forall i, \\
 L_{yro} &\leq \tilde{y}_{ro} \leq U_{yro}, \quad \forall r, \\
 u_r, v_i &\geq \varepsilon, \quad \forall r, i, \\
 u_o &\text{ free,}
 \end{aligned}
 \tag{EQ 4-2}$$

where $\tilde{y}_{ro}$ (outputs of new DMU), $\tilde{x}_{io}$ (inputs of new DMU), u_r , and v_i are variables. Notice that in this model unlike any other DEA model, inputs and outputs are also variables. The objective function is to maximize the efficiency of the new DMU, while the weights must be feasible for all other units and factors can vary within the specified ranges. To have an improved unit, the efficiency score of the new unit is set to be greater than or equal to 1.0. DEA models which result in an efficiency score of more than 1.0 has been reported in the literature. Andersen and Petersen [Ande93] developed modified versions of the DEA models for ranking efficient units in which the unit, super efficient unit, could obtain an efficiency score of more than one by excluding the subject unit from the analysis.

In this research an upper limit, $(1+\delta)$, is considered in the P-DEA model for the efficiency of the new unit otherwise the model would be unbounded. The amount of possible increase in the efficiency of an empirically efficient unit, designated as δ , can be specified by management (for example: 5%). This is an estimate and does not mean that the efficiency improvement for all efficient units will necessarily be 5%. Based on the P-DEA model results, for some units it will be more or less than 5% while for some units there might be no improvements possible and that is the reason for the Practical Frontier "touching" the empirical one.

The P-DEA model, (EQ 4-2), can be transformed to a linear fractional programming model by substituting $\tilde{y}_{ro}.u_r$ and $\tilde{x}_{io}.v_i$ by new variables p_r and q_i , respectively, and replacing $L_{xio} \leq \tilde{x}_{io} \leq U_{xio}$ and $L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}$ with $v_i.L_{xio} \leq q_i \leq v_i.U_{xio}$ and $u_r.L_{yro} \leq p_r \leq u_r.U_{yro}$, correspondingly. Then the linear fractional program can be transformed to a linear program [Char62], which is shown in (EQ 4-3), so that the linear programming method can be applied to solve the case. The process is relatively straightforward. The objective function is a

fraction or ratio, therefore, in maximizing it the relative magnitude of the numerator and denominator is important not their individual values. It is possible to achieve the same effect by setting the denominator equal to a constant (for example 1) and maximizing the numerator. The linear program will be as follows:

$$\begin{aligned}
 \text{Max.} \quad & \sum_{r=1}^s p_r + u_o \\
 \text{s.t.} \quad & \sum_{i=1}^m q_i = 1, \\
 & \sum_{r=1}^s y_{rj} u_r + u_o - \sum_{i=1}^m x_{ij} v_i \leq 0, \quad \forall j, \\
 & \sum_{r=1}^s p_r + u_o - \sum_{i=1}^m q_i \geq 0, \quad (\text{EQ 4-3}) \\
 & \sum_{r=1}^s p_r + u_o - \sum_{i=1}^m q_i (1 + \delta) \leq 0, \\
 & u_r L_{yro} \leq p_r \leq u_r U_{yro}, \quad \forall r, \\
 & v_i L_{xio} \leq q_i \leq v_i U_{xio}, \quad \forall i, \\
 & u_r, v_i \geq \varepsilon, \quad \forall r, i, \\
 & u_o \text{ free.}
 \end{aligned}$$

By solving the above model, $\tilde{x}_{io}^* = \frac{q_i^*}{v_i}$ and $\tilde{y}_{ro}^* = \frac{p_r^*}{u_r}$ can be calculated.

These values are the inputs and outputs of the new unit. In order to define the practical frontier, the P-DEA model must be run for each efficient unit.

4.2. Methodology

The proposed procedure for improving the efficient unit and finding the practical frontier has three stages. Figure 4-1 summarizes the proposed methodology. In the first stage, we evaluate the efficiency of all the units using conventional DEA methodology and find the efficient and inefficient units. This first stage may itself comprise of several steps starting with an unrestricted DEA run, followed by constrained models until a satisfactory model is developed.

In the second stage, we have to obtain the ranges within which the inputs and outputs of efficient units can vary. We chose to obtain the magnitude of the possible improvement for already efficient DMUs by interviewing management. Then, using this information we solve the proposed P-DEA model for each efficient unit in order to find the inputs and outputs of the new "improved" DMUs, which together with a few empirical units will form the Practical Frontier.

Finally, in the last stage, running the DEA model with all the real and new "improved" DMUs together, including any constraints to define the new frontier. This new frontier envelops or touches the old one but will not cross it. This means that the managers of the empirically efficient DMUs may more readily accept that their new input/output targets will be within the bounds of believability and can be seen as realistic. Nevertheless, it may be a struggle for some to accept these indicated changes, in spite of the fact that their own managers established the extent of the variance.

FIGURE 4-1: METHODOLOGY

4.3. Limitation of the Model

Acquiring management opinion is a lengthy and time-consuming process; however, when the desired information is acquired the results from the DEA analysis may be more acceptable by those measured. In some applications, it might be difficult to find an expert or someone who is acknowledged to be an expert by those being measured. Furthermore, different experts could have different opinions, which can be frustrating if not properly handled. In situations where there are more than one expert, one of them can be chosen as the key individual from whom the information is acquired. Then the information can be presented to other experts for critiques [Igni91]. An alternative strategy is to gather all experts in one room, let them argue out their assessments and come to a consensus conclusion [Hart89]. The reader is referred to [Lieb98] for details on different techniques for collaborative knowledge acquisition methodologies.

4.4. Log-linear frontier

The production function considered in the previous sections was piecewise linear. Another model proposed by Charnes et al. [Char82] is piecewise log-linear, which can be formulated as:

$$\begin{aligned} \text{Max. } h_o &= \frac{u_o \prod_{r=1}^s y_{ro}^{u_r}}{\prod_{i=1}^m x_{io}^{v_i}} \\ \text{s.t. } \frac{u_o \prod_{r=1}^s y_{ro}^{u_r}}{\prod_{i=1}^m x_{io}^{v_i}} &\leq 1, \end{aligned} \quad (\text{EQ 4-4})$$

$$u_r, v_i \geq 1, \quad \forall r, i,$$

$$u_o \text{ free.}$$

The idea of improving the performance of DEA efficient units in the preceding sections for a piecewise linear frontier is also applicable for a piecewise log-linear frontier. Here is the mathematical model:

$$\begin{aligned}
 \text{Max. } h_o &= \frac{u_o \prod_{r=1}^s \tilde{y}_{ro}^{u_r}}{\prod_{i=1}^m \tilde{x}_{io}^{v_i}} \\
 \text{s.t. } \frac{u_o \prod_{r=1}^s y_{rj}^{u_r}}{\prod_{i=1}^m x_{ij}^{v_i}} &\leq 1, \\
 1 \leq \frac{u_o \prod_{r=1}^s \tilde{y}_{ro}^{u_r}}{\prod_{i=1}^m \tilde{x}_{io}^{v_i}} &\leq 1 + \delta,
 \end{aligned}
 \tag{EQ 4-5}$$

$$L_{xio} \leq \tilde{x}_{io} \leq U_{xio}, \quad \forall i,$$

$$L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}, \quad \forall r,$$

$$u_r, v_i \geq 1, \quad \forall r, i,$$

$$u_o \text{ free.}$$

Taking the natural logarithm, the model will be written as:

$$\begin{aligned}
 \text{Max.} \quad & \ln u_o + \sum_{r=1}^s \ln \tilde{y}_{ro}^{u_r} - \sum_{i=1}^m \ln \tilde{x}_{io}^{v_i} \\
 \text{s.t.} \quad & \ln u_o + \sum_{r=1}^s \ln y_{rj}^{u_r} - \sum_{i=1}^m \ln x_{ij}^{v_i} \leq 1, \quad \forall j \\
 & \ln u_o + \sum_{r=1}^s \ln \tilde{y}_{ro}^{u_r} - (1 + \delta) \sum_{i=1}^m \ln \tilde{x}_{io}^{v_i} \leq 1 \quad (\text{EQ 4-6}) \\
 & \ln u_o + \sum_{r=1}^s \ln \tilde{y}_{ro}^{u_r} - \sum_{i=1}^m \ln \tilde{x}_{io}^{v_i} \geq 1 \\
 & L_{xio} \leq \tilde{x}_{io} \leq U_{xio}, \quad \forall i, \\
 & L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}, \quad \forall r, \\
 & u_r, v_i \geq 1, \quad \forall r, i, \\
 & u_o \text{ free.}
 \end{aligned}$$

Then substituting the nonlinear terms $\ln \tilde{y}_{ro}^{u_r}$ and $\ln \tilde{x}_{io}^{v_i}$ by the new variables p_r and q_i , and replacing $L_{xio} \leq \tilde{x}_{io} \leq U_{xio}$ and $L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}$ with $L_{xio}^{v_i} \leq e^{q_i} \leq U_{xio}^{v_i}$ and $L_{yro}^{u_r} \leq e^{p_r} \leq U_{yro}^{u_r}$ correspondingly will result in a linear program.

$$\begin{aligned}
 & \text{Max.} \quad \ln u_o + \sum_{r=1}^s p_r - \sum_{i=1}^m q_i \\
 & \text{s.t.} \quad \ln u_o + u_r \ln \sum_{r=1}^s y_{rj} - v_i \sum_{i=1}^m x_{ij} \leq 1, \quad \forall j, \\
 & \quad \ln u_o + \sum_{r=1}^s p_r - (1 + \delta) \sum_{i=1}^m q_i \leq 1 \\
 & \quad \ln u_o + \sum_{r=1}^s p_r - \sum_{i=1}^m q_i \geq 1 \quad \quad \quad (\text{EQ 4-7})
 \end{aligned}$$

$$\begin{aligned}
 & L_{xio}^{v_i} \leq e^{q_i} \leq U_{xio}^{v_i}, \quad \forall i, \\
 & L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}, \quad \forall r, \\
 & u_r, v_i \geq 1, \quad \forall r, i, \\
 & u_o \text{ free.}
 \end{aligned}$$

After $[u^* \ v^* \ p^* \ q^*]$ are solved from the linear program, $\tilde{x}_{io}^* = e^{q_i^* / v_i^*}$ and $\tilde{y}_{ro}^* = e^{p_r^* / u_r^*}$ can be calculated. These values become the inputs and outputs of the new unit. The linear program must be solved for each efficient unit in order to find all the new units that define the practical frontier.

CHAPTER 5

Data, analysis and Results

One of the goals of this thesis is to develop a DEA model based on real data in order to test the proposed model and methodology and define targets for efficient units. This chapter presents the complete process of developing the DEA model using bank branch data from defining the factors and initial analysis of data until running the model and finding the efficiency scores for each branch. Preparing the questionnaire to gather management opinions and the difficulties related to it are also discussed. Management's opinion about the relative importance of weights, factors' bounds and the amount of possible increase in efficiency of the best practice branches are solicited. This information is then used to define targets for DEA efficient units using stage2 and stage3 of the proposed methodology. Finding the new units (stage 2) and establishing the new frontier (stage 3) are explained in the last two sections of this chapter.

5.1. Preliminary test

To investigate if the model and methodology are appropriate for our purpose, which was finding targets for DEA efficient units, a preliminary test was performed using the sample data in [Kao94]. This data set was a small sample with 17 DMUs. Therefore, it was easier to go through each stage and make necessary changes to the model and the methodology. In fact, the P-DEA model explained in chapter 4 was completed based on the preliminary test.

The efficiency scores of seventeen forests in Taiwan were evaluated in [Kao94] using DEA. The inputs of the model were budget (\$/hectare), initial stocking ($\text{m}^3/\text{hectare}$), labor (person/10,000 hectare) and area (1000 hectare). The outputs were main product, i.e. timber harvested, ($\text{m}^3/\text{hectare}$) average stocking ($\text{m}^3/\text{hectare}$), and recreation (visit/100 hectare). Each year, some growth is accumulated to the initial stocking, deducting the harvests results in a final stocking.

Based on the tabulated data in [Kao94], in which each entry is an average of 10-years of data from 1978 to 1988, the efficient forests were found. Ten out of the seventeen forests were on the frontier. In the preliminary test, targets were found for these efficient forests based on the information provided in [Kao94] and some assumptions. The level of initial stocking and area are given and cannot be altered. It is supposed that budget and labor can be reduced to as much as 180 and 140 units, respectively, harvest is not allowed to exceed 60 units, and tourists can be attracted within a bound of 200 units. The possible increase in efficiency of the efficient forests was set to 2%. Based on these assumptions, stage 2 of the methodology was solved for efficient forests and ten new units were defined. In stage 3, DEA was solved for all the units, old (17) and new (10), and a new frontier was defined. Eight of the new units were on the new frontier, and the efficiency of those two new units which were not on the new frontier was

improved from the efficiency of the empirically efficient units from which they were derived.

5.2. Bank Branch Data

The results of a DEA analysis relies on the availability and quality of the data. Collecting the data is an important step in a DEA analysis and it is often the longest time component in the analysis. The main problems which analyst would face in collecting the data are explained in Chapter 3.

After completing the preliminary test using forests data and getting encouraging results, the bank branch data of a large Canadian bank was obtained. It was a large data set consisting of 1265 branches all across Canada. The branches were sorted by their transit number and the region (Province) of each branch was indicated. The data included the amount of different types of services, sales and number of staff (number of full-time equivalent number of employees) for each branch.

This data set was used for other efficiency studies at CMTE so it was a good clean database. In all the previous studies a large number of the branches were found to be efficient. This shows that the bank does work well, however, because of the competitive environment in which Canadian banks are operating, management looks for ways to improve the efficiency of even the best practice branches. Therefore, this problem is an appropriate application to test these theories. And since the objective was to prove the theory (proposed model and methodology), working on a subset of the data was sufficient, hence the data for the province of Alberta was chosen. There were two other reasons that a subset of data was considered for analysis. One of the bank VPs from whom the managerial input was acquired had been working in Alberta for several years. The other reason was that Alberta has a stronger economy than some other provinces and it

was not fair to compare bank branches in Alberta with branches in other provinces.

5.3. The Robustness of the DEA Model

Although the robustness of the DEA model is of crucial importance, especially when the model is to be used for managerial guidance, there is little guidance available in the literature for the DEA user. Pedraja-Chaparro et al. [Pedr99] highlighted four important issues that have great influence in model results: the distribution of the true efficiencies of DMUs; the size of the sample; the number of inputs and outputs included in the analysis; and the degree of correlation between inputs and outputs.

The number of DMUs included in the analysis is very important in enabling DEA to discriminate between good and poor performance. All the other factors being the same, the discriminating power of DEA increases when a larger number of DMUs is used. On the other hand, by increasing the number of inputs and outputs in the analysis, other things being equal, the discriminating power of DEA will be reduced.

Correlation between inputs and outputs has the same influence on the model as increasing the number of factors, since they contribute less information than uncorrelated inputs and outputs.

They concluded that there are no simple rules of thumb to offer to DEA users on the quality and validity of their model and the simple rule proposed by Banker et al., that the number of DMUs should be at least three times more than the sum of the number of inputs and outputs, just emphasizes two of the key issues they raised.

5.4. DEA Production Model

It is notable that two types of models have been used by researchers in banking applications: Production models and Intermediation models, which were explained in Chapter 2. In this study a DEA production model was developed based on branch sales data. Three types of FTEs (Full Time Equivalent number of employees) were considered as the inputs of the model. The inputs are the resources for each branch. The aim is to study branch sales, therefore, those factors from the data set were chosen as the output that reflect this objective. Loans, Mortgages, Registered Retired Saving Plans (RRSPs) and Letters of Credit were considered as the outputs of the model. The number of DMUs included in the analysis were 79 branches located in Alberta, Canada. Based on [Pedr99] the number of DMUs is very important in DEA discriminating power and as a result in the robustness of the DEA analysis. In this analysis the sum of inputs (3) and outputs (4) is 7, and the number of DMUs (79) is large enough to enable DEA to discriminate between best and poor performers. Figure 5-1 shows the inputs and outputs used in the DEA production model.

FIGURE 5-1: DEA PRODUCTION MODEL

The statistics characterizing the data set are given in Table 5-1.

	Minimum	Maximum	Mean	Standard deviation
<u>Inputs</u>				
FTE sales	0.76	49.52	8.83	8.89
FTE support	0.00	40.93	2.67	6.01
FTE other	0.00	7.21	0.35	1.04
<u>Outputs</u>				
Loans	0.00	308.00	54.42	55.62
Mortgages	0.00	137.00	6.48	16.83
RRSPs	6.00	1090.00	278.78	212.22
Letters of Credit	11.00	429.00	67.91	67.91

TABLE 5-1: DATA STATISTICS

5.5. Initial Analysis of the Data

Once the data was collected and the factors have been chosen for the DEA model, another important issue was to analyze the correlation between inputs and outputs. A high correlation between inputs (or outputs) could imply that the two variables may represent the same thing and this will decrease the discriminating power of DEA, hence, one of the highly correlated variables can be eliminated from the model; although, care must be taken because a mathematical correlation could imply logical or causal correlation. On the other hand, very low correlation between one variable and all the other variables could indicate that it does not fit into the model.

Correlation analyses were done for all inputs and outputs. The results are shown in Table 5-2. It can be observed that no low correlations were found between these variables, however, some of the variables were highly correlated, for example FTE sales and RRSPs. Highly correlated variables were not eliminated from the model for two reasons. Although the correlation between inputs and outputs decreases DEA's discriminating power, in this analysis the number of DMUs is considerably more than the sum of inputs and outputs, hence this effect is not serious. The second reason is that the interpretation of the model and presentation of the results to management is more meaningful when considering all the expected types of FTEs and sales' volumes.

	FTEsales	FTEsupport	FTEother	RRSP	LC	LOANS	MORT
FTEsales	1.00	0.87	0.81	0.92	0.74	0.88	0.85
FTEsupport	0.87	1.00	0.89	0.82	0.90	0.66	0.79
FTEother	0.81	0.89	1.00	0.79	0.76	0.57	0.67
RRSP	0.92	0.82	0.79	1.00	0.72	0.85	0.82
LC	0.74	0.90	0.76	0.72	1.00	0.60	0.77
LOANS	0.88	0.66	0.57	0.85	0.60	1.00	0.80
MORT	0.85	0.79	0.67	0.82	0.77	0.80	1.00

TABLE 5-2: INPUTS AND OUTPUTS CORRELATION RESULTS

Figure 5-2 shows the scatter plot between the two input variables FTE Sales and FTE Support.

FIGURE 5-2: SCATTER PLOT OF FTE SALES AND FTE SUPPORT

The scatter plot for other variables are presented in Appendix B.

5.6. DEA Results

In this stage, the BCC input oriented model was used, without any constraints on weights. Input orientation is consistent with management's objective of improving staff efficiency at the current level of outputs [Scha97].

Using the DEA-Solver-PRO software, the above production model with 3 inputs and 4 outputs for 79 branches was solved and the efficiency scores were obtained. The efficiency scores are shown in Table 5-3.

TABLE 5-3: BASIC DEA – EFFICIENCY SCORES

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1.00	41	649	0.45
2	29	0.38	42	659	1.00
3	39	0.71	43	679	0.74
4	49	1.00	44	699	0.64
5	59	1.00	45	739	0.58
6	79	0.27	46	749	0.61
7	89	0.54	47	769	0.64
8	99	0.46	48	779	0.48
9	109	0.59	49	789	1.00
10	139	0.33	50	799	0.64
11	159	1.00	51	809	1.00
12	179	0.53	52	819	0.94
13	189	0.77	53	829	0.91
14	209	1.00	54	849	1.00
15	229	0.91	55	859	1.00
16	239	0.57	56	879	0.38
17	249	1.00	57	889	0.80
18	269	0.68	58	909	0.46
19	279	1.00	59	919	1.00
20	299	0.63	60	929	0.62
21	309	1.00	61	939	0.51
22	319	0.55	62	949	1.00
23	329	1.00	63	969	1.00
24	339	1.00	64	979	0.69
25	349	0.76	65	989	1.00
26	379	0.79	66	1019	0.65
27	389	0.65	67	1049	0.47
28	399	0.78	68	1069	0.80
29	429	1.00	69	1079	1.00
30	449	0.88	70	1089	1.00
31	459	0.62	71	1179	1.00
32	479	1.00	72	1229	0.77
33	509	0.81	73	1309	1.00
34	519	1.00	74	1319	1.00
35	529	0.75	75	1419	0.38
36	539	1.00	76	1859	1.00
37	549	0.45	77	1899	1.00
38	579	0.36	78	1939	1.00
39	599	1.00	79	1969	0.50
40	609	0.55			

Table 5-4 gives a summary of the efficiency results: the percentage of efficient units, minimum, median and average of efficiency scores.

Model	# DMUs	% efficient units	Minimum efficiency scores	Average efficiency scores	Median
DEA	79 - real	41	0.27	0.77	0.79

TABLE 5-4: EFFICIENCY RESULTS – BASIC DEA

As it is shown 41% (32 of 79) of the branches were efficient relative to others. So far, no restrictions were considered on input and output weights and the multipliers were allowed to vary freely. However, the weights assigned to the units may be unreasonable when critically examined by management.

The efficiency score distribution is presented in Figure 5-3. A large number of inefficient branches, about 60% (28 out of 47), have an efficiency score greater than 0.5 and less than 0.8. Only 28% (13 branches out of 47) of the inefficient branches have an efficiency score less than 0.5.

FIGURE 5-3: EFFICIENCY SCORE DISTRIBUTION

Efficient branches can be split into two groups of robustly efficient units and marginally efficient units based on the number of times they appear in the reference set of inefficient branches. The separation of inefficient units into two groups of marginally inefficient units and distinctly inefficient units is according to their efficiency score. These four groups are explained in Chapter 3. Figure 5-4 shows the number of units in each group.

FIGURE 5-4: NUMBER OF UNITS IN EACH GROUP

Most of the efficient branches appeared in one or two reference sets, therefore, the number of branches in the marginally efficient group (19 branches) is more than the number of branches in robustly efficient group (13 branches), as is shown in Figure 5-4. These two groups form the efficient group together. The robustly efficient branches are the most successful ones and are managing their resources well.

The marginally inefficient branches are those which can reach the frontier and raise their efficiency easier than other inefficient ones. There are only three branches in this group and most inefficient branches (32 branches) are among the distinctly inefficient ones. These units would have difficulty in making themselves efficient in the short term. They are obviously not succeeding and the reasons for that should be investigated by management.

5.7. Incorporating Management Opinion

In the second stage of the methodology, in order to develop the artificial units, management input is required. Both the bank VPs and I agreed that a face-to-face discussion was necessary to ensure that the right information was acquired. Hence, a meeting was set and a questionnaire was prepared. It was a structured meeting and the main questions were about:

- Experts' opinion of the production model I constructed - should we consider adding or deleting any input or output variables?
- Judgement about the importance of each input and output - Do the variables have the same weight?
- If the efficiency score of the best practice branches can be increased - how much of an increase would be reasonable?
- Whether each input or output is fixed or can be changed?
- Allowable changes (a range of + or -) - Is it the same magnitude for all branches or can changes be variable with respect to different groups (type of branch, urban-rural)?

A brief presentation about DEA, the research, how it is related to their work and the potential usefulness for them, and the information I needed was given at the beginning of the meeting. The questionnaires were handed out at the end of the meeting. Two VPs participated in the questionnaire and they discussed it together prior to returning it. Their perspective to the questions were as follows:

- Production Model: It was suggested to include revenue as the output of the model rather than sales' volume; it would be a stronger decision

making tool. However, they recognized that it is not possible for me to include revenue in the model as the data was not provided.

- **Weighting:** In terms of inputs they suggested to weight them as follows: FTE Sales - 50%, FTE Support – 30%, FTE Other – 20%. The following weighting could be applied to the outputs: Loans, Mortgages, RRSPs – 30% each, and Letters of Credit – 10%.
- **Efficiency score of the best practice branches:** They felt that a 2-4% increase in efficiency would be a realistic expectation on an annual basis.
- **Inputs/ Outputs:** All inputs and outputs should be able to be changed.
- **Allowable ranges:** For inputs, no more than 5% increase and 20% decrease would be allowable. For outputs: they would set the increase to no more than 50% and the decrease to no more than 10%.
- **No groupings** were identified based on branch size.

5.8. DEA Model with Multiplier Constraints

In the basic DEA model discussed so far, the weights were allowed to vary freely and this flexibility made the unit appear at its best; however, based on management opinion the model can be more realistic considering the relative importance of the weights. The relative importance of the weights were expressed as percentages by management. They were converted to constraints as ratios and added to the basic model to get a refined measure of efficiency. The mathematical form of these constraints are shown below:

$$\frac{v2 (FTE Support)}{v1 (FTE Sales)} = \frac{3}{5}$$

$$\frac{v3 (FTE Other)}{v1 (FTE Sales)} = \frac{2}{5} \quad (EQ 5-2)$$

$$\frac{u3 (Loans)}{u1 (RRSPs)} = 1$$

$$\frac{u4 (Mortgages)}{u1 (RRSPs)} = 1$$

$$\frac{u2 (Letters of Credit)}{u1 (RRSPs)} = \frac{1}{3}$$

These constraints were added to the basic DEA model and the Assurance Region Model (AR-V-I) was solved using DEA-Solver-PRO software. The efficiency results of the basic model and the restricted model are summarized and compared in Table 5-5. The efficiency scores of the restricted model are given in Table 5-6.

Model	# DMUs	% efficient units	Minimum efficiency scores	Average efficiency scores	Median
Basic DEA	79 - real	41	0.27	0.77	0.79
DEA with weight restriction	79 - real	10	0.25	0.64	0.59

TABLE 5-5: EFFICIENCY RESULTS – COMPARISON

TABLE 5-6: STAGE1 - DEA MODEL WITH WEIGHT RESTRICTION - EFFICIENCY SCORES

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1.00	41	649	0.37
2	29	0.31	42	659	0.77
3	39	0.60	43	679	0.68
4	49	0.81	44	699	0.44
5	59	0.46	45	739	0.38
6	79	0.25	46	749	0.55
7	89	0.48	47	769	0.54
8	99	0.38	48	779	0.41
9	109	0.45	49	789	1.00
10	139	0.26	50	799	0.56
11	159	0.33	51	809	0.92
12	179	0.50	52	819	0.61
13	189	0.51	53	829	0.84
14	209	0.71	54	849	1.00
15	229	0.82	55	859	1.00
16	239	0.37	56	879	0.29
17	249	0.31	57	889	0.56
18	269	0.38	58	909	0.40
19	279	0.85	59	919	1.00
20	299	0.38	60	929	0.50
21	309	0.52	61	939	0.47
22	319	0.44	62	949	0.87
23	329	1.00	63	969	0.95
24	339	0.97	64	979	0.48
25	349	0.70	65	989	0.88
26	379	0.74	66	1019	0.58
27	389	0.59	67	1049	0.31
28	399	0.57	68	1069	0.77
29	429	0.86	69	1079	0.95
30	449	0.73	70	1089	0.84
31	459	0.59	71	1179	0.76
32	479	0.80	72	1229	0.73
33	509	0.71	73	1309	0.86
34	519	1.00	74	1319	0.93
35	529	0.58	75	1419	0.37
36	539	1.00	76	1859	0.53
37	549	0.35	77	1899	0.97
38	579	0.33	78	1939	0.60
39	599	0.97	79	1969	0.47
40	609	0.50			

Although the minimum of efficiency score in the restricted model has not changed much as compared to the basic model, the number of efficient branches has decreased significantly and 10% of the units (8 branches out of 79) remained efficient.

The comparison of the efficiency score distribution of the basic model and restricted model is shown in Figure 5-5.

FIGURE 5-5: EFFICIENCY SCORE DISTRIBUTION – BASIC DEA AND RESTRICTED DEA

Adding the weight constraints to the DEA model increased the discriminating power of DEA and as it is shown in Figure 5-5 the distribution of efficiency scores in the restricted model is skewed towards the lower efficiency scores.

The number of branches in each group of robustly efficient, marginally efficient, marginally inefficient and distinctly inefficient units of restricted DEA model is compared to those of the basic DEA model in the Figure 5-6.

FIGURE 5-6: NUMBER OF UNITS IN EACH GROUP - COMPARISON

Since the number of branches on the frontier has decreased in the restricted DEA model when compared to the basic DEA model, the number of branches in each group of robustly efficient and marginally efficient units have decreased accordingly. On the other hand, the number of units in each group of marginally inefficient and distinctly inefficient units of the restricted DEA model has increased compared to those of the basic DEA model.

5.9. Detecting Outliers

In the restricted DEA analysis of bank branches, a large number of units (45 branches) were found to be distinctly inefficient. This could be the result of outliers, for example having commercial branches in the data set which form the efficient frontier but their operations are completely different from other branches. Outliers are atypical observations and should be deleted from data set.

To deal with the problem of outliers in DEA, the efficient observations can be deleted until efficiency estimates stabilized [Wils93].

The frontier was “peeled off”; first by deleting the robustly efficient units and then the marginally efficient units from the data set and the DEA analysis was redone to detect outliers. Figure 5-7 and 5-8 show the efficiency score distribution and the number of units in each group of robustly efficient, marginally efficient, marginally inefficient and distinctly inefficient units.

FIGURE 5-7: PEELING THE FRONTIER -NUMBER OF UNITS IN EACH GROUP – COMPARISON

Peeling off the frontier resulted in increasing efficiency scores and therefore the distribution is skewed towards higher scores, which is rational because eliminating the robustly efficient and marginally efficient units reduces the number of units in the analysis and therefore decreases the discriminating power of DEA.

FIGURE 5-8: PEELING THE FRONTIER - NUMBER OF UNITS IN EACH GROUP - COMPARISON

As it is shown there is not much change in the number of units in each group which eliminates the possibility of having outliers.

5.10. Finding the New Units – Stage 2

Once management opinion was acquired, it was possible to replace the required parameters in our model and find the inputs and outputs of the new units. These parameters were: possible increase in efficiency of the best practice units (δ); input and output allowable ranges of variation ($L_{xio} \leq \tilde{x}_{io} \leq U_{xio}$, $L_{yro} \leq \tilde{y}_{ro} \leq U_{yro}$). Equation 5-1 shows the desired replacement:

$$\delta = 0.04$$

$$(1 - 0.20) * x_{io} \leq \tilde{x}_{io} \leq (1 + 0.05) * x_{io}, \quad \forall i \quad (\text{EQ 5-1})$$

$$(1 - 0.10) * y_{ro} \leq \tilde{y}_{ro} \leq (1 + 0.50) * y_{ro}, \quad \forall j$$

Then, the proposed P-DEA model was solved for each efficient unit, which has scored 1.0 in the first stage-DEA analysis (DEA with restricted weights), using Excel's solver. The same weight constraints as Stage1 were also used here. The inputs and outputs of 8 new units were found, and are shown in Table 5-7.

TRANS.	FTE SALES	FTE SUPPORT	FTE OTHER	RRSP	LC	LOANS	MORT
9new	36.27	32.74	4.07	236.70	123.30	841.50	496.60
329new	16.27	2.09	0.70	128.70	10.50	495.90	280.50
519new	4.06	0.00	0.00	30.80	0.90	290.45	70.50
539new	8.03	0.55	0.00	107.10	12.00	486.23	61.50
789new	1.05	0.44	0.00	9.00	1.50	93.00	97.50
849new	0.61	0.00	0.00	1.50	6.00	46.50	34.50
859new	8.35	1.52	0.00	46.80	3.00	565.56	69.30
919new	28.28	9.44	1.66	192.60	24.30	981.00	202.50

TABLE 5-7: STAGE 2 – INPUTS AND OUTPUTS OF NEW UNITS

Note that the reason that the "FTE other" is zero for 5 (out of 8) new units is because their value was zero for the source units.

5.11. Establishing the Practical Frontier – Stage 3

In the last stage, the DEA-Solver-PRO software was used to solve the production model for the 79 real branches and 8 new units altogether considering the same weight restrictions as Stage1 and Stage2. The efficiency score of units are given in Table 5-8.

TABLE 5-8: STAGE 3 - RESTRICTED DEA - EFFICIENCY SCORES FOR ALL UNITS

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1.00	36	539	0.79	71	1179	0.52
2	29	0.21	37	549	0.22	72	1229	0.59
3	39	0.56	38	579	0.21	73	1309	0.65
4	49	0.60	39	599	0.91	74	1319	0.73
5	59	0.43	40	609	0.34	75	1419	0.30
6	79	0.17	41	649	0.23	76	1859	0.45
7	89	0.37	42	659	0.51	77	1899	0.72
8	99	0.28	43	679	0.52	78	1939	0.40
9	109	0.27	44	699	0.29	79	1969	0.34
10	139	0.19	45	739	0.27	80	9new	1.00
11	159	0.24	46	749	0.34	81	329new	0.97
12	179	0.37	47	769	0.42	82	519new	1.00
13	189	0.32	48	779	0.30	83	539new	1.00
14	209	0.53	49	789	0.72	84	789new	1.00
15	229	0.61	50	799	0.34	85	849new	1.00
16	239	0.25	51	809	0.68	86	859new	0.996
17	249	0.21	52	819	0.41	87	919new	1.00
18	269	0.24	53	829	0.64			
19	279	0.65	54	849	0.80			
20	299	0.25	55	859	0.77			
21	309	0.34	56	879	0.20			
22	319	0.33	57	889	0.36			
23	329	0.94	58	909	0.26			
24	339	0.91	59	919	1.00			
25	349	0.52	60	929	0.39			
26	379	0.56	61	939	0.31			
27	389	0.40	62	949	0.61			
28	399	0.34	63	969	0.71			
29	429	0.65	64	979	0.28			
30	449	0.68	65	989	0.52			
31	459	0.44	66	1019	0.41			
32	479	0.60	67	1049	0.21			
33	509	0.54	68	1069	0.59			
34	519	0.74	69	1079	0.71			
35	529	0.37	70	1089	0.61			

The efficiency score results of the models in Stage1 and Stage3 are shown in Table 5-9.

Model	Restricted DEA (Stage1)	Restricted DEA (Stage3)
# DMUs	79 -real	87- real and new
% efficient Units	10	9
Minimum efficiency score	0.25	0.17
Average efficiency score	0.64	0.52
#New units on the frontier	-	6
#New units – improved	-	2
#Old units on the frontier	-	2

TABLE 5-9: EFFICIENCY RESULTS – FIRST AND THIRD STAGE COMPARISON

The comparison of the efficiency score distribution of the DEA model from Stage 1 and Stage 3 is presented in Figure 5-9. The number of units in the analysis of Stage3 has increased compared to Stage1 thus increasing the discriminating power of DEA. This can be noticed in Table 5-9 where the %of efficient units, the average and the minimum of the efficiency score of the model in Stage3 have decreased compared to those of the model in Stage1; and in Figure 5-9 where the distribution of the efficiency score in the third stage model is skewed to the left (lower efficiency scores).

FIGURE 5-9: EFFICIENCY SCORE DISTRIBUTION – STAGE1 AND STAGE3

The number of robustly efficient branches and marginally inefficient branches have decreased in the DEA model of Stage3 compared to those of the DEA model in Stage1, while the number of marginally efficient and distinctly inefficient branches have increased. These results are shown in Figure 5-10.

FIGURE 5-10: NUMBER OF UNITS IN EACH GROUP – STAGE1 AND STAGE3

The number of real and new units in each group of efficient and inefficient units are shown in Figure 5-11.

FIGURE 5-11: NUMBER OF REAL AND NEW UNITS IN EACH GROUP

As it is shown in Figure 5-11, all the units on the new frontier (8 efficient units) are either real units (2) or new units found in stage two (6). For those real units which are still on the frontier no further improvement is indicated by this study.

The important result is presented in Figure 5-12. A large number of the new units (6 units) are on the new frontier and received an efficiency score of 1.0. It is notable that those new units (2 units) which are not on the new frontier still improved over their source unit. Therefore, the new units can be considered as targets for real efficient branches. Table 5-10 shows the inputs, outputs, and efficiency scores of the generated units along with their source units. It shows that the proposed methodology and model work and a new frontier, which is practical since it is based on management opinion, can be found. Table 5-11 shows the reference set for inefficient branches.

FIGURE 5-12: THIRD STAGE RESULT FOR NEW UNITS

TRANS.	FTE SALES	FTE SUPPORT	FTE OTHER	RRSP	LC	LOANS	MORT	SCORE
9	45.34	40.93	5.09	263	137	935	429	1
9new	36.27	32.74	4.07	236.70	123.30	841.50	496.60	1
329	16.06	1.99	0.67	143	7	551	187	0.94
329new	16.27	2.09	0.70	128.70	10.50	495.90	280.50	0.97
519	3.87	0	0	34	1	227	47	0.74
519new	4.06	0.00	0.00	30.60	0.90	290.45	70.50	1
539	7.65	0.52	0	119	8	366	41	0.79
539new	8.03	0.55	0.00	107.10	12.00	486.23	61.50	1
789	1	0.42	0	6	1	62	65	0.72
789new	1.05	0.44	0.00	9.00	1.50	93.00	97.50	1
849	0.76	0	0	1	4	31	23	0.8
849new	0.61	0.00	0.00	1.50	6.00	46.50	34.50	1
859	7.95	1.45	0	52	2	432	77	0.77
859new	8.35	1.52	0.00	46.80	3.00	565.56	69.30	0.996
919	35.35	11.8	2.07	214	27	1090	225	1
919new	28.28	9.44	1.66	192.60	24.30	981.00	202.50	1

TABLE 5-10: INPUTS, OUTPUTS AND EFFICIENCY SCORES – OLD AND NEW UNITS COMPARISON

TABLE 5-11: STAGE 3 - RESTRICTED DEA - REFERENCE SET FOR INEFFICIENT UNITS

TRANS.	REF.	LAMBDA	REF.	LAMBDA	TRANS.	REF.	LAMBDA	REF.	LAMBDA
29	519n	0.27	789n	0.73	649	789n	0.74	849n	0.26
39	539n	0.64	919n	0.36	659	519n	0.21	789n	0.79
49	539n	0.99	919n	0.01	679	519n	0.76	539n	0.24
59	539n	0.15	919n	0.85	699	789n	0.06	849n	0.94
79	519n	0.23	789n	0.77	739	789n	0.28	849n	0.72
89	519n	0.86	539n	0.14	749	519n	0.07	789n	0.93
99	519n	0.59	789n	0.41	769	519n	0.74	539n	0.26
109	519n	0.01	789n	0.99	779	519n	0.53	789n	0.47
139	519n	0.42	789n	0.58	789	789n	0.42	849n	0.58
159	519n	0.43	789n	0.57	799	789n	0.93	849n	0.07
179	519n	0.62	789n	0.38	809	539n	0.99	919n	0.01
189	519n	0.10	789n	0.90	819	519n	0.26	789n	0.74
209	519n	0.83	789n	0.17	829	519n	0.31	539n	0.69
229	519n	0.74	789n	0.26	849	849n	1.00		
239	789n	0.58	849n	0.42	859	519n	0.36	539n	0.64
249	789n	0.15	849n	0.85	879	789n	0.21	849n	0.79
269	519n	0.09	789n	0.91	889	789n	0.67	849n	0.33
279	519n	0.82	539n	0.18	909	519n	0.16	789n	0.84
299	789n	0.55	849n	0.45	929	519n	0.46	539n	0.54
309	519n	0.17	789n	0.83	939	519n	0.22	789n	0.78
319	519n	0.95	539n	0.05	949	519n	0.37	789n	0.63
329	539n	0.69	919n	0.31	969	519n	0.16	539n	0.84
339	539n	0.29	919n	0.71	979	789n	0.93	849n	0.07
349	519n	0.66	789n	0.34	989	519n	0.01	789n	0.99
379	519n	0.98	539n	0.02	1019	519n	0.38	789n	0.62
389	519n	0.25	789n	0.75	1049	789n	0.22	849n	0.78
399	519n	0.03	789n	0.97	1069	519n	0.73	539n	0.27
429	519n	0.95	539n	0.05	1079	519n	0.95	789n	0.05
449	539n	0.45	919n	0.55	1089	849n	1.00		
459	519n	1.00	789n	0.00	1179	789n	0.15	849n	0.85
479	519n	0.97	789n	0.03	1229	539n	0.92	919n	0.08
509	519n	0.35	539n	0.64	1309	519n	0.27	539n	0.73
519	519n	0.56	789n	0.44	1319	519n	0.58	539n	0.42
529	789n	0.69	849n	0.31	1419	849n	1.00		
539	519n	0.49	539n	0.51	1859	539n	0.86	919n	0.14
549	789n	0.80	849n	0.20	1899	519n	0.14	539n	0.86
579	519n	0.15	789n	0.85	1939	519n	0.27	789n	0.73
599	539n	0.60	919n	0.40	1969	519n	0.47	789n	0.53
609	519n	0.31	789n	0.69					

The analysis results were presented to the Bank VPs in two parts. The first part of the results consisted of the efficiency scores of each branch, the reference set and targets for the inefficient branches. It also included the number of branches in each group of robustly efficient, marginally efficient, marginally inefficient and distinctly inefficient units. This part of the results was handed out along with the questionnaires. The second part of the results which included the inputs and outputs of the new units for each efficient branch was presented to the two VPs who participated in the questionnaire for their comments.

5.12. Management Usage of the Results

DEA is a unique way of analyzing and comparing data. It compares the input and output data of a production unit to the data of other similar units. The three required data components for a DEA study are: a set of similar units, their inputs and outputs. The complete process of using DEA is explained in Chapter 3. The benefits of a DEA performance measuring system for management can be summarized as:

- A better understanding of the process within each unit of organization,
- A means for better control,
- Providing useful information for decision-making.

An important result that can be obtained from DEA is the efficiency measure. Units with the efficiency measure of 1.0 are the best practice units and form the empirical frontier. Efficiency measure for inefficient units, which have an efficiency score of less or greater than 1.0 based on the model orientation, indicates their distance to the frontier.

The reference sets for the inefficient units are one of the most important pieces of information obtained from a DEA analysis. The reference set provides

the target values for inputs and outputs of the inefficient units to improve its efficiency.

One of the short-term actions by management, based on DEA results, is to set achievable targets for inefficient units. However, the progress of inefficient units should be checked in the subsequent period(s) and new targets should be examined, and these are the long-term usages of DEA. If the organization changes over time, the DEA model should reflect the changes and the relationship between the model factors should be reanalyzed.

In this thesis the ability of DEA was extended to provide targets for empirically efficient units by developing the mathematical models that can attain a new frontier. This new frontier is created based on the inclusion of managerial input into new mathematical developments. Since the value of the parameters in the model are acquired from management, it ensures that targets for efficient units are practical. The progress of empirically efficient units can be checked by management to find out if they were achievable and new targets can be set over time.

In some applications when management needs to choose the best project from a group of efficient projects; or choose a number of efficient projects based on the limited budget, a form of ranking for the efficient units is required for fair and equitable decision-making. This new frontier provides adjusted efficiency scores for all units, which can be useful in ranking the best practice units.

CHAPTER 6

Sensitivity Analysis

Sensitivity analysis, which tests how the results might change with perturbation in the data, is used in engineering, and operations research, as well as in other disciplines.

Sensitivity analysis has taken a variety of forms in the DEA literature: adding or deleting DMUs, adding or deleting inputs or outputs, and increasing or decreasing the number of inputs and outputs. Some studies in this area were referred to in Chapter 2. The objective of this chapter is not to analyze the sensitivity of the DEA model, instead the sensitivity of the proposed P-DEA model is our concern.

It is shown how sensitive the new frontier is to the parameters defined by management and used in the P-DEA model. These parameters are: input and output bounds; and possible increase in efficiency of an already efficient unit (δ). The influence of changing the factor bounds on the results is explained in the first section of this chapter. The second section deals with the variations in the possible increase in efficiency of a stage1 efficient unit.

6.1. Sensitivity to Input and Output Bounds

Bounds (allowable ranges of variation) were assigned based on management judgement to each input and output in the P-DEA model as discussed in Chapter 5. To see if the results are sensitive to these bounds, we varied the factor bounds in the P-DEA model, found the new units, defined the new frontier, and compared the frontier created from a model with the new bounds to the frontier of the model with the original bounds. Two new models were considered: model 1 (with wider bounds than the original model), and model 2 (with tightened bounds than the original model).

The input and output bounds, based on management's opinion, in the original model are as follows:

- For inputs: allowable increase is no more than 5% and allowable decrease is no more than 20%.
- For outputs: 50% increase and 10% decrease is acceptable.

These bounds can mathematically be shown as Equation 6-1:

$$\begin{aligned}(1 - 0.20) * x_{io} &\leq \tilde{x}_{io} \leq (1 + 0.05) * x_{io}, \quad \forall i \\ (1 - 0.10) * y_{ro} &\leq \tilde{y}_{ro} \leq (1 + 0.50) * y_{ro}, \quad \forall j\end{aligned}\tag{EQ 6-1}$$

In model 1, the range of bounds was made wider for allowing more variation by increasing the upper bound and decreasing the lower bound of inputs and outputs. For inputs and outputs a 5% margin above and below the upper and lower bounds was considered acceptable. Equation 6-2 shows the bounds used in model 1.

$$\begin{aligned}(1 - 0.25) * x_{io} &\leq \tilde{x}_{io} \leq (1 + 0.10) * x_{io}, \quad \forall i \\ (1 - 0.15) * y_{ro} &\leq \tilde{y}_{ro} \leq (1 + 0.55) * y_{ro}, \quad \forall j\end{aligned}\quad (\text{EQ 6-2})$$

In model 2, factor bounds were tightened by decreasing the value of upper bound and increasing the value of lower bound of inputs and outputs. For inputs, no increase and 15% decrease were set to be allowable. While for outputs, 45% increase and 5% decrease were considered acceptable. These new bounds are presented in Equation 7-3.

$$\begin{aligned}(1 - 0.15) * x_{io} &\leq \tilde{x}_{io} \leq (1 + 0) * x_{io}, \quad \forall i \\ (1 - 0.05) * y_{ro} &\leq \tilde{y}_{ro} \leq (1 + 0.45) * y_{ro}, \quad \forall j\end{aligned}\quad (\text{EQ 6-3})$$

Model 1 (the P-DEA model with wider bounds) and model 2 (the P-DEA model with tightened bounds) were then solved, and two sets of new units were defined. In the third stage of the methodology, two new frontiers were established by solving the DEA model with real branches and each set of these new (artificial) units. These two new frontiers were then compared to the new frontier created from the original model.

A summary of the results is presented in Table 6-1.

Model in Stage 2	P-DEA Original Bounds for I/O	P-DEA Wider Bounds for I/O (Model 1)	P-DEA Tightened Bounds for I/O (Model 2)
Model in Stage 3	Restricted DEA	Restricted DEA	Restricted DEA
# DMUs	87 (real + new)	87 (real + new)	87 (real + new)
% Efficient Units	9	9	6
Minimum efficiency score	0.170	0.164	0.169
Average Efficiency score	0.524	0.515	0.50
# New units on the new frontier	6	6	5
# New units – improved	2	2	3
# Old units on the new frontier	2	2	1

TABLE 6-1: SUMMARY OF RESULTS – CHANGING THE INPUT AND OUTPUT BOUNDS

As the result of widening and tightening the allowable range of inputs and outputs, the minimum and average efficiency scores have not changed much. The efficiency score distribution of the model with original bounds, wider bounds and tighter bounds is presented in Figure 7-1.

FIGURE 6-1: EFFICIENCY SCORE DISTRIBUTION COMPARISON – CHANGING THE FACTOR BOUNDS

The distribution of the model with wider bounds and tightened bounds are close to that of the model with the original bounds. Note that these bounds are input and output bounds not weight bounds, which is normally used in DEA analysis.

Changing the upper and lower bounds for inputs and outputs of an efficient unit in the P-DEA model resulted in changes in the inputs and outputs of the generated unit in the second stage. Table 6-2 shows the inputs and outputs

generated for Transit 329 in the second stage for P-DEA model with original bounds, wider bounds and tightened bounds.

	<i>Trans.</i>	<i>FTEsales</i>	<i>FTEsupport</i>	<i>FTEother</i>	<i>RRSP</i>	<i>LC</i>	<i>LOANS</i>	<i>MORT</i>
<i>P-DEA with original bounds</i>	329	16.06	1.99	0.67	143	7	551	187
	329new	12.85	2.09	37.88	214.50	6.30	826.50	168.30
<i>P-DEA with tightened bounds</i>	329	16.06	1.99	0.67	143	7	551	187
	329new	13.65	1.69	27.32	135.85	6.65	798.95	177.65
<i>P-DEA with wider bounds</i>	329	16.06	1.99	0.67	143	7	551	187
	329new	12.05	2.19	32.66	121.55	5.95	854.05	158.95

TABLE 6-2: INPUTS AND OUTPUTS GENERATED FROM DIFFERENT MODELS FOR TRANSIT 329 IN THE SECOND STAGE

The new units found in Stage2 from models with different bounds are different. Therefore, the set of units in Stage3 is not the same for these three models. The results of the third stage show that the new efficient units from each of the three models are either on the new frontiers or their efficiency score is increased, which is presented in Figure 6-2. This indicates that changing the allowable bounds for input and output variation affects the targets for empirically efficient units, which means different units will be generated in Stage2, however, all of these new units are either on the frontier and have an efficiency score of 1.0 or they have an efficiency score greater than their source unit. This proves the robustness of the P-DEA model.

FIGURE 6-2: COMPARISON OF REAL AND ARTIFICIAL UNITS – CHANGING THE FACTOR BOUNDS

6.2. Sensitivity to Efficiency Increase (δ)

The possible amount of increase in efficiency of an empirically efficient unit (δ) is another parameter in the P-DEA model, for which management input is required. In the analysis described in Chapter 5, delta was set to 0.04, which means that a 4% increase in efficiency of the empirically efficient branches is realistic according to management. To find out how sensitive the results are to this parameter, we examined its variation. The P-DEA model was solved with different deltas and a set of new units were found for each variation. Based on the set of new units from each model, new frontiers were established and compared to the new frontier from original model ($\delta = 4\%$). Table 6-3 summarizes the results when the value of delta was increased to 6% and decreased to 2%.

Model in Stage 2	P-DEA $\delta=4\%$ (original)	P-DEA $\delta=6\%$	P-DEA $\delta=2\%$
Model in Stage 3	Restricted DEA	Restricted DEA	Restricted DEA
# DMUs	87 (real + new)	87 (real + new)	87 (real + new)
% Efficient Units	9	9	9
Minimum efficiency score	0.170	0.160	0.164
Average efficiency score	0.524	0.505	0.514
#New units on the new frontier	6	6	6
# New units – improved	2	2	2
# Old units on the new frontier	2	2	2

TABLE 6-3: SUMMARY OF RESULTS – CHANGING THE VALUE OF δ

The minimum and average efficiency scores have changed as a result of changing the value of δ , however, the changes are not significant. Figure 6-3 shows the efficiency score distribution of the P-DEA model with different δ values.

FIGURE 6-3: EFFICIENCY SCORE DISTRIBUTION COMPARISON – INCREASING THE VALUE OF DELTA

The distribution of the P-DEA models with higher and lower values for δ are not very different from that of the P-DEA model with the original δ . Although, the new units generated from the P-DEA models with the lower and higher δ s are different, they have similar characteristics to the new efficient units generated from the original (management applied δ) P-DEA model using the bounds obtained from management. The units from the P-DEA models with the lower and higher δ s are on the new frontier or their efficiency score is increased from that of their source unit (the empirical DEA frontier units), offering an indication that the P-DEA model is robust.

CHAPTER 7 *Conclusions and Recommendations*

This chapter presents the conclusions of this research and suggests recommendations for areas of future work.

7.1. Conclusion

Although, it is important to know the specific productivity of an organization so that it can be compared to other organizations, the most important objective in productivity measurement is improvement. The motivation for this research came from the very real need for management to establish improvement targets for their best performing DMUs. In many cases, banking being one, a substantial portion of the DMUs are DEA efficient, even after multiplier bounds are applied. If 20-40% of the DMUs under study are found to be on the empirical frontier, what opportunities can be found to offer to the manager of these DMUs for productivity improvement?

Managers typically do not like to be measured because the criteria used and the methods employed can easily be found unfair and inequitable. Hence, without some authoritative management input, DMU managers tend to push back and generally refuse to follow the targets set for them. It is postulated that expert opinions by their superiors, or even by a subset of their own peers help in them accepting these new goals.

With the above in mind, a new DEA technique which combines a rigorous mathematical development with managerial input was created. The Practical Frontier approach discussed in this thesis answers these needs fully.

The objectives of this thesis were to define a new frontier in Data Envelopment Analysis which provides targets for empirically efficient units; and to test the solution approach with real data. Both objectives were achieved.

Consequently, a linear programming model, P-DEA, and a methodology were developed to define a new frontier in DEA. This new frontier, which is "above" the empirical one, is called the Practical Frontier because the potential improvement in already efficient DMUs is based on management input. This development extends the DEA theory, hence broadening its application. Acquiring management input is the hard part and takes considerable time. This was highlighted as the limitation of the model along with the difficulty of finding the appropriate expert or experts in some applications.

Then, the approach was extended to the multiplicative model, where the efficiency is measured as the multiplicative combination of outputs divided by multiplicative combination of inputs, and its mathematical formulation was also presented.

In order to validate the model, a preliminary test was performed using the sample data from [Kao94]. The efficiency scores of seventeen forests in Taiwan were evaluated and targets were provided for the efficient ones. Validation of the

model was successful and this indicated the need to go ahead with a real application.

The financial services industry was chosen as the industry to test the new model and methodology on.

- The bank branch data was collected, inputs and outputs were chosen and a DEA model was developed to assess the branches' sales.
- DEA was solved for all branches and 41% of them were found to be efficient.
- Management's opinion about the DEA production model, inputs and outputs, and the two parameters in the P-DEA model were acquired.
- Using expert input, the DEA model could be made more realistic by considering different weights for inputs and outputs. These weights were added to the model and the restricted model was solved for all branches. This time only 10% of them were on the frontier.
- In the restricted DEA analysis, a large number of units were found to be distinctly inefficient which could be the result of outliers. The possibility of having outliers was examined by "peeling off" the frontier. First the robustly efficient units and then the marginally efficient units were excluded from the data set. Peeling off the frontier increased the efficiency scores; however, there was not much change in the number of distinctly inefficient units which eliminated the possibility of having outliers.
- The P-DEA model was solved for each efficient unit and a set of new units, which can be considered as the targets for them, was defined based on the information provided by management.

- Then, the DEA model was solved for the real units and the new units altogether and a new frontier was established.

The sensitivity of the results to the parameters defined by management in the P-DEA model was also examined.

- First, the sensitivity of the results to input and output factors was explored by widening and tightening the factor bounds. Two new models, a model with wider bounds and a model with tighter bounds, were developed and two sets of new units were generated. Two new frontiers were established based on the two sets of new units and compared to the frontier created from the new units of model with original bounds. The two new sets of units formed part of the new frontiers or their efficiency score were higher than that of their source units.
- The second part of the sensitivity analysis was to investigate changes in the δ value. The P-DEA model was solved with different values of δ , new sets of units were created and new frontiers were established.

It was found that the new units generated from each model were either on the new frontier or improved, which proves the robustness of the proposed P-DEA model.

In summary, new theoretical and mathematical developments in DEA were introduced, thus overcoming the technique's limitation in offering improvements to empirically efficient units. The model and methodology were found to be successful in defining a new frontier in DEA while incorporating management input. This new frontier envelops or touches the DEA frontier and thus indicates targets for most empirically efficient units. It also provides a means of ranking of best practice units based on their adjusted efficiency scores. It offers valuable

insight to management in what can be expected for their DMUs. Therefore, it can be useful in many applications which, up to now, did not lend themselves to DEA treatment.

7.2. Recommendations

This research provides a framework for indicating targets for empirically efficient units. The following is a list of recommendations for future research, which can further explore real applications of this technique and investigate the advances in DEA developed in this work.

- Extending the approach for other applications, especially those applications in which prioritizing the efficient units is the objective, should be investigated. The results of this approach can be compared to the other ranking methods in DEA.
- In this work, the P-DEA model was solved for each efficient branches in order to find the new units. The possibility of grouping the efficient units and defining targets for each group, which will reduce the amount of work in the second stage, can be further investigated.
- The possibility of having multiple solutions for the proposed P-DEA model (Stage2) can be further investigated.
- Management input in a DELPHI type feedback setting may offer better expert opinion.

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Glossary

<i>Additive Model</i>	DEA model in which the projection of inefficient unit to the envelopment surface involves reduction in inputs with a simultaneous reduction in outputs.
<i>Allocative Efficiency</i>	A measure of the organization's ability of using the optimal mix of inputs to produce outputs.
<i>BCC Model</i>	DEA model which allows variable returns to scale and estimates the technical efficiency of DMUs.
<i>CCR Model</i>	DEA model which allows constant returns to scale and estimates the overall technical and scale efficiency.
<i>CRS</i>	Constant Returns to Scale. A proportionate increase in inputs result in the same proportionate increase in outputs.
<i>DEA</i>	Data Envelopment Analysis. A non-parametric approach based on linear programming method for measuring the relative efficiency of a group of similar units.
<i>DMU</i>	Decision Making Unit. A unit included in the DEA analysis.
<i>Effectiveness</i>	The ability of an organization to attain its pre-determined goals and objectives.
<i>Efficiency</i>	The ability to attain the outputs with a minimum level of resources.

<i>Empirical Frontier/ Envelopment Surface</i>	A frontier or surface determined from the best observed or best practice units.
<i>Input Oriented Model</i>	DEA model in which the objective is to minimize inputs while producing at least the given output level.
<i>LP model</i>	Linear Programming Model. A mathematical programming model in which the objective function and all the constraints are linear.
<i>Multiplicative Model</i>	DEA model in which a multiplicative combination instead of an additive combination of inputs and outputs are used to achieve virtual inputs and outputs. It has a piecewise log-linear envelopment surface.
<i>Output Oriented Model</i>	DEA model which aims to maximize outputs while not exceeding the observed input levels.
<i>Overall Efficiency</i>	Efficiency measured as the product of technical and allocative efficiency.
<i>Peer Group</i>	A set of efficient unit to which the inefficient unit is compared.
<i>Price Efficiency</i>	The efficiency of the organization to purchase the inputs that meet the quality standard at the lowest price.
<i>Production Function</i>	Function in which outputs are defined as functions of inputs.
<i>Productivity</i>	Ratio of outputs to inputs.
<i>Scale Efficiency</i>	Examines whether the unit is operating in its optimal size.
<i>Sensitivity Analysis</i>	Analysis which examines the sensitivity of the results to perturbation in data.
<i>Slack Variable</i>	A standard LP terminology for additional variables added to the model in order to convert inequality

	constraints to equality constraints. In DEA refers to addition improvements which is possible in specific inputs or outputs.
<i>Technical Efficiency</i>	The efficiency in converting the inputs to outputs.
<i>Theoretical Frontier</i>	Frontier of best possible production.
<i>Translation Invariance</i>	Property of a DEA model in which translating the original inputs or outputs has no effect on the efficiency score.
<i>Unconstrained/ Unbounded Model</i>	DEA model in which there are no constraints/bounds on input and output weights.
<i>Units Invariance</i>	Property of a DEA model in which the efficiency scores are independent of the units in which inputs and outputs are measured.
<i>Variable</i>	Any inputs or outputs in the DEA model.
<i>VRS</i>	Variable Returns to Scale. A proportionate increase in inputs results in a non-proportionate increase in outputs.
<i>Weight/ Multipliers</i>	Relative importance of the inputs and outputs in the DEA model.
x_{io}	Input i of DMU _{o}
y_{ro}	Output r of DMU _{o}
x_{ij}	Input i of DMU _{j}
y_{jo}	Output r of DMU _{j}
v_i	Multiplier of input i

u_r	Multiplier of output r
λ_j	Coefficient which determines the combination of efficient units that comprise the projection of an inefficient unit to the frontier
θ	Efficiency measure, $0 < \theta \leq 1$
ϕ	Efficiency measure, output oriented and ≥ 1
δ	Possible improvement in the efficiency of an already efficient unit
s_i	input slack
s_r	output slack

Appendix A *Bank Branch Data*

The data used in the research to assess the branches' sales is presented here.

TRANS.	FTESALES	FTESUPP	FTEOTHER	RRSP-OP	LC-ISSUE	LOANS	MORT
9	45.34	40.93	5.09	263	137	935	429
29	9.02	1.34	0.1	42	6	176	32
39	26.12	8.24	1.01	130	20	679	101
49	10.94	4.87	1.03	134	37	437	80
59	49.52	32.28	7.21	308	46	726	227
79	10.82	1.09	0	27	2	181	36
89	11.52	1.98	0	44	5	337	47
99	8.11	3.91	0	34	1	245	33
109	5.08	0	0	20	2	142	40
139	9.96	5.26	0	29	2	202	49
159	9.86	1.01	0	67	10	161	52
179	7.49	1	0	34	0	249	36
189	4	1.58	0	42	2	159	17
209	5.78	1.52	0.26	85	1	196	78
229	4.87	1.05	0	52	4	237	52
239	2.93	1.97	0	6	2	127	18
249	3.34	0	0	9	5	60	31
269	5.99	0.97	0	61	0	133	24
279	6.61	0.87	0.79	28	0	375	37
299	2.96	1.58	0	21	2	103	23
309	5.3	0	0	25	4	168	38
319	9.84	5.02	0	55	1	301	50
329	16.06	1.99	0.67	143	7	551	187
339	25.06	7.76	0.05	151	13	808	211
349	5.31	1.06	0.06	35	3	250	40
379	6.46	1.59	0	37	3	323	35
389	4.4	0.91	0.33	28	2	178	42
399	3.63	0	1.23	21	1	161	24
429	6.16	0.75	0	34	6	227	142
449	29.22	6.66	1.29	135	13	760	161
459	8.46	0.67	0.87	48	1	293	50
479	4.87	2.65	0.35	41	6	313	30
509	10.69	3.17	0	93	3	393	77
519	3.87	0	0	34	1	227	47
529	2.69	0.45	0	22	0	112	30
539	7.65	0.52	0	119	8	366	41
549	4.81	1.05	0	16	2	142	18
579	7.54	1.17	0	29	1	164	36
599	17.11	5.86	0	93	24	684	162
609	5.91	0.66	0	40	3	177	42
649	4.24	1.08	0	21	0	107	42
659	3.67	0	0	55	2	162	22
679	8.33	2.39	0	54	4	347	53
699	2.21	0.06	0	5	0	74	13

Trans.	FTESALES	FTESUPP	FTEOTHER	RRSP-OP	LC-ISSUE	LOANS	MORT
739	3	0	0	18	1	77	21
749	3.71	1.17	0.12	12	2	148	52
769	10.1	3.53	0.64	76	7	329	54
779	7.79	2.33	0.09	39	1	207	55
789	1	0.42	0	6	1	62	65
799	3.2	0.97	0	13	1	140	39
809	12.05	0.9	0.08	69	2	410	186
819	4.55	0.17	0.73	36	5	171	42
829	9.42	1.88	1	59	3	420	97
849	0.76	0	0	1	4	31	23
859	7.95	1.45	0	52	2	432	77
879	3.52	0.4	0	12	2	57	39
889	3	0	0	8	1	134	20
909	6.22	0.95	0	37	0	135	59
919	35.35	11.8	2.07	214	27	1090	225
929	14.77	2.66	0.01	36	9	425	73
939	6.12	0	0.14	28	1	176	38
949	3.81	0.02	0	49	1	180	42
969	10.46	0.68	0	73	0	461	83
979	3.72	1.22	0	33	1	136	23
989	2	1	0	18	5	157	26
1019	5.42	0.63	0	42	2	199	31
1049	3.03	0.95	0	14	1	79	16
1069	7.75	1.81	0	39	2	369	56
1079	4.53	1.66	0	19	1	337	25
1089	1	0	0	2	1	31	36
1179	1.25	0	0.33	0	1	38	64
1229	15.79	2.44	1	120	10	464	127
1309	9.83	1.95	0.09	118	1	359	109
1319	7.97	0.12	0.03	60	1	301	142
1419	2	0.1	0	1	1	6	11
1859	20.42	10.19	0.83	107	16	408	238
1899	9.75	1.76	0	47	3	511	63
1939	5.04	0	0.03	31	3	189	30
1969	7.17	0.95	0	40	1	207	43

Appendix B *Correlation Analyses Results*

Appendix C *Sensitivity Analyses Results*

STAGE 2 – MODEL WITH WIDER BOUNDS – INPUTS AND OUTPUTS OF NEW UNITS

Trans.	FTE sales	FTE support	FTE other	RRSP	LC	Loans	Mort
9n	34.01	30.70	3.82	223.55	116.45	794.75	541.22
329n	17.72	2.19	0.74	221.65	5.95	468.35	256.95
519n	4.26	0.00	0.00	28.90	0.85	351.85	39.95
539n	8.42	0.57	0.00	184.45	12.40	445.93	34.85
789n	1.10	0.46	0.00	9.30	1.55	96.10	100.75
849n	0.57	0.00	0.00	1.55	6.20	48.05	35.65
859n	8.75	1.60	0.00	80.60	3.10	547.24	65.45
919n	26.51	8.85	1.55	181.90	22.95	926.50	191.25

STAGE 3 – DEA RESULTS – REAL UNITS AND UNITS FROM THE MODEL WITH WIDER BOUNDS

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1	38	579	0.203996	75	1419	0.276699
2	29	0.201483	39	599	0.924486	76	1859	0.448858
3	39	0.570087	40	609	0.331909	77	1899	0.730838
4	49	0.606035	41	649	0.230844	78	1939	0.390663
5	59	0.436037	42	659	0.495858	79	1969	0.323218
6	79	0.164472	43	679	0.499633	80	9n	1
7	89	0.347464	44	699	0.268465	81	329n	0.99
8	99	0.267678	45	739	0.255416	82	519n	1
9	109	0.265829	46	749	0.326988	83	539n	1
10	139	0.180905	47	769	0.399597	84	789n	1
11	159	0.229917	48	779	0.287202	85	849n	1
12	179	0.356889	49	789	0.70498	86	859n	0.998249
13	189	0.310797	50	799	0.33832	87	919n	1
14	209	0.504466	51	809	0.688676			
15	229	0.581585	52	819	0.398052			
16	239	0.244222	53	829	0.647034			
17	249	0.199468	54	849	0.750823			
18	269	0.232533	55	859	0.771566			
19	279	0.617557	56	879	0.190092			
20	299	0.249957	57	889	0.35912			
21	309	0.324708	58	909	0.250687			
22	319	0.315956	59	919	1			
23	329	0.95351	60	929	0.38833			
24	339	0.925124	61	939	0.300511			
25	349	0.498764	62	949	0.587199			
26	379	0.528762	63	969	0.7185			
27	389	0.382712	64	979	0.287666			
28	399	0.333293	65	989	0.519287			
29	429	0.611271	66	1019	0.390477			
30	449	0.68988	67	1049	0.199807			
31	459	0.418994	68	1069	0.571706			
32	479	0.573834	69	1079	0.673559			
33	509	0.543705	70	1089	0.57			
34	519	0.70795	71	1179	0.485994			
35	529	0.367492	72	1229	0.590679			
36	539	0.779876	73	1309	0.657031			
37	549	0.215798	74	1319	0.713926			

STAGE 2 – MODEL WITH TIGHTENED BOUNDS – INPUTS AND OUTPUTS OF NEW UNITS

TRANS.	FTE SALES	FTE SUPPORT	FTE OTHER	RRSP	LC	LOANS	MORT
9n	38.54	34.79	4.33	249.85	130.15	932.69	407.55
329n	16.06	1.99	0.67	135.85	6.65	585.44	177.65
519n	3.87	0.00	0.00	32.30	0.95	307.15	44.65
539n	7.65	0.52	0.00	113.05	11.60	492.44	38.95
789n	1.00	0.42	0.00	8.70	1.45	89.90	94.25
849n	0.65	0.00	0.00	1.45	5.80	44.95	33.35
859n	7.95	1.45	0.00	49.40	2.90	626.40	73.15
919n	30.05	10.03	1.76	203.30	39.15	1035.50	262.55

**STAGE 3 – DEA RESULTS – REAL UNITS AND UNITS FROM THE MODEL WITH
TIGHTENED BOUNDS**

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1	38	579	0.21124	75	1419	0.313592
2	29	0.207909	39	599	0.777034	76	1859	0.33968
3	39	0.474	40	609	0.342155	77	1899	0.655053
4	49	0.535528	41	649	0.230055	78	1939	0.403176
5	59	0.391056	42	659	0.512575	79	1969	0.332198
6	79	0.16989	43	679	0.492975	80	9n	1
7	89	0.351488	44	699	0.313228	81	329n	0.8189
8	99	0.274666	45	739	0.278023	82	519n	0.994875
9	109	0.271492	46	749	0.339786	83	539n	0.934232
10	139	0.186088	47	769	0.392213	84	789n	1
11	159	0.236446	48	779	0.294927	85	849n	1
12	179	0.366048	49	789	0.741072	86	859n	1
13	189	0.322518	50	799	0.329418	87	919n	1
14	209	0.516316	51	809	0.608109			
15	229	0.595727	52	819	0.41082			
16	239	0.249234	53	829	0.590813			
17	249	0.225606	54	849	0.850933			
18	269	0.241338	55	859	0.709653			
19	279	0.618143	56	879	0.210863			
20	299	0.256594	57	889	0.361247			
21	309	0.33606	58	909	0.259512			
22	319	0.32281	59	919	0.961513			
23	329	0.781971	60	929	0.361794			
24	339	0.818502	61	939	0.310516			
25	349	0.511348	62	949	0.604571			
26	379	0.540414	63	969	0.645212			
27	389	0.395095	64	979	0.280097			
28	399	0.346339	65	989	0.530349			
29	429	0.624556	66	1019	0.401974			
30	449	0.597831	67	1049	0.221255			
31	459	0.428297	68	1069	0.559561			
32	479	0.586686	69	1079	0.688762			
33	509	0.499507	70	1089	0.646			
34	519	0.726628	71	1179	0.548606			
35	529	0.368894	72	1229	0.472197			
36	539	0.730025	73	1309	0.597447			
37	549	0.213453	74	1319	0.678882			

STAGE 2 – MODEL WITH DELTA = 6% – INPUTS AND OUTPUTS OF NEW UNITS

TRANS.	FTE SALES	FTE SUPPORT	FTE OTHER	RRSP	LC	LOANS	MORT
9n	36.272	32.744	4.072	236.7	123.3	841.5	491.1962
329n	16.62	2.09	0.70	128.70	10.50	495.90	280.50
519n	4.0635	0	0	51	1.5	340.5	42.3
539n	8.0325	0.546	0	178.5	12	410.5815	61.5
789n	1.05	0.441	0	9	1.5	93	97.5
849n	0.608	0	0	1.5	6	46.5	34.5
859n	8.3475	1.5225	0	78	1.8	530.0513	69.3
919n	28.28	9.44	1.656	192.6	24.3	981	202.5

**STAGE 3 – DEA RESULTS – REAL UNITS AND UNITS FROM THE MODEL WITH
DELTA = 6%**

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1	38	579	0.201203	75	1419	0.295146
2	29	0.195228	39	599	0.915413	76	1859	0.450006
3	39	0.565254	40	609	0.319972	77	1899	0.714623
4	49	0.607857	41	649	0.231223	78	1939	0.37879
5	59	0.428128	42	659	0.484789	79	1969	0.306806
6	79	0.160189	43	679	0.458087	80	9n	1
7	89	0.315568	44	699	0.29176	81	329n	0.9639
8	99	0.251934	45	739	0.267583	82	519n	1
9	109	0.26499	46	749	0.328197	83	539n	1
10	139	0.172474	47	769	0.367758	84	789n	1
11	159	0.218926	48	779	0.2714	85	849n	1
12	179	0.335144	49	789	0.7254	86	859n	0.993497
13	189	0.309819	50	799	0.335	87	919n	1
14	209	0.468471	51	809	0.694037			
15	229	0.542392	52	819	0.386042			
16	239	0.247548	53	829	0.625342			
17	249	0.213213	54	849	0.800878			
18	269	0.23197	55	859	0.74224			
19	279	0.560259	56	879	0.201121			
20	299	0.254115	57	889	0.361388			
21	309	0.319402	58	909	0.246882			
22	319	0.290684	59	919	1			
23	329	0.947027	60	929	0.37045			
24	339	0.909997	61	939	0.293197			
25	349	0.467337	62	949	0.562462			
26	379	0.487345	63	969	0.701738			
27	389	0.371676	64	979	0.284844			
28	399	0.336677	65	989	0.517649			
29	429	0.562503	66	1019	0.373765			
30	449	0.680452	67	1049	0.211207			
31	459	0.386506	68	1069	0.527221			
32	479	0.529876	69	1079	0.62253			
33	509	0.523423	70	1089	0.608			
34	519	0.667254	71	1179	0.518948			
35	529	0.369425	72	1229	0.595306			
36	539	0.741647	73	1309	0.636687			
37	549	0.215348	74	1319	0.671785			

STAGE 2 – MODEL WITH DELTA = 2% – INPUTS AND OUTPUTS OF NEW UNITS

TRANS.	FTE SALES	FTE SUPPORT	FTE OTHER	RRSP	LC	LOANS	MORT
9n	36.272	32.744	4.072	236.7	205.5	841.5	463.7962
329n	16.62	2.09	0.70	128.70	10.50	495.90	280.50
519n	4.0635	0	0	30.6	0.9	340.5	42.3
539n	8.0325	0.546	0	178.5	12	435.1815	36.9
789n	1.05	0.441	0	9	1.5	93	97.5
849n	0.608	0	0	1.5	6	46.5	34.5
859n	8.3475	1.5225	0	78	1.8	530.0513	69.3
919n	28.28	9.44	1.656	192.6	24.3	981	202.5

**STAGE 3 – DEA RESULTS – REAL UNITS AND UNITS FROM THE MODEL WITH
DELTA = 2%**

# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE	# DMU	TRANS.	SCORE
1	9	1	38	579	0.205227	75	1419	0.295146
2	29	0.201193	39	599	0.915413	76	1859	0.450006
3	39	0.565254	40	609	0.330729	77	1899	0.719651
4	49	0.607857	41	649	0.231223	78	1939	0.390211
5	59	0.428128	42	659	0.497008	79	1969	0.320007
6	79	0.16459	43	679	0.492119	80	9n	1
7	89	0.342264	44	699	0.29176	81	329n	0.9639
8	99	0.264092	45	739	0.267583	82	519n	1
9	109	0.265584	46	749	0.33141	83	539n	1
10	139	0.179433	47	769	0.393581	84	789n	1
11	159	0.227927	48	779	0.283824	85	849n	1
12	179	0.351783	49	789	0.7254	86	859n	0.993497
13	189	0.314084	50	799	0.335	87	919n	1
14	209	0.494984	51	809	0.694037			
15	229	0.571647	52	819	0.397629			
16	239	0.247548	53	829	0.637161			
17	249	0.213213	54	849	0.800878			
18	269	0.235065	55	859	0.759809			
19	279	0.608296	56	879	0.201121			
20	299	0.254115	57	889	0.361388			
21	309	0.326297	58	909	0.252039			
22	319	0.308843	59	919	1			
23	329	0.947027	60	929	0.382426			
24	339	0.909997	61	939	0.300947			
25	349	0.491182	62	949	0.583553			
26	379	0.517236	63	969	0.707505			
27	389	0.382526	64	979	0.284844			
28	399	0.338413	65	989	0.518809			
29	429	0.597564	66	1019	0.387939			
30	449	0.680452	67	1049	0.211207			
31	459	0.410003	68	1069	0.563094			
32	479	0.561751	69	1079	0.659621			
33	509	0.535418	70	1089	0.608			
34	519	0.698872	71	1179	0.518948			
35	529	0.369425	72	1229	0.595306			
36	539	0.76803	73	1309	0.646999			
37	549	0.215348	74	1319	0.703113			