Robust State Estimation and Control of Highway Traffic Systems

by

Rashid R. Kohan

A thesis submitted in conformity with the requirements for the Degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

© Copyright by Rashid R. Kohan
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.
Abstract

Robust State Estimation & Control of Highway Traffic Systems

Rashid R. Kohan

Degree of Doctor of Philosophy 2001
Department of Electrical & Computer Engineering
University of Toronto

In this thesis, a modified second-order continuum model is used to describe the traffic behaviour along highways. The model is identified and verified using several sets of traffic measurements collected from a major highway in metropolitan Toronto, Canada.

A robust nonlinear sliding mode observer is developed to generate estimates of average velocity and density for a segment of a highway within a corridor, given loop detector measurements at the end-points of the segment. The sliding mode approach has several advantages over other estimation techniques such as the Kalman Filtering including proof of estimate convergence and simplified computations. However, the primary advantage is the robustness of the observer with respect to unmodelled dynamics and disturbances. Unmodelled dynamics are associated with the traffic factors whose effects cannot be captured (proposed in the traffic flow models, e.g., road geometry and weather conditions. On the other hand, model disturbances such as unavailable (not measured) traffic flow at a ramp or measurements provided by a faulty detector can also create unpredictable traffic states. Based on the presented traffic model, a systematic design procedure is developed to make the observer robust with respect to the modelling uncertainties and unavailable traffic states. Simulation and experimental results show the effectiveness of the proposed observer in estimating the states of a highway traffic system.

Moreover, a new decentralized state feedback linearizing controller for ramp metering using variable structure control is presented. The main aim is to develop a robust controller to locally stabilize freeway traffic despite the presence of disturbances and modelling errors. Simulation results show that the proposed controller provides improved performance in achieving the design objectives over other existing ramp control strategies such as neural network and linear feedback controllers.
Acknowledgments

The completion of this thesis would not have been possible without the help and support of Professors Edward J. Davison and Manfredi Maggiore. I would like to express my sincere gratitude and appreciation to both for their insightful guidance and helpful discussions during the final months of the preparation of this dissertation. I am also grateful to Professor Vadim I. Utkin who kindly agreed to act as the external appraiser and whose sagacious commentary led to substantive refinements in the final draft.

I also wish to thank my fellow students and friends in the System Control Group with whom I shared a unique research environment and student life, in particular, Amir Aghdam for his amusing comments and adventures, Steve Postma for his sincere friendship and the proof-reading of the thesis, and Peyman Gohari for being patient and helpful!

Special thanks to Mrs. Nita Rajcoomar at the Ministry of Transportation of Ontario (MTO) for providing numerous traffic data sets used throughout the thesis.

Finally, I would like to thank my parents and brothers for their endless encouragement and love. I am dedicating this thesis to Sandra; if not for her patience, understanding and love, I would not be able to come this far.

Toronto, June 2001

Rashid R. Kohan
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>List of Figures</strong></td>
<td>6</td>
</tr>
<tr>
<td><strong>List of Tables</strong></td>
<td>9</td>
</tr>
<tr>
<td><strong>1 Introduction</strong></td>
<td>10</td>
</tr>
<tr>
<td>1.1 Outline</td>
<td>13</td>
</tr>
<tr>
<td><strong>2 Dynamic Modelling of Traffic Flow on Freeways</strong></td>
<td>15</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>15</td>
</tr>
<tr>
<td>2.2 Microscopic Models</td>
<td>16</td>
</tr>
<tr>
<td>2.2.1 Linear Car-Following Models</td>
<td>16</td>
</tr>
<tr>
<td>2.2.2 Nonlinear Car-Following Models</td>
<td>18</td>
</tr>
<tr>
<td>2.3 Macroscopic Models</td>
<td>19</td>
</tr>
<tr>
<td>2.3.1 Continuum First-Order Models</td>
<td>23</td>
</tr>
<tr>
<td>2.3.2 Second-Order Models</td>
<td>24</td>
</tr>
<tr>
<td>2.4 Model Identification</td>
<td>30</td>
</tr>
<tr>
<td>2.4.1 A Case Study: Highway 401</td>
<td>30</td>
</tr>
<tr>
<td>2.4.2 Parameter Estimation</td>
<td>33</td>
</tr>
<tr>
<td>2.4.3 The Choice of Initial Values for Parameters</td>
<td>35</td>
</tr>
<tr>
<td>2.4.4 The Optimal Parameter Set and Model Validation</td>
<td>37</td>
</tr>
<tr>
<td>2.5 Transferability of the model parameters</td>
<td>39</td>
</tr>
<tr>
<td><strong>3 Robust State Estimation using Sliding Mode Techniques</strong></td>
<td>44</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>44</td>
</tr>
<tr>
<td>3.2 Variable Structure Control Theory</td>
<td>46</td>
</tr>
<tr>
<td>3.2.1 Sliding Mode Dynamics</td>
<td>48</td>
</tr>
<tr>
<td>3.2.2 Sliding Mode Observers</td>
<td>49</td>
</tr>
<tr>
<td>3.3 Design of a Sliding Mode Observer</td>
<td>53</td>
</tr>
<tr>
<td>3.3.1 Continuous Realization of the Observer</td>
<td>60</td>
</tr>
<tr>
<td>3.3.2 Experimental Results</td>
<td>61</td>
</tr>
<tr>
<td>3.3.3 Robustness Analysis with respect to Parameter Variations</td>
<td>71</td>
</tr>
<tr>
<td>3.4 State Estimation for a Traffic System: The General Case</td>
<td>72</td>
</tr>
</tbody>
</table>
## Contents

3.4.1 Sliding Mode Observers based on the Equivalent Control Method 73  
3.4.2 Observer Design for a Traffic System with an Arbitrary Number  
    of Segments .......................................................... 75  
3.5 Design of a Robust Sliding Mode Observer .......................... 78  
3.5.1 Robustness Analysis ............................................... 84  
3.5.2 Simulation results ............................................... 87  
3.5.3 Experimental Results ........................................... 89  
3.6 Robust State Estimation of Multi-lane Traffic Systems .......... 94  
3.6.1 Modelling of Multi-lane Traffic Systems ....................... 94  
3.6.2 Robust Observer Design ......................................... 96  
3.6.3 Experimental Results ........................................... 100  

4 Robust Ramp Metering Control of Highway Traffic Systems using  
    Feedback Linearizing Techniques ................................ 108  
4.1 Introduction .................................................................. 108  
4.2 Robust Feedback Linearizing Control ............................... 110  
4.3 Robust control of Traffic Flow ...................................... 114  
    4.3.1 A Global Change of Coordinates .............................. 116  
    4.3.2 The Nonlinear Controller ...................................... 118  
    4.3.3 Robustness Analysis ........................................... 121  
4.4 Ramp metering using neural network & linear feedback controllers  
    4.4.1 A neural network controller ................................... 123  
    4.4.2 A linear feedback controller ................................... 126  
4.5 Simulation results .................................................... 128  

5 Conclusion ....................................................................... 136  
5.1 Concluding Remarks ................................................... 136  
5.2 Suggested Future Work ................................................ 137  

A The Simplex Optimization Algorithm of Nelder and Mead ........ 139  
B Computing the Observability Matrix $Q(x)$ using Mathematica  
    .................................................................................... 143  
C Proof of the Theorems and Corollaries of Chapter 4 .......... 145  

Bibliography ....................................................................... 154
List of Figures

2.1 Positions of the leader and follower for a linear car-following model ........................................ 17
2.2 A segment of a roadway ...................................................................................................................... 20
2.3 Speed-density characteristics ............................................................................................................ 21
2.4 Sample speed-density data from Highway 401 ................................................................................. 22
2.5 Positions of the downstream and upstream vehicles for a second-order macroscopic model .... 25
2.6 A Freeway system divided into $N$ sections ....................................................................................... 27
2.7 The Freeway Network in Metropolitan Toronto ................................................................................. 31
2.8 The westbound portion of the study site. ............................................................................................. 32
2.9 The eastbound portion of the study site. ............................................................................................... 32
2.10 Sample data from the detector station 401DW0100DWS for Sept. 22, 1997 .................. 33
2.11 The Flow diagram of the optimization algorithm .......................................................................... 36
2.12 Traffic velocity versus density at the detector station 401DW0070DWS ..................................... 37
2.13 Parameter estimation using Simplex Algorithm (case 1a) ............................................................... 40
2.14 Parameter estimation using Simplex Algorithm (case 3b) ............................................................... 41
2.15 Transferability of the parameters for the westbound station, May 23, 1997 ......................... 42
2.16 Transferability of the parameters for the eastbound station, May 26, 1997 ......................... 43
3.1 Faulty measurement recorded by the station 401DW0070DWS, Sept. 26, 1997 ......................... 47
3.2 Sliding condition .................................................................................................................................. 51
3.3 A traffic system divided into $M$ subsystems of $m$ sections ($L_{tm} < L_{(t+1)m}$) ...................... 53
3.4 A traffic system with $N = 3$ sections ................................................................................................. 54
3.5 Selected study site from Highway 401 for experiments ................................................................. 61
3.6 Measured and estimated velocities at stations B(top) and C for Site (a), Sept. 24, 1997 ........ 65
3.7 Measured and estimated volume at stations B(top) and C for Site (a), Sept. 24, 1997 .................. 66
### List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.8</td>
<td>Mean velocity errors for Site (a), Sept. 24, 1997</td>
</tr>
<tr>
<td>3.9</td>
<td>Estimated section densities for Site (a), Sept. 24, 1997</td>
</tr>
<tr>
<td>3.10</td>
<td>Measured and estimated velocities at stations B(top) and C for Site (b), May 26, 1997</td>
</tr>
<tr>
<td>3.11</td>
<td>Measured and estimated volume at stations B(top) and C for Site (b), May 26, 1997</td>
</tr>
<tr>
<td>3.12</td>
<td>Mean velocity errors for Site (b), May 26, 1997</td>
</tr>
<tr>
<td>3.13</td>
<td>Estimated section densities for Site (b), May 26, 1997</td>
</tr>
<tr>
<td>3.14</td>
<td>Sensitivity of EKF (-) and SMO (-.) with respect to parameter variations</td>
</tr>
<tr>
<td>3.15</td>
<td>A freeway system divided into $N = 2$ sections</td>
</tr>
<tr>
<td>3.16</td>
<td>Lipschitz condition for the speed-density characteristic</td>
</tr>
<tr>
<td>3.17</td>
<td>Robust sliding mode observer: simulation results for velocity, scenario 1</td>
</tr>
<tr>
<td>3.18</td>
<td>Robust sliding mode observer: simulation results for density, scenario 1</td>
</tr>
<tr>
<td>3.19</td>
<td>Robust sliding mode observer: simulation results for velocity, scenario 2</td>
</tr>
<tr>
<td>3.20</td>
<td>Robust sliding mode observer: simulation results for density, scenario 2</td>
</tr>
<tr>
<td>3.21</td>
<td>Robust sliding mode observer: experimental results for Sept. 22, 1997</td>
</tr>
<tr>
<td>3.22</td>
<td>Robust sliding mode observer: experimental results for Sept. 24, 1997</td>
</tr>
<tr>
<td>3.23</td>
<td>A segment of a multi-lane roadway</td>
</tr>
<tr>
<td>3.24</td>
<td>A three-lane highway with two sections</td>
</tr>
<tr>
<td>3.25</td>
<td>Individual lane velocities for section 1 (left plots) and section 2 (right plots), Sept. 22, 1997</td>
</tr>
<tr>
<td>3.26</td>
<td>Estimated and measured traffic flow for individual lanes, section 1 (left plots) and section 2 (right plots) for Sept. 22, 1997</td>
</tr>
<tr>
<td>3.27</td>
<td>Estimated density for individual lanes, section 1 (left plots) and section 2 (right plots), Sept. 22, 1997</td>
</tr>
<tr>
<td>3.28</td>
<td>Individual lane velocities for section 1 (left plots) and section 2 (right plots), Sept. 24, 1997</td>
</tr>
<tr>
<td>3.29</td>
<td>Estimated and measured traffic flow for individual lanes, section 1 (left plots) and section 2 (right plots) for Sept. 24, 1997</td>
</tr>
<tr>
<td>3.30</td>
<td>Estimated density for individual lanes, section 1 (left plots) and section 2 (right plots), Sept. 24, 1997</td>
</tr>
<tr>
<td>4.1</td>
<td>A freeway system for ramp metering control</td>
</tr>
<tr>
<td>4.2</td>
<td>A processing element</td>
</tr>
<tr>
<td>4.3</td>
<td>Two-layer feedforward neural network</td>
</tr>
<tr>
<td>4.4</td>
<td>The neural network controller</td>
</tr>
<tr>
<td>4.5</td>
<td>A two-layer neural network used in the traffic system controller</td>
</tr>
<tr>
<td>4.6</td>
<td>Section density and velocity using different ramp metering strategies, scenario 1</td>
</tr>
<tr>
<td>4.7</td>
<td>Ramp metering rates, scenario 1</td>
</tr>
<tr>
<td>4.8</td>
<td>Section density and velocity using different ramp metering strategies, scenario 2</td>
</tr>
</tbody>
</table>
List of Figures

4.9  Ramp metering rates, scenario 2. ........................................ 131
4.10 Section density and velocity, scenario 3. .............................. 132
4.11 Section density and velocity, scenario 4(a). ........................... 134
4.12 Ramp metering rates, scenario 4(a). ................................. 134
4.13 Section density and velocity, scenario 4(b). .......................... 135
4.14 Ramp metering rates, scenario 4(b). ................................. 135

A.1  Simplex search algorithm ................................................... 140
List of Tables

1. Parameter values for the model of Karaaslan et al.
2. Initial values for the parameters of the speed-density characteristic
3. Initial values for the rest of the model parameters
4. Optimal parameter values for the traffic flow model using SOA
5. Parameter Transferability indices
6. System parameters used in the SMO
7. SMO Gains used in experiments
8. $L_2$-norm of the error of vehicle volume and velocity using the SMO and EKF for the study site (a), Sept 24, 1997
9. $L_2$-norm of the error of vehicle volume and velocity using the SMO and EKF for the study site (b), May 26, 1997
10. Elements of the gain matrix $M$
11. Design parameters used by the robust sliding mode observer
12. Performance indices of estimators for Sept. 22 and 24, 1997, respectively
13. Performance indices for multi-lane robust SMO for Sept. 22, 1997 (top) and Sept. 24, 1997 (bottom)
Chapter 1

Introduction

Traffic networks continue to play an important role in the modern societies of the 21st century. Despite increasing interest in public transit systems, the automobile is still the dominant form of transportation. Never before have conditions been so favorable for conducting research on modelling, control and optimization of traffic flow on roadways. First, the rapid growth of world population demands the implementation of efficient transportation solutions. Currently there are more than half a billion registered vehicles world-wide, and it is projected that this number will double by the year 2030 [1]. Secondly, century-old conventional solutions to increasing the capacity of traffic networks such as road-building, do not address the modern concerns over air quality and consumption of fossil fuels. On an annual basis, vehicles currently consume 25% of the world’s energy production and produce 15% of the global CO₂ emissions. In countries like Germany, the construction of additional highways is prohibited, and, therefore, there is an immediate need for other accommodations to cope with the steady increase of traffic flow.

It is, therefore, important to be able to design and operate highways with the greatest efficiency. In achieving such a goal, traffic flow modelling, estimation of traffic states, and control of traffic flow are essential tools which are addressed mostly from a robustness perspective in this work.

Availability of reasonably accurate models of traffic flow dynamics is a prerequisite for advanced highway traffic management, surveillance, and control schemes. Traffic
flow models seek to describe traffic behaviour by application of the laws of physics and mathematics. At present, there is no unified theory of traffic flow, instead there are several proposed models to describe the phenomena. At the applied level, the benefits of mathematical models can be numerous: Mathematical models can be used to predict the traffic behaviour under certain conditions. They can provide an assessment of the sensitivity of traffic flow performance to environmental conditions, various driving patterns and roadway parameters, and they can provide a basis as to how we might study, control, observe, or interpret traffic phenomena. Depending on the field of application, a large number of static and dynamic traffic flow models may be found in the literature [2], [3]. Moreover, based on the degree of refinement, the models are divided into two categories: microscopic and macroscopic. Throughout this work, only macroscopic models which use aggregate variables will be employed.

In the first macroscopic model presented by Lighthill and Witham [4] in 1955, traffic density was the only state variable. As a result, traffic behaviour was poorly modeled particularly in transient conditions. By adding another differential equation representing the momentum dynamics of the average velocity, Payne [5] in 1971 remedied the shortcomings of the first-order model. The second-order model was later improved according to further empirical and theoretical results by numerous authors, e.g., Payne [6], Cremer [7], Cremer and May [8], Papageorgiou [3], Lyrintzis et al. [9], and Karaaslan et al. [10] to name a few.

The development of an estimation strategy using a traffic flow model and measurements collected from highways is the next stage in operating an automated traffic management system. The amount of attention devoted to this line of research has been relatively small compared to the modelling stage. A number of researchers, Nahi [11] and Trivedi [12] have considered the problem of processing data at a fixed spatial location to produce estimates of spatial average quantities. With good initial conditions, Nahi's method shows the ability to estimate the density closely in homogeneous situations. Motivated by stochastic effects of traffic dynamics and measurements, Kalman Filtering techniques have been widely used by various authors: Gazis and Knapp (1971) [13] and Knapp (1973) [14] proposed a method which utilizes the time series of speed and flow data from each detector and generates crude
estimates of vehicles count. These estimates are then filtered using a Kalman Filter. Since this approach requires substantial data storage, it is not considered practical and later was revised by Ghosh and Knapp [15] via applying Kalman Filtering to a linear first-order model. Since the flow measurement is a nonlinear function of the state variables and necessitates linearization about nominal trajectories of the states, Extended Kalman Filters have also been utilized in traffic state estimation using second-order models [3]. The measurement and model uncertainties are assumed to be white noise with normal distributions. Although this may be valid for the measurements, the uncertainties involved in the model equations such as disturbances and modelling errors cannot be simply regarded as white noise. In dealing with these classes of uncertainties, the notion of robust state estimation arises which requires further investigation reinforced by experimental results.

One measure to obtain maximum operating efficiency and maintain desirable traffic conditions is to optimize the traffic flow on the existing freeways. Highway congestion is mainly a mismatch between the designed highway capacity and the amount of traffic entering the network [16]. One approach to address this problem, as mentioned previously, is to increase the highway capacity by adding more lanes. An alternative approach consists of applying feedback control theory to the highway system via variable speed signs [17], appropriate speed commands [18], [19], and on-ramp metering [20, 21, 22]. The ramp metering strategies are well-known to reduce the occurrence and the extent of recurrent and non-recurrent congestion as well as total travel time [21], [23].

In this thesis, a modified second-order continuum model is used to describe the traffic behaviour along highways. The model will be identified and verified using several sets of traffic data collected from a major highway. State estimation of single- and multi-lane traffic systems using measurements at fixed spatial locations will be presented. The main contribution of the work is the design of a sliding mode state estimator which is robust with respect to bounded modelling errors and disturbances. Robustness analysis and proof of stability of the observer dynamics as well as simulation and experimental results will be included. The results of robust observation can then be employed in an automated highway system to control the traffic flow.
To synthesize a traffic control system, it is assumed that we have full access to the system states. Feedback linearizing techniques are finally utilized to design a robust on-ramp metering controller.

1.1 Outline

The scope of this work is as follows. In Chapter 2, continuum models of traffic flow will be reviewed. In particular, a second-order macroscopic model will be chosen and identified using real traffic data from Highway 401 in Toronto, Canada. The optimization algorithm used in the parameter estimation is presented and, finally, the transferability of the model parameters will be discussed.

Chapter 3 is dedicated to the robust state estimation of single- and multi-lane highway traffic systems. For this purpose, the sliding mode techniques are employed due to their well-known robustness properties with respect to unstructured dynamics. Initially, the background material on the design procedure of sliding mode observers is presented. The design of a sliding mode observer for a single-lane system is then followed by several experimental results. To compare the performance of the proposed observer with other estimation methods, an Extended Kalman Filter is also designed and implemented in parallel. Robustness of the sliding mode observer with respect to disturbances and modelling errors will then be addressed and verified through undertaking a number of simulation and experimental scenarios. The chapter will be concluded by presenting the robust state estimation of multi-lane systems. First, the single-lane model is modified to provide a simple multi-lane model which describes the lane-changing process of vehicles. Design of a robust observer for such systems is then followed. Experimental results illustrate the effectiveness of the robust observer in providing estimates of traffic states for each individual lane.

In Chapter 4, a new state feedback linearizing controller for on-ramp metering strategies is presented. The main aim is to develop a robust controller to locally stabilize freeway traffic despite the presence of disturbances and modelling errors. Stability proof and robustness analysis for the controller are included. Results from a number of simulation scenarios show that the proposed controller provides improved
performance in achieving the design objectives over other existing ramp control algorithms such as a neural network controller and a linear feedback controller.

Finally, the thesis closes with Chapter 5, the concluding remarks and suggested future work. It is concluded that the sliding mode observers can be successfully applied in highway traffic management. Besides providing more accurate state estimation compared to the Kalman Filtering techniques, the sliding mode observers are shown to have robustness properties. This has been verified experimentally as well as in a number of simulation scenarios. The proposed robust feedback linearizing controller is also found to stabilize the traffic system faster and more accurately when tested in parallel with other ramp metering strategies.
Chapter 2

Dynamic Modelling of Traffic Flow on Freeways

2.1 Introduction

The main purpose of this chapter is to present the basic features of traffic flow on freeways and present first- and second-order mathematical models proposed by various researchers in the past. Regardless of the approach adopted, the development of a traffic model suitable for practical purposes has the following four stages: Conceptualization, Mathematical formulation, Calibration, and Validation. Conceptualization is likened to the generation of various hypotheses made by the modeller. Decisions should be made to consider those hypotheses that are significant in the modelling process. Mathematical formulation defines the factors that are to be included in the model as variables or parameters and entails the derivation of the functional relationship between the model variables. Calibration determines appropriate numerical values for the model parameters using real data collected from a roadway. Finally, model validation tends to assess the model performance using traffic data. In this chapter, Sections (2.2) and (2.3) mainly discuss the conceptualization and formulation phases, while Sections (2.4) and (2.5) present the calibration and validation stages.
2.2 Microscopic Models

There are two main genres of traffic models: Microscopic and Macroscopic. Microscopic or car-following models describe the behavior of each vehicle following another vehicle or a string of vehicles. In particular, they provide quantitative values over time of the acceleration of vehicles in a single lane. Car-following models were first proposed by Reusche [24] and Pipes [25] in the 1950's. Their work was then extended theoretically and experimentally by Herman and others [26] - [29]. The model of Herman [30] is based on the concept that a driver accelerates, or decelerates, in response to a stimulus he receives from his driving environment. The general form is derived from the psycho-physical principle of stimulus response:

\[
\text{response}(t) = \text{sensitivity} \times \text{stimulus}(t - \tau). \tag{2.1}
\]

As indicated above, the response is made at some time lag \(\tau\) after the stimulus occurs and its magnitude is proportional to the magnitude of the stimulus. In general, the sensitivity factor may have a nonlinear form as discussed later in this section.

Since the development of the most widely used macroscopic models, e.g., the model of Payne [5], is based on early microscopic formulations, the principles of microscopic theory is briefly reviewed.

2.2.1 Linear Car-Following Models

Consider a string of vehicles on a single-lane roadway in which the \((n+1)\)th vehicle follows the \(n\)th vehicle. At a steady-state, the driver of vehicle \((n+1)\) keeps a desired distance \(s(t)\) such that he can avoid a collision if vehicle \(n\) were to make a stop or sudden change in velocity. Figure 2.1.a shows the pair of vehicles prior to the deceleration of vehicle \(n\) and Figure 2.1.b depicts the positions of the vehicles at rest [31]. Here, \(x_i(t)\) is the position of vehicle \(i\) at time \(t\). Referring to Figure 2.1, one can write the following relation between the stopping distances \(\Delta_n(t)\), \(\Delta_{n+1}(t)\), spacing \(s(t)\), and \(L\), the distance from the front bumper of vehicle \(n\) to the front bumper of vehicle \(n+1\) at rest

\[
s(t) + \Delta_n(t) = \tau \dot{x}_{n+1}(t + \tau) + \Delta_{n+1}(t) + L, \tag{2.2}
\]
Section 2.2. Microscopic Models

where $\tau$ is the reaction time of the driver of vehicle $(n+1)$ and $\dot{x}_{n+1}(t+\tau)$ is the velocity of vehicle $(n+1)$ before initializing deceleration. With the assumption that the deceleration of vehicles is uniform, one can determine the stopping distances $\Delta_i(t)$ according to the equation

$$2\ddot{x}_i(t)\Delta_i(t) = \dot{x}_i^2(t). \quad i = n, n+1$$

Therefore, the spacing at steady-state is given by

$$s(t) = \tau\dot{x}_{n+1}(t+\tau) + \Delta_{n+1}(t) - \Delta_n(t) + L$$

$$= \tau\dot{x}_{n+1}(t+\tau) + \frac{\dot{x}_{n+1}^2(t+\tau)}{2\ddot{x}_{n+1}(t)} - \frac{\dot{x}_n^2(t)}{2\ddot{x}_n(t)} + L. \quad (2.3)$$

Furthermore, if the deceleration and the initial velocities of the vehicles are identical, equation (2.3) becomes

$$x_n(t) - x_{n+1}(t) = \tau\dot{x}_{n+1}(t+\tau) + L.$$

Differentiating with respect to time yields the deceleration equation for vehicle $(n+1)$

$$\ddot{x}_{n+1}(t+\tau) = \frac{1}{\tau} \left( \dot{x}_n(t) - \dot{x}_{n+1}(t) \right), \quad (2.4)$$

which is in the form of equation (2.1). Equation (2.4) is referred to as the basic equation of the car-following models and simply shows that deceleration, or acceleration,
of a vehicle is proportional to its relative speed to the preceding vehicle with the sensitivity factor $\lambda_s = \frac{1}{\tau}$.

Herman et al. [29] considered two forms of stability associated with the car-following model (2.4): local and asymptotic stability. Local stability is concerned with the response of a following vehicle after a change in velocity of the preceding vehicle, while asymptotic stability refers to the process in which a perturbation is propagated through a stream of vehicles. Although the car-following model (2.4) has shown to possess local stability, it suffers from asymptotic instability. Should drivers behave according to equation (2.4), a fluctuation in the velocity of the lead vehicle would amplify and propagate through the stream of vehicles until it results in a collision. Therefore, the linear car-following model cannot be regarded as a realistic model of driver behaviour along a roadway.

### 2.2.2 Nonlinear Car-Following Models

Based on the reasonable assumption that the sensitivity of the stimulus-response relationship is inversely proportional to the vehicle spacing, rather than being constant, a new nonlinear car-following model of the form

$$\ddot{x}_{n+1}(t + \tau) = \frac{\alpha_0}{x_n(t) - x_{n+1}(t)} \left( \dot{x}_n(t) - \dot{x}_{n+1}(t) \right),$$

(2.5)

was introduced [28]. Here $\alpha_0$ is a constant positive parameter. The typical values for the time delay $\tau$ and $\alpha_0$ are 1.5 (seconds) and 35(km/h), respectively [28]. A number of car-following experiments performed at the General Motors test track have confirmed the stability of the model in the asymptotic sense [29]. Further examination of the theory led by Gazis and others [29] suggested a generalized form of the car-following equation as

$$\ddot{x}_{n+1}(t + \tau) = \alpha_0 \frac{\dot{x}_{n+1}^m(t + \tau)}{[x_n(t) - x_{n+1}(t)]^l} \left( \dot{x}_n(t) - \dot{x}_{n+1}(t) \right),$$

(2.6)

where $m$ and $l$ are positive constants. Various nonlinear models are characterized by pairs of specific $(m, l)$. For instance, the linear model (2.4) is obtained by setting $m = l = 0$. 
2.3 Macroscopic Models

Macroscopic theories of traffic flow treat the traffic stream as a continuous fluid in a pipe. While the behaviour of individual drivers is ignored, one is only concerned with the behaviour of aggregated traffic variables. In this section the relationship between the fundamental variables of the traffic flow theory, namely density, flow, and velocity, will be investigated and a macroscopic model will be derived. Modified versions of the model will then be employed in Chapters 3 and 4 to design the proposed observers and controllers.

An early influential contribution to the macroscopic theory of traffic flow was made by Lighthill and Whitham [4] in 1955. Their first-order model includes the fundamental equation for traffic flow and the continuity equation. To derive the model, we shall consider a segment of infinitesimal length $dx$ along a roadway as shown in Figure 2.2. Let $\rho(x,t)$ denote the average aggregate density in units of (veh/km), i.e., vehicles per unit length at time $t$, and $v(x,t)$ denote the average velocity of vehicles in the section in units of (km/h). Also define the traffic flow $q(x,t)$ as the number of vehicles exiting the section during the time period $dt$. At this stage, all the aggregate variables should be understood as mathematical definitions as they cannot have physical interpretations for infinitesimal $dx$ and $dt$.

Throughout the thesis, unless otherwise specified, we only consider single-lane traffic systems. For a multi-lane freeway the above terms refer to the average across all lanes. Moreover, instead of standard MKS units, we employ units which are commonly found in the traffic control literature, such as (km/h) instead of (m/s). By definition, the relationship existing between the aforementioned variables is exactly

$$q(x,t) = \rho(x,t) \ v(x,t),$$

as in fluid dynamics. In spite of this correspondence, intuition and driving experience on freeways suggest that $v(x,t)$ is directly influenced by the traffic density. As the number of vehicles increases in the section the average velocity monotonically decreases, and vice versa. This observation has motivated a simple traffic flow-model in
which density and velocity are related by a static, monotonically-decreasing function

\[ v(x, t) = V\left(\rho(x, t)\right). \]  

(2.8)

Several functional forms for \(V(.)\) can be found in the literature [3, 31, 32]. Greenshields [33] proposed a linear relationship between flow and density

\[ V\left(\rho(x, t)\right) = V_f\left(1 - \frac{\rho(x, t)}{\rho_j}\right), \]  

(2.9)

where \(V_f\) is the free-flow speed and \(\rho_j\) is the jam density, i.e., the density at which the velocity is zero (Figure 2.3.a). It can be shown that the macroscopic relationship (2.9) is a special case of the nonlinear microscopic model (2.6) with \(m = 0, \ l = 2\). Equations (2.7) and (2.9) define a parabolic flow-density model

\[ q(x, t) = V_f\left(1 - \frac{\rho(x, t)}{\rho_j}\right)\rho(x, t). \]  

(2.10)

According to this equation, the maximum flow of \(q_{\text{max}} = \frac{1}{4} \rho_j V_f\) occurs at the density \(\rho_c = \frac{1}{2} \rho_j\) called the critical density. Another family of speed-density characteristics has been derived from the car-following models

\[ V(\rho) = V_f\left(1 - \left(\frac{\rho}{\rho_j}\right)^t\right)^m, \]  

(2.11)

which is a generalization of the Greenshields model [34, 35] (Figure 2.3.b). Here, the dependency of the variables on \(x\) and \(t\) has been dropped for simplicity.
Of particular interest to this work is the logarithmic speed-density characteristic presented by Greenberg [36]

\[ V(\rho) = V_m \ln(\frac{\rho_f}{\rho}). \]  

(2.12)

where \( V_m \) is the speed at maximum flow. We employ a multi-regime speed-density model depicted in Figure 2.3.c which is a modified version of (2.12):

\[ V(\rho) = \begin{cases} 
V_f(1 - \frac{\rho}{\rho_{sf}}), & 0 \leq \rho \leq \rho_f \\
\frac{V_f}{\ln(\frac{\rho_f}{\rho})} \ln(\frac{\rho}{\rho_f}), & \rho_f \leq \rho \leq \rho_{ic} \\
0, & \rho_{ic} \leq \rho 
\end{cases} \]  

(2.13)

Here \( \rho_{ic}, \rho_f, \) and \( V_{sf} \) are positive real constants. For the speed-density curve to be continuous, these parameters should relate as \( V_{sf} = V_f(1 - \frac{\rho_f}{\rho_{sf}}) \), or \( \rho_f = \rho_{sf}(1 - \frac{V_f}{V_{sf}}) \). Therefore, the speed-density curve of (2.13) can be characterized by four parameters \( V_f, \rho_f, V_{sf}, \) and \( \rho_{sf} \). As (2.13) suggests, we assume that average velocity decreases more rapidly for high densities \( (\rho > \rho_f) \) than for relatively lower densities \( (\rho < \rho_f) \).
The structural form of (2.13) is in fact motivated by an investigation on real traffic data collected from Highway 401 in Toronto, Canada. A typical speed-density data set measured by a detector station along the highway is depicted in Figure 2.4. As the distribution of the data points suggests, one may represent the speed-density curve in the form of (2.13) (refer to Section 2.4 for further discussion).

A general flow-density curve also known as the fundamental diagram is shown in Figure 2.3.d. As the diagram suggests, the traffic volume increases for any increase in density until the critical density $\rho = \rho_{cr}$ is reached. Thereafter, the volume decreases until $q(\rho_f) = 0$ is achieved. The A – B portion of the relationship is normally classified as the free flow condition. Numerous studies have confirmed that the free flow is essentially a linear relationship [37, 38, 39]. While the B – C portion is known as the stable flow region, the C – D part is referred to the breakdown or unstable flow condition. Instability means that once an over-critical density has been reached, the traffic becomes more and more congested until the jam density occurs. From the microscopic point of view, an unautomated traffic system lacks the asymptotic stability in the C – D region. A stable automated traffic system, on the other hand, can operate at any point of the flow-density characteristic without becoming unstable. This
will be discussed further in Chapter 4 where traffic density and velocity controllers are presented.

Both first- and second-order models are based on the conservation equation of vehicles on the road. In the sequel, the derivation of this fundamental relationship between traffic flow and density follows.

2.3.1 Continuum First-Order Models

Let us consider the traffic on a segment of a freeway with several on- and off-ramps. Referring to Figure 2.2, the total number of cars present at time \( t \) on the road in the interval \( a < x < b \) is given by

\[
N(t) = \int_a^b \rho(x, t)dx.
\]

The net increase of cars over the time period \( (t, t+\Delta t) \) can be derived to the first-order approximation by

\[
N(t + \Delta t) - N(t) = \left( q(a, t) - q(b, t) \right) \Delta t.
\]

Taking the limit as \( \Delta t \to 0 \), we obtain the equation

\[
\frac{dN}{dt} + \left( q(b, t) - q(a, t) \right) = 0,
\]

or

\[
\frac{d}{dt} \int_a^b \rho(x, t)dx + \int_a^b \frac{\partial}{\partial x} q(x, t)dx = 0. \tag{2.14}
\]

Over the fixed interval \([a, b]\), one can write (2.14) as

\[
\int_a^b \left( \frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} q(x, t) \right) dx = 0.
\]

Since this must hold for any arbitrary interval \([a, b]\), we conclude that

\[
\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} q(x, t) = 0, \tag{2.15}
\]

which is referred to as the conservation equation of vehicles. In obtaining (2.15), we assumed that there is no on- or off-ramps present in the section. The equation can
be modified to include ramp flows:

\[
\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} q(x, t) = g(x, t),
\]

(2.16)

where \( g(x, t) \) is the ramp generation term in units of (veh/h/km).

Equations (2.7) and (2.16) form the simple continuum model of Lighthill and Whitham which is essentially steady state in nature and does not take into account the non-equilibrium traffic flow dynamics. Numerous experimental investigations of traffic flow and empirical data collected from highway traffic systems [40], [41] have shown that, unfortunately, the static relationship between the velocity and flow is not sufficiently accurate to describe the evolution of the traffic at all times. This can be remedied by introducing a second-order continuum formulation which takes into account acceleration by replacing (2.8) with a dynamic equation. Perhaps the most widely used model of this kind is the model proposed by Payne [5], [42] in 1971 and used by several other researchers: Cremer [8], [43], Willsky et. al, [44], and Papageorgiou [45], [46] to name a few.

### 2.3.2 Second-Order Models

The second-order model of Payne is based on both microscopic and macroscopic flow considerations. Let \( x \) be a fixed point along the road and \( v(x, t) \) be the velocity of vehicles passing point \( x \) at time \( t \). Here \( t \) and \( x \) are independent variables. Then

\[
v(x, t) = V\left( \rho(x, t) \right),
\]

(2.17)

where \( V(.) \) is the static speed-density characteristic. To obtain the momentum equation, introduce two constant parameters in (2.17); a small reaction time or delay \( \tau \) in the speed term and a small space increment \( \Delta x \) in the speed-density characteristic:

\[
v(x + \delta x, t + \tau) = V\left( \rho(x + \Delta x, t) \right),
\]

(2.18)

where \( x + \delta x \) is the new position of the vehicle and \( \rho(x + \Delta x, t) \) is the density at \( x + \Delta x \) down the road at time \( t \) as illustrated in Figure 2.5. Equation (2.18) implies that the density of vehicles down the road \( (x + \Delta x) \) at time \( t \) affects the upstream
velocity with a time delay \((t + \tau)\). Note that \(\Delta x\) is a model parameter while \(\delta x\) can be estimated as

\[
\delta x \simeq \tau v(x, t). \tag{2.19}
\]

Equation (2.18) simply indicates that the velocity of the vehicles with a time delay depends on the traffic density downstream. Now expand the left-hand side with respect to \(\delta x\) and \(\tau\) and the right-hand side with respect to \(\Delta x\) about the point \((x, t)\) in a Taylor series:

\[
v(x, t) + \tau \frac{\partial v(x, t)}{\partial t} + \delta x \frac{\partial v(x, t)}{\partial x} = V\left(\rho(x, t)\right) + \Delta x \left(\frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x}\right)_{(x, t)}.
\]

Substituting for \(\delta x\) from (2.19) in the equation above yields

\[
v(x, t) + \tau \left(\frac{\partial v(x, t)}{\partial t} + v(x, t) \frac{\partial v(x, t)}{\partial x}\right) = V\left(\rho(x, t)\right) + \Delta x \left(\frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x}\right)_{(x, t)}. \tag{2.20}
\]

Generally speaking, the space increment \(\Delta x\) is dependent on the road geometry and the traffic density. For our purposes here, we assume that \(\Delta x\) is a constant model parameter. Furthermore, the speed gradient \(\frac{\partial V}{\partial \rho}\) is assumed to be constant up to the first order. Therefore, equation (2.20) becomes

\[
\frac{\partial v(x, t)}{\partial t} = \frac{1}{\tau} \left(V(\rho(x, t)) - v(x, t)\right) - v(x, t) \frac{\partial v(x, t)}{\partial x} - \nu \frac{\partial \rho(x, t)}{\partial x} \tag{2.21}
\]

where \(\nu = -\frac{\Delta x}{\tau} \frac{\partial V}{\partial \rho}\). Each term in the right-hand side of equation (2.21) has a physical interpretation and makes this second-order model a good candidate for practical purposes. The first term represents relaxation to the equilibrium speed. It describes the
effect of drivers adjusting their speeds to the equilibrium speed-density characteristic $V(\rho(x, t))$ according to the time constant $\tau$. In case the time delay $\tau$ is very small, equation (2.21) reduces to the static equation (2.8). The relaxation term is also the most dominant term in the equation since the other terms manifest their effects only when there are (occasional) fluctuations in the traffic velocity and density upstream or downstream. The convection term represents the influence of incoming upstream traffic on the average velocity. Finally, the anticipation term characterizes the effect of drivers reacting to downstream traffic density: drivers tend to accelerate if the density downstream is lower, and vice versa.

Equation (2.21) can be slightly modified to include the effect of the traffic flow into and out of the mainstream at ramps. This is achieved by introducing a friction term [9] in the form of

$$G = -\delta g(x, t) v(x, t),$$

where $\delta$ (not to be confused with $\delta x$) is a constant parameter depending on the road geometry and $g(x, t)$ is the generation term in (2.16). The complete second-order continuum model for a highway system is then given by equations (2.7), (2.16), and

$$\frac{\partial v(x, t)}{\partial t} = \frac{1}{\tau} \left( V(\rho(x, t)) - v(x, t) \right) - v(x, t) \frac{\partial v(x, t)}{\partial x} - \nu \frac{\partial \rho(x, t)}{\partial x} - \delta g(x, t) v(x, t). \tag{2.22}$$

The next step is to discretize (2.7), (2.16), and (2.22) in space to obtain a set of ordinary differential equations. These continuous-time equations will then be employed to synthesize the state observers and ramp controllers. Consider a freeway system divided into $N$ sections, each of length $L_i$, $i = 1 \ldots, N$, as shown in Figure 2.6. Let us define the following space-discretized aggregated traffic variables for the section $i$:

- $\rho_i(t)$, the traffic density in section $i$ (veh/km);
- $v_i(t)$, the mean speed of traffic in section $i$ (km/h);
- $q_i(t)$, the traffic flow leaving section $i$ (veh/h); and
- $r_i(t)$ and $p_i(t)$, the on-ramp and off-ramp flows of section $i$ (veh/h).
To discretize the algebraic equation (2.7), we note that the traffic flow exhibits an anticipatory behaviour, meaning flow at point $x$ depends on density and velocity downstream. Thus, we approximate the flow leaving section $i$ as the convex combination

$$q_i(t) = \alpha \rho_i(t) v_i(t) + (1 - \alpha) \rho_{i+1}(t) v_{i+1}(t),$$

as in [3], [43], or

$$q_i(t) = \alpha \rho_i(t) v_{i+1}(t) + (1 - \alpha) \rho_{i+1}(t) v_{i+2}(t),$$

for $i = 1, \ldots, N - 1$ and $q_N(t) = \rho_N(t) v_N(t)$ as in [47]. Here $0 < \alpha < 1$ is a model parameter to be identified from traffic data. We will employ either (2.23a) or (2.23b) depending on our specific problem. One would expect $\alpha$ to be close to 1 since the number of vehicles leaving section $i$ is mainly a function of the section states rather than the traffic conditions further downstream. To specify the boundary conditions, we may choose stationary conditions for the entrance and exit sections. Since the traffic information downstream of segment $N$ and upstream of segment 1 is not available, we assume $v_0 = v_{in}, v_{N+1} = v_{out}$, and $\rho_N = \rho_{N+1}$.

To obtain a set of ordinary differential equations for each section, we discretize (2.16) and (2.22) in space. Since the average density $\rho_i(t)$ and velocity $v_i(t)$ are no longer functions of space for the section $i$, the partial derivatives $\frac{\partial \rho_i}{\partial t}$ and $\frac{\partial v_i}{\partial t}$ are transformed into ordinary derivatives. Moreover, the partial derivatives in space,
Section 2.3. Macroscopic Models

namely $\frac{\partial \rho}{\partial x}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial q}{\partial x}$, are simply substituted by forward or backward differences yielding two ordinary differential equations for each section

$$\frac{d\rho_i(t)}{dt} = \frac{1}{L_i} \left( q_{i-1}(t) - q_i(t) + r_i(t) - p_i(t) \right).$$

(2.24)

$$\frac{dv_i(t)}{dt} = \frac{1}{\tau} \left( V(\rho_i(t)) - v_i(t) \right) + \frac{\eta}{L_i} v_i(t) \left( v_{i-1}(t) - v_{i+1}(t) \right) - \frac{\nu}{L_i} \left( \rho_{i+1}(t) - \rho_i(t) \right) -$$

$$\frac{\delta}{L_i} \left( r_i(t) + p_i(t) \right) v_i(t).$$

(2.25a)

The convection term in (2.25a) has been discretized, according to [47], in such a way to include the effect of traffic velocity both downstream and upstream of section $i$. Moreover, to further refine the convection term and compensate for the discretization errors a new parameter $\eta$ is added to the term. Taking into account the effect of the upstream velocity only [3, 5, 43], the convection term may be discretized as $\frac{\partial}{\partial t} v_i(t)[v_{i-1}(t) - v_i(t)]$, yielding the following momentum equation

$$\frac{dv_i(t)}{dt} = \frac{1}{\tau} \left( V(\rho_i(t)) - v_i(t) \right) + \frac{\eta}{L_i} v_i(t) \left( v_{i-1}(t) - v_i(t) \right) - \frac{\nu}{L_i} \left( \rho_{i+1}(t) - \rho_i(t) \right) -$$

$$\frac{\delta}{L_i} \left( r_i(t) + p_i(t) \right) v_i(t).$$

(2.25b)

As mentioned before, the design of the proposed observers and controllers are based on equations (2.23), (2.24), and (2.25). To verify the robustness of the proposed observers and controllers with respect to structural changes in the model dynamics, a different model presented by Karaaslan et al. [10] will be used to generate the system states. For simulation purposes, this model will be also employed when no empirical traffic data is available:
\[ q_i(t) = \alpha \rho_i(t)v_i(t) + (1 - \alpha)\rho_{i+1}(t)v_{i+1}(t) \]  

\[ \frac{d\rho_i(t)}{dt} = \frac{1}{L_i} \left( q_{i-1}(t) - q_i(t) + r_i(t) - \rho_i(t) \right) \]  

\[ \frac{dv_i(t)}{dt} = \frac{1}{\tau} \left( V(\rho_i(t)) - v_i(t) \right) + \frac{1}{L_i} \frac{\rho_{i-1}(t)}{\rho_i(t) + \chi} v_{i-1}(t) \left( \sqrt{v_{i-1}(t)v_i(t)} - v_i(t) \right) - \frac{\mu(t)}{\tau} \frac{1}{L_i} \frac{\rho_{i+1}(t) - \rho_i(t)}{\rho_i(t) + \chi}, \]  

where \[ \mu(t) = \begin{cases} \frac{\mu_1}{\rho_i - \rho_{i+1}(t) + \sigma} & \text{if } \rho_{i+1}(t) \geq \rho_i(t), \\ \mu_2 & \text{otherwise} \end{cases} \]

In this model, the static speed-density characteristic is the generalized Greenshields equation (2.11). The modellers believe that the convection term in the form of \( v_i(t)[v_{i-1}(t) - v_i(t)] \) has a strong effect in the momentum equation specially when the density in the section \( i - 1 \) is smaller than the density in section \( i \). This term is then modified to include a density-dependent coefficient \( \frac{\rho_{i-1}(t)}{\rho_i(t) + \chi} \) and a weaker passing speed \( v_{i-1} \rightarrow \sqrt{v_{i-1}v_i} \). Moreover, the anticipation term is found to be strong when the downstream traffic is less dense. To resolve this incongruity, two different coefficients have been introduced in the anticipation term based on the section densities.

The nominal parameter values used in the model are listed in Table 2.1 [10].

<table>
<thead>
<tr>
<th>( V_i )</th>
<th>( \rho_i )</th>
<th>( l )</th>
<th>( m )</th>
<th>( \alpha )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>93.1 (km/h)</td>
<td>110 (veh/km)</td>
<td>1.86</td>
<td>4.05</td>
<td>0.95</td>
<td>40 (veh/km)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \zeta )</th>
<th>( \sigma )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 (km/h)</td>
<td>12 (km²/h)</td>
<td>6 (km²/h)</td>
<td>120 (veh/km)</td>
<td>35 (veh/km)</td>
<td>0.0057 (h)</td>
</tr>
</tbody>
</table>
2.4 Model Identification

The purpose of this section is to present results from several identification experiments. In doing so, the numerical values of the model parameters will be determined by an off-line optimization procedure which fits the model to the collected traffic data. These parameters are: $V_f$, $p_j$, $V_j$, $p_{ij}$, $\alpha$, $\tau$, $\eta$, $\nu$, and $\delta$.

2.4.1 A Case Study: Highway 401

The heart of the highway traffic network in Metropolitan Toronto is a 36-km section of Highway 401 shown in Figure 2.7. Its great popularity as a major commuter corridor within the city and as a primary route for inter-province traffic in Southern Ontario are the main causes of heavy traffic volumes. To accommodate the high demand of traffic, the highway consists of up to 16 directional lanes as well as collector and express facilities. The traffic capacity of the highway has been pushed far beyond its original design capacity for the past two decades; traffic volumes have increased to 320,000 average annual daily traffic (AADT) in some sections in 1994 [48] compared to 230,000 AADT in 1984 [49]. This is well beyond the design capacity of 180,000 AADT, making the Highway 401 the second-heaviest traveled section of roadway in North America [48]. As a result, recurrent congestion due to high demand exceeding the highway capacity is a common phenomenon during morning and afternoon rush hours, as well as non-recurrent congestion due to incidents throughout the same peak periods.

A Freeway Traffic Management System (FTMS) consisting of traffic surveillance, automatic incident detection, closed-circuit camera television, variable message signs, and a driver information center has been installed by the Ontario Ministry of Transportation (MTO) to detect and respond to the formation of congestion and smooth out the traffic flow along the highway [49]. The system employs induction loop detectors embedded in the roadway approximately every 600 meters to measure and report traffic flow, occupancy, and, in case of dual loop detectors, average speed data for every $T_d = 20$ seconds. Here, occupancy represents the percentage of time the
detector is on, i.e., registering the presence of a car, during the 20-second sampling interval.

For the purposes of this study, two locations near a major bottleneck on the highway were selected. In selecting these study sites, the primary criterion was the availability of traffic volume data as well as velocity data along the roadway. This provides an excellent comparison ground for evaluating the performance of the proposed estimators in Chapter 3. Moreover, it was believed that some sections, in particular the eastbound section immediately after highway 427, which experience high flow per lane are major bottleneck areas under normal conditions. The intention was to choose one detector station downstream from the bottleneck, one immediately upstream, and one far enough upstream from the primary bottleneck. Therefore, the available traffic data from the sites covers a broad spectrum of the traffic conditions from free flow to congestion. As a result, if the performance of the proposed estimators are acceptable using this data, it is believed that similar, if not better, results will be obtained for other highway locations.

Data was collected over a total of 4 days in the months of March and September 1997. Most data represented typical weekday morning and afternoon commute conditions covering free-flow to heavy-volume traffic scenarios. Schematic diagrams
Section 2.4. Model Identification

Figure 2.8: The westbound portion of the study site.

Figure 2.9: The eastbound portion of the study site.

of the study sites with the lane configurations and detector locations are shown in Figures 2.8 and 2.9. All the detector stations, except the ramp units, are dual loop sensors providing flow as well as average traffic velocity per lane. For each detector station, the average traffic velocity and flow will then be obtained from the individual sensors.

Sample traffic flow, speed and occupancy measurements obtained from the detector station 401DW0100DWS (Figure 2.8) are shown in Figure 2.10. The measurements are characterized by large variations over time. These typical measurement sets are used by COMPASS surveillance algorithm to balance the traffic flow between the collector and the express lanes.
2.4.2 Parameter Estimation

The parameter estimation problem for the model (2.23)-(2.25) is now formulated and solved using the optimization algorithm presented in [50]. Referring to Figure 2.6, measurement sets provided by loop detectors located at both ends of each section are available. The average traffic velocity $\omega$ at each detector station, also called time mean speed, is the harmonic mean of the velocities of the individual vehicles passing the station [3], [51] and is related to the aggregated section velocities through

$$\omega_i = \alpha v_i + (1 - \alpha)v_{i+1}, \quad i = 1, 2, \ldots, N - 1$$

(2.27)

and $\omega_0 = v_m, \omega_N = v_N$. Let us define a state vector $x$ including the aggregated velocity and density of the sections

$$x = \begin{bmatrix} \rho_1 & v_1 & \ldots & \rho_i & v_i & \ldots & \rho_N & v_N \end{bmatrix}^T, \quad x \in \mathbb{R}^{2N}$$

(2.28)

and consider the traffic flow and velocity at both ends of the site as well as the flow at on- and off-ramps as the system inputs which are measured and uncontrollable

$$u = \begin{bmatrix} q_m & v_m & \tau_1 & p_1 & \ldots & \tau_N & p_N & q_N & v_N \end{bmatrix}^T, \quad u \in \mathbb{R}^{2N+4}$$

(2.29)
The output of the system can be considered the traffic volume and time mean speed at any station $i$

$$y = [q_i, \omega_i]^T, \quad y \in \mathbb{R}$$  \quad (2.30)

and the parameter vector is

$$\beta = [\alpha, \tau, V_f, \rho, \nu, \eta, \delta]^T, \quad \beta \in \mathbb{R}$$  \quad (2.31)

Equations (2.23), (2.24), and (2.25) are then written in the general form of a nonlinear dynamic system

$$\dot{x} = f(x, u, \beta), \quad (2.32)$$

$$y = g(x, \beta). \quad (2.33)$$

For identification purposes, we assume that the initial condition $x(0) = x_0$ is known. In fact, if one initiates the identification procedure when the traffic is at a steady state level, e.g., free flow condition, the state values (2.28) can be directly approximated from the detector measurements. Another possibility is to treat the initial conditions as unknown parameters included in the parameter vector $\beta$. The identification procedure then approximates the initial condition as well as the parameters.

As the model equations (2.32) and (2.33) are nonlinear in both states and parameters, the estimation process is not a trivial problem. The most conventional approach in this case is the least-square mean error algorithm which minimizes the difference between the model output and the measured data [3], [51]. The formulation of the optimization problem is presented below.

Given the initial condition $x_0$, the input measurement $u$, and the output measurement $y_m$ from an internal detector station over the time slot $0 \leq t \leq T_o$, find the parameter set $\beta$ which minimizes the least square criterion

$$J(\beta) = \frac{1}{T_o} \int_0^{T_o} \left( y(t) - y_m(t) \right)^T Q \left( y(t) - y_m(t) \right) dt, \quad (2.34)$$

where $Q$ is a positive-definite $2 \times 2$ scaling matrix. Here, $y(t)$ is the traffic volume and velocity at the internal station $i$ ($1 < i < N$) produced by the model equations for a specific value of $\beta$. The time duration $T_o$ is chosen to include a broad spectrum
of traffic conditions from the real data set. In order to balance the volume errors against the speed errors, the weighting matrix $Q$ is chosen to be

$$Q = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_q^2 \end{bmatrix},$$

(2.35)

where $\sigma_v^2$ and $\sigma_q^2$ are the variations of the measured traffic velocity and volume, respectively, at station $i$.

A direct search method called the Simplex Optimization Algorithm (SOA) of Nelder and Mead [50], [52] was used to minimize the performance index (2.34) which has the advantage of not requiring any explicit evaluation of derivatives. The SOA is briefly reviewed in Appendix A. It is to be noted that the function (2.34) may have a large number of local minima over the parameter space $[3, 8]$, and, hence, a number of different starting points in the optimization procedure was used to obtain the optimal solution.

The optimization structure is depicted in Figure 2.11. The algorithm starts with an initial value for the parameter set $\beta$ randomly chosen in the acceptable region of the parameter space. Several different sets of initial parameters are used to assure the independency of the optimal parameter set to the initial values. In each iteration, the cost function is evaluated using the collected traffic data and the parameter set. The parameter set is then appropriately updated by the algorithm if further improvement in the minimization of the cost function is possible (refer to Appendix A).

### 2.4.3 The Choice of Initial Values for Parameters

To enhance the chance of obtaining an optimal parameter set and to reduce the time required for the optimization procedure, the initial values for the model parameters are chosen within the admissible parameter space and close to their expected values.

The parameters associated with the static speed-density characteristic (2.13), namely $V_f$, $\rho_f$, $V_{sf}$, $\rho_{sf}$, can be estimated off-line using a least-square fit on the measured traffic velocity and density. The data recorded on Sept. 26, 1997 between 5:00 AM and 10:00 PM at the station 401DW0070DWS is shown in Figure 2.12. Here the
Section 2.4. Model Identification

Figure 2.11: The Flow diagram of the optimization algorithm

Table 2.2: Initial values for the parameters of the speed-density characteristic

<table>
<thead>
<tr>
<th></th>
<th>$V_f$</th>
<th>$\rho_{ij}$</th>
<th>$V_{ij}$</th>
<th>$\rho_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>121 (km/h)</td>
<td>110 (veh/km)</td>
<td>104.5 (km/h)</td>
<td>80 (veh/km)</td>
</tr>
</tbody>
</table>

density is calculated from the speed and flow measurements using the microscopic relation $\rho = q/v$ at the detection zone. It is interesting to see that the measured data points form a family of hyperbolic curves perhaps due to the round-off error of the sensors. The nominal speed-density characteristic is also shown in the figure using the numerical values of Table 2.2. These values are then used as the initial parameter values in the SOA.

Initial values for the remaining parameters $\alpha$, $\tau$, $\eta$, $\nu$, and $\delta$ are taken from the literature [9], [53] and listed in Table 2.3.
Section 2.4. Model Identification

2.4.4 The Optimal Parameter Set and Model Validation

The traffic data from Highway 401 study sites (Figures 2.8, 2.9) are now used in the outlined optimization procedure. The westbound section was divided into 6 segments with the traffic sensors located at the boundaries of the adjacent segments as well as at both ends of the site. One advantage of this configuration is that any detector station can be taken as the internal site \( i \) \( (1 < i < N) \) in computing \( J(\beta) \) in equation (2.34). The average length of the sections is 575(m). The eastbound portion is divided into 5 sections with an average length of 525(m).

To consider the effect of merging traffic in the mainstream into the parameter estimation, it was decided to process the data from two internal stations. One station is chosen prior to and another station immediately after a major ramp or bifurcation. These stations are 401DW0060DWS and 401DW0080DWS for the westbound site.

Table 2.3: Initial values for the rest of the model parameters

<table>
<thead>
<tr>
<th>( \alpha ) (km/h)</th>
<th>( \tau ) (h)</th>
<th>( \eta )</th>
<th>( \nu ) (km(^3)/h(^2)/veh)</th>
<th>( \delta ) (km/veh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>9.7e-3</td>
<td>0.2</td>
<td>6.5e-2</td>
<td>1e-4</td>
</tr>
</tbody>
</table>

Figure 2.12: Traffic velocity versus density at the detector station 401DW0070DWS
Table 2.4: Optimal parameter values for the traffic flow model using SOA

<table>
<thead>
<tr>
<th>Case</th>
<th>Case la (lb)</th>
<th>Case 2a (2b)</th>
<th>Case 3a (3b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data from the westbound site, station 401DW0060DWS</td>
<td>Data from the westbound site, station 401DW0060DWS</td>
<td>Data from the eastbound site, station 401DW0060DES</td>
</tr>
<tr>
<td></td>
<td>(401DW0080DWS) for Sept. 24, 1997 between 4:00 and 7:00 PM</td>
<td>(401DW0080DWS) for Sept. 22, 1997 between 4:00 and 7:00 PM</td>
<td>(401DW0040DWS) for Mar. 13, 1997 between 8:00 and 11:00 AM</td>
</tr>
<tr>
<td></td>
<td>Data from the eastbound site, station 401DW0060DES</td>
<td>Data from the eastbound site, station 401DW0060DES</td>
<td>Data from the eastbound site, station 401DW0060DES</td>
</tr>
<tr>
<td></td>
<td>(401DW0040DWS) for Mar. 13, 1997 between 8:00 and 11:00 AM</td>
<td>(401DW0040DWS) for Mar. 13, 1997 between 8:00 and 11:00 AM</td>
<td>(401DW0040DWS) for Mar. 13, 1997 between 8:00 and 11:00 AM</td>
</tr>
</tbody>
</table>

The evolution of the objective function and the parameters for the cases (1a) and (3b) are also shown in Figures 2.13 and 2.14, respectively. Convergence to optimal values was achieved after approximately 1000 iterations for most of the cases. Based on these identified values, the following remarks can be made:

**Remark 1.** The same traffic model was used for all the highway segments regardless of the differences in their geometry and traffic capacity. Therefore, the identified values represent a model which describes an average traffic behaviour for Highway 401.

**Remark 2.** The optimal values of the parameters $V_f$, $\rho_{jf}$, $V_j$, $\rho_j$ are all in their
expected range and in agreement with the primary results of Table 2.2. While the free speed \( V_f \) is comparable with values reported in [3] and [48], the critical density \( \rho_c \) is substantially smaller than the average value of 110 (veh/km). This may be due to the different (Canadian) driving experience and the special choice of the fundamental diagram (2.13).

### 2.5 Transferability of the model parameters

Based on the results obtained in the model identification, the optimal parameter sets are expected to represent the traffic behaviour for a broad range of traffic situations, different times of day, and different locations along the highway. Here, the transferability of the parameters is further investigated using the data from different days and other detector stations.

To quantify the parameter transferability property, the mean square variation between the velocity and flow generated by the model and the corresponding measured data are used as an indication of the model performance. If the measured quantities are denoted by \( v_m \) and \( q_m \) and the model variables by \( v \) and \( q \), then the fitness criteria are defined as

\[
J_v = 1 - \frac{1}{T_o} \int_0^{T_o} \frac{1}{v_m^2} (v - v_m)^2 \, dt
\]

\[
J_q = 1 - \frac{1}{T_o} \int_0^{T_o} \frac{1}{q_m^2} (q - q_m)^2 \, dt,
\]  

where \([0 \; T_o]\) is the time interval of the simulation.

The identified parameter set for case (1a) from Table 2.4 is used throughout the rest of this section. Table 2.5 lists the performance indices \( J_v \) and \( J_q \) at the westbound station 401DW0090DWS and the eastbound station 401DW0070DES (Figures 2.8 and 2.9) for May 23, and 26, 1997, respectively.

The perfect match between the model states and the measurements is obtained when the fitness indices are \( J_v = 1 \) and \( J_q = 1 \). It can be seen from Table 2.5 that the optimal choice of parameters is fairly transferable.
Figure 2.13: Parameter estimation using Simplex Algorithm (case 1a)
Figure 2.14: Parameter estimation using Simplex Algorithm (case 3b)
Section 2.5. Transferability of the model parameters

Figures 2.15 and 2.16 show the speed and volume profiles of the model and measurements. The agreement between the two indicates that the model is able to compute the states of the traffic and predict congestion quite effectively when the optimized set of parameters is used.

Table 2.5: Parameter Transferability indices

<table>
<thead>
<tr>
<th></th>
<th>401DW0090DWS</th>
<th>401DW0070DES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_v$</td>
<td>0.8123</td>
<td>0.9132</td>
</tr>
<tr>
<td>$J_q$</td>
<td>0.8709</td>
<td>0.8534</td>
</tr>
</tbody>
</table>
Section 2.5. Transferability of the model parameters

Figure 2.16: Transferability of the parameters for the eastbound station, May 26, 1997
Chapter 3

Robust State Estimation using Sliding Mode Techniques

3.1 Introduction

Increasing population and globalization of trade have increased the demand placed on highway traffic networks. Existing highways are now utilized beyond their designed capacity, compromising safety and increasing transport time, and so politically there is a genuine need to address this concern. Traffic density on highways can be reduced by either physically increasing the size of the highway network or by improving the management of the existing network. Either approach will increase the inherent network capacity, but the costs of implementing the competing approaches are quite disparate. If the network is physically enlarged, then the environmental and fiscal costs can be quite large. On the other hand, improvements in efficiency can be obtained with a comparatively meagre investment in new sensing equipment and research, although management costs may incrementally increase.

In order for a new management policy to be effective, traffic conditions at critical points along the highway must be actively monitored, especially during times of peak usage. Most current highway traffic control policies and incident detection techniques require accurate knowledge of traffic states such as average velocity and density [10, 19, 44]. Such information can be obtained by embedding a sufficiently large number
of sensors. Therefore, accurate measurement is not infeasible. However, to reduce this cost, researchers have focused on the development of model-based management policies which are capable of providing accurate estimates of traffic states using a smaller number of sensors.

The theory of deterministic (Luenberger) and stochastic (Kalman) filters or "state estimators" for LTI systems is well documented in the literature, e.g., [54]. The nonlinear nature of the traffic flow makes these linear approaches inappropriate and necessitates a nonlinear design methodology. Walcott & Zak [55], and Misawa & Hedrick [56] give an overview of existing nonlinear observer techniques. Some of these, like the Extended Kalman Filter (EKF), involve excessive on-line computation, and, therefore, are not always practical for real-time control.

Motivated by the apparently stochastic nature of traffic dynamics and measurements, Kalman Filtering Techniques have been widely used to produce estimates of traffic states [3, 43, 44]. Along this line of research, the model and measurement uncertainties are assumed to be white noise with a normal distribution. Although this may be a valid assumption for the measurement vector, the uncertainties such as disturbances, modelling errors and unmodelled dynamics in the model equations require more investigation and cannot be simply considered as white noise.

In this chapter, we take a more deterministic point of view. A robust nonlinear sliding mode observer is developed to generate estimates of average velocity and density for a segment of a highway within a corridor, given loop detector measurements at the end-points of the segment. The observer is based on the nonlinear model of traffic flow developed in Section 2.3. The sliding mode structure approach has several advantages over Kalman Filtering, including proof of estimate convergence and simplified computations. However, the primary advantage is the robustness with respect to unmodelled dynamics and model parameters: Based on the presented traffic model, we develop a systematic design procedure to make the observer invariant with respect to the modelling uncertainties and unavailable (not measured) traffic states.

The proposed nonlinear observer is then experimentally verified and compared to the Extended Kalman Filter using data collected from Highway 401. The objective is to estimate the traffic states in the areas between the loop detectors. This could
potentially result in a reduction in the number of sensors which are expensive to install, maintain, and repair. According to the Ministry of Transportation of Ontario (MTO), the average cost of a loop detector is about CND $1000 excluding the installation and any probable repair and maintenance\textsuperscript{1}. Moreover, loop detectors are sensitive to pavement resurfacing and maintenance and can be easily perturbed and, thus, will produce faulty measurements. As an example, the traffic data collected on Sept. 26, 1997 by the detector station 401DW0070DWS at Highway 401 and 427 intersection (Figure 2.8) shows measurement corruption between 4:00 and 10:00 PM as depicted in Figure 3.1. The repetitive patterns and often negative recordings are the imminent sign of faulty measurements. For this particular case, the cause was the resurfacing and construction carried on lane 1 of the express lanes\textsuperscript{2}.

The organization of this chapter is as follows. In Section 3.2 some background material on variable structure control and sliding mode observers is provided. Section 3.3 outlines the steps taken in designing a sliding mode observer for a traffic system with $N = 3$ sections along with simulation and experimental results. Design of an observer for a system with an arbitrary numbers of sections is presented in Section 3.4. Design of a robust observer and robustness analysis will be addressed in Section 3.5. And finally, Section 3.6 will be devoted to the robust state estimation problem for multi-lane traffic systems.

### 3.2 Variable Structure Control Theory

The control of nonlinear and uncertain systems has been devoted a considerable amount of research and attention for the past few decades. One approach to the problem is to linearize the system about a nominal operating point or trajectory and then apply well-known linear control strategies. A major drawback of this method is that the linear control laws are valid only when the system states are in the vicinity of the nominal trajectories. To overcome this problem, gain scheduling algorithms may be invoked which require the system to be linearized about a number of nominal

\textsuperscript{1}Personal talk with Felix Tam and David Tsui, MTO, Nov. 30, 2000

\textsuperscript{2}Personal talk with Nita Rajcoomar, MTO, Oct. 10, 1997
Section 3.2. Variable Structure Control Theory

Figure 3.1: Faulty measurement recorded by the station 401DW00/0DWS, Sept. 26, 1997

trajectories. Another approach is to find a nonlinear transformation which converts the system into controller canonical form or an equivalent linear form where the control design is easily facilitated using classical control theory. The disadvantage of this approach is that the construction of an appropriate transformation is nontrivial in most cases.

One of the most attractive methods which can be applied to a broad class of nonlinear systems resulting in controllers that are robust to modeling errors and unknown disturbances is the variable structure control. The development of variable structure control, the control of dynamical systems with discontinuous state feedback controllers, is essentially due to Utkin [57, 58, 59] and represents a fundamental advance in the design of switching control systems. The theory rests on the concept of changing the structure of the controller in the response to the changing states of the system to obtain a desired response. This is accomplished by the use of a high speed switching control law which forces the system trajectories onto a chosen manifold where they remain thereafter. The system is insensitive to certain parameter variations and disturbances while the trajectories are kept on the manifold. If the
state vector is not accessible, then a suitable state observer must be utilized.

### 3.2.1 Sliding Mode Dynamics

Consider a nonlinear dynamical system of the form

\[ \dot{x} = f(x) + g(x)u \quad (3.1) \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \) and \( f \) and \( g \) are smooth functions of \( x \). The design idea of sliding mode control \cite{57} implies that the control elements \( u_i, \ i = 1, \ldots, m \) undergo discontinuities in the smooth surfaces \( \sigma_i = \{ x : s_i(x) = 0 \} \) in the system state space, i.e.,

\[ u_i(x) = \begin{cases} u_i^+(x), & \text{if } s_i(x) > 0 \\ u_i^-(x), & \text{if } s_i(x) < 0 \end{cases} \quad (3.2) \]

where the control functions \( u_i^+(x) \) and \( u_i^-(x) \) are continuous functions. These functions are designed to force the state onto a so-called sliding manifold, making it attractive and invariant. The motion along the sliding manifold is described by a system of \((n - m)th\) order. The manifold itself is designed such that this reduced-order system has properties such as stability at the origin.

The design of switching control systems of the type (3.1) and (3.2) is greatly facilitated by the introduction of sliding modes. If there exists a submanifold \( M \) of any intersection of discontinuous surfaces, that is, \( s_i(x) = 0 \) for \( i = 1, \ldots, p \leq m \), such that \( s_i \dot{s}_i < 0 \) in the neighborhood of almost every point in \( M \), then the system trajectory once entering \( M \) remains there. \( M \) is called a sliding manifold and the motion in \( M \) is called a sliding mode.

Variable structure control system design entails specification of the switching surfaces \( s_i(x) \) and the control elements \( u_i^+(x) \) and \( u_i^-(x) \). A key observation is that the sliding mode dynamics depend on the geometry of \( M \) which is specified by the switching functions \( s_i(x) \). The sliding can be induced on a desired manifold \( M \) by designing the control functions \( u_i^+(x) \) and \( u_i^-(x) \) to guarantee that \( M \) is attractive. Thus, the design is a two step process: Design of the "sliding mode" dynamics by the choice of switching surfaces, and design of the "reaching" dynamics by the specification of the control functions.
Section 3.2. Variable Structure Control Theory

In dealing with systems (3.1) and (3.2), the method of equivalent control [57] is useful for characterizing the sliding dynamics. If a trajectory lies on a sliding manifold \( M \), then \( s = 0 \) and all time derivatives of \( s(x) \) also vanish. The control is not defined by (3.2) when \( s = 0 \). Instead, it is defined by continuation: Let \( u_{eq} \) denote the equivalent control which is obtained while the trajectory remains in the manifold \( M \). Then, \( u_{eq} \) is defined by

\[
\dot{s} = S(x)\dot{x} = S(x)\{f(x) + g(x)u_{eq}\} = 0,
\]

where \( S(x) = \frac{\partial}{\partial x}s(x) \). Assuming \( \det \{ S(x)g(x) \} \neq 0 \), we have

\[
u_{eq} = -[S(x)g(x)]^{-1} S(x)f(x).
\]

Motion in the sliding mode is then defined by

\[
\dot{x} = \{I_{n \times n} - g(x)[S(x)g(x)]^{-1} S(x)\} f(x),
\]

with \( x(0) \) satisfying \( s(x(0)) = 0 \). Note that the equivalent control as defined by (3.4) depends only on the switching surface \( s(x) \) and not on the control functions \( u_i^*(x) \).

### 3.2.2 Sliding Mode Observers

As in the design of a variable structure controller, there are two basic steps in the design of a sliding mode observer:

- The design of the switching surface so that the behaviour of the system has certain prescribed properties on the surface, e.g., the state estimation error becomes asymptotically stable on the surface.

- The design of the estimator gain to steer the error dynamics to the switching surface and maintain it on the surface.

Consider the autonomous nonlinear system

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n
\]
and a vector of measurements of the states
\[ y = h(x), \quad y \in \mathbb{R}^m \]  
(3.6)
Assume \( h(x) = [h_1(x) \ldots h_m(x)]^T \), and define the directional derivative of \( h_j \) along the vector field \( f \) as
\[ L_jh_j(x) = \frac{\partial h_j}{\partial x} f(x), \]
with
\[ L_j^{k}h_j(x) = L_j \left( L_j^{k-1}h_j(x) \right), \quad k = 2, 3, \ldots \]
Now introduce the following notations based on the directional derivatives of the system (3.5) and (3.6)
\[ H_i(x) = [h_i(x) L_jh_i(x) \ldots L_j^{\kappa_i-1}h_i(x)]^T, \quad i = 1, \ldots, m \]  
(3.7)
where \( \kappa_i \) (\( i = 1, \ldots, m \)) is an integer number. Finally, define
\[ H(x) = [H_1(x) H_2(x) \ldots H_m(x)]^T. \]  
(3.8)
If there exist \( \kappa_1, \ldots, \kappa_m \) such that \( \sum_{i=1}^{m} \kappa_i = n \) and the mapping \( H(x) \) is a global diffeomorphism, then the system is said to be completely observable. One can also define the observability matrix of the nonlinear system (3.5) and (3.6) as in [60]
\[ Q(x) = \frac{\partial H(x)}{\partial x}. \]  
(3.9)
Functions \( f(x) \) and \( h(x) \) are assumed to be smooth enough such that all the partial derivatives introduced above exist and are continuous. Moreover, for a given domain \( X_0 \subset \mathbb{R}^n \) of initial conditions of the system (3.5), which is assumed to be bounded, all the solutions of the system belong to an open domain \( X \subset \mathbb{R}^n \), for all \( t \geq 0 \). Note that a necessary condition for \( H(x) \) to be a diffeomorphism is that \( Q(x) \) is non-singular for \( \forall x \in X \). Consequently, the observer design may be considered for the states of \( x \) from the measurements \( y \). For the rest of this section, we assume that the output vector is linearly related to the states \( y = Cx \).

Following the steps of the observer design, one may define a sliding surface \( s \) as the error between the estimated value \( \hat{y}_i \) and the measurement \( y_i \) in the form
\[ s = [s_1 \ldots s_m]^T = \Gamma(y - C\bar{x}), \]
Section 3.2. Variable Structure Control Theory

where \( \hat{x} \in \mathbb{R}^n \) is an estimate for \( x \), and \( \Gamma \in \mathbb{R}^{m \times m} \) is a gain matrix to be specified. The surface \( s \) is attractive if

\[
s \hat{s} > 0.
\]

(3.10)

A sufficient condition for (3.10) is

\[
s, \hat{s} < 0. \quad i = 1, \ldots, m
\]

(3.11)

Inequality (3.11) constrains trajectories to point towards the surface \( s(t) = 0 \), as illustrated in Figure 3.2, and is referred to as the sliding condition. If this is satisfied, then sliding along \( s = 0 \) is guaranteed and the error \( (y_i - \hat{y}_i) \) converges to zero.

To guarantee the sliding condition to occur, switching terms are added to the traditional observer structure

\[
\dot{\hat{x}} = \hat{f}(\hat{x}) + K\Gamma(y - C\hat{x}) + \Lambda I_s,
\]

(3.12)

where \( \hat{f} \) is our model of \( f \), \( K \) and \( \Lambda \) are \( n \) by \( m \) gain matrices and the vector \( I_s \) represents the switching terms

\[
I_s = [\text{sign}(s_1) \ldots \text{sign}(s_m)]^T.
\]

(3.13)

The dynamics of the state error vector \( \bar{x} = x - \hat{x} \) is then given by

\[
\dot{\bar{x}} = f(x) - \hat{f}(\hat{x}) - K\Gamma(C\bar{x}) - \Lambda I_s.
\]

(3.14)

Figure 3.2: Sliding condition
For simplicity, the following notation will be used from now on
\[
\Delta f(x, \hat{x}) = f(x) - \hat{f}(\hat{x}).
\] (3.15)

The value of $\Delta f$ depends both on the modeling complexity and the error magnitude. To simplify the observer design, known nonlinear terms in the system can also be treated as bounded errors and included in $\Delta f$ as modeling errors.

During sliding, the switching terms in (3.13) act to keep $s \equiv 0$. Therefore,
\[
\dot{s} = \Gamma C \dot{x} \equiv 0
\] (3.16)
and $\tilde{I}_s$, the equivalent switching vector, can be obtained by substituting (3.14) in (3.16)
\[
\Gamma C(\Delta f - \Lambda \tilde{I}_s) = 0,
\]
thus,
\[
\tilde{I}_s = (\Gamma C \Lambda)^{-1} \Gamma C \Delta f.
\] (3.17)

With an appropriate choice of $\Gamma$ and $\Lambda$, the $m \times m$ matrix $\Gamma C \Lambda$ is invertible. The equivalent dynamics on the reduced order manifold is obtained from (3.14) and (3.17):
\[
\dot{x} = \begin{pmatrix} I_{n \times n} - \Lambda(\Gamma C \Lambda)^{-1} \end{pmatrix} \Gamma C \Delta f,
\] (3.18)
\[
\Gamma C \ddot{z} = 0.
\] (3.19)

A proper design of the sliding manifold shall guarantee that (3.18) is asymptotically stable. The system dynamics is then reduced from $n$th to $(n-m)$th order. Equation (3.18) shows that the linear gain $K$ does not influence the dynamics of the errors on the sliding surface. It only affects the capture phase by decreasing the time required to reach the sliding manifold.

Any further analysis requires that the structure of the modelling error $\Delta f$, and, therefore, our modeling of $f$, to be known.
3.3 Design of a Sliding Mode Observer

In this section, a sliding mode observer for a highway traffic system modeled by (2.23)-(2.25) will be developed. The system is divided into $N$ sections as in Figure 2.6 with traffic sensors located at the end of every $m$ sections as well as the ramp locations. The mainstream sensors provide traffic velocity and volume while the ramp sensors report only volume every $T$ seconds. The observer receives the measurements from the sensors and produces estimates of the traffic states for each section. Given this configuration for the sensors, the highway system is divided into approximately $M = N/m$ subsystems as shown in Figure 3.3. The system division is performed in such a way that the subsystem $l+1$ has little influence in the dynamics of the subsystem $l$, $l = 1, \ldots, M - 1$. This is achieved by requiring that the length of the last section of the subsystem $l$ small compared to the length of the first section of the subsequent subsystem $l+1$. In this case, the effect of the anticipation term in the momentum equation for the last section of the subsystem $l$ is negligible and one can consider the decentralized problem of state estimation for each subsystem at a time. Here, the observer presentation is for $m = 3$ shown in Figure 3.4. In the next section, generalization of the observer design for an arbitrary $m$ will be discussed.

The states of the system are the average velocity and density of the sections, $\mathbf{x} = [x_1, x_2, \ldots, x_s]^T = [\rho_1, v_1, \ldots, \rho_3, v_3]^T \in \mathbb{R}^6$, and the outputs are the measured velocity and flow at the end point $D$. As the length of the last section, $i = 3$, is small, one can assume a microscopic relation between flow, velocity and density for this section and can compute the average velocity and density of the

![Figure 3.3: A traffic system divided into $M$ subsystems of $m$ sections ($L_{im} << L_{(i+1)m}$)]
Assumptions 1. The velocities of the sections and the outgoing traffic velocity \( v_{\text{out}} \) are strictly positive:

\[
v_1, v_2, v_{\text{out}} \geq V_{\text{min}} > 0.
\]

As it will be discussed later in this section, assumptions 1 will be also required to obtain a stable reduced-order error dynamics for the observer.

Rewriting (2.23b), (2.24), (2.25a), and (3.20) for the traffic system of Figure 3.4 yields

\[
\dot{x}_1 = \frac{1}{L_1} \left( q_{\text{in}} + r_1 - p_1 - \alpha x_1 x_4 - (1 - \alpha) x_3 x_6 \right)
\]

\[
\dot{x}_2 = \frac{1}{\tau} \left( V(x_1) - x_2 \right) + \frac{\eta}{L_1} x_2 (v_{\text{in}} - x_4) + \frac{\nu}{L_1} (x_3 - x_1) - \frac{\delta}{L_1} (r_1 + p_1) x_2
\]

\[
\dot{x}_3 = \frac{1}{L_2} \left( r_2 - p_2 + \alpha x_1 x_4 + (1 - 2\alpha) x_3 x_6 - (1 - \alpha) x_5 v_{\text{out}} \right)
\]
Section 3.3. Design of a Sliding Mode Observer

\[
\begin{align*}
\dot{x}_4 &= \frac{1}{\tau} \left( V(x_4) - x_4 \right) + \frac{\eta}{L_2} x_4 (x_2 - x_4) - \frac{\nu}{L_2} (x_2 - x_3) - \frac{\delta}{L_2} (r_2 + p_2) x_4 \quad (3.24) \\
\dot{x}_5 &= \frac{1}{L_3} \left( r_3 - p_3 + \alpha x_3 x_6 + (1 - \alpha) x_3 v_{out} - q_{out} \right) \quad (3.25) \\
\dot{x}_6 &= \frac{1}{\tau} \left( V(x_6) - x_6 \right) + \frac{\eta}{L_3} x_6 (x_4 - v_{out}) - \frac{\delta}{L_3} (r_3 + p_3) x_6, \quad (3.26) \\
y &= [y_1 \ y_2]^T = [x_5 \ x_6]^T. \quad (3.27)
\end{align*}
\]

It should be mentioned that while \( q_{out} \) and \( v_{out} \) form the output, \( q_{in} \) and \( v_{in} \), \( p_i \), and \( r_i \), \( i = 1, 2, 3 \), are measured external inputs. Later in Section (3.5) where the design of a robust observer is presented, these input variables will be considered as disturbances.

Prior to the observer presentation, it should be investigated whether the system (3.21)-(3.27) is observable. First an acceptable domain \( X \) for the states \( x \) is derived based on simple physical considerations. Since the density of each section is positive and, on average, less than the jam density, we have

\[ 0 \leq x_{2i-1} \leq \gamma_d \rho_i, \quad i = 1, 2, 3 \]

where \( \gamma_d \geq 1 \). Moreover, according to Assumptions 1

\[ 0 < x_{2i} \leq \gamma_v V_f. \]

Investigation on several speed-density characteristics, e.g., Figure 2.12, provides nominal values for the parameters \( \gamma_d \) and \( \gamma_v \):

\[ \gamma_d = 1.12, \quad \gamma_v = 1.1. \]

Therefore,

\[ X = \left\{ x \in \mathbb{R}^6 \mid x_{2i-1} \in [0, \gamma_d \rho_i], \ x_{2i} \in (0, \gamma_v V_f), \ i = 1, 2, 3 \right\} \quad (3.28) \]

For the system to be completely observable, \( H(x) \) as defined in (3.8) should be a diffeomorphism over \( X \). At this point of the development, we will make the assumption that the system (3.21)-(3.27) is in fact completely observable.

Justification for this assumption is based on numerical experiments carried out on the system. In particular, the observability matrix \( Q(x) \) (3.9) was computed for the
optimal parameter values of Tables 2.2 and 2.3 using Mathematica, and is listed in Appendix B. In this case $Q(x)$ was found to have rank $n = 6$ for a large number of arbitrary vectors $x$ chosen in $X$. It is to be noted that checking the rank condition on a large finite of points does not guarantee that the rank condition is satisfied on the entire set $X$; however, if the number of points chosen is substantially large, we may conjecture that $Q(x)$ is invertible for all $x$ in $X$ and hence the system is locally observable. The invertibility of $Q(x)$, $\forall x \in X$, however is a necessary but not sufficient condition for complete observability.

Let $\hat{x}_i$ denote the estimate of $x_i$, define the estimate error as $e_i = x_i - \hat{x}_i$, and consider the following observer structure:

\begin{align*}
\dot{x}_1 &= \frac{1}{L_1} \left( q_{i1} + r_1 - p_1 - \alpha \hat{x}_4 \hat{x}_4 - (1 - \alpha)\hat{x}_3 \hat{y}_2 \right) + \Lambda_1 I_z + a_1(\hat{x}, y) \\
\dot{x}_2 &= \frac{1}{\tau} \left( V(\hat{x}_1) - \hat{x}_2 \right) + \frac{\eta}{L_1} \hat{x}_2 (r_1 - p_1) \hat{x}_4 - \frac{\nu}{L_1} (\hat{x}_3 - \hat{x}_1) - \frac{\delta}{L_1} (r_1 + p_1) \hat{x}_2 + \Lambda_2 I_z + a_2(\hat{x}, y) \\
\dot{x}_3 &= \frac{1}{L_2} \left( r_2 - p_2 + \alpha \hat{x}_1 \hat{x}_4 + (1 - 2 \alpha) \hat{x}_3 \hat{y}_2 - (1 - \alpha) y_1 v_{out} \right) + \Lambda_3 I_z + a_3(\hat{x}, y) \\
\dot{x}_4 &= \frac{1}{\tau} \left( V(\hat{x}_3) - \hat{x}_4 \right) + \frac{\eta}{L_2} \hat{x}_4 (\hat{x}_2 - y_2) - \frac{\nu}{L_2} (y_1 - \hat{x}_3) - \frac{\delta}{L_2} (r_2 + p_2) \hat{x}_4 + \Lambda_4 I_z + a_4(\hat{x}, y) \\
\dot{x}_5 &= \frac{1}{L_2} \left( r_3 + p_3 - q_{out} + \alpha \hat{x}_3 \hat{y}_2 + (1 - \alpha) y_1 v_{out} \right) + \Lambda_5 I_z + a_5(\hat{x}, y), \\
\dot{x}_6 &= \frac{1}{\tau} \left( V(y_1) - y_2 \right) + \frac{\eta}{L_3} \hat{y}_2 (\hat{x}_4 - v_{out}) - \frac{\delta}{L_3} (r_3 + p_3) y_2 + \Lambda_6 I_z + a_6(\hat{x}, y). \tag{3.34}
\end{align*}

Here, $\Lambda_1, \ldots, \Lambda_6 \in \mathbb{R}^{1 \times 2}$ and $a_1(\hat{x}, y), \ldots, a_6(\hat{x}, y)$ are the observer gains and injection terms yet to be determined in order to make the origin of the error dynamics globally exponentially stable (GES). The error dynamics is obtained from (3.21)-(3.26) and (3.29)-(3.34)

\begin{equation}
\dot{e}_i = -\frac{\alpha}{L_1} (x_i - \hat{x}_i \hat{x}_4) - \frac{1 - \alpha}{L_1} y_2 e_3 - \Lambda_1 I_z - a_1(\hat{x}, y) \tag{3.35}
\end{equation}
Following the design steps of Section 3.2, define the sliding manifold and switching terms as

\[
\begin{align*}
\dot{s} &= \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}^T = \Gamma \begin{bmatrix} e_3 \\ e_6 \end{bmatrix}^T, \\
\Gamma &= \text{diag}(\frac{L_3}{\alpha}, 1).
\end{align*}
\]

Proposition 1. The switching surface \( s = 0 \) is attractive, i.e., \( s_1 \dot{s}_1 < 0 \) and \( s_2 \dot{s}_2 < 0 \) for all \( s_1 \neq 0, s_2 \neq 0 \), if

\[
\begin{bmatrix}
\Lambda_3 \\
\Lambda_6
\end{bmatrix} = 
\begin{bmatrix}
\frac{\alpha}{L_3} y_2 k_1 & 0 \\
0 & k_2
\end{bmatrix},
\]

\[
a_3(\hat{x}, y) = -\frac{\alpha}{L_3} \dot{x}_3 y_2 + k_3(y_1 - \hat{x}_3),
\]

\[
a_6(\hat{x}, y) = -\frac{\eta}{L_3} \dot{x}_4 y_2 + k_4(y_2 - \hat{x}_6),
\]
Section 3.3. Design of a Sliding Mode Observer

where the gains $k_1, \ldots, k_4$ satisfy

$$k_1 > \gamma_d \rho_j, \quad k_2 = \frac{n}{L_3} \gamma_e^2 V_f^2, \quad k_3 > 0, \quad k_4 > 0.$$

Proof. Computing $\dot{s}_1$ and $\dot{s}_2$ from (3.25), (3.26), (3.33), (3.34), (3.41) and (3.43), we have

$$\dot{s}_1 = y_2(x_3 - k_1 \text{sign}(s_1)) - k_2 s_1, \quad (3.47)$$

$$\dot{s}_2 = \frac{n}{L_3} y_2 x_4 - k_2 \text{sign}(s_2) - k_4 s_2. \quad (3.48)$$

The system states are bounded according to (3.28), in particular, $y_2 > 0$ and $x_3 \leq \gamma_d \rho_j$.

By setting $k_1 > \gamma_d \rho_j$ and $k_3 > 0$, we guarantee that

$$y_2(x_3 - k_1 \text{sign}(s_1)) < k_2 s_1, \quad (s_1 > 0)$$

and

$$y_2(x_3 - k_1 \text{sign}(s_1)) > k_2 s_1, \quad (s_1 < 0)$$

which imply

$$s_1 \dot{s}_1 < 0. \quad (s_1 \neq 0)$$

Similarly, since $0 \leq x_4, y_2 \leq \gamma_e V_f$, for $k_2 = \frac{n}{L_3} \gamma_e^2 V_f^2$ and $k_4 > 0$, one has

$$s_2 \dot{s}_2 < 0. \quad (s_2 \neq 0)$$

It should be noted that the gains $k_1, \ldots, k_4$ can always be chosen to satisfy the conditions required in proposition 1: The maximum possible values for the density ($\gamma_d \rho_j$) and velocity ($\gamma_e V_f$) determine the lower bounds for $k_1$ and $k_2$ while $k_3$ and $k_4$, the gains associated with the linear parts in the observer dynamics, are chosen to be positive.

Once the sliding condition $s = 0$ is satisfied, the dimension of the error dynamics is effectively reduced from $n = 6$ to $n - m = 4$, and the reduced-order dynamics
can be approximated using the equivalent control method [61]. During sliding, the switching term $I_s$ is acting to keep $s \equiv 0$, hence, $\dot{s} \equiv 0$. Therefore, one can obtain the switching vector along the sliding manifold from equations (3.47) and (3.48):

$$I_s = \begin{bmatrix} \text{sign}(s_1) \\ \text{sign}(s_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{k_1} x_3 \\ \frac{n}{k_2} \frac{1}{k_3} y_2 x_1 \end{bmatrix}. \quad (4.49)$$

Under the satisfaction of assumptions 1, we can derive observer gains $\Lambda_1, \ldots, \Lambda_4$ and output injection terms $a_1, \ldots, a_4$ which render the origin of the reduced-order error dynamics, defined by $s \equiv 0$, asymptotically stable.

**Proposition 2.** The origin of the reduced-order error dynamics is asymptotically stable if

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_1} (1 - \alpha) k_1 y_2 & -\frac{\alpha}{\tau_1} k_2 \frac{1}{\tau_2} \hat{x}_1 \\ -\frac{\nu}{\tau_1} k_1 & -\frac{\tau_1}{\tau_2} k_2 \frac{1}{\tau_2} \hat{x}_2 \\ -\frac{1}{\tau_2} (2\alpha - 1) k_1 k_2 y_2 & \frac{\alpha}{\tau_2} \frac{\tau_1}{\tau_2} k_2 \frac{1}{\tau_2} \hat{x}_1 \\ \frac{\nu}{\tau_2} k_1 & \frac{\tau_1}{\tau_2} k_2 \frac{1}{\tau_2} \hat{x}_2 + \frac{\tau_1}{\tau_2} k_2 \frac{1}{\tau_2} \hat{x}_3 \end{bmatrix},$$

and

$$\begin{bmatrix} a_1(\hat{x}, y) \\ a_2(\hat{x}, y) \\ a_3(\hat{x}, y) \\ a_4(\hat{x}, y) \end{bmatrix} = \begin{bmatrix} \frac{\nu}{\tau_1} \hat{x}_3 + \frac{\tau_1}{\tau_2} \hat{x}_4 \\ \frac{\nu}{\tau_1} \hat{x}_3 + \frac{\tau_1}{\tau_2} \hat{x}_4 \\ -\frac{\nu}{\tau_2} \hat{x}_2 \hat{x}_4 + \frac{1}{\tau_2} (2\alpha - 1) k_1 y_2 \hat{x}_3 \\ -\frac{\nu}{\tau_2} \hat{x}_2 \hat{x}_4 - \frac{\tau_1}{\tau_2} \hat{x}_3 - k_4 \hat{x}_4 \end{bmatrix},$$

where $k_s < 1$ and $k_s > -\frac{1}{\tau}$.
Section 3.3. Design of a Sliding Mode Observer

Proof. Substituting the equivalent value for $I_4$ from equation (3.49) into the error dynamics (3.35)-(3.38), and using the above definitions for $A_1, \ldots, A_4$ and $a_1, \ldots, a_4$ yield the following reduced-order observer dynamics:

$$
\dot{e}_1 = -\frac{\alpha}{L_1} x_4 e_1 \tag{3.50}
$$

$$
\dot{e}_2 = -\left(\frac{1}{\tau} + \frac{\delta}{L_1} (r_1 + p_1) + \frac{\eta}{L_1} (x_4 - v_{in})\right) e_2 + \frac{1}{\tau} \left(V(x_4) - V(\hat{x}_4)\right) + \frac{\nu}{L_1} e_1 \tag{3.51}
$$

$$
\dot{e}_3 = -\frac{1}{L_2} (2\alpha - 1)(1 - k_s) y_2 e_3 + \frac{\alpha}{L_2} x_4 e_1, \tag{3.52}
$$

$$
\dot{e}_4 = -\left(k_s + \frac{1}{\tau} + \frac{\eta}{L_2} y_2 + \frac{\delta}{L_2} (r_2 + p_2)\right) e_4 + \frac{1}{\tau} \left(V(x_4) - V(\hat{x}_3)\right) + \frac{\eta}{L_2} x_4 e_2. \tag{3.53}
$$

Since $x_4 > 0$, the origin of (3.50) is globally exponentially stable (GES). This implies that equations (3.51)- (3.53) represent GES error dynamics with vanishing perturbations provided that the gains $k_s$ and $k_e$ are properly chosen. The origin of the (3.51) is GES if $\psi > 0$. Since $r_1, p_1 > 0$ this is guaranteed for any allowable $0 < x_4, v_{in} \leq \gamma_v V_f$ when

$$
\frac{1}{\tau} - \frac{\eta}{L_1} \gamma_v V_f > 0,
$$

or

$$
L_1 > \eta \tau \gamma_v V_f. \tag{3.54}
$$

The constraints on $k_s$ and $k_e$ in turn imply that the origins of (3.52) and (3.53) are also GES.

It should be emphasized that the gains $\Lambda_1, \ldots, \Lambda_4$ and terms $a_1, \ldots, a_4$ as provided by proposition 2 are design criteria, not assumptions on the system dynamics.

3.3.1 Continuous Realization of the Observer

The main drawback of the sliding mode strategy is that the resulting observer dynamics is discontinuous on the switching surface and, consequently, undesired chattering will be generated at a theoretically infinite frequency. This problem can be
remedied by replacing the switching function by a proper continuous function in the neighborhood of the sliding surface:

\[
\text{conti}(s_i) = \begin{cases} 
\text{sign}(s_i), & |s_i| > \mu \\
\frac{s_i}{\mu}, & |s_i| \leq \mu 
\end{cases}
\]  

(3.55)

or

\[
\text{conti}(s_i) = \frac{s_i}{|s_i| + \mu},
\]  

(3.56)

where \(\mu\) is a small positive number. For the simulation purposes throughout the remaining of this chapter and Chapter 4, we employ the continuous realization in (3.56) with \(\mu = 0.02\).

### 3.3.2 Experimental Results

To investigate the effectiveness of the proposed sliding mode observer, it is more informative to perform a number of experimental scenarios and verify the observer performance in real-life conditions rather than present simulation results. For this
Section 3.3. Design of a Sliding Mode Observer

purpose, we consider the eastbound portions of Highway 401 between Highway 427 and Eglinton Avenue (site a) and between Dixon Road and Highway 427 (site b) to be our study sites as shown in Figure 3.5. Each site is divided into three sections approximately 550(m) in length with only one off-ramp in the first site and two on-ramps in the second site. Two sets of loop detector stations, labeled A and D, at the entrance and exit of the segments, provide the measured external inputs \((q_m, v_m)\) and measured outputs \((q_{out}, v_{out})\) while the single detectors O, P, and Q report the on and off-ramp vehicle volumes \(p_1, r_1,\) and \(r_2,\) respectively. Detectors at stations labeled B and C also measure the mean velocities

\[
v_b = \alpha v_1 + (1 - \alpha)v_2, \quad (3.57)
\]

\[
v_c = \alpha v_2 + (1 - \alpha)v_3, \quad (3.58)
\]

and flows \(q_b\) and \(q_c,\) but this data is not used by the observer. Rather, it is used to test the accuracy of the observer's estimates at these points.

<table>
<thead>
<tr>
<th>Table 3.1: System parameters used in the SMO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>(V_f)</td>
</tr>
<tr>
<td>(\rho_{jf})</td>
</tr>
<tr>
<td>(V_{jf})</td>
</tr>
<tr>
<td>(\rho_j)</td>
</tr>
<tr>
<td>(\alpha)</td>
</tr>
</tbody>
</table>

The identified values for the parameters of the system given in Table 2.4 are averaged and listed in Table 3.1. These values are then used in the implementation of the observer dynamics (3.29)-(3.34). The observer gains \(k_1, \ldots, k_6\) are also listed in Table 3.2. Besides satisfying propositions 1 and 2, the gains are chosen such that no significant improvement in the performance of the observer is achieved by increasing them. Moreover, the condition on the length of section one given by (3.54) is satisfied for both sites.

For the purposes of comparison, a continuous-discrete Extended Kalman Filter (EKF) was also designed and simulated using the same traffic data. The EKF equa-
Table 3.2: SMO Gains used in experiments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1 = 1.1 \gamma_o \rho_j$</td>
<td>95.5 veh/km</td>
<td>$k_4 &gt; 0$</td>
<td>540 (1/h)</td>
</tr>
<tr>
<td>$k_2 = 2.8 \gamma_o^2 V_f^2$</td>
<td>26450 km/h$^2$</td>
<td>$k_5 &lt; 1$</td>
<td>0.5</td>
</tr>
<tr>
<td>$k_3 &gt; 0$</td>
<td>0.002 (km)$^3$/((h$^2$ veh))</td>
<td>$k_6 &gt; -\frac{1}{r}$</td>
<td>360 (1/h)</td>
</tr>
</tbody>
</table>

Estimations can be summarized as:

\[
\dot{\hat{x}} = f(\hat{x}) \quad kT \leq t \leq (k+1)T
\]
\[
\dot{\hat{y}} = h(\hat{x}) = [\hat{x}_s \quad \hat{x}_v]^{T}
\]
\[
\dot{P} = F(\hat{x})P + PF^{T}(\hat{x}) + Q
\]
\[
\hat{x}_k^+ = \hat{x}_k^- + K_k(\hat{y} - \hat{y})
\]
\[
P_k^+ = (I - K_kH_k^-)P_k^-
\]
\[
K_k = P_kH_k^- (H_k^-P_k^-H_k^- + R)^{-1},
\]

where $T = 20$ (sec) is the sampling period of the measurements, $f$ represents the right-hand side of (3.21)-(3.26), $F = \partial f/\partial x$, $H = \partial h/\partial x$, $R = \text{diag}[1300 130]$, and $Q = 100 \cdot \text{diag}[13 17 13 17 13 17]$ were tuned to minimize the estimate error. The EKF was initialized at $\hat{x}(0) = [17 110 17110 17 110]^T$ and $P(0) = \text{diag}[100 1.3 100 1.3 100 1.3]$ for all experiments.

Estimates of the mean speed and flow using the Sliding Mode Observer (SMO) and the EKF, as well as the measured data from detector stations B and C, are depicted in Figures 3.7, 3.6 for site (a) for a period of two hours during the September 24, 1997 afternoon rush hour and in Figures 3.11, 3.10 for site (b) for a period of three hours of May 23, 1997. Both estimators describe traffic volume with satisfactory accuracy. However, the SMO produces better estimates of velocity at both stations. This is made clear in Figures 3.8 and 3.12, which show the error between the estimated velocities and the real data at points B and C. Also, depicted in Figures 3.9 and 3.13 are the estimates of the average densities. As there is no direct measurement for densities, these estimates cannot be used to determine the performance of the estimators. However, since the overall performance of the SMO is better, it is
Section 3.3. Design of a Sliding Mode Observer

Table 3.3: $L_2$-norm of the error of vehicle volume and velocity using the SMO and EKF for the study site (a), Sept 24, 1997

<table>
<thead>
<tr>
<th>$L_2$-norm error</th>
<th>Sliding Mode Observer</th>
<th>Extended Kalman Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>e_{vB}</td>
<td>$ (veh/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vC}</td>
<td>$ (veh/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vB}</td>
<td>$ (km/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vC}</td>
<td>$ (km/h)</td>
</tr>
</tbody>
</table>

Table 3.4: $L_2$-norm of the error of vehicle volume and velocity using the SMO and EKF for the study site (b), May 26, 1997

<table>
<thead>
<tr>
<th>$L_2$-norm error</th>
<th>Sliding Mode Observer</th>
<th>Extended Kalman Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>e_{vB}</td>
<td>$ (veh/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vC}</td>
<td>$ (veh/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vB}</td>
<td>$ (km/h)</td>
</tr>
<tr>
<td>$</td>
<td>e_{vC}</td>
<td>$ (km/h)</td>
</tr>
</tbody>
</table>

believed that the estimates of density produced by the SMO are closer to their real values.

To quantify the difference in performance, we computed the $L_2$-norm, defined as

$$
||e||_2 = ||x_m - \hat{x}||_2 = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (x_m(t) - \hat{x}(t))^2 \, dt},
$$

(3.59)

for the estimate errors in flow and velocity at points B and C. Here, $x_m$ is the measured quantity and $\hat{x}$ is an estimate for $x_m$. Tables 3.3 and 3.4 summarize these results. While both algorithms show similar results for $||e_{vB}||_2$, SMO provides smaller values for $||e_{vB}||_2$ and $||e_{vC}||_2$. 
Figure 3.6: Measured and estimated velocities at stations B (top) and C for Site (a), Sept. 24, 1997
Figure 3.7: Measured and estimated volume at stations B(top) and C for Site (a), Sept. 24, 1997

Figure 3.8: Mean velocity errors for Site (a), Sept. 24, 1997
Figure 3.9: Estimated section densities for Site (a), Sept. 24, 1997
Figure 3.10: Measured and estimated velocities at stations B (top) and C for Site (b), May 26, 1997
Section 3.3. Design of a Sliding Mode Observer

Figure 3.11: Measured and estimated volume at stations B (top) and C for Site (b), May 26, 1997

Figure 3.12: Mean velocity errors for Site (b), May 26, 1997
Figure 3.13: Estimated section densities for Site (b), May 26, 1997
3.3.3 Robustness Analysis with respect to Parameter Variations

In addition to the experimental estimation results obtained by employing the Sliding Mode Observer and the Extended Kalman Filter, the sensitivity of the two algorithms with respect to parameter variations has also been investigated. Consider the cost function

\[ J(\theta) = J_u(\theta) + \frac{\sigma_u}{\sigma_q} J_q(\theta), \]

where \( \theta = [\alpha \, \tau \, V_f \, \rho_1 \, V_{ij} \, \rho_{ij} \, \eta \, v \, \delta]^T \). \( J_u \) and \( J_q \) are the standard deviation of the mean velocity error and volume error, respectively, and \( \sigma_u \) and \( \sigma_q \) are the standard deviation of the measured mean speed and volume at the detector station B of site (a).

The parameters of the model are changed about the nominal parameter set \( \bar{\theta} \) (defined in Table 3.1) one at a time in the range \( 0.75 \bar{\theta}_i < \theta_i < 1.25 \bar{\theta}_i \) while all other parameters are kept at their nominal (identified) values. There are several ways to measure the parameter sensitivity of the estimators. Here, we consider the normalized deviation in the cost function \( J \) from its nominal value which is the most direct and linear way to quantify the changes. In doing so, the expression

\[ \left( \frac{J(\theta) - J(\bar{\theta})}{J(\bar{\theta})} \right), \]

was computed and plotted in Figure 3.14 for each estimator.

Note that the EKF is especially sensitive to changes in \( V_f \). In fact, the EKF generates unrealistic results when \( V_f \) is increased beyond 25% of its nominal value. Both observers are relatively insensitive to changes in \( \delta \) and \( \nu \). This is probably because the test site contains only one on-ramp, making the generation term \( G = -\delta g v \) in (2.22) small, and because the diffusion term \( \partial \rho / \partial x \) is also relatively small. Interestingly, the SMO is considerably more robust with respect to the changes in \( \tau \) and \( \alpha \), especially when these parameters are smaller than their nominal values.
Figure 3.14: Sensitivity of EKF (-) and SMO (-.) with respect to parameter variations.

3.4 State Estimation for a Traffic System: The General Case

The sliding mode observer designed in section (3.3) estimates the traffic states of a system with $N = 3$ subsections. It requires traffic measurements from two detector stations located at the entrance and exit sections in addition to the on- and off-ramp sensors. Referring to Figure 3.4, the number of sensors used in the main stream is then reduced by half while the observer produces the traffic states fairly accurately.

As the number of segments increases, the performance of such an estimator de-
teriorates especially when trying to estimate the states of the sections close to the entrance. This is simply due to the fact that the traffic state of the upstream sections are weakly affected by the traffic conditions downstream.

In the following, first, a systematic design methodology for sliding mode observers based on the Equivalent Control Method [61], [62] is presented and generalized for multi-output systems. Then, the state estimation problem for a traffic system with $N > 3$ sections using the proposed observer will be presented. It should be mentioned that the design procedure of section (3.3) is not appropriate when $N > 3$ due to the lack of direct coupling between the states of the system at the exit section (system outputs) and the downstream sections.

### 3.4.1 Sliding Mode Observers based on the Equivalent Control Method

Consider the nonlinear system (3.5), (3.6)

\[
\begin{align*}
\dot{x} &= f(x), \quad x \in \mathbb{R}^n \\
y &= h(x), \quad y \in \mathbb{R}^m
\end{align*}
\]

and assume that the number of states satisfies $n = k.m$ for some integer $k$. This is the case for the systems we consider throughout the thesis.

As presented in Section 3.2.2, let us introduce the following notations based on the Lie derivatives of the system

\[
H_i(x) = [h_{i1}(x) \ h_{i2}(x) \ldots \ h_{im}(x)]^T, \quad i = 1, \ldots, k \quad (3.60a)
\]

where

\[
\begin{align*}
H_i(x) &= h(x), \\
h_{j(i+1)}(x) &= L_j h_i(x) = \frac{\partial h_i(x)}{\partial x} f(x), \\
j &= 1, \ldots, m, \quad i = 1, \ldots, k - 1
\end{align*}
\]
and define
\[ H(x) = [H_1(x) \ H_2(x) \ \ldots \ H_k(x)]^T, \quad (3.60c) \]
\[ Q(x) = \frac{\partial H(x)}{\partial x}. \quad (3.60d) \]

Assume \( H(x) \) is a global diffeomorphism, as in Section 3.2.2, which implies that \( Q(x) \) is nonsingular for every \( x \) in \( X \subset \mathbb{R}^n \).

To estimate the states of (3.5) from the measurements (3.6) we consider an observer of the form
\[ \hat{x} = \left( \frac{\partial H(\hat{z})}{\partial \hat{x}} \right)^{-1} M \text{sign}\left( V(t) - H(\hat{x}) \right), \quad (3.61) \]
where \( M \) is a gain matrix with positive elements in the form of \( M = \text{diag}(M_i) \), \( M_i = \text{diag}(m_{ij}) \), \( j = 1, \ldots, m \), \( i = 1, \ldots, k \) yet unspecified. The vector \( V \) is defined as
\[
\begin{align*}
V(t) &= [V_1(t) \ V_2(t) \ \ldots \ V_k(t)]^T, \\
V_i(t) &= [v_{i1}(t) \ v_{i2}(t) \ \ldots \ v_{im}(t)]^T, \quad i = 1, \ldots, k
\end{align*}
\]
with
\[
\begin{align*}
V_1(t) &= y(t), \\
v_{j(i+1)}(t) &= m_j \text{sign}\left( v_{ij}(t) - h_j(\hat{x}) \right). \\
j = 1, \ldots, m, \quad i = 1, \ldots, k - 1
\end{align*}
\]
Since the map \( H: \mathbb{R}^n \to \mathbb{R}^n \) is an injection, it is sufficient to show that the modified observation error
\[
\begin{align*}
E(t) &= [E_1(t) \ E_2(t) \ \ldots \ E_k(t)]^T = H(x(t)) - H(\hat{x}(t)), \\
E_i(t) &= [e_{i1}(t) \ e_{i2}(t) \ \ldots \ e_{im}(t)]^T, \quad i = 1, \ldots, k
\end{align*}
\]
converges to zero. From (3.5), (3.6) and (3.61) it follows that
\[
\begin{align*}
\dot{E}(t) &= \frac{dH(t)}{dt} - \frac{\partial H(x)}{\partial x} \hat{x} \\
&= \frac{dH(\hat{x})}{dt} - M \text{sign}\left( V(t) - H(\hat{x}) \right), \quad (3.65)
\end{align*}
\]
or

\[ \dot{e}_{j_i}(t) = h_{j_{i(i+1)}}(\mathbf{x}) - m_{j_i} \text{sign}\left(v_{j_i}(t) - h_{j_i}(\bar{\mathbf{x}})\right). \quad (3.66) \]

\( j = 1, \ldots m, \quad i = 1, \ldots k \)

Since \( V_1(t) = y(t) = H_1(\mathbf{x}(t)) \) and \( E_1(t) = V_1(t) - H_1(\bar{\mathbf{x}}(t)) \), sliding occurs for the first set of equations in (3.66), i.e., \( i = 1 \), if \( m_{j_1} \geq |h_{j_2}(\mathbf{x})|, \quad j = 1, \ldots, m \).

During sliding, \( E_1(t) = 0 \) and according to the Equivalent Control Method

\[ H_2(\mathbf{x}) = \left( m_{j_i} \text{sign}(V_1(t) - H_1(\bar{\mathbf{x}})) \right)_{eq}, \]

which follows from (3.62)

\[ V_2(t) = H_2(\mathbf{x}(t)), \]

\[ E_2(t) = V_2(t) - H_2(\bar{\mathbf{x}}(t)). \]

Similarly, the next step requires \( m_{j_2} \geq |h_{j_{23}}(\mathbf{x})| \) for sliding to occur in the second set of equations in (3.66). In general, in step \( i, \quad i = 1, \ldots, k \) the sliding occurs if

\[ m_{j_i} \geq |h_{j_{i(i+1)}}(\mathbf{x})|. \quad j = 1, \ldots, m \]

The gain matrix \( M \) is, therefore, defined by the upper bounds of the functions \( h_{j_{i(i+1)}}(\mathbf{x}) \).

### 3.4.2 Observer Design for a Traffic System with an Arbitrary Number of Segments

Consider the traffic system of Figure (2.6) with \( N \) subsections. The system states and outputs are defined as before

\[ \mathbf{x} = [x_1 \ldots x_{2N}]^T = [\rho_1 \ldots \rho_N \; v_1 \ldots v_N]^T, \]

\[ \mathbf{y} = [y_1 \; y_2]^T = [\rho_N \; v_N]^T, \]
where \( n = 2N, \) \( m = 2, \) and \( k = N. \)

From (2.23b), (2.24), (2.25a), the system equations are

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{L_1} \left( q_{n} + r_i - p_i - \alpha x_1 x_4 - (1 - \alpha) x_3 x_6 \right) \\
\dot{x}_2 &= \frac{1}{\tau} \left( V(x_1) - x_2 \right) + \frac{\eta}{L_1} x_2 (v_m - x_4) - \frac{\nu}{L_1} (x_3 - x_1) - \frac{\delta}{L_1} (r_i + p_i) x_2 \\
\vdots \\
\dot{x}_{2N-1} &= \frac{1}{L_N} \left( \alpha x_{2N-3} x_{2N} + (1 - \alpha) x_{2N-1} v_{out} - q_{out} - p_N + r_N \right) \\
\dot{x}_{2N} &= \frac{1}{\tau} \left( V(x_{2N-1}) - x_{2N} \right) + \frac{\eta}{L_N} x_{2N} (x_{2N-2} - v_{out}) - \frac{\delta}{L_N} (r_N + p_N) x_{2N},
\end{align*}
\]

(3.67) (3.68) (3.69) (3.70)

\[
h(x) = [h_1(x) \ h_2(x)]^T = [x_{2N-1} \ x_{2N}]^T.
\]

(3.71)

Following the design procedure, first the vectors \( H_i(x) \) will be computed

\[
H_i(x) = \begin{bmatrix}
h_{1i}(x) \\
h_{2i}(x)
\end{bmatrix} = \begin{bmatrix}
x_{2N-i} \\
x_{2N}
\end{bmatrix}, \quad i = 1, \ldots, k - 1
\]

\[
H(x) = [H_1(x) \ldots H_N(x)]^T.
\]

Here \( f \) represents the right-hand side of (3.67) - (3.70). To obtain the observer dynamics (3.61), the auxiliary terms \( V_i(t) \) and the matrix gain \( M \) should be specified. This is effected by initializing

\[
\begin{align*}
V_1(t) &= [v_{11}(t) \ v_{21}(t)]^T \\
&= [h_{11}(x) \ h_{21}(x)]^T
\end{align*}
\]

and writing the first set of (modified) error dynamics (3.66)

\[
\begin{align*}
e_{11}(t) &= h_{11}(x) - h_{11}(\bar{x}), \\
\dot{e}_{11}(t) &= \frac{dh_{11}(x)}{dt} - m_{11} \text{sign} \left( v_{11}(t) - h_{11}(\bar{x}) \right).
\end{align*}
\]
If $m_{11} > \left| \frac{dh_{11}(x)}{dt} \right|$ or $m_{11} > |\dot{x}_{2N-1}|$, then $e_{11} \dot{e}_{11} < 0$. The conservation equation for section $N$ is

$$\dot{x}_{2N-1} = \frac{1}{L_N} (q_{N-1} - q_{\text{out}} + r_N - p_N).$$

Assuming the ramp and mainstream flows satisfy

$$r_N, p_N, q_{N-1}, q_{\text{out}} \leq q_{\text{max}},$$

where $q_{\text{max}}$ is the maximum highway capacity, one derives

$$|\dot{x}_{2N-1}| \leq \frac{2}{L_N} q_{\text{max}}.$$

Therefore, $m_{11}$ is chosen to satisfy

$$m_{11} > \frac{2}{L_N} q_{\text{max}}.$$

While $e_{11} = 0$, one should set $h_{12}(x)$ to

$$h_{12}(x) = \frac{dh_{11}(x)}{dt} = \left( m_{11} \, \text{sign}(v_{11} - h_{11}(\tilde{x})) \right).$$

Similarly,

$$e_{21}(t) = h_{21}(x) - h_{21}(\tilde{x}),$$

$$\dot{e}_{21}(t) = \frac{dh_{21}(x)}{dt} - m_{21} \, \text{sign} \left( v_{21}(t) - h_{21}(\tilde{x}) \right).$$

When $m_{21} > \left| \frac{dh_{21}(x)}{dt} \right|$ or $m_{21} > |\dot{x}_{2N}|$, sliding occurs for $e_{21}$. Therefore, $m_{21}$ is set to the maximum microscopic deceleration $b_{\text{m}}$ which, in general, is greater than its macroscopic counterpart $|\dot{x}_{2N}|$.

While $\dot{e}_{21} = 0$, $h_{22}(x)$ is set to

$$h_{22}(x) = \frac{dh_{21}(x)}{dt} = \left( m_{21} \, \text{sign}(v_{21} - h_{21}(\tilde{x})) \right).$$
Now define
\[ e_{12}(t) = h_{12}(x) - h_{12}(\tilde{x}), \]
\[ v_{12}(t) = m_{i_1}(v_{i_1}(t) - h_{11}(x)) 
  = h_{12}(x), \]
to obtain
\[ \dot{e}_{12}(t) = \frac{dh_{12}(x)}{dt} - m_{i_2}\text{sign}\left( v_{12}(t) - h_{12}(\tilde{x}) \right). \]
For \( m_{i_2} > |\frac{dh_{12}(x)}{dt}| \), the error \( e_{21} \) will also converge to zero. But
\[ \frac{dh_{12}(x)}{dt} = \frac{\alpha}{L_N} x_{2N} \dot{x}_{2N-3} + \frac{\alpha}{L_N} \dot{x}_{2N} x_{2N-3} + \frac{1 - \alpha}{L_N} \dot{\psi}_{out} \dot{x}_{2N-1}. \]
Consequently,
\[ m_{i_2} > \frac{\alpha}{L_N}(\gamma_v V_f)(\frac{2}{L_N - 1} q_{max}) + \frac{\alpha}{L_N}(\gamma_d \rho_d) \beta \] 
\[ + \frac{1 - \alpha}{L_N}(\gamma_v V_f)(\frac{2}{L_N} q_{max}). \]
In a similar fashion, one can derive the modified error dynamics \( e_{ji} \), auxiliary signals \( v_{ji} \), and the matrix elements \( m_{ji} \). Nevertheless, a generalized form for the gain elements \( m_{ji} \) cannot be presented due to the high complexity of the system. Table 3.5 summarizes the lower bounds for the first four elements of the gain matrix \( M \).

Table 3.5: Elements of the gain matrix \( M \)

<table>
<thead>
<tr>
<th>( m_{i_1} )</th>
<th>( \frac{2}{L_N} q_{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{21} )</td>
<td>( b_m )</td>
</tr>
<tr>
<td>( m_{i_2} )</td>
<td>( \frac{2}{L_N} \gamma_v V_f q_{max} (\frac{\alpha}{L_N - 1} + \frac{1 - \alpha}{L_N}) + \frac{\alpha}{L_N} \gamma_d \rho_d \beta )</td>
</tr>
<tr>
<td>( m_{22} )</td>
<td>( \frac{1}{\tau}(\frac{V_f}{\rho_f L_N} q_{max} + b_m) + \frac{2b_m}{L_N}(\eta \gamma V_f + \delta q_{max}) )</td>
</tr>
</tbody>
</table>

### 3.5 Design of a Robust Sliding Mode Observer

A primary advantage of sliding mode techniques in state estimation and control is their robustness with respect to disturbances and modelling errors. Other observer structures will provide biased estimates when there is model uncertainty. This can
be seen, for instance, from the estimates produced by the Extended Kalman Filter in Figures 3.6 and 3.7. Sliding mode observers can compensate model uncertainties as it will be discussed in this section. Based on the design methodology of Section (3.2), a robust sliding mode observer for highway traffic systems is designed. The stability proof and the robustness analysis of the observer are also presented. Finally, the performance of the proposed robust observer will be verified through both simulation and experimental results.

Figure 3.15: A freeway system divided into $N = 2$ sections

Let us consider a stretch of a highway divided into $N = 2$ sections with an off-ramp in the first section as in Figure 3.15. The presence of any other on- or off-ramp in the configuration does not affect the generality of the problem. According to (2.24) and (2.25b), the system equations are

\[ \dot{x}_1 = \frac{1}{L_1} \left( q_{in} - p_1 - \alpha x_1 x_2 - (1 - \alpha) x_3 x_4 \right) \]  
(3.72)

\[ \dot{x}_2 = \frac{1}{r} \left( V(x_1) - x_2 \right) + \frac{\eta}{L_1} x_2 (v_{in} - x_2) - \frac{v}{L_1} (x_3 - x_3) - \frac{\delta}{L_1} p_1 x_2 + \phi_e \]  
(3.73)

\[ \dot{x}_3 = \frac{1}{L_2} \left( \alpha x_1 x_2 + (1 - \alpha) x_3 x_4 - q_{out} \right) \]  
(3.74)

\[ \dot{x}_4 = \frac{1}{r} \left( V(x_3) - x_4 \right) + \frac{\eta}{L_2} x_4 (x_2 - x_4), \]  
(3.75)

where $\phi_e$, the modelling error in the momentum equation, is considered bounded:
\[ |\phi_s| < \Phi_m \] for constant \( \Phi_m \). The conservation equations are assumed to be exact with no modelling errors. Moreover, the following assumptions are made in the derivation of the robust observer.

**Assumptions S1.**

- \( q_{out} \) and \( v_{out} \) are measured.
- The velocities of the sections and \( v_{out} \) are strictly positive
  \[ v_1, v_2, v_{out} \geq V_{min} > 0. \]
- Section 2 is short enough so that the measured outputs \( q_{out} \) and \( v_{out} \) provide
  \[ y_1 = \frac{q_{out}}{v_{out}} = x_3, \]
  \[ y_2 = v_{out} = x_4. \]
- The velocity and flow of the incoming traffic and ramp flow are not available and considered as disturbances
  \[ 0 \leq q_{in}, p_1 \leq q_{max}, \]
  \[ 0 \leq v_{in} \leq V_{max}. \]
- The model parameters are set to their nominal values and the nominal speed-density characteristic \( V(.) \) is assumed to satisfy the Lipschitz condition:

\[
|V(\rho) - V(\dot{\rho})| \leq \kappa |\rho - \dot{\rho}|. \quad (3.76)
\]

Since the speed-density characteristic (2.13) is continuous and differentiable at every point \( 0 \leq \rho \leq \rho_f \) except \( \rho = \rho_f \), the Lipschitz constant \( \kappa \) can be computed from the maximum unidirectional slope of \( V(.) \) shown in Figure 3.16:

\[
\kappa = \left| \frac{\partial V(\rho)}{\partial \rho} \right|_{\rho_f} \\
= \left| - \frac{V_f}{\ln(\frac{\rho_f}{\rho_f})} \right| \\
= 4 \text{ km}^2/(\text{veh.h}). \]
Define $\phi_d = \frac{1}{L_1} (q_{in} - p_1), \phi_0 = \eta v_{in} - \delta p_1$ and rewrite (3.72)-(3.75) as

\begin{align*}
\dot{x}_1 &= -\frac{\alpha}{L_1} x_1 x_2 - \frac{1 - \alpha}{L_1} q_{out} + \phi_d \\
\dot{x}_2 &= -\frac{1}{\tau} x_2 - \frac{\eta}{L_1} x_2^2 + \frac{1}{\tau} V(x_1) + \frac{\nu}{L_1} x_1 - \frac{\nu}{L_1} x_3 + \frac{1}{L_1} \phi_0 x_2 + \phi_v \\
\dot{x}_3 &= \frac{\alpha}{L_2} x_1 x_2 - \frac{\alpha}{L_2} q_{out}, \\
\dot{x}_4 &= -(\frac{1}{\tau} - \frac{\eta}{L_2} x_2) x_4 - \frac{\eta}{L_2} x_4^2 + \frac{1}{\tau} V(x_3). \tag{3.80}
\end{align*}

The observer structure will then follow

\begin{align*}
\dot{\hat{x}}_1 &= -\frac{\alpha}{L_1} \hat{x}_1 \hat{x}_2 - \frac{1 - \alpha}{L_1} q_{out} + \Lambda_1 I_s, \tag{3.81} \\
\dot{\hat{x}}_2 &= -\frac{1}{\tau} \hat{x}_2 - \frac{\eta}{L_1} \hat{x}_2^2 + \frac{1}{\tau} V(\hat{x}_1) + \frac{\nu}{L_1} \hat{x}_1 - \frac{\nu}{L_1} y_1 + \frac{\eta}{L_1} V_{max} \hat{x}_2 + \Lambda_2 I_s, \tag{3.82} \\
\dot{\hat{x}}_3 &= \frac{\alpha}{L_2} \hat{x}_1 \hat{x}_2 - \frac{\alpha}{L_2} q_{out} + \Lambda_3 I_s, \tag{3.83} \\
\dot{\hat{x}}_4 &= -(\frac{1}{\tau} - \frac{\eta}{L_2} \hat{x}_2) y_2 - \frac{\eta}{L_2} y_2^2 + \frac{1}{\tau} V(y_1) + \Lambda_4 I_s, \tag{3.84}
\end{align*}

where $\phi_0$ is replaced by its maximum value in (3.82). The sliding manifold and
switching terms are defined as

\[
\begin{align*}
    s &= [s_1 \ s_2]^T = [e_3 \ e_4]^T, \\
    I_s &= [\text{sign}(s_1) \ \text{sign}(s_2)]^T.
\end{align*}
\]

The first step is to choose \( A_3 \) and \( A_4 \). Define

\[
\begin{bmatrix}
    A_3 \\
    A_4
\end{bmatrix} =
\begin{bmatrix}
    k_3 & 0 \\
    0 & k_4
\end{bmatrix}, \tag{3.85}
\]

to have the following error dynamics along the sliding surface:

\[
\begin{align*}
    \dot{s}_1 &= \frac{a}{L_2} (x_1 x_2 - \dot{x}_1 \dot{x}_2) - k_3 \text{sign}(s_1), \\
    \dot{s}_2 &= \frac{n}{L_2} y_2 (x_2 - \dot{x}_2) - k_4 \text{sign}(s_2).
\end{align*}
\]

For

\[
\begin{align*}
    k_3 &> \frac{a}{L_2}(|\dot{x}_1 \dot{x}_2| + q_{\text{max}}), \tag{3.86} \\
    k_4 &> \frac{n}{L_2}(|\dot{x}_2| + V_{\text{max}}) V_{\text{max}}. \tag{3.87}
\end{align*}
\]

the sliding surface \( s \) is attractive. The equivalent switching term is found to be

\[
I_s =
\begin{bmatrix}
    \frac{a}{L_2} k_3 (x_1 x_2 - \dot{x}_1 \dot{x}_2) \\
    \frac{n}{L_2} k_4 (x_2 - \dot{x}_2) y_2
\end{bmatrix}. \tag{3.88}
\]

The next step is to compute the observer gains \( A_1, A_2 \) to render the origin of the reduced-order error dynamics stable and make the observer robust with respect to \( \phi_d \) and \( \phi_v \) in (3.77) and (3.78). The dependency on the system states is ignored by the assumed model of uncertainty. Although this is true for \( \phi_d = \frac{1}{L_1} (q_m - p) \) and \( \phi_v = \eta u_m - \delta p \), the modelling uncertainty in the momentum equation \( \phi_\delta \) can be a function of the states and may cause larger errors in density and velocity estimates than the expected values.

**Proposition 3.** The origin of the reduced-order error dynamics

\[
\begin{align*}
    \dot{e}_1 &= \dot{x}_1 - \dot{x}_1, \\
    \dot{e}_2 &= \dot{x}_2 - \dot{x}_2,
\end{align*}
\]
is stable if

\[
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2
\end{bmatrix} = \begin{bmatrix}
k_{11} & k_{12} \\
0 & k_{22}
\end{bmatrix},
\]  
(3.89)

with the following conditions

\[
\begin{align*}
k_{11} & > 0 \\
k_{12} & = -\alpha k_4 \left( \frac{L_2}{L_1} + \eta \frac{k_{11}}{k_3} \right) \hat{x}_i, \\
k_{22} & > k_4 \frac{L_2}{\eta y_2} \left( \frac{\eta}{L_1} V_{max} - \frac{\eta}{L_1} \frac{1}{\tau} \right).
\end{align*}
\]

\[\square\]

**Proof.** Substituting (3.88) and (3.89) in the reduced-order error dynamics and rearranging the terms yield

\[
\begin{align*}
\dot{e}_1 &= -\frac{\alpha}{L_1} \left( 1 + \frac{k_{11} L_1}{k_3 L_2} \right) (x_1 x_2 - \hat{x}_1 \hat{x}_2) - \frac{k_{12}}{k_4} \frac{\eta}{L_2} (x_2 - \hat{x}_2) + \phi_d, \\
\dot{e}_2 &= -\frac{1}{\tau} \dot{e}_2 - \frac{\eta}{L_1} (x_2 + \hat{x}_2 + \frac{k_{22}}{k_4} \frac{L_1}{L_2} y_2) e_2 + \frac{\nu}{L_1} e_1 + \frac{\eta}{L_1} V_{max} e_2 + \frac{1}{L_1} (\phi_0 - \eta V_{max}) x_2 + \frac{1}{\tau} \left( V(x_1) - V(\hat{x}_1) \right) + \phi_v.
\end{align*}
\]

Now set

\[
\frac{k_{12}}{k_4} \frac{\eta}{L_2} y_2 = -\frac{\alpha}{L_1} \left( 1 + \frac{k_{11} L_1}{k_3 L_2} \right) \hat{x}_1,
\]

or

\[
k_{12} = -\alpha k_4 \left( \frac{L_2}{L_1} + \frac{k_{11}}{k_3} \right) \hat{x}_1,
\]

(3.92)

and define a new parameter

\[
k = \frac{\eta}{L_1} \hat{x}_2 + \frac{k_{22}}{k_4} \frac{\eta}{L_2} y_2,
\]

(3.93)

to have

\[
\begin{align*}
\dot{e}_1 &= -\frac{\alpha}{L_1} \left( 1 + \frac{k_{11} L_1}{k_3 L_2} \right) x_2 e_1 + \phi_d, \\
\dot{e}_2 &= -\left( \frac{1}{\tau} + k + \frac{\eta}{L_1} x_2 - \frac{\eta}{L_1} V_{max} \right) e_2 + \frac{\nu}{L_1} e_1 + \\
&\frac{1}{L_1} (\phi_0 - \eta V_{max}) x_2 + \frac{1}{\tau} \left( V(x_1) - V(\hat{x}_1) \right) + \phi_v.
\end{align*}
\]

(3.94)

(3.95)
Thus, for

\( k_{11} > 0, \) \hspace{1cm} (3.96)

\( k > \frac{\eta}{L_i} V_{\text{max}} - \frac{1}{\tau}, \) \hspace{1cm} (3.97)

the origin of (3.94) and (3.95) is stable.

Finally, using (3.93) and (3.97), one can obtain the constraint on the gain \( k_{22} \)

\[ k_{22} > \frac{L_2}{L_1} \left( L_1 \frac{1}{\eta} \right) \left( V_{\text{max}} - \dot{x}_2 - \frac{L_1}{\eta} \right). \] \hspace{1cm} (3.98)

\[ \Box \]

### 3.5.1 Robustness Analysis

When properly chosen, the observer gains \( k_{11}, k_{12}, \) and \( k_{22} \) not only stabilize the error dynamics, but guarantee the robustness of the observer dynamics with respect to the modelling errors \( \phi_d, \phi_o, \) and \( \phi_v. \) To this end, let us consider the following design criteria (not assumptions) for the estimation errors:

\[ |e_1| < \epsilon_d, \] \hspace{1cm} (3.99)

\[ |e_2| < \epsilon_v. \] \hspace{1cm} (3.100)

In other words, the errors in density and velocity are required not to exceed \( \epsilon_d \) and \( \epsilon_v, \) respectively.

Note that equations (3.94) and (3.95) are in the form of first-order differential equations. We take advantage of the following lemma in the robustness analysis.

**Lemma 3.1.** Consider the first-order differential equation

\[ \dot{z}(t) = b(t) z(t) + c(t), \quad z(t) \in \mathbb{R} \]

where \( z(0) = z_0 \) and the coefficients \( b(t) \) and \( c(t) \) satisfy

\[ b(t) \leq B < 0, \]
Then for a bounded initial condition $z_0$, the solution converges to

$$\limsup_{t \to \infty} |z(t)| \leq \frac{C}{|B|}.$$  

**Proof.** Let us consider the Lyapunov function candidate

$$V_L(t) = \frac{1}{2} z(t)^2$$

with the time derivative

$$\dot{V}_L(t) = z(t) \dot{z}(t) = b(t) z(t)^3 + c(t) z(t)$$

$$\leq -|B| z(t)^2 + C |z(t)|$$

$$\leq -\frac{1}{2} |B| z(t)^2 - (\frac{1}{2} |B| z(t)^2 - C |z(t)| + \frac{C^2}{2|B|}) + \frac{C^2}{2|B|}$$

$$\leq -\frac{1}{2} |B| z(t)^2 - \left( \sqrt{\frac{1}{2} |B| |z(t)|} - \frac{C}{\sqrt{2|B|}} \right)^2 + \frac{C^2}{2|B|},$$

$$\leq -\frac{1}{2} |B| z(t)^2 + \frac{C^2}{2|B|}.$$  

Therefore,

$$\dot{V}_L(t) \leq -|B| V_L(t) + \frac{C^2}{2|B|},$$

which implies

$$V_L(t) \leq e^{-|B|t} V_L(0) + \frac{C^2}{2B^2}.$$  

After sufficiently long period of time, one obtains

$$\limsup_{t \to \infty} V_L(t) \leq \frac{C^2}{2B^2},$$

or

$$\limsup_{t \to \infty} |z(t)| \leq \frac{C}{|B|}.$$  

Since $x_2 \geq V_{\min}$ and $|\phi_d| \leq \frac{1}{L_1} q_{\max}$, the right-hand side of (3.94) yield

$$-\frac{\alpha}{L_1} (1 + \frac{k_{ii} L_1}{k_3 L_2}) x_2 \leq -\frac{\alpha}{L_1} (1 + \frac{k_{ii} L_1}{k_3 L_2}) V_{\min} < 0. \quad (3.101)$$
Using lemma 1, we have the following bound on the estimation error

$$|e_1| \leq \frac{\frac{1}{L_1} q_{\text{max}}}{\frac{1}{L_1} \left(1 + \frac{k_{11}}{k_3 L_2}\right) V_{\text{min}}}.$$ 

If $k_{11}$ satisfies

$$k_{11} > \frac{k_3 L_2}{L_1} \left(\frac{q_{\text{max}}}{\alpha \varepsilon_d V_{\text{min}}} - 1\right),$$

(3.102)

criterion (3.99) is fulfilled.

Similarly, the right-hand side of equation (3.95) provides

$$-\left(\frac{1}{\tau} + k + \frac{\eta}{L_1} x_2 - \frac{\eta}{L_1} V_{\text{max}}\right) \leq -\left(\frac{1}{\tau} + k - \frac{\eta}{L_1} V_{\text{max}}\right) \leq 0,$$

$$\frac{\nu}{L_1} e_1 + \frac{1}{L_1} (\phi_d - \eta V_{\text{max}}) x_2 + \frac{1}{\tau} \left(V(x_i) - V(\hat{x}_i)\right) + \phi_e \leq \frac{\nu}{L_1} \varepsilon_d + \frac{\eta}{L_1} V^2 + \frac{\xi}{\tau} \varepsilon_d + \Phi_m.$$

Consequently,

$$|e_2| \leq \frac{\frac{\nu}{L_1} \varepsilon_d + \frac{\eta}{L_1} V^2 + \frac{\xi}{\tau} \varepsilon_d + \Phi_m}{\frac{1}{\tau} + k - \frac{\eta}{L_1} V_{\text{max}}}.$$

Finally, for (3.100) to be satisfied, one may choose $k$ such that

$$\frac{\left(\frac{\nu}{L_1} + \frac{\xi}{\tau}\right) \varepsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m}{\frac{1}{\tau} + k - \frac{\eta}{L_1} V_{\text{max}}} \leq \varepsilon_v,$$

or

$$k > \frac{\eta}{L_1} V_{\text{max}} - \frac{1}{\varepsilon_v} + \frac{1}{\nu} \left(\left(\frac{\nu}{L_1} + \frac{\xi}{\tau}\right) \varepsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m\right).$$

(3.103)

Using (3.93) and (3.103), the constraint on $k_{22}$ can be obtained

$$k_{22} > k_4 \frac{L_2}{L_1} \frac{1}{y_2} \left[\frac{V_{\text{max}} - \hat{x}_2}{\eta \tau} + \frac{1}{\eta} \left(\left(\frac{\nu}{L_1} + \frac{\xi}{\tau}\right) \varepsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m\right)\right].$$

(3.104)

Constrains (3.102) and (3.104) on $k_{11}$ and $k_{22}$ guarantee the uniform ultimate boundedness (UUB) of the estimation errors $e_1$ and $e_2$ in the presence of the modeling errors $\phi_d$, $\phi_o$, and $\phi_e$. ```
The results of the stability and robustness analysis for the proposed sliding mode observer are summarized here. The switching gains $\Lambda_1, \ldots, \Lambda_4$ in terms of the design parameters $k_{11}, k_{12}, k_{22}, k_3,$ and $k_4$ as in (3.85) and (3.89) are computed according to the following steps.

- First, the gains $k_3$ and $k_4$ are determined according to

\[
\begin{align*}
k_3 &> \frac{\alpha}{L_2} (|\hat{x}_1 \hat{x}_2| + q_{max}), \\
k_4 &> \frac{\eta}{L_2} (|\hat{x}_2| + V_{max}) V_{max},
\end{align*}
\]

so that the sliding surface is attractive.

- Then, the origin of the reduced-order error dynamics is made stable by setting the gains $k_{11}, k_{12}$ and $k_{22}$ according to

\[
\begin{align*}
k_{11} &> 0 \\
k_{12} &= -\alpha k_4 \left( \frac{L_2}{L_1} + \eta k_{11} \right) \frac{\hat{x}_1}{y_2} \\
k_{22} &= k_4 \left( \frac{L_2}{L_1} \frac{1}{y_2} (V_{max} - \hat{x}_2 - \frac{L_1}{\eta} \frac{1}{}\tau) \right).
\end{align*}
\]

- In addition to stability, the gains $k_{11}$ and $k_{22}$ in the previous step can be further tuned as

\[
\begin{align*}
k_{11} &> k_3 \frac{L_2}{L_1} \left( \frac{q_{max}}{\alpha \epsilon_d V_{min}} - 1 \right), \\
k_{22} &> k_4 \frac{L_2}{L_1} \frac{1}{y_2} \left[ V_{max} - \hat{x}_2 - \frac{L_1}{\eta} \frac{1}{\tau} + \frac{L_1}{\eta} \frac{1}{\epsilon_d} \left( \frac{\nu}{L_1} + \frac{k_4}{\tau} \right) \epsilon_d + \frac{\eta}{L_1} \frac{V_{max}^2}{\epsilon_d} + \Phi_m \right],
\end{align*}
\]

to fulfill the design criteria $|e_1| < \epsilon_d$ and $|e_2| < \epsilon_d$, and to make the observer robust with respect to the modelling uncertainties.

3.5.2 Simulation results

The robust sliding mode observer is now implemented to estimate the states of a highway traffic system. The robustness of the observer with respect to the disturbances as well as the modelling structure is investigated through a series of simulation
Table 3.6: Design parameters used by the robust sliding mode observer

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa )</td>
<td>Lipschitz constant</td>
<td>4 km/(veh.h)</td>
</tr>
<tr>
<td>( q_{\text{max}} )</td>
<td>maximum mainstream flow</td>
<td>2981 veh/h</td>
</tr>
<tr>
<td>( V_{\text{max}} )</td>
<td>maximum velocity</td>
<td>133 km/h</td>
</tr>
<tr>
<td>( V_{\text{min}} )</td>
<td>minimum velocity</td>
<td>5 km/h</td>
</tr>
<tr>
<td>( \Phi_m )</td>
<td>modelling error in the momentum equation</td>
<td>1300 km/h(^2)</td>
</tr>
<tr>
<td>( \epsilon_d )</td>
<td>maximum allowable error in density</td>
<td>10 veh/km</td>
</tr>
<tr>
<td>( \epsilon_v )</td>
<td>maximum allowable error in velocity</td>
<td>18 km/h</td>
</tr>
</tbody>
</table>

scenarios. In doing so, the dynamics of the traffic flow is generated by the model of Karaaslan et al. (2.26a)-(2.26c) with the parameter values listed in Table 2.1 while the observer dynamics as derived in Section 3.5 is based on our model (2.24) and (2.25b). It should be mentioned that the traffic model (2.26a)-(2.26c) satisfy all the assumptions required in the design procedure (Assumptions S1). The system has no on- or off-ramps and the section lengths are \( L_1 = L_2 = 500 \) (m). Moreover, for the system to represent realistic traffic conditions, the incoming traffic quantities, i.e., \( u_n \) and \( q_n \), are taken from Highway 401 detector measurements. For this purpose, traffic data at the detector station 401DW080DES recorded during the morning rush-hour (scenario 1) and the afternoon rush-hour (scenario 2) of March 13, 1997 is used. It should be once more emphasized that this data is not used by the observer. The outputs of the system, namely \( q_{\text{out}} \) and \( v_{\text{out}} \), are the only information used by the observer; there is no other a priori assumption about the system available to the observer. The observer dynamics (3.81)-(3.84) is implemented using the identified parameter values of Table 3.1 and design criteria listed in Table 3.6. Also, the initial conditions of the observer dynamics are set to be different from those of the actual states.

The system velocities and the observer estimates are shown in Figures 3.17 and 3.19. It is clear that the velocity of section 2 can be perfectly regenerated by the observer as long as the sliding condition of (3.87) is satisfied for the gain \( k_4 \). Moreover, the velocity error for section 1 is bounded by \( \epsilon_v = 18 \) (km/h) upon choosing \( k_{22} \), according
to (3.104). In fact,
\[ \max\{|e_2|\} = 11 \text{ (km/h)}, \]
which is much smaller than \( \epsilon_v \) for both scenarios.

Figures 3.18 and 3.20 show the density profiles of the system and the observer. Again, the density of section 2 is perfectly estimated while the density error for section 1 does not exceed \( \epsilon_v = 10 \) (veh/km) upon satisfying (3.102):
\[ \max\{|e_1|\} = 8 \text{ (veh/km)}. \]

### 3.5.3 Experimental Results

The performance of the robust sliding mode observer is also tested in generating the states of a real traffic system and is compared to the results obtained from the Extended Kalman Filter. The study site consists of two sections between the detector stations 401DW0080DWS and 401DW0100DWS along the westbound portion of Highway 401 in Figure 2.8.

The design parameters of Table 3.6 and the identified model parameters of Table 2.4 are utilized for the implementation of the robust observer.

The error indices described in Section 3.3 for each estimator is summarized in Table 3.7. Quantitatively speaking, the robust observer has a better performance in estimating the velocity and volume; the velocity estimate has a moderate improvement of 54% over the EKF. Qualitative improvements are also apparent in Figures 3.21 and 3.22 which depict the velocity, volume, and density profiles of the proposed observer, EKF, and measurements. As can be seen, the EKF fails to provide acceptable estimation when the incoming traffic information is not available. On the contrary, the sliding mode observer is fairly robust in this respect as it produces the fluctuations in the velocity and volume with reasonable accuracy.
Section 3.5. Design of a Robust Sliding Mode Observer

Figure 3.17: Robust sliding mode observer: simulation results for velocity, scenario 1

Figure 3.18: Robust sliding mode observer: simulation results for density, scenario 1
Section 3.5. Design of a Robust Sliding Mode Observer

Figure 3.19: Robust sliding mode observer: simulation results for velocity, scenario 2

Figure 3.20: Robust sliding mode observer: simulation results for density, scenario 2
Figure 3.21: Robust sliding mode observer: experimental results for Sept. 22, 1997
Figure 3.22: Robust sliding mode observer: experimental results for Sept. 24, 1997
3.6 Robust State Estimation of Multi-lane Traffic Systems

The design methodology of the robust sliding mode observer for single-lane systems can be generalized to include multi-lane traffic systems as well. The macroscopic modelling of traffic flow on a multi-lane highway should describe the lane-changing process lacking in the single-lane models. In this section, the equations describing such systems are presented based on the traffic model (2.24) and (2.25) derived in Section 2.3.2. The robust observer design will then be followed by the experimental results.

3.6.1 Modelling of Multi-lane Traffic Systems

The early deterministic multi-lane models developed by Munjal and Pipes [35] were first-order models based on the conservation equations without considering the generation of traffic due to sinks and sources directly. Due to several oversimplifying assumptions, the applicability of these models to real traffic situations is limited.

For the purposes of this work, a modified version of the macroscopic multi-lane models developed by Michalopoulos et. al. [63] is used. The evolution of the traffic flow in each lane is described by a second-order model based on the conservation and momentum equations in order to take into account the acceleration and inertia effects of the fusion process.

Let us consider a segment of a multi-lane highway as shown in Figure 3.23. The velocity and density variables for each lane are defined as in Section 2.3. Here, indices

Table 3.7: Performance indices of estimators for Sept. 22 and 24, 1997, respectively

<table>
<thead>
<tr>
<th>$L_2$-norm error</th>
<th>Robust Sliding Mode Observer</th>
<th>Extended Kalman Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td></td>
<td>e_{g1}</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>e_{g2}</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>e_{g3}</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>e_{g4}</td>
</tr>
</tbody>
</table>
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

Figure 3.23: A segment of a multi-lane roadway

$m$, $n$, and $k$ are associated with the lane numbers. According to the multi-lane model developed in [63], the conservation equation (2.16) for lane $n$ should be modified to include the exchange of vehicles between adjacent lanes $m$ and $k$

$$\frac{\partial q_n(x,t)}{\partial x} + \frac{\partial \rho_n(x,t)}{\partial t} = g(t) + Q_{kn}(x,t) + Q_{mn}(x,t).$$  \hspace{1cm} (3.107)

The lane generation terms $Q_{mn}$ and $Q_{kn}$ are defined as

$$Q_{mn}(x,t) = \alpha_0[(\rho_m(x,t) - \rho_n(x,t)) - (\rho_{me} - \rho_{ne})]$$  \hspace{1cm} (3.108)

$$Q_{kn}(x,t) = \alpha_0[(\rho_k(x,t) - \rho_n(x,t)) - (\rho_{ke} - \rho_{ne})],$$  \hspace{1cm} (3.109)

where $\alpha_0$ is the sensitivity parameter describing the intensity of the flow exchange, and $\rho_{ne}$, $\rho_{me}$, and $\rho_{ke}$ are the equilibrium densities of the lanes. Throughout this section, we assume the equilibrium quantities are the same for all lanes. The system is conserved with respect to the lane generation terms, i.e.,

$$\sum_{all \text{ lanes}} Q_{ij} = 0.$$  

The momentum equation (2.22) for lane $n$ can be also changed to accommodate the momentum exchange between lanes due to the lateral flow:
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

Figure 3.24: A three-lane highway with two sections

\[
\frac{\partial v_n(x, t)}{\partial t} = \frac{1}{\tau} \left( V(\rho_n(x, t)) - v_n(x, t) \right) - v_n(x, t) \frac{\partial v_n(x, t)}{\partial x} - \nu \frac{\partial \rho_n(x, t)}{\partial x} - \delta g(x, t) v_n(x, t) + M_{kn}(x, t) + M_{mn}(x, t),
\]

(3.110)

with the momentum generation terms

\[
M_{kn}(x, t) = \beta_0[(v_k(x, t) - v_n(x, t)) - (v_{ke} - v_{ne})]
\]

\[
M_{mn}(x, t) = \beta_0[(v_m(x, t) - v_n(x, t)) - (v_{me} - v_{ne})],
\]

where \(\beta_0\) is a positive parameter and index \(e\) is associated with the equilibrium condition. The conservation of momentum is also maintained within the section by requiring

\[
\sum_{\text{all lanes}} M_{ij} = 0.
\]

3.6.2 Robust Observer Design

For the observer design, we consider a three-lane system with two sections and one off-ramp as shown in Figure 3.24. Define the state and output vectors \(x = \)
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

\[
\begin{bmatrix}
\rho_{11} & \cdots & \rho_{13} \\
\rho_{21} & \cdots & \rho_{23} \\
\rho_{31} & \cdots & \rho_{33}
\end{bmatrix}^T, \quad
y = \begin{bmatrix}
\rho_{11} & \rho_{12} & \rho_{13} \\
\rho_{21} & \rho_{22} & \rho_{23} \\
\rho_{31} & \rho_{32} & \rho_{33}
\end{bmatrix}^T.
\]

The generalization of the problem for systems with more lanes and ramps is straightforward. Using equation (3.107) and (3.110), one can write the space-discretized state equations as

\[
\begin{align*}
\dot{x}_1 & = -\frac{\alpha}{L_1} x_1 x_2 + \frac{\alpha_0}{L_1} x_3 - \frac{1}{L_1} q_{s1} + \phi_{d11} \\
\dot{x}_2 & = -\left(\frac{1}{\tau} + \beta_0\right) x_2 - \frac{\eta}{L_1} x_2^2 + \frac{1}{\tau} V(x_1) + \frac{\nu}{L_1} x_1 - \frac{\nu}{L_1} x_7 + \beta_0 x_4 + \frac{1}{L_1} \phi_{s11} x_2 + \phi_v \\
\dot{x}_3 & = -\frac{\alpha}{L_1} x_3 x_4 - 2\alpha_0 x_3 + \alpha_0 x_1 + \alpha_0 x_5 - \frac{1}{L_1} q_{s2} + \phi_{d12} \\
\dot{x}_4 & = -\left(\frac{1}{\tau} + 2\beta_0\right) x_4 - \frac{\eta}{L_1} x_4^2 + \frac{1}{\tau} V(x_3) + \frac{\nu}{L_1} x_3 - \frac{\nu}{L_1} x_8 + \beta_0 x_6 + \beta_0 x_2 + \frac{1}{L_1} \phi_{s12} x_4 + \phi_v \\
\dot{x}_5 & = -\frac{\alpha}{L_1} x_5 x_6 + \alpha_0 x_3 - \alpha_0 x_5 - \frac{1}{L_1} q_{s3} + \phi_{d13} \\
\dot{x}_6 & = -\left(\frac{1}{\tau} + \beta_0\right) x_6 - \frac{\eta}{L_1} x_6^2 + \frac{1}{\tau} V(x_5) + \frac{\nu}{L_1} x_5 - \frac{\nu}{L_1} x_11 + \beta_0 x_4 + \frac{1}{L_1} \phi_{s13} x_4 + \phi_v \\
\dot{x}_7 & = \frac{\alpha}{L_2} x_1 x_2 + \alpha_0 x_9 - \alpha_0 x_7 - \frac{\alpha}{L_2} q_{s1} \\
\dot{x}_8 & = -\left(\frac{1}{\tau} + \beta_0 - \frac{\eta}{L_2} x_3\right) x_8 - \frac{\eta}{L_2} x_8^2 + \frac{1}{\tau} V(x_7) + \beta_0 x_{10} \\
\dot{x}_9 & = \frac{\alpha}{L_2} x_3 x_4 + \alpha_0 x_7 + \alpha_0 x_{11} - 2\alpha_0 x_9 - \frac{\alpha}{L_2} q_{s2} \\
\dot{x}_{10} & = -\left(\frac{1}{\tau} + \beta_0 - \frac{\eta}{L_2} x_4\right) x_{10} - \frac{\eta}{L_2} x_{10}^2 + \frac{1}{\tau} V(x_9) + \beta_0 x_8 + \beta_0 x_{12} \\
\dot{x}_{11} & = \frac{\alpha}{L_2} x_5 x_6 + \alpha_0 x_9 - \alpha_0 x_{11} - \frac{\alpha}{L_2} q_{s3} \\
\dot{x}_{12} & = -\left(\frac{1}{\tau} + \beta_0 - \frac{\eta}{L_2} x_6\right) x_{12} - \frac{\eta}{L_2} x_{12}^2 + \frac{1}{\tau} V(x_{11}) + \beta_0 x_{10},
\end{align*}
\]

which are the counterparts of a single-lane equations (3.77)-(3.80). Here, \(\phi_v\) is the modelling error in the momentum equations and

\[
\begin{align*}
\phi_{d11} &= \frac{q_{s1}}{L_1}, \quad \phi_{e11} = \eta v_{s1}, \quad \phi_{d12} = \frac{q_{s2}}{L_1}, \\
\phi_{e12} &= \eta v_{s2}, \quad \phi_{d13} = \frac{1}{L_1}(q_{s3} - p_1), \quad \phi_{e13} = \eta v_{s3} - \delta p_1.
\end{align*}
\]

The same assumptions discussed in Section 3.5 for single-lane systems are considered here for the development of the observer. The dynamics of the observer will then
follow with introducing the switching terms. Adopting the same notations for the estimated variables and errors, we have

\[
\begin{align*}
\dot{x}_1 &= -\frac{\alpha}{L_1} \dot{x}_1 \dot{x}_2 + \alpha_0 \dot{x}_3 - \alpha_0 \dot{x}_1 - \frac{1 - \alpha}{L_1} q_{a1} + \Lambda_1 I_s \\
\dot{x}_2 &= \left(\frac{1}{\tau} + \beta_0\right) \dot{x}_2 - \frac{\eta}{L_1} \dot{x}_2 + \frac{1}{\tau} V(\dot{x}_1) + \frac{\nu}{L_1} \dot{x}_1 - \frac{\nu}{L_1} y_1 + \beta_0 \dot{x}_4 + \frac{\eta}{L_1} V_{\text{max}} \dot{x}_2 + \Lambda_2 I_s \\
\dot{x}_3 &= -\frac{\alpha}{L_1} \dot{x}_3 \dot{x}_4 - 2\alpha_0 \dot{x}_3 + \alpha_0 \dot{x}_1 + \alpha_0 \dot{x}_5 - \frac{1 - \alpha}{L_1} q_{a2} + \Lambda_3 I_s \\
\dot{x}_4 &= \left(\frac{1}{\tau} + 2\beta_0\right) \dot{x}_4 - \frac{\eta}{L_1} \dot{x}_4 + \frac{1}{\tau} V(\dot{x}_3) + \frac{\nu}{L_1} \dot{x}_3 - \frac{\nu}{L_1} y_3 + \beta_0 \dot{x}_6 + \beta_0 \dot{x}_2 + \frac{\eta}{L_1} V_{\text{max}} \dot{x}_4 + \Lambda_4 I_s \\
\dot{x}_5 &= -\frac{\alpha}{L_1} \dot{x}_5 \dot{x}_6 + \alpha_0 \dot{x}_3 - \alpha_0 \dot{x}_5 - \frac{1 - \alpha}{L_1} q_{a3} + \Lambda_5 I_s \\
\dot{x}_6 &= \left(\frac{1}{\tau} + \beta_0\right) \dot{x}_6 - \frac{\eta}{L_1} \dot{x}_6 + \frac{1}{\tau} V(\dot{x}_5) + \frac{\nu}{L_1} \dot{x}_5 - \frac{\nu}{L_1} y_5 + \beta_0 \dot{x}_4 + \frac{\eta}{L_1} V_{\text{max}} \dot{x}_6 + \Lambda_6 I_s \\
\dot{x}_7 &= \frac{\alpha}{L_2} \dot{x}_7 \dot{x}_2 + \alpha_0 y_3 - \alpha_0 y_1 - \frac{\alpha}{L_2} q_{a1} + \Lambda_7 I_s \\
\dot{x}_8 &= \left(\frac{1}{\tau} + \beta_0\right) \dot{x}_8 - \frac{\eta}{L_2} \dot{x}_2 + \frac{1}{\tau} V(y_1) + \beta_0 y_4 + \Lambda_8 I_s \\
\dot{x}_9 &= \frac{\alpha}{L_2} \dot{x}_3 \dot{x}_4 + \alpha_0 y_1 + \alpha_0 y_3 - 2\alpha_0 y_3 - \frac{\alpha}{L_2} q_{a2} + \Lambda_9 I_s \\
\dot{x}_{10} &= \left(\frac{1}{\tau} + \beta_0\right) \dot{x}_{10} - \frac{\eta}{L_2} y_2 + \frac{1}{\tau} V(y_3) + \beta_0 y_2 + \beta_0 y_5 + \Lambda_{10} I_s \\
\dot{x}_{11} &= \frac{\alpha}{L_2} \dot{x}_8 \dot{x}_2 + \alpha_0 y_3 - \alpha_0 y_5 - \frac{\alpha}{L_2} q_{a3} + \Lambda_{11} I_s, \\
\dot{x}_{12} &= \left(\frac{1}{\tau} + \beta_0\right) \dot{x}_{12} - \frac{\eta}{L_2} y_6 + \frac{1}{\tau} V(y_5) + \beta_0 y_4 + \Lambda_{12} I_s.
\end{align*}
\]

The sliding manifold and switching terms are defined as

\[
\begin{align*}
s &= [s_1 \ s_2 \ \cdots \ s_6]^T = [e_7 \ e_8 \ \cdots \ e_{12}]^T, \\
I_s &= [\text{sign}(s_1) \ \text{sign}(s_2) \ \cdots \ \text{sign}(s_6)]^T.
\end{align*}
\]

The observer gains \( \Lambda_i \in R^{1 \times 6}, \ i = 1, 2, \ldots, 12 \) can be determined according to the design procedure presented in Section 3.5. The origin of the error dynamics along the
sliding surface is GES by choosing the gains \( \Lambda_7, \Lambda_8, \ldots, \Lambda_{12} \) as

\[
\begin{bmatrix}
\Lambda_7 \\
\Lambda_8 \\
\vdots \\
\Lambda_{12}
\end{bmatrix}
= \text{diag}(K_7, K_8, \ldots, K_{12}),
\]

with the following conditions

\[
\begin{align*}
K_7 &> \frac{\alpha}{L_2} (|\dot{z}_1^1 \dot{z}_2^1| + q_{\text{max}}), & K_8 &> \frac{\eta}{L_2} (|\dot{z}_2^1| + V_{\text{max}}) V_{\text{max}}, \\
K_9 &> \frac{\alpha}{L_2} (|\dot{z}_3^1 \dot{z}_4^1| + q_{\text{max}}), & K_{10} &> \frac{\eta}{L_2} (|\dot{z}_4^1| + V_{\text{max}}) V_{\text{max}}, \\
K_{11} &> \frac{\alpha}{L_2} (|\dot{z}_5^1 \dot{z}_6^1| + q_{\text{max}}), & K_{12} &> \frac{\eta}{L_2} (|\dot{z}_6^1| + V_{\text{max}}) V_{\text{max}}.
\end{align*}
\]

The equivalent switching term is found to be

\[
I_s = \begin{bmatrix}
\frac{\alpha}{L_2} K_7 (x_1 x_2 - \dot{x}_1 \dot{x}_2) \\
\frac{\eta}{L_2} K_8 (x_2 - \dot{x}_2)y_2 \\
\frac{\alpha}{L_2} K_9 (x_3 x_4 - \dot{x}_3 \dot{x}_4) \\
\frac{\eta}{L_2} K_{10} (x_4 - \dot{x}_4)y_4 \\
\frac{\alpha}{L_2} K_{11} (x_5 x_6 - \dot{x}_5 \dot{x}_6) \\
\frac{\eta}{L_2} K_{12} (x_6 - \dot{x}_6)y_6
\end{bmatrix}.
\]

Moreover, the origin of the reduced-order error dynamics is stable upon setting

\[
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\Lambda_3 \\
\Lambda_4 \\
\Lambda_5 \\
\Lambda_6
\end{bmatrix}
= \begin{bmatrix}
k_{11} & k_{12} & 0 & 0 & 0 & 0 \\
0 & k_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & k_{33} & k_{34} & 0 & 0 \\
0 & 0 & 0 & k_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & k_{55} & k_{56} \\
0 & 0 & 0 & 0 & 0 & k_{66}
\end{bmatrix},
\]

with the constraints

\[
\begin{align*}
k_{11} &> 0, \\
k_{12} &= -\alpha K_8 \left( \frac{L_2}{L_1} + \eta \frac{k_{11}}{K_7} \right) \frac{\dot{x}_1}{y_2}, \\
k_{22} &> K_8 \frac{L_2}{\eta y_2} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \dot{x}_2 - \frac{1}{\tau} - \beta_0 \right)
\end{align*}
\]
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

\[ k_{33} > 0 \]
\[ k_{34} = -\alpha K_{10} \left( \frac{L_2}{L_1} + \eta \frac{k_{33}}{K_9} \right) y_4 \]
\[ k_{44} > K_{10} \frac{L_2}{\eta} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \frac{\dot{x}_4}{L_1} - \frac{1}{\tau} - \beta_0 \right) \]
\[ k_{35} > 0 \]
\[ k_{55} = -\alpha K_{12} \left( \frac{L_2}{L_1} + \eta \frac{k_{55}}{K_{11}} \right) \dot{x}_5 \]
\[ k_{66} > K_{12} \frac{L_2}{\eta} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \frac{\dot{x}_6}{L_1} - \frac{1}{\tau} - \beta_0 \right) \]

In addition to the stability, the robustness of the observer can be also guaranteed when the gains \( k_{11}, k_{22}, \ldots, k_{55}, k_{66} \) are properly determined. As before, consider the design criteria for the density and velocity errors as

\[ |e_i| < \epsilon_d, \quad i = 1, 3, 5 \]
\[ |e_j| < \epsilon_v, \quad j = 2, 4, 6 \]

It can be shown that these specifications are met if

\[ k_{11} > k_7 \frac{L_2}{L_1} \left( \frac{q_{\text{max}}}{\alpha \epsilon_d V_{\text{min}}} - 1 \right) \]
\[ k_{22} > k_8 \frac{L_2}{\eta} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \frac{\dot{x}_2}{L_1} - \frac{1}{\tau} + \frac{1}{\epsilon_v} \left[ \left( \frac{\nu}{L_1} + \frac{\kappa}{\tau} \right) \epsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m \right] \right) \]
\[ k_{33} > k_9 \frac{L_2}{L_1} \left( \frac{q_{\text{max}}}{\alpha \epsilon_d V_{\text{min}}} - 1 \right) \]
\[ k_{44} > k_{10} \frac{L_2}{\eta} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \frac{\dot{x}_4}{L_1} - \frac{1}{\tau} + \frac{1}{\epsilon_v} \left[ \left( \frac{\nu}{L_1} + \frac{\kappa}{\tau} \right) \epsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m \right] \right) \]
\[ k_{55} > k_{11} \frac{L_2}{L_1} \left( \frac{q_{\text{max}}}{\alpha \epsilon_d V_{\text{min}}} - 1 \right) \]
\[ k_{66} > k_{12} \frac{L_2}{\eta} \left( \frac{\eta}{L_1} V_{\text{max}} - \frac{\eta}{L_1} \frac{\dot{x}_6}{L_1} - \frac{1}{\tau} + \frac{1}{\epsilon_v} \left[ \left( \frac{\nu}{L_1} + \frac{\kappa}{\tau} \right) \epsilon_d + \frac{\eta}{L_1} V^2 + \Phi_m \right] \right) \]

To show the performance of the proposed observer, experimental results from a multi-lane section of Highway 401 are presented in the following section.

### 3.6.3 Experimental Results

The westbound express lanes between the detector stations 401DW0080DWS and 401DW0100DWS (Figure 2.8) form a three-lane traffic system suitable for our exper-
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

Implementation. As in Figure 3.24, the system includes two sections of length $L_1 = 574$ (m) and $L_2 = 571$ (m) with one off-ramp in the first section. The traffic velocity and flow are measured for each lane at the exit and comprise the output vector $y$.

The design parameters listed in Table 3.6 along with the typical values for $\alpha_o$ and $\beta_o$ as in [63]

$$\alpha_o = \beta_o = 360 \text{ (1/h)}$$

are employed by the robust observer.

The measurements from the intermediate station are used to assess the accuracy of the observer estimates. The estimated density, velocity, and flow for each lane are depicted in Figures 3.25-3.27 and 3.28-3.30 for the traffic data recorded on Sept. 22 and 24, 1997. The performance indices defined in equation (3.59) are also computed and listed in Table 3.8. When compared with the results obtained in Section 3.5 for the single-lane robust observer, one concludes that the robust multi-lane sliding mode observer can provide similar degrees of accuracy in estimation of traffic velocity, volume, and density of each lane.

Table 3.8: Performance indices for multi-lane robust SMO for Sept. 22, 1997 (top) and Sept. 24, 1997 (bottom)

<table>
<thead>
<tr>
<th>$L_2$-norm error</th>
<th>Robust Sliding Mode Observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|e_{q11}|$</td>
<td>447.3 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q12}|$</td>
<td>444.1 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q13}|$</td>
<td>422.0 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q11}|$</td>
<td>13.9 (km/h)</td>
</tr>
<tr>
<td>$|e_{q12}|$</td>
<td>10.5 (km/h)</td>
</tr>
<tr>
<td>$|e_{q13}|$</td>
<td>10.4 (km/h)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L_2$-norm error</th>
<th>Robust Sliding Mode Observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|e_{q11}|$</td>
<td>433.6 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q12}|$</td>
<td>441.8 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q13}|$</td>
<td>434.4 (veh/h)</td>
</tr>
<tr>
<td>$|e_{q11}|$</td>
<td>16.7 (km/h)</td>
</tr>
<tr>
<td>$|e_{q12}|$</td>
<td>12.4 (km/h)</td>
</tr>
<tr>
<td>$|e_{q13}|$</td>
<td>11.3 (km/h)</td>
</tr>
</tbody>
</table>
Section 3.6. Robust State Estimation of Multi-lane Traffic Systems

Figure 3.25: Individual lane velocities for section 1 (left plots) and section 2 (right plots), Sept. 22, 1997
Figure 3.26: Estimated and measured traffic flow for individual lanes, section 1 (left plots) and section 2 (right plots) for Sept. 22, 1997
Figure 3.27: Estimated density for individual lanes, section 1 (left plots) and section 2 (right plots), Sept. 22, 1997
Figure 3.28: Individual lane velocities for section 1 (left plots) and section 2 (right plots), Sept. 24, 1997.
Figure 3.29: Estimated and measured traffic flow for individual lanes, section 1 (left plots) and section 2 (right plots) for Sept. 24, 1997
Figure 3.30: Estimated density for individual lanes, section 1 (left plots) and section 2 (right plots), Sept. 24, 1997
Chapter 4

Robust Ramp Metering Control of Highway Traffic Systems using Feedback Linearizing Techniques

4.1 Introduction

Traffic congestion on a highway segment within a corridor is mainly due to the mismatch between the designed highway capacity and the amount of traffic admitted to the network from on-ramps. Increasing the capacity of the highway by adding more traffic lanes is a solution which is not always feasible.

An alternative approach entails employing control over the flow of vehicles. The major control strategies employ variable speed limits and message signs, traffic rerouting techniques and metering rates which can be monitored and controlled in real time. Among these, ramp metering strategies have been recognized as one of the most efficient and practical ways to optimize corridor flow, eliminate recurrent and nonrecurrent traffic congestion by regulating mainstream density and velocity, and reduce total travel time including waiting times on ramps [21], [23].

Several attempts have been made toward the development of freeway metering control strategies both in local and area-wide levels [64] – [67]. In a local metering system, the metering rate is based on locally measured traffic conditions in an isolated
Section 4.1. Introduction

section of the network. Two widely used strategies are demand-capacity control [64] and linear quadratic feedback control [64], [65]. In demand-capacity control the traffic volume upstream the merge area is measured and the metering rate is controlled through real-time comparison of the upstream volume to the downstream volume. On the other hand, in linear quadratic feedback control the deviations of the mainstream traffic volume from the nominal values are minimized. The resulting control law is relatively simple and requires only one local measurement; the downstream volume of the merge area. In area-wide metering approaches, the control parameters for a set of controllers are optimized in order to achieve a system-level objective.

Most of the existing ramp metering algorithms mentioned above are linearly based and/or rely on steady state traffic conditions which can be violated in real traffic systems. Such algorithms often fail to drive the system to its desired state when the system is perturbed by severe traffic fluctuations like flow disturbances. Moreover, traffic systems sometimes show unpredictable behaviour that cannot be captured through exact modelling. These factors raise robustness concerns in the design of ramp metering controllers.

This chapter presents a static state feedback linearizing controller for local ramp metering strategies and addresses robustness of the controller with respect to modelling uncertainties. Based on the results of the previous chapter, it is assumed that the states of the traffic system are known. Therefore, the design of the robust controller can be considered in parallel with that of the observer for real-time highway operations over all expected traffic conditions.

Section 4.2 provides background material on feedback linearizing control [68], [69]. In Section 4.3 a robust ramp metering controller will be designed for a freeway system with one on-ramp followed by the robustness analysis. Two other metering strategies based on neural network and linear feedback control theory are presented in Section 4.4. Finally, a number of simulation scenarios compare the performance of the feedback linearizing controller with the neural network and linear feedback controllers in Section 4.5.
4.2 Robust Feedback Linearizing Control

This section is devoted to the design of robust state feedback linearizing controllers for systems with modelling uncertainties. The main goal here is to identify classes of systems for which such controllers exist and to provide a constructive design procedure. The required background material [69] for Section 4.3 is also reviewed. Proof of all the theorems and corollaries are presented in Appendix C.

Let us first consider the feedback linearizing problem for single input nonlinear systems with no uncertainties

\[ \dot{x} = f(x) + g(x) u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R} \]  

in the neighborhood \( U_{x_e} \subset \mathbb{R}^n \) of an equilibrium point \( x_e \) corresponding to \( u = 0 \), i.e., \( f(x_e) = 0 \) and \( g(x_e) \neq 0 \). Functions \( f \) and \( g \) are assumed to be smooth vector fields and, without loss of generality, \( x_e \) is assumed to be the origin (one can always perform a change of coordinate \( z = x - x_e \) so that \( z = 0 \) is an equilibrium point).

**Definition 3.1** The systems

\[ \dot{x}_1 = f_1(x_1) + g_1(x_1) u_1, \]
\[ \dot{x}_2 = f_2(x_2) + g_2(x_2) u_2, \]

are feedback equivalent if there exist a state feedback

\[ u_1 = k(x_1) + \beta(x_1) u_2, \]

where \( k \) and \( \beta \) are two smooth functions with \( k(0) = 0, \ \beta(0) \neq 0 \) and a local diffeomorphism in a neighborhood of the origin \( U_0 \subset \mathbb{R}^n \)

\[ x_2 = T(x_1), \quad T(0) = 0 \]

such that the system

\[ \dot{x}_1 = f_1(x_1) + (k g_1)(x_1) + (\beta g_1)(x_1) u_2, \]
in $x_2$ coordinate becomes

$$\dot{x}_2 = \frac{dT}{dx_1}(f_1 + k g_1) \circ T^{-1}(x_2) + \left(\frac{dT}{dx_1}(\beta g_1) \circ T^{-1}(x_2)\right) u_2$$

$$= f_2(x_2) + g_2(x_2) u_2.$$

\[\square\]

**Definition 3.2** The nonlinear system (4.1) is said to be locally state feedback linearizable if it is locally feedback equivalent to a system in Brunovsky controller form

$$z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \quad z \in \mathbb{R}^n \quad (4.2)$$

\[\square\]

**Definition 3.3** An $r$-dimensional map $D$ on $W$, an open subset of $\mathbb{R}^n$, is called a distribution if it assigns to each $p \in W$ an $r$-dimensional subspace of $\mathbb{R}^n$ such that for each $p_0 \in W$ there exist a neighborhood $U$ of $p_0$ and $r$ smooth vector fields $f_1, \ldots, f_r$ with the following properties

i) $f_i(p), \ldots, f_r(p)$ are linearly independent for every $p \in U$,

ii) $D = \text{span}\{f_1(p), \ldots, f_r(p)\}, \forall p \in U.$

\[\square\]

**Definition 3.4** A Distribution $D$ is called involutive if for any two vector fields $f$ and $g$ belonging to $D$ their Lie bracket $[f, g]$ also belongs to $D$.

\[\square\]

The following theorem identifies the nonlinear systems which are locally feedback linearizable by means of necessary and sufficient conditions.

**Theorem 3.1** The system (4.1) is locally state feedback linearizable if in a neighborhood of the origin $U_0$
i) \( \text{span}\{g, \ldots, ad^{n-1}_tg\} = \mathbb{R}^n \),

ii) the distribution \( \mathcal{G}_0 = \text{span}\{g, \ldots, ad^{n-2}_tg\} \) is involutive and of constant rank \( n-1 \),

or, equivalently, if

iii) the distributions

\[
\mathcal{G}_i = \text{span}\{g, \ldots, ad^i_1g\}, \quad 0 \leq i \leq n-1
\]

are involutive and of constant rank \( i+1 \).

It should be mentioned that the linearizing transformation for a feedback linearizable system is not unique.

**Definition 3.5** The system

\[
\begin{align*}
\dot{x}_i &= x_{i+1} + \phi_i(x_1, \ldots, x_i), \quad 1 \leq i \leq n-1 \\
\dot{x}_n &= \phi_n(x_1, \ldots, x_n) + u,
\end{align*}
\]

(4.3)

is said to be in triangular form.

**Corollary 3.1** The systems in the triangular form of

\[
\begin{align*}
\dot{x}_i &= x_{i+1} + \phi_i(x_1, \ldots, x_i), \quad 1 \leq i \leq n-1 \\
\dot{x}_n &= \phi_n(x_1, \ldots, x_n) + u,
\end{align*}
\]

for which \( \phi_1, \ldots, \phi_n \) are smooth functions and \( \phi_i(0) = 0, \quad 1 \leq i \leq n \), are feedback linearizable.

We now consider single input uncertain systems

\[
\dot{x} = f(x) + g(x)u + q(x, \theta(t)),
\]

(4.4)

for which \( x \in \mathbb{R}^n, \ u \in \mathbb{R}, \ \theta \in \Omega \subset \mathbb{R}^p, \ f, \ g \) and \( q \) are assumed to be smooth known vector fields. \( \theta(t) \) is a vector of unknown time-varying disturbances, uncertain parameters or unmodelled dynamics for which only functional bounds are known. The
Section 4.2. Robust Feedback Linearizing Control

origin \( x = 0 \) is assumed to be an equilibrium point of the nominal system \((f, g)\).

**Theorem 3.2** The system (4.4) is locally feedback equivalent to

\[
\begin{align*}
\dot{z}_j &= z_{j+1} + \phi_i(z_1, \ldots, z_j, \theta(t)), \quad 1 \leq j \leq n - 1 \\
\dot{z}_n &= v + \phi_n(z_1, \ldots, z_n, \theta(t)).
\end{align*}
\]  

if

i) the nominal system \((f, g)\) is locally feedback linearizable,

ii) the strict triangularity assumption

\[
ad_q \mathcal{G}_i \subset \mathcal{G}_i \quad 0 \leq i \leq n - 2
\]

are satisfied in \( U_0 \).

**Definition 3.6** A static state feedback

\[
u = k(x),
\]

where \( k \) is a smooth function mapping \( \mathbb{R}^n \to \mathbb{R} \) and \( k(0) = 0 \), is said to be a local robust state feedback stabilizing control for the system (4.4) if the origin of the closed loop system

\[
\dot{x} = f(x) + k(x) g(x) + q(x, \theta(t))
\]

is locally asymptotically stable for any \( \theta \in \Omega' \subset \Omega \).

**Theorem 3.3** (Robust state feedback stabilization) For system (4.4), assume that:

i) the nominal system \((f, g)\) is locally feedback linearizable,

ii) \( ad_q \mathcal{G}_i \subset \mathcal{G}_i \quad 0 \leq i \leq n - 2 \),

iii) \( \Omega \) is a known compact set.

Then, there exists a static state feedback which is locally a robust stabilizing control.
4.3 Robust control of Traffic Flow

Consider a portion of a highway of length $L$ with one on-ramp and one off-ramp as shown in Figure 4.1. The model of traffic flow (2.23a), (2.24), (2.25b) for the section is

\[
\dot{\rho} = \frac{1}{L} \left( q_{in} - \alpha \rho v - (1 - \alpha) \rho_s v_s + q_r - q_s \right)
\]

\[
\dot{v} = \frac{1}{\tau} \left( V(\rho) - v \right) + \frac{\eta}{L} \left( v_{in} - v \right) + \frac{\nu}{L} (\rho - \rho_s) - \frac{\delta}{L} v q_s + \psi_o,
\]

where $\rho_s$ and $v_s$ are the traffic density and velocity of the downstream section. $\psi_o$ represents the modelling error in the momentum equation. The on-ramp is assumed to be long enough so that the friction term due to vehicle merging can be ignored. The incoming traffic flow $q_{in}$ is uncontrollable by local ramp metering and, thus, regarded as a disturbance. Furthermore, $v_{in}$, $\rho_{in}$, and $v_s$ are unknown and can be considered as model uncertainties. The only controllable input to the system is the on-ramp volume $q_r$ which is regarded the control command. The linear model of Greenshields (2.9) [33]

\[ V(\rho) = V_f(1 - \frac{\rho}{\rho_j}) \]

is used for the speed-density characteristic where $V_f$ and $\rho_j$ are the free flow speed and jam density. The parameters $L$, $V_f$, $\rho_j$, $\tau$, $\alpha$, $\eta$, $\nu$ and $\delta$ are assumed to be known.

Our objective is to control the ramp flow to achieve a desired level of highway capacity by maintaining optimal traffic density $\rho_d$ and velocity $v_d$ profiles. Investigation on several flow-density characteristics shows that the optimal highway capacity:

![Figure 4.1: A freeway system for ramp metering control](image-url)
will be achieved in a neighborhood of the critical density $\rho_c = \frac{1}{2}\rho_j$ for which the traffic flow leaving the section becomes maximum (refer to Figure 2.3.d) [3]. The desired density and capacity are assumed to be piecewise constant functions of time and satisfy

$$V(\rho_d) = \nu_d.$$  \hspace{1cm} (4.8)

As already mentioned, it is assumed that the states of the system, namely section density $\rho$ and velocity $\nu$, are available to the controller. By performing a linear change of coordinates, we shift the equilibrium point of the nominal system (4.6), (4.7) to the origin. Let $x_1$ and $x_2$ denote the tracking errors $x_1 = \rho - \rho_d$, $x_2 = \nu - \nu_d$, and write the system equations as:

$$\dot{x}_1 = -\alpha l x_1 x_2 - p_1 x_1 - p_2 x_2 + \phi + u,$$  \hspace{1cm} (4.9)

$$\dot{x}_2 = -m_1 x_1 - m_2 x_2 - \eta l x_2^2 + \theta x_2 + \psi,$$  \hspace{1cm} (4.10)

where

$$u = \frac{1}{L} q_r,$$  \hspace{1cm} (4.11)

is the normalized control signal and

$$l = \frac{1}{L}, \quad p_1 = \alpha \frac{\nu_d}{L}, \quad p_2 = \alpha \frac{\rho_d}{L},$$

$$m_1 = \frac{1}{\tau} \frac{V_r}{\rho_j} - \frac{\nu}{L}, \quad m_2 = \frac{1}{\tau} + \frac{2\eta\nu_d}{L},$$

are known positive constants while

$$\phi = \frac{1}{L} q_{in} - \frac{\alpha}{L} \rho_d \nu_d - \frac{1 - \alpha}{L} \rho_j \nu_o - \frac{1}{L} q_s,$$  \hspace{1cm} (4.12)

$$\theta = \frac{\eta}{L} \frac{v_{in}}{q_s},$$  \hspace{1cm} (4.13)

$$\psi = \frac{\nu}{L} (\rho_d - \rho_o) + \frac{\eta}{L} \nu_d (v_{in} - \nu_d) - \frac{\delta}{L} q_s + \psi_0,$$  \hspace{1cm} (4.14)

are regarded as uncertain parameters. Since the incoming traffic flow $q_{in}$ and velocity $v_{in}$ as well as the downstream density $\rho_o$ and velocity $\nu_o$ are bounded according to $0 \leq q_{in} \leq q_{max}$, $0 \leq v_{in} \leq V_{max}$, and $0 \leq \rho_o \nu_o \leq q_{max}$, the uncertain parameters $\phi$ and $\theta$ as defined in (4.12), (4.13) belong to a known compact set:

$$-\frac{\alpha}{L} \rho_d \nu_d - \frac{1 - \alpha}{L} q_{max} - \frac{1}{L} q_{sm} \leq \phi \leq \frac{1}{L} q_{max} - \frac{\alpha}{L} \rho_d \nu_d,$$  \hspace{1cm} (4.15)
\[ \frac{-\delta}{L} q_{sm} \leq \theta \leq \theta_m, \quad \theta_m = \eta \frac{V_{\text{max}}}{L}, \tag{4.16} \]

where \( q_{sm} \) is the maximum allowable off-ramp flow and assumed to be proportional to \( q_{\text{max}} \):

\[ q_{sm} = \gamma q_{\text{max}} \quad 0 < \gamma < 1 \]

The range of the variation of \( \psi \) as defined in (4.14) is unknown due to the modelling error \( \psi_q \). Later in the robustness analysis, we will find an acceptable range for \( \psi \) according to the design criteria. For the time being, we assume \( \psi \) is bounded according to

\[ |\psi| \leq \psi_m, \tag{4.17} \]

for some constant \( \psi_m > 0 \).

### 4.3.1 A Global Change of Coordinates

Let us first find a global change of coordinate in which system (4.9), (4.10) can be written in the triangular form according to Definition 3.3. Define the nominal system \((f, g)\) and the uncertain part \(q\) as

\[
\begin{align*}
    f(x) &= (-\alpha l x_1 x_2 - p_1 x_1 - p_2 x_2) \frac{\partial}{\partial x_1} + (-m_1 x_1 - m_2 x_2 - l \eta x_2^2) \frac{\partial}{\partial x_2}, \\
    g(x) &= \frac{\partial}{\partial x_1}, \\
    q(x, \theta) &= \phi \frac{\partial}{\partial x_1} + (\theta x_2 + \psi) \frac{\partial}{\partial x_2},
\end{align*}
\]

where \( \theta = [\phi \ \psi]^T \). According to the Nonlinear Feedback Linearizing Theorem 3.1, the nominal system \((f, g)\) is locally feedback linearizable since

i) \( adf g = (\alpha l x_2 + p_1) \frac{\partial}{\partial x_1} + m_1 \frac{\partial}{\partial x_2}, \quad m_1 \neq 0 \). Therefore,

\[ \text{span}\{g, adf g\} = \mathbb{R}^2, \]

ii) the distribution \( \mathcal{G}_0 = \text{span}\{g\} \) is involutive and of constant rank 1.

Moreover, the strict triangularity assumption

\[ adf \mathcal{G}_0 \subset \mathcal{G}_0, \]
Section 4.3. Robust control of Traffic Flow

is satisfied in any neighborhood of the origin \( x = [x_1, x_2] = 0 \). Therefore, we can apply Frobenius' Theorem (Appendix C, Theorem C.1) which guarantees the existence of a smooth function \( h(x) \), \( h(0) = 0 \) such that in a neighborhood of the origin

\[
< dh, \text{ad}_{(-f)} g > \neq 0, \quad (4.18)
\]

\[
< dh, g > = 0, \quad (4.19)
\]

and a local diffeomorphism

\[
z = [z_1, z_2] = T(x), \quad T(0) = 0
\]

in which the system (4.9), (4.10) is locally feedback equivalent to

\[
\dot{z}_1 = z_2 + \Phi_1(z_1, \theta), \quad (4.20)
\]

\[
\dot{z}_2 = v + \Phi_2(z_1, z_2, \theta). \quad (4.21)
\]

Computing

\[
\text{ad}_{(-f)} g = -(\alpha l x_2 + p_1) \frac{\partial}{\partial x_1} - m_1 \frac{\partial}{\partial x_2},
\]

(4.18), (4.19) become

\[
-(\alpha l x_2 + p_1) \frac{\partial h}{\partial x_1} - m_1 \frac{\partial h}{\partial x_2} \neq 0,
\]

\[
\frac{\partial h}{\partial x_1} = 0.
\]

Since \( m_1 \neq 0 \), any invertible function of \( x_2 \), \( h(x_2) \), satisfies these conditions. If we choose \( h(x) = x_2 \), the feedback linearizing transformation, \( z = (h(x), L_j h(x)) \), is

\[
z_1 = x_2, \quad (4.22)
\]

\[
z_2 = -m_1 x_1 - m_2 x_2 - \eta l x_2. \quad (4.23)
\]

Define the state feedback

\[
u = -\frac{L_j^2 h(x)}{L_j L_j h(x)} + \frac{1}{L_j L_j h(x)} v_0
\]

\[
= \alpha l x_2 + p_1 x_1 + p_2 x_2 - \frac{1}{m_1} v_0. \quad (4.24)
\]
Section 4.3. Robust control of Traffic Flow

The system (4.9), (4.10) under the change of coordinate (4.22), (4.23) with definition (4.24) can be written in the triangular form of (4.20), (4.21):

\[
\begin{align*}
\dot{z}_1 &= z_2 + \theta z_1 + \psi, \\
\dot{z}_2 &= v - m_2 \theta z_1 - 2\eta l \theta z_1^2 - 2\eta l \psi z_1 + \xi_0,
\end{align*}
\]

with the new control command and uncertainty parameter

\[
\begin{align*}
v &= v_0 - m_2 z_2 - 2\eta l z_1 z_2, \\
\xi_0 &= -m_1 \phi - m_2 \psi.
\end{align*}
\]

4.3.2 The Nonlinear Controller

A robust controller is now developed for the systems (4.25) and (4.26). The first step is to design the controller for the first-order system

\[
\dot{z}_1 = v_1 + \theta z_1 + \psi,
\]

which corresponds to equation (4.25). Here \( v_1 \) is a fictitious control command which has replaced \( z_2 \) in (4.25). Define the static state feedback

\[
v_1 = -k_1 z_1 - \theta_m z_1 + \pi(z_1), \quad (k_1 > 0)
\]

where \( \theta_m \) denotes the maximum of \( \theta \) and \( \pi(z_1) \) is an injection term yet to be determined. Let us consider the Lyapunov function candidate

\[
V_1 = \frac{1}{2} z_1^2.
\]

Its time derivative is

\[
\dot{V}_1 = z_1(-k_1 z_1 - \theta_m z_1 + \pi(z_1) + \theta z_1 + \psi)
\]

\[
= -k_1 z_1^2 + (\theta - \theta_m) z_1^2 + (\pi(z_1) + \psi) z_1
\]

\[
\leq -k_1 z_1^2
\]

for

\[
\pi(z_1) = -\psi_m \text{sign}(z_1),
\]

(4.31)
which proves that the origin $z_i = 0$ of equation (4.29) is globally exponentially stable. Now consider the system (4.25), (4.26) and perform the global change of coordinates

$$
\begin{align*}
\dot{z}_1 &= z_1, \\
\dot{z}_2 &= z_2 - v_1 = z_2 + (k_i + \theta_m)z_1 - \pi(z_i),
\end{align*}
$$

(4.32) (4.33)

to obtain

$$
\begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 - (k_i + \theta_m)\tilde{z}_1 + \pi(\tilde{z}_1) + \theta \tilde{z}_1 + \psi, \\
\dot{\tilde{z}}_2 &= v + \tilde{z}_i(-2\eta l \theta \tilde{z}_i - m_2 \theta - 2\eta l \psi) + \xi_0 + \\
&\quad (k_i + \theta_m)\left(\tilde{z}_2 - (k_i + \theta_m)\tilde{z}_1 + \pi(\tilde{z}_1) + \theta \tilde{z}_1 + \psi\right).
\end{align*}
$$

(4.34) (4.35)

Furthermore, with the new control

$$w = v + (k_i + \theta_m)\left(\tilde{z}_2 - (k_i + \theta_m)\tilde{z}_1 + \pi(\tilde{z}_1)\right),
$$

(4.36)

which consists of control command $v$ and all other known terms in equation (4.35), equations (4.34) and (4.35) become

$$
\begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 - k_i \tilde{z}_1 + (\theta - \theta_m)\tilde{z}_1 + \pi(\tilde{z}_1) + \psi, \\
\dot{\tilde{z}}_2 &= w + \tilde{z}_i h(\tilde{z}_1, \theta) + \xi,
\end{align*}
$$

(4.37) (4.38)

where

$$
\begin{align*}
h(\tilde{z}_1, \theta) &= (k_i + \theta_m - m_2 - 2\eta l \tilde{z}_i)\theta - 2\eta l \psi, \\
\xi &= \xi_0 + (k_i + \theta_m)\psi \\
&= -m_1 \phi + (k_i + \theta_m - m_2)\psi.
\end{align*}
$$

(4.39) (4.40)

An upper bound for the function $h(\tilde{z}_1, \theta)$ will be needed in the stability proof and is devised here. Recalling (4.16) and (4.17), one has

$$
|h(\tilde{z}_1, \theta)| \leq (k_i + \theta_m - m_2 - 2\eta l \tilde{z}_i) \max\{\theta_m, \frac{\delta}{L q_{sm}}\} + 2\eta l \psi_m,
$$

(4.41)

where the gain $k_i$ is chosen such that

$$k_i > m_2 - \theta_m + 2\eta l \max\{|\tilde{z}_1|\}.
$$

(4.42)
Since $\ddot{z}_i = z_i = x_2 = v - u_d$, and $u_d$ is in the proximity of the critical velocity $v_c \simeq \frac{1}{2} V_{\text{max}}$, we have $\max\{|\dot{z}_i|\} = \frac{1}{2} V_{\text{max}}$. Therefore,

$$2\eta l \max\{|\dot{z}_i|\} = \eta l V_{\text{max}} = \theta_m,$$

and (4.42) becomes

$$k_i > m_2. \quad (4.43)$$

Numerical values of the parameters guarantee that $\theta_m > \frac{\delta}{\xi} q_m$. Therefore, (4.41) yields

$$|h(\ddot{z}_i, \Theta)| \leq \zeta(\ddot{z}_i), \quad (4.44)$$

where

$$\zeta(\ddot{z}_i) = (k_i + \theta_m - m_2 - 2\eta l \ddot{z}_i)\theta_m + 2\eta l \psi_m.$$

For the sake of simplicity in the presentation of the stability proof, let us also assume that the uncertainty $\xi$ as defined in (4.40) satisfies

$$-\xi_m \leq \xi \leq \xi_M, \quad (4.45)$$

where $\xi_m$ and $\xi_M$ are positive constants. This is a valid assumption since $\xi$ is a function of the uncertainties $\phi$ and $\psi$. Therefore, one can always determine these bounds in terms of $\phi_m$ and $\psi_m$ defined in (4.15) and (4.17). This will be further discussed in the robustness analysis in Section 4.3.3.

The final stage is to design the control command $\omega$ to globally exponentially stabilize the system (4.37),(4.38) in spite of the presence of the uncertainty $\Theta$.

We consider the Lyapunov function candidate

$$V_2 = \frac{1}{2} \dot{z}_1^2 + \frac{1}{2} \dot{z}_2^2,$$

with the time derivative

$$\dot{V}_2 = \ddot{z}_1 (\ddot{z}_2 - k_i \ddot{z}_1 + (\theta - \theta_m)\dot{z}_1 + \pi(z_1) + \psi) + \ddot{z}_2 (w + \ddot{z}_1 h(\ddot{z}_1, \Theta) + \xi)$$

$$ = -k_i \ddot{z}_1^2 + (\theta - \theta_m)\dot{z}_1^2 + (\pi(z_1) + \psi) \ddot{z}_1 + \ddot{z}_1 \ddot{z}_2 + (w + \xi) \ddot{z}_2 +$$

$$\ddot{z}_1 \ddot{z}_2 h(\ddot{z}_1, \Theta).$$
Employ (4.16), (4.31), (4.44) with the definition $\tilde{Z}_2 = [\tilde{z}_1, \tilde{z}_2]^T$ to obtain

$$\dot{V}_2 \leq -k_1 \tilde{z}_1^2 + \tilde{z}_1 \dot{z}_2 + (w + \xi) \tilde{z}_2 + |\tilde{z}_2| ||\tilde{Z}_2|| \zeta(\tilde{z}_1),$$

and define

$$w = -k_2 \tilde{z}_2 - \frac{1}{4} \tilde{z}_2 \zeta(\tilde{z}_1)^2 + \varepsilon(\tilde{z}_2),$$

(4.46)

to have

$$\dot{V}_2 \leq -(k_1 - 1) \tilde{z}_1^2 - (k_2 - 1) \tilde{z}_2^2 - ||\tilde{Z}_2||^2 - \frac{1}{4} \tilde{z}_2^2 \zeta(\tilde{z}_1)^2 + |z_2| ||\tilde{Z}_2|| \zeta(\tilde{z}_1) +$$

$$(\varepsilon(\tilde{z}_2) + \xi) \tilde{z}_2$$

$$\leq -(k_1 - 1) \tilde{z}_1^2 - (k_2 - 1) \tilde{z}_2^2 - \left(||\tilde{Z}_2|| - \frac{1}{2} |z_2| \zeta(\tilde{z}_1)\right)^2 + (\varepsilon(\tilde{z}_2) + \xi) \tilde{z}_2.$$

Finally, by setting

$$\varepsilon(\tilde{z}_2) = -\frac{1}{2} \xi_m \left(1 + \text{sign}(\tilde{z}_2)\right) + \frac{1}{2} \xi_m \left(1 - \text{sign}(\tilde{z}_2)\right),$$

(4.47)

we obtain

$$\dot{V}_2 \leq -(k_1 - 1) \tilde{z}_1^2 - (k_2 - 1) \tilde{z}_2^2 - \left(||\tilde{Z}_2|| - \frac{1}{2} |z_2| \zeta(\tilde{z}_1)\right)^2,$$

which guarantees that the origin $(\tilde{z}_1, \tilde{z}_2) = 0$ is globally exponentially stable for $k_1 > 1$ and $k_2 > 1$.

**Remark.** It is important to note how one can implement the real control command to the system $q_r$ (on-ramp traffic flow) according to the proposed design procedure. First, the final control command $w$ given by (4.46) is constructed. The intermediate control signals $v$ and $v_o$ are then computed using (4.36) and (4.27), respectively. Finally, the state feedback equation provided by (4.24) along with the definition (4.11) yield the flow command $q_r$.

### 4.3.3 Robustness Analysis

In this section we discuss the robustness properties of the proposed controller and find an acceptable range for the system modelling errors.
Since the origin of (4.37), (4.38) is globally exponentially stable, recalling the change of coordinate (4.32), (4.33) we have

\[
\begin{align*}
    z_1 &= 0, \\
    z_2 &= (k_1 + \theta_m)z_1 - \pi(z_1) = 0,
\end{align*}
\]

at steady state. By virtue of (4.22), (4.23), and (4.31), we obtain

\[
\begin{align*}
    |x_1| &= |\rho - \rho_d| = \frac{\psi_m}{m_1}, \\
    x_2 &= u - v_d = 0,
\end{align*}
\]

which implies that the desired velocity can be achieved perfectly and the steady state error in the traffic density depends on \(\psi_m\).

Given the following design criterion for the traffic density at steady state:

\[
    |\rho - \rho_d| < \epsilon_d. \tag{4.50}
\]

we should have

\[
    \psi_m < m_1 \epsilon_d.
\]

Recall the definition of \(\psi\) and \(\psi_m\) from (4.14) and (4.17), respectively, to conclude

\[
    |\frac{\nu}{L}(\rho_d - \rho_o) + \frac{\eta}{L}v_d(u_{in} - v_d) - \frac{\delta}{L}q_s + \psi_0| \leq m_1 \epsilon_d.
\]

Since \(0 \leq \rho_o \leq \rho_j\) and \(0 \leq u_{in} \leq V_{\text{max}}\), then,

\[
    -m_1 \epsilon_d - \frac{\nu}{L} \rho_d - \frac{\eta}{L}v_d(V_{\text{max}} - v_d) \leq \psi_0 \leq m_1 \epsilon_d + \frac{\nu}{L}(\rho_j - \rho_d) + \frac{\eta}{L}v_d^2 + \frac{\delta}{L}q_{\text{max}}, \tag{4.51}
\]

which gives the permissible range of the modelling error \(\psi_0\) for the traffic density to satisfy (4.50). As expected, \(\psi_0\) is a function of \(\epsilon_d\) as well as the desired values \(\rho_d\) and \(v_d\).
Section 4.4. Ramp metering using neural network & linear feedback controllers

One can also determine the range of the auxiliary signal $\xi$ used in the control command (4.46). From (4.15) and (4.40), we can write

\[
\frac{m_1}{L}(\alpha \rho_d v_d - q_{max}) + (k_i + \theta_m - m_z)\psi \leq \xi \leq \frac{m_1}{L}(\alpha \rho_d v_d + (1 - \alpha)q_{max} + \gamma q_{max}) + (k_i + \theta_m - m_z)\psi,
\]

which yields

\[
\begin{align*}
\xi_m &= \frac{m_1}{L}(q_{max} - \alpha \rho_d v_d) + (k_i + \theta_m - m_z)m_1\epsilon_d, \\
\xi_M &= \frac{m_1}{L}(\alpha \rho_d v_d + (1 - \alpha)q_{max} + \gamma q_{max}) + (k_i + \theta_m - m_z)m_1\epsilon_d,
\end{align*}
\]

since $k_i + \theta_m - m_z \geq 0$ according to (4.43).

An instance of the robust linearizing controller for ramp metering will be presented in Section 4.5.

4.4 Ramp metering using neural network & linear feedback controllers

The performance of the proposed control system will be tested in a number of simulation scenarios. For the purposes of comparison, a neural network controller and a linear feedback controller ALINEA [64] are also designed and implemented at a local metering level.

4.4.1 A neural network controller

An artificial neural network attempts to achieve a high level of computational performance via certain parallel structure of simple processing elements. The most widely used processing element, the sigmoidal feed-forward or Adaline, is shown in Figure 4.2. The scalar output $y$ is the nonlinear sigmoidal function $\sigma$ of the weighted sum of the inputs $x_i$, $i = 1, \ldots, n$ and a possible constant bias

\[
y = \sigma(\sum_{i=1}^{n} w_i x_i + w_{n+1}),
\]
where $w_i$, $i = 1 \ldots, n+1$ are adjustable weights.

The sigmoidal operator can be expressed by many nondecreasing functions which satisfy $\sigma(-\infty) = 0$, $\sigma(\infty) = 1$ such as

$$\sigma(x) = \frac{1}{1 + e^{-ax}}.$$

Neural networks are specified by the net topology, node characteristics, and training or learning rules. The process elements are connected together to form what is known as a feed-forward layer, shown in Figure 4.3. Many layers then can be cascaded, with the output of one layer connected to the inputs of the next layer, to form a network. It has been shown [70], [71] that a network consisting of only two layers of processing elements can be used to approximate any nonlinear function with an arbitrary degree of accuracy.

There exist several algorithms to train a layered neural network. The error backpropagation algorithm has been recognized as one of the most appropriate neural network algorithms for control applications [72], [73] where little a priori information of the system and its properties is necessary. Figure 4.4 shows the structure of a neural network controller for a single input nonlinear system $(f, g)$ with disturbance vector...
Section 4.4. Ramp metering using neural network & linear feedback controllers

Figure 4.3: Two-layer feedforward neural network

\[ d \text{ and input signal } u. \] The back-propagation algorithm converges to a set of weights that represents a local minimum for the mean-square tracking error \( e \) between the plant output \( y \) and the desired output \( y_d \). This is achieved by updating the network weights \( w \) according to

\[ \dot{w}_i = -2\zeta e^T \frac{\partial e}{\partial w_i}, \]

Figure 4.4: The neural network controller
Section 4.4. Ramp metering using neural network & linear feedback controllers

where \( \zeta > 0 \) is the learning rate [74]. To compute the term \( \frac{\partial e}{\partial w_i} \), we use the chain rule

\[
\frac{\partial e}{\partial w_i} = \frac{\partial e}{\partial u} \frac{\partial u}{\partial w_i} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial w_i}
\]

assuming \( y_d \) is independent of \( u \). At this stage the learning algorithm involves the Jacobian of the plant \( \frac{\partial y}{\partial u} \) which is unknown. A simple way to overcome this problem is to approximate the partial derivatives by their signs which can be known when we have some information about the orientation in which the control parameters influence the outputs of the plant. Therefore,

\[
\dot{w}_i = -2 \zeta e^T \frac{\partial u}{\partial w_i} \text{sign}(\frac{\partial y}{\partial u}).
\] (4.52)

If the network converges, it will drive the error \( e \) to zero.

The neural network controller of Figure 4.4 is now implemented for the traffic system (4.6), (4.7) where \( y = [\rho \ v]^T \) is the output signal, \( y_d = [\rho_d \ v_d]^T \) contains the reference trajectories, and \( u \) is the ramp flow command. The external disturbance \( d \) consists of the incoming traffic velocity and flow, off-ramp flow as well as the downstream traffic velocity and density \( d = [v_i \ q_i \ p_o \ v_o]^T \). Figure 4.5 shows a two-layer feedforward neural network used in the controller.

To implement (4.52), we notice that the ramp flow \( u \) affects the traffic states \( (\rho, v) \) in different ways: an increase in \( u \) will result in an increase in \( \rho \) but a decrease in \( v \), and vice versa. Therefore,

\[
\text{sign}(\frac{\partial \rho}{\partial u}) = +1,
\]

\[
\text{sign}(\frac{\partial v}{\partial u}) = -1.
\]

4.4.2 A linear feedback controller

The ALINEA ramp controller is supported by classical control theory. It has proven to be relatively simple compared to other known algorithms and highly robust.
Section 4.4. Ramp metering using neural network & linear feedback controllers

Figure 4.5: A two-layer neural network used in the traffic system controller with respect to disturbances and model inaccuracies. It also requires a minimal amount of traffic measurements from the detector stations.

Unlike the feedback linearizing and the neural network controllers, ALINEA is based on the first-order traffic model (2.24)

\[
\dot{\rho}(t) = \frac{1}{L} \left( q_{in}(t) + \tau(t) - q_s(t) - q_{out}(t) \right),
\]

where \( q_{out} = \rho \nu = V_j (1 - \frac{\rho}{\rho_j}) \rho \). This equation is first linearized around a nominal steady state point \((\bar{\rho}, \bar{\tau}, \bar{q}_{in}, \bar{p})\) to get

\[
\Delta \dot{\rho}(t) = \frac{1}{L} \left( \Delta q_{in}(t) + \Delta \tau(t) - \Delta q_s(t) - Q \Delta \rho(t) \right),
\]

where

\[
Q = \frac{dq_{out}}{dt}\Big|_{\rho=\bar{\rho}} = V_j \left(1 - \frac{2\bar{\rho}}{\rho_j}\right),
\]

and \( \Delta q_{in} = q_{in} - \bar{q}_{in}, \ \Delta q_s = q_s - \bar{q}_s, \ \Delta \tau = r - \bar{\tau}, \) and \( \Delta \rho = \rho - \bar{\rho} \). An integral feedback law in the form

\[
\Delta \tau = -k_c (\Delta \rho - \Delta \rho_d),
\]

is then applied to the system where \( \rho_d \) is the desired density and \( \Delta \rho_d = \rho_d - \bar{\rho} \). Note that the positive gain \( k_c \) determines the location of the poles of the closed-loop system. Assuming \( \Delta q_{in} = \Delta q_s = 0 \), the transfer function of the closed-loop system
can be derived as follows

\[ H_a(s) = \frac{\Delta \rho_a(s)}{\Delta \rho(s)} = \frac{k_c/L}{s^2 + \frac{Q}{L} + \frac{k_c}{L}}, \]

where the poles are located at \( s_{1,2} = \frac{1}{2}(-\frac{Q}{L} \pm \sqrt{\frac{Q^2}{L^2} - \frac{4k_c}{L}}). \)

### 4.5 Simulation results

In this section, several traffic scenarios are considered and the performance of the robust controller is compared with the performance of the neural network controller and ALINEA based on the simulation results. For this purpose, a section of a single-lane roadway of length \( L = 500 \text{m} \) with one on-ramp as shown in figure 4.1 is considered. The desired density \( \rho_d \) is chosen close to the critical density \( \rho_c = \frac{1}{2} \rho_j \), e.g., \( \rho_d = 0.9 \rho_c \). Therefore, according to (4.8) \( v_d = \frac{11V_f}{20} \). The disturbances are also set to \( v_m = 0.8V_f, q_m = 0.16 \rho_j V_f, q_s = 0.01 \rho_j V_f, \rho_s = 0.4 \rho_j, \) and \( q_s = 3V_f, \) all in their acceptable domains.

The gains for feedback linearizing controller are chosen \( k_1 = k_2 = 1.2 \) to satisfy the design requirements. The maximum allowable density error is \( \epsilon_d = 5 \text{(veh/km)} \) and the uncertainty parameter in the momentum equation (4.7) is set to \( \psi_0 = -0.2 \text{(m/s^2)} \) within the set given by (4.51).

The weights of the neural network controller \( \omega_{ij}, b_i, s_i, t_i \) \((i = 1, 2, j = 1, 2, 3)\) in Figure 4.5 are initialized randomly in the interval \([0, 1]\). The adaptation gain \( \zeta \) in the back-propagation is set to 3 and the constant \( c \) in the nonlinear operator \( \sigma(\cdot) \) is \( c = 0.6 \).

For an uncontrolled traffic system, the on-ramp flow is generally a function of public demand and mainstream traffic density. For simulation purposes here, we restrict the ramp metering rate to the interval \([0, 3600] \text{ (veh/hour)} \). In other words, a maximum of one car per second is allowed to enter the highway.

With the exception of the final simulation scenario (scenario 4), the model (4.6) and (4.7) is used in all the simulations. In the first simulation (scenario 1), the system initial conditions are chosen as \( \rho(0) = \frac{1}{2} \rho_c, u(0) = \frac{3}{4} V_f \), which are in the
stable region of the flow-density diagram. All controllers stabilize the traffic system around the desired trajectories with satisfactory results (see Figure 4.6). However, the feedback linearizing controller responds considerably faster than ALINEA and the neural network; while ALINEA and neural network controllers settle down after 800 and 400 seconds, respectively, the feedback linearizing controller converges after 100 seconds. Moreover, the amount of overshoot in both density and velocity profiles is considerably smaller for the feedback linearizing controller when compared to the other controllers. Figure 4.7 shows the metering rates for the controllers. The metering rate for the feedback linearizing controller oscillates between the extreme values faster than the rates of other controllers. This explains the relatively fast convergence rate of the controller. Moreover, it indicates that the proposed controller is practically easier to implement as it can be directly used to command the traffic light mounted on the on-ramp.

Figure 4.8 and 4.9 show the simulation results for the second scenario where the initial conditions are chosen in the congested region of the flow-density diagram, namely, \( \rho(0) = \frac{3}{2} \rho_c, \) \( v(0) = \frac{1}{4} V_f. \) Similar results are obtained as in the first scenario with the conclusion that the feedback linearizing controller has less settling time and overshoot. As shown in 4.9, the feedback linearizing controller exerts a larger control authority during the transient than other controllers (approximately 30 seconds before ALINEA and 70 seconds before the neural network controller). As a result, the response of the system to the feedback linearizing controller is faster.

In the third simulation scenario, the robustness of the feedback linearizing controller is examined with respect to parameter variations. For this purpose, the controller is designed using the nominal values for the parameters \( \beta = [V_f, \rho_c, \alpha, \tau, \eta, \nu, \delta] \). The parameters of the model equations (4.6), (4.7) are then perturbed about their nominal values by 15%. The initial conditions are chosen as in scenario 1. The simulation results (Figure 4.10) show that the performance of the feedback linearizing controller is little affected by the new condition. The vehicle density error increases to 1 (veh/km) well within the allowable range of \( \epsilon_d = 5 \) (veh/km), while the settling time and the overshoot remain unchanged.

Finally in scenario 4, the robustness of the proposed controller is further investi-
Section 4.5. Simulation results

Figure 4.6: Section density and velocity using different ramp metering strategies, scenario 1.

Figure 4.7: Ramp metering rates, scenario 1.
Section 4.5. Simulation results

Figure 4.8: Section density and velocity using different ramp metering strategies, scenario 2.

Figure 4.9: Ramp metering rates, scenario 2.
Section 4.5. Simulation results

Figure 4.10: Section density and velocity, scenario 3.

gated with respect to the structural changes in the traffic modelling. For this purpose, the designed controllers which are based on the model equations (4.6) and (4.7) are applied to a different system (Karaaslan’s model) (2.26a)-(2.26c) with the parameters listed in Table 2.1. Once more, two different sets of initial conditions are considered: in case (a) we choose \( \rho(0) = \frac{1}{2}\rho_c, \ v(0) = \frac{3}{4}V_f \) (stable region) and in case (b) \( \rho(0) = \frac{3}{2}\rho_c, \ v(0) = \frac{1}{4}V_f \) (unstable region). The values for the controllers gains
Section 4.5. Simulation results

and disturbances are the same as in scenarios 1 and 2. Simulation results are shown in Figures 4.11- 4.14 for cases (a) and (b), respectively. The feedback linearizing controller stabilizes the system with satisfactory performance while the ALINEA and neural network controllers fail to achieve the design objectives in both cases and lead the system to unstable operating regions. In case (a), as shown in Figure 4.11, the proposed controller results in tracking the desired velocity trajectory almost perfectly as discussed before. However, the section density reaches 92% of the desired value after a settling time of 100 seconds. Again, this is within the acceptable range defined by the design criterion $\epsilon_* = 5$ (veh/km). The ALINEA and neural network controllers bring the system to jammed conditions regardless of the desired values.

For case (b), the settling time of the system using feedback linearizing controller increases to 300 seconds, the desired velocity is perfectly achieved, and the density settles within the expected region. As Figure 4.14 suggests the ramp command of the proposed controller is off, i.e. $u = 0$, as long as the system is in the unstable region of the fundamental curve. This corresponds to the first 270 seconds of the simulation time. Only when the system trajectories reach the vicinity of the desired values the controller activates the ramp command and maintains the system states in that region afterwards. In this sense, both ALINEA and neural network controllers fail to recognize the critical condition of the system and activate the ramp in the wrong time, thus, causing instability.
Section 4.5. Simulation results

Figure 4.11: Section density and velocity, scenario 4(a).

Figure 4.12: Ramp metering rates, scenario 4(a).
Section 4.5. Simulation results

Figure 4.13: Section density and velocity, scenario 4(b).

Figure 4.14: Ramp metering rates, scenario 4(b).
Chapter 5

Conclusion

This chapter is devoted to the concluding remarks, the contributions of this work, and suggestions for possible future research.

5.1 Concluding Remarks

In this work three major research areas associated with highway traffic management systems have been addressed: modeling, state estimation, and control. The emphasis was on the robustness properties of the proposed observers and controller.

In Chapter 2, a family of macroscopic traffic flow models have been considered for design and implementation of the robust state estimators and controllers. First, a multi-regime speed-density model has been found to fit the traffic measurements with satisfactory results. The derivation of a second-order dynamic model which incorporates several driving factors such as traffic anticipatory behaviour, friction due to ramp flows, and effect of upstream traffic was then presented. All of the experiments performed in this thesis were based on the traffic data collected from Highway 401. The first series of experiments were carried out to estimate the model parameters. For this purpose, a number of different traffic conditions for different locations along the highway were selected and a direct optimization search called Simplex algorithm was employed. It has been observed that the results of the parameter estimation are fairly independent of traffic conditions, time of day, and site location.
The problem of robust state estimation of traffic systems using sliding mode techniques was presented in Chapter 3. First, a sliding mode observer was designed for a traffic system with $N = 3$ sections. The experimental results confirmed the effectiveness of the proposed state estimation when compared to the Extended Kalman Filter. Moreover, robustness analysis with respect to parameter variations showed that the sliding mode observer had lower sensitivity than the Extended Kalman Filter. Generalization of the proposed observer to traffic systems with arbitrary number of sections was then accomplished. The robustness properties of the observer with respect to unmodelled dynamics, disturbances, and modelling errors was achieved by properly adjusting the observer gains. Simulation and experimental results for the robust state estimation showed significant improvements in the performance of the estimator compared to the Extended Kalman Filter. The results of the robust state observation for single-lane systems were then generalized to multi-lane systems.

Finally, in Chapter 4, it has been shown that the feedback linearizing techniques can be successfully applied to develop a new ramp metering control strategy. The main contribution of the work is to show the robustness properties of the proposed controller with respect to the modelling errors, disturbances, parameter variations as well as modelling structure. In all of the simulation scenarios studied here, the performance of the feedback linearizing controller surpasses the performance of other metering strategies previously reported through maintaining a stable operating condition, faster response and achieving the design objectives.

### 5.2 Suggested Future Work

The macroscopic traffic flow model used in this work is a modified version of Payne’s model [5]. Along the line of model identification, one may investigate the possibility of using different and perhaps more sophisticated models, such as the model of Karaaslan et al. [10]. Also, the effect of applying different optimization algorithms on parameter estimation can be considered.

In this work, it has been decided to design and analyze the sliding mode observers and the feedback linearizing controller for the continuous-time models. This is mainly...
due to the fact the theory for continuous-time observers and controllers has been developed substantially compared to their discrete counterparts. Therefore, it may be of interest to consider the same problems for discrete models.

The problem of local ramp metering control consisted of a set of decentralized controllers has been considered in this thesis. Each controller functions independently and responds to changes in local traffic conditions. It would be of interest to generalize the results of the robust feedback controller design for local ramp metering to include area-wide metering control in which the control module is centralized.

Finally, it may also be of interest to integrate a closed-loop control system consisting of measurement devices, a robust state estimation algorithm, and a robust control strategy. This will be the core of a robust freeway traffic management system which not only requires substantially smaller number of traffic sensors but provides reliable estimates of traffic states and produces effective control commands.
Appendix A

The Simplex Optimization Algorithm of Nelder and Mead

Iterative optimization techniques are divided into two main categories; direct search methods and gradient methods. Direct search methods, e.g., the Simplex Optimization Algorithm (SOA), are those which do not require explicit evaluation of any partial derivatives of the objective function, but instead rely on values of the function and information obtained in past iterations. On the other hand, gradient methods such as steepest descent algorithm rely on the computation of the partial derivatives of the objective function. In the following, the original Simplex method devised by Spendley et al. [75] is presented. Then, the revised version of Nelder and Mead [52] will be described.

Let us consider the problem of optimizing an objective function $f$ of $n$ independent variables $\mathbf{x} = [x_1, \ldots, x_n]^T$. In the Simplex optimization algorithm the function $f$ is evaluated at $n + 1$ mutually equidistant points in the space of the $n$ independent variables. These points are said to form the vertices of a regular simplex. A regular simplex in two dimensions, for instance, consists of an equilateral triangle while in three dimensions is a pyramid.

The method is initiated by evaluating the function at each vertex of the simplex according to the following rules:
A.

The simple, Optimization Algorithm of Nelder and Mead

Figure A.1: Simplex search algorithm

i) The vertex at which the function takes on the largest value is determined and reflected in the centroid of the remaining \( n \) vertices. Thus, a new simplex is found.

ii) The function is evaluated at the new vertex and step (i) is repeated.

If the new vertex happened to be the vertex of the function greatest value in the new simplex, the search procedure would cease to progress and mere oscillation between the last two simplexes would occur. To prevent this, a new rule is introduced:

iii) If, at any stage, the vertex selected by (i) is the most recently introduced vertex of the simplex, the vertex of next largest function value is reflected instead.

A typical performance of the procedure with a function of two variables is illustrated in Figure A.1 in which the numbers denote the order in which the vertices were generated. Rule (iii) has been used in the simplex comprising points 5, 7, and 8. Since point 7 is closer to the minimum, the simplex revolves around this point until vertex 12, which coincides with point 9, is reached. Thus, the simplexes repeat themselves and the search falls into an infinite loop. The following rule is, therefore, introduced:
Appendix A. The Simplex Optimization Algorithm of Nelder and Mead

iii) If one vertex of the simplex remains unchanged for more than \( M \) consecutive iterations, reduce the size of the simplex by halving the distance of the remaining vertices from this vertex.

As Spendley \textit{et al.} suggested, \( M \) should depend on the number of independent variables and can be set to

\[
M = 1.65n + 0.05n^2.
\]

The search is terminated when the size of the simplex has been reduced by a specific factor.

While the Simplex method works quite well for many problems, in certain situations its progress is very slow. Such a situation occurs in which the simplex initially has to contract rapidly, e.g., in the case of a narrow valley. This can be efficiently remedied by introducing a simplex with adaptive size. The Simplex method of Nelder and Mead which incorporates such a flexibility is one of the most powerful sequential techniques in direct search algorithms. In this method three basic operations, namely reflection, expansion, and contraction are introduced in dealing with simplexes.

Let \( v_i \) and \( f_i \), \( i = 1, 2, \ldots, n + 1 \), denote the vertices of the simplex and the corresponding function values, respectively. Also, let \( g, h, \) and \( s \) be, respectively, the indices of the vertices of the largest, second largest, and smallest function values and let \( v \) be the centroid of all the vertices excluding \( v_s \). Initially, \( v_s \) is reflected in \( v \) to give a new vertex \( v_r \)

\[
v_r = (1 + \alpha) v - \alpha v_s,
\]

where \( \alpha \) is a positive constant called \textit{reflection coefficient}. \( v_r \) is, therefore, on the line joining \( v_s \) and \( v \) such that \( \overline{v_r v} = \alpha \overline{v_s v} \) in which \( \overline{v_r v} \) denotes the distance between \( v_r \) and \( v \), for instance.

If \( f_h > f_r > f_s \), then \( v_r \) replaces \( v_s \) and the basic iteration continues. If the reflection produces a new minimum, that is \( f_r < f_s \), then it seems worthwhile to investigate whether a further step in this direction will also be successful. Therefore a new point, \( v_e \), along the extended line \( v_s v_r \) is formed where

\[
v_e = \gamma v_r + (1 - \gamma) v,
\]
in which the expansion coefficient \( \gamma \) is defined by \( \gamma = \frac{v_{y}}{v_{x}} > 1 \). If the new function value \( f_{x} \) is smaller than \( f_{y} \), the expansion has been successful and \( v_{x} \) is replaced by \( v_{y} \). Otherwise it will be considered a failure and \( v_{y} \) is replaced by \( v_{x} \).

If reflection gives a point for which \( f_{r} > f_{x} \), the new point corresponds to the largest function value in the new configuration and a contraction is called instead. If \( f_{r} < f_{y} \), \( v_{r} \) replaces \( v_{y} \). A new point \( v_{c} \) is created on the line joining \( v_{y} \) and \( v_{x} \) with \( f_{c} \) the contraction coefficient is defined as \( \beta = \frac{v_{y} - v_{x}}{v_{y} - v_{c}} \), \( 0 < \beta < 1 \). If \( f_{c} < f_{y} \), the contraction is considered successful and \( v_{c} \) replaces \( v_{y} \) and the basic process continues.

Nelder and Mead found that useful values for the new coefficients were \( \alpha = 1 \), \( \beta = \frac{1}{2} \), and \( \gamma = 2 \) which correspond to a simple reflection, halving when a failure occurs, and doubling when a successful direction is located.
Appendix B

Computing the Observability Matrix $Q(x)$ using Mathematica

(* The following mathematica program checks the observability criterion rank(Q)= n for a highway system including three sections. The total number of states is n=6 and the output vector is the traffic velocity and flow at the exit section. *)

(* States *)

x = {x1, x2, x3, x4, x5, x6};

(* speed-density characteristics *)

Ve1 = Vf*(1 - x1/kj); Ve2 = Vf*(1 - x3/kj); Ve3 = Vf*(1 - x5/kj);

(* Section 1 and 2 outgoing flows *)

q1 = alpha*x1*x4 + (1-alpha)*x3*x6;
q2 = alpha*x3*x6 + (1-alpha)*x5*vout;

(* System dynamics *)

fx = {1/L1*(qin - p1 - q1),
     1/tau*(Ve1-x2) + eta/L1*x2*(vin - x4) - nu/L1*(x3-x1),
     1/L2*(q1 - q2),
     ...}
1/\( \text{tau} \)\((\text{Ve2}-x4) + \text{eta}/\text{L2} \times x4 \times (x2 - x6) - \text{nu}/\text{L2} \times (x5-x3) - \text{delta}/\text{L2} \times \text{p1} \times x4, \\
1/\text{L3} \times (q2 - \text{qout}), \\
1/\text{tau} \times (\text{Ve3}-x6) + \text{eta}/\text{L3} \times x6 \times (x4-\text{vout}) \} ; 

(* Output vector *)

hx = \{x5, x6\} ;

(* Form the observability matrix \( Q(x) \) *)

Fx = \text{Outer}[D,fx,x] ;

H1 = \text{Outer}[D,hx,x] ;

H21 = H1[[1]] . Fx + \text{Outer}[D,H1[[1]],x] . fx ;
H22 = H1[[2]] . Fx + \text{Outer}[D,H1[[2]],x] . fx ;

H31 = H2[[1]] . Fx + \text{Outer}[D,H2[[1]],x] . fx ;
H32 = H2[[2]] . Fx + \text{Outer}[D,H2[[2]],x] . fx ;

H41 = H3[[1]] . Fx + \text{Outer}[D,H3[[1]],x] . fx ;
H42 = H3[[2]] . Fx + \text{Outer}[D,H3[[2]],x] . fx ;

H51 = H4[[1]] . Fx + \text{Outer}[D,H4[[1]],x] . fx ;
H52 = H4[[2]] . Fx + \text{Outer}[D,H4[[2]],x] . fx ;

H61 = H5[[1]] . Fx + \text{Outer}[D,H5[[1]],x] . fx ;
H62 = H5[[2]] . Fx + \text{Outer}[D,H5[[2]],x] . fx ;

Q = \{H1, H21, H22, H31, H32, \\
H41, H42, H51, H52, H61, H62\} ;

(* Check the rank condition through computing the null space *)

NQ = \text{NullSpace}[Q]

NQ = \[] ;
Appendix C

Proof of the Theorems and Corollaries of Chapter 4

The proof of the theorems and corollaries presented in Chapter 4 as the background material for the design of the feedback linearizing controller is presented here [69].

**Theorem C.1 (Frobenius)** Let $D$ be an $r$-dimensional distribution on $W$, an open subset of $R^n$. Around any point $p \in W$ there exists a coordinate neighborhood $(U, x_1, \ldots, x_n)$ with $x_i = \phi_i(p)$, $1 \leq i \leq n$, such that $d\phi_i \in D^+$, i.e., for any $f \in D$

$$\langle d\phi_i, f \rangle = 0, \quad \forall x \in U, \quad r + 1 \leq i \leq n$$

if and only if $D$ is an involutive distribution. \hfill \Box

**Proof of Theorem 3.1**

Under the assumptions (i) and (ii), Frobenius' Theorem C.1 can be applied which guarantees the existence of a smooth function $h : R^n \to R$ such that in a neighborhood of the origin

$$\langle dh, ad_{(-i)}^{n-1}g \rangle \neq 0, \quad \text{(C.1)}$$

$$\langle dh, ad_{(-i)}^i g \rangle = 0, \quad 0 \leq i \leq n - 2 \quad \text{(C.2)}$$

145
and $h(0) = 0$. In other words (i) and (ii) guarantee the existence of a solution $h$ for the set of linear partial differential equations

$$(dh, g), (dh, ad_{(-f)} g), \ldots, (dh, ad_{(-f)}^{n-1} g) = (0, \ldots, 0, \gamma(x)) \quad \text{(C.3)}$$

for some smooth function $\gamma(x)$, $\gamma(0) \neq 0$. Define $n$ functions $h(x), L_f h(x), \ldots, L_f^{n-1} h(x)$, and transformation:

$$z = (z_1, z_2, \ldots, z_n) = (h(x), L_f h(x), \ldots, L_f^{n-1} h(x)) \triangleq T(x). \quad \text{(C.4)}$$

$T(x)$ is, in fact, a local diffeomorphism in a neighborhood of the origin. This can be shown by applying the Inverse Function Theorem and verifying that the Jacobian $dT/dx$ is nonsingular. Define the $n \times n$ matrix

$$N(x) = \frac{dT}{dx} \begin{bmatrix} g & \ldots & ad_{(-f)}^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} (dh, g) & \ldots & (dh, ad_{(-f)}^{n-1} g) \\
(d(L_f h), g) & \ldots & (d(L_f h), ad_{(-f)}^{n-1} g) \\
\vdots & \vdots & \vdots \\
(d(L_f^{n-1} h), g) & \ldots & (d(L_f^{n-1} h), ad_{(-f)}^{n-1} g) \end{bmatrix}.$$ 

By definition (C.3), the first row of the matrix $N(x)$ is given by $[0, \ldots, 0, \gamma(x)]$. Now, apply Leibniz’s formula and (C.3) to have

$$\langle d(L_f h), g \rangle = L_f \langle dh, g \rangle + \langle dh, ad_{(-f)} g \rangle = 0$$
$$\langle d(L_f h), ad_{(-f)} g \rangle = L_f \langle dh, ad_{(-f)} g + \langle dh, ad_{(-f)}^2 g \rangle = 0$$
$$\vdots$$
$$\langle d(L_f^{n-1} h), ad_{(-f)}^{n-1} g \rangle = L_f \langle dh, ad_{(-f)}^{n-2} g + \langle dh, ad_{(-f)}^{n-1} g \rangle = \gamma(x),$$

so that the second row of matrix $N(x)$ is $[0, \ldots, 0, \gamma(x), \langle dh, ad_{(-f)}^{n-1} g \rangle]$. Applying Leibniz’s theorem repeatedly, the remaining rows of matrix $N(x)$ can be computed as

$$N(x) = \begin{bmatrix} 0 & \ldots & 0 & \gamma(x) \\
0 & \ldots & \gamma(x) & \langle d(L_f h), ad_{(-f)}^{n-1} g \rangle \\
\vdots & \vdots & \vdots & \vdots \\
\gamma(x) & \ldots & \langle d(L_f^{n-2} h), ad_{(-f)}^{n-2} g \rangle & \langle d(L_f^{n-1} h), ad_{(-f)}^{n-1} g \rangle \end{bmatrix}.$$
which shows that $N(x)$ is nonsingular in a neighborhood of the origin, and, therefore, $z = T(x)$ is a local diffeomorphism. The system

$$\dot{x} = f(x) + g(x)u$$

is expressed in $z$-coordinates as

$$\dot{z}_i = L_i^0 h(x) + L_i^1 L_i^{-1} h(x)u, \quad 1 \leq i \leq n. \quad (C.5)$$

From (C.3), applying Leibniz's formula yields

$$L_s L^i h(x) = 0, \quad 0 \leq i \leq n - 2$$

$$L_s L^n h(x) = \gamma(x),$$

and equation (C.5) becomes

$$\dot{z}_i = z_{i+1}, \quad 1 \leq i \leq n - 1$$

$$\dot{z}_n = L^n h(x) + L_n L^{n-1} h(x)u. \quad (C.6)$$

Now, define the state feedback

$$u = -\frac{L^n h(x)}{L_s L^n h(x)} + \frac{1}{L_s L^{n-1} h(x)} v$$

$$\triangleq k(x) + \beta(x) v, \quad (C.7)$$

and note that $k(0) = 0$ since $f(0) = 0$, and $\beta(0) \neq 0$ since $L_s L^{n-1} h(0) = \gamma(0) \neq 0$. Substituting (C.7) in (C.6) we obtain the Brunovsky controller form (4.2). While condition (i) implies that rank $G_i = i + 1$ for $0 \leq i \leq n - 1$ in $U_s$, from inspecting the matrix $N(x)$ we conclude that $dh \ldots, d(L^i h)$ are linearly independent for $0 \leq i \leq n - 2$, which proves, applying Frobenius's Theorem, that the distributions $G_i$ are involutive for $0 \leq i \leq n - 2$. \hfill \Box

**Proof of Corollary 3.1**

The feedback linearizing transformation may be directly computed as

$$z_1 = x_1$$

$$z_2 = x_2 + \phi_1(x_1)$$

$$z_3 = x_3 + \phi_2(x_1, x_2) + \frac{\partial \phi_1}{\partial x_1}(x_2 + \phi_1(x_1)) \triangleq x_3 + \tilde{\phi}_2(x_1, x_2),$$
and in general
\[
\begin{aligned}
z_{i+1} &= x_{i+1} + \phi_i(x_1, \ldots, x_i) + \sum_{j=1}^{i-1} \frac{\partial^j \phi_i}{\partial x_j}(x_{j+1} + \phi_j(x_1, \ldots, x_j)) \\
&= x_{i+1} + \tilde{\phi}_i(x_1, \ldots, x_i), \quad 3 \leq i \leq n - 1
\end{aligned}
\]

Therefore, the new control command
\[
v = \phi_n(x_1, \ldots, x_n) + u + \sum_{j=1}^{n-1} \frac{\partial^j \phi_n}{\partial x_j}(x_{j+1} + \phi_j(x_1, \ldots, x_j)).
\]

transforms the system (4.3) into the Brunovsky controller form. \hfill \Box

**Proof of Theorem 3.2**

Since the nominal system \((f, g)\) is locally feedback linearizable, Theorem 3.1 implies:

(a) the distributions \(G_i = \text{span}\{g, \ldots, ad^ig\}, \quad 0 \leq i \leq n - 1\) are involutive and of constant rank \(i + 1\) in a neighborhood of the origin \(U_0\);

(b) there exists a function \(h(x)\) such that in a neighborhood of the origin
\[
\langle dh, ad^{n-1}_f g \rangle \neq 0
\]
\[
\langle dh, G_{n-2} \rangle = 0
\]

\[z = (h(x), \ldots, L^{n-1}_f h(x)) = T(x)\]

is a local change of coordinates in which the nominal system \((f, g)\) becomes

\[
\begin{aligned}
\dot{z}_j &= z_{j+1}, \quad 1 \leq j \leq n - 1 \\
\dot{z}_n &= L_n^nh(x) + L_n^{n-1}h(x)u \triangleq v
\end{aligned}
\]

with
\[
u = -\frac{L_n^nh(x)}{L_nL_{n-1}^{n-1}h(x)} + \frac{1}{L_nL_{n-1}^{n-1}h(x)}v;
\]

(c) in \(z\)-coordinates
\[G_i = \text{span}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-i}}\}, 0 \leq i \leq n - 1.\]
Hence, the closed-loop system in z-coordinates is

\[
\begin{align*}
\dot{z}_j &= z_{j+1} + L_q L_q^{-1} h_i, \quad 1 \leq j \leq n - 1 \\
\dot{z}_n &= v + L_q L_q^{n-1} h_i,
\end{align*}
\]

and \( q \) is expressed as

\[
q = \sum_{j=1}^{n} L_q L_q^{n-1} h_i \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} \phi_j \frac{\partial}{\partial z_j}.
\]

The strict triangularity assumption (ii) in z-coordinates becomes

\[
\sum_{j=1}^{n} (\frac{\partial}{\partial z_k} \phi_j) \frac{\partial}{\partial z_j} \in \text{span}\left\{ \frac{\partial}{\partial z_i}, \ldots, \frac{\partial}{\partial z_{n-i}} \right\} \quad n - i \leq k \leq n, \quad 0 \leq i \leq n - 2
\]

which implies

\[
\frac{\partial}{\partial z_k} \phi_j = 0, \quad 1 \leq j \leq n - i - 1, \quad n - i \leq k \leq n, \quad 0 \leq i \leq n - 2
\]

i.e.,

\[
\frac{\partial}{\partial z_k} \phi_j = 0, \quad j + 1 \leq k \leq n, \quad 1 \leq j \leq n - 1
\]

from which

\[
\phi_j = \phi_j(z_1, \ldots, z_i, \theta), 1 \leq j \leq n.
\]

Therefore, the closed-loop system in z-coordinates is given by (4.5). \[ \square \]

In presenting the proof of Theorem 3.3, we make use of the following Lemma.

**Lemma C.1**

If there exists \( i - 1 \) smooth functions \( v_1(z_1), \ldots, v_{i-1}(z_1, \ldots, z_i) \) and a smooth stabilizing state feedback control \( v = v_i(z_1, \ldots, z_i) \) with

\[
v_j(0, \ldots, 0) = 0, \quad 1 \leq j \leq i
\]

for the system

\[
\begin{align*}
\dot{z}_j &= z_{j+1} + \phi_j(z_1, \ldots, z_i, \theta(t)), \quad 1 \leq j \leq i - 1 \\
\dot{z}_i &= v + \phi_i(z_1, \ldots, z_i, \theta(t)),
\end{align*}
\]
such that

\[ V_i = \frac{1}{2} \sum_{j=2}^{i} \left( z_j - v_{j-1}(z_1, \ldots, z_{j-1}) \right)^2 + \frac{1}{2} z_1^2, \]

has time derivative

\[ \dot{V}_n \leq -k_i z_i^2 - \sum_{j=2}^{n} (k_j - n + 1) \left( z_j - v_{j-1}(z_1, \ldots, z_{j-1}) \right)^2, \]

for \( k_i > 0, k_j > i, \ 2 \leq j \leq i \). Then, there exists a smooth stabilizing state feedback control \( v = v_{i+1}(z_1, \ldots, z_{i+1}) \) with \( v_{i+1}(0, \ldots, 0) = 0 \), for the system

\[
\begin{align*}
\dot{z}_j &= z_{j+1} + \phi_j(z_1, \ldots, z_j, \theta(t)), \quad 1 \leq j \leq i \\
\dot{z}_{i+1} &= v + \phi_{i+1}(z_1, \ldots, z_{i+1}, \theta(t)),
\end{align*}
\]

such that

\[ V_{i+1} = \frac{1}{2} \sum_{j=2}^{i+1} \left( z_j - v_{j-1}(z_1, \ldots, z_{j-1}) \right)^2 + \frac{1}{2} z_i^2, \]

has time derivative

\[ \dot{V}_{i+1} \leq -k_i z_i^2 - \sum_{j=2}^{i} (k_j - n) \left( z_j - v_{j-1}(z_1, \ldots, z_{j-1}) \right)^2, \]

for \( k_{i+1} > i \).

\( \square \)

**Proof of Lemma C.1**

Perform the global change of coordinates

\[ \ddot{z}_i = z_1, \ \ddot{z}_j = z_j - v_{j-1}(z_1, \ldots, z_{j-1}), \quad 2 \leq j \leq i + 1 \]

for the system (C.8), and derive the time derivative of

\[ V_{i+1} = \frac{1}{2} \sum_{j=2}^{i+1} \tilde{z}_j^2, \]

which satisfies

\[ \dot{V}_{i+1} \leq -k_i \dot{z}_i^2 - \sum_{j=2}^{i} (k_j - i + 1) \dot{z}_j^2 + \dot{z}_i \ddot{z}_{i+1} + \ddot{z}_i \ddot{z}_{i+1} \]

(C.9)
Appendix C. Proof of the Theorems and Corollaries of Chapter 4

with

\[ \dot{z}_{i+1} = u + \phi_{i+1}(z_1, \ldots, z_{i+1}, \theta(t)) - \sum_{j=1}^i \frac{\partial v_i}{\partial z_j} \dot{z}_j \]

\[ = u + \phi_{i+1}(z_1, \ldots, z_{i+1}, \theta(t)) - \sum_{j=1}^i \frac{\partial v_i}{\partial z_j} \left( z_{j+1} + \phi_j(z_1, \ldots, z_j, \theta(t)) \right) \]

\[ \triangleq u + \phi_{i+1}(Z_{i+1}, \theta(t)) \quad (C.10) \]

where \( Z_{i+1} = [z_1, \ldots, z_{i+1}]^T \). Since \( \phi_{i+1}(Z_{i+1}, \theta(t)) \) is a smooth function and \( \phi_{i+1}(0, \theta) = 0, \forall \theta \in \Omega \), on has

\[ \phi_{i+1}(Z_{i+1}, \theta(t)) = \sum_{j=1}^{i+1} z_j \psi_j(Z_{i+1}, \theta(t)) \]

with \( \psi_j, 1 \leq j \leq i+1 \), smooth functions. Since \( \theta \in \Omega \) there exists a smooth function \( \alpha_{i+1}(Z_{i+1}) \) such that

\[ |\psi_j(Z_{i+1}, \theta(t))| \leq \frac{\alpha_{i+1}(Z_{i+1})}{i+1}, \forall \theta \in \Omega, 1 \leq j \leq i+1. \quad (C.11) \]

Substituting (C.10) in (C.9) with (C.11), one obtains

\[ \dot{V}_{i+1} \leq -k_i \dot{z}_i^2 - \sum_{j=2}^i (k_j - i + 1) \dot{z}_j^2 + \dot{z}_{i+1}(\dot{z}_i + v) + \sum_{j=1}^{i+1} |z_j||\psi_j(Z_{i+1}, \theta(t))| \]

\[ \leq -k_i \dot{z}_i^2 - \sum_{j=2}^i (k_j - i + 1) \dot{z}_j^2 + \dot{z}_{i+1}(\dot{z}_i + v) + \dot{z}_{i+1}\|Z_{i+1}\|\alpha_{i+1}(Z_{i+1}). \quad (C.12) \]

Now define the state feedback control

\[ v = v_{i+1}(z_1, \ldots, z_{i+1}) \]

\[ = -\ddot{z}_i - (k_{i+1} - i + 1)\dot{z}_{i+1} - \frac{1}{4}\ddot{z}_{i+1}\alpha_{i+1}^2(Z_{i+1}), \quad (C.13) \]

and substitute (C.13) in (C.12) to have

\[ \dot{V}_{i+1} \leq -k_i \dot{z}_i^2 - \sum_{j=2}^i (k_j - i + 1) \dot{z}_j^2 + \dot{z}_{i+1}\|Z_{i+1}\|\alpha_{i+1}(Z_{i+1}) - \frac{1}{4}\ddot{z}_{i+1}\alpha_{i+1}^2(Z_{i+1}) \]

\[ \leq -k_i \dot{z}_i^2 - \sum_{j=2}^i (k_j - i) \dot{z}_j^2 - \left( \|Z_{i+1}\| - \frac{1}{2}\alpha_{i+1}|\dot{z}_{i+1}| \right)^2 \]

\[ \leq -k_i \dot{z}_i^2 - \sum_{j=2}^i (k_j - i) \dot{z}_j^2 \]
which concludes the proof.

\[\square\]

**Proof of Theorem 3.3**

Since the nominal system \((f, g)\) is assumed to be locally state feedback linearizable and the strict triangularity assumption holds, Theorem 3.2 applies which guarantees the existence of a local change of coordinates and the state feedback transforming the system into

\[
\begin{align*}
\dot{z}_i &= z_{i+1} + \phi_i(z_1, \ldots, z_i, \theta(t)), \quad 1 \leq j \leq n - 1 \\
\dot{z}_n &= v + \phi_n(z_1, \ldots, z_n, \theta(t)).
\end{align*}
\]  

\[(C.14)\]

First, the case \(n = 1\) is investigated, i.e., for the system

\[
\dot{z}_1 = v + \phi_1(z_1, \theta(t)).
\]

Since \(\phi_1(z_1, \theta(t))\) is a smooth function and \(\phi_1(0, \theta) = 0, \forall \theta \in \Omega\), one can write

\[
\phi_1(z_1, \theta(t)) = z_1 \psi_1(z_1, \theta(t))
\]

with \(\psi_1\) a smooth function. Since by assumption (iii) \(\Omega\) is a known compact set, there exists a smooth function \(\alpha_1(z_1)\) such that

\[
\psi_1(z_1, \theta(t)) \leq \alpha_1(z_1), \quad \forall \theta \in \Omega.
\]

Define the control

\[
v = -k_i z_i - z_i \alpha_1(z_1)
\]

with \(k_i > 0\) and consider the function

\[
V_i = \frac{1}{2} z_i^2.
\]

Its time derivative is

\[
\dot{V}_i = z_i \left(-k_i z_i - z_i \alpha_1(z_i) + z_i \psi_1(z_1, \theta(t))\right)
\]

\[
= -k_i z_i^2 - z_i^2 \left(\alpha_1(z_i) - \psi_1(z_1, \theta(t))\right)
\]

\[
\leq -k_i z_i^2
\]
which proves that the origin $z_i = 0$ is globally exponentially stable and, therefore, $x_i = 0$ is globally asymptotically stable.

Applying Lemma C.1 $n - 1$ times, a smooth state feedback control is iteratively built so that for the closed loop system the positive definite function

$$V_n = \frac{1}{2} z_i^2 + \frac{1}{2} \sum_{j=2}^{n} \left( z_j - \psi_{j-1}(z_1, \ldots, z_{j-1}) \right)^2$$

has negative definite time derivative

$$\dot{V}_n \leq -k_1 \dot{z}_i^2 - \sum_{j=2}^{n} (k_j - n + 1) \left( z_j - \psi_{j-1}(\tilde{z}_1, \ldots, \tilde{z}_{j-1}) \right)^2.$$

Therefore, the equilibrium point $(z_1, \tilde{z}_2, \ldots, \tilde{z}_n) = 0$ is globally asymptotically stable, which implies that the origin $x = 0$ is globally asymptotically stable. \qed
Bibliography


Bibliography


