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Justifying and Proving
in Secondary School Mathematics

by

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for the degree of Doctor of Philosophy
Department of Curriculum, Teaching and Learning
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Abstract

One of the central features of mathematics that distinguishes it from other academic disciplines is the manner in which it develops and accepts new knowledge. Mathematical proof is one of the defining features of mathematics and as such it should be a major part of the secondary mathematics curriculum.

This study examined the beliefs that students have about proof. These beliefs underpin what students believe constitutes acceptable proof. This study also explored their abilities in proof construction and how their beliefs impact on this ability.

Two survey instruments were used to collect data and information. One instrument provided data about beliefs whilst the other provided students with opportunities to evaluate sample proofs and to construct their own.

This study confirms that senior secondary school students have a generally positive view of the role of proof in the curriculum. Although students have difficulty constructing proofs, they do appreciate the value of general arguments and of proofs that offer explanation. However, students also have many misconceptions about proof, its nature and its role.

The results of this study suggest that more curricular emphasis on proof’s different functions and on the process of proving would improve the proving capabilities of our students. It is also evident that there is a great need for rich mathematical tasks that create, in students’ minds, a reason for proof. There is also a need for more study about how proofs that explain can assist our students’ mathematical understanding.
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# Table of Contents

Abstract ........................................................................................................................................ ii  
Acknowledgements .................................................................................................................... iii  

Chapter One .................................................................................................................................. 1  
  Introduction ................................................................................................................................. 1  
  Purpose of the Study ..................................................................................................................... 4  
  The Research Questions .............................................................................................................. 5  
  Significance of the Study ............................................................................................................ 6  
  Limitations of the Study ............................................................................................................. 6  
  Plan of the Thesis ....................................................................................................................... 7  
  Personal Background of the Researcher ..................................................................................... 7  
  Conclusion .................................................................................................................................. 10  

Chapter Two: The Literature Review ........................................................................................... 11  
  Introduction ................................................................................................................................. 11  
  What is Proof? ............................................................................................................................ 11  
  The Role of Proof ....................................................................................................................... 14  
  Summary .................................................................................................................................... 17  
  Proof as Verification or Convincing ......................................................................................... 18  
  Proof as Systemisation .............................................................................................................. 21  
  Proof as Explanation .................................................................................................................. 24  
  Proof as Discovery ...................................................................................................................... 27  
  Proof as Communication of Mathematical Knowledge ............................................................ 28  
  Pedagogical Aspects of Proof .................................................................................................. 29  
    Introduction ............................................................................................................................... 29  
    Student Difficulties with Proof ............................................................................................... 29  
    Student Beliefs about Proof .................................................................................................... 32  
    Student Competency Levels of Proving ............................................................................... 34  
    Making Proof Authentic for Students .................................................................................. 37  
    Constructivism, Discourse and Proof .................................................................................... 38  
    Visual Proof ............................................................................................................................. 39  
    Technology in the Aid of Proof .............................................................................................. 41  
    Proof as Problem Solving ....................................................................................................... 46  
    Further Implications for Teaching ......................................................................................... 48  
    Summary .................................................................................................................................. 51  

Chapter Three: Method ................................................................................................................ 53  
  Research Approach and Design ............................................................................................... 53  
  Development of the Instruments ............................................................................................... 54  
    Student Survey on Justifying and Proving in School Mathematics – Part I ....................... 54  
      Rationale behind the instrument ....................................................................................... 54  
      Pilot of the Instrument ....................................................................................................... 59
Table 6: Comparison of Student Constructed Proofs that Are Algebraic or Prose ........ 74
Table 7: Comparison of Student Ratings of Sample Proofs by Type:
   Algebraic, Prose or Visual ............................................................................ 75
Table 8: Comparison of Student Constructed Geometry Proofs:
   Two Column Format .................................................................................... 77
Table 9: Comparison of Student Ratings of Sample Proofs:
   Two Column versus Other Formats ............................................................ 78
Table 10: Summary of Student Responses to Whether a New Proof Was Necessary
   Given Changes to the Original Proof’s Premise ............................................ 79
Table 11: Summary of Student Ratings of Sample Proofs that are Visual ............... 81
Table 12: Summary of Student Ratings of Sample Algebra Proofs ......................... 83
Table 13: Summary of Student Ratings of Sample Geometry Proofs ...................... 84
Table 14: Summary of Student Ratings of Sample Proofs of False Propositions ........ 85
Table 15: Student Ratings Regarding Proof Using Technology .............................. 87
Table 16: Comparison of Student Responses to Proof Construction
   Items by Gender ............................................................................................. 89
Table 17: Comparison of Student Views on the Role of Proof in
   Convincing them of Validity .......................................................................... 90
Table 18: Comparison of Student Views about Mathematical Understanding ............ 92
Table 19: Summary of the Findings ..................................................................... 94
Table 20: Comparison of this Study’s Findings (T) with
   the London Study’s Findings (L) ................................................................ 109

List of Figures

Figure 1. Diagram of Survey Item G3 ................................................................. 67
Figure 2. Diagram of Survey Item G7 ................................................................. 67

List of Appendices

Appendix A: Student Survey on Justifying and Proving
   in School Mathematics – Part I ................................................................. 122
Appendix B: Student Survey on Justifying and Proving
   in School Mathematics – Part II ............................................................... 132
Appendix C: Instructions for Administering the Survey ..................................... 151
Appendix D: Ethical Review Protocol ............................................................... 156
Chapter One

Introduction

*In most sciences, one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.*

- Hermann Hankel

Mathematics is a subject unlike any other. Its nature is uniquely its own. Is it a science? Or, is it an art? The answer is both. Sullivan (1960) argued that although mathematics has benefited from its association with science and from its useful applications, it is truly significant because it is also an art:

Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our minds it informs us of much that depends on our minds. (p. 2021)

Sullivan traced the real beginnings of mathematics to Pythagoras arguing that the concept of proof signifies the beginning of what constitutes mathematics. Although the Egyptians used geometrical formulas, primarily for utilitarian purposes like land surveying, they were derived empirically. Sullivan stated that these formulae were often wrong because they did not have the concept of proof.

Certainly mathematics serves many functions and it can be both an applied and a pure discipline. Much of its subject matter has been developed in the search to solve a particular problem. Many problems have come to be solved as a result of exploring mathematical patterns for the sheer enjoyment of mathematical exploration. Proof is central to what constitutes mathematics. Often, individuals have particular views of mathematics that are limiting and misrepresentative of the subject. For example, many mistakenly see mathematics as a collection of algorithms used to solve problems, both real and contrived. However, this view of mathematics is one that limits mathematics to a finished product of already determined knowledge. It is a view that ignores the creative “art” of mathematics, the developmental nature of how mathematics comes to be known.
A prover can be likened to an explorer charting new territory. Just as an explorer is uncertain of where he might be headed, so often is the proof maker. Whilst instincts and prior experience can inform both the explorer and the prover, there are elements of unchartedness in both activities that demand creativity and perseverance, insight and skill, and sometimes a bit of good fortune. As an explorer charts new lands or universes, proof plays a pivotal role in validating the creation of new frontiers of mathematical knowledge. As Hoyles (1997) argued:

Proof lies at the heart of mathematics. It has traditionally separated mathematics from the empirical sciences as an indubitable method of testing knowledge which contrasts with natural induction from empirical pursuits. (p.7)

Thus, proof has the role in mathematics of providing a means to judge the correctness of developing mathematics. In its role of verifier, proof has a parallel to science. In science, it is accepted that experiments are the means to test and validate new scientific knowledge. Proof has such a function in mathematics. However, proof is also a creative activity. Constructing a proof requires a certain amount of ingenuity and creativity. In this sense, proof leans towards the art. As mathematics is both science and art, and proof is at the heart of mathematics, it is no surprise then, that proof is both art and science.

Mathematical proof is not limited to the verification of knowledge. Proof also plays a role in helping students to understand why the various algorithms and procedures work. This understanding is essential if students are to adapt and apply the knowledge that they have learned to new situations. In this sense, proof is an enabler. Proof enables students to see the connections between the how and the why of mathematics.

Proof has several other functions. In addition to verification and explanation, proof contributes to the discovery of new results. In the development of a proof, new mathematical knowledge and novel strategies are often developed. This is another creative aspect of proof.

Proof contributes to the organisational structure of mathematics by providing a means to systemise the body of knowledge. Proofs by their nature utilise other known results thus creating a connected web of mathematical knowledge.
Further, proof is the way mathematicians communicate not only the result but also the reasoning behind the theorem. Proof is the manner in which a mathematician demonstrates and communicates his or her work.

Secondary mathematics curriculum planners, recognising the pivotal role that proof plays in the development of mathematics, have quite rightly placed proof as a cornerstone of mathematics curricula. This is reflected in curriculum documents which argue eloquently for its infusion throughout the curriculum. Whilst the treatment of proof varies from textbook to textbook, it is most often given a prominent role. However, this pride of place, the intended curriculum, often has given way to an implemented curriculum that is largely empirical, whether it be the mastery of techniques or the over-reliance on investigations. Consequently, far too often, proof is either non-existent in the school experience or it is limited to the upper secondary school levels and to only the advanced-level student.

Unfortunately, a significant body of research supports the notion that students find proof to be difficult. Further, it seems that the teaching of proof is not approached in a systematic and deliberate fashion. It is often not central to the learning process in many classrooms. It may be prominent in parts of the course but not in others or it may be seen as an enrichment activity only for the most able students. Moore (1994) reported that students' transition to proof is abrupt:

Many students begin their upper-level mathematics courses having written proofs only in high school geometry and having seen no general perspective of proof or methods of proof. Furthermore, at many colleges and universities students are expected to write proofs in real analysis, abstract algebra, and other advanced courses with no explicit instruction in how to write proofs. This abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses. (p. 249).

Is proof difficult for students because they have not had enough developmental experiences with both the process of proving and the nature of proving? Or, is proof just difficult? Anyone who has tried to prove a theorem and has experienced difficulty (in other words, any mathematician!) will agree that writing up a proof is often difficult. However, proof is made more difficult for students when both the nature and concepts of proving are not part of a systematic curricular approach. Too often, proof suddenly appears as part of a senior advanced secondary mathematics course. Students are
expected to appreciate both its nature and role as well as develop some proficiency, all without the benefit of curricular experiences which would help them. The more difficult the subject matter, the more attention that must be made to developing teaching strategies to teach it. Proof warrants such attention.

Currently, there exists a great gap between what seems to be the intention of curriculum writers and what is actually attempted and accomplished by teachers, textbook authors, and students. The good intentions of curriculum documents seem to be rarely realised in our schools. Students need to be part of a classroom in which the daily routine involves justification and proof. If such activities are part of the regular routine they will be seen as a natural and necessary part of doing mathematics.

Given proof’s central and pivotal role in mathematics, it is essential that we provide curricular experiences that help students to develop both an appreciation for the value of proof and strategies for developing proofs. Efforts to do this are underway in current curriculum documents. A greater emphasis on communication and writing-to-learn activities often focus on providing justification for mathematical knowledge. Justification is a word now often linked to proof and perhaps is a step in the development towards proof. There is still much work that needs to be done if the teaching of proof is to be deemed successful.

**Purpose of the Study**

As previously argued, proof has had and will continue to have a central role in the mathematics curriculum. In Ontario, the extent to which proof plays a central role in the everyday classroom experiences of students is ambiguous. Despite being highlighted as one of two process components in the senior division (along with problem solving), proof is often treated by teachers and textbook authors in a haphazard way. The relegation of geometry to a lesser role has further diminished the opportunities for students to learn the process and nature of proof. Consequently it is not clear what students understand about proof’s role in mathematics. It is also not clear what kinds of proving experiences students need to have so that they can develop both an appreciation for the nature of proof but also a level of facility with the process of proof.
The purpose of the study is to examine the state of proof in the Ontario curriculum in grade 12 and the Ontario Academic Credits (OAC) by investigating the ability of students to recognise the qualities of a correct mathematical proof. As well as their ability to construct correct mathematical proofs, the study also examines the students' attitudes and beliefs about proof and their view of proof and its role in mathematics.

Another aim of the study is to make comparisons with a similar study. That study, funded by the Economic and Social Research Council and conducted by the Institute of Education at the University of London, set out to determine the impact of England's National Curriculum on students' views of and competencies with mathematical proof. The London study participants were high-achieving grade-ten students while this study uses older students as the participants.

This study is a similar, but smaller, study focussed on the Ontario curriculum and on students about two or three years older. I believe that this study will add to what has already been learned from the London study and will provide mathematics educators with further insights into how to improve our students' views of and competencies with proof.

**The Research Questions**

This study attempted to answer the following questions:

1. What are the qualities of a mathematical argument that students recognise as a valid mathematical proof?
2. Are students generally proficient in proving and in what ways do students attempt to construct proofs?
3. What is the nature of students' views of proof? For example, what are their views about a proof's form, its roles such as convincing and explaining, and are these views affected by the use of technology?
4. What are the similarities and the differences in the results with respect to the London study?
Significance of the Study

This study seeks to explore what students think about proof. It also seeks to analyse both their understanding of what constitutes acceptable proof and how well they are able to construct proofs.

The study may be useful to curriculum planners who are interested in how to improve the curricular treatment of proof. Because this study attempts to connect student beliefs with what they can actually do, perhaps it will motivate curriculum planners to develop experiences that deal with the nature of proof. What students believe constitutes proof has a significant impact on their proving competency.

The study should also be useful to teachers who are trying to teach proof. A cross-curricular component is perhaps more difficult to implement than a specific content one. Proof should be incorporated throughout the curriculum and as such it has no specific time allotment as, for example, algebra would have. Some of the questions in the instruments could be useful to teachers who are attempting to explain what constitutes acceptable proof. It is perhaps more important to be aware of the various beliefs and misconceptions that students hold, if one is to be successful in the teaching of proof.

Finally, as the study was based on the London study, it may be useful to mathematics educators who are specialists in the domain of proof. There are many similar conclusions and some differences which may add to what has been learned from the London study and offer motivation for further such studies.

Limitations of the Study

The findings in this study are based on a sample of 70 students. These students came from two classes of OAC students and one class of grade 12 advanced-level students in two secondary schools in the same school board. Hence, the conclusions may not generalise to every student in every part of the province.

Whilst there are parallels drawn to the London study, it should be noted that the London study was a funded study on a much larger scale. Whilst comparisons can be instructive, they cannot be definitive due to the differences in the scales of the two studies. Further, the London study’s participants were, on average, two to three years younger than the Ontario students used as participants for this study.
Nevertheless, as discussed in the Methods chapter, the findings may be very
typical of what senior secondary students in Ontario perceive proof to be and what they
are able to do when asked to construct a proof.

Plan of the Thesis

The remaining chapters of the thesis are organised in the following manner:

Chapter Two is the literature review which examines the various roles that proof
plays in mathematics and the aspects of proof which are germane to its teaching in
secondary schools.

Chapter Three discusses the method of research employed including the
development of the instruments used to gather the data. A small pilot study of one of the
instruments is also discussed. The method of analysis of the data gleaned from the
instruments is also discussed.

Chapter Four reports the major findings of the study.

Chapter Five revisits the research questions and uses the findings of the study to
provide answers. The discussion of the research questions is related to the literature
whenever appropriate. Implications for further research are also discussed.

Personal Background of the Researcher

Clandinin (1992) said that teachers have personal practical knowledge that is not
always easy to define but significantly influences their teaching practice. I believe that
this idea can be extended to the teacher-researcher. My personal practical knowledge may
be difficult to articulate but it has certainly influenced many of the ideas in this thesis. As
Schon (1983) argued, this knowledge, which he called, knowing-in-action, is an
unstatable, tacit knowledge that drives our actions. It is worthwhile, therefore, to tell my
story. It is in telling my story, that as Mattingly (1991) argued, one can begin to know the
deep beliefs and assumptions that are implicit in my actions. Often these cannot be stated
explicitly as qualities of one's personality but can be implicitly inferred from the events
and actions of one's professional life.

Mathematics was always a subject which I found to be interesting and most of the
time not too difficult. One of the qualities of mathematics that made me enjoy it is the
fact that one does not have to memorise a lot of facts. I was never good at memorising and found some subjects to be a real chore because there seemed to be a great emphasis on recalling information. Mathematics was not like that – if you could remember a small body of facts and definitions you could deduce from these most of the mathematics that you would need. One’s power of reasoning and a little patience and perseverance was all that was usually necessary.

I learned early on that mathematics was a subject like no other. Not only did it seem to not require a lot of memorising, those who tried to memorise would eventually be found out. It was possible to get by for a while by memorising but eventually you had to understand the concepts. I also soon learned that if you could do mathematics well you were special. The other subjects did not seem to be too choosy when choosing their stars but mathematics seemed to possess a certain standard that was not easily met by many. This became startlingly clear to me in grade eleven. Up until this time, several students would do as well as I would in mathematics. I thought that some of them were better than I was as some would often get better marks on the tests. It wasn’t until we met geometry that I realised I had only one other serious mathematical peer.

In geometry, as Menelaus said, “there is no royal road”. Several of my high-achieving peers could not do as well in geometric proof. I remember that this was the first time I realised that I really had some ability in mathematics and that this ability was not commonplace. I also began to think about how one learns mathematics. I began to realise that students who memorise well can succeed to a certain level in mathematics but eventually fall short. I also came to know that some aspects of mathematics cannot be memorised. Understanding mathematics was more than just learning how to do a procedure.

As I was good at mathematics and seemed to have a knack for explaining it (probably because I struggled always to understand it), I decided on a career as a secondary school teacher. I soon became aware of what Skemp (1976) referred to as the mismatch between student and teacher. It is no surprise that I had students in my class who learned by memorising and since I didn’t teach this way, there were some adjustments that students had to make. Further, I found it difficult to teach students who had been mostly taught to memorise. I soon began to realise that the way in which a
previous teacher has taught the students has a significant influence on how well they and I are going to do together.

Consequently, after three years of teaching, I became a department head. I had decided that if I was to continue teaching, I needed to have more influence on how mathematics is taught. I set out to change the way teachers taught mathematics. I tried to influence teachers to “teach for understanding” even though it was not always easy to define just what I meant by that! As a result, I encouraged teachers to do less telling and more asking. I tried to improve my own practice by working on my questioning ability so that I could draw out more from the students themselves. In my department, we tried to work together, to develop in students the ability to reason, to question, to think critically about what they were learning. To achieve these goals, we emphasised communication and problem solving.

Despite having achieved measures of success in my own teaching practice and also having had influence on other teachers’ practices, I was dissatisfied. There had to be more that could be done to improve the learning of mathematics. I knew that for many students mathematics was a difficult subject and I also knew that my department was working very hard. We needed to work smarter.

This led me to my studies at the Ontario Institute for Studies in Education at the University of Toronto. I was searching for answers on how to improve my efficacy as both a teacher and a leader of other teachers.

My original interest centred on the use of technology. I was interested in how I could use technology to deepen a student’s understanding of mathematics. I soon concluded that technology was useful, if handled correctly, but may not lead to the improvements in understanding that I sought. I then turned my focus towards communication, more specifically, writing. Although I found the writing-to-learn literature interesting, I also became aware that while writing about your thinking processes can help you metacognitively, I didn’t see it as in the mainstream of “doing mathematics”. Finally, I became interested in proof. Proof is perhaps the highest form of communication in mathematics and is certainly in the mainstream of mathematics. However, what really turned me on to proof was the notion of proof as explanation. Even though I had been doing proofs since I was a grade eleven student, I had never thought of
proof in those terms and the use of proof as a way of explaining intrigued me. Perhaps proof was the window with which to study how to deepen my students' understanding of mathematics.

Conclusion

Early in my education, I came to realise that mathematics is unique. I now know that proof is one of the aspects of mathematics that makes it unique. I soon learned that to be really good at mathematics requires that you understand deeply; that memorising does not get you very far. I first became aware that I was a really good mathematics student when our class learned about geometric proof. You could not memorise your way through it; you had to understand it. My dissatisfaction with the type of teaching that leads to students memorising led me to become a department head and my sense of wanting to improve the teaching of mathematics led to my doctoral studies. It was in these studies, that proof resurfaced. Just as I had seen it in grade eleven as unique, I now see that it has a unique place in mathematics and that its role as an explanatory tool has not been fully realised.
Chapter Two: The Literature Review

Mathematics is the most exact science, and its conclusions are capable of absolute proof. But this is so only because mathematics does not attempt to draw absolute conclusions. All mathematical truths are relative, conditional.

- Charles Proteus Steinmetz

Introduction

This literature review begins by discussing what mathematical proof is and what its various functions are. Proof has several roles and its multi-faceted functions have created confusion about its role in mathematics education. The literature review attempts to clarify these functions and provide a background for teaching proof in secondary schools. The final part of the literature review deals specifically with the issues surrounding the teaching and learning of proof.

What is Proof?

Mathematical proof means many things to us. One reason for its multitude of meanings is its different connotations in everyday conversation. For example, in a courtroom, a lawyer must convince or offer proof to the jury that his client is innocent. The standard of “beyond a reasonable doubt” has parallels but also limitations upon comparison with mathematical proof. Other uses of proof include: an impression taken for correction (print); a trial print from a negative (photography); the relative strength of liquor; invulnerable or impervious as in fireproof or waterproof.

Another reason for confusion may also be the different standards of proof in mathematics and its sister discipline of science. Almeida (1996) stated that the standards of proof in theoretical physics do not always meet the standard held by mathematicians. He cited the example of a theorem in quantum mechanics, by a Nobel-prize co-winner in physics, that had to be more rigorously proved by mathematicians before it was acceptable to the mathematical community. Polya (1960) highlighted the difference in proof in science and mathematics:

In mathematics as in the physical sciences we may use observation and induction to discover general laws. But there is a difference. In the physical sciences, there
is no higher authority than observation and induction but in mathematics there is such an authority: rigorous proof. (p. 1982)

In general, a scientist accepts as proof a sufficiency of empirical evidence. To test a hypothesis, she sets up experiments to generate evidence. If sufficient evidence is accumulated to support the hypothesis, and none is found to contradict it, she has proven the hypothesis. Scientific theories developed in this manner are accepted as provisionally true. They are subject to modification as new evidence or theories which challenge the hypothesis come to light. A scientific proof is in this way similar to that of a courtroom in that a reasonable amount of doubt is implicit.

Mathematical proof differs from scientific and judicial proof in that it must meet a more stringent criterion of "truth". Yet, within the mathematical community itself, there exist much debate about "what is truth?" and hence "what is proof?". As the knowledge base of mathematics grows and becomes more specialised, there is less communication between the specialties. Almeida (1996) reported that standards of proof vary within the mathematical community due to the sometimes autonomous development of mathematical specialities and their subsequent isolation from each other. Kleiner and Movshovitz-Hadar (1990) said that the manner in which a mathematician proves is often a reflection of his or her "overall view of mathematics" (p. 28). As there are different views of mathematics then it follows that there will be different notions of and standards of proof within the mathematical community. Notions of the bedrock position of mathematics as certitude continue to be challenged. New methods and strategies, in particular the role of technology and the use of probabilistic proofs, further challenge the argument that mathematics is truth. Hersh (1990) argued that:

there will continue to be different varieties of proof, different levels of rigor (p. 394)

It is not surprising that as new strategies for doing mathematics emerge, the role of proof is much debated. As proof is central to the act of doing mathematics, changes in the way mathematics is done, will necessarily challenge our view of proof. Challenges to proof are in fact testimony of its central role in mathematics.

The culture of the mathematical community is evolving and so too is its standard of acceptable proof. Almeida (1996) pointed out that the mathematical community
determines what level of detail is necessary, what methods are allowed, what axioms can be assumed without proof, and so on. These standards are subject to change whenever the community feels it advantageous. In this sense, notions of proof as absolutist or non-fallibilist are not in keeping with proof in the practice of mathematicians.

Consequently, defining mathematical proof is fraught with ambiguity and controversy. Further complicating the issue is the multitude of types of proof. Some more common types include direct proofs by deduction, by exhaustion, by induction, by construction, as well as indirect proof by contradiction which can include methods such as infinite descent. Each proof type has its own peculiarities necessitating a general definition.

I shall consider that a proof is a reasoned argument from accepted truths. Accepted truths are those notions, axioms, theorems and mathematical conventions and methods accepted as true by the mathematical community.

What then constitutes acceptable proof? The answer is it depends. Proofs in formal logic or pure mathematics differ from proofs in graph theory which differ again from proofs in the physical sciences. What constitutes acceptable proof for a grade five student is different than for a secondary school student. Proof in schools serves a different function than proof in the larger domain of the mathematical community. Maher and Martino (1996) argued that in mathematics education we are more interested in student understanding. This is achieved through a continuum of explanation via argument and justification to proof and “the distinctions between them are not always sharp” (p. ).

As this is a thesis about students and proof, what should proof be to students? Austin (1995) expected that a successful mathematics education is one in which the student appreciates that:

- a mathematical proof is an argument just like other everyday arguments – it involves everyday, sensible thinking
- and
- that a mathematical proof is an argument unlike all other arguments – it involves an absolute certainty that other arguments lack. (p. 76)

Austin summarised the views of many in the mathematical community. Whereas there are commonalities between mathematical proof and other disciplines’ proofs, proof in mathematics holds a special and privileged place. As previously stated, proof “lies at the
heart of mathematics" and many argue that proof sets mathematics apart from other disciplines. Gardiner and Moreira (1999) claimed that:

Mathematics is not proof; mathematics is not spotting patterns; mathematics is not calculation. All are necessary; none is sufficient. (p. 19)

Therefore, unless proof is an integral part of the mathematical experiences of students, their mathematical experience will be, at best, incomplete, inauthentic and limiting. Proof, while not the whole of mathematics, is necessary for the full and complete mathematical experience.

The Role of Proof

As previously argued, proof plays a central role in mathematics. But what is its role? Within the mathematical community, there is debate about its role. Until recently, the common view of proof was as a mathematical truth. Davis and Hersh (1981) presented a dialogue between an “ideal mathematician” and his student. They defined the ideal mathematician as “the most mathematician-like mathematician”. They stated that despite his work being unappreciated and misunderstood by the general population, nevertheless he maintained a high regard for his own work:

However, the mathematician regards his work as part of the very structure of the world, containing truths which are valid forever, from the beginning of time, even in the most remote corner of the universe. (p. 34)

Many mathematicians believe that each proof contributes to a body of knowledge, mathematics, which is absolute. There is great appeal in such an argument; it is comforting to know that within the confines of a mathematical system there is certainty. This is juxtaposed against almost every other discipline in which knowledge is less certain. Kline (1953) disagreed:

Mathematics is a body of knowledge. But it contains no truths...not only is there no truth in the subject, but some theorems in some branches contradict theorems in others. For example, some theorems established in geometries created during the last century contradict those proved by Euclid in his development of geometry. (p. 9)

Resnik (1992) argued that it is:
a popular view that no result has been fully demonstrated until it has been derived from an accepted set of axioms within an accepted formal system. (p. 12)

Resnik (1992) disagreed with the "popular view" noting that even within an idealised axiomatic system problems occur which threaten the chain of truth. He referenced Russell and Whitehead who conceded that their axioms of infinity and irreducibility were not "intuitively plausible".

An emphasis on proof as truth is linked to the notion of proof as rigorous proof. Otte (1994) argued that rigorous proof with its heavy emphasis on logic and logical symbols gives a false view of mathematics. He argued that the fact that one could, within the language of logic, prove theorems independent of any outside reality, did not negate the fact that mathematics is integrated with reality. Mathematics in this context is seen as a style of reasoning outside of any physical meaning. Otte challenged this view with the argument that mathematics is related to objects and hence is contextual and as such its truth is provisional.

Hence, this view of proof as truth, upon greater scrutiny, is found wanting. Tarski (1968) argued that the axiomatic method grew out of a desire to restrict the use of intuitive evidence which is fallible. Tarski pointed out that until the end of the 19th century, the only criterion that proofs had to meet was to be intuitively convincing. Confidence in proof was shaken probably upon the discovery of non-Euclidean geometries which undermined what was thought to be known about geometry. Tarski (1968) stated:

The belief that formal proof can serve as an adequate instrument for establishing truth of all mathematical statements has proved to be unfounded.... On the other hand, the notion of proof has not lost its significance either. Proof is still the only method used to ascertain the truth of sentences within any specific mathematical theory. (p. 77)

Chazan (1990) characterised the debate as one in which rationalist philosophers have perpetuated a stereotypical view of the subject of mathematics and hence proof:

The views of mathematics knowledge and of mathematics presented above are over-simplifications of philosophical rationalism. While they reflect certain truths about mathematics, they misrepresent mathematicians' reports of other aspects of mathematical experience – its social dynamics, creativity, and intellectual beauty. As a result, mathematicians and historians of mathematics have criticized
rationalist philosophers for not accurately reflecting the uncertainty, irrationality, intuition, and exploration which characterize the everyday lives of mathematicians. (p. 14)

Chazan went on to suggest a quasi-empirical approach is more in keeping with the current practice of mathematicians. He argued that proof is a social process of agreement after a great deal of inquiry and conjecturing.

One can engage in a spirited debate between absolutism and fallibilism. Suffice to say, that proof has a role in determining mathematical truth, whatever mathematical truth is. It remains, as Tarski (1968) noted, the best method to certify newly discovered mathematics.

Notwithstanding these arguments that proof is not absolute, there is an aesthetic argument to be made for proof. Proof has a role as an object of beauty. Like a painting in a gallery, a polished proof can be beautiful. An elegant proof can be appreciated, like the painting, as an example of human achievement. In the introduction to *The Moment of Proof—Mathematical Epiphanies*, Donald Benson (1999) gives four criteria for elegance. An elegant proof is one in which the mathematics has importance; the proof is subtle but not obscure; it is not overly long; and it must have an element of surprise. Number theorist G. H. Hardy certainly appreciated both the elegance and substance of proof. He described a good proof in the following way, (Hoffman, 1998):

There is a very high degree of unexpectedness, combined with inevitability and economy. The argument takes so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching consequences; but there is no escape from the conclusions. (p. 43)

The mathematician Paul Erdös, considered by many as the most prolific mathematician after Euler, had doubts about the existence of God but he was resolute in his belief in God’s Book. The Book contained the best proofs of all mathematical theorems, proofs that are elegant and perfect. His highest compliment about another mathematician’s proof, was the statement “It’s straight from the Book”. Erdös appreciated the beauty and aesthetic qualities that a mathematical proof offered and this was his way of showing approval to a colleague’s work. He would go on to say, “You don’t have to believe in God, but you should believe in the Book”.


Summary

In summary, proof's lofty place as the holder of mathematical truth may be tenuous because of the debate about the notion of mathematical truth. However, if we accept that there can be mathematical truth, surely proof is the ultimate evidence of such truth. Finally, proofs can be beautiful and beauty is often justification enough for the inclusion of subject matter in realms such as the arts and literature. Hence, proof could be justified as a worthy curricular topic on that basis alone.

However, proof is much more than that. Its several other functions which will be discussed in greater detail in the following sections. Luthuli (1996) enumerated the several functions of proof:

Proof in mathematics serves a number of functions or roles and the following functions of proof can be listed:

i) Verification – certification of correctness
ii) Explanation – answering the question, Why?
iii) Convincing – settling doubt
iv) Systemisation – formalizing mathematical knowledge
v) Discovery – the invention of new results
vi) Communication – the transmission of mathematical knowledge
vii) Meeting intellectual challenge – Satisfying curiosity and; attempting to answer the questions like: Fine, but how can it be done differently? Is this the best way to do it? Is there an easier way of doing it? In this proof there are some gaps, can I provide the missing details? In this proof there are some loose arguments, can I supply a tighter argument?; etc.

(p. 19)

This delineation of the functions of proof is a useful way to organise a discussion about proof. It should be noted, however, that although these functions can be distinct, within a specific proof, several functions are often served. Using these functions of proof, as enumerated by Luthuli, the next part of the literature review is organised in the following manner:
Proof as Verification or Convincing
Proof as Systemisation
Proof as Explanation
Proof as Discovery
Proof as Communication of Mathematical Knowledge
The final function, meeting intellectual challenge, is discussed in several parts of the thesis, but does not have its own section. The literature review continues:

**Proof as Verification or Convincing**

Proof as verification is also sometimes included in the literature as proofs that convince or proofs that justify. Hersh (1993) stated that "proof is convincing argument, as judged by qualified judges" (p. 389). Writing proofs is the way that the mathematical community inculcates new mathematical knowledge. Hersh stated that most often the only way that mathematicians have to test their work is to submit their proof and have the mathematical community scrutinise it. This is part of the verification aspect of proof. Sierpinska (1994) stated that:

proof aims at increasing the degree of firmness with which we accept a fact as basis for our understanding. (p. 75)

de Villiers (1990) explained that the traditional function of proof is almost exclusively one of verification. A proof removes doubt and silences skeptics. He provided a selection of views on the role of proof:

only a testing process that we apply to these suggestions of our intuitions (Wilder, 1944)

only meaningful when it answers the student's doubts, when it proves what is not obvious (Kline, 1973)

the necessity, the functionality, of a proof can only surface in situations in which the students meet uncertainty about the truth of mathematical propositions (Alibert, 1988)

an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing (Hanna, 1989)

(de Villiers, 1990, p. 17)

It is clear that a significant aspect of proof's role in the realm of mathematics is that of verification of mathematical results. This has historical roots dating back at least to the time of the ancient Greeks. In *Mathematics in Western Culture*, Morris Kline (1953) explained that the ancient Greeks were gifted in philosophy, they loved the art of
reason and this delight in mental gymnastics distinguished them as a people. They, therefore, insisted that “all mathematical conclusions be established only by deductive reasoning” (p. 46).

This insistence ignored results which were derived by any non-deductive means. In one sense limiting but in another enlightening, these historical roots of mathematical culture have led us to a unique discipline of mathematical reasoning. This discipline has as its primary tool, proof. Proof was (and continues to be) the means for building and verifying a body of mathematical knowledge.

Proof as verification is often associated with “formal proof”. Blewitt and Costello (1996) reported that for most of the last century mathematics has been dominated by formalism. Formal proof has had an exalted place in the mathematical culture, often considered “real proof” that is rigorous and certain. It is here that the distinction between *proof as verification* and *proof as convincing* can be seen. Formal proofs may provide certitude but they are often unconvincing on an intuitive level. Often steeped in logical symbols and operators, their notation and style prevent, for many, an internal or personal conviction. It is for this reason that many have advocated less formalisation. Hanna (1983) stated that rigorous proof, that is, “proof that satisfies the requirements of proof in an axiomatic system” (p. 66), has never, in professional mathematical practice, been as important as understanding and significance. Blum and Kirsch (1991) argued in favour of what they called “pre-formal proofs” as an alternative to formal proof. A pre-formal proof is a “chain of correct, but not formally represented conclusions which refer to valid, non-formal premises” (p. 187). Whilst such a proof has to be a valid and rigorous proof, it is also more natural and utilises intuitive concepts so that it is more satisfying as a convincing proof. Kline (1980) argued that all the discussion about rigor is problematic in that we do not have a rigorous definition of rigor. He wondered how many fundamental assumptions are made without explicit acknowledgement and hence the whole notion of rigorous proof is compromised by our inability to be absolutely explicit within an axiomatic system. At some point we come to what Euclid called “common notions” and undefinable terms.

Hence, the role of proof as verification of mathematical results is limiting. If presented as the only or primary role of proof it is also a fundamentally dishonest
rationale for proof. Hersh (1993) stated that we have other means than proof for our conviction that something is true:

Belief can come from examples and special cases, from analogy with other results, from an expected symmetry or an unexpected elegance, even from an inexplicable feeling of rightness. All these illogical logics may tell us "it is true!" (p. 395)

Segal (1999) stated that one normally arrives at conviction by other means than logical proof. She made a distinction between conviction and validity. One convinces oneself, by a variety of means including but not limited to proof, but a result is validated by a formal approach known as proof. Fischbein (1982) stated that frequently the conviction that comes from a "formally certain proof" does not result in more subtle feelings of "I feel it must be so". de Villiers (1995) discussed the various roles of proof stating that the verification aspect of proof is not as important as it might first appear. He stated that one usually has some form of a priori conviction before attempting a proof. One usually does not try to prove something unless one is reasonably convinced that it is true. Kline (1973) also argued that proof's place was at the end, not the beginning. He saw proof as removing doubt, if there was any, and in most cases, there is not.

Therefore, if one justification for including proof in the curriculum is because it is part of authentic mathematical practice, then pretending that the truth of the result to be proven is unknown, is inauthentic. However, providing curriculum experiences in which part of the process is to establish whether or not a proposition is valid can be a worthwhile way to allow students to experience what doing real mathematics is like. Whilst school mathematics is traditionally "beyond doubt", providing students with situations in which the pattern fails after several successes would go a long way towards making their proving experiences more authentic.

Blum and Kirsch (1991) argued that naive empiricism, that is, the use of specific examples, convinces and is a necessary first-step towards developing a proof. However, Chazan (1993) pointed out that an a priori conviction can lead to problems. He discussed the misconception that "evidence is proof". In other words, several examples are enough to convince someone of the veracity of the result, but is this a proof? The answer is, of course, no. Yet, Chazan has pointed out a pitfall in presenting proof solely as verification. If one is convinced with examples, and proof is "being convinced", then are not examples
enough? He also explained that there are also misconceptions of "proof as simply evidence". Stated differently, one believes that the proof is only valid for the associated diagram. One does not know that the diagram is simply a representation of all the figures that satisfy the given information. He cited Schoenfeld's observation that students often correctly prove a result only to violate the result in subsequent work. They either did not really believe the result or did not recognise the meaning of having proved it. Chazan explained that students often are mistrustful of a proof to the extent that they check it empirically.

Proof as verification is therefore a bit dichotomous. Proof remains the principal way that mathematical knowledge is developed, verified and communicated. As such, its inclusion in the curriculum is necessary if the mathematics curriculum is to mirror the reality of how mathematicians do mathematics. And yet, proof is more than verification. If proof is presented to students as only verification, it will be seen as redundant because most proving activity in schools is limited to results that are already known to be true. It is therefore, necessary to present students with mathematical tasks that create in their view a need for proof.

** Proof as Systemisation

Another role of proof is its function as a means of connecting mathematical results into an integrated body of knowledge. Thurston (1995) stated a series of questions whose aim was to get at what it is that mathematicians accomplish. Throughout, the value of proof is implicit: "that progress made by mathematicians consists of proving theorems" (p. 29).

Proof plays a central role in the building up of mathematical knowledge, that Thurston (1995) claimed is increasing at an amazing rate. This is:

because we have a high standard for clear and convincing thinking and because we place a high value on listening to and trying to understand each other, we don't engage in interminable arguments and endless redoing of our mathematics. We are prepared to be convinced by others. Intellectually, mathematics moves very quickly. (p. 33)

This process and the accepted conventions of reasoning, begun by the ancient Greeks and perfected by mathematicians ever since, provides mathematicians with a way
to develop and communicate mathematical knowledge. Thurston detailed his own experience working with proof in the area of 3-dimensional manifolds. The process of developing the proof and then presenting it involved a considerable amount of effort and both self- and community-verification. Not surprisingly, not unlike Andrew Wiles’ need to prove the Shimura-Taniyama conjecture in order to prove Fermat’s Last Theorem, Thurston’s proof is used to continue the building up of mathematical knowledge:

Mathematicians were actually very quick to accept my proof, and to start quoting it and using it based on what documentation there was, based on their experience and belief in me, and based on acceptance by opinions of experts with whom I spent a lot of time communicating the proof. (p. 37)

In actual practice, although proof clearly has a verification role, as de Villiers (1990) noted:

the main objective clearly is not to check whether certain statements are really true, but to organize logically unrelated individual statements which are already known to be true, into a coherent unified whole. (p. 21)

Proof functions as a means to systemise known results into a logical system of definitions, axioms and theorems. Some of the most important functions of systemisation as noted by de Villiers (1990) are:

it helps with the identification of inconsistencies, circular arguments and hidden or not explicitly stated assumptions

it unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results

it provides a useful global perspective or bird’s eyview of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived

it is helpful for applications within and outside mathematics, since it aids checking the applicability of a whole complex structure or theory simply by evaluating the suitability of its axioms and definitions

it often leads to alternative deductive systems which provide new perspectives and/or are more economical, elegant, and powerful than existing ones

(de Villiers, 1990, p. 20)
There are several examples of theorems whose proof serves more the need for systemisation than for verification. de Villiers cited the Intermediate Value Theorem as a result which appears self-evident upon simple examination of a representative graph. For many, the proof is not necessary as a verification of the result, but as a way of completing and organising the body of mathematical knowledge. Proof, therefore, serves not only a function of verification, but also of systemisation.

Jones (2000) stated that proof as systemisation provides an overview of the subject as a field of intricately related structures. However, Rav (1999) discussed the systemisation of mathematics by contrasting two models. The skyscraper model is the view of mathematics built upon secure foundations, each level of the skyscraper is built upon the previous layer. The other model, the spaceship model, is based upon the assemblage of several components each of which has been separately tested.

Rav argued in favour of the latter model stating that the skyscraper model "has lost its credentials" given its foundation on logical axioms which may or may not be true. However, it would seem that both models have a place as models of systemisation. The skyscraper model seems apt for a specialisation within mathematics, like planar geometry for example. Certainly, the work of Euclid has not been rendered invalid and his Elements, albeit with some flaws, remains a masterpiece of systemisation. The spaceship model seems appropriate when several branches of mathematics come together to prove a result. In the proving of Fermat's Last Theorem, several seemingly unconnected branches of mathematics were assembled to prove the theorem. Who could foresee that semi-elliptical curves would, with the aid of the Shimura-Taniyama conjecture, be connected to that old chestnut of number theory, Fermat's Last Theorem?

Rav (1999) argued:

In general, from the perspective of systemic cohesiveness, in constructing a proof we avail ourselves of constituents whose reliability has already been tested, parts of existing mathematical knowledge whose coherence results from previous proofs. Thanks to the logical components of proofs, the outcome of the new proof can now be safely added *qua warranted assertions* to the fabric of mathematical knowledge. (p. 30)

The achievement of Andrew Wiles is really an achievement of the mathematical community of the last three centuries. Wiles' achievement rests upon the work of several
mathematicians who went before him. The proof of Fermat's Last Theorem, is a high-profile and recent example of the role of proof in the systemisation of mathematics. Its achievement could not have occurred without the careful systemisation of mathematics.

**Proof as Explanation**

Perhaps, the most significant function of proof is its role in answering the question why? Hanna (1990) stated that:

A proof is valued for bringing out essential mathematical relationships rather than for merely demonstrating the correctness of a result. (p. 8)

Hanna used the terminology "proofs that explain" to signify a proof that proves but also explains why and "proofs that prove" as proofs that prove but do not have a high degree of explanation. Proofs that explain are more valuable to the mathematical community than proofs that merely prove. As argued previously, proofs that serve only a verification role have limited value. Proofs that explain are also of much more interest to mathematics educators. Proofs that do not explain well would generally be of little use in secondary school mathematics classrooms. As Hersh (1993) stated:

There are different kinds of explanation. A proof is a complete explanation. Sometimes a partial explanation suffices. Sometimes we skip the proof, if a lemma or theorem seems clear enough on its own. (p. 397)

This sentiment is summed up by the oft-quoted Manin (1977, quoted in Coe and Ruthven (1994)):

A good proof makes us wiser. (p. 42)

Not all proofs are explanatory. Certainly many formal proofs, while serving the function of verification and/or systemisation, do not generally, explain well. We may know that the result is true but we have very little notion of why. The proof of the four-colour problem by Appel and Haken, using a computer to verify sub-cases, convinces most people that the result is true. However, it is unsatisfying because it fails to explain why the result is true. Hersh (1993) quoting Paul Halmos discussing the Appel and Haken proof:
I do not find it easy to say what we learned from all that. We are still far from having a good proof of the four-colour theorem. I hope as an article of faith that the computer missed the right concept and the right approach. Hundred years from now the map theorem will be, I think, an exercise in a first year graduate course, provable in a couple of pages by means of appropriate concepts which will be completely familiar by then. The present proof relies in effect on an Oracle and I say down with Oracles! They are not mathematics. (p. 393)

Halmos, like Manin, considers a good proof to be mathematically significant because it makes us more knowledgeable. Proofs that only prove, as in the Appel and Haken proof, do not hold much value either aesthetically or informatively. The Appel-Haken proof's claim of verification is also not as strong as other proofs because it is impossible for other mathematicians to verify the proof line by line.

Similarly, if for no other reason than its length, Wiles' proof of Fermat's Last Theorem would not be considered explanatory. Whilst it has been verified, only a handful of mathematicians have waded through the entire proof, and so its validity is less sure than other shorter proofs. One might be tempted to argue, as Halmos did, that perhaps a good proof is still out there waiting to be found. If not Fermat's "proof", perhaps a more explanatory version than Wiles has provided.

de Villiers (1990) argued that one can achieve a level of conviction without proof and the implication is that if proof serves only to convince but not to explain, it is not very satisfying:

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (e.g. accurate constructions and measurement; numerical substitution, etc.), this generally provides no satisfactory explanation why it may true. It merely confirms that it is true, and even though the consideration of more and more examples may increase one's confidence even more, it gives no psychological satisfactory sense of illumination, i.e. an insight or understanding into how it is the consequence of other familiar results. (p. 19)

What is an explanatory proof? Steiner (1978, page 143) cited by Hanna (1989) described an explanatory proof in the following way:

an explanatory proof makes reference to the characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property. It must be evident, that is, that if we substitute in the proof a different object from the same domain, the theorem collapses; more,
we should be able to see as we vary the object how the theorem changes in response. (p. 48)

Hanna (1998) observed that the need for proofs that explain, as opposed to simply convince, was expressed as early as the 17th century. Arnauld and Nicole, who studied and admired Euclid, also criticized him. In their book *La logique ou l'art de penser* (first published in 1674), they argued that the majority of proofs in *The Elements* are proofs which *convainquent* (convince) rather than proofs which *éclairent* (clarify). Proofs which clarify have the same meaning as proofs that explain. Barbeau (1990), used slightly different terminology than Hanna but argued similarly for the value of proofs that explain. He cited Euler who routinely offered several proofs of the same result. Certainly, several of Euler’s proofs meet the criteria for elegance discussed earlier in this review (see The Role of Proof). Euler, and other mathematicians like him, are not content to have a proof that only convinces. As Barbeau (1990) stated:

What accounts for this tendency to keep grappling with a result that should have been settled once and for all? I believe that it is the true intention of the researcher to establish not just the correctness of a result but also its significance. It is not enough to believe the result; one must also be convinced that it is worth knowing. (p. 26)

It is this sense of being *worth knowing* that meets Steiner’s and Hanna’s criteria of connecting to the mathematical relationships of which it is a part. In the case of the four-colour problem and Fermat’s Last Theorem, we continue to “keep grappling” hoping that some day, a better proof will be found.

Proofs that explain have particular significance for school mathematics. Certainly, one hopes that school mathematics encompasses mathematics that is worth knowing. Moreover, proofs that explain should enjoy more prominence as a result of a shift in emphasis in the way mathematics is taught and learned. Hersh (1993) stated:

Mathematical proof can convince, and it can explain. In mathematical research, its primary role is convincing. At the high-school or undergraduate level, its primary role is explaining. (p. 398)

Hence, in school mathematics, the main purpose of proof is understanding. Defining what it means to understand is an elusive concept. We know it when we see it but have difficulty in its articulation. We do not fully understand what it means “to
understand but we can certainly teach in a way which fosters understanding. Proofs that explain play a significant role in fostering understanding.

The implications of proofs that explain and their role in fostering understanding and how they can be used in the mathematics curriculum, particularly in the secondary school, will be explored in a later section.

Proof as Discovery

Lakatos (1976) saw proof as a means to discover new mathematical results. Proofs that serve only to verify do not very often lead to new knowledge but proofs which seek to explain can also lead us in directions originally not thought of. The growth of informal mathematical theories follows from a process of conjecturing, trying to prove, finding counterexamples and refining the tentative proof. Davis and Hersh (1981) argued that good proofs lead inevitably to the development of new mathematics:

Proof, in its best instances, increases understanding by revealing the heart of the matter. Proof suggests new mathematics. The novice who studies proofs gets closer to the creation of new mathematics. (p. 151)

Rav (1999) provided two case histories, one in number theory and the other in set theory to illustrate his contention that proofs play an intricate role in the development of new mathematics:

They illustrate the intricate role of proofs in generating mathematical knowledge and understanding, going way beyond their purely logical-deductive function. (p. 6)

Resnick (1992) cited Euclid’s Theorem and the proof of the infinitude of primes, as an example of opening up new avenues of mathematical knowledge. He said that the proof reformulates the data in a way that encourages us to draw new conclusions. Another example is the history of the solution of Fermat’s Last Theorem. Several new theories were conjectured and proven in the quest for this proof. Fermat’s legacy is not the theorem itself; rather, it is the quest for its proof that has generated so much mathematical knowledge and insight.

Hence, proof serves as a catalyst for the growth of mathematical knowledge. As is often the case, in looking for the proof, we find other interesting mathematics.
Proof as Communication of Mathematical Knowledge

Thurston (1995) stated that proofs are the mathematicians way of communicating mathematical ideas. He argued for taking greater care in the construction of the proofs, not only with definitions and theorems, but also with our ways of thinking. In such a way, a proof will communicate not only a result but the thinking that led to it:

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrisation conjecture for Haken manifolds. (p. 37)

In its best form, a proof explains the reason behind the theorem, the thinking that came together to create the proof. Cuoco, Goldenberg and Mark (1996) stated that a proof shows how mathematical ideas are connected. It is in making connections that one explains the result and this is both intellectually satisfying and often leads to new results. A proof also communicates our "habits of mind" that led us to the result. As Thurston (1995) has argued, this is of greater importance than verification. Beaulieu (1990) discussed the contribution of the Bourbaki. She stated that contrary to the view that the Bourbaki were overly concerned with formalism, their original goal was to summarize mathematical knowledge into a reference book:

Bourbaki's original goal, and the reason why the group was created, was to present mathematicians with the tools of their trade in the form of a treatise of analysis which could serve both as a teaching-learning book and as a reference for mathematical researchers, producers and users of mathematics alike. (p. 37)

Whereas the Bourbaki may never have realized their original goal it is interesting to note that their original goal was the communication of mathematical knowledge. In fact, one can draw a parallel to Euclid. Certainly Euclid's *Elements* are as much about mathematical communication as about systemisation or verification. Euclid, through the *Elements*, succeeded in organising and communicating the known mathematics of his time.

Hence, communicating mathematical results as well as the techniques and habits of mind that were used to arrive at them is another role of proof. Rav (1999) likened the statement of the theorem to a tag, that is, a label for the proof, like a headline for a news story. It is the proof that conveys the mathematical knowledge:
The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systemisation of results — the entire mathematical know-how is embedded in proofs. (p. 20)

**Pedagogical Aspects of Proof**

**Introduction**

Velleman (1994) made a comparison between how one does a jigsaw puzzle and how one constructs a proof:

Proofs are a lot like jigsaw puzzles. There are no rules about how jigsaw puzzles should be solved. The only rule concerns the final product: All the pieces must fit together, and the picture must look right. The same holds for proofs. (p. 82)

The teaching of proof is complicated by the fact that it is not easy to communicate to students a strategy for proving. Making proof accessible to students is the focus of this section. Proofs multi-faceted functions as discussed in earlier sections will underpin the discussion that ensues.

An outline of the subtopics in this section follows:

- Student Difficulties with Proof
- Student Beliefs about Proof
- Student Competency Levels of Proving
- Making Proof Authentic for Students
- Constructivism, Discourse and Proof
- Visual Proof
- Technology in the Aid of Proof
- Proof as Problem Solving
- Further Implications for Teaching

The literature review continues:

**Student Difficulties with Proof**

There is no shortage of literature indicating that most students are not successful with proof. Fischbein (1982) concluded, based on a study of 400 high school students, that less than 15% of them really understood what constitutes a mathematical proof. This led him to make the rather provocative statement:
The concept of formal proof is completely outside the mainstream of behaviour. (p. 17)

Senk (1985) reported on a study of 2,699 students in 99 geometry classes from 13 schools in 4 states in the United States. Approximately 62% of the participants were at the grade ten level with about 19% in each of grades nine and eleven. She found that:

At the end of a full-year course in geometry in which proof writing is studied, about 25 percent of the students have virtually no competence in writing proofs, another 25 percent can do only trivial proofs, about 20 percent can do some proofs of greater complexity; and only 30 percent master proofs similar to the theorems and exercises in standard textbooks. (pp. 453-454)

More recently, the situation appears to not have changed too much. Healy and Hoyles (1998) reported, after a study of 2,459 Year 10 students from 94 classes in 90 schools in England and Wales that:

High-attaining Year 10 students show a consistent pattern of poor performance in constructing proofs. (p. 2)

The story is not much different when one considers university students. Moore (1994) found that even trivial proofs proved to be major challenges for university undergraduate students. Jones (2000) quoted several studies which showed that university students, including those being educated to be mathematics teachers, have difficulties with understanding proofs, appreciating their value and nature, as well as constructing them. He lamented the "vicious circle" of newly graduated secondary school mathematics teachers who do not understand the nature and role of proof, being charged with the responsibility of teaching secondary students about proof.

Hence it is clear that proof is a problem for almost all students and possibly their teachers. What factors have led to this problem and what can be done to improve the situation?

One of the major problems is the de-emphasis of proof in many national curricula. Many countries have revised their curriculum so that proof has virtually disappeared from it. In England, Gardiner and Moreira (1999) among others, have reported that:

mathematics, based on exact calculation, logical reasons and proof has given way to a kind of experimental mathematics. (p. 17)
In other countries, if proof remains, it might only be found within Euclidean geometry and often this curriculum area does not enjoy the prominence it once had. Senk (1985) argued that even with a full-year geometry course which had a significant portion devoted to proof writing, there exists a “severe mismatch between the intentions of the geometry curriculum and what students actually learn” (p. 456). The geometry strand is indeed troubled, as Howson and Kahane (1986) have pointed out and its reduction in curricular emphasis has further lessened the opportunity for students to experience proof.

A major influence on mathematics curriculum in both the United States and Canada is the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (1989). It argued, in Standard 3 (Mathematics as Reasoning) that only some students should be required to construct proofs:

- make and test conjectures; formulate counterexamples; follow logical arguments;
- judge the validity of arguments; construct simple valid arguments
- so that, in addition, college-intending students can –
- construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction. (p. 143)

It can, therefore, be no surprise, that without curricular experiences with proof, that student difficulties with it are compounded. However, the situation is improving. Recently, the National Curriculum in England and Wales has been revised to address the concerns about a de-emphasis on proof. The revised Principles and Standards for School Mathematics make a stronger statement about proof. Proof is explicitly mentioned along with reasoning as one of the standards, this time, for *all* students:

**Reasoning and Proof**

Instructional programs from pre-kindergarten through grade 12 should enable all students to –

- recognise reasoning and proof as fundamental aspects of mathematics,
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof

(National Council of Teachers of Mathematics, 2000)

Another reason that proof is difficult for students is that proof encompasses so many concepts and competencies that act as hurdles students must jump before arriving
at the finished product of a proof. Proofs are not generally easily found. Bell (1976) discussed the rather extensive mathematical knowledge required in a typical proof in school geometry. Which knowledge is relevant and which is not and how do you assemble it coherently, are issues that must be considered in arriving at the proof. Van Dormolen (1977) described arriving at a proof as a disorderly trial and error process that afterwards leads to a tidying up phase. A combination of technical expertise, intuition, creativity, perseverance and confidence are often necessary to construct the proof. Carlson (1999) reported that the need for perseverance is more acutely necessary than what has been previously reported in the literature. Citing a study of successful mathematics graduate students, she found exceptional degrees of persistence and confidence when these students were faced with challenging mathematical tasks.

Student Beliefs about Proof

Notwithstanding the double blows of reduced curricular emphasis and proof’s inherent complexities, there is also the issue of student beliefs about the function and role of proof. Coe and Ruthven (1994) pointed out that even when students were prompted to discuss the nature of proof they appeared to find it difficult to talk about proof in any explicit way. The students did not seem to view proof as a distinct entity in its own right but saw it only in some particular mathematical context. Vinner and Kopelman (1998) argued that since proof was outside the mainstream of behaviour and hence artificial, it must be explicitly taught even, if necessary, by artificial means. Perhaps the traditional approach of intuitive approaches leading later to more formal means needs rethinking. Harel and Sowder (1998) concluded that students had no overview of proof and teachers are not always aware of what constitutes proof to the student. As a result, students experience major difficulties understanding, appreciating and constructing proofs.

Edwards (1998) taught a module on proof to her secondary students and found made the following conclusions: students must believe a statement is true before they will attempt to construct its proof; given such a belief, students may believe that one or more examples are sufficient to verify its truth; however, students may believe, that a proof is based on more than a series of examples. The last two statements seem contradictory, but
perhaps, this is the nature of proof itself. Given its multiplicity of meanings and uses, it is quite possible that students see proof as serving different functions.

Chazan (1993) argued that:

The evidence ... indicates that students have good reasons to believe that evidence is proof. (p. 383)

There seems to be growing evidence that for many students proof serves primarily a role as certifier and is largely redundant if one is already certain. Coe and Ruthven (1994) reported on a study of students in their first year of sixth form college. They found that the students' strategies for proving consisted mainly of empirical ones; there were a small percentage of students who used anything resembling a deductive proof. Lampert (1993) expressed the view that students believe something is valid based either on the authority of their teacher or their textbook or because it is evident from examining a few examples.

Schoenfeld (1985) has argued that in problem solving students often do not use the mathematical knowledge that they have. Even though they have demonstrated knowledge of it in other ways, he argued that their beliefs about mathematics prevent them from utilising their knowledge. Schoenfeld said that for most students proof lacks purpose because they see it only as confirming what they already know to be true either because it is obvious to them or because they accept it based on authority. Balacheff (1991) attributed this to the students being practical persons. In other words, they do not see the need for argumentation that leads to conclusions that are already established by other less time-consuming means.

It is clear then that students have a view of proof as primarily a tool to verify mathematical results. If verification is linked with conviction in this view of proof, then it seems quite logical that naïve empiricism is a dominant method of proof for many students.

Segal (1999) made distinctions between conviction and validity and the private and public aspects of proof. Convincing oneself is not the same as convincing someone else. Even though students may accept a proof as valid and correct, Segal found that they still want to check it empirically in order to convince themselves. This indicates, once again, that for many, conviction comes from evidence. Therefore, arguments which are
valid may not convince and conversely arguments which convince (usually empirical ones) are not usually valid.

Students can also be convinced by the look and feel of a proof. Students are also cognisant of the form that mathematical proofs take and will often judge a proof's validity on the basis of form alone. Balacheff (1991) cited an example of a student who viewed a proof as valid only if it had a certain form:

a student I observed ... after having found the solution, in quite a good manner, then engaged in an odd process just to formulate his solution mathematically. Unfortunately, the solution he produced no longer looked like anything coherent. (p. 176)

Coe and Ruthven (1994) studied rather extensively student beliefs about proof. They found that students were primarily concerned with verification and usually this took the form of testing out a conjecture using a few examples. They demonstrated mostly empirical proving strategies with little in the way of deductive approaches. They also found that

few students were concerned to explain why rules or patterns occurred, or to locate them within some wider mathematical system. (p. 51)

Although some students were able to see that proof could offer explanation, Coe and Ruthven found that for most students knowing why was not that important.

**Student Competency Levels of Proving**

A well-known model of psychological development is the Van Hiele model of development. Similar to Piaget in its hierarchical notion of psychological development, it is distinct from Piaget in that the van Hiele's model is tied to progress in Euclidean geometry. At level 2, a student can deduce and recognise properties. At level 3, a student is able to construct proofs, they can see more than one way to prove, they recognise the distinction between a statement and its converse. At the level 4, the highest level, the learner is able to work in a variety of axiomatic systems including non-Euclidean ones.

It is at level 2 that students can begin to reason. At level 2, or informal deduction, students are able to follow a formal proof but they are unable to see how the logical order of the statements could be changed or how a related proof from different premises could
be constructed. Based on the literature about student performance with proof, it would seem that most of the students are operating at this level of development.

Both Piaget and the Van Hiele’s suggested that instruction must help students progress through lower levels before they are able to achieve the higher levels. Battista and Clements (1995) argued that dealing with formal proof within axiomatic systems, before students are ready can lead to confusion about the role of proof. They concluded that:

...the explicit study of axiomatic systems is unlikely to be productive for the vast majority of students in high school geometry. (p. 50)

Balacheff (1991) recognised four levels of proof. In increasing order of development they are: naïve empiricism, the crucial experiment, the generic example, the thought experiment. The first two levels were what Balacheff called pragmatic proofs: those proofs that resorted to actual actions or showings. In generic example, the proof rests upon the properties, the example is a generalisation of a class, not a specific example. Finally, the highest level is the thought experiment. At this level, a student is able to distance him/herself from the action and to make logical deductions based upon an awareness of the properties and relationships of the situation.

Implicit in these hierarchies is the notion that students will progress from one level to another and that one must move through one level before reaching another. Whilst this is undoubtedly true, it should also be stressed that students will move back and forth depending on the task that they are involved with. In other words, a student capable of a thought experiment in one situation may regress to naïve empiricism in another. Once again, we see that finding a theory of how students learn to prove is rather elusive.

Miyazaki (2000) described six levels of proof. Contrasting the method of representation (functional language or visual representations) with inductive or deductive reasoning, Miyazaki developed a model that showed two paths through which students may progress. She argued, using Piaget’s terminology of formal and concrete operations, that students may progress to formal reasoning only by way of concrete operations.

These levels suggest strategies for teaching. Once students have been assessed at a particular level, curricular activities must be designed to move them to the next level.
For example, Miyazaki (2000), suggested that when students are operating in inductive reasoning with visual models, teachers need to call into question the generality of their actions. This will help them recognise the need for deductive reasoning. The problem for teachers is twofold: first, diagnosing the level of each student; second, managing the diverse curricular experiences that may be necessary given that students in the same classroom will often be working at different levels. Similarly Balacheff (1991) argued for a Lakatos-like approach of refutations. The teacher must create the conditions necessary for the students to become aware of a contradiction. The expertise of the teacher becomes a critical factor as it is clear that the role of the teacher is paramount in creating the conditions in which students will question and refine their conjectures and proofs. Balacheff (1991) said:

The teacher’s intervention will be fundamental. The way he or she manages the teaching situation may bring the students to see that their knowledge and the rationality of their conjectures must be questioned and perhaps modified, because no ad-hoc adaptation of a particular solution, or its radical rejection, can by itself lead to a conceptual advance. (p. 109)

Leonard (1997) argued that perhaps our expectations for students are too ambitious. He suggested that an acceptable goal for secondary school might be having students acquire the concept of proving without getting bogged down in all the technicalities. He thought that there were two competencies necessary for first-year university students to have:

1. The need to question a statement, in other words, to have a feeling for when something needs proving. …
2. The ability (a) to follow a proof – not necessarily to reproduce it word by word – and (b) to recognise the appropriateness of the logic, that is, the order of the argument, or lack therein. (p. 204)

Leonard raised an important point. How much is possible and are we too ambitious? Does our quest to develop in students a proving capability cause us to neglect developing in them an appreciation for what constitutes a proof? Battista and Clements (1995) made similar arguments, tying them to research into levels of proof. They argued that the most effective way of developing meaningful use of proof is to, for the most part, avoid formal proof: A focus on justifying ideas in a classroom culture of discourse was advocated.
Making Proof Authentic for Students

There are several justifications for an emphasis on proof in the curriculum. One justification for inclusion is “it is what mathematicians do”. Then proofs that are included in the curriculum should respect the function that proof has within the professional mathematics community. In other words, we should be authentic in our use of proof.

Simpson (1995) juxtaposed “two routes to proof” in our schools. One he characterised as “proof through reasoning”, the other as “proof through logic”. Proof through reasoning is comprised of the functions: explore, discover, find patterns, explain, and, justify. These lead eventually to a formalisation. The other route involves sums, drill and test, techniques, calculations which lead to a predicate/ propositional calculus. Simpson clearly saw the former as the natural (or authentic) route:

A “natural” learner always attempts to “make sense” of their experiences by connecting them immediately to their existing mental structures, looking for explanations and reasons based on those connections. (pp. 41-42)

So the route to proof via “reasoning” puts more emphasis on exploration, on the discovery of patterns and on the development of a language in which to write those discoveries down. Perhaps too much emphasis in the past on formal logic proofs has led to the de-emphasis in proof in some countries.

As argued previously in this review, authentic mathematical practice does not require excess formalism. Even in mathematical practice, proofs that merely verify are not as useful as proofs which systemise or explain or enable new mathematical discoveries.

Burton (1999) interviewed seventy practising professional mathematicians to determine how they “come to know” mathematics. Coming to know was characterised as a feeling of rightness, an intuitive knowing, not necessarily the conviction that may come from proof. However, for these mathematicians, knowing was helped by making connections. Almost all of the mathematicians believed that being able to connect their work to the work of others, improved their knowing and feeling of rightness. Proof as systematisation is a powerful motivation for the professional mathematician. Even when their work is disconnected, the mathematicians were aware of the many historical
examples of disconnectedness leading eventually to connections. They, therefore, actively seek out these connections.

Another important observation that Burton (1999) made was that mathematical practice is more likely to be a blend of community and individual efforts rather than solely one or the other:

The experiences of these mathematicians help, I think, to emphasise the flow and inter-dependence in meaning making between socio-cultural formulations and individual acquisition. Learning is neither solely individual nor wholly social. (p. 139)

**Constructivism, Discourse and Proof**

Recent curriculum reform has been influenced greatly by constructivism. Constructivism is a theory of how one learns. Whilst there are different forms of constructivism; all share two common principles: that knowledge is actively constructed by the learner and that coming to know is the process of organising and adapting to the world as experienced by the learner. Constructivism is not a teaching strategy although it obviously informs but does not explicitly define our teaching practice. Steffe and D’Ambrosio (1995) stated that “the basic tenets of constructivism are orienting” (p. 146). Further they claimed that there are:

three principal currencies of a mathematics teacher: the posing of situations, the encouragement of reflection, and interactive communication. (p. 156)

In this view of constructivism the role of the teacher is still seen as important. The N.C.T.M. *Professional Standards for Teaching Mathematics* (1991) call for the teacher to “orchestrate discourse”. For students to construct knowledge, Confrey (1990) argued that they must be prodded to reflect and to make modifications to their knowledge:

reflection as an “objectification” of a construct, functions as the bootstrap by which the mathematician pulls her/himself up in order to stabilize the current construction and to obtain the position from which the next construct can be created (p. 109)

The orchestration of discourse and the encouragement of reflection are necessary components of an effective mathematics program and these can be very helpful to the teaching of proof. Sometimes however constructivism has been interpreted as reducing the role of the teacher in the classroom. Open-ended tasks and discovery learning without
sufficient teacher interventions are not conducive to the teaching of proof. The teacher plays a vital and non-passive role in the teaching of proof.

A good example of "orchestrating discourse" in the teaching of proof is the work of Imre Lakatos. In *Proofs and Refutations*, Lakatos (1976) illustrated how a proof can be criticised and how the criticisms are used to improve the conjecture and sometimes the proof itself:

I hope that now all of you see that proofs, even though they may not prove, certainly do help to improve our conjecture. (p. 37)

Lakatos advocated a heuristic approach, similar to Polya's four step plan, that can be used to refine a proof. His four stages were: Primitive conjecture; Rough proof (thought experiment); Global counterexamples; Proof re-examined. These stages could be extended but the essential method was to critically analyse the initial proof and look for counterexamples that could be used to improve or extend the proof. Lakatos (1976) argued:

Discovery does not go up or down, but follows a zig-zag path, prodded by counterexamples, it moves from the naive conjecture to the premises and then turns back again to delete the naive conjecture and replace it by the theorem. Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end product. (p. 42)

Through the process of discovery, informal theories were developed. Lakatos illustrated that the role of the teacher as expert is fundamental to the teaching of proof. He or she must orchestrate the necessary discourse that will bring to light refutations or counterexamples. These refutations are the bootstrap, using Confrey's (1990) terminology, with which we are pulled up to modify the theorem and the proof.

**Visual Proof**

Implicit in the much of the discussion so far about proof is the assumption that the proof is presented in a written form. Davis (1993) lamented what he saw as an over-reliance on only one aspect of mathematical practice and that this unbalanced emphasis on written proof has deprived us of other forms of mathematical communication. He discussed *visual theorems*, which he saw as a graph or diagram that:
the eye organises into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation. (p. 333)

Likewise, Barwise and Etchemendy (1996) have argued that a reliance on a "universal scheme of representation – linguistic, graphical, or diagrammatic – is a mistake" (p. 180). They argued that a large part of doing a proof well is finding the best way to represent it:

As long as the purported proof really does clearly demonstrate that the information represented by the conclusion is implicit in the information represented by the premises, the purported proof is valid. (p. 180)

In the book, *Proofs without words*, Nelson (1993), presented a battery of proofs. The typical proof is a diagram or a series of diagrams which sometimes contains a mathematical equation or formula. Whilst there may be some question about such proofs, their validity would depend on your view of the role of a proof. Many of these proofs are extremely explanatory and would meet the criteria of proof as explanation. Edwards (1998) pointed out that visual proofs or diagrams can hold great explanatory value for students:

these representations held explanatory power for the students, since they used them to justify their decisions about the odd and even statements. In this sense, although the visual arguments were in no sense full or formal mathematical proofs, they were a kind of “proof that explains” for the students who created them. (p. 502)

Brown (1999) discussed that mathematicians look for two qualities in a proof: evidence and insight. He suggested that many see visual proof as a trade-off between these two qualities: what one loses in rigour, one gains in insight. Brown argued that this is an over-simplification. A visual proof does provide evidence and if it is a good one it also makes us wiser:

Pictures often yield insight, but that is not essential. The examples I have given are mainly a form of evidence – a different form, to be sure, than verbal/symbolic proofs; but they have the same ability to provide justification, sometimes with and sometimes without the bonus of insight and understanding. (p. 42)
Often a visual proof is based on a diagram which can be conceived of as a generic example of the proposition. Peled and Zaslavsky (1997) discussed the problems with generic examples. They argued that what the teacher perceives as generic may be seen by students as only an example. Hence, a diagram may constitute a proof for someone who sees it as generic, but only an example for someone who does not. Similar arguments were expressed in Hoyles and Healy (1999) who found that the majority of students saw a visual representation as specific rather than general. They wrote:

Clearly many students were unfamiliar with the power of visual representations and their potential to serve as generic examples, a conclusion supported by our process date: for example, the presence of a picture in a request for a geometry proof confused some students about whether it was general or not. Visual arguments were also frequently attributed lower status than other forms, and described as 'simple' or 'daft', even by those who acknowledged their explanatory power. (p. 107)

Another problem with visual proof, is that it is often left to the beholder to determine its meaning. In a traditional proof, the path from premise to conclusion is usually clear, whereas this is often not clear in a visual proof. Critics of visual proof may argue that it is incomplete, as discussed by Hoyles and Healy (1999), but Hanna (1991) reminded us that a proof is never complete. However, finding good visual proofs is not easy and even when they are found it is not always clear what the proof maker intended. Further, we have such a tradition of written proofs that visual proofs suffer by comparison. Visual proofs lack a tradition of practice and this impedes their acceptance. For these reasons, visual proof can be problematic for teachers and their students.

**Technology in the Aid of Proof**

Reform movements have posited many reasons for the inclusion of technology in the mathematics curriculum. One argument in favour is the authentic one: it is increasingly being used by professional mathematicians. Another reason is as the society becomes more technologically based, so too will schools, as schools are microcosms of society. One can no longer argue that students need to learn skills because calculation devices and computers are not available to them. Perhaps more significantly, it is also clear that our society has come to rely on such devices in the work place. More and more, the perceived needs of the marketplace influence schools. From the 1957 success of the
Soviet Sputnik space program, through the marketplace dominance of the Japanese in the 1980s to the emerging realities of technological change on almost every fabric of our society, it is clear that technology will continue to impact educational change.

According to Howson and Kahane (1986) curriculum must change to reflect the changing demands and expectations of society. Further, the types of mathematics required of the general population are changing. As our society has become more technological (mathematical), many of the “basic skills” of mathematics are less used and consequently seem less important. Keitel (1989) stated this paradox in the following way:

No modern society can exist without mathematics, but, the overwhelming majority of people in a modern society can and do live quite well while doing hardly any mathematics. (p. 9)

While the reality of our society is more and more structured by mathematics, the average individual is more and more relieved of the need to do mathematics. However, the type of mathematics that is being replaced is mathematics that is cognitively of the lowest order. The types of mathematics that are increasing important are ones which assist people to reason, interpret and reflect. These skills of reasoning, interpreting and reflecting are necessary for the development of proof.

Technology can be a real aid in the beginning processes of proving, primarily in the conjecturing phase. Demana and Waits (1990) argued that technology will enable students to view numerous examples and formulate conjectures based upon the examples. Technology can be an aid in exploration and discovery as it makes the process less tedious; students can spend more of their time considering hypotheses and drawing conclusions instead of a disproportionate amount of time spent on carrying out the computations. It is also likely to lead to students studying a sufficient number and breadth of examples to make it possible to make a realistic conjecture.

Hoffer (1987) stated that:

Technology-supported learning aids users by performing tasks that require a low level of thought or that might be time-consuming and thereby enables users to extend or increase their level of thought and to apply their energies to more important tasks. (p. 638)
Further, a reduction in time spent on algorithms and manipulations, made possible with the use of technology, provides the much-needed classroom time to find and test conjectures and to develop the proving capabilities of students.

Luthuli (1996) reported that a historical look at textbooks indicates that the presentation and treatment of geometry is primarily preoccupied with primarily verification and to some extent with explanation. He argued that geometry is a curriculum area rich in opportunity for students to engage in their own discovery of mathematical truths.

Many of the more significant uses of technology in the mathematics curriculum that are pertinent to proof involve the study of geometry. It is significant to note that Fey and Good (1985) and Howson and Kahane (1986) describe the geometry strand as "troubled" and the loss of Euclidean geometry as "a sad development". Perhaps, the use of technology in this strand of curriculum will return it to its previous prominence. However, we must avoid what Lampert (1993) has cautioned against:

Common methods of teaching geometry and the assumptions they express about student learning mix a formalist philosophy of mathematics with a reliance on teaching authority as the source of mathematical knowledge and truth. (p. 149)

The richness that Luthuli (1996) argued in favour of will only be possible if old methods of the teacher’s authority as the sole means of justification give way to classrooms in which rich tasks are coupled with active discussions and argumentation.

Palmiter (1991) stated the Geometric Supposers can increase students’ van Hiele levels of understanding from the first level (the visual level in which the student sees a geometric figure as an indivisible whole) to the second level (in which the student can distinguish properties of the figure). The majority of students are at the first level upon entry to high school and the third level seems to be necessary to be able to do proofs. The Geometric Supposers allow students to make and test conjectures about the properties of geometric figures. The software simulates the types of drawings and the measurements that a mathematician would make as he or she searched for a pattern. It encourages inductive reasoning and it enables proof by facilitating the prerequisite stages. Fey (1989) reported that this software resembles the pedagogy more commonly found in a science
laboratory class. Students are actively involved in constructing geometric theorems rather than passively listening to a geometry lecture from their teacher.

Jones (1995) argued that dynamic geometry makes proof accessible to students. To be successful in geometry and geometric proof, students must first know the basic geometric definitions. The software enables students to distinguish, as Laborde (1993) has distinguished, the difference between a drawing and a figure. In dynamic software, a drawing appears to be a figure but upon dragging it no longer appears to be. However, a figure upon dragging maintains the properties that define it. This is a powerful way for students to understand the essential defining characteristics of the various figures that they will need to be conversant with if they are going to be successful with proof. Once students have a good understanding of the definitions of geometric figures then can begin to work with them in an effective way.

Schumann and de Villiers (1993) argued that an authentic practice of mathematicians is “theorem finding”. Dynamic software such as Cabri or Geometer’s Sketchpad allows for the continuous modification of geometric configurations allowing for a medium in which theorem finding can legitimately occur. Students can determine which characteristics are invariant and which are not. They go on to provide fourteen examples of how the software can be used to develop theorem finding expertise. The use of the software does not remove the need for proof. However, it does change the function of proof from verification to explanation, discovery and communication. de Villiers (1998) and Olive (1998) argued that the Geometer’s Sketchpad, for example, removes the need for proof as conviction but it does not provide the explanation. Proof as explanation is still important. Consequently, Schumann and de Villiers (1993) argued that attempts to interest students in proof as verification are not successful but students find it quite satisfactory to then view a proof as an attempt at explanation, rather than verification (p. 18).

Hoyles and Jones (1998) have argued that the inclusion of dynamic geometry software has made the gap between induction and deduction more difficult to bridge. Similar to Schumann and de Villiers (1993), they have argued that a traditional approach to proof as verification in this context falls flat and may lead to a largely empirical curriculum of investigation and conjecturing. They suggested that students can use the
software to study a *generic example* which then leads to a study of the implicit relationships and hence a backdrop for developing a proof. Proof as explanation again becomes the rationale for proof. Clearly, as argued by Hoyles and Jones (1998), there is much work to be done to ensure that proof is a natural part of this process.

It is particularly effective to introduce students to examples which upon proving the result lead to other surprising results. It is in this way, that proof serves to facilitate discovery of new results. In searching for a proof one comes to the essential characteristic upon which it depends and this leads to further generalisations. de Villiers (1999) illustrated many examples of how *The Geometer's Sketchpad* be very helpful in this regard. One such example is a kite (quadrilateral ABCD with adjacent sides AB = AD and CB = CD) whose sides’ midpoints are joined and the quadrilateral formed is a rectangle. By continuous variation, we can easily be convinced that this is always true. However, as de Villiers argued, this does not provide us with a satisfactory explanation as to why the midpoint quadrilateral of a kite is a rectangle. It is only upon producing a deductive proof do we come to the essential characteristic of the original figure. What is essential is that the diagonals of quadrilateral ABCD are perpendicular. It is not necessary for ABCD to be a kite. de Villiers (1999) in *Rethinking Proof with the Geometer's Sketchpad* illustrated how the use of technology can be used to help students recognise the various roles of proof.

Technology, then is an invaluable aid in the early stages of proof development, making the conjecturing process one in which students can actively participate. It can also provide, in students’ minds, a reason for proof as de Villiers’ example of the kite’s midpoint quadrilateral demonstrates. Technology can also be useful in aiding students to develop processes for constructing proofs. As reported by Silver (1998), sophisticated computer software programs, like *Proofchecker* can be useful in helping students learn if – then constructs as well as fill-in-the-gap type proofs. These are, of course, useful skills to have but hardly supplant the reasoning and creativity required to put together a good proof. Reasoning and creativity remain solely a human endeavour and since they are at the heart of proof, so, too, remains proof.

At present there has not been too much research into how technology can be useful beyond the early stages of investigating, conjecturing and theorem finding. At the
end of the process it can be useful in verification of a result. As proof is essentially about reasoning, can technology be expected to help us with this aspect? Hoyles and Healy (1999) found the use of the computer can help students make the difficult transition from conjecture to proof:

Our results show that the computer-integrated teaching experiment were largely successful in helping students widen their view of proof and in particular link informal argumentation to formal proof – a transition known to be problematic. (p. 112)

There needs to be more research into how technology can aid students in making the transition from informal to formal proof. At the present time there are few examples of how this can be achieved through technology. Reasoning and creativity are integral to proof and it remains unclear just how much technology can help us with these crucial aspects.

**Proof as Problem Solving**

Problem solving has been a major focus of mathematics curriculum for the last twenty years. The 1980’s saw the N.C.T.M. make several calls for increased emphasis. Resnick (1989) observed that to become a good mathematical problem solver one had to acquire the habits of interpretation and sense making in the discipline of mathematics. Participation in the activity of doing mathematics is fundamental to acquiring the practices that produce good mathematics. Mason, Burton and Stacey (1982) outlined four processes which they said embodied mathematical thinking. The four processes are: specialising which means looking at examples to learn about the question; generalising or looking for a pattern; conjecturing which involves making a reasonable, but as yet, unproven, statement; convincing or proof. This is quite similar to Polya’s four step plan of understanding the problem, devising a plan, carrying out the plan and looking back. Mariotti and Maracci (1999) investigated the transition from conjecturing to convincing and found that:

the relationship between the process of constructing a conjecture and that of constructing its proof presents a complexity which must not be underevaluated. (p. 272)
They pointed out that students often have difficulty formulating their conjecture in a clear and precise manner and this hampers their subsequent attempts to find a proof.

Farrell (1987) made a distinction between proof-doing and proof-writing. There is a process of finding the proof which is often not modelled for students. She advocated "talking aloud" to model the process of arriving at a proof. This is similar to Schoenfeld's (1985) problem-solving classes in which students bring problems for Schoenfeld to struggle with in front of the class. In this public problem solving, Schoenfeld models the problem solving process. By talking aloud, Farrell models that proof-doing is "a form of problem solving with its own set of heuristics" (p. 245). Similarly, Lakatos (1976) spoke about the zig-zag discovery necessary to developing a proof.

Often students with good problem solving skills are good at proof-doing but may lack the mechanical or organisational facilities at proof-writing. By separating proof-doing from proof-writing, Farrell pointed out that often students see only the finished product, the proof-writing, and hence have a misshapen view of proving. Fawcett (1938) designed a high-school geometry course in which students developed their own axiomatic system from beginning notions. Students were encouraged to formulate conjectures and the class worked together to prove them. Fawcett, although obviously a knowledgeable member of the class, expected the class to challenge him, as they would anyone else, to explain any proofs or conjectures that he put forward for class scrutiny.

Epp (1994) rejected the view that proof was the formal end product following and separated from intuition. She pointed out that the proof process involves a lot of deductive and intuitive reasoning of which we may not even be aware. Evoking this intuition is necessary if students are to acquire some facility in deductive reasoning. Mental images are necessary for students to grasp meaning and significance, as the proof develops. Greeno (1994) agreed with Epp, adding that an interaction between formal and informal reasoning would probably be one "in which presentation of proofs is accompanied by rich uses of diagrams" (p. 277). Edwards (1998) similarly argued that when students spontaneously develop their own visual representations these help them to better express their proofs. The diagrams enable the students to see more clearly the argument they are in the process of formulating.
Luthuli's (1996) final function of proof, *meeting intellectual challenge*, certainly is met by proof as the final stage of a problem solving process. Many of the questions he posed as motivating the refinements that one can make are illustrations of what Barbeau (1990) called grappling to improve a proof. This heuristic process of refinement, well-known in problem solving contexts, has value also in proof.

**Further Implications for Teaching**

Much of the previous discussion centred on the pedagogy of proof. Notwithstanding that this discussion has already touched on the role of the teacher, there are many specific implications for the teaching of proof. One implication is that teachers themselves are aware of the many functions of proof. Teacher education programs do not always address proof explicitly and the university mathematics that the teacher studied, if any, may take for granted that the student is aware of the reasons for proof. Mathematics teacher education must make the pedagogical aspects of proof a priority. Jones (2000) has discussed this problem in greater detail.

Putting the teacher education issues aside, let us turn now to the issues of student beliefs. Sowder and Harel (1998) provided several suggestions of how the teacher could counteract many of the erroneous or incomplete beliefs about proof that students have. They characterised student beliefs about proof into externally based proof schemes (authoritarian, ritual and symbolic); empirical (perpetual, examples-based); analytic (transformational, axiomatic).

Authoritarian proof schemes are proofs which are based on the authority of the teacher, the textbook, or a more knowledgeable classmate. To counteract this, Sowder and Harel (1998) recommended that the teacher use instructional approaches like orchestrating discourse and using group work to lessen the impact of the teacher as the sole authority. A Lakatos style of refining conjectures and proofs, although perhaps too ambitious for many teachers, would enable students to see proof as a natural part of meaning making. In the ritual proof scheme, students accept a proof based on its form alone (also Balacheff (1991)) . Teachers need to allow for different representations of proofs as long as these meet the criteria of a reasoned argument. Perhaps, teachers can encourage the use of written prose from time to time. There is a tendency in mathematics
classes to not want to write sentences and paragraphs because this is seen as un-
mathematical. This is also laziness because writing good arguments is time consuming.
These tendencies must be avoided if students are to see that mathematical proofs can take
many forms. Farrell (1987) suggested having half the students write a geometry proof in
standard two-column format and the other half in a paragraph form. The symbolic proof
scheme can be a bad thing when students manipulate them with little understanding of
their meaning. Sowder and Harel (1998) suggested emphasising the meaning behind the
symbols may alleviate this problem.

As Fischbein (1982) argued that formal proof is not a natural undertaking, Vinner
and Kopelman (1998) have stated that perhaps the popular practice of expecting students
to manage the move along a continuum from intuitive informal reasoning towards
formalism without direct teaching has been misguided. A step towards providing explicit
teaching is providing students with models and principles of proving which help them to
organise their approaches.

Koedinger (1998) stated that students are able to make conjectures fairly easily
although often the conjectures are trivial. When asked for an argument to support the
conjecture, most often the response is that they haven’t found any counterexamples.
Clearly, this is a view of proof as verification. This is similar to Chazan’s (1993)
observeration that “evidence is proof”. Koedinger (1998) proposed a model to prompt
students to think about deductive proof because they did not do this spontaneously. The
model has both inductive and deductive elements and recognises that the ability to
produce a deductive argument also requires a specific knowledge of how to argue and
these need to be explicitly taught:

- Drawing conclusions
  - If I find a counterexample, the conjecture is false.
  - If I find a proof, the conjecture is true

- Switching strategies:
  - If many attempts to find a counterexample fail, perhaps the conjecture is true,
    and I should try to prove it.
  - If I can’t find a proof, perhaps the conjecture is false, and I should look for a
counterexample.

- Checking for errors:
  - If I’ve found a counterexample, I should still check that I cannot find a proof.
• If I've found a proof, I should still check extreme examples to make sure there is no counterexample.
• If I seem to have a proof and a counterexample, perhaps I've formulated the proof problem wrong or made incorrect measurements in my counterexample.

(Koedinger, 1998, p. 335)

Dreyfus and Hadas (1987) acknowledged that students experience difficulty with proof. They formulated six principles, that when explicitly taught, can help students to understand the nature and scope of proof:

Principle 1: A theorem has no exceptions. A mathematical statement is said to be correct only if it is correct in every conceivable instance.

Principle 2: Even "obvious" statements have to be proved. In particular, a proof may not be built on the apparent features of a figure.

Principle 3: A proof must be general. One or more particular cases cannot prove a general statement. However, one counterexample is sufficient to refute it.

Principle 4: The assumptions of a theorem must be clearly identified and distinguished from the conclusions.

Principle 5: The converse of a correct statement is not necessarily correct.

Principle 6: Complex figures consist of basic components whose identification may be indispensable in a proof.

(Dreyfus and Hadas, 1987, p. 49)

They went on to give several curricular examples of how these principles can be reinforced. Proof may be part of the school experience but it is often not taught in a systematic and explicit way. Consequently there are many misconceptions about proof that are never addressed. These models suggest that it is worthwhile to discuss proof-doing as well as proof-writing.

These models suggested are examples of what Hoyles (1998) has called a culture of proving. Arguing against proof as a hierarchically structured activity to be achieved after the inductive processes, she instead advocated a back and forth approach to proof that mixes induction with deduction:

Thus the scenario we envisage is one where students construct the objects for themselves on the computer, conjecture about the relationships between them, and check the truth of their conjecture with the tools available. This hopefully leads them to be able to justify their conjectures by reflection on their constructions and computer feedback. This sequence is not undertaken in a linear fashion but one which is spiral and iterative. (p. 172)
She cited an example of an investigation into the congruency conditions for triangles. Students used dynamic geometric software to construct several examples. For each, they completed a sheet with the columns: Diagram, Prediction, Result, Check, Justification. As in the other models, this approach provides students with a framework to include justification and proof into the inductive process of studying examples and conjecturing.

Whilst models of proving can help students keep proof at the forefront of their mathematical work, it is also necessary to pose good mathematical tasks that illicit the need for proof. The curriculum is often overloaded with algorithms and techniques and this can kill interest in the subject. Movshovitz-Hadar (1988) stated that mathematics theorems contain an endless source of surprises, and it is these surprises that are interesting and provide a motivation for proof. She enumerated ten types of mathematical surprises that can be used throughout the curriculum. These surprises can provide the motivation for proof that is often lacking in our instruction:

Intellectual surprise usually gives us a sense of fulfilment, an appreciation of some wisdom, a joy from its wittiness, and a drive to find some more. Making mathematical findings appear unexpected, or even contra-expected, is the secret of teaching mathematics the surprise-way. (p. 35)

Leonard (1997) expressed similar views. He cited three examples which showed how the teacher can create a classroom environment in which students actually see proof as convincing argument. For students to really see proof in this way there must first be a challenge to their initial conceptions.

**Summary**

In the previous sections, it has been demonstrated that the teaching of proof is a challenging task. Its teaching is complicated by factors such as the misunderstanding about its several roles. What teachers and students believe constitutes proof significantly influences its curricular treatment. Whilst there are several barriers to the successful teaching of proof, much research has been done in the past ten years which is encouraging. Current secondary school curriculum documents argue that justification and proof are to be a significant part of the culture of mathematics classrooms. A greater emphasis on communication and discourse in the mathematics classroom has led to more opportunities for students to construct and justify their own mathematical arguments. In
addition to providing such opportunities it is necessary to provide students with heuristics and models for the development of a proof. These can give students support as they attempt to justify and prove their mathematical conjectures. The greater availability and use of technology, especially software such as Capri or The Geometer’s Sketchpad have increased the opportunities for students to participate in formulating conjectures and investigating them. It is still unclear how technology can be helpful in the proof construction stage. However, technology can be useful in motivating the need for proof because it often convinces us but does not offer explanation. Used judiciously, technology can create a need in students’ minds for an explanation and as such it nicely sets the stage for proof.
Chapter Three: Method

When we mean to build, We first survey the plot, then draw the model, And when we see the figure of the house, Then must we rate the cost of the erection.

- William Shakespeare (King Henry IV – Part II)

In Chapter One, several research questions were proposed for investigation. These questions pertain to students’ views of what constitutes valid mathematical proof and to ascertaining student proficiencies with the construction of proofs. In this chapter, the research approach and design of the study are elaborated on.

Research Approach and Design

The study uses both quantitative and qualitative research methods. Primarily quantitative methods were used to collect data to determine the state of proof in the Ontario mathematics curriculum. A descriptive study was used to determine “what is”. However, to understand, why, some qualitative methods were used. Although there is no clear boundary between the two methods, of the two instruments used to collect data, one was more quantitative, the other more qualitative.

Borg and Gall (1989) stated that there has been in the past much debate about the efficacy of each method. Each method has strengths and weaknesses and the argument that the two paradigms are incompatible is now frequently rejected as overly simplistic. In fact, the emergent view is that both methods used in combination can be beneficial. They quoted Reichardt and Cook noting that what they say about evaluation research is equally valid for educational research:

The solution, of course, is to realise that the debate is inappropriately stated. There is no need to choose a research method on the basis of a traditional pragmatic stance. Nor is there any reason to pick between two polar-opposite paradigms. Thus, there is no need for a dichotomy between the method-types and there is every reason (at least in logic) to use them together to satisfy the demands of evaluation research in the most efficacious manner possible. (p. 382)
Development of the Instruments

Two instruments were used to collect the data for the study. These can be found in the appendices A and B. The two instruments and their development are discussed in greater detail in the next two sections.

Student Survey on Justifying and Proving in School Mathematics – Part I

The first instrument, Student Survey on Justifying and Proving in School Mathematics – Part I, (Appendix A) consists of a series of open-ended questions which attempt to ascertain student beliefs about proof. The responses to each of the questions are in prose and as such the methods to analyse the data are qualitative.

Primarily, this instrument will provide the data which will help to answer the research question 3:

What is the nature of students’ views of proof? For example, what are their views about a proof’s form, its roles such as convincing and explaining, and are these views affected by the use of technology?

It will also be helpful providing insight into the other research questions, especially question 1:

What are the qualities of a mathematical argument that students recognise as a valid mathematical proof?

A detailed analysis of the rationale behind each of the questions follows. For each question of the survey, a rationale is provided for the question. When appropriate, reference to the professional literature is also made. The original questionnaire is presented below, single-spaced and block indented with the rationale for each question in regular format immediately following.

Rationale behind the instrument

STUDENT SURVEY ON JUSTIFYING AND PROVING IN SCHOOL MATHEMATICS– part I
Instructions: Answer each question as completely as you can giving specific examples to make your point of view as clear as possible. There are no right or wrong answers. However, complete answers are more useful than partial ones.
Questions 1, 2 and 3 deal with students' perceptions of proof with regards to its role in justifying or verifying mathematical results. de Villiers (1996) stated the following about the role of proof as verification:

Traditionally from a strict logical viewpoint, the function of proof has been seen almost exclusively in terms of its verification function; i.e. checking the correctness of mathematical statements. The dominant idea has been that proof is used mainly to remove doubt (i.e. personal or those outside sceptics) and has strongly influenced teaching practice and most discussions and research on the teaching of proof. (p. 24)

There is some debate then about the role of proof. Its use as a method of verification is perhaps overstated. How do students view proof? Do they see proof as necessary to their acceptance of the validity of the mathematics? Questions 1, 2 and 3 attempted to answer those questions.

1. Think of the last mathematical procedure you were taught. How do you know that the procedure is a valid method? How were you convinced?

In this question, I sought to focus students on a particular example and have them provide me with some insight into how they come to accept a procedure as valid.

2. In general, when new mathematics is introduced to you are you easily convinced? What does it take to convince you of the validity of the mathematical results?

In this question, I sought to extend their focus in question 1 from a specific example to their learning of mathematics in general.

3. Has there ever been a time when you haven't been convinced of results that were used in mathematics class? If so, were you eventually convinced? What did it take to convince you?

In this question, I sought to find out if they are ever unconvinced and what they need to have happen for them to be convinced. I also think that this question probes more completely how easily they are convinced. In other words, if there are no examples provided it would seem to indicate that they are easily convinced. This question is a way of "checking" their honesty in questions 1 and 2.
There has been intense debate about mathematics education over the last twenty years. Mathematics should be “socially constructed” and the classroom should be a “learning community” in which it is possible to discover and develop mathematical concepts. Hanna (1996) stated that this climate is one which supports the inclusion of mathematical proof as a form of explanation:

This would seem to be the right climate to make the most of proof as an explanatory tool, as well as to exercise it in its role as the ultimate form of mathematical justification. (p. 12)

So proof has a broader role than solely justification or verification. Its greater importance in secondary schools will result from its role as an explanation and hence its value as an aid in developing students’ understanding of mathematical concepts.

4. The goal of all teaching and learning should be developing understanding. In mathematics, understanding has been distinguished as two types: understanding how and understanding how and why.
   a) Why do you believe such a distinction has been made?
   b) Can you give an example that would illustrate the two types of understanding?
   c) Which type of understanding is usually preferable? Explain. Give an example.
   d) In general, how important is understanding how to you?
   e) In general, how important is understanding how and why to you?
   f) Which type is more important to you? Explain. Give an example.

Skemp (1976) made a distinction between these two types of mathematical understanding. This question is necessary to make the distinction of understanding how and understanding why clear to students. Many students would, as Skemp argued, view understanding solely as understanding how. Any discussion of proof as understanding is contingent on the notion of understanding how and why.

The multiple parts of this question sought to lead students through a logical sequence of argument that will help to focus their thoughts. Parts (a) and (b) are necessary to help students make the distinction between how and how and why for themselves. In parts (c) through (f), I sought to gauge students need for explanation. How important is it for students to know the why behind the how of mathematics? This
question helped to frame the remaining questions. It provided the backdrop to asking further questions about proof as explanation.

5. When a new topic is introduced to you by your mathematics teacher
   a) How much convincing of its truth or validity do you normally require?
   b) Can you be convinced without full understanding (how and why)?
   c) What does it take to explain a result so that you understand both how and why?

With this question, I sought to make a link between the earlier thoughts expressed by the student about his or her need for convincing and the thoughts expressed in question 4 about understanding. Part (a) and (b) tried to make this link while part (c) set the stage for a more complete discussion about proof.

6. In mathematics, the word proof is used from time to time.
   a) How familiar are you with proof? In previous mathematics courses, what has been your experience with proof?
   b) Why do you think we need proof?
   c) What do you think constitutes a mathematical proof?
   d) Can you give an example of an instance in your mathematics class when a mathematical proof was developed? If so, explain what the topic was and to the best of your recollection what was involved in the proof?
   IF YOU COULD NOT GIVE AN EXAMPLE SKIP PART E AND PROCEED TO PART F
   e) If you gave an example in part d): What did this proof contribute to your understanding of mathematics? What type of understanding did the proof help with, if at all?
   f) In general, how does proof contribute to your mathematical understanding?

This question again presented the student with a logical progression of thoughts so that their views on mathematical proof can be focussed and some insights gleaned from them. The treatment of proof in the curriculum varies according to the teacher. Part (a) asked students if they knew what proof was and how prevalent proof has been in their mathematics courses. Part (b) sought to tie their view of proof to their earlier thoughts about verification and understanding. Part (c) extended part (a) to a more specific question of what the student thinks a proof is. Part (d) was a checkpoint for the earlier parts. If students could not provide an adequate example of proof was proof really a part
of their experiences or was it limited to a specific curriculum area like deductive
gometry? Finally, parts (e) and (f) asked students to express their views on the value of
the proofs that they have experienced. How has proof contributed to their understanding?

7. Graphing calculators and computers are being used more often
in mathematics classes. In what ways have you found a
calculator/computer helps you to develop understanding? Give
examples.
a) Does a calculator/computer improve understanding **how**? In
what ways?
b) Does a calculator/computer improve understanding **how and
why**? In what ways?
c) Does a calculator/computer help you develop or understand
proofs better? In what ways?
d) In general, is the use of calculators/computers a positive or
negative development in the learning of mathematics?
Explain and give examples to support your viewpoint.

The use of technologies has become increasingly prevalent in teaching and
learning. Mathematics classrooms have undergone significant change with respect to the
role of calculators and computers. Claims are often made in the literature about the role
of technology in aiding understanding. For example, Giamati (1995) stated:

The use of *The Geometer's Sketchpad* in this type of exploration was invaluable.
The students gained a deeper understanding of the problem by using their scripts
to explore it and make conjectures than if the results had merely been explained to
them. Naturally, the exploration did not replace the proof, but it became a solid
foundation on which to build the proof. (pp. 457-8)

Students today seem highly dependent on technology. Does it help them to
understand better? What are their views on the use of technology and how it enables them
to gain a fuller understanding? Is there a distinction between the value of a
calculator/computer with the two types of understanding? What is the connection
between the use of technology and the development and understanding of a proof? Each
of the parts of the above question sought to lead students through a thoughtful
examination of the role of technology in their own learning of mathematics.

8. Write down anything else that you know about proof.

This final question gave students an opportunity to comment on any other aspect
of proof that has not already been commented on.
Pilot of the Instrument

The instrument, *Student Survey on Justifying and Proving in School Mathematics – Part I*, was tested with two classes of grade twelve students. In a couple of instances, the questions were slightly reworded to make them a bit clearer. The answers that these students gave to the questions were most useful in determining how to categorise the possible responses. This enabled a coding to be included in the survey instrument. The students who participated in the pilot were not the same students who participated in the main study.

Summary

This survey was developed to answer questions about student beliefs about proof. The intention of some questions was to educate students about terminology so that answers to subsequent questions can be given with an understanding of what the intention of the questions was. Asking completely open questions was rejected because it may have led to students answering unintended questions and I may have misinterpreted the student responses. On the other hand, I tried to be careful not to inject my own bias into the questions and thus lead the students to a conclusion that I was expecting. It is hoped that the right balance between these two extremes was found.

Student Survey on Justifying and Proving in School Mathematics - Part II

The second instrument, *Student Survey on Justifying and Proving in School Mathematics – Part II*, (Appendix B) consists of the presentation of proofs in several forms and across two curriculum areas (number theory/algebra and geometry). The format of the instrument is that of one used in the Institute of Education, University of London study (Healy and Hoyles, 1998). Permission to use whole or part of the instrument was granted by the Institute of Education at the University of London.

The survey was divided into two curriculum areas (arithmetic/algebra and geometry). Each part had 7 sections. Some of the sections involved already constructed “proofs”. In these sections, students chose which proof they thought was the best and which proof they thought their teacher would give the best mark to. Some of these sections had a further part which required students to critique all of the proofs. The “best”
proof would receive a rating of 5 whilst the "worst" proof would receive a rating of 1. Proofs could be rated from 1 to 5. An average rating was computed for each proof using the student ratings.

Further, in each curriculum area there were two opportunities for students to construct their own proof. These were related to the theme of the other sections.

The data accumulated from this instrument was used to answer the research questions. The ratings that students provided for each of the sample proofs provided a window on their views about what constitutes proof and hence was helpful in answering research questions 1 and 3:

What are the qualities of a mathematical argument that students recognise as a valid mathematical proof?

What is the nature of students' views of proof? For example, what are their views about a proof's form, its roles such as convincing and explaining, and are these views affected by the use of technology?

As the instrument provides four opportunities for students to actually construct their own proof, the data accumulated from these four items was useful in answering research question 2:

Are students generally proficient in proving and in what ways do students attempt to construct proofs?

In addition to the data from the first instrument which asked students about their view of the usefulness of technology, the second instrument provided a proof by technology. Students were asked to critique this proof and answer several questions about it. Therefore, the information provided helped to answer part of the third research question above:

....and are these views affected by the use of technology?

Finally, as the second instrument is an adaptation of another instrument a comparison of the results obtained between this study and the London study led to the answer of the last research question:

What are the similarities and the differences in the results with respect to the London study?
Comparison with the London Study Instrument

One of the two instruments used for this study had the same format as the London study. The same two curriculum areas were used. As well, some of the items included sample proofs and some of the items involved student constructed proofs. In general, the formats of the two surveys were almost identical.

This study was designed for students in either grade 12 or OAC. It was necessary to change most of the content of the items to make it more suitable for older students. However, some of the items of the London study were also used. In general, the proofs that were the harder ones of the London study became the easier ones of this study. Five of the fourteen items of the London study were used in this study.

In addition to the changes in the content of most of the instrument items, there were other changes:

1. In the London study, in each curriculum section, there is a question about the generality of proof. The students were given a proven statement and then another statement. They were to determine whether or not a new proof was necessary. In both of the examples of the London study, the modified statement was also true. In this study, these sections were changed to include four new statements. Two statements were still valid. However, the other two statements could not be deduced to be valid based upon the given theorem. One of these also involved students’ understanding the difference between a proposition and its converse proposition. These modifications were made to provide a more comprehensive view of students’ understanding of the generality of proof.

2. The London study asked students to consider a visual proof separate from the other proofs of the same section. In this study, visual proofs were included in the section with the other proofs. Whilst I was interested in students' impressions about visual proofs, I was concerned about students having enough time to complete the survey.

3. In the geometry section of this study, a proof using geometric software was included. This could be considered similar to including a visual proof but it involved several examples of the figure under consideration as it was dragged. Students were asked to comment on the solution in a similar fashion to the other
sections. Whilst, this study does not study the impact of technology in an extensive way, I was interested to know how students thought it might help them with proof.

With the second instrument, *Student Survey on Justifying and Proving in School Mathematics – Part II*, there was not the same need to pilot it. As it was an adaptation of a funded and published study, its format was deemed to be without any major flaws. Much of the content was changed to make its use with senior students feasible. Several of the questions that were to be used for sample proofs (i.e. section A4 for example) were posed to students in my own classes as open-ended, construct a proof-type problems. Note that these were not the same students who participated in the main study. The various answers that students gave served as inspiration for developing sample proofs, both valid and erroneous. Therefore, the sample proofs used in the instrument are typical of what secondary students produce when asked to construct a proof. Again, these were not the same students who participated in the main study.

The sample proofs that form the core of the instrument were intended to show a wide range of styles of proof. They were also intended to show a wide range of proofs that students might do if themselves. Therefore, some proofs are seriously flawed, others have minor flaws, and some are almost perfect. Of course, there is an element of subjectivity in determining whether a proof is perfect, flawed or seriously flawed. With such a wide spectrum of proofs it is not always possible to know why a student selected a particular proof. Was the style or the content of the particular proof the reason for selection? It is beyond the scope of this instrument and this thesis to make deductions about why a particular student selected a particular proof. It is however, within the scope of this study, to make general observations and conclusions about the typical student.

**Summary**

This instrument’s inspiration was the survey used in the University of London/Institute of Education’s study of grade ten students’ competencies with proof. Generally, this study used the same format, changing the study items to make them more suitable for older students. A few modifications were made: more emphasis on the
generality of proof, less emphasis on visual proof, and the inclusion of an item involving technology.

Main Study

Participants

The target population was grade 12 and Ontario Academic Credit (OAC) students. Seventy students from three classes of mathematics students studying grade 12 or OAC mathematics (1 grade 12 advanced, 1 OAC Calculus and 1 OAC Algebra and Geometry class) participated in the study. Two classes came from a large secondary school of about 1800 students while the other class came from a school of approximately 1100 students. These two schools were typical of a large comprehensive secondary school that would be found in a city of about 250 000. It should also be noted that these schools, whilst located in the city, include county students. This school board does not have any schools that could be characterised as county schools. Hence each secondary school has enrolment comprised of city and county dwellers. As such, the students who participated in the study, could be considered as typical of the senior advanced-level Ontario student.

I was not the teacher of any of the students who participated in the study, nor was I a teacher at the schools that participated.

Data Collection

All of the data collected came from the two instruments. The first instrument, Justifying and Proving in School Mathematics – Part I was administered over a 50 minute period of time. The second instrument, Justifying and Proving in School Mathematics – Part II was administered during a 75 minute classroom period. For each of the three classes which participated in the survey, detailed administrative instructions were followed by the classroom teacher who administered the surveys to his class. These instructions are included in Appendix B. The second instrument was administered either the following day or the day afterwards. All three classes were administered the instruments in the first part of June 1999.
Data Analysis

The confidentiality of participants was assured at the outset of the process on the consent forms. Consent forms are included in Appendix D. To link the two instruments each participant was coded A101, A102 etc. Hence, the conclusions drawn from instrument two for a particular participant could be compared with the conclusions from instrument one. The teachers used a form which listed the student name and code. It was decided to list the names to ensure that the teacher made sure the same student got the same code for each instrument. I was the only person who had access to the list of names after they had been submitted.

The data from Part II was entered into Microsoft Excel work sheets using the student codes. This software allowed for computing average ratings on the hundreds of items of the second instrument. The data from Part I, all written answers to questions about the nature of proof, were entered into Microsoft Word. This allowed each question’s responses to be collated together. Conclusions about the data were much more easily accomplished with the data for each question collated separately.

The data from Part I was coded by hand using the coding developed from the pilot study. This enabled an examination of the first instrument’s data independently of the second instrument. Many of the conclusions about attitudes and beliefs about proof came from the analysis of the data from the Part I instrument.

Ethical Issues

To gain access to schools, the director of education for the school board was contacted by letter. Subsequent to approval, a letter to the teachers, detailing the administrative procedures was also sent. The survey administrative procedures are included as Appendix C. Ethical review documentation and the letter sent to the director of education is included as Appendix D.
Chapter Four: The Findings

Mathematicians are like lovers ... Grant a mathematician the least principle, and he will draw from it a consequence that you must also grant him, and from this consequence another.

- Fontenelle

This chapter provides the results of the data analysis of the two survey instruments. The two survey instruments are included as appendices A (Part I) and B (Part II). Conclusions are discussed in detail with specific rationale and reference to the instruments to support the conclusion. Most of the conclusions are derived from the analysis of the second instrument (Part II) which consisted of two curriculum areas. In the following discussion, the instrument questions are referred to as A1 for question 1 in the algebra section or as G4 as question 4 in the geometry section. These can be found in Appendix B.

**Finding 1: Students have great difficulty in constructing proofs of all types. However, they are better at constructing geometric proofs than proofs which could be classified as arithmetical or algebraic.**

In each of the algebra and the geometry sections there were two opportunities for students to try to construct their own proofs. The results of the survey are summarised in Table 1:

Table 1

| Percentage Comparisons of Students’ Responses to Proof Construction Items |
|-----------------------------|----------------|----------------|----------------|----------------|
| Correct                     | 11.4           | 5.7            | 23.9           | 15.5           |
| Almost Correct              | 14.3           | 0              | 14.1           | 15.5           |
| Incorrect attempt           | 67.2           | 54.3           | 43.7           | 42.3           |
| No attempt                  | 7.1            | 40.0           | 18.3           | 26.8           |

The first opportunity to prove a result was the following:

**A2:** If p and q are any two odd numbers \((p+q)(p-q)\) is always a multiple of 4.
This should be a fairly easy proposition for a senior advanced mathematics student to prove. Only 11.4% of the students could correctly prove this result. Another 14.3% incorrectly assumed the numbers were consecutive or had some other minor flaw, but generally they almost assembled a logical proof. Therefore, counting those proofs which were close to being correct, only 25.7% of the students were able to prove a result which should not have been too difficult for senior secondary students. Of the remaining 67.2% who attempted to prove the result, (7.1% made no attempt), “proofs” ranged from overly-complicated and nonsensical algebraic manipulations to simply plugging in numbers. Several students attempted to use mathematical induction and one student tried an indirect proof.

The second opportunity to prove a result came at the end of the algebra section:

A6: The square of a two digit number is decreased by the square of the number formed by reversing its digits. Prove that the number formed in this way is divisible by the sum of the digits of the original number.

This is a harder proposition, but still not beyond what should be expected of a senior advanced student. There were several difficulties evident. Many students (40%) did not attempt this problem as they ran out of time for the first section and were instructed to proceed to the next section. Some students who did attempt the problem did not read it carefully enough and hence did not understand what they were supposed to do. Only 5.7% of students could correctly prove this result. There were no other proofs that were close to being correct.

Students who used solely numerical examples to justify the result constituted 22.8% of the sample. Of these, 62.5% (constituting 14.3% overall) were unable to make sense of what they were trying to prove.

In the geometry section, students again had two opportunities to try to construct their own proof. The first such opportunity was to prove the following:

G3: A is the centre of a circle and AB is a radius. C is a point on the circumference where the perpendicular bisector of AB crosses the circle. Prove whether or not triangle ABC is always equilateral.
Students were provided with the above diagram for this question. This is a fairly simple proposition for a senior advanced level student and yet, as in the algebra section, the results are disappointing. In assessing the geometry proofs, it is necessary to be a bit more flexible with approaches. A proof was judged correct if it was perfect or it had a few minor flaws that did not compromise the logical progression of the proof. For example, using the parallel lines theorems without first stating which lines are parallel, would not disqualify a proof from being considered correct. Using this criteria of correctness, 23.9% of students correctly prove this result. Another 14.1% had produced a proof that was flawed but demonstrated that they understood what was required to produce an acceptable proof. Of the remaining students, 18.3% made no attempt, 18.3% produced nonsensical proofs and 25.4% demonstrated some concept of what was expected but couldn’t quite pull it all together.

The second geometric proof construction exercise consisted of the following:

G7: RSTW and SXTW are (overlapping) parallelograms. Prove triangles RSW and SXT are congruent.
Students were again furnished with the diagram. As expected, the second proposition, produced results which were slightly worse. It could be argued that this proposition is no more difficult than the first one. Many students complicated the proof by referencing several angles when the proposition can be easily proven by the side-side-side postulate. However any correct proof was deemed acceptable. On that basis the results are as follows: 15.5% produced a correct proof and another 15.5% produced a proof with minor flaws that did not compromise the essence of the proof. Consequently, 31% of the students could produce a proof which was either correct or nearly correct. Of the 41.3% who produced incorrect attempts, 12.7% had some idea of how to proceed and 29.6% produced nonsense. The remaining 26.8% made no attempt.

From the results in Table 1, it can be seen that students are better at geometric proofs. More than twice as many students were able to produce the easier geometric proof as were able to produce the easier algebra proof (23.9% versus 11.4%). Further, almost three times as many students were able to produce the second geometric proof as were able to produce the second algebraic proof (15.5% versus 5.7%). This suggests that students have more experience with proving in geometry than in other realms. Perhaps students have a been taught a framework for proving in geometry and are therefore able to produce a reasonable attempt at a proof.

However, it should also be noted that comparing no attempts for A2 and G3, it can be observed that more students are unwilling to attempt a geometric proof than an algebraic one. Perhaps this can again be linked to prior experience. If the experience was a bad one (i.e. geometry proofs), some students may tend to avoid it whereas more students would be willing to try the algebra proof since they did not have any experience with it (and hence not a bad experience).

It is difficult to make any meaningful comparisons between A6 and G7 which came at the end of the sections. The large number of non-attempts for A6 may be due to the instruction for students to move on to the geometry section after 30 minutes.
**Finding 2:** Students who perform better at algebra proof construction are much more likely to believe that proof is necessary to convince themselves of the validity of a proposition. Such a connection does not appear in geometric proving.

When students were asked question 1 (Survey, Part I):

Think of the last mathematical procedure you were taught. How do you know that the procedure is valid? How were you convinced?

slightly more than half (52.9%), said that proof was necessary. However, when the answer "proof" was compared to how well students did on the proof construction items, there is a relationship between ability in proof construction and belief that proof is important. The relationship is more pronounced in algebraic proving than in geometric proving as shown in table 2.

**Table 2**

Comparison of Proof Construction Ability with Belief that Proof Is Important

<table>
<thead>
<tr>
<th>Question</th>
<th>Correct or Almost Correct</th>
<th>Number of Correct or Almost Correct</th>
<th>Number who believe proof is necessary</th>
<th>Percentage of Correct/Almost who believe proof is necessary</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>Correct</td>
<td>7</td>
<td>6</td>
<td>85.7</td>
</tr>
<tr>
<td>A2</td>
<td>Almost</td>
<td>9</td>
<td>6</td>
<td>66.7</td>
</tr>
<tr>
<td>A6</td>
<td>Correct</td>
<td>3</td>
<td>2</td>
<td>66.7</td>
</tr>
<tr>
<td>A6</td>
<td>Almost</td>
<td>0</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>G3</td>
<td>Correct</td>
<td>16</td>
<td>9</td>
<td>56.3</td>
</tr>
<tr>
<td>G3</td>
<td>Almost</td>
<td>10</td>
<td>5</td>
<td>50.0</td>
</tr>
<tr>
<td>G7</td>
<td>Correct</td>
<td>11</td>
<td>5</td>
<td>45.5</td>
</tr>
<tr>
<td>G7</td>
<td>Almost</td>
<td>11</td>
<td>8</td>
<td>72.7</td>
</tr>
</tbody>
</table>

Note: Only students who answered question 1 of Survey Part I were compared with their proof construction results.
Of those students who answered question 1 of the Survey Part I, 52.9% said that proof was the whole or part of what convinced them of the validity of a theorem or procedure. Some students gave more than one answer. For example, several students said how it relates to other things they know (i.e. systemisation). However, if a student made mention of proof, they were included in the 52.9%.

When this result is compared to students who were successful or almost successful at the proof construction items, a stronger belief in proof's role in convincing emerges. The relationship is more pronounced in algebra, as 85.7% of those students who were successful with proving A2 also said that proof was important to determine a proposition's validity. When students who constructed almost correct proofs are considered the result is somewhat less, but still significantly greater than 52.9%, at 66.7%. Only a small number were successful with A6.

Somewhat surprisingly, this need for validity and conviction, does not appear to have the same connection with proving ability in geometry. In geometry, when correct responses to G3 are considered, 56.3% also said proof was important. Correct responses to G7 produced 45.5% who also said proof was important. Hence, the need for conviction and validity from a proof seems to be less of a factor in geometry. In geometry, when one considers students' belief about the need for proof to validate, there appears to be little difference between students who were successful with constructing proofs and those who were not successful. A possible explanation is that geometric proving can have an algorithmic feel to it. Students can approach it in the same manner as solving equations because the format can be set. Unlike the algebra proving, a known framework may have led them to be convinced as result of form as opposed to substance. Another explanation is that geometry is more visual and students may already be convinced by the diagram. It is also possible that the algebra items were more unfamiliar to them and so the value of proof was greater. This is consistent with the view of proof as removing doubt, if there is any. There may have been doubt in the algebra items but not so in the geometry items.
**FINDING 3: A significant percentage of students, when asked to construct a proof, resort to a “proof by examples”. However, students appreciate the value of a general argument instead of specific examples even if they would not be able to construct the general argument proof themselves.**

When students are asked to construct their own proof, a proof-by-examples is the proof provided by almost 40% of students who provided any proof. The table of the previous section is further refined (Table 3) to show what proportion of incorrect proofs could be classified as proofs-by-examples. The columns *Percentage of attempts* adjusts the categories by only considering the students who made an attempt. For example, in A2, 25.7% of the proofs were correct or almost correct. Since 92.9% of students made an attempt, this constitutes $25.7/92.9$ or 27.7%.

Table 3

<table>
<thead>
<tr>
<th>Proof by Examples with Other Types of Proofs</th>
<th>Algebra A2</th>
<th>Percentage of attempts</th>
<th>Algebra A6</th>
<th>Percentage of Attempts</th>
<th>Average of adjusted A2 and A6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct or Almost Correct</td>
<td>25.7</td>
<td>27.7</td>
<td>5.7</td>
<td>9.5</td>
<td>18.6</td>
</tr>
<tr>
<td>Incorrect Proof by examples</td>
<td>35.7</td>
<td>38.4</td>
<td>22.8</td>
<td>38.2</td>
<td>38.3</td>
</tr>
<tr>
<td>Incorrect Proof by other means</td>
<td>31.5</td>
<td>33.9</td>
<td>31.5</td>
<td>52.8</td>
<td>43.4</td>
</tr>
<tr>
<td>No attempt</td>
<td>7.1</td>
<td>-----</td>
<td>40.3</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>Percentage of Incorrect that are proof by examples</td>
<td>53.1</td>
<td>53.1</td>
<td>42.0</td>
<td>42.0</td>
<td>-----</td>
</tr>
</tbody>
</table>

In their proof of A2, a significant percentage of students (35.7%) used anywhere from one to several particular solutions to argue that the result had been proved. Included in the 35.7%, about one-fifth of the students or 7.1%, tried first to use an algebraic
argument, but when that failed to work, they reverted to examples. This indicates, perhaps, an understanding that particular examples alone are not a proof.

In the proofs of A6, as stated earlier, 22.8% thought that a few examples could be used to prove the result. This result is consistent with the previous result for A2, when only those students who attempted the problem are considered. For A2 there were 23 out of 65 attempts (35.4%) and for A6, there were 16 out of 42 attempts (38%). In fact, when no attempts are taken out of the calculations, the results for each of A2 and A6 are almost identical, about 38% of students produced proofs by example.

However, the results are a little better when students are asked to select from among sample proofs. In several of the items in the survey students are asked to rate sample proofs. A rating of 1 through 5, 5 being the best, was given to each sample proof. These ratings are then averaged to arrive at a student rating for each sample proof. In Table 4, two sample proof-by-examples are compared:

Table 4
Comparison of Student Ratings of Sample Algebra Proofs that Are Proofs by Example

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof by Examples</th>
<th>Average Rating</th>
<th>Percentage of students who would choose as theirs</th>
<th>Percentage of students who gave the highest rating of 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>Bonnie</td>
<td>3.0</td>
<td>25.0</td>
<td>11.8</td>
</tr>
<tr>
<td>A5</td>
<td>Nisha</td>
<td>3.2</td>
<td>28.1</td>
<td>11.4</td>
</tr>
</tbody>
</table>

Whilst sample proofs that are merely examples were ranked about average (on a scale of 1 being worst and 5 being best), they are not ranked very highly as the best proof (see the last column). Their rating as what students would do themselves is higher recognising, perhaps, that students recognise the limitations but will provide such a proof when they cannot do any better.

It should be noted that constructing a proof by examples seems to be limited to proofs involving algebra or arithmetic. In the geometry proofs G3 and G7, no such examples of "proving by example" were noted. It should also be noted that the lack of "proving by example" in geometry may be more a result of students’ inability to see specific cases in the geometric realm when these are obviously easily seen in arithmetic and to a lesser extent algebra.
Further, in geometry, students tend not to rate proofs by example highly. They tend to be less convinced by example proofs in geometry than they would be in algebra. The results of the geometry sections G1, G2 and G4 are summarised in the Table 5.

Table 5

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof by Examples</th>
<th>Average Rating (of ratings from 1 through 5)</th>
<th>Percentage of students who would choose as theirs</th>
<th>Percentage of students who gave the highest rating of 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>Irene</td>
<td>2.3</td>
<td>7.1</td>
<td>1.4</td>
</tr>
<tr>
<td>G2</td>
<td>Kobi</td>
<td>2.2</td>
<td>2.8</td>
<td>4.2</td>
</tr>
<tr>
<td>G4</td>
<td>Dylan</td>
<td>2.3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Dylan’s (G4) proof consisted of four examples of measurements. It received an average rating of only 2.3. Kobi’s (G2) consisted of drawing the diagram and moving a point to different locations and taking measurements and received an average rating of 2.2. Irene’s (G1) proof consisted of a diagram with several different line segments drawn. It also received only 2.3. Unlike algebra proofs, in geometry students do not value proofs by example as highly.

Thus, proofs by example are constructed in significant numbers by students but only in algebra/arithmetic. They are also selected, with some reservations, as valid proof. In geometry, students do not construct proofs by example, nor do they seem to select them as valid proof as much as they do in algebra/arithmetic.
**Finding 4:** In proving arithmetical or algebraic results, there is a tendency amongst students to attempt to construct an algebraic argument over one that could be described as prose. When students are asked to select from among several types of proofs they tend to select algebraic formal proofs over other less formal but equally valid proofs.

Students tend to think that an algebraic formal proof is more mathematically valid than less formal (but equally valid) proofs. When students are asked to construct a proof, they tend to try to use algebra even when algebra is unnecessary and when their lack of skill inhibits them.

The results of the two opportunities to prove in the algebra section are summarised in Table 6:

<table>
<thead>
<tr>
<th>Question</th>
<th>Percentage Correct</th>
<th>Percentage Correct Algebraic Argument</th>
<th>Percentage Correct Prose Argument</th>
<th>Percentage Almost Correct</th>
<th>Percentage Almost Correct Algebraic Argument</th>
<th>Percentage Almost Correct Prose Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>11.4</td>
<td>5.7</td>
<td>5.7</td>
<td>14.3</td>
<td>12.9</td>
<td>1.4</td>
</tr>
<tr>
<td>A6</td>
<td>5.7</td>
<td>5.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Of those students who correctly proved A2, it is roughly split down the middle between an algebraic argument and a prose argument. Of those students who almost had a proof, an overwhelming percentage (90%) used an algebraic argument. This suggests that one of the limiting factors in achieving a successful proof is the lack of strong algebraic skills.

For the proofs of A6, only one student attempted to prove it using prose and was unsuccessful. As expected, as results become more complicated, algebra can be more efficient and effective.

However, of the 34.3% who tried to use an algebraic solution, most were nonsensical arguments, which either were complicated by the introduction of too many variables, or were defeated by a lack of algebraic skill.

When students are asked to select from among several types of proofs they tend to select algebraic formal proofs thinking they are more mathematically valid than less
formal (but equally valid) proofs. Consequently, they tend to select proofs that are algebraic over those which can be characterised as prose. The results are summarised in Table 7.

In A1, students were asked to rate several proofs of:

when you add the squares of any two consecutive numbers, your answer is always odd.

Table 7

**Comparison of Student Ratings of Sample Proofs by Type: Algebraic, Prose or Visual**

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof</th>
<th>Type of Proof</th>
<th>Average Rating</th>
<th>Percentage of students who would choose as theirs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>Arthur</td>
<td>Algebraic</td>
<td>3.1</td>
<td>7.8</td>
</tr>
<tr>
<td>A1</td>
<td>Bonnie</td>
<td>Examples</td>
<td>3.0</td>
<td>25.0</td>
</tr>
<tr>
<td>A1</td>
<td>Ceri</td>
<td>Algebraic</td>
<td>3.9</td>
<td>48.4</td>
</tr>
<tr>
<td>A1</td>
<td>Duncan</td>
<td>Prose</td>
<td>3.0</td>
<td>6.3</td>
</tr>
<tr>
<td>A1</td>
<td>Eric</td>
<td>Prose</td>
<td>3.5</td>
<td>12.5</td>
</tr>
<tr>
<td>A1</td>
<td>Fatima</td>
<td>Visual</td>
<td>1.7</td>
<td>0</td>
</tr>
<tr>
<td>A5</td>
<td>Mike</td>
<td>Algebraic</td>
<td>2.9</td>
<td>17.2</td>
</tr>
<tr>
<td>A5</td>
<td>Nisha</td>
<td>Examples</td>
<td>3.2</td>
<td>28.1</td>
</tr>
<tr>
<td>A5</td>
<td>Otis</td>
<td>Algebraic</td>
<td>3.3</td>
<td>23.4</td>
</tr>
<tr>
<td>A5</td>
<td>Petra</td>
<td>Prose</td>
<td>3.7</td>
<td>31.3</td>
</tr>
</tbody>
</table>

In A1, there were six sample proofs from which students could choose. Using a rating scale of 1 (worst) to 5 (best), students rated each proof. A correct formal (algebraic) proof (Ceri’s) received the highest average rating of 3.9. Almost half (48.4%) of the students said this would be the proof that they would have done if they had been asked to construct it. An equally correct, and perhaps more explanatory proof, (Eric’s) received the next highest rating of 3.5. An incorrect, but formal proof (Arthur’s) ranked third with a rating of 3.1. It was encouraging that a correct prose argument was higher rated than the incorrect formal proof. However, another almost correct prose-like argument, (Duncan’s), came fifth with a rating of 2.4.

In A5, students were asked to rate four proofs of:

If k is an even number then 48 is always a factor of \( k^3 - 4k \).

Of the four proofs the two primarily correct ones can be classified as a primarily algebraic one and another that is a mixture of algebra and prose. The incorrect proofs consist of a proof by examples and an algebraic proof. In this section, students correctly
rated the two more correct proofs more highly than the incorrect proofs. The algebraic/prose proof (Petra’s) had an average of 3.7 whilst the primarily algebraic proof (Otis’) had an average of 3.3. However, the results for all four proofs are very close. The incorrect proofs rated 3.2 (Nisha’s proof by examples) and 2.9 (Mike’s algebraic answer).

Because of the expression is algebraic to begin with, proofs favouring algebra are to be expected. Both Mike’s incorrect answer and Otis’ correct answer introduce an additional variable. Both of these proofs are rated almost as highly as Petra’s more explanatory proof which uses the minimum amount of algebra. If students think an algebraic formal argument is more mathematically sound they choose such proofs and they will attempt to construct such when asked to develop a proof. Even erroneous, nonsensical proofs like Arthur’s answer are chosen over more valid, but less formal proofs. For many students, the existence of lots of variables and the illusion of sophisticated manipulations seem to be sufficient to justify the validity of a proof. Proof by examples was chosen by fewer students than would have probably supplied such a proof if asked to construct it themselves.
**FINDING 5:** In geometry, two-column proofs are highly favoured by students as the format they use when asked to construct a proof of their own. Further, when asked to select from several types of proofs, students gravitate towards proofs that are of the two-column format even if they are invalid proofs.

In the construction of geometric proofs, there is an overwhelming preference for proofs which could be characterised as a two column (statement / justification) format. Students had two opportunities to construct proofs in geometry. The results are summarised in Table 8:

Table 8: Comparison of Student Constructed Geometry Proofs: Two Column Format

<table>
<thead>
<tr>
<th>Question</th>
<th>Percentage who attempted a proof</th>
<th>Percentage of attempted proofs that were two column type</th>
<th>Percentage of attempted proofs that were other formats</th>
</tr>
</thead>
<tbody>
<tr>
<td>G3</td>
<td>81.7</td>
<td>82.1</td>
<td>17.9</td>
</tr>
<tr>
<td>G7</td>
<td>73.3</td>
<td>81.6</td>
<td>18.4</td>
</tr>
</tbody>
</table>

Of those students who attempted a proof of G3, 82.1% used a two-column or a close variation of a two-column proof. This format of making a statement followed by the theorem or reason was also used by 81.6% of the students who attempted a proof of G7.

This format seems to be stressed in school and is a good way to present a geometric proof. Students who did not use this format were likely to make statements without justification and this was not considered an adequate proof. Further, students also tend to select two-column proofs from among sample proofs as shown in Table 9.
In geometry, students tend to favour two-column proofs, even if these are wrong, over more valid but less traditional formats. In G1, Frank’s two-column proof that the incorrect statement is true received an average rating of 2.8, making it third. Jacob’s correct two-column proof had the highest rating. In G2, Natalie’s incorrect two-column proof was rated second with an average rating of 3.3. Linda’s correct two-column proof was first with 4.6. In G4, Barry’s essentially one example written up as a two-column proof received a rating of 3.6 which was very close to Ewan’s visual proof (3.7) and Cynthia’s correct two-column proof (4.1).

The form of a two-column proof, even an incorrect one, is enough to convince many students of its validity.
FINDING 6: STUDENTS CORRECTLY RECOGNISE BOTH THE GENERALITY OF A PROVEN RESULT AND ITS LIMITATIONS.

HOWEVER, A MAJORITY OF STUDENTS ARE UNABLE TO DISTINGUISH BETWEEN A PROPOSITION AND ITS CONVERSE AND THE NEED FOR AN ADDITIONAL PROOF TO ESTABLISH THE CONVERSE RESULT.

Two questions, one in each domain, explored students’ ability to recognise the generality and the limitations of a proof. Given a “change” in the premise, students were asked to determine if the previous proof is sufficient or whether another proof (or possibly, a refutation) would be necessary. The results are summarised in Table 10.

Table 10
Summary of Student Responses to Whether a New Proof Was Necessary Given Changes to the Original Proof’s Premise

<table>
<thead>
<tr>
<th>Question</th>
<th>Statement</th>
<th>Change made to original</th>
<th>Requires new Proof? Yes or No</th>
<th>Percentage who say first proof suffices</th>
<th>Percentage who say a new proof is necessary</th>
</tr>
</thead>
<tbody>
<tr>
<td>A3</td>
<td>2</td>
<td>odd consecutive numbers</td>
<td>No</td>
<td>81.2</td>
<td>18.8</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>even numbers</td>
<td>Yes</td>
<td>13.0</td>
<td>87.0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>odd perfect squares</td>
<td>No</td>
<td>58.0</td>
<td>42.0</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>converse proposition</td>
<td>Yes</td>
<td>60.9</td>
<td>39.1</td>
</tr>
<tr>
<td>G5</td>
<td>2</td>
<td>rectangle</td>
<td>No</td>
<td>80.3</td>
<td>19.7</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>trapezoid</td>
<td>Yes</td>
<td>7.0</td>
<td>93.0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>converse proposition</td>
<td>Yes</td>
<td>56.3</td>
<td>43.7</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>rhombus</td>
<td>No</td>
<td>54.9</td>
<td>45.1</td>
</tr>
</tbody>
</table>

In A3, 81.1% of students recognised that the result was still valid if the premise was changed to consecutive odd numbers. Less (60.0%) were sure that the result was still valid when the premise was changed to odd perfect squares. In G5, 80.3% of students knew that the result continued to be valid if the premise changed from a parallelogram to a rectangle. However, only 54.9% were sure that a new proof was not required if the premise changed to a rhombus. It is not clear from the research whether or not this result is indicative of the understanding of generality or whether it is a simple as students not
knowing what a rhombus is. Whilst, one would expect that a seventeen year old student would know what a rhombus is, it is quite possible that many do not.

It can be concluded that students recognise, with some limitations, that particular cases of a more general already proven result do not require new proofs.

Students are also able to recognise the limitations of proven results. In A3, students were told that it has been correctly proven that:

If p and q are any two odd numbers, \((p+q)(p-q)\) is always a multiple of 4.

They were then asked whether or not a new proof needed to be constructed given certain changes. When the premise was changed from odd numbers to even numbers, 87.0% of students correctly concluded that a new proof was necessary.

Similarly in G5, students were told that it had been correctly proven that:

The opposite angles of a parallelogram are congruent.

If the figure is changed from a parallelogram to a trapezoid, 93.0% of students correctly deduced that a new proof was necessary.

Students are not aware of the difference between a proposition and its converse. They do not see the need for additional proof suggesting that they do not understand the structure of an if – then proposition. In A3, only 39.1% of the students were able to correctly determine that a new proof was required if the statement was changed to:

If \((p+q)(p-q)\) is a multiple of 4, p and q are any two odd numbers.

The result in the geometry section G5 is somewhat better. When the statement was changed so that premise is that figure is a quadrilateral with the equal opposite angles and the conclusion is that it is therefore a parallelogram, 43.7% of the students determined that a new proof was required.

This would seem to indicate that there is a lack of attention to theorems and their converses in the mathematics curriculum. The modified statement, while phrased somewhat awkwardly, perhaps was not seen to be different in a mathematical sense by the majority of students.

Students do not seem to be aware of the structure of the if-then proposition and hence do not see its converse as a different proposition requiring a new proof.
FINDING 7: STUDENTS DO NOT SEE VISUAL PROOFS AS VALID MATHEMATICAL PROOF. THEY ARE MORE INCLINED TOWARDS VISUAL PROOFS IN GEOMETRY THAN IN OTHER MATHEMATICAL DOMAINS LIKE ARITHMETIC AND ALGEBRA.

There were two examples of visual proof, one in each of the domains, from which students could choose as a sample proof as shown in Table 11:

Table 11
Summary of Student Ratings of Sample Proofs that are Visual

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proofs that are Visual</th>
<th>Average Rating</th>
<th>Percentage of students who would choose as theirs</th>
<th>Percentage of students who gave the highest rating of 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>Fatima</td>
<td>1.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>G4</td>
<td>Ewan</td>
<td>3.7</td>
<td>27.8</td>
<td>29.2</td>
</tr>
</tbody>
</table>

Fatima’s answer provided a proof in visual form. It might be argued that this too is not a valid proof since it does not provide a “general” conclusion. It is also somewhat flawed in that it does not clearly show the connection between the consecutive numbers. However, if it is seen solely as examples it could be considered as valid as Bonnie’s answer. It was rated the lowest at 1.7. Not one student chose Fatima’s answer as the answer that they would give. Further no one rated it as worthy of the highest rating of 5. It might be concluded that visual proofs do not constitute a significant part of the curriculum as students have overwhelmingly rejected this form of proof.

Hence, in algebra, students are wary of proofs which are visual. Even though they are often quite willing to accept proofs based on a few specific examples, when these examples are visually demonstrated, they are not convinced that they constitute a valid proof.

However in geometry, students are more inclined to accept a visual proof as acceptable. Ewan’s answer consisted of drawing a tessellation of parallelograms to prove that the opposite interior angles of a parallelogram are congruent. This proof received the second highest rating (average of 3.7), with 29.2% stating that Ewan’s solution was worth a maximum mark of 5 and perhaps surprisingly 27.8% saying it was the proof they would have provided if asked. The highest average rating of 4.1 went to Cynthia’s correct traditional two column proof. Further evidence of this, is the relatively good rating
received by Gail’s answer in section G1 (see table 9). A diagram proof in which Gail constructed an arc with compass to prove that the shortest distance from a point to a line segment is not necessarily to the midpoint, received the second highest average rating of 3.1. Jacob’s correct two column proof received the highest average rating of 4.0.

Comparing the ratings of Fatima’s and Ewan’s solutions, it can be argued that students are more accepting of visual proofs in geometry than in algebra. Almost 30% thought Ewan’s solution was the best whilst no one thought Fatima’s deserved a mark of 5. Hence, visual proofs are deemed by students to be more acceptable to students in geometry than in algebra. Students rate visual geometric proofs almost as highly as a traditional two-column proof.
FINDING 8: **Students recognise the power of an explanatory proof even if they would have difficulty constructing one.**

Students do not always construct proofs to be explanatory but they seem to appreciate the value of an explanatory proof. The results for algebra are summarised in Table 12:

Table 12

**Summary of Student Ratings of Sample Algebra Proofs**

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof</th>
<th>Average Rating (ranking 1 to 5)</th>
<th>Percentage of students who would choose as theirs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>Arthur</td>
<td>3.1</td>
<td>7.8</td>
</tr>
<tr>
<td>A1</td>
<td>Bonnie</td>
<td>3.0</td>
<td>25.0</td>
</tr>
<tr>
<td>A1</td>
<td>Ceri</td>
<td>3.9</td>
<td>48.4</td>
</tr>
<tr>
<td>A1</td>
<td>Duncan</td>
<td>3.0</td>
<td>6.3</td>
</tr>
<tr>
<td>A1</td>
<td>Eric</td>
<td>3.5</td>
<td>12.5</td>
</tr>
<tr>
<td>A1</td>
<td>Fatima</td>
<td>1.7</td>
<td>0</td>
</tr>
<tr>
<td>A5</td>
<td>Mike</td>
<td>2.9</td>
<td>17.2</td>
</tr>
<tr>
<td>A5</td>
<td>Nisha</td>
<td>3.2</td>
<td>28.1</td>
</tr>
<tr>
<td>A5</td>
<td>Otis</td>
<td>3.3</td>
<td>23.4</td>
</tr>
<tr>
<td>A5</td>
<td>Petra</td>
<td>3.7</td>
<td>31.3</td>
</tr>
</tbody>
</table>

In A1, both Ceri’s (formal) and Eric’s (informal) proofs are highly explanatory. Eric’s answer is perhaps the more explanatory and was rated high enough (3.5) to indicate that students see the value of such a proof. However, only 6.3% of students said it would be the proof that they would provide if asked for a proof.

Explanatory proofs can take many forms and each proof, is in some way, explanatory. A prose argument tends to be more conversational and perhaps for this reason may be seen as more explanatory. Whilst students still prefer the formal algebraic proof and tend to provide these when asked for proof, a higher percentage of students recognised the validity of Eric’s answer even if they wouldn’t have written up the proof in that manner.

A similar result was found in A5. The two correct answers (Otis’ and Petra’s) combined algebra with prose and hence had a high level of explanatory power. These were rated highly by students (3.3 and 23.4%; 3.7 and 31.3%).
In A4, in which the correct solution was to prove a proposition false, a more explanatory correct disproof (Iris’s proof, rating 3.2) was rated slightly more highly than a more direct proof by counterexample (Ken’s proof, rating 2.9). Ken’s proof was rated more highly as the proof students would do themselves.

Hence, in algebra, students recognise the power of an explanatory proof. They tend to see an explanatory proof as more valid than other less explanatory proofs, even if they would not provide such a proof if asked to construct it themselves.

In geometry, students tend to construct two-column geometric proofs. Students also tend to also pick two-column geometric proofs as the proof that is the most valid. These proofs could also be characterised as more explanatory since each statement is explained by stating a reason or theorem. The results are summarised in Table 13.

An explanatory non two-column proof was also rated reasonably well to support this conclusion, although there is a strong tendency towards two-column proofs. Marty’s solution (G2), is an explanatory prose argument that received a rating of 3.1 compared to a correct two-column proof (Linda, 4.6) and an incorrect two-column proof (Natalie, 3.3). Table 13

Summary of Student Ratings of Sample Geometry Proofs

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof</th>
<th>Average Rating (rating 1 to 5)</th>
<th>Percentage of students who would choose as theirs</th>
</tr>
</thead>
<tbody>
<tr>
<td>G2</td>
<td>Kobi</td>
<td>2.2</td>
<td>3.2</td>
</tr>
<tr>
<td>G2</td>
<td>Linda</td>
<td>4.6</td>
<td>87.3</td>
</tr>
<tr>
<td>G2</td>
<td>Marty</td>
<td>3.1</td>
<td>4.8</td>
</tr>
<tr>
<td>G2</td>
<td>Natalie</td>
<td>3.3</td>
<td>4.8</td>
</tr>
</tbody>
</table>
FINDING 9: **Students are able to recognise the falsity of a proposition. However, they will often select proofs of erroneous results if they are either algebraic enough (Algebra) or if they follow a two column format (Geometry).**

Students are able to recognise when a result to be proven is false. Two sections, one in algebra, A4, and one in geometry, G1, provided students with sample proofs of a proposition that is false. Some of the samples were refutations and some were incorrect proofs. The results of these two sections are summarised in Table 14.

**Table 14**

Summary of Student Ratings of Sample Proofs of False Propositions

<table>
<thead>
<tr>
<th>Question</th>
<th>Sample Proof</th>
<th>Incorrect “proof” or refutation</th>
<th>Average rating</th>
<th>Percentage of students who would choose as theirs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A4</td>
<td>Hassan</td>
<td>proof</td>
<td>2.3</td>
<td>20.3</td>
</tr>
<tr>
<td>A4</td>
<td>Iris</td>
<td>refutation</td>
<td>3.2</td>
<td>30.4</td>
</tr>
<tr>
<td>A4</td>
<td>Julie</td>
<td>proof</td>
<td>2.2</td>
<td>7.2</td>
</tr>
<tr>
<td>A4</td>
<td>Ken</td>
<td>refutation</td>
<td>2.9</td>
<td>31.9</td>
</tr>
<tr>
<td>A4</td>
<td>Lenore</td>
<td>proof</td>
<td>2.5</td>
<td>10.1</td>
</tr>
<tr>
<td>G1</td>
<td>Harriet</td>
<td>proof</td>
<td>1.4</td>
<td>9.2</td>
</tr>
<tr>
<td>G1</td>
<td>Irene</td>
<td>refutation</td>
<td>2.3</td>
<td>7.7</td>
</tr>
<tr>
<td>G1</td>
<td>Frank</td>
<td>proof</td>
<td>2.8</td>
<td>20.0</td>
</tr>
<tr>
<td>G1</td>
<td>Jacob</td>
<td>refutation</td>
<td>4.0</td>
<td>46.2</td>
</tr>
<tr>
<td>G1</td>
<td>Gail</td>
<td>refutation</td>
<td>3.1</td>
<td>16.9</td>
</tr>
</tbody>
</table>

In A4, students were provided with proofs or counter-proofs of an erroneous result:

*When you add the squares of any three consecutive numbers, your answer is always even.*

The two refutations received average ratings of 3.2 and 2.9. Sixty two (62.3) percent of the students said they would provide one of the correct refutations which proved the result to be not valid. The results were almost equally split between a counter-example (2.9 and 31.9%, Ken’s proof) and a more traditional argument (3.2 and 30.4%, Iris’s proof). A more traditional argument appeared slightly more valid even though the counter-example is clearly easier and more concise. The counter-example was slightly favoured as the proof the students would do themselves. However, a significant number of students (37.7%) selected erroneous proofs rated 2.5, 2.3 and 2.2. Whilst all of these
ratings are below an okay rating of 3, there was less variation in the ratings of these proofs, suggesting the introduction of a false proposition created some blurring of the distinctions between the proofs. Perhaps, as one might expect, the students have not had much experience with proofs of false propositions.

Similar conclusions can be drawn in geometry. In GI, students picked refutations that argued that the statement was false for their top two choices (Jacob, 4.0 and Gail, 3.1). A proof that incorrectly argued the proposition was true (Frank's two column proof, 2.8) was rated third over another refutation of the proposition (Irene, 2.3). Hence, students are able to determine truth or falsity of a proposition in geometry without too much difficulty. Again, an incorrect two-column proof has some appeal to students.

Hence, students are able to recognise when a result to be proven is false; although it would seem that they are not accustomed to checking for the truthfulness of the result. However, students seem to be prone to falling into the trap of recognising the form of an acceptable proof without too much concern for whether its content is valid. In algebra, despite evidence of the falseness of the proposition, a significant number of students are willing to accept erroneous proofs if they are algebraic enough.

Even though Julie's incorrect proof by examples (A4) was not seen as acceptable, two other more algebraic but also incorrect approaches (Hassan's proof, 2.3, 20.2% and Lenore's proof, 2.5, 10.1%) were rated higher. It would seem that the absence of a correct algebraic proof led some students to select an algebraic proof even if it wasn't correct. Even when a counter-example is given indicating that the proposition is false, a significant number of students will select a proof that the proposition is true if it is algebraic enough.

Similarly in geometry, despite evidence of the falseness of a proposition, an incorrect proof of its correctness is sometimes chosen if it is a two column proof format. In GI, Frank's two column proof of a false result received an average rating of 2.8 and was chosen as best by 17.1% (10.0% ranked it 4 and 24.2% ranked it a 3). The format of a two column proof leads some students to select a proof of a false proposition even when evidence exists to suggest the proposition is false. However, this seems to be less of a phenomenon in geometry than in algebra where students are more likely to be taken in by lots of algebraic symbols and manipulations.
**Finding 10:** Students find technology proofs convincing but do not see them as a valid general argument. They are unsure about their explanatory power.

In the geometry section, students had an opportunity to view a proof by technology. The results of their views are summarised in Table 15.

**Table 15**

**Student Ratings Regarding Proof Using Technology**

<table>
<thead>
<tr>
<th>G6:</th>
<th>Average Rating (rating from 1 to 5).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kay’s Answer</td>
<td>3.3</td>
</tr>
<tr>
<td>Statements to which students Agree (1), Don’t Know (2) or Disagree (3):</td>
<td>(rating from 1 to 3)</td>
</tr>
<tr>
<td>Question: Kay’s answer</td>
<td></td>
</tr>
<tr>
<td>Has a mistake in it</td>
<td>2.6</td>
</tr>
<tr>
<td>Shows that the statement is always true</td>
<td>2.3</td>
</tr>
<tr>
<td>Only shows that the statement is true for some parallelograms</td>
<td>1.7</td>
</tr>
<tr>
<td>Shows you why the statement is true</td>
<td>2.1</td>
</tr>
<tr>
<td>Is an easy way to explain to someone in your class who is unsure</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Proofs by technology tend to be rated as an average attempt at a proof.

In G6, Kay uses a geometry software package to prove whether a proposition is true or false. By constructing the figure, measuring and dragging the figure, she concludes that the result is valid. Students gave this a teacher mark with averaged 3.25 (3 signifies okay, 5 is best).

Hence, a proof by technology does not appear to be considered extremely valid by students. However, the several examples generated by the software are rated more highly than proofs by example that are done by hand.

Students are convinced by technology proofs.

In G6, students disagreed with the statement that Kay’s proof has a mistake in it. With a rating scale of (1: agree, 2: don’t know, 3: disagree), this statement had an average of 2.6. Only 3.2% of the students agreed with the statement, whilst 38.1% didn’t know and 58.7% disagreed. Students are convinced of the correctness of the proposition by the use of technology.
Students tend to think that proofs by technology lack generality.

In G6, students disagreed with the statement that Kay's proof shows that the statement is always true. Over half (52.3%) of the students disagreed with the statement while 20.6% didn't know and 27.0% agreed. If the don't knows are removed, twice as many students disagreed as agreed.

Somewhat more (60.3%) of the students agreed with the statement that Kay's proof only shows that the statement is true for some parallelograms. Less are undecided (11.1%) about this statement, while 28.6% disagreed.

Students are unsure about the explanatory power of a technology proof.

In G6, when asked to rate the statement that Kay's proof shows you why the statement is true, the average rating was 2.1 with the results almost evenly split between agree (33.3%), don't know (25.4%) and disagree (41.3%). When asked whether it is an easy way to explain to someone in your class who is unsure, similar results are found. The average rating is 2.0 with agree (42.9%), don't know (15.9%), disagree (41.3%). Whilst there are fewer students undecided about this question, as a group, they are quite divided on the question of whether or not technology offers explanation.
**FINDING 11: THERE APPEARS TO BE NO SIGNIFICANT CORRELATION BETWEEN GENDER AND THE ABILITY TO CONSTRUCT A MATHEMATICAL PROOF.**

Of the 70 students who participated in the survey, 28 or 40% were female whilst 42 or 60% were male. Table 16 considers the results from the students who were either able to construct or almost construct a correct proof of A2 and A6.

**Table 16**

Comparison of Student Responses to Proof Construction Items by Gender

<table>
<thead>
<tr>
<th>Gender</th>
<th>Percentage Participation</th>
<th>Number of Correct Proofs A2</th>
<th>Number of Almost Correct Proofs A2</th>
<th>Number of Correct Proofs A6</th>
<th>Number of Almost Correct Proofs A6</th>
<th>Total Correct or Almost Correct A2 &amp; A6</th>
<th>Total Correct or Almost Correct weighted by participation percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Female</td>
<td>40</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>8</td>
<td>$8 \times 100/40 = 20$</td>
</tr>
<tr>
<td>Male</td>
<td>60</td>
<td>5</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>14</td>
<td>$14 \times 100/60 = 23.3$</td>
</tr>
</tbody>
</table>

From the calculations in Table 16, it can be seen when correct and almost correct proofs are considered, the females produced 8 correct proofs while the males produced 14. However, when we consider the participation rate of the females was 40% and weight the results in light of this, we can see that the results are really 20 versus 23.3. It can be argued, therefore, that there is no significant difference in the ability to construct a proof when gender is considered.
FINDING 12: PROOF APPEARS TO PLAY A MINIMAL ROLE IN CONVINCING STUDENTS OF THE TRUTH OF A MATHEMATICAL RESULT.

When students were asked the questions:

In general, when new mathematics is introduced to you are you easily convinced? What does it take to convince you of the validity of the mathematical results?

they responded that they are easily convinced, as shown in Table 17.

Table 17
Comparison of Student Views on the Role of Proof in Convincing them of Validity

<table>
<thead>
<tr>
<th>Percentage easily convinced</th>
<th>Percentage</th>
<th>What convinces them?</th>
</tr>
</thead>
<tbody>
<tr>
<td>64.2$^\dagger$</td>
<td>10.4</td>
<td>Proof</td>
</tr>
<tr>
<td></td>
<td>26.9</td>
<td>Belief in the Teacher/Textbook</td>
</tr>
<tr>
<td></td>
<td>20.9</td>
<td>Examples</td>
</tr>
<tr>
<td></td>
<td>7.4</td>
<td>Other reasons</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Percentage not easily convinced</th>
<th>Percentage</th>
<th>What convinces them?</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.8$^\dagger$</td>
<td>11.9</td>
<td>Proof</td>
</tr>
<tr>
<td></td>
<td>20.9</td>
<td>Their own understanding</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>Belief in the Teacher/Textbook</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>Examples</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>Other reasons</td>
</tr>
</tbody>
</table>

Note: The percentages in the rightmost column do not sum to 64.2% or 35.8% since many students did not give examples. Also, some students gave several answers and these were included in the figures.

The comments made by students who said they were not easily convinced indicates that their lack of conviction is often a result of their perception that they do not understand. More than half of those, reflecting an overall percentage of 20.9% said that they were not convinced because they did not understand the mathematics. It would seem that a large majority (85.1%) of students are either easily convinced or relate the notion of conviction with their own level of mastery of the mathematics being taught.

Proof does not appear to play a significant role in convincing students whether or not they are easily convinced or not. Only 22.3% of the students mentioned proof in their responses to what it took to convince them of a result. The percentages are almost evenly split between those students who are easily convinced and those who are not. Further, it
should be noted that since students were told that this was a survey about proof, some of them might have been tempted to give the answer they thought the researcher was looking for. Perhaps, the percentages are even lower! The following comment from a student perhaps points out some of the issues surrounding proof:

When new mathematics is introduced it takes a while to sink in. Usually it makes sense at first but then as I think about it, I begin to question the concept and its procedures as I begin to work through it. Although proofs are sometimes helpful, sometimes they are based on other concepts that are hard to grasp, so they do not always help.

For many students, it seems that proofs could be helpful but because they are built on previous concepts and knowledge, they may be in accessible and consequently not very helpful. However, for some students, proof offers a complete explanation:

Again, to convince me of the validity of mathematical results, I tend to need to see the complete proof of the procedure and its derivation and then understand the concepts behind it. Otherwise, without understanding how the procedure operates, I find it is no use to me when it comes time to apply the procedures to problems encountered.

Whereas, proof is not necessary for many students to be convinced of a result, a large number of students are convinced solely by the authority of the teacher or the textbook. Not surprisingly the vast majority of those convinced by authority are easily convinced. Of those students who were easily convinced, 41.9% (26.9% overall) cited their belief in their teacher as the reason for being convinced (a very small number cite the textbook). Another 32.6% (20.9%) mentioned that examples were the reason. When students are not easily convinced, these reasons are given much smaller credence. Interestingly, proof becomes more important, relative to the other reasons, when students are not easily convinced. This, of course, mirrors the use of proof in the mathematical community. Often, a proof is taken for granted when the result seems fairly obvious. As the veracity of a result is called into question, the role of proof as conviction increases.
FINDING 13: STUDENTS ARE NOT ONLY INTERESTED IN HOW TO DO THE MATHEMATICS, THEY HAVE A GREAT NEED TO UNDERSTAND WHY IT WORKS.

The majority of students appreciate the need to understand both the how and the why of mathematics. Using Skemp's (1976) definitions of the two types of mathematics, students overwhelmingly stated that they preferred to understand both how and why. The results are compared in Table 18.

Table 18
Comparison of Student Views about Mathematical Understanding

<table>
<thead>
<tr>
<th>Which type of understanding is preferable</th>
<th>Percentage which is most important to them</th>
<th>Percentage who rated Very Important</th>
<th>Percentage who rated Somewhat Important</th>
<th>Percentage who rated Not Important</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand how</td>
<td>15.1</td>
<td>87.7</td>
<td>9.6</td>
<td>2.7</td>
</tr>
<tr>
<td>Understand how and why</td>
<td>84.9</td>
<td>90.4</td>
<td>4.1</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Whilst there will be always some students who believe that knowing the reasons why something works is a waste of time, it was surprising that so many (84.9%) responded that how and why was the type of understanding which was more important to them. The following comment from a student captures the sentiment:

Definitely how and why understanding is preferable even though some students act like it's a pain in the butt when they are taught not only the theory but how to apply it. It really helps to know why you need something.

Not surprisingly, students rated both types of understanding as very important. How is considered very important by 87.7% of the students. The remaining students said it was either somewhat or not important. Some of these students were really answering the question of which type is more important and because they believed that how and why was more important, they said that how was not as important. Most students recognised that understanding how is the first stage of understanding and as such is very important but that understanding how and why is more important. A small number, 5.5% (but larger than for how) stated that how and why was not important because the only thing that mattered was getting the answer or the marks on the test. However, the
prevailing view amongst students is that how and why offers a more complete understanding. A typical response is the following:

I need to know why in order to be taught how. How and why gives me a better understanding of everything because it shows exactly how to do a question and why you do it in such a way. It makes more sense to show how and why.

Many students have a great need to know the why behind the mathematics so that they can gain an understanding of why they are learning it. They also found it was easier to remember if they understood why. Several more students also saw that it is only in understanding how and why that one can adapt to new situations and solve related problems. Comments like the following came up several times:

Why is important because it gives you a greater understanding of the problem and allows you to grasp the more complicated problems.

Understanding how and why is very important because for certain problems in class you can apply the how and get the correct answer but if you come across a variation in a similar problem during a test or in homework then oftentimes you will need the “why” to do independent thinking and solve it.

Extremely important! I want to know how to do a question but I like to know why it is so, because if I know why then I’ll understand the logic then I’ll remember how to do a question further.

Students recognise that knowing how is important but that only knowing how is limiting and ultimately defeating. To really understand mathematics, one must also know why something is true. The why-we-hold-it-to-be-so aspect of understanding mathematics is very important to students.
Summary

The findings of this study are summarised in table 19.

Table 19

Summary of the Findings

<table>
<thead>
<tr>
<th>Finding Number</th>
<th>Statement of the Finding</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Students have great difficulty in constructing proofs of all types. However, they are better at constructing geometric proofs than proofs which could be classified as arithmetical or algebraic.</td>
</tr>
<tr>
<td>2</td>
<td>Students who perform better at algebra proof construction are much more likely to believe that proof is necessary to convince themselves of the validity of a proposition. Such a connection does not appear in geometric proving.</td>
</tr>
<tr>
<td>3</td>
<td>A significant percentage of students, when asked to construct a proof, resort to a “proof by examples”. However, students appreciate the value of a general argument instead of specific examples even if they would not be able to construct the general argument proof themselves.</td>
</tr>
<tr>
<td>4</td>
<td>In proving arithmetical or algebraic results, there is a tendency amongst students to attempt to construct an algebraic argument over one that could be described as prose. When students are asked to select from among several types of proofs they tend to select algebraic formal proofs over other less formal but equally valid proofs.</td>
</tr>
<tr>
<td>5</td>
<td>In geometry, two-column proofs are highly favoured by students as the format they use when asked to construct a proof of their own. Further, when asked to select from several types of proofs, students gravitate towards proofs that are of the two-column format even if they are invalid proofs.</td>
</tr>
<tr>
<td>6</td>
<td>Students correctly recognise both the generality of a proven result and its limitations. However, a majority of students are unable to distinguish between a proposition and its converse and the need for an additional proof to establish the converse result.</td>
</tr>
<tr>
<td>7</td>
<td>Students do not see visual proofs as valid mathematical proof. They are more inclined towards visual proofs in geometry than in other mathematical domains like arithmetic and algebra.</td>
</tr>
<tr>
<td>8</td>
<td>Students recognise the power of an explanatory proof even if they would have difficulty constructing one.</td>
</tr>
<tr>
<td>9</td>
<td>Students are able to recognise the falsity of a proposition. However, they will often select proofs of erroneous results if they are either algebraic enough (algebra) or if they follow a two column format (geometry).</td>
</tr>
<tr>
<td>10</td>
<td>Students find technology proofs convincing but do not see them as a valid general argument. They are unsure about their explanatory power.</td>
</tr>
<tr>
<td>11</td>
<td>There appears to be no significant correlation between gender and the ability to construct a mathematical proof.</td>
</tr>
<tr>
<td>12</td>
<td>Proof appears to play a minimal role in convincing students of the truth of a mathematical result.</td>
</tr>
<tr>
<td>13</td>
<td>Students are not only interested in how to do the mathematics, they have a great need to understand why it works.</td>
</tr>
</tbody>
</table>
Chapter Five: Interpretation of the Findings

Although to penetrate into the intimate mysteries of nature and hence to learn the true causes of phenomena is not allowed to us, nevertheless it can happen that a certain fictive hypothesis may suffice for explaining many phenomena.

- Leonhard Euler

Introduction

This discussion will be organized in the following manner. First, the research questions will be answered. Then, as appropriate, the research questions will be discussed in the context of the current literature. Finally, a discussion of the implications and suggestions for future research will be offered.

The Research Questions

This study focused on the research questions posed at the outset. They are:

1. What are the qualities of a mathematical argument that students recognise as a valid mathematical proof?
2. Are students generally proficient in proving and in what ways do students attempt to construct proofs?
3. What is the nature of students’ views of proof? For example, what are their views about a proof’s form, its roles such as convincing and explaining, and are these views affected by the use of technology?
4. What are the similarities and the differences in the results with respect to the London study?

Each research question will now be discussed in light of the findings of this study. In addition, reference will be made to the relevant literature whenever possible.

1. What are the qualities of a mathematical argument that students recognise as a valid mathematical proof?

From their school experiences, students have learned that proofs often follow a particular format. For example, a geometric proof is often a two-column format and other types of proof tend to involve some form of algebraic manipulation. Consequently,
students put a high value on such proofs and sometimes see them as valid as a consequence of their form alone. In finding 4, I stated that:

When students are asked to select from among several types of proofs they tend to select algebraic formal proofs over other less formal but equally valid proofs.

There is a tendency for students to judge a proof as valid on the basis of its form. Erroneous, nonsensical proofs are often judged as valid because they use algebra. The conclusions are similar when one considers geometry. In finding 5, I stated that:

Further, when asked to select from several types of proofs, students gravitate towards proofs that are of the two-column format even if they are invalid proofs.

Whether the curriculum topic is algebra or geometry, students judge the validity of a proof, in part, from its form. As in algebra, there is a tendency amongst students to select nonsensical geometric proofs because they are in a two-column format.

Students judging a proof's validity based on its form is consistent with the literature. Balacheff (1991) discussed the effect that student conceptions about acceptable proof forms has on their proving capabilities. Perhaps this is the result of what Simpson (1995) characterised as a route to proof through logic. Otte (1994) argued that students who view proving as solely an exercise in the manipulation of logical symbols will find it difficult to know when a proof is valid.

Yet students are able to recognise the value of a general argument. They understand that for an argument to be valid it must represent all possible cases. In finding 3, I stated that:

However, students appreciate the value of a general argument instead of specific examples even if they would not be able to construct the general argument themselves.

This result is somewhat contrary to the arguments of Chazan (1993) who stated that for many students evidence is proof. The distinctions are not always clear for several reasons. First, since proof serves many functions, as discussed in de Villiers (1990) and Luthuli (1996), can we really be too upset when students show differing views of proof? Dreyfus (1999) discussed in detail the problematic nature of proof given its multiplicity of meanings and interpretations. He argued that it is no surprise that students are confused about proof because it appears that mathematicians and mathematics educators
are just as confused. This view is also supported by Harel and Sowder (1998) who argued that teachers do not always know what students believe constitutes a proof. Second, it is important, I believe, to make a distinction here between what students do and what they may intend to do. A student may be quite aware that a proof by examples is not valid, but may not be able to assemble the finished product of a proof. Bell (1976) and Van Dormelen (1977) discussed the difficulties in pulling a proof together. The results of the two survey instruments taken together show me that students do see proof as general argument but may not always demonstrate this as Chazan (1993) had observed.

In rating proofs by example as only average, students are perhaps indicating that these proofs convince them but do not satisfy their need for an explanation for their validity. As such it provides students with evidence that the proposition is valid, but not enough reasons why it is so. However, it does not satisfy their need for an explanation for its validity. The role of proof as conviction for students is minimal as confirmed by the literature. Kline (1973) was one of the first of many to conclude that proof as conviction is only meaningful if there was a priori doubt. Fischbein (1982) stated that a formally valid proof may not convince and Hersh (1993) outlined several different ways in which one can be convinced that have little to do with proof. Segal (1999) made a distinction between what it takes to convince oneself and someone else. Proofs that provided a general argument and hence, a more explanatory one, were considered more valid by students. Students demonstrated that they saw proof as a means of explanation and that this explanatory aspect of proof was important to their acceptance of the validity of the proof. In finding 8, I stated that:

Students recognise the power of an explanatory proof even if they would have difficulty constructing one.

Despite the fact that students tended to favour proofs that were heavily algebraic, they also favoured proofs that combined algebra with prose. Proofs that were highly explanatory tended to be selected over proofs that were less so. Hence, it can be argued that students see proof as explanation and will, when asked, select one that is explanatory, over one that is less explanatory. In geometry, the two-column proof, with its statement and reason format, tends to be potentially highly explanatory. It is unclear from my study whether students selected two-column proofs because of form, capacity to
explain, or a combination of the two. However, the role of explanation in students' views of the validity of a proof seems to be a significant factor given findings 11 and 12:

Proof appears to play a minimal role in convincing students of the truth of a mathematical result.

Students are not only interested in how to do the mathematics, they have a great need to understand why it works.

When students were asked what it takes to convince you of the validity of a mathematical result, less than one-quarter of them stated that proof convinces them. Conviction comes from several sources including the teacher, the textbook, worked examples, as well as the student's own level of competency and comfort with the material. When asked to compare understanding how and understanding how and why, overwhelmingly, students rated understanding how and why as more important to them. This indicates a desire on their part to understand why something is true. Proofs that are explanatory, then, serve to fulfil this need to understand why. Hanna (1990,1995,1998) and Hersh (1990, 1993) have argued that proofs that explain are more intellectually satisfying to students. The students who participated in this study lend further support to this argument.

Interestingly, there was a difference in how students viewed the validity of proofs, across the two curriculum areas of algebra and geometry. Whereas in algebra, students were prepared to accept proofs by example as marginally valid, in geometry, they overwhelmingly rejected them. Perhaps students have been taught that a diagram can be deceiving and hence they do not accept an example in diagram form as readily.

Students are also able to recognise both the generality and limitations of a result. In the two items that tested this, one in algebra and the other in geometry, students were able to correctly determine when changes to the premises required a new proof or when the original proof sufficed. In finding 6, I found that:

Students correctly recognise both the generality of a proven result and its limitations.

However, a majority of students are unable to distinguish between a proposition and its converse and the need for an additional proof to establish the converse result.
This recognition of generality appeared to be consistent across the two curriculum areas with no significant differences. Just as students were equally good in either algebra or geometry in recognising generality they were also equally bad across the two curriculum areas with regards to converse propositions. The students did not appear to be aware of the necessity for a new proof when the converse proposition is considered. Perhaps students have not had enough experience with converse propositions. My own experience as a teacher of mathematics supports this notion. As an example, a well-known theorem such as the Pythagorean theorem is often only taught as a one-way theorem.

Students do not see visual proof as valid although there is more of a tendency to accept it in geometry than in algebra. In finding 7, I stated:

Students do not see visual proofs as valid mathematical proof. They are more inclined towards visual proofs in geometry than in other mathematical domains like arithmetic and algebra.

The visual proof in algebra was rated very low and not a single participant chose it as one that students themselves would do nor one that would earn the highest mark. Visual proofs were rated higher in geometry but the rating were still only average when compared to other non-visual proofs. Previously, I argued that the form of a proof was a significant factor in how students view a proof’s validity. Perhaps this is also a factor with regards to visual proof as students have not had much experience with this type of proof and hence its form is somewhat foreign to them. The results of this study are not surprising given the literature (Barwise and Etchemendy, 1996; Peled and Zaslavsky, 1997) on visual proof which indicates that this form of proof is still in its infancy and may not be accepted by either students or their teachers.

Summary

The following factors are viewed by students as important when judging a proof’s validity. First, form is important – a proof is judged more valid if it is of a familiar form. Second, visual proof does not generally familiar to students and hence is not seen as valid. Third, generality and limitations of arguments are understood by students. Fourth, general arguments that explain are considered more valid than examples that do not explain. Fifth, notwithstanding the comments about form, explanatory arguments are
considered more valid and help to address students' need for understanding why something is true. Finally, a proof's validity in students' eyes has little to do with convincing; it has more to do with its explanatory power.

2. Are students generally proficient in proving and in what ways do students attempt to construct proofs?

In general, I found that students had great difficulties with proof construction. I stated in finding 1 that:

Students have great difficulty in constructing proofs of all types. However, they are better at constructing geometric proofs than proofs which could be classified as arithmetical or algebraic.

Even when almost correct proofs were considered, the student attempts on the four proof construction items were poorly done. Students do not seem to be able to get started even on proof items that should have been fairly easy given that they were part of the London study which was administered to participants on average two to three years younger. I would have to conclude that students are generally not proficient with proving in the two curriculum areas of this study. However, as stated in the finding, they are better in the geometry strand. They seem to be able to start a geometric proof, perhaps because, proving is very much a part of the geometry curriculum experience. Whereas it may not be a part of the algebra curriculum experience. (Davis, 1986). As a result they have more familiarity with proof in geometry and the two-column format gives them a way of expressing their proof. It should be stressed that these items (A2, A6, G3, G7) are really about proof writing. These items measured a student's ability to “write up the proof”. Hence, having a framework for expression may have led to the better results in geometry. Even though A2 was not a difficult proposition, many students clearly did not know how to begin.

This conclusion is well supported by the literature. Bell (1976), Chazan (1990), Fischbein (1982), Senk (1985), Healy and Hoyles (1998), Gardiner and Moreira (1999), among others, all in varying degrees, lamented the proving capabilities of students. Although the students in this study performed better in the geometry strand, they still have not performed well and this, too, is consistent with most of the literature. Howson
and Kahane (1986), Fey and Good (1985), Senk (1985), de Villiers (1998) and Healy and Hoyles (1998), among others, have commented on the difficulties students have with geometry and geometric proving. In particular, Senk (1985) stated that:

about 30 percent of the students in full-year geometry courses that teach proof reach a 75 percent mastery of proof. (p. 448)

This result does not seem too far off the results found in my study.

Even though students recognise the value of a general argument, they often resorted to naïve empiricism when constructing their own proofs. As stated in finding 3:

A significant percentage of students, when asked to construct a proof, resort to a “proof by examples”. However, students appreciate the value of a general argument instead of specific examples even if they would not be able to construct the general argument proof themselves.

Almost 40% of students who provided any proof, provided a proof by examples. It is unclear how many of these students really believe their proof is valid or whether it is just the best they could do under the circumstances. Further, it calls into question the students’ view of the role of proof. If they see proof as convincing, then under this definition, their examples are convincing and so this constitutes a proof in their eyes. Again, Chazan (1990), as well as Balacheff (1991), support this argument. It would be necessary to conduct interviews to determine exactly why students resorted to naïve empiricism. However, it seems likely that the students are aware of the limitations of their proof by examples, given the results reported in the previous section. Students have demonstrated an interest in understanding why, and in explanatory proofs. Maher and Martino (1996) cited a five-year case study in which they concluded that students were interested in proof because they had an innate need to make sense of the mathematics at hand. Various authors (especially Hanna (1989) and Hersh (1993)) have argued that proof as explanation satisfies one’s desire for a complete explanation.

Students have also selected sample proofs that are more general and have generally not given sample proofs by examples high ratings. Hence, it is reasonable to conclude that, for most students, proof by examples, may be the starting point in constructing a proof but it is not seen by them as satisfactory. Similarly, Blum and Kirsch (1991) have argued that naïve empiricism is a necessary first step in arriving at a proof.
Students tend towards using algebra even when this is unnecessary or when they are not sufficiently skilled to make it work for them. In finding 4, I stated:

In proving arithmetical or algebraic results, there is a tendency amongst students to attempt to construct an algebraic argument over one that could be described as prose.

In using algebra, students are, as previously argued, recognising a prescribed form that they have seen modelled in class. Proof writing involves many skills and concepts that must come together to produce a proof. It is not always clear why a student did not manage to "pull it all together" to arrive at the finished product of a proof. Perhaps, an over-reliance on algebraic manipulation led to poorer results in proof construction than if students wrote a paragraph. It also seemed to have some impact on students' ability to construct proofs that were explanatory as stated in finding 8:

Students recognise the power of an explanatory proof even if they would have difficulty constructing one.

A view that a proof consists of algebraic manipulations can sometimes lead students to algebraic manipulations which are nonsensical and, of course, non-explanatory. This type of proof is often fraught with problems from the outset as students are either unsure of why they are using algebra or are unable to muster the necessary skill. Without a clear conception of what they are doing with the algebra the students are unable to assemble a proof.

Summary

The following conclusions pertain to students' ability to construct proofs. First, students are not proficient in proving. Second, they are slightly better with geometric proofs than with algebraic proofs. Third, proof by examples is a last resort for many students and should not be seen as evidence that they do not know what constitutes proof. Finally, students' views on the form that a proof must have can lead them to do erroneous and nonsensical symbolic manipulations.
3. What is the nature of students’ views of proof? For example, what are their views about a proof’s form, its roles such as convincing and explaining, and are these views affected by the use of technology?

Even though students often will provide a proof by examples when asked to construct a proof, it seems that they are only providing such a proof because it is “better than nothing”. As previously argued, students do have great difficulty constructing proofs, and when such difficulty ensues, they resort to proofs by examples. However, they do see the value in a general argument as reported in finding 3:

However, students appreciate the value of a general argument instead of specific examples even if they would not be able to construct the general argument proof themselves.

Students also have the view that proofs should be in a prescribed format. In proofs involving number theory, students had a tendency to favour proofs that were algebraic. Both in their own proof constructions and in selecting from sample proofs, students tended towards algebraic proofs as stated in finding 4:

In proving arithmetical or algebraic results, there is a tendency amongst students to attempt to construct an algebraic proof over one that could be described as prose. When students are asked to select from among several types of proofs they tend to select algebraic formal proofs over other less formal but equally valid proofs.

Even when the proof is valid, if it is written in a narrative or prose form, the students tended to be more impressed by proofs that contained algebra. The presence of lots of algebra and algebraic manipulations was enough to convince some students of a proof’s validity. In this sense, students have a view of proof that values its form, sometimes more than its substance.

This was also evident in geometric proving. The two column format, long a staple in geometry classrooms and textbooks, is highly favoured by students. Again, even when the proof is nonsensical, the appearance in a two-column format influences students as stated in findings 4 and 8:

In geometry, two-column proofs are highly favoured by students as the format they use when asked to construct a proof of their own. Further, when asked to select from several types of proofs, students gravitate towards proofs that are of the two-column format even if they are invalid proofs.
Students are able to recognise the falsity of a proposition. However, they will often select proofs of erroneous results if they are either algebraic enough (algebra) or if they follow a two column format (geometry).

So it can be concluded that form plays a major role in students’ view of proof. Clearly, there are formats which are more frequently used than others and it is perfectly natural for students to select those proofs which are consistent with their previous experience. What is unnatural is that students will select proofs that are wrong, and sometimes clearly so, simply because they possess the correct form. Related to this view of correct form, are students’ views about visual proof. Without much reservation it can be stated that visual proofs of non-geometric theorems are rejected out of hand by students. Whilst there is some tolerance in geometry, perhaps because of its already visual nature, there is little tolerance for such an approach in other realms. In summation, here is finding 7:

Students do not see visual proofs as valid mathematical proof. They are more inclined towards visual proofs in geometry than in other mathematical domains like arithmetic and algebra.

Davis (1993) pointed to the bias that mathematicians have against visualisation, preferring to have written proofs. He stated that there needs to be a recognition that the eye has legitimacy as an “organ of discovery and inference” (p. 342). It should come as no surprise then that students would reject visual proof given that they have probably never seen it used before.

In general, students do not view proof as satisfying their need for conviction. Students are convinced for many reasons. Certainly, in most cases, seeing several examples is enough to convince most people (including many mathematicians) of the correctness of a result. Further, school curricula usually present mathematics that is tried and true and students know this. There isn’t much room for doubt because they are already convinced. Therefore, it is not much of a surprise that proof plays a very minor role in convincing as reported in finding 12:

Proof appears to play a minimal role in convincing students of the truth of a mathematical result.

Further, when students were asked to consider the last mathematical procedure that they learned and what it took to convince them of this result, only 52.9% indicated
proof as necessary to convince them of its validity. However, students who did well in the proof construction items, were more influenced by proof than the average student who participated in this study. In finding 2, I stated:

Students who perform better at algebra proof construction are much more likely to believe that proof is necessary to convince themselves of the validity of the proposition. Such a connection does not appear in geometric proving.

We can conclude, therefore, that students’ beliefs about the importance of proof does have an impact on their ability to write up a proof. However, this belief seems only an issue with algebraic proving. It may be, as argued previously in chapter four, that students have more doubts about the propositions’ validity in the algebra section and hence proof played more of a role there than in geometry. Perhaps, the familiar format of the geometric proofs, offset their need for conviction.

However, students do see proof as a method of explanation. Proofs that are relatively more explanatory were favoured over other less explanatory proofs. I stated in finding 8:

Students recognise the power of an explanatory proof even if they would have difficulty constructing one.

Students, by their preferences, indicated that proofs that explain are more valuable to them. Clearly, since students do not need proof to convince them that a result is true, the function that proof serves is one of helping to explain. Students are very interested in knowing the why behind the mathematics that they are learning. Their need to understand as fully as possible is stated in finding 13:

Students are not only interested in how to do the mathematics, they have a great need to understand why it works.

Whilst some students indicated that understanding why was not important to them, the vast majority believed it not only improved their understanding, it also helped them to adapt to problems which are extensions of the basic concept. If they do not understand why, they find it more difficult to work through more complex problems. Students recognised this and since they saw proof as enabling their understanding, it can be argued that proof contributes to their ability to solve problems.
The use of technology is encouraged in the mathematics curriculum. Students usually are highly interested in its use both inside and outside of the classroom. However, it is often difficult to assess just what students are learning through the use of technology. Several questions were posed to students to determine their view of technology and its usefulness with respect to proof. In addition, a sample proof using technology was presented and students asked to assess its validity.

Students found a proof by technology to be a reasonable attempt at a proof. Whilst such a proof is not considered the most valid, it does rate better than examples generated by hand. The multitude of examples available through the technology gives it a higher validity rating. Further, the more examples that satisfy the proposition, the more convincing it is. Students find proofs by technology to be a very convincing argument. de Villiers (1991, 1992) observed that students' need for conviction is more than met by the use of technology. The use of technology provides students with a certainty whether or not they think the proof is valid. As a result, students think that a proof by technology lacks generality. They correctly reason that just a proof by examples, albeit with many more examples. Consequently, they are ambivalent towards a technology proof as a method of explanation. It may very well convince them, but does it explain why the theorem is valid? For many students, the answer is no.

Summary

The following factors form the basis of students' views of the overall role of proof. First, students do see proof as a general argument. Second, in contrast, they are also influenced by a proof's form. Third, alternate forms of proof, like visual proof, if not in the realm of their experience, are not viewed favourably. Fourth, proof appears to play a minimal role in convincing students. Fifth, students who do well in proof construction are more likely to view proof as playing a more significant role in conviction or validity. Sixth, students tend to see proof as way of explaining mathematics. Their interest in knowing why is helped by proofs that explain. Finally, with respect to technology, for the majority of students, a proof by technology is an average attempt at proof, it is convincing, it lacks generality and it does not explain too well. These conclusions are summarised by finding 10:
Students find technology proofs convincing but do not see them as a valid general argument. They are unsure about their explanatory power.

4. What are the similarities and the differences in the results with respect to the London study?

In this section, I will state the major findings of the London study which are pertinent to my study and then make comparisons to my own findings. Some of the findings of the London study were related to the previous mathematics attainment of the students, the performance of schools and the characteristics of the teachers. These issues were not examined in my study. The findings from the London study will be indented and numbered as in the technical report with the prefix L.

L1. High-attaining Year 10 students show a consistent pattern of poor performance in constructing proofs.

Similarly to finding L1, I also found that students have great difficulty in constructing proofs of all types. The students who participated in this study would be on average two to three years older than the participants in the London study. The Ontario students are also high-attaining as they were either advanced grade twelve or Ontario Academic Credit students. Therefore, it can be concluded that the "type" of student in both studies is similar but of different ages. The findings of the two studies are similar.

L2. Students' performance is considerably better in algebra than in geometry in both constructing and evaluating proofs.

I found the opposite of this finding. In finding 1, I stated that in proof construction, students in my study were better in geometry than they were in algebra. I found that, in geometry, students were able to organise their approach better and less likely to resort to using empirical arguments. I suspect that the reason for the differences in the two studies may be that the curricular emphasis on geometry, weak as it is, is still greater in Ontario, than in England and Wales. It may also have to do with the age of the students and their mathematical majority. The van Hiele theory of development would support the notion that students who are two to three years older would also be better.
Unlike the London study, I found no discernible differences between students’ ability to evaluate proofs across the two curriculum areas.

L3. Most students appreciate the generality of a valid proof.

I found similar to the London study that students knew the difference between general and specific arguments. In finding 6, I stated that students recognise generality and limitations of an argument. I also found in finding 3, that even though they may construct a proof using only empirical methods, they selected proofs that were more general when asked to evaluate proofs.

L4. Students are better at choosing a valid mathematical argument than constructing one, although their choices are influenced by factors other than correctness, such as whether they believe the argument to be general and explanatory and whether it is written in a formal way.

I found, similar to the London study, that students were better at evaluating already constructed proofs than at constructing them from scratch. Even though, as reported in finding 3, students would sometimes construct a proof by giving examples, they rated proofs that were more general more highly than empirical proofs.

I also found that students were influenced by the way a proof was presented. I found, as reported in finding 4, that students had a preference for proofs that used more algebraic symbols or used the traditional two-column format in geometry. Equally valid proofs that were in formats that were a bit different were not as highly rated as proofs that had a more familiar look to them.

I also found that students evaluated explanatory proofs more highly than non-explanatory proofs. Similar to the London findings, in finding 8, I stated that students recognised the power of explanatory proof even if this would not be how they would construct the proof for themselves.


My study did find a relationship between students’ views of proof and their proof construction ability. As I stated in finding 2, students who saw proof as fulfilling their
need for conviction and validity, were better at proof construction. This difference was more pronounced in algebra whilst in geometry it was not a great factor. In algebraic proving, students who had a need for proof to validate were more likely to assemble a correct or an almost correct proof than students who used other means for conviction.

L7. In algebra, girls and boys perform significantly differently, with girls constructing better proofs than boys and choosing different forms of arguments.

In finding 11, with respect to proof construction, I found that there was no significant difference between the genders. Unlike the London study, I did not investigate how gender affected one’s evaluation of sample proofs.

**Summary**

To summarise the parallel findings of the two studies, the similarities and differences can be found in Table 20.

Table 20

**Comparison of this Study’s Findings (T) with the London Study’s Findings (L)**

<table>
<thead>
<tr>
<th>Similarities</th>
<th>Differences</th>
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<tbody>
<tr>
<td>Students consistently perform poorly in constructing proofs</td>
<td>In proof construction, L found students better in algebra whilst T found students better in geometry</td>
</tr>
<tr>
<td>Students recognise the generality of arguments</td>
<td>In proof evaluation, L found students better in algebra whilst T found no difference between algebra and geometry</td>
</tr>
<tr>
<td>Students are better at evaluating than constructing proofs</td>
<td>Regarding gender; L found girls better in constructing algebra proofs whilst T found no significant difference</td>
</tr>
<tr>
<td>In evaluating proofs, students are influenced by factors such as form and the explanatory power of a proof</td>
<td></td>
</tr>
<tr>
<td>Students’ views of proof did influence their ability to construct proofs</td>
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</tbody>
</table>
Implications for Further Research

I believe that I have just scratched the surface of what is possible with the data that I have collected. There is much more that I could say if time, my own energy, and the patience of my thesis committee were not finite. Here are some of the questions or issues that could be explored further:

1. The whole connection between beliefs and proving competencies

   There is much more to be said here. When comparisons were made between students’ view of proof as serving to validate and their ability to construct a proof, a relationship was found. Are there similar relationships between a view in the belief in the teacher and accepting proofs as valid on the basis of form? Or, does one who believes that examples convince also tend to pick proofs that are empirical? There are many questions such as this and many papers that could be written after further analysis of the data.

2. The seeming differences between the two curriculum areas of algebra and geometry.

   There seemed to be several differences, some conflicting with the London study. Some questions that come to mind are: Did students do better in geometry because they were older than the London participants or did they do better because they had more curricular exposure to geometry? How much of an influence does having a known format help students to construct proofs? For example, would using models as outlined earlier in this thesis in the section Further Implications for Teaching, similarly ameliorate the algebra proving? Why do students see proof by examples not the least bit valid in geometry but will tolerate it in algebra?

3. Gender issues

   In this study, it was inconclusive whether gender played a role in one’s proving capabilities. Perhaps, with larger numbers of participants, this question could be addressed. The London study found girls to be better. Does age again play a role? Other questions that come to mind: Does one’s preferred method of expressing a proof relate to
gender in any way. For example, girls tend to like to write more than boys. Would girls prefer prose or narrative arguments to algebraic ones? Is there more of a preference for visual representations amongst boys?

4. Teacher issues

In this study, no attempt was made to connect teacher beliefs and teacher preparation (education) to the student results. Interestingly, the participants came from two schools within the same school board and yet there appear to be differences not only in curriculum but also beliefs about proof. What influence does the teacher have? I suspect the influence is profound. Issues for further exploration include developing proper teacher education experiences that would assist teachers to know the various functions and roles of proof. For instance, there is much work that could be done on developing the whole notion of proof as explanation and how this role can be fully exploited in the classroom.

Concluding Remarks

Proof lies at the heart of what it means to do mathematics. Proof continues to play an important role, not only in the practice of professional mathematicians, but also in mathematics education. Its role as a complete explanation is perhaps its most important contribution to the education of secondary school students.

This study has attempted to provide a window on proof in secondary school mathematics. Because of proof's several roles and functions, proof can be a difficult concept to pin down. There are misconceptions about what constitutes acceptable proof. Students need to have curricular experiences that help them to see the varied roles that proof plays.

Proof's role as a complete explanation is perhaps its most important contribution to the education of secondary school students. Not unlike the slippery slope of assessing mathematical understanding, proofs that explain can be recognised but their explanatory powers are difficult to assess. Notwithstanding these difficulties, the educational value of proofs that explain has not been fully utilised in our secondary schools.
I believe that mathematics achievement and enjoyment will be greatly improved if the regular classroom routine is one in which proof, in all its forms and roles, plays a prominent role. What is needed is a classroom *culture of proving* that values and encourages the justification and proof of mathematical arguments.
References


Olive, J. (1998). Opportunities to explore and integrate mathematics with the geometer’s sketchpad. In R. Lehrer & D. Chazan (Eds.), Designing learning environments for
developing understanding of geometry and space. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.


Appendix A

Student Survey on Justifying and Proving in School Mathematics – Part I

Code: ____________________________

Please complete the following information:

Birthdate: _________/_________/________
            day       month       year

Gender:     Female       Male

Instructions:

You are taking part in a research study into students’ ideas about proof.

The results are very important and the results will be used to shape future curriculum.

We are interested in individual results so you need to complete the survey on your own. Your identity will be kept confidential.

In this part of the survey, there are a number of open-ended questions. There are no right or wrong answers in this section.

It is important, however, to provide as complete an answer as possible. Whenever possible, explain your answer with examples.

The survey is double sided, so when your teacher tells you to begin, you will start on the other side of this page.
1. Think of the last mathematical procedure you were taught. How do you know that the procedure is a valid method? How were you convinced?

2. In general, when new mathematics is introduced to you are you easily convinced? What does it take to convince you of the validity of the mathematical results?
3. Has there ever been a time when you haven’t been convinced of results that were used in mathematics class? If so, were you eventually convinced? What did it take to convince you?

4. The goal of all teaching and learning should be developing understanding. In mathematics, understanding has been distinguished as two types: understanding how and understanding how and why.

   a) Why do you believe such a distinction has been made? Can you give an example that would illustrate the two types of understanding?
b) Which type of understanding is usually preferable? Which type is more important to you? Explain. Give an example.


c) In general, how important is understanding how to you? Explain.


d) In general, how important is understanding how and why to you? Explain.
5. When a new topic is introduced to you by your mathematics teacher

   a. How much convincing of its truth or validity do you normally require?

   ________________________________
   ________________________________
   ________________________________
   ________________________________
   ________________________________

   b. Can you be convinced without full understanding (*how and why*)?

   ________________________________
   ________________________________
   ________________________________
   ________________________________
   ________________________________

   c. What does it take to explain a result so that you understand both *how and why*?

   ________________________________
   ________________________________
   ________________________________
   ________________________________
   ________________________________
6. In mathematics, the word *proof* is used from time to time.

   a. How familiar are you with *proof*? In previous mathematics courses, what has been your experience with proof?

   Please do not write in this space.

   6a. VF
   6a. MF
   6a. LF
   6a. NF
   6a. AL
   6a. GE
   6a. OT

   b. Why do you think we need proof?

   6b. VE
   6b. UN
   6b. OT

   c. What do you think constitutes a mathematical proof?
d. Can you give an example of an instance in your mathematics class when a mathematical proof was developed? If so, explain what the topic was and to the best of your recollection what was involved in the proof?

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

IF YOU COULD NOT GIVE AN EXAMPLE SKIP PART E AND PROCEED TO PART F

e. If you gave an example in part d): What did this proof contribute to your understanding of mathematics? What type of understanding did the proof help with, if at all?

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

________________________________________________________________________________________

6e. WH
6e. HW
6e. EX
6e. NH
6e. OT
f. In general, how does proof contribute to your mathematical understanding?

7. Graphing calculators and computers are being used more often in mathematics classes.

a. In what ways have you found a calculator/computer helps you to develop understanding? Give examples.
b. Does a calculator/computer improve understanding how? In what ways?


c. Does a calculator/computer improve understanding how and why? In what ways?


d. Does a calculator/computer help you develop or understand proofs better? In what ways?


e. In general, is the use of calculators/computers a positive or negative development in the learning of mathematics? Explain and give examples to support your viewpoint.

8. Write down anything else that you know about proof:
Appendix B

Student Survey on Justifying and Proving in School Mathematics – Part II

Code: ________________________________

Please complete the following information:

Birthdate: _______ / _______ / _______
day month year

Gender: Female Male

Instructions:

You are taking part in a research study into students’ ideas about proof.

The results are very important and the results may be used to shape future curriculum.

We are interested in individual results so you need to complete the survey on your own. Your identity will be kept confidential.

In this part of the survey, there are several types of questions. One type will involve a mathematical statement followed by several answers which were worked out by students who were trying to determine whether or not the statement was true or false (student proofs). Some answers may be right and some may be wrong, but there is never only one right answer. You will be asked a series of questions about these answers. Another type of question will ask you to construct a proof of your own. These questions will be similar to the already constructed proofs. You are allowed calculators and other mathematical tools like rulers and compasses.

This part of the survey is divided into two parts of the curriculum. After about 30 minutes, your teacher will ask you to move onto the second part of the curriculum, if you haven’t already done so.

The survey is double sided, so when your teacher tells you to begin, you will start on the other side of this page.
A1. Arthur, Bonnie, Ceri, Duncan, Eric and Fatima were trying to prove whether the following statement was true or false: When you add the squares of any two consecutive numbers, your answer is always odd.

Arthur’s answer:
a is any odd number and b is the next even number,
\[ a^2 + b^2 = c^2 \]
a^2 = c^2 - b^2
b^2 = c^2 - a^2
\[ a^2 + b^2 = c^2 - b^2 + c^2 - a^2 \]
\[ = 2c^2 - (a^2 + b^2) \]
\[ = 2c^2 \text{ is even and } a + b \text{ is odd so its square } \]
\[ (a + b)^2 \text{ is also odd and an even number minus an odd number is odd.} \]
So, Arthur says it’s true.

Bonnie’s answer:
1^2 + 2^2 = 1 + 4 = 5 odd
2^2 + 3^2 = 4 + 9 = 13 odd
3^2 + 4^2 = 9 + 16 = 25 odd
4^2 + 5^2 = 16 + 25 = 41 odd
So Bonnie says it’s true.

Duncan’s answer:
Even numbers end in 0 2 4 6 or 8
Squaring these numbers they end in 0 4 6 6 or 4
Odd numbers end in 1 3 5 7 or 9
Squaring these numbers they end in 1 9 5 9 or 1
Adding these squares they end in 1 3 1 5 or 5
So Duncan says it’s true.

Ceri’s answer:
Let n be a whole number, then n + 1 is the next whole number.
\[ n^2 + (n + 1)^2 = n^2 + n^2 + 2n + 1 \]
\[ = 2n^2 + 2n + 1 \]
\[ = 2(n^2 + n) + 1 \]
So Ceri says it’s true.

Eric’s answer:
If the first number is even, then its square is an even number multiplied by an even number which is even. The second number is an odd number multiplied by an odd number and its square would be odd. Adding an even and an odd number results in an odd number.
So Eric says it’s true.

Fatima’s answer:
Fatima drew the following picture to answer the question:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
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<tr>
<td>1</td>
<td>5</td>
<td>13</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

So Fatima says it’s true.

From the above answers choose one that would be the closest to what you would do if you were asked to answer this question:
I would choose

_________________________’s answer.

In the chart at the right, circle the mark out of 5 that your teacher would probably give each answer.
**A1.** For each of the following, circle whether you agree (A), Don't know (DK) or disagree (D).

<table>
<thead>
<tr>
<th></th>
<th>Agree</th>
<th>Don't Know DK</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Arthur's answer:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Has a mistake in it</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
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</tr>
<tr>
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<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Is an easy way to explain to someone in your class who is unsure</td>
<td>1</td>
<td>2</td>
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</tr>
<tr>
<td><strong>Bonnie's answer:</strong></td>
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<tr>
<td><strong>Ceri's answer:</strong></td>
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<td></td>
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</tr>
<tr>
<td><strong>Duncan's answer:</strong></td>
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<td></td>
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<td></td>
<td></td>
</tr>
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<td>3</td>
</tr>
<tr>
<td><strong>Fatima's answer:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Has a mistake in it</td>
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<td>3</td>
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<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Please do not write in this space.
A2. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

If \( p \) and \( q \) are any two odd numbers, \((p + q) \times (p - q)\) is always a multiple of 4.

*Write your answer in the space below:*

*Answer:*
A3. Suppose that the following statement (statement 1) has now been proven:

**Statement 1:** If $p$ and $q$ are any two odd numbers, $(p + q) \times (p - q)$ is always a multiple of 4.

For each of statements 2, 3, 4 or 5 below, Gemma asks what needs to be done to prove each statement.
For each statement check either choice A: Gemma doesn’t need to do anything; Statement 1 already proved this.
or choice B: Gemma needs to construct a new proof.

For each statement, check either A or B in the table below:

<table>
<thead>
<tr>
<th>Statement 2</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>If $p$ and $q$ are consecutive odd numbers, $(p + q) \times (p - q)$ is always a multiple of 4</td>
<td>Gemma doesn’t need to do anything</td>
</tr>
<tr>
<td>Statement 3</td>
<td>If $p$ and $q$ are two even numbers, $(p + q) \times (p - q)$ is always a multiple of 4</td>
<td></td>
</tr>
<tr>
<td>Statement 4</td>
<td>If $p$ and $q$ are odd perfect squares, $(p + q) \times (p - q)$ is always a multiple of 4</td>
<td></td>
</tr>
<tr>
<td>Statement 5</td>
<td>If $(p + q) \times (p - q)$ is a multiple of 4, $p$ and $q$ are any two odd numbers</td>
<td></td>
</tr>
</tbody>
</table>
A4. Hassan, Iris, Julie, Ken and Lenore were asked to prove whether the following statement is true or false.

When you add the squares of any three consecutive numbers, your answer is always even.

**Hassan's answer:**

x is any whole number,

\[ x^2 + (x+1)^2 + (x+2)^2 \]

\[ = x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 \]

\[ = 3x^2 + 6x + 5 \]

\[ 3 + 6 + 5 = 14 \]

14 is divisible by 2

So Hassan says it's true.

**Iris's answer:**

If the first number is even, its square will be even. The second number must be odd and its square is odd. The third number is even and its square is even. This combination will always add up to an odd number.

So Iris says it's false.

**Julie's answer:**

\[ 1^2 + 2^2 + 3^2 = 14 \]

\[ 5^2 + 6^2 + 7^2 = 110 \]

\[ 11^2 + 12^2 + 13^2 = 434 \]

\[ 23^2 + 24^2 + 25^2 = 1730 \]

\[ 35^2 + 36^2 + 37^2 = 3890 \]

So Julie says it's true.

**Ken's answer:**

\[ 6^2 + 7^2 + 8^2 \]

\[ = 36 + 49 + 64 \]

\[ = 149 \]

So Ken says it's false.

**Lenore's answer:**

Suppose the first number is even, say 2x. The sum of the squares of the three consecutive numbers is

\[ (2x)^2 + (2x+1)^2 + (2x+2)^2 = 4x^2 + 4x^2 + 4x + 1 + 4x^2 + 8x + 4 \]

\[ = 8x^2 + 12x + 5 \]

\[ = 2(4x^2 + 6x + 2) + 1 \]

Since \( 2(4x^2 + 6x + 2) \) is even, \( 2(4x^2 + 6x + 2) + 1 \) is odd

So Lenore says it's true.

From the above answers, choose one which would be closest to what you would do if you were asked to answer this question.

I would choose ________________________'s answer.

**Circle the mark out of 5 that your teacher would probably give each answer:**

<table>
<thead>
<tr>
<th></th>
<th>worst</th>
<th>okay</th>
<th>best</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hassan's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Iris's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Julie's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Ken's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Lenore's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Mike, Nisha, Otis and Petra were asked to prove whether the following statement is true or false.

If \(k\) is an even number then 48 is always a factor of \(k^3 - 4k\).

**Mike’s answer:**
\[
k^3 - 4k = 48x \\
k(k^2 - 4) = 48x \\
k(k+2)(k-2) = 48x \\
k(k+2)(k-2) = x \\
48 \\
so Mike says it's true.
\]

**Nisha’s answer:**
I made a chart.

<table>
<thead>
<tr>
<th>k</th>
<th>(k^3 - 4k)</th>
<th>does 48 divide?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 -4(0) = 0</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>2^3 - 4(2) = 8</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>4^3 - 4(4) = 48</td>
<td>Yes</td>
</tr>
<tr>
<td>6</td>
<td>6^3 - 4(6) = 192</td>
<td>Yes</td>
</tr>
</tbody>
</table>

So Nisha says it’s true.

**Otis’s answer:**
Let \(k = 2t\), then 
\[
k^3 - k = (2t)^3 - 4(2t) \\
= 8t^3 - 8t \\
= 8t(t^2 - 1) \\
= 8t(t-1)(t+1)
\]

48 = 16 X 3 and since \(t(t-1)(t+1)\) has 6 as a factor if \(t\) is at least 2, then 48 is a factor. If \(t\) is 1 or 0, the expression is 0 and 48 is a factor of 0.

So Otis says it's true.

**Petra’s answer:**
\(k^3 - 4k\) has a common factor of \(k\) so if I factor out the \(k\) from \(k^3 - 4k\), I get \(k^2 - 4\) which is a difference of squares and this can be further factored into \((k-2)(k+2)\). Therefore, I have the equivalent expression \(k(k-2)(k+2)\). This is 3 consecutive even numbers. At least one is divisible by 6. If we consider \(k\) values of at least 4 so that the product is not zero, the other two are divisible by at least 8. So 48 is a factor. If the product is zero, 48 is also a factor.

So Petra says it’s true.

From the above answers, choose one which would be closest to what you would do if you were asked to answer this question.

I would choose ________________________’s answer.

**Circle the mark out of 5 that your teacher would probably give each answer:**

<table>
<thead>
<tr>
<th></th>
<th>worst</th>
<th>okay</th>
<th>best</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Nisha’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Otis’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Petra’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
A5. Mike, Nisha, Otis and Petra were asked to prove whether the following statement is true or false.

If $k$ is an even number then 48 is always a factor of $k^3 - 4k$.

<table>
<thead>
<tr>
<th></th>
<th>Agree</th>
<th>Don't Know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mike's answer:</strong></td>
<td>A</td>
<td>DK</td>
<td>D</td>
</tr>
<tr>
<td>Has a mistake in it</td>
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<td>3</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>DK</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nisha's answer:</strong></td>
<td></td>
<td>DK</td>
<td>D</td>
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<tbody>
<tr>
<td><strong>Otis's answer:</strong></td>
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<tr>
<th></th>
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<th>D</th>
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<tbody>
<tr>
<td><strong>Petra's answer:</strong></td>
<td></td>
<td></td>
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<tr>
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<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Shows you why the statement is true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Is an easy way to explain to someone in your class who is unsure</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
A6. Prove whether the following statement is true or false. Write down your answer in the way that would get the best mark you can.

The square of a two digit number is decreased by the square of the number formed by reversing its digits. Prove that the number formed in this way is divisible by the sum of the digits of the original number.

Write your answer here:
G1. Frank, Gail, Harriet, Irene and Jacob were trying to prove whether the following statement is true or false:
The shortest distance between any point P and a line segment AB is the line joining P to C, where C is the midpoint of AB.

Harriet's answer:
A straight line is always the shortest distance between two points.
So Harriet says it's true.

Irene's answer:
I drew some more lines joining P to AB. I can see that some of them are shorter than PC.
So Irene says it's false.

Jacob's answer:
E is any point on BC and D is any point on AC.

\[
\begin{align*}
\text{Statement} & \quad \text{Reason} \\
\text{If } \angle PCE > 90^\circ & \quad \text{Sum of the angles in a triangle } = 180^\circ \\
\text{angle } PEC < 90^\circ & \quad \text{Longest side of triangle is opposite largest angle} \\
\text{and } PE > PC & \quad \text{But if } \angle PCE > 90^\circ \\
\text{angle } PCD < 90^\circ & \quad \text{Sum of angles on a straight line } = 180^\circ \\
\text{angle } PDC \text{ can be } > 90^\circ & \quad \text{Sum of angles in a triangle } = 180^\circ \\
\text{PD can be } < PC & \quad \text{Longest side of triangle is opposite largest angle} \\
\therefore \text{PC is not always the shortest distance} & \\
\text{So Jacob says it's false.}
\end{align*}
\]

Frank's answer:

\[
\begin{align*}
\text{E is any point on BC and D is any point on AC.} \\
\text{Statement} & \quad \text{Reason} \\
\text{AC } =\text{ BC} & \quad \text{C is the midpoint} \\
\text{CE}^2 + \text{PC}^2 = \text{PE}^2 & \quad \text{Pythagoras theorem} \\
\text{CD}^2 + \text{PC}^2 = \text{PD}^2 & \quad \text{Pythagoras theorem} \\
\text{PC } \leq \text{ PE} & \quad \text{CE is greater than 0} \\
\text{PC } \leq \text{ PD} & \quad \text{CD is greater than 0} \\
\therefore \text{PC is the shortest distance} & \\
\text{So Frank says it's true.}
\end{align*}
\]

Gail's answer:
I drew an arc with my compass using P as the centre and so that the arc just touched the line AB. The line from P to C crossed the circle showing that PC was not the shortest line.
So Gail says it's false.

From the above answers, choose one which would be closest to what you would do if you were asked to answer this question.
I would choose ________________'s answer.

In the chart below, circle the mark out of 5 that your teacher would probably give each answer.

<table>
<thead>
<tr>
<th></th>
<th>worst</th>
<th>okay</th>
<th>best</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frank's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Gail's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Harriet's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Irene's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Jacob's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
G2. C is any point on the perpendicular bisector of AB. Kobi, Linda, Marty and Natalie were trying to prove whether the following statement is true or false:

Triangle ABC is always isosceles.

**Kobi's answer:**
I moved C to different places on the perpendicular bisector and measured AC and BC. They were always the same so the triangles were all isosceles.

*So Kobi says it's true.*

**Linda's answer:**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>AD = BD</td>
<td>Bisector</td>
</tr>
<tr>
<td>ADC = 90°</td>
<td>Perpendicular line</td>
</tr>
<tr>
<td>BDC = 90°</td>
<td>Perpendicular line</td>
</tr>
<tr>
<td>DC = DC</td>
<td>Same line</td>
</tr>
<tr>
<td>ΔADC = ΔBDC</td>
<td>Two sides and included angle the same</td>
</tr>
</tbody>
</table>

∴ AC = BC

*So Linda says it's true.*

**Marty's answer:**

Because CD bisects AB at right angles, B is a reflection of A. So you could think of ABC as made up of two right angle triangles which are reflections of each other. This means the sides AC and BC will be the same length.

*So Marty says it's true.*

From the above answers, choose one which would be closest to what you would do if you were asked to answer this question.

I would choose ______________________’s answer.

In the chart below, circle the mark out of 5 that your teacher would give each answer:

<table>
<thead>
<tr>
<th>Answer</th>
<th>worst</th>
<th>okay</th>
<th>best</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kobi's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Linda's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Marty's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Natalie's answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
G2. For each of the following, circle whether you agree, don't know or disagree.

The statement is: Triangle ABC is always isosceles.

<table>
<thead>
<tr>
<th>Kobi's answer:</th>
<th>agree</th>
<th>don't know</th>
<th>disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Shows that the statement is always true</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Only shows that the statement is true for some positions of C</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Shows you why the statement is true</td>
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<td>2</td>
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<tr>
<td>Is an easy way to explain to someone in your class who is unsure</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Linda's answer:</th>
<th>agree</th>
<th>don't know</th>
<th>disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
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<table>
<thead>
<tr>
<th>Marty's answer:</th>
<th>agree</th>
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<th>disagree</th>
</tr>
</thead>
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<tr>
<td>Has a mistake in it</td>
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<td>3</td>
</tr>
<tr>
<td>Shows that the statement is always true</td>
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<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Natalie's answer:</th>
<th>agree</th>
<th>don't know</th>
<th>disagree</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>3</td>
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<tr>
<td>Shows that the statement is always true</td>
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</tr>
</tbody>
</table>
G3. A is the centre of a circle and AB is a radius. C is a point on the circumference where the perpendicular bisector of AB crosses the circle. Prove whether the following statement is true or false. Write your answer in a way that would get you the best mark you can.

Prove: Triangle ABC is always equilateral.

Write your answer here
Amanda, Barry, Cynthia, Dylan and Ewan were trying to prove whether the following statement is true or false.

The opposite interior angles of a parallelogram are congruent.

**Amanda’s answer:**
I drew a parallelogram and cut it out of the paper.
I then folded it over so that one half of it was on top of the other. It matched perfectly.

So Amanda says it’s true.

**Barry’s answer:**
I drew a parallelogram with \( \angle ABC = 65 \) degrees

<image>

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB \parallel CD</td>
<td>given</td>
</tr>
<tr>
<td>( \angle BCD = 180 - 65 )</td>
<td>interior angles</td>
</tr>
<tr>
<td>= 115</td>
<td>are supplementary</td>
</tr>
<tr>
<td>AD \parallel BC</td>
<td>given</td>
</tr>
<tr>
<td>( \angle BAD = 180 - 65 )</td>
<td>interior angles</td>
</tr>
<tr>
<td>= 115</td>
<td>are supplementary</td>
</tr>
<tr>
<td>So ( \angle BCD = \angle BAD )</td>
<td></td>
</tr>
</tbody>
</table>

So Barry says it’s true.

**Cynthia’s answer:**
I drew a parallelogram and drew in a diagonal AD.

<image>

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \angle BAC = \angle ACD )</td>
<td>AB \parallel CD, alt. ( \angle )</td>
</tr>
<tr>
<td>( \angle BCA = \angle DAC )</td>
<td>AD \parallel BC, alt. ( \angle )</td>
</tr>
<tr>
<td>AC = AC</td>
<td>common side</td>
</tr>
<tr>
<td>( \triangle BAC \equiv \triangle DCA )</td>
<td>ASA</td>
</tr>
<tr>
<td>( \angle ABC = \angle CDA )</td>
<td>congr. tri ( \angle )’s.</td>
</tr>
<tr>
<td>( \angle DAB = \angle DCB )</td>
<td>adding stmts 1 and 2</td>
</tr>
</tbody>
</table>

So Cynthia says it’s true.

**Dylan’s answer:**
I accurately measured the angles of all sorts of parallelograms and made a table to illustrate.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>70</td>
<td>110</td>
<td>70</td>
</tr>
<tr>
<td>93</td>
<td>87</td>
<td>93</td>
<td>87</td>
</tr>
<tr>
<td>66</td>
<td>114</td>
<td>66</td>
<td>114</td>
</tr>
<tr>
<td>32</td>
<td>148</td>
<td>32</td>
<td>148</td>
</tr>
</tbody>
</table>

In each case \( a = c, \ b = d \).

So Dylan says it’s true.

**Ewan’s answer:**
I drew a tessellation of parallelograms and used the parallel lines theorem to mark equal angles.
I used the F pattern and the Z pattern to mark the equal angles. From this I can see that in each parallelogram the opposite angles are equal.

<image>

So Ewan says it’s true.
G4. The opposite interior angles of a parallelogram are congruent. From the previous answers on the last page, choose one which be closest to what you would do if you were asked to answer this question.

I would choose _______________________'s answer.

In the chart below, circle the mark out of 5 that your teacher would probably give each answer:

<table>
<thead>
<tr>
<th></th>
<th>worst</th>
<th>okay</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Amanda’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Barry’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Cynthia’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Dylan’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Ewan’s answer</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

For each of the following, circle whether you agree, don’t know or disagree.

<table>
<thead>
<tr>
<th>Amanda’s answer:</th>
<th>Agree</th>
<th>Don’t Know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
<td>1</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Barry’s answer:</th>
<th>Agree</th>
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<th>Disagree</th>
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</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
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</table>

<table>
<thead>
<tr>
<th>Cynthia’s answer:</th>
<th>Agree</th>
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<th>Disagree</th>
</tr>
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<tbody>
<tr>
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</table>

<table>
<thead>
<tr>
<th>Dylan’s answer:</th>
<th>Agree</th>
<th>Don’t Know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
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<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ewan’s answer:</th>
<th>Agree</th>
<th>Don’t Know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it</td>
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</tr>
</tbody>
</table>
G5. Suppose it has now been proved that:

**Statement 1:** The opposite angles of parallelogram are congruent.

For each of statements 2, 3, 4 or 5 below, Ferdinand asks what needs to be done to prove each statement.

For each statement check either choice A or choice B:

**Choice A:** Ferdinand doesn’t need to do anything; statement 1 has already proved this.

**Choice B:** Ferdinand needs to construct a new proof.

For each statement, check either A or B in the table below:

<table>
<thead>
<tr>
<th>Statement</th>
<th>A Ferdinand doesn’t need to do anything</th>
<th>B Ferdinand needs to construct a new proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statement 2</td>
<td>The opposite interior angles of a rectangle are congruent</td>
<td></td>
</tr>
<tr>
<td>Statement 3</td>
<td>The opposite interior angles of a trapezoid are congruent</td>
<td></td>
</tr>
<tr>
<td>Statement 4</td>
<td>If the opposite interior angles of a quadrilateral are congruent then the quadrilateral is a parallelogram</td>
<td></td>
</tr>
<tr>
<td>Statement 5</td>
<td>The opposite interior angles of a rhombus are congruent</td>
<td></td>
</tr>
</tbody>
</table>
Kay was asked to construct squares on each side of a parallelogram. She was then asked to prove whether the following statement is true or false.

If squares are constructed on each side of a parallelogram, then their centres form the vertices of another square.

Kay used a geometry software package to perform the constructions and she measured the resulting figure's angles and sides to see if indeed it was a square. She then "dragged" the figure to create several variations. The figures that she dragged are shown in the space below. All of these preserved the properties of a square.

Kay dragged the figures several more times. So Kay says it's a square.

Continued on the next page.
G6. On the previous page, Kay was trying to prove that the vertices formed by the centres of the squares formed on the sides of a parallelogram also formed a square. She constructed the figure using a geometry software package and then dragged the figure several times. She measured the sides and angles of the figure and concluded that it was a square.

Answer the following questions considering Kay's work:

<table>
<thead>
<tr>
<th>Kay's answer</th>
<th>Agree</th>
<th>Don't Know</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Has a mistake in it.</td>
<td>1</td>
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<td>3</td>
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</tbody>
</table>

The mark your teacher would give Kay’s work is

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>worst</td>
<td>okay</td>
<td>best</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What do you think are the strengths and weaknesses of Kay’s proof?

**Strengths:**

**Weaknesses:**
G7. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

Given: RSTW and SXTW are parallelograms.
Prove: Triangles RSW and SXT are congruent.

Write your answer here:
Appendix C

Instructions for Administering the Survey

INSTRUCTIONS FOR THE TEACHER:

Once again, thank you for helping me by allowing your students to participate in the survey.

1. The survey is to be given to two parts.
   Part I is a general survey which seeks to solicit information regarding student attitudes and beliefs about mathematical proof. There is also a component which inquires about student views of the role of technology in mathematical proof. Part II is more specific. It seeks to have students demonstrate in more tangible ways their knowledge of proof in specific situations.

2. You will need 50 minutes to complete part I and 70 minutes for part II.

3. Please conduct the survey in a formal way.

4. Please do NOT help the students or intervene in their work.

5. You will need to give a brief introduction (no more than 2 minutes).

6. The surveys are coded so that it is possible to make connections between a student’s responses to part I and the same student’s responses to part II. It will be necessary to have a class list and to record the code that students have for part I so that for part II students will receive the same code on their survey sheets. The form should be submitted with the surveys.

7. The coding form as well as the surveys themselves will be kept in locked files for the duration of the survey. The coding form will be destroyed upon completion of the research.

8. On the next page are the instructions that should be read to students prior to the completion of the surveys.

Note: There are further teacher instructions for the completion of the survey found on page 3.

continued on page 2
INSTRUCTIONS FOR STUDENTS

Teacher: Please read these instructions to the students. Only Part I instructions are to be read prior to students completing part I and Part II instructions prior to students completing part II.

Instructions for Part I

a) You are taking part in a research study into students’ ideas about proof.
b) The results are very important and the results will be used to shape future curriculum.
c) We are interested in individual results so you need to complete the survey on your own, but their your identity will be kept confidential.
d) In part I, there are a number of open-ended questions. There are no right or wrong answers in this section. It is important, however, to provide as complete an answer as possible. Whenever possible, explain your answer with examples.

Instructions for Part II

Repeat a), b) and c) above and:

e) In part II, there will be several types of questions. One type will involve a mathematical statement followed by several answers which were worked out by students who were trying to work out or whether or not the statement was true or false (student proofs). Some answers may be right and some may be wrong, but there is never only one right answer. You will be asked a series of questions about these answers. Another type of question will ask you to construct a proof of your own. These questions will be similar to the already constructed proofs. You are allowed calculators and other mathematical tools like rulers and compasses.
f) Part II is divided into two parts of curriculum. After 30 minutes has elapsed, I will tell you to move onto the second part if you haven’t already done so.

Teacher Instructions continued on page 3
FURTHER TEACHER INSTRUCTIONS

9. If any student finishes early in part II, then tell them in this order to:
   a) check through their answers
   b) write what they now think proof in mathematics is for. They can write this on
      the back of the last sheet in part II.

10. At the end of the time for the survey tell the students to stop working. Then, tell
    them to go through their sheets and where they have left a question blank write in
    the space provided for the question either No response if they could not do it, OR
    No time if they had run out of time.

11. Please ensure that students have completed the minimal demographic information
    required.

12. After both parts of the survey have been completed, please do the following: a)
    complete the teacher administration survey and include it along with b) the
    student coding form and c) the survey forms for part I and II in the envelope
    provided. Please use the cheque enclosed to cover postage costs.

13. Thank you!
### STUDENT CODING FORM

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</table>
Teacher Administration Survey

Teacher Name: ________________________________________________

Please record the following information:

1. The surveys were completed on the following dates:
   
   Part I __________________________
   Part II _________________________

2. The time taken to complete the surveys was as follows:

   Part I __________________________
   Part II _________________________

   Was the time allowed to complete the surveys adequate for the majority of students?

   Were the instructions followed without any difficulty? If there was difficulty, please elaborate.

   Were there any parts of the survey which you believe your students could not be expected to adequately address? Please elaborate if this was the case.

Thank you for the time you have taken to complete this as well as the administration of the survey.
Appendix D

Ethical Review Protocol

To be completed by Principal investigators for all studies which
- involve the use of human subjects, and/or
- involve the analysis of data collected from/on human subjects where such data are not in the public domain.

Title of Project/Thesis: Justifying and Proving in Secondary School Mathematics

Principal Investigator(s) or
Student and Faculty Supervisor:
Frank Leddy, Student
Gila Hanna, Supervisor

Department in which project/Thesis will be housed: Curriculum, Learning and Teaching

Objectives of Study: The study’s objectives are to answer the following questions:
1. What are the features of proofs and the processes of justification that senior secondary students recognise and can demonstrate?
2. In what ways do students construct proofs?
3. What beliefs about proof do students have and how do these effect their proving capabilities?
4. What role can technology play in the development of students’ proving capabilities? What are its strengths and what are its limitations?

1. **Data Collection**

(a) What data are being collected? (achievement scores, attitude scores, experimental test results, etc.)

The data will be collected through the use of two instruments. The first instrument is a series of open-ended questions designed to explore student beliefs about proof. See attachment 1. The second instrument will provide students with the opportunity to demonstrate their competencies with mathematical proof. The data collected will be responses to multiple choice presentations of various sample proofs as well as the students’ own construction of proofs. See attachment 2.

(b) How will the data be collected? (Survey, questionnaire, structured interviews, observation, participant observation)

The data will be collected during the equivalent to two class periods. Students will complete the questionnaire instruments in the given time periods. The two surveys will be coded so that a student’s responses to beliefs about proof may be used to inform the researcher about his/her responses in the demonstration survey. The questionnaires will be administered by a small number of people who will follow prescribed procedures.
(c) Procedures: Please outline procedures to be followed in (a) and (b) above.

The surveys will be conducted during two class periods. The survey on beliefs will be conducted first, followed immediately or shortly thereafter by the demonstration survey. Students will work independently for the allotted time period after they have been given instructions from the survey administrator. A common set of procedures will be followed at each site by the administrators. See attachment 3.

(d) Instruments: Please list all questionnaires, tests, observation schedules, interview schedules, etc. to be used. Attach copies where possible.

The following are the instruments to be used:
Attachment 1 Justifying and Proving in Senior Secondary School Mathematics I
Attachment 2 Justifying and Proving in Senior Secondary School Mathematics II

Other attachments included:
Attachment 3 Instructions for administering the surveys
Attachment 4 Letters of permission and administrative consent

(e) Indicate what information will be taken from existing records (e.g. school records, hospital records).

None.

(f) Curriculum Materials: Where the study involves field testing of curriculum materials, please describe the materials (i.e. the substantive content) which are to be developed and tested.

Not Applicable.

2. Subjects

(a) Describe the subject population and give the age/level and the affiliation as appropriate (e.g. school, university/college students, school board employees, hospital employees, members of the public). Indicate the number of subjects to be included in the study.

Senior secondary (grades 11 and 12) students in the advanced (university intending) stream of mathematics. Approximately 250 students will be surveyed.

(b) How will the subjects be selected for inclusion in the study?

Several schools in various locations will be used. The principal investigator will use his contacts in several school districts to gain access. The administrator at each site will select the classes to be surveyed based upon the criteria in Section 2 (a) above.
3. Data Access, Uses and Interpretation

(a) Who will have access to the raw data?

Five administrators, the principal investigator and the faculty supervisor will have access to the raw data. The administrators will collect the surveys after they have been completed and send them directly to the principal investigator. After they have been received, only the principal investigator and the supervisor will have access to the raw data.

(b) How will confidentiality and/or anonymity of the raw data be maintained? (e.g. will names be deleted and replaced by codes known only to the investigators; will data be stored in locked files?)

No names will be used. The data will be coded in two ways. Each school will have a code designation. This coding is only as a safeguard in case of the need for clarification about the administration of the survey. The surveys will also be coded so that student A who completes part I can be linked to his/her responses in part II.

(c) What disposition will be made of the raw data at the end of the study? (e.g. to be stored in data archives).

After the completion of the study, the raw data will be destroyed.

(d) What feedback will be given to subjects and/or to those individuals who provided informed or administrative consent?

A copy of the final research report (which will provide general conclusions) will be provided to the participating schools and school districts.

(e) What steps will be taken to maintain anonymity of subjects and test sites in written reports?

No names will be used in any reports leading up or following the final research thesis. General background information about the subjects will be only describe the general characteristics of the population studied. No specific information about the students, teachers, schools or school districts will appear in the written reports.

(f) What steps will be taken to alert participants to possible evaluative interpretation and to give them an opportunity to withdraw from the study? (By evaluative interpretation is meant, for example, the indirect evaluation of a teacher's professional performance or of a student's academic performance, as the result of participating in the study, where such evaluation is not an objective of the study).

The letters of permission and administrative consent clearly state that the participants are able to withdraw from the survey at any time. There will be no evaluative interpretation of any students, teachers, schools or school districts. Only general conclusions about the state of proof in the secondary school mathematics curriculum shall be made.
4. **Informed Consent**

(a) Will informed consent be obtained from all participants?

Yes  No  X

(b) Will administrative consent be obtained?

Yes  X  No

(c) What steps will be taken to obtain individual informed consent and/or administrative consent?

Administrative consent will be sent to directors of education or the equivalent. Further, a letter will also be sent to the teachers of the classes chosen to participate in the research study.

(d) Will the informed/administrative consent be written? Yes  X  No If not, why not?

(e) What information will be given to subjects and/or others who are providing informed consent? Please attach a copy of each letter to be sent to potential participants. This letter should describe the study in lay terms, outline potential benefits/risks to participants, indicate that participants are free to withdraw at any time, outline what safeguards will be taken to maintain the confidentiality of the data and to protect participants from possible evaluation on the basis of the written report.

Not applicable. This study meets the criteria for administrative consent only.

*Administrative Consent*

Administrative consent may be deemed sufficient:

a) for studies which have as their intent and focus the acquisition of statistical information and where the collection of data presents

   (i) no invasion of personal privacy,
   (ii) no potential social or emotional risk;

b) for studies which have as their intent and focus the development and evaluation of curriculum materials, resources, guidelines, test items and program evaluation rather than the observation and evaluation of persons as individuals.

_________________________________________  ____________________
Signature of investigator(s)  Date

Or

Student and Faculty Supervisor
April 2, 1999

Director of Education,

Dear

I am a doctoral student at the Ontario Institute of Education at the University of Toronto. I am currently conducting research in mathematics education to complete my thesis requirement. My topic of research is Justifying and Proving in Secondary School Mathematics.

I would like to collect some of my research data in some of your secondary schools during the period May 15 through June 15. I have had preliminary discussions with the following heads of mathematics departments and have received their approval subject to it being acceptable to you.

The research study consists of students completing two surveys about mathematical proof. The first survey asks students to articulate their beliefs about mathematical proof and its place in the curriculum. The second survey gives students the opportunity to demonstrate their proving capabilities. The surveys can be conducted during two 70 minute instructional periods. No disruption to the regular routine of the schools is required.

The data collected will be kept confidential; no names of students, teachers, schools or school boards will be published in the thesis or any subsequent research reports. You may, of course, at any time, withdraw from the research study.

I am confident that the study will prove to be a positive experience for both students and teachers and I look forward to your favourable response. Could you return, one copy of this letter with your signature below indicating that I have administrative consent to proceed.

I thank you for your consideration. I may be contacted at the above address or by e-mail.

Yours sincerely,

Frank Leddy

frank.leddy@ecolint.ch

_________________________________________________________

Frank Leddy has my permission to conduct research in the above named schools according to the terms outlined above.

_________________________________  ________________
Signature                                      Date