WHEN THE CHROMATIC NUMBER IS CLOSE TO THE MAXIMUM DEGREE

by

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A thesis submitted in conformity with the requirements for the degree of Master of Science
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Abstract

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Let $\Delta(G)$ and $\chi(G)$, respectively, be the maximum degree and the chromatic number of a graph $G$. The $c$-colorability problem asks to determine whether there is a proper $c$-coloring of $G$. We know that $G$ is $(\Delta + 1)$-colorable and the classical theorem of Brooks states that $G$ is not $\Delta$-colorable only if $G$ has a $(\Delta + 1)$-clique or $\Delta = 2$ and $G$ contains an odd cycle. Reed extended Brooks’ Theorem by showing that there exists a $\Delta_0$ such that if $\Delta(G) \geq \Delta_0$ then $G$ is not $(\Delta - 1)$-colorable if and only if $G$ contains a $\Delta$-clique.

In this thesis, we extend Reed’s characterization of $(\Delta - 1)$-colorable graphs. We present a naive coloring procedure and a modification of this procedure. The modified procedure along with a structural decomposition plays an important role in our characterization of graphs with $\chi$ close to $\Delta$. We characterize $(\Delta - 2)$, $(\Delta - 3)$, $(\Delta - 4)$ and $(\Delta - 5)$-colorable graphs, for sufficiently large $\Delta$, and prove a general structure for graphs with $\chi$ close to $\Delta$. We give a linear time algorithm to check the $(\Delta - k)$-colorability of a graph, for sufficiently large $\Delta$ and any constant $k$. Also, we solve the asymptotic version of an open problem on graphs without large complete subgraphs.

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Contents

1 Introduction
   1.1 Statement of the problem .................................. 1
   1.2 Preliminaries .................................................. 1
   1.3 Overview ...................................................... 2

2 A structural decomposition ........................................ 3
   2.1 A naive coloring procedure .................................... 3
   2.2 The decomposition ............................................. 5
   2.3 Reed’s variant of the naive coloring procedure ............... 8
   2.4 The chromatic number of a dense set ........................ 9

3 Characterization of graphs with $\chi \geq \Delta - 2$ ................. 12
   3.1 A Lemma ...................................................... 12
   3.2 Graphs with chromatic number at least $\Delta - 1$ .......... 13
   3.3 Graphs with chromatic number at least $\Delta - 2$ .......... 16

4 An approach to a general characterization ....................... 22
   4.1 A bound on the size of edge-critical dense subgraphs ........ 22
   4.2 Graphs with chromatic number at least $\Delta - 3$ .......... 24
   4.3 Graphs with chromatic number at least $\Delta - 4$ .......... 25
   4.4 A lowerbound for the number of $(\Delta - 5)$-edge critical graphs .... 31

5 Further results ...................................................... 32
   5.1 $(\Delta - k)$-colorability ................................... 32
   5.2 Graphs without a complete subgraph .......................... 35

Bibliography .................................................................. 38
Chapter 1

Introduction

1.1 Statement of the problem

In 1941, Brooks proved that every graph $G$ is $(\Delta + 1)$-colorable and he characterized those graphs with $\chi = \Delta + 1$.

Theorem 1.1 (Brooks) For every graph $G$, $\chi(G) \leq \Delta + 1$. Moreover, $\chi(G) = \Delta + 1$ if and only if $\Delta(G) \neq 2$ and $G$ contains a $(\Delta + 1)$-clique, or $\Delta(G) = 2$ and $G$ contains an odd cycle.

Remark 1.2 It is easy to see that, in fact, $G$ has a $(\Delta + 1)$-clique if and only if it has a $(\Delta + 1)$-clique as a connected component. The same is true when $\Delta = 2$ for odd cycles.

Brooks' characterization of graphs with $\chi = \Delta + 1$ yields an efficient algorithm to check $\Delta$-colorability of a graph. In 1998, Reed [19] proved that for sufficiently large $\Delta$, a graph of maximum degree $\Delta$ contains a $\Delta$-clique. This fact will turn $(\Delta - 1)$-colorability to an easy problem.

Our main goal in this thesis, is to obtain similar characterizations for graphs with chromatic number close to $\Delta$ in the sense that we can check $c$-colorability of graph efficiently for $c$ sufficiently close to $\Delta$.

1.2 Preliminaries

In this thesis, we use the definitions and notations from [24]. Note that, throughout the whole thesis, by the word graph we mean a simple undirected graph. A graph $G = (V, E)$ is called critical or vertex-critical if $\chi(G') < \chi(G)$ for every proper subgraph $G'$ of $G$. $G$
is called edge-critical if $\chi(G') < \chi(G)$ for every graph $G' = (V, E')$ where $E' \subseteq E$. $G$ is called c-(edge)critical if it is (edge-)critical and $\chi(G) = c$. So, a c-edge critical graph is c-critical, by definition. A vertex $v \in V$ is called a dominating vertex if $N(v) = V - \{v\}$.

The join of disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined to be graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in V_2\}$. The join of a graph $G_1 = (V_1, E_1)$ and a set of vertices $V_2 = \{v_1, \ldots, v_t\}$ is defined as the join of $G_1$ and $G_2 = (V_2, \emptyset)$. A vertex $v$ is joined to the set of vertices $V = \{v_1, \ldots, v_t\}$ if $V \subseteq N(v)$.

The following facts will frequently be used in our proofs.

**Lemma 1.3** Every c-chromatic graph contains a c-(edge)critical subgraph.

**Lemma 1.4** If $G$ is a c-critical graph then $\delta(G) \geq c - 1$.

**Theorem 1.5** If $G$ is the join of two disjoint graphs $G_1$ and $G_2$ then $\chi(G) = \chi(G_1) + \chi(G_2)$. Moreover, $G$ is (edge-)critical if and only if $G_1$ and $G_2$ are (edge-)critical.

### 1.3 Overview

The main results of this thesis appear in Chapters 3, 4 and 5. We start by presenting a naive coloring procedure and a modification of this procedure in Chapter 2. We will see how the modified procedure can help us to characterize graphs with chromatic number close to $\Delta$. We will explain the idea of a structural decomposition of a graph into $\Delta$-clique like subgraphs. At the end of Chapter 2, we will state some lemmas and theorems which we need for our later proofs.

In Chapter 3, we will extend Brooks' and Reed's characterizations of graphs with $\chi \geq \Delta + 1$ and $\chi \geq \Delta$ to graphs with $\chi \geq \Delta - 1$ and $\chi \geq \Delta - 2$, by using the structural decomposition from Chapter 2. We will show that the number of non-isomorphic $(\Delta - 1)$-edge critical and $(\Delta - 2)$-edge critical graphs are 2 and 4, respectively.

Chapter 4 provides a general structure for graphs with chromatic number close to their maximum degree. Applying this theorem, we will characterize graphs with $\chi \geq \Delta - 3$ and $\chi \geq \Delta - 4$ by showing that there are 27 and 421 non-isomorphic $(\Delta - 3)$-edge critical and $(\Delta - 4)$-edge critical graphs, respectively. Also, we will give a lowerbound on the number of $(\Delta - 5)$-edge critical graphs.

Chapter 5 deals with algorithmic aspects of our results. There, we provide a linear time algorithm to check $(\Delta - k)$-colorability of a graph, for small constant $k$. We will also show that our main results in Chapters 3 and 4 will solve the asymptotic version of an open problem in graph coloring.
Chapter 2

A structural decomposition

We will start this chapter by presenting a naïve procedure for graph coloring which has many significant applications. Then we introduce a more powerful modification of this procedure. This modification is made by use of a structural decomposition of the graph into subgraphs $D_1, \ldots, D_t$ and $S$ where each $D_i$ is very close to being a $\Delta$-clique. We will give a procedure which obtains such a decomposition for all graphs. We will then study these dense sets and examine their structure, since they play a crucial role in the main results of this thesis.

2.1 A naïve coloring procedure

Finding an optimal coloring of a graph is a very difficult problem. Even determining whether the chromatic number of a graph is three is NP-complete. This fact will give us a taste of the complexity of the coloring problem.

We know that if a graph $G$ is of maximum degree $\Delta$ then $\chi(G) \leq \Delta + 1$. In fact, a $(\Delta + 1)$-coloring of a graph can be easily found using a greedy algorithm. Consider an arbitrary order of the vertices and color each vertex in turn such that it receives a color other than those which appear in its neighborhood. This algorithm works since each vertex has at most $\Delta$ colors in its neighbors and thus there would be at least one color which did not appear there. The above greedy algorithm is simple and straightforward but it might not come close to an optimal coloring.

There is also a naïve random coloring procedure which seems to be very powerful. The procedure which was first introduced by Kahn [6] is as follows:

- choose a color for each vertex uniformly at random,
- uncolor those vertices which have the same color as one of its neighbors,
• complete the resulting proper partial coloring, e.g., by iterating, or with a greedy approach.

In a related application, it can be shown, using probabilistic methods, that with a positive probability the proper partial coloring can be extended to a coloring of the whole graph. The basic idea of this naive procedure is that we wish to have many repeated colors in the neighborhood of any uncolored vertex. This will make it possible to complete the partial coloring with fewer than $\Delta$ colors. In particular, consider a graph $G$ of maximum degree $\Delta$ and a partial coloring of $G$ with only $\Delta - k + 1$ colors. We color the rest of the graph greedily. If we have at least $k$ colors which have appeared more than once in the neighborhood of a vertex $v$ then $v$ has at most $\Delta - k$ different colors in its neighborhood and so there is at least one feasible color remaining for $v$. Thus, if the naive procedure produces a partial coloring such that there are many repeated colors in the neighborhood of each vertex then that partial coloring can be completed properly. In general, we prove that a sufficient number of feasible colors will remain for each vertex so that it can be colored in the extension of the partial coloring.

If we have many edges in the neighborhood of a vertex, then the chance of having many repeated colors is low. That is because if two neighbors of a vertex receive the same color, they will get uncolored if there is an edge between them. It is obvious that if the neighborhood of a vertex is a clique, there will be no repeated colors after the partial coloring. Furthermore, if the neighborhood of the vertex contains very few non-adjacent pairs of vertices then very few pairs of vertices have a chance to retain the same color. This explains why the procedure tends to work well for sparse graphs.

As mentioned, variants of this very simple coloring procedure have many important applications. Johanssen's result [5] that the chromatic number of a triangle-free graph of maximum degree $\Delta$ is at most $O(\frac{\Delta}{\ln \Delta})$ is based on a modification of the iterated version of the above naive coloring procedure. In each iteration, each vertex may be activated at random to receive a color. Then, a color will be chosen from the color list of each activated vertex and those vertices involved in a conflict will be uncolored. The main trick in this approach is to make a non-uniform distribution on the lists of colors for each vertex. This trick, along with the activation probability, makes it possible to gain a positive probability for the completion of the partial coloring.

Kahn's proof for the asymptotic version of the list coloring conjecture is the first significant application of the naive coloring procedure [6]. The list-chromatic index of a graph $G$, $\chi'_l(G)$, is the least $t$ such that for any assignment of lists of colors to the edges of $G$, where each list has size $t$, there is a proper edge-coloring of $G$ such that the color
of each edge is in its list. The conjecture is that for a graph $G$ of maximum degree $\Delta$, $\chi'_f(G) = \chi'(G) \leq \Delta + 1$. He proved that $\chi'_f(G) = \Delta + o(\Delta)$. The iterative version of the naive coloring procedure will color edges instead of vertices. At each iteration, an uncolored edge is assigned a color which is chosen from its list at random. An edge is uncolored if it has been assigned the same color as one of its neighbors. At the end of each iteration, the lists will be updated, essentially by removing the color of an edge from its neighbors' lists. It is shown that the process can be repeated sufficiently many times such that after the last iteration the few remaining uncolored edges can be colored greedily. If we consider the line graph of $G$, the problem is now to color the vertices. It can be seen that the neighborhood of a vertex in the line graph of a $\Delta$-regular graph is sparse, since the maximum degree in the line graph is $2\Delta - 2$, and it can be verified that the neighborhood of a vertex in the line graph has at most $\Delta^2$ edges. Thus, by our rule of thumb that the naive coloring procedure works well on sparse graphs, it is not surprising that it works well on line graphs. But, it should be noted that sparseness alone was not enough for it to work as well as it did.

2.2 The decomposition

As we described, the above naive procedure works well on sparse graphs. In 1998, Reed[19] presented a coloring procedure which colors dense parts of the graph individually, and uses the same naive coloring procedure for the rest of the graph. Applying random permutations on the colors of each colored dense subgraph will fix the possible conflicts among these subcolorings.

We clarify the concepts mentioned above. It will be shown that the vertex set of every graph can be partitioned into disjoint sets $D_1, \ldots, D_t, S$, such that each $D_i$ is dense, $\Delta$-clique-like and every vertex in $S$ is sparse. First, we give formal definitions for such a decomposition. A vertex $v$ in a graph $G$ of maximum degree $\Delta$ is \textit{d-dense} if there are at least $\binom{\Delta}{2} - d\Delta$ edges in the subgraph induced by its neighborhood. Otherwise, $v$ is \textit{d-sparse}. Note that if $d$ is sufficiently small and $v$ is $d$-dense, this implies that $v$ has nearly $\Delta$ neighbors and very few missing edges in its neighborhood.

\textbf{Definition 2.1} A \textit{d-dense decomposition} of a graph $G$ of maximum degree $\Delta$ is a partition of $V(G)$ into disjoint sets $D_1, \ldots, D_t, S$, such that:

(a) a vertex $v$ is in $D_i$ if and only if it has at least $\frac{3\Delta}{4}$ neighbors in $D_i$;

(b) there are at most $8d\Delta$ edges between each $D_i$ and the rest of the graph;
(c) the number of vertices in each $D_i$ is between $\Delta + 1 - 8d$ and $\Delta + 1 + 4d$;

(d) if a vertex $v$ is in $S$ then it is $d$-sparse.

We are interested in such decompositions since it has been proved that if we can color each dense set $D_i$ with $\Delta - k$ colors, where $k$ is sufficiently small, a $(\Delta - k)$-coloring can also be found for the whole graph (see Theorem 2.3).

The following theorem [19] shows that for appropriate values of $d$, there exists a $d$-dense decomposition for all graphs.

**Theorem 2.2** A graph $G$ of maximum degree $\Delta$ has a $d$-dense decomposition, for all $d \leq \frac{\Delta}{100}$.

The constructive proof of the above theorem is based on the following algorithm to obtain dense sets of a $d$-decomposition of a graph. By definition, a vertex $v$ is in $S$ only if it is $d$-sparse. Therefore, we may initially define a set $D_v$ for each dense vertex and set it as $N(v) + v$. Then, we modify each $D_i$ using a 2 step process as follows:

Step 1. While there is a vertex $u$ in $D_v$ such that $|N(u) \cap D_v| < \frac{3\Delta}{4}$, delete $u$ from $D_v$.

Step 2. While there is a vertex $u$ not in $D_v$ such that $|N(u) \cap D_v| \geq \frac{3\Delta}{4}$, add $u$ to $D_v$.

Note that in the above procedure we first perform all deletions, and then add new vertices. Thus, it can be seen that $D_v$ is uniquely defined, since the order of deleting vertices in the first step does not make any difference in the produced dense sets, nor does the order of adding the vertices in the second step. We will prove that the above process yields a $d$-dense decomposition of the graph.

**Proof:** We will show that these sets will define a partition of $d$-dense vertices, that is, either they contain the same subset of $d$-dense vertices or they are disjoint. We first prove that they satisfy the properties (a), (b), (c) and (d).

To prove that (a) is satisfied, it can be seen from the above procedure that every vertex in $D_v$ has at least $\frac{3\Delta}{4}$ neighbors in $D_v$, and every vertex outside $D_v$ has fewer than $\frac{3\Delta}{4}$ neighbors in $D_v$. It is obvious that every vertex outside $D_v$ has fewer than $\frac{3\Delta}{4}$ neighbors in $D_v$. Otherwise it would be added in the second step of the process. For the first part, note that if a vertex has fewer than $\frac{3\Delta}{4}$ neighbors in $D_v$, it would be deleted in the first step and so the remaining vertices have at least $\frac{3\Delta}{4}$ neighbors in $D_v$. In the second step we will add only those vertices with at least $\frac{3\Delta}{4}$ neighbors in $D_v$. Thus, a vertex in $D_v$ has at least $\frac{3\Delta}{4}$ neighbors in it.
For (b), we claim that every $d$-dense vertex has at least $\Delta - 2d$ neighbors. Suppose $v$ to be a $d$-dense vertex with fewer than $\Delta - a$ neighbors. Then $|E(N(v))| \leq \left(\frac{\Delta - a}{2}\right)$. Since $v$ is $d$-dense we have $\left(\frac{\Delta - a}{2}\right) \geq \left(\frac{\Delta}{2}\right) - d\Delta$. This inequality implies that $a \leq 2d$.

Next we show that in the deletion phase of the algorithm, at most $6d$ vertices are deleted from $D_u$. Suppose that more than $6d$ vertices are deleted from $D_u$. Thus, each deleted vertex has fewer than $\frac{3\Delta}{4}$ edges to the vertices which remain in $N(v)$. Thus, $|E(N(v))| < \left(\frac{6d}{2}\right) + \frac{3\Delta}{4}6d + \left(\frac{\Delta - 6d}{2}\right) < \left(\frac{\Delta}{2}\right) - d\Delta$, for $d \leq \frac{\Delta}{100}$. Contradiction.

We just showed that every $d$-dense vertex $v$ has at least $\Delta - 2d$ neighbors and at most $6d$ of them may be deleted, thus $|D_u| = \Delta + 1 - 8d$. Also, since $|E(N(v), G - N(v))| \leq 2d\Delta$ and adding a vertex in the second phase will decrease it by at least $\frac{2\Delta}{4}$, fewer than $4d$ vertices can be added in the second phase, thus $|D_u| \leq \Delta + 1 + 4d$, and (c) holds.

To prove (d), it would be sufficient to show that $v \in D_u$. $v$ has at least $\Delta - 2d$ neighbors and at most $6d$ of them may be deleted. Thus, $v$ has at least $\Delta - 8d > \frac{3\Delta}{4}$ neighbors in $D_u$ and so $v \in D_u$.

It only remains to show how to use the above procedure to obtain a partition of $d$-dense vertices and consequently $V(G)$. We show that if $u$ and $v$ are $d$-dense vertices and $D_u$ intersects $D_v$, then $u \in D_v$ and $v \in D_u$. Consider some $x \in D_u \cap D_v$. Since $x \in D_u$, $x$ has at least $\frac{3\Delta}{4}$ neighbors in $D_u$ where $|D_u - N(u)| \leq 4d \leq \frac{\Delta}{25}$. So $x$ is adjacent to at least $\frac{2\Delta}{3}$ neighbors of $u$. Similarly, $x$ is adjacent to at least $\frac{2\Delta}{3}$ neighbors of $v$, and $x$ has at most $\Delta$ neighbors. Thus, $|N(u) \cap N(v)| \geq \frac{\Delta}{3}$; we will improve this bound below.

We know that $|E(N(u) \cap N(v), N(v) - N(u))| \leq |E(N(u), G - N(u))| \leq 2d\Delta \leq \frac{\Delta^2}{50}$. Since $v$ is $d$-dense, the subgraph induced on $N(v)$ and consequently any subgraph of the above induced graph has at most $d\Delta \leq \frac{\Delta^2}{100}$ missing edges. Thus, $|E(N(u) \cap N(v), N(v) - N(u))| \geq |N(u) \cap N(v)| \times |N(v) - N(u)| - \frac{\Delta^2}{100}$. Thus, $|N(u) \cap N(v)| \times |N(v) - N(u)| \leq \frac{\Delta^2}{25}$. So, recalling that $|N(u) \cap N(v)| \geq \frac{\Delta}{3}$, we now have $|N(v) - N(u)| \leq \frac{\Delta}{8}$. This yields the improved bound $|N(u) \cap N(v)| \geq |N(v)| - |N(v) - N(u)| \geq \frac{5\Delta}{6}$.

With the same argument as in part (d), we conclude that $u \in D_v$ and $v \in D_u$ since $|N(u) \cap N(v)| \geq \frac{5\Delta}{6}$.

Now, we define the sequence $D_1, \ldots, D_t$ and $S$ as follows:
set $i = 0$;

- while there is a $d$-dense vertex $v$ which has not appeared in $D_1, \ldots, D_i$, set $i = i + 1$ and $D_i = D_v$;

- set $t = i$ and $S$ to be the set of the remaining vertices which have not appeared in $D_1, \ldots, D_i$. These vertices will all be $d$-sparse.

To show that the above procedure yields a partition of the vertices of $G$, it is sufficient to show that all $D_i$'s are disjoint. Suppose that $D_i$ and $D_j$, $i < j$, correspond to $d$-dense sets $D_u$ and $D_v$ respectively and they intersect. Thus, $v \in D_u = D_i$. Contradiction. The sequence $D_1, \ldots, D_t$ and $S$ is thus a $d$-dense decomposition of $G$.

### 2.3 Reed's variant of the naive coloring procedure

Reed [19] used the above decomposition to modify the naive coloring procedure, discussed in 2.1.

Suppose that $G$ is a graph of maximum degree $\Delta$ with a $d$-dense decomposition $D_1, \ldots, D_t, S$, where $d \leq \frac{\Delta}{5000}$. Suppose further that we have a partition of each $D_i$ into independent sets $\{P^j_1, \ldots, P^j_{l_i}\}$ (i.e. a $l_i$-coloring of $D_i$). Typically, our goal is to color $G$ with $c$ colors for some particular $c \geq \max_1 \leq i \leq t l_i$.

Reed modified the naive coloring procedure as follows:

- Choose a color for each vertex in $S$ uniformly at random from $\{1, \ldots, c\}$.
- Choose a permutation of $\{1, \ldots, c\}$ for each dense set $D_i$ uniformly at random. Color the vertex or vertices in $P^j_1$ with the $j$'th color in the permutation, for $1 \leq j \leq l_i$.
- uncorder any vertex which has the same color as one of its neighbors.
- complete the resulting proper partial coloring, e.g., with a greedy approach.

In [19], Reed applied the above variant of the naive coloring procedure to bound the chromatic number of a graph by a convex combination of its clique number and its maximum degree plus 1. In fact, he proved that if $G$ has maximum degree $\Delta$ sufficiently large, and clique number less than or equal to $\omega$, where $\omega$ is sufficiently close to $\Delta$, $\chi(G) \leq \frac{\Delta + 1 + \omega}{2}$.

Molloy and Reed [13] applied this method to prove that there is a constant $c$ such that the total chromatic number of any graph $G$ is bounded by $\Delta(G) + c$. 
In another application, Molloy and Reed [17] proved that any graph $G$ of maximum degree $\Delta$ has a proper $(\Delta + 1)$-coloring such that for every color $c$ and every vertex $v$, $c$ appears on at most $O(\frac{\log \Delta}{\log \log \Delta})$ neighbors of $v$. Such a coloring is called frugal. Alon (see [17]) had earlier provided an example of a graph which shows that $O(\frac{\log \Delta}{\log \log \Delta})$ is best possible.

### 2.4 The chromatic number of a dense set

In this section, we will study the structure of a dense set with a given chromatic number close to $\Delta$ where $\Delta$ is sufficiently large. The reason that we are interested in the chromatic number of the dense sets of a graph is that if we can color each dense set with $\Delta - k$ colors where $k$ is sufficiently small, then we can obtain a $(\Delta - k)$-coloring of the graph itself. In fact, Molloy and Reed [15] proved the following theorem.

**Theorem 2.3** There is a constant $\Delta_0$ such that: Consider any graph $G$ of maximum degree $\Delta \geq \Delta_0$, and any $d$-dense decomposition of $G$ for $d = 1000\sqrt{\Delta}$. If $k^2 + k < \Delta - k$ then $G$ is $(\Delta + 1 - k)$-colorable if and only if the subgraph induced on each dense set $D_i$ is $(\Delta + 1 - k)$-colorable.

**Corollary 2.4** For any $\Delta > \Delta_0$ and $k$ such that $k^2 + k < \Delta - k$, there exists an $N = N(\Delta)$ such that: If $\chi(G) > \Delta + 1 - k$, then there exists a subgraph $H$ of $G$ with at most $N$ vertices, such that $\chi(H) > \Delta + 1 - k$.

The upper bound for the size of a $d$-dense set is $\Delta + 1 + 4d$ and so we can set $N(\Delta) = \Delta + 4000\sqrt{\Delta}$. So, the above corollary is deduced from Theorem 2.3. In fact, Molloy and Reed showed that the above corollary holds for $k^2 + k < \Delta$. This upper bound on $k$ in Theorem 2.3 is tight, as shown by the following theorem from [18] which is proved using a construction from [2].

**Theorem 2.5** For every $\Delta$, $k$ and $N$ with $k < \Delta - 2$, and $k^2 + k \geq \Delta$, there exists a graph $G$ of maximum degree $\Delta$ such that $\chi(G) > \Delta + 1 - k$, $|V(G)| \geq N$, and for every proper subgraph $H$ of $G$, $\chi(H) \leq \Delta + 1 - k$.

Molloy and Reed [18] bounded the $N(\Delta)$ in Corollary 2.4 by $\Delta + 1$, as follows:

**Theorem 2.6** For $\Delta > \Delta_0$ and $k$ such that $k^2 + k < \Delta - k$, if $\chi(G) > \Delta + 1 - k$, then there exists a subgraph $H$ of $G$ containing a dominating vertex such that $\chi(H) > \Delta + 1 - k$. 


Of course, since $H$ has a dominating vertex, $|H| \leq \Delta + 1$.

We call an optimal coloring of a graph $G$ strongly optimal if the number of color classes of size 2 is the maximum among all optimal colorings. Considering an optimal coloring, we call a vertex $v$ unicolored in that coloring if it forms a color class of size one, that is, there is no other vertex with the same color as $v$.

**Lemma 2.7** In an optimal coloring of graph $G$, if $v$ is a unicolored vertex then every color appears in $N(v)$.

**Proof:** Consider $c$ to be a color which does not appear in $N(v)$. Color $v$ with $c$ and we will have a proper coloring of $G$ with one less color. Contradiction.

Thus,

**Corollary 2.8** In an optimal coloring of a graph, every two unicolored vertices are adjacent.

**Lemma 2.9** If the graph induced on a $d$-dense set $D$ is of maximum degree at most $\Delta$ and $\chi(D) \geq \Delta - k$, then it contains a clique of size at least $\Delta - 2k - 4d - 1$.

**Proof:** In an optimal coloring of $D$, at least $\Delta - 2k - 4d - 1$ colors appear on only one vertex. Otherwise, there would be at most $\Delta - 2k - 4d - 2$ unicolored vertices and thus, at least $(\Delta - k) - (\Delta - 2k - 4d - 2) = k + 4d + 2$ color classes of size at least two. Thus, $|D_i| \geq (\Delta - 2k - 4d - 2) + 2(k + 4d + 2) = \Delta + 4d + 2$ which contradicts the fact that $|D_i| \leq \Delta + 4d + 1$. Now, by the above corollary, these vertices should form a clique.

Thus, a dense set with chromatic number close to $\Delta$ consists of a clique and a few vertices. Therefore, in an optimal coloring of the dense set, most of the color classes have size 1 and very few of them have size greater than 1.

**Lemma 2.10** In a strongly optimal coloring of a graph induced by a $d$-dense set, for each vertex $t$ and color $c$, if $t$ has color $c$, and the color class of $c$ has size greater than two, then $t$ is adjacent to all unicolored vertices.

**Proof:** Consider a unicolored vertex $w$ which is not adjacent to $t$. Color $t$ with the same color as $w$. This results in another optimal coloring with at least one more color class of size 2. Contradiction.

**Lemma 2.11** In a $(\Delta - k)$-coloring of a graph induced by a $d$-dense set $D$, where $k$ is sufficiently small and $\Delta$ is sufficiently large, each vertex of $D$ is adjacent to more than $\frac{2\Delta}{3}$ unicolored vertices.
Figure 2.1: In a color class of size at least 2, one of the vertices is connected to all unicolored vertices.

**Proof:** Suppose $v \in D$. $v$ has at least $\frac{3\Delta}{4}$ neighbors in $D$ where $|D| \leq \Delta + 1 + 4d$ and at least $\Delta - 2k - 4d - 1$ of the vertices in $D$ are unicolored. So, there are at most $2k + 8d + 2$ vertices which are not unicolored. $v$ has at least $\frac{3\Delta}{4}$ neighbors in $D$, thus $v$ is adjacent to more than $\frac{2\Delta}{3}$ unicolored vertices.

**Lemma 2.12** If there is a color class $C$ of size at least 2 in an optimal coloring of a graph $G$, then at least one of the two vertices is adjacent to all unicolored vertices.

**Proof:** Let $C = \{v_1, v_2, \ldots, v_m\}$, where $m \geq 2$. Suppose that $v_i$ is not adjacent to $w_i$, for $1 \leq i \leq m$. Then by coloring $v_i$ with the color of $w_i$, for $1 \leq i \leq m$, we get a proper coloring with one less color (fig. 2.1). Contradiction.
Chapter 3

Characterization of graphs with $\chi \geq \Delta - 2$

In Chapter 1, we saw that there is a greedy algorithm to color a graph $G$ of maximum degree $\Delta$ with $\Delta + 1$ colors. In fact, it turned out to be easy to characterize those graphs with $\chi \geq c$ when $c = \Delta + 1$, since Brooks proved that $\chi(G) = \Delta + 1$ if and only if $G$ has a $(\Delta + 1)$-clique or in the case $\Delta = 2$, $G$ has an odd cycle. In [19], Reed proved that this characterization extends to $c \geq \Delta$, as long as $\Delta$ is large.

**Theorem 3.1** For sufficiently large $\Delta$, if $G$ is a graph of maximum degree $\Delta$, then $\chi(G) \geq \Delta$ if and only if $G$ contains a $\Delta$-clique as a subgraph.

In this chapter, we will obtain a similar characterization of graphs with chromatic number at least $\Delta - 1$ and $\Delta - 2$. Using Theorem 2.3, we can focus on $d$-dense sets of a graph instead of the graph itself, for $d \leq \frac{\Delta}{100}$.

### 3.1 A Lemma

In this section, we prove a lemma which will be used to prove the main theorems of this chapter.

Consider a graph $G$ of maximum degree $\Delta$, where $\Delta \geq \Delta_0$ from Theorem 2.3. Suppose that $\chi(G) \geq \Delta - k$, where $k^2 + k < \Delta - k$. Define $d = 1000\sqrt{\Delta} \leq \frac{\Delta}{100}$. By Theorem 2.3, we know that in a $d$-dense decomposition of $G$, there exists a $d$-dense set $D$ where the chromatic number of the induced subgraph on $D$ is at least $\Delta - k$, otherwise, $G$ can be colored with fewer than $\Delta - k$ colors. Now, assume that $H$ is a minimal subgraph of the induced graph on $D$ with chromatic number $\Delta - k$. That is, removing any vertex or
edge from $H$ will lessen $\chi(H)$. We call $H$ a $(\Delta - k)$-edge critical dense subgraph of $G$. Recall that $|D| \leq \Delta + 1 + 4d$ and note that $|H| \geq \Delta - k$, since $\chi(H) = \Delta - k$. So, $V(H)$ is almost $D$ except for very few vertices.

**Lemma 3.2** Consider a graph $G$ of maximum degree $\Delta$ with $\chi(G) \geq \Delta - k$, where $k^2 + k < \Delta - k$. There is a $(\Delta - k)$-edge critical dense subgraph $H$ of $G$ such that if $C$ is a strongly optimal $(\Delta - k)$-coloring of $H$ which contains $\lambda_i$ color classes of size $i$, for $i \geq 1$, then

a) there is a dominating vertex in $H$;

b) $\sum_{i \geq 2} \lambda_i (i - 1) \leq k + 1$;

c) there cannot be only one class of size $> 1$

d) there is no color class of size greater than $k + 1$ in $C$;

**Proof:**

By Theorem 2.6, there is a subgraph $H_0$ with a dominating vertex $v_d$ where $\chi(H_0) \geq \Delta - k$. Thus, $H_0 - \{v_d\}$ contains a $(\Delta - k - 1)$-edge critical subgraph. Join $v_d$ to this subgraph to form $H$. So, $H$ is $(\Delta - k)$-edge critical and contains a dominating vertex.

To prove b), suppose $v_d$ to be a dominating and consequently a unicolored vertex. We have $\text{deg}(v_d) = (\sum_{i \geq 1} i \lambda_i) - 1 \leq \Delta$ and $\sum_{i \geq 1} \lambda_i = \Delta - k$. By subtracting the above equalities, we get $\sum_{i \geq 2} \lambda_i (i - 1) \leq k + 1$.

For c), assume that $H$ has exactly one color class $c$ of size $t$, $t \geq 2$. Lemma 2.13 implies that there is a vertex in color class $c$ joined to the $(\Delta - k - 1)$-clique. This forms a $(\Delta - k)$-clique, and so $H$ is not critical. Contradiction.

To show d), part b) implies that there is no color class of size greater than $k + 2$, and if there is a class of size $k + 2$ then it is the only class of size greater than 1. Thus parts b) and c) imply d).

As a special case of the part b) of the above Lemma, we have the following corollary:

**Corollary 3.3** There are at most $k + 1$ color classes of size 2;

### 3.2 Graphs with chromatic number at least $\Delta - 1$  

**Theorem 3.4** Consider a graph $G$ of maximum degree $\Delta$, where $\Delta$ is sufficiently large. $\chi(G) \geq \Delta - 1$ if and only if $G$ contains a subgraph isomorphic to either
Figure 3.1: There is no color class of size > 2 and exactly two color classes of size 2.

(i) a \((\Delta - 1)-\)clique or
(ii) a \((\Delta - 4)-\)clique joined to a \(C_5\) (a cycle on 5 vertices).

Proof: If \(\chi(G) > \Delta - 1\) then by Theorem 3.1, \(G\) contains a \((\Delta - 1)-\)clique. Assume that \(\chi(G) = \Delta - 1\) and \(G\) does not contain a \((\Delta - 1)-\)clique. We will show that \(G\) contains a \((\Delta - 4)-\)clique joined to a \(C_5\). Suppose that \(H\) is a \((\Delta - 1)-\)edge critical dense subgraph of \(G\) and \(C\) is a strongly optimal \((\Delta - 1)-\)coloring of \(H\).

By Lemma 3.2.d, the color classes of \(C\) have size at most 2 and by Corollary 3.3 the number of these classes is at most 2.

If there is no color class of size two, then \(H\) is a clique of size at least \(\Delta - 1\). Contradiction. By Lemma 3.2.c, there cannot be only one color class \(c\) of size 2. Thus, there are exactly two color classes of size 2.

Therefore, the coloring of graph \(H\) consists of \(\Delta - 3\) unicolored vertices and two color classes of size 2. We will prove that \(H\) is a \((\Delta - 4)-\)clique joined to a \(C_5\). Suppose that \(c_1 = \{u_1, v_1\}\) and \(c_2 = \{u_2, v_2\}\) are the two color classes of size 2 (fig. 3.1).

By Lemma 2.12, we can assume that \(u_1\) and \(u_2\) are adjacent to all unicolored vertices.

It can be seen that \(u_1\) is not adjacent to \(u_2\), otherwise \(u_1, u_2\) and the \((\Delta - 3)-\)clique form a \((\Delta - 1)-\)clique. Contradiction.

Lemma 3.5 Neither \(v_1\) nor \(v_2\) is adjacent to all unicolored vertices.

Proof: \(v_1\) and \(v_2\) cannot both be connected to all unicolored vertices. Otherwise, there will be no edge between \(c_1\) and \(c_2\) (any such edge would form a \((\Delta - 1)-\)clique). Thus, \(H\) would be \((\Delta - 2)-\)colorable. Now, without loss of generality, assume that \(v_1\) is adjacent to all the vertices in the \((\Delta - 3)-\)clique, and that there is a unicolored vertex.
Figure 3.2: Neither $v_1$ nor $v_2$ is adjacent to all unicolored vertices.

$w$ where $v_2$ is not adjacent to $w$. $v_1$ is not adjacent to $u_2$, otherwise we would have a $(\Delta - 1)$-clique. Now, since $u_1$ is not adjacent to $u_2$ we may recolor $H$ with one fewer color as indicated in figure 3.2. Contradiction.

Thus, there exist unicolored vertices $w_1$ and $w_2$ such that $v_i$ is not adjacent to $w_i$, for $i = 1, 2$. If $w_1 \neq w_2$, then the graph can be colored properly with one fewer color, since $u_1$ is not adjacent to $u_2$ (fig. 3.3). So, $w_1 = w_2 = w$.

Thus, the graph is in fact a $(\Delta - 4)$-clique joined to a graph $H_1$ on 5 vertices, where $\chi(H_1) = 3$ (fig. 3.4). Since $\chi(H_1) = 3$, $H_1$ is not bipartite, so it contains an odd cycle. It may not contain $C_3$. Otherwise, we would have a $(\Delta - 1)$-clique. Therefore, $H_1$ includes $C_5$ since $|H_1| = 5$. $H_1$ is edge-critical, thus $H_1$ is $C_5$. 

3.3 Graphs with chromatic number at least $\Delta - 2$

Theorem 3.6 Consider a graph $G$ of maximum degree $\Delta$, where $\Delta$ is sufficiently large. $\chi(G) \geq \Delta - 2$ if and only if it contains a subgraph isomorphic to

(i) a $(\Delta - 2)$-clique or

(ii) a $(\Delta - 5)$-clique joined to a $C_5$ or

(iii) a $(\Delta - 6)$-clique joined to one of the three graphs shown in figure 3.5.

Proof: Suppose that $H$ is a $(\Delta - 2)$-edge critical dense subgraph of $G$, and $C$ is a strongly optimal $(\Delta - 2)$-coloring of $H$. Assume that $H$ is of maximum degree $\Delta$. Otherwise, we can apply Theorem 3.4 on $H$ where $\chi(H) \geq \Delta(H) - 1$ and show that $H$ contains either a $(\Delta - 2)$-clique or a $(\Delta - 5)$-clique joined to a $C_5$.

Suppose that $H$ does not contain a $(\Delta - 2)$-clique. It would be sufficient to prove that $H$ contains one of the graphs shown in figure 3.5.

Lemma 3.7 There is no color class of size greater than 2.

Proof: By Lemma 3.2.d, color classes of $C$ have size at most 3 and there is at most one color class of size 3. By Lemma 3.2.c and the above fact, if there is a color class of size 3, then there is at least one color class of size 2. By applying Lemma 3.2.b, we conclude that if there is a color class of size 3, then there is exactly one color class of size 2.

Let $c_1 = \{u, v\}$ and $c_2 = \{t_1, t_2, t_3\}$ be the color classes of sizes 2 and 3, respectively. By Lemma 2.10, $t_1, t_2$ and $t_3$ are adjacent to every unicolored vertex. Also, by Lemma 2.12, let $u$ be adjacent to every unicolored vertex (fig 3.6).
Figure 3.5: The only two edge-critical graph on 7 vertices with $\chi = 4$ with no dominating vertex. The one at the bottom is $\overline{C}_7$.

Figure 3.6: If there is a color class of size 3 then, there is also a color class of size 2.
Figure 3.7: There is a unicolored vertex \( w \) which is nonadjacent to \( v \). There is no edge from \( u \) to color class \( c_2 \). So, \( u \) may get the color \( c_2 \) and \( v \) the same color as \( w \) which lessens the number of colors by one.

Vertex \( u \) is adjacent to no \( t_i \), for \( 1 \leq i \leq 3 \), otherwise there would be a \((\Delta - 2)\)-clique. If \( v \) is adjacent to all unicolored vertices, then there would be no edge from color class \( c_1 \) to color class \( c_2 \), as otherwise there would be a \((\Delta - 2)\)-clique. Thus, there is some unicolored vertex \( w \) which is not adjacent to \( v \). Figure 3.7 shows that \( H \) can be recolored properly with one fewer color. Contradiction.

Thus, we only have color classes of size 2. By Corollary 3.3 there are at most three color classes of size 2. By our assumption, \( H \) is of maximum degree \( \Delta \), and by Lemma 3.2.a there is a dominating vertex in \( H \). Thus, there are exactly three color classes of size two.

So, there are \( \Delta - 5 \) unicolored vertices, and 3 color classes \( c_i = \{u_i, v_i\}, 1 \leq i \leq 3 \), of size 2. By Lemma 2.12, assume \( u_i \), for \( 1 \leq i \leq 3 \), are adjacent to all unicolored vertices.

**Lemma 3.8** There is at least one unicolored vertex which is not joined to \( \{u_1, v_1, u_2, v_2, u_3, v_3\} \).

**Proof:** Let \( H' \) be the graph induced on \( u_1, v_1, u_2, v_2, u_3, v_3 \). Suppose that every unicolored vertex is adjacent to all of \( H' \), so \( H \) is a \((\Delta - 5)\)-clique joined to \( H' \). \( H' \) is 3-edge critical, since \( H \) is \((\Delta - 2)\)-edge critical. \( \chi(H') = 3 \), thus \( H' \) includes an odd cycle. Thus, \( H' \) contains either a 3-cycle or a 5-cycle as a subgraph. But \( |H'| = 6 \), thus \( H' \) is not critical. Contradiction.
By the above lemma, without loss of generality, let $v_1$ be a vertex that is not adjacent to a unicolored vertex $w$.

**Lemma 3.9** $v_2$ and $v_3$ cannot both be joined to all the unicolored vertices.

**Proof:** Assume that $w_1, \ldots, w_m$ are those unicolored vertices which are not adjacent to $v_1$. Suppose that $v_2$ and $v_3$ are adjacent to all unicolored vertices. Since $u_1, u_2$ and $u_3$ are adjacent to all unicolored vertices, $H$ can be considered as a $(\Delta - m - 5)$-clique joined to the graph induced on $u_1, v_1, u_2, v_2, u_3, v_3$ and $w_1, \ldots, w_m$ (fig. 3.8). $H$ is critical, so, removing $v_1$ from $H$ will lessen $\chi(H)$ by at least one. Thus, $H - \{v_1\}$ can be colored with $\Delta - 3$ colors. In this coloring, $w_1, \ldots, w_m$ are again unicolored since they are dominating vertices in $H - \{v_1\}$. Now, adding the vertex $v_1$, we can color it with the color of $w_1$, since it does not appear on any other vertex. Thus, there is a proper coloring of $H$ with $\Delta - 3$ colors. Contradiction.

**Lemma 3.10** There is at most one unicolored vertex which is not joined to $\{u_1, v_1, u_2, v_2, u_3, v_3\}$.

**Proof:** We assumed that $v_1$ is not adjacent to a unicolored vertex. By Lemma 3.9, without loss of generality, we can assume that $w_1$ and $w_2$ are not adjacent to $v_1$ and $v_2$, respectively. We will prove that $w_1 = w_2$. Let $H'$ be the induced graph on $u_1, v_1, u_2, v_2, u_3, v_3$, etc.
$w_1$ and $w_2$. $H'$ contains neither a dominating vertex nor a 5-clique. $\chi(H') \geq 5$ otherwise $\chi(H) \leq \Delta - 3$. $\chi(H') = 5$ since $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are independent sets. Thus $H'$ contains a 5-edge critical subgraph. Toft [23] proved that there are two 5-edge critical graphs on 8 vertices, both of them contain a dominating vertex. He also proved that $K_2$ joined to a $C_5$ is the only 5-edge critical graph on 7 vertices and there is no 5-edge critical graph on 6 vertices. Thus, we may have one of the following cases:

Case 1: $H'$ contains a 5-edge critical subgraph on 8 vertices. Thus, $H'$ should have a dominating vertex. Contradiction.

Case 2: $H'$ contains a 5-edge critical subgraph on 7 vertices which is a $K_2$ joined to a $C_5$. It can be seen that either of $w_1$ and $w_2$ is dominating vertex in the above 5-edge critical graph. Without loss of generality, assume that $w_1$ is dominating in the 5-edge critical graph. So, $w_1$ is dominating vertex in $H - \{v_1\}$. There is a $(\Delta - 3)$-coloring of $H - \{v_1\}$ since $H$ is $(\Delta - 2)$-edge critical. Color of $w_1$ does not appear on any other vertex. Thus we can color $v_1$ with the color of $w_1$ and get a $(\Delta - 3)$-coloring of $H$. Contradiction.

Lemmas 3.8 and 3.10 yield the following corollary.

**Corollary 3.11** There is exactly one unicolored vertex which is not joined to $\{u_1, v_1, u_2, v_2, u_3, v_3\}$.

**Lemma 3.12** $H$ is a $(\Delta - 6)$-clique joined to a 4-edge critical graph $H'$ on 7 vertices with no dominating vertex.

**Proof:** By Corollary 3.11, let $H'$ be the induced graph on $u_1, v_1, u_2, v_2, u_3, v_3$ and unicolored vertex $w$, where $w$ is not joined to $\{u_1, v_1, u_2, v_2, u_3, v_3\}$. $H'$ is joined to $H - H'$ which are all unicolored vertices. By Corollary 2.8, $H - H'$ is a clique since it contains only unicolored vertices. $\chi(H) = \Delta - 2$ and $\chi(H') = 4$, so $|H - H'| = \Delta - 6$. $H'$ is edge-critical otherwise $H$ would not be edge-critical. It can be seen that there is no dominating vertex in $H$ since $\{u_1, v_1\}, \{u_2, v_2\}$ and $\{u_3, v_3\}$ are independent sets and $w$ is not joined to them.

Thus, it is sufficient to characterize the 4-edge critical graphs on 7 vertices which contain no dominating vertex.

**Lemma 3.13** If $v \in V(H')$, then $3 \leq \deg(v) \leq 4$.

**Proof:** $\deg(v) \geq 3$, since $\chi(H') = 4$, and $H'$ is critical. There is no dominating vertex in $H$, thus $\deg(v) \leq 5$. We prove that $\deg(v) \neq 5$. Assume that $\deg(v) = 5$, and let $u$ be the vertex which is not adjacent to $v$ (fig. 3.9). Since $H'$ is critical, $H' - \{u\}$ is 3-colorable.
Figure 3.9: No vertex of degree 5

Now, $v$ is a dominating vertex in $H' - \{u\}$, thus the color of $v$ has not appeared on any other vertex. Since $u$ is not adjacent to $v$, this coloring can be extended to a proper coloring of $H$ by coloring $u$ with the color of $v$. Contradiction. 

Krivelevich [9] proved that for every $k \geq 4$ and $n > k$ a $k$-critical graph on $n$ vertices has at least $(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)})n$ edges. Thus, $H'$ has at least 12 edges. This along with Lemma 3.13 make three possible cases for $H'$:

**Case i:** Every vertex in $H'$ has degree 4. Sadeghi et. al [10] proved that if a graph $G$ with $\chi(G) = r$ is $r$-regular, then $|V(G)| \geq 2r - 1$, and equality holds only for $r = 4, 6$ and 8. They showed that for $r = 4$, $\overline{C_7}$ is the only solution. It can be seen that $\overline{C_7}$ is not edge-critical, since it contains the first of the 2 graphs in figure 3.5.

**Case ii:** There are five vertices of degree 4 and two of degree 3. Assume that $H'$ is a 4-edge critical graph of the above degree sequence. Add a dominating vertex to $H'$. The resulting 5-edge critical graph has degree sequence $6, 5, 5, 5, 4, 4$, but none of the two 5-edge critical graphs on 8 vertices (listed in the proof of 3.10) has such degree sequence[23]. Thus there is no 4-edge critical graph in this case.

**Case iii:** There are three vertices of degree 4 and four of degree 3. There are two graphs on 7 vertices with the above degree sequence, and both of them are 4-edge critical as indicated in Figure 3.5.

Thus, $H'$ is one of the graphs shown in figure 3.5.
Chapter 4

An approach to a general characterization

In the last chapter, we characterized graphs with chromatic number greater than or equal to \( \Delta - 1 \) and \( \Delta - 2 \) where \( \Delta \) is the maximum degree of the graph and is sufficiently large. As we saw, such graphs contain a subgraph \( H \) which is a clique of size close to \( \Delta \) joined to an edge-critical graph \( H' \) with relatively few vertices.

In this chapter, we show that for sufficiently small \( k \), a \( (\Delta - k) \)-chromatic graph contains a similar structure, that is a clique of size close to \( \Delta \) joined to an edge-critical graph \( H' \). We begin by proving an upperbound on the size of \( H' \) which is linear in terms of \( k \). Using this upperbound, we will characterize graphs with chromatic number greater than or equal to \( \Delta - 3 \) and \( \Delta - 4 \). We close this chapter by giving a lowerbound for the number of non-isomorphic \( (\Delta - 5) \)-edge critical graphs.

4.1 A bound on the size of edge-critical dense subgraphs

In this section, we will prove a theorem on the structure of \( (\Delta - k) \)-edge critical dense subgraphs of a graph \( G \) of maximum degree \( \Delta \).

We use the following Lemma from [11] in our proof.

**Lemma 4.1** For any graph \( G \) with \( \chi(G) > \frac{|G|}{2} \), there is some \( X \subseteq V(G) \) such that \( G - X \) is the join of graphs \( H_1, H_2, \ldots, H_t, t > |X| \), and where

(a) \( \chi(H_i) \leq \lceil \frac{|H_i|}{2} \rceil \), \( 1 \leq i \leq t \);

(b) \( \sum_{i=1}^{t} \chi(H_i) = \chi(G) \).
From the above lemma we conclude that $\chi(G - X) = \chi(G)$. Thus, if $G$ is vertex-critical, then $X = \emptyset$.

**Corollary 4.2** For any vertex-critical graph $G$ with $\chi(G) > \frac{|G|}{2}$, $G$ is the join of graphs $H_1, H_2, \ldots, H_t$, where

(a) $\chi(H_i) \leq \lceil \frac{|H_i|}{2} \rceil$, $1 \leq i \leq t$;

(b) $\sum_{i=1}^{t} \chi(H_i) = \chi(G)$.

Now, we prove the following theorem, which bounds the size of $H'$.

**Theorem 4.3** Consider a graph $G$ of maximum degree $\Delta$ with $\chi(G) \geq \Delta - k$, where $k^2 + k < \Delta - k$, and $\Delta$ is sufficiently large. If $H$ is a $(\Delta - k)$-edge critical dense subgraph of $G$, then $H$ is a clique $Q$ joined to a graph $H'$, where $|H'| \leq \frac{5}{2}(k + 1)$ and $\chi(H') \leq \frac{3}{2}(k + 1)$.

**Remark 4.4** This upper bound on $|H'|$ is the best possible. In the proof, it will be seen that equality holds if and only if $H'$ is the join of $\frac{k+1}{2}$ disjoint 5-cycles. We have $|H'| = \frac{5}{2}(k + 1)$ and $\chi(H') = \frac{3}{2}(k + 1)$ since for each 5-cycle, 3 colors should be devoted. Let $H$ be a $(\Delta + 1 - \frac{5}{2}(k+1))$-clique joined to $H'$. $H$ has maximum degree $\Delta$, $\chi(H) = \Delta - k$ and $H$ is a clique joined to an edge-critical graph of size $\frac{5}{2}(k + 1)$.

**Proof:** It can be seen that $\chi(H) > \lceil \frac{|H|}{2} \rceil$, and $H$ is vertex-critical. Assume any $(\Delta - k)$-coloring of $H$. Applying Corollary 4.2, $H$ is the join of graphs $H_1, H_2, \ldots, H_t$, where $t > 0$. If $|H_i| = 1$ then it would be a dominating vertex in $H$.

Note that since $H$ is edge-critical, there can be no two vertices with the same neighborhood. Otherwise, removing either of them would not affect the chromatic number. This implies that no $H_i$ could be an independent set. Also, in each $H_i$ there would be no two vertices with the same neighborhood. Otherwise, they would have the same neighborhood in $H$.

Applying the above facts, $|H_i| \neq 2$. Also if $|H_i| = 3$, then $H_i$ could not be a triangle. Thus, there is at least one edge missing, therefore there are two vertices with the same neighborhood. Thus, $|H_i| \neq 3$. By a similar argument, $|H_i| \neq 4$. Otherwise there would be two vertices with the same neighborhood in $H$. Thus, if $|H_i| \neq 1$, then $|H_i| \geq 5$, and we have $\chi(H_i) \leq \lceil \frac{|H_i|}{2} \rceil \leq \frac{3}{5}|H_i|$. Thus,

$$\chi(H) = \sum_{i=1}^{t} \chi(H_i) \leq |\{H_i : |H_i| = 1\}| + \sum_{|H_i| \geq 5} \frac{3}{5}|H_i|.$$  

Note that each $H_i$ of size 1 corresponds to a dominating vertex in $H$. Also, the dominating vertices in $H$ form a clique which is joined to the rest of $H$, say $H'$. So, $|H'| = \sum_{|H_i| \geq 5} \frac{3}{5}|H_i|$ and $\chi(H) = \text{number of dominating vertex in } H + \chi(H')$. Thus,
\[ \chi(H') \leq \frac{3}{5} |H'|. \]

**Lemma 4.5** \(|H'| - \chi(H') \leq k + 1.\)

**Proof:** Assume that \(|H'| - \chi(H') > k + 1.\) By Theorem 2.6, there is a dominating vertex \(v\) in \(H.\) Since \(\chi(H) = \Delta - k,\) by Lemma 2.7, \(\deg(v) > (\Delta - k - 1) + k + 1 = \Delta.\) Contradiction.

Applying the above lemma to \(\chi(H') \leq \frac{3}{5} |H'|,\) we get \(|H'| \leq \frac{5}{2}(k + 1)\) and \(\chi(H') \leq \frac{3}{2}(k + 1).\)

### 4.2 Graphs with chromatic number at least \(\Delta - 3\)

**Theorem 4.6** Consider a graph \(G\) of maximum degree \(\Delta,\) where \(\Delta\) is sufficiently large. \(\chi(G) \geq \Delta - 3\) if and only if \(G\) contains a subgraph isomorphic to one of the following 27 graphs:

- i) a \((\Delta - 3)\)-clique;
- ii) a \((\Delta - 6)\)-clique joined to a \(C_5;\)
- iii) a \((\Delta - 7)\)-clique joined to one of the graphs shown in figure 3.5;
- iv) a \((\Delta - 6)\)-clique joined to a \(C_7;\)
- v) a \((\Delta - 7)\)-clique joined to one of the graphs shown in figure 4.1;
- vi) a \((\Delta - 8)\)-clique joined to one of the graphs shown in figure 4.2;
- vii) a \((\Delta - 9)\)-clique joined to the graphs shown in figure 4.3.

**Proof:** Suppose that \(\chi(G) \geq \Delta - 3,\) and \(H\) is a \((\Delta - 3)\)-edge critical dense subgraph of \(G\) and \(C\) is a strongly optimal \((\Delta - 3)\)-coloring of \(H.\) If \(H\) has maximum degree less than \(\Delta,\) then applying Theorem 3.6, \(H\) contains a subgraph isomorphic to one of the graphs stated in i), ii) or iii).

So assume that \(H\) is of maximum degree \(\Delta\) and does not contain a \((\Delta - 3)\)-clique. We will show that \(H\) contains a subgraph isomorphic to one of the graphs stated in iv), v), vi) and vii).

By Lemma 4.3, \(H\) is a clique \(Q\) joined to a graph \(H'\) where \(|H'| \leq 10,\) and \(H'\) has no dominating vertex. Since \(H'\) is critical, \(\chi(H') \geq 3.\) Since \(|H| \leq \Delta + 1,\) and \(H\) has a vertex of degree \(\Delta,\) \(|H| = \Delta + 1.\) Also, \(\chi(H) = \Delta - 3,\) so:
\[ |Q| + |H'| = \Delta + 1 \]

\[ \chi(Q) + \chi(H') = \Delta - 3 \]

Subtracting the above equations yields \( \chi(H') = |H'| - 4 \), since \( \chi(Q) = |Q| \). Thus, we may have one of the following cases:

Case 1: \( \chi(H') = 3 \). This implies that \( H' \) is a 3-edge critical graph on \( \chi(H') + 4 = 7 \) vertices. Therefore, \( H' \) is a 7-cycle (fig. 3.1);

Case 2: \( \chi(H') = 4 \). This implies that \( |H'| = \chi(H') + 4 = 8 \). Toft proved that there are 5 non-isomorphic 4-edge critical graphs on 8 vertices [23]. One of the five contains a dominating vertex. The others are shown in fig. 3.2;

Case 3: \( \chi(H') = 5 \). This implies that \( |H'| = \chi(H') + 4 = 9 \). Jensen and Royle showed that there are 21 non-isomorphic 5-edge critical graphs on 9 vertices [3]. Four of them contain a dominating vertex. The others are shown in fig. 3.3;

Case 4: \( \chi(H') = 6 \). This implies that \( |H'| = \chi(H') + 4 = 10 \). By Remark 3.5, the graph shown in fig. 3.4 is the only 6-edge critical graph on 10 vertices (fig. 3.4).

\[ \square \]

4.3 Graphs with chromatic number at least \( \Delta - 4 \)

Theorem 4.7 Consider a graph \( G \) of maximum degree \( \Delta \geq \Delta_0 \). \( \chi(G) \geq \Delta - 4 \) if and only if \( G \) contains a subgraph isomorphic to one of 421 non-isomorphic \( (\Delta - 4) \)-edge critical graphs.
Figure 4.2: Four 4-edge critical graph on 8 vertices with no dominating vertex.
Figure 4.3: 5-edge critical graphs on 9 vertices with no dominating vertex. (continued)
Figure 4.3: 5-edge critical graphs on 9 vertices with no dominating vertex. (continued)
Figure 4.3: 5-edge critical graphs on 9 vertices with no dominating vertex.
The proof of Theorem 4.7 relies heavily on a list computed by Royle [21] of all small edge-critical graphs.

**Proof:** Suppose that $H$ is a $(\Delta - 4)$-edge critical dense subgraph of $G$ and $C$ is a strongly optimal $(\Delta - 4)$-coloring of $H$. If $H$ has maximum degree less than $\Delta$, then applying Theorem 4.6, $H$ contains a subgraph isomorphic to one of the 27 non-isomorphic $(\Delta - 4)$-edge critical graphs of maximum degree less than $\Delta$. If $H$ is of maximum degree $\Delta$ then by Theorem 4.3, $H$ is a clique joined to a subgraph of size at most 12. Since $H$ is of maximum degree $\Delta$, it may be one of the following:

1) a $(\Delta - 8)$-clique joined to one of 4-edge critical graphs on 9 vertices with no dominating vertex. There are 21 non-isomorphic 4-edge critical graphs on 9 vertices[21] and none of them has a dominating vertex;

2) a $(\Delta - 9)$-clique joined to one of 5-edge critical graphs on 10 vertices with no dominating vertex. There are 162 non-isomorphic 5-edge critical graphs on 10 vertices[21] and 141 of them have no dominating vertices;

3) a $(\Delta - 10)$-clique joined to one of 6-edge critical graphs on 11 vertices with no dominating vertex. There are 393 non-isomorphic 6-edge critical graphs on 11 vertices[21] and 230 of them have no dominating vertices;

4) a $(\Delta - 11)$-clique joined to one of 7-edge critical graphs on 12 vertices with no dominating vertex. There are 395 non-isomorphic 7-edge critical graphs on 12 vertices[21] and only 2 of them have no dominating vertices;

Considering all the above cases, when $\Delta$ is sufficiently large, there are 421 non-isomorphic $(\Delta - 4)$-edge critical graphs.
4.4 A lowerbound for the number of $(\Delta - 5)$-edge critical graphs

In the previous sections, we characterized $(\Delta - 2)$, $(\Delta - 3)$, $(\Delta - 4)$ and $(\Delta - 5)$-colorable graphs, where $\Delta$ is the maximum degree of the graph and is sufficiently large. In order to characterize $(\Delta - 6)$-colorable graphs, we have to figure out all $(\Delta - 5)$-edge critical graphs. The list computed by Royle [21] of all small edge-critical graphs shows that there are 17036 non-isomorphic 6-edge critical graphs on 12 vertices, each of which can be joined to a $(\Delta - 11)$-clique to form a $(\Delta - 5)$-edge critical graph. So, we have the following table:

<table>
<thead>
<tr>
<th>Chromatic number</th>
<th>number of edge-critical graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>1</td>
</tr>
<tr>
<td>$\Delta - 1$</td>
<td>2</td>
</tr>
<tr>
<td>$\Delta - 2$</td>
<td>4</td>
</tr>
<tr>
<td>$\Delta - 3$</td>
<td>27</td>
</tr>
<tr>
<td>$\Delta - 4$</td>
<td>421</td>
</tr>
<tr>
<td>$\Delta - 5$</td>
<td>$&gt;17000$</td>
</tr>
</tbody>
</table>

Table 4.1: Number of $(\Delta - k)$-edge critical graphs for $k = 1, 2, 3$ and 4.
Chapter 5

Further results

In general, determining whether a graph $G$ of maximum degree $\Delta$ is $c$-colorable is NP-hard though for some specific pairs of $c$ and $\Delta$ it is easy. We know that every graph $G$ is $(\Delta + 1)$-colorable. It is also easy to determine whether $G$ is $\Delta$-colorable. Using Brooks' Theorem, we only need to search for a component which is a $(\Delta + 1)$-clique and, if $\Delta = 2$, we will also check for components which are odd cycles.

In [18], Molloy and Reed proved the following theorem:

**Theorem 5.1** For any constant $\Delta \geq \Delta_0$ and $k$ where $k^2 + k \leq \Delta$, $(\Delta - k)$-colorability of graphs with maximum degree $\Delta$ can be determined in linear time.

**Proof:** Consider a $d$-dense decomposition of $G$, where $d = 1000\sqrt{\Delta}$. We can find such a decomposition in linear time using the procedure described in 2.2. By Theorem 2.3, it would be sufficient to prove that $(\Delta - k)$-colorability of a $d$-dense set $D$ can be determined in constant time. By Definition 2.1, $|D| \leq \Delta + 1 + 4d$. Since both $\Delta$ and $k$ are constants, $(\Delta - k)$-colorability of the graph induced on $D$ can be checked in constant time. Thus, $(\Delta - k)$-colorability of $G$ can be determined in linear time.

In this chapter, we will apply the structural decomposition stated in 2.2 and Theorem 4.3 to prove that for sufficiently large $\Delta$ and sufficiently small constant $k$, $(\Delta - k)$-colorability of $G$ can be determined in linear time, even if $\Delta \neq O(1)$.

We close this chapter by showing that our results from Chapter 3 and 4 will solve the asymptotic version of an open problem from Jensen and Toft [4].

### 5.1 $(\Delta - k)$-colorability

In this section, we will show that for $\Delta \geq \Delta_0$ and sufficiently small constant $k$, $(\Delta - k)$-colorability of $G$ can be determined in linear time.
**Theorem 5.2** Consider a $d$-dense set $D$, where $d \leq \frac{\Delta}{100}$ and $\Delta$ is sufficiently large. If $\chi(D) \geq \Delta - k$, where $k^2 + k \leq \Delta - k$, then $D$ contains a clique of size greater than $\Delta - \frac{5}{2}(k + 1)$.

**Proof:** By Theorem 4.3, $D$ contains a $(\Delta - k)$-edge critical dense subgraph $H$ which is a clique $Q$ joined to a graph $H'$ where $\chi(H') \leq \frac{3}{2}(k + 1)$. Since $\chi(H) = \Delta - k$, $|Q| = \chi(Q) > \Delta - \frac{5}{2}(k + 1)$. $\blacksquare$

**Theorem 5.3** Consider a $d$-dense set $D$, where $d \leq \frac{\Delta}{100}$ and $\Delta$ is sufficiently large. If $\chi(D) \geq \Delta - k$, where $k^2 + k \leq \Delta - k$, then $|D| \leq \Delta + \frac{5}{2}(k + 2)$.

**Proof:** By Theorem 2.6, $D$ contains a $(\Delta - k)$-edge critical dense subgraph $H$, where $|H| \leq \Delta + 1$. Every vertex in $H$ has degree at least $\Delta - (k + 1)$ since $H$ is edge-critical. Thus $|E(H, D - H)| \leq (k + 1)(\Delta + 1)$. Consider any $v \in D - H$. By Definition 2.1, $v$ has at least $\frac{3\Delta}{4}$ neighbors in $D$ and $|D| \leq \Delta + 4d + 1$. So, $v$ is adjacent to at least $\frac{2\Delta}{3}$ vertices of $H$. Thus $|D - H| \leq |E(H, D - H)|/\frac{2\Delta}{3} \leq \frac{5}{2}(k + 2)$. $\blacksquare$

**Theorem 5.4** For sufficiently large $\Delta$ and any $k$ such that $k^2 + k \leq \Delta - k$, the number of $(\Delta - k)$-edge critical graphs of maximum degree less than or equal to $\Delta$ is less than $2^{O(k^2)}$.

**Proof:** By Theorem 4.3, if $H$ is a $(\Delta - k)$-edge critical graph of maximum degree less than or equal to $\Delta$ then $H$ is a clique $Q$ joined to a graph $H'$, where $|H'| \leq \frac{5}{2}(k + 1)$. Now, $Q$ and consequently $H$ can be specified by $H'$. Thus, the number of $(\Delta - k)$-edge critical graphs is less than $2^{O(k^2)}$. $\blacksquare$

**Theorem 5.5** For any constant $k$, there is a linear time algorithm which will test $(\Delta(G) - k)$-colorability of any input graph $G$ with $\Delta(G) \geq \max(\Delta_0, k^2 + 2k)$.

**Proof:** Consider a $d$-dense decomposition of $G$, where $d = 1000\sqrt{\Delta}$. We can find such a decomposition in linear time using the procedure described in 2.2. Since $k^2 + 6 \leq \Delta - k$, by Theorem 2.3, it would be sufficient to prove that $(\Delta - k)$-colorability of a $d$-dense set $D$ can be determined in $O(\Delta^2)$ since the number of edges in $D$ is of order $\Theta(\Delta^2)$.

We will show that given a $(\Delta - k)$-edge critical dense graph $H$, it can be determined in $O(\Delta^2)$ whether $D$ contains a subgraph isomorphic to $H$. The theorem will then be proved since by Theorem 5.4, the number of $(\Delta - k)$-edge critical dense graphs is of order $O(1)$. 

Lemma 5.6 If $D$ contains a clique of size at least $\Delta - \frac{5}{2}(k + 1)$ then a clique of size at least $\Delta - 25(k + 1)^2$ can be found in $O(\Delta^2)$ time.

Proof: Suppose that $Q$ is a clique of size $\Delta - \frac{5}{2}(k + 1)$ in $D$. Note that every vertex in $Q$ has degree at least $\Delta - \frac{5}{2}(k + 1) - 1$ in $D$. We repeatedly delete all the vertices in $D$ with degree less than $\Delta - \frac{5}{2}(k + 1) - 1$ to form $D'$. Thus $D'$ still contains all the vertices of $Q$. Let $v$ be a vertex in $D'$.

If $v \in Q$ then $v$ is joined to $Q - \{v\}$ and fewer than $4k + 10$ other vertices since $|D'| \leq |D|$ and by Theorem 5.3, $|D| \leq \Delta + \frac{3}{2}(k + 2)$. Each of these $4k + 10$ vertices may be non-adjacent to at most $\frac{5}{2}(k + 1)$ vertices in $N_{D'}(v)$. We delete the endpoints of all missing edges from $N_{D'}(v)$. Thus, at most $(4k + 10) \times \frac{5}{2}(k + 1) \leq 25(k + 1)^2 - \frac{5}{2}(k + 1) - 1$ vertices will be deleted from $N_{D'}(v)$. Recalling that $v$ had at least $\Delta - \frac{5}{2}(k + 1) - 1$ neighbors in $D'$, the remaining neighborhood of $v$ is a clique of size at least $\Delta - 25(k + 1)^2$.

If $v \notin Q$ then this procedure will fail to find a clique of the desired size. But, there are at most $4k + 10$ vertices in $D'$ which are not contained in $Q$. Thus, we will repeat this procedure on at most $4k + 11$ vertices of $D'$ before finding a clique.

Since $k$ is a constant, we could find a clique of size at least $\Delta - 25(k + 1)^2$ in $O(\Delta^2)$ time.

Having proven Lemma 5.6, we return to the proof of Theorem 5.5 by presenting an $O(\Delta^2)$ time algorithm which determines whether $D$ contains a subgraph isomorphic to $H$.

By Theorem 4.3, our edge-critical graph $H$ is a clique $Q$ joined to a graph $H'$ where $|H'| \leq \frac{5}{2}(k+1)$. If $D$ contains a subgraph isomorphic to $H$ then $D$ contains a clique of size $\Delta - \frac{5}{2}(k + 1)$. By the above lemma, we can find a clique $C$ of size at least $\Delta - 25(k + 1)^2$ in $D$ in $O(\Delta^2)$ time. By Theorem 5.3, $|D - C|$ is of order $O(k^2) = O(1)$.

Step 1. Search for a clique $C$ of size $\Delta - 25(k + 1)^2$ in $D$. If no clique is found then HALT and output "$D$ does not contain a subgraph isomorphic to $H$".

Assume that $D$ contains a copy of $H$, and let $H_0$ be one such copy. Now, let $B = H_0 - C$. $B \subset D - C$. Since the number of subgraphs of $D - C$ is of order $2^{O((\frac{k^2}{2}))} = O(1)$, it would be sufficient to show that for each subgraph $B$ of $D - C$, it can be determined in constant time whether there is at least one $C' \subseteq C$ such that $H$ is isomorphic to $C' \cup B$.

Step 2. For every $B \subset D - C$ do as follows: Let $T$ be the graph induced on $B \cup C$, and $T'$ be the graph induced on the endpoints of missing edges in $T$. If $T'$ contains a subgraph isomorphic to $H'$ and $|T - T'| \geq |Q|$ then HALT and output "$D$ contains a subgraph isomorphic to $H$".

Step 3. HALT and output "$D$ does not contain a subgraph isomorphic to $H$".
Since $D$ contains a clique of size $\Delta - 25(k+1)^2$, $|T'| < 30k^2 \in O(1)$. Thus it can be determined in $O(1)$ time whether $T'$ contains $H'$ as a subgraph. Since $T'$ is joined to $T - T'$, if $T'$ contains a subgraph isomorphic to $H'$ and $|T - T'| \geq |Q|$ then $T$ contains $H$ as a subgraph. Furthermore, if $T \supset H$ then $T' \supset H'$, so if no $H$ is found in Step 2, then $D$ does not contain a subgraph isomorphic to $H$. 

**5.2 Graphs without a complete subgraph**

The relation between the maximum degree, chromatic number and clique number of a graph, has raised many interesting problems. Finding the best possible lower bound for the maximum degree $\Delta$ of a graph $G$ in terms of $\chi(G)$ when $G$ does not contain a $(\chi(G) - k)$-clique as a subgraph, is one of them. The cases $k = 0$ and $k = 1$ are discussed individually in “Graph Coloring Problems” [4] as follows:

**Problem 5.7** Does there exist a $\Delta$-chromatic graph without a $\Delta$-clique as a subgraph and of maximum degree $\Delta$ for any value of $\Delta \geq 9$? (This is problem 4.8 of [4]).

and

**Problem 5.8** Find the best possible lower bound in terms of $k$ for the maximum degree $\Delta(G)$ of a $k$-chromatic graph $G$ not containing a $(k-1)$-clique. (This is problem 4.7 of [4]).

Reed proved that for sufficiently large $\Delta$, every graph with $\chi \geq \Delta$ contains a $\Delta$-clique, and thus $\Delta \geq \chi + 1$ is the best possible. Thus the answer to Problem 5.6 is 'No' for sufficiently large $\Delta$.

For $k = 1$, Brooks’ Theorem implies that $\Delta \geq \chi$, which is not the best possible lower bound. We show that Theorems 3.1, 3.4 and 3.6 yield that $\Delta \geq \chi + 3$ is the best possible lower bound for $\Delta$ in terms of $\chi$ when the graph does not contain a $(\chi - 1)$-clique. Thus the solution for the open problem 4.7 in [4] for $\Delta \geq \Delta_0$ is $\Delta \geq \chi + 3$.

**Theorem 5.9** Consider a graph $G$ of maximum degree $\Delta \geq \Delta_0$. If $G$ does not contain a $(\chi - 1)$-clique, then $\Delta \geq \chi + 3$ and this lower bound is best possible.

**Proof:** We prove $\chi \leq \Delta - 3$. For $\chi \geq \Delta$, Theorem 3.1 implies that $G$ contains a clique of size $\Delta = \chi$.

For $\chi \geq \Delta - 1$, note that a $(\Delta - 4)$-clique joined to a $C_5$ contains a $(\Delta - 2)$-clique. Thus, Theorem 3.4 implies that $G$ contains a clique of size $\Delta - 2 = \chi - 1$. 

All the graphs shown in Fig. 3.5 contain a triangle. So, if $\chi \geq \Delta - 2$, Theorem 3.6 implies that $G$ contains a clique of size $\Delta - 3 = \chi - 1$.

To show that this is the best possible, let $G$ be a $(\Delta - 9)$-clique joined to the join of two $C_5$'s (fig. 5.1). It can be seen that $\chi(G) = \Delta - 3$. Also, there is no 5-clique in the join of two $C_5$'s, and so $\omega(G) = \Delta - 5 = \chi - 2$.

I propose the following more general problem:

**Open Problem 5.10** Consider a graph $G$ of maximum degree $\Delta$. Find the best possible upper bound in terms of $\Delta$ for the chromatic number $\chi(G)$ of a graph $G$ not containing a $(\chi - k)$-clique, where $k$ is a sufficiently small constant.

The structure of the graph in Figure 5.1 can be generalized to show that the upper-bound in Problem 5.10 must be at least $\Delta - (2k + 1)$.

**Theorem 5.11** For all $k$ and all $\Delta$ sufficiently large, there is a graph $G$ with $\chi = \Delta - (2k + 1)$ where $G$ does not contain a $(\chi - k)$-clique.

**Proof:** Suppose that $C_{k+1,5}$ is the join of $k + 1$ disjoint $C_5$'s. Let $G$ be a $(\Delta + 1 - 5(k + 1))$-clique joined to $C_{k+1,5}$. $\chi(G) = \Delta - 5(k + 1) + 1 + 3(k + 1) = \Delta - (2k + 1)$ and $\omega(G) = \Delta - 5(k + 1) + 1 + 2(k + 1) \leq \Delta - 3k - 2 < \chi(G) - k$.

Using the list of small edge-critical graphs computed by Royle [21], it can be seen that for $k = 2$, the best possible upper bound sought for in Problem 5.10 is indeed $\chi \leq \Delta - (2k + 1)$. However, for larger values of $k$, the bound must be even higher.
Theorem 5.12 For any $k$ sufficiently large and $\Delta$ sufficiently large, there exists a graph $G$ with $\chi \geq \Delta - (k + o(k))$ where $G$ does not contain a $(\chi - k)$-clique.

The proof uses the following exercise from [16]:

Lemma 5.13 If $n$ is sufficiently large then there exists a graph on $n$ vertices with chromatic number at least $\frac{n}{2}$ and with clique number at most $n^{\frac{3}{7}}$.

Proof: By the above lemma, there is a graph $G_c$ on $n$ vertices, where $\frac{n}{2} - n^{\frac{3}{7}} > k$, such that $G_c$ has chromatic number at least $\frac{n}{2}$ and clique number less than $n^{\frac{3}{7}}$. Let $G$ be the join of a $t$-clique to $G_c$, where $t$ is sufficiently larger than $|G_c|$. $\chi(G) \geq t + \frac{n}{2}$ and $\omega(G) < t + n^{\frac{3}{7}}$. So, $G$ does not contain a $(\chi(G) - k)$-clique. But $\chi \geq \Delta - (k + o(k))$ since $\Delta = t + n - 1$. $\blacksquare$
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