Three Essays on the Basis Risk of Fixed Income Securities

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Management
University of Toronto

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Three essays on the basis risk of fixed income securities

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ABSTRACT

The three essays can be regarded as studies on the basis risk of fixed income securities. They investigate the spreads among different bonds. The first essay, Market Risk and Credit Risk in a General Equilibrium Model, assumes perfect liquidity and focuses on the credit spread. By incorporating credit risk into the standard asset pricing models, it provides one of the first studies on how credit spread relates to market risk, including equity risk, interest risk, and inflation risk. The second essay, Illiquidity and Expected Return of Treasury Securities, focuses on Treasury bonds with zero default risk. The yield spreads among the bonds are solely due to liquidity difference. We derive, quantitatively, how this spread is related to the bid-ask spread, brokerage fee, bond maturity, and investors' expected holding period. It is one of the first theoretical models on the liquidity of treasury securities. The third essay, An Indirect Estimation of the Transaction Costs of Corporate Bonds, is an empirical estimation of the transaction costs of corporate bonds. It is observed that bonds with less liquidity tend to be the ones with lower credit rating quality. Liquidity risk and credit risk are thus intertwined. We are able to separate their effects and obtain estimates for liquidity spreads and credit spreads. In summary, the first essay studies credit risk; the second studies liquidity risk, and the third, as an empirical study, investigates both issues. They jointly contribute to the understanding of the basis risk of fixed income securities.
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I. Introduction to the thesis

This Ph.D. dissertation contains three essays,

1. Market Risk and Credit Risk in a General Equilibrium Model
2. Illiquidity and Expected Return of Treasury Securities
3. An Indirect Estimation of the Transaction Costs of Corporate Bonds

The three essays can be read independently, and hence, as is customary, each essay contains an introductory section motivating the paper and putting it into the context of the literature. In this section, we will summarize each essay respectively, and then discuss their internal relationship.

A. Essay 1: Market Risk and Credit Risk in a General Equilibrium Model

In the current credit risk literature, there are mainly two types of models. The first type, structural form models, usually starts from exogenous assumptions on the process of the firm asset value. For example, Merton (1974) assumes that the asset value $F_t$ follows a geometric Brownian motion in the risk-neutral world:

$$\frac{dF_t}{F_t} = r dt + \sigma dW_t, \quad (1)$$

where $r$ is the interest rate, $\sigma$ the volatility, and $W_t$ a standard Brownian motion. The corporate debt value is then calculated as a contingent claim on the value of the firm. The dynamic of the credit spread, i.e., the yield spread of corporate debt over that of the Treasury bonds with the same maturity, can be studied.

In order to know how credit spread is related to other macro variables, such as interest rate or inflation, we have to exogenously assume the correlation between firm asset value
and these macro factors, which is not clear in the current literature. This reveals a very important limitation of the structural form models: they are partial equilibrium models in nature.

The second type, reduced form models, assumes that the default events are governed by a Poisson process and the focus is to model the default intensity rate of the jump process. As we have to exogenously specify factors driving the default intensity rate, they are again partial equilibrium models.

On the other hand, in the asset pricing literature, equity is usually unlevered and credit risk is not considered in the equilibrium setting. There is thus an important gap between the two schools of literature.

Essay 1 intends to fill in this gap by integrating credit risk into the asset pricing literature. We develop a monetary general equilibrium model where all the financial prices, including that of credit risky bonds, are endogenously determined.

Such an approach helps to shed new light on three important issues. First, we propose a concept of dynamic capital structure irrelevance. That is, Modigliani and Miller's (1958) capital structure irrelevance holds, but, given the current outstanding amount of debt, the capital structure changes endogenously because the production affects the value of equity and debt disproportionately. This will produce a counter-cyclical pattern for both the leverage ratio and the equity premium. Because the credit spread is also negatively driven by production, this induces a positive correlation between the equity premium and credit spread, consistent with empirical evidence. By adding a capital structure, we not only provide a new way to capture the counter-cyclical equity premium, but also are able to study the relationship between equity risk and credit risk.

We show that the dynamic capital structure also causes trouble to empirical studies. The stock beta is very volatile and it captures the counter-cyclical pattern of the capital structure that is not going to be explained by the market portfolio. This implies that the usual two-pass approach will yield a biased estimate of systematic risk. On the
other hand, any variable that is counter-cyclical has hope to pick up the variation of systematic risk due to capital structure. These variables include equity value (i.e., size), leverage ratio, dividend yield, credit spread, and possibly, book-to-market ratio.

Second, we find that credit spreads can react differently to interest rate shocks, depending on investors' risk aversion and the maturity of debt. Under some circumstances, credit spreads for different maturities can change in opposite directions. However, for relatively long maturity, the credit spread and the interest rate are always negatively correlated. This has important implication for the risk management of corporate bond portfolios or credit derivatives.

Third, we provide one of the first models to ever study the role of inflation on credit risk. We find that, while a higher money supply reduces credit risk, the relationship between the expected inflation and the credit spread is complicated. When the expected inflation changes due to the monetary shocks, it will move with the credit spread in opposite directions. However, when the expected inflation changes due to a shock from the production growth rate, it can move with the credit spread in the same direction. This is because a higher production growth can reduce both expected inflation and default risk.

B. Essay 2: Illiquidity and Expected Return of Treasury Securities

In Assay 1, we assume that the whole yield spread of corporate bond over Treasury bonds is due to credit risk. This implies that the market is perfectly liquid. Assay 2 studies the illiquidity of fixed income securities. In particular, we focus on the Treasury bonds, where the credit risk is zero. The yield spread among these Treasury securities is solely due to the difference in liquidity.

In their groundbreaking equilibrium model, Amihud and Mendelson (1986) show that the return of common stocks is a decreasing and concave function of the bid-ask
spreads, and the portfolios that are expected to be held for a longer time will have a higher bid-ask spread. Ever since then, the trade-off between liquidity and expected return for equity has been studied extensively in the past decade. In addition, similar relationship for Treasury bonds is also widely investigated empirically.

In sharp contrast, similar theoretical analysis for fixed income securities has rarely, if ever, been done. Because fixed income securities are very different from equities, it is important to develop a theoretical model to investigate (1) whether similar results in Amihud and Mendelson (1986) hold for Treasury bonds; and (2) what are the features specific to fixed income securities.

The study on the illiquidity of fixed income securities is not only interesting for intellectual reasons. It is common for many central banks to routinely update their bond holding to the most liquid benchmark. A thorough understanding of the liquidity spreads for debts with different maturities and ages will help the central banks to control debt service costs. In addition, the liquidity spreads can be explored as arbitrage opportunities. According to the Economist, one of the major strategies of Long Term Capital Management was to short on-the-run US Treasury bonds and buy off-the-run Treasury bonds and bet that the return would converge in the short run.

In Essay 2 we develop a simple model where the investors face the trade-off among price, bid-ask spread, brokerage fee, maturity, and expected investment horizon. We are able to show that (1) The (ask and bid) price is a decreasing and convex function of the bid-ask spread. Alternatively, the expected return is an increasing and concave function of the spread; (2) An investor with a longer expected holding period will require a lower return from a given discount bond than an investor with a shorter expected holding period does; (3) The clientele effect: an investor with longer expected holding period is more willing to hold bonds with larger bid-ask spreads, holding everything else constant; and (4) The yield spread of bonds with different bid-ask spreads is a decreasing and convex function of the time to maturity. In particular, the yield spread
is a linear function of the reciprocal of the time to maturity. Such a pattern is observed by Amihud and Mendelson (1991). They find that the correlation between the yield spread and the reciprocal of the time to maturity is as high as 0.97. But no theoretical reason is provided in their paper. In our model we prove the proposition with an elegant expression.

C. Essay 3: An Indirect Estimation of the Transaction Costs of Corporate Bonds

While an accurate estimate of transaction costs for corporate bonds is crucial for studying investment strategies and understanding market liquidity, such an estimate is lacking in the current empirical studies. This largely owes to the lack of transparency in the corporate bond market. Most of the bonds are traded infrequently over the counter, and only bid prices, instead of bid/ask prices, are quoted. In addition, during the life of a particular bond, its trading will decline as it is increasingly held by institutional investors for long term investments. How can we obtain, in spite of all these obstacles, an intuitive and comprehensive measure of the transaction costs? What is the expected transaction cost for a typical investor trading corporate bonds? What are the determinants of the costs? Is illiquidity priced into the bond yields? These are the major issues we intend to address in Essay 3.

The paper contributes to the literature in three important aspects. First, methodologically, to our knowledge, we provide the first LDV model to estimate the transaction costs for corporate bonds. This method only needs the time series data of corporate bonds that update the prices whenever a trade occurs. As such, it can include bonds with very infrequent trading, which are usually discarded in previous studies. As liquidity is the major subject of research, the method presents an important improvement by including illiquid bonds. Our median estimate of the round-trip transaction costs is 0.59%, much higher than those in the previous studies. However, when only the more
liquid half of the data are included, the median is 0.23%, very close to 0.26% obtained by Schultz (2001).

Second, we enrich the understanding of the determinants for transaction costs and bond liquidity. The estimated transaction costs are found to be positively related to volatility and duration, and negatively related to trading frequency. In addition, bonds with a higher rating react more to interest rate shocks and less to the stock market shocks. The opposite is true for low quality bonds. Moreover, contrary to Hong and Warga (2000), aging does not contribute monotonically to higher transaction costs. In fact, for bonds that are relatively young (e.g., less than 10 years old), liquidity actually increases with the age. However, when the bonds are old (e.g., more than 13 years old), their liquidity can drop quickly and they gradually leave the market.

It is well known that the average transaction cost for equity is negatively related to the firm's capitalization (Demsetz (1968), Benston and Hagerman (1974), Copeland and Galai (1983), Stoll and Whaley (1983), Roll (1984), and Lesmond, Ogden and Trzcinka (1999)). Similarly, the transaction cost for bonds is negatively related to bond size (e.g., Hong and Warga, 2000). The unaddressed question is, which factor, the firm size or the bond size, is more important for the bond's liquidity? We show that the firm size dominates the bond size in determining the liquidity of bonds. This is consistent with the hypothesis that bonds issued by larger firms have better publicity and are better distributed, resulting in a higher trading activity and liquidity. In fact, the non-callable bonds in our sample are much more liquid than the callable bonds even though their bond sizes are very similar, primarily because larger firms tend to issue non-callable bonds and smaller firms tend to issue callable bonds.

Third, theoretically, Aniuhud and Mendelson (1986) and Chen (2001) show that investors require a higher return for securities with higher transaction costs. While previous studies estimate transaction costs for corporate bonds, they stop short of investigating whether illiquidity is priced in bond yields. We observe that bonds with a
smaller outstanding amount tend to be the ones with lower rating quality, and exhibit a higher spread over riskless bonds. The issues of credit risk and illiquidity are thus intertwined. It is very important to know how each part contributes to the total spread. This knowledge will help to predict the evolution of a particular bond's spread over time because credit risk and illiquidity have different dynamics. In addition, in the credit risk literature, default probability is usually recovered from the total spread and used to evaluate other credit-sensitive instruments. Clearly, the liquidity spread part in the total spread should have been excluded because it does not reveal the default probability of the firm. We show that both credit rating and trading frequency significantly contribute to the total spreads. In particular, for a median yield spread of 1.58%, about 1.43% is credit spread and 0.15% is liquidity spread. In other words, about 9.5% of the total spread is due to illiquidity. Holding the credit spread at 1.43%, the liquidity spread can be as high as 0.54%.

II. Relationship among the three essays

The three essays can be regarded as studies on the basis risk of fixed income securities. In other words, they investigate the spreads among different bonds. Essay I assumes perfect liquidity and focuses on the credit spread. By incorporating credit risk into the standard asset pricing models, it provides one of the first studies on how credit spread relates to market risk, including equity risk, interest risk, and inflation risk.

Essay 2 focuses on Treasury bonds with zero default risk. The yield spread among the bonds are solely due to liquidity difference. We derive, quantitatively, how this spread is related to the bid-ask spread, brokerage fee, bond maturity, and investors' expected holding period. It is one of the first theoretical models on the liquidity of treasury securities.

Essay 3 empirically estimates the transaction costs of corporate bonds. It is observed
that bonds with less liquidity tend to be the ones with lower credit rating quality. Liquidity risk and credit risk are thus intertwined. We are able to separate their effects and obtain estimates for liquidity spreads and credit spreads.

In a nutshell, the first essay studies credit risk, the second studies liquidity risk, and the third, as an empirical study, investigates both issues. They jointly contribute to the understanding of the basis risk of fixed income securities.

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Chen, Long, 2001, Illiquidity and expected return of Treasury securities, working paper, University of Toronto.


Market Risk and Credit Risk in a General Equilibrium Model

Long Chen

Market Risk and Credit Risk in a General Equilibrium Model

ABSTRACT

This paper intends to integrate credit risk into the asset pricing literature. With a monetary equilibrium model where the credit risky bonds are endogenously determined, we are able to (1) produce a stochastic and counter-cyclical equity premium positively correlated with credit spread; (2) show the correlation between the interest rate and credit spread can be either positive or negative. Credit spreads for different maturities can move in opposite directions; (3) study the role of inflation on credit risk. A higher money supply reduces default risk. Positive monetary shocks can move expected inflation and credit spread in opposite directions. However, production growth rate changes can move expected inflation and credit spread in the same direction.
I. Introduction to the first essay

It has been common for empirical studies to use credit spreads to explain asset returns and equity premium (see, for example, Chen (1991), Chen, Roll, and Ross (1986), Fama and French (1989, 1993), and Campbell and Ammer (1993)). In fact, Gerter and Lown (1990) find that credit spreads have outperformed other leading financial indicators in measuring the business cycle in U.S. since the mid-1980s. Given the wide acceptance of credit spreads in empirical studies, there are, however, few theoretical papers that have successfully incorporated credit risk into the equilibrium asset pricing literature. In standard general equilibrium models, equity is usually unlevered and credit risk is not considered. On the other hand, the current credit risk literature usually starts from exogenous specifications of the asset or default rate processes and is of partial equilibrium in nature. This unsatisfactory fact is emphasized by both Campbell (2000) and Sundaresan (2000) in their review of finance for the past two decades. Campbell (2000) states that “it will be helpful to use information from government and corporate bond markets as well as from stock markets”. Similarly, Sundaresan (2000) acknowledges that “we have not developed a satisfactory general equilibrium model of default in a continuous-time-setting.”

This paper intends to incorporate credit risk into the asset pricing literature. We develop a general equilibrium model where all financial prices, including credit-risky instruments, are endogenously determined. Analytical formulae show clearly how state variables are priced into credit spreads and other risk premia through the marginal rate of substitution, which provides theoretical support for using credit spreads to explain other risk premia and measure the health of economy. More important, this paper studies how market risk and credit risk interact with each other with the internal consistency of the equilibrium setting. In particular, we are able to contribute to the literature by addressing the following important issues:

1. A stochastic equity risk premium. We assume the capital structure irrelevance
holds. For given debt structure, because the production change will have a much bigger impact on the equity value than on the debt value, the leverage ratio is stochastic and counter-cyclical. It in turn produces a stochastic and counter-cyclical equity premium whose dynamic is solely driven by the endogenous leverage ratio. This way of capturing the stochastic equity premium is different from other papers, including Barberis, Huang, and Santos (1998) and Campbell and Cochrane (1999). In addition, this paper predicts that the equity premium should be a function of the maturity of corporate debt, which is very intuitive. Furthermore, because the credit spread is also negatively driven by the production, the equity premium and credit spread are positively correlated. While empirically this positive correlation is usually interpreted as showing that credit spreads can capture business conditions, we show theoretically that equity premium and credit spreads are tightly related because the equity and corporate bonds are both contingent claims on the firm's production. The two risk premia are bound to correlate due to the capital structure effect. Moreover, our analysis shows that the relationship between equity premium and credit spread is non-linear, inconsistent with the linear regression practice in many empirical studies.

Finally, and more important, our simple model shows theoretically why the book-market ratio and the size might work in explaining equity premium, the question left by Fama and French's convincing empirical work (1992, 1993, 1995, 1996). The equity beta, driven by the endogenous leverage ratio, is counter-cyclical and can not be explained by the market portfolio premium, which is a constant in our model. Instead, any variable that is counter-cyclical can pick up some variation in the equity beta. These variables include the size, dividend/equity price ratio, leverage ratio, and possibly the book-market ratio. This is consistent with Hecht (2000) who shows that the role of book-market ratio and the size are much mitigated in the cross section of firm level returns.

(2) The "flight to quality" phenomenon. "Flight to quality" refers to the situation wherein investors intend to substitute risky assets with riskless bonds when the market
is down. This will drive the riskless bond price up and the equity value down. While this pattern has been explained by previous studies (e.g., Barsky (1989)), we enrich the picture by showing that both the equity premium and credit spreads will increase at such a crisis time.

(3) To study the correlation between the interest rate and credit spreads in an endogenous way. In the current credit risk literature, this correlation is usually specified exogenously. For example, in a typical structural form model such as Merton (1974), the asset value $F_t$ is assumed to follow a geometric Brownian motion in the risk-neutral world:

$$\frac{dF_t}{F_t} = rdt + \sigma dW_t,$$

where $r$ is the interest rate, $\sigma$ the volatility, and $W_t$ a standard Brownian motion. A larger interest rate is thus interpreted as a higher growth rate of the asset value in the risk-neutral world, which leads to a higher expected future asset value and lower default probability. It also implies that interest rate changes have no contemporaneous effect on the current asset value, which is clearly against empirical evidence. Longstaff and Schwartz (1995) extend the model by assuming a stochastic interest rate and some constant correlation between the interest rate and the asset value. However, in a general equilibrium setting, both Barsky (1989) and Abel (1988) show that the correlation between the unlevered asset and interest rates can be in either direction. The structural form models can not derive this correlation endogenously.\(^2\) Similar problem arises for the reduced form models where the correlation between the interest rate and the default hazard rate has to be specified exogenously.\(^3\) The empirical evidence on this issue is not quite clear either. Longstaff and Schwartz (1995) and Duffee (1996, 1998, 1999) all

\(^2\)Collin-Dufresne and Goldstein (2001) point out that the negative correlation between the credit spread and the interest rate in a typical structural form model might not hold in a dynamic setting where the current asset value is affected by the interest rate. However they are not able to address the problem within their partial equilibrium framework.

\(^3\)See Sundaresan (2000) for a review of the rich literature of reduced form models.
report a negative correlation between the interest rates and credit spreads. On the other hand, Fridson, Garman, and Wu (1997) find some evidence that the real interest rate and the trailing default probability are positively correlated.

The foregoing argument motivates us to develop an equilibrium model where the interest rate and the asset price are endogenously derived. Our result shows that the comovement of the credit spread and interest rate can be either positive or negative, depending on investors' risk aversion and the maturity of the risky debt structure. In particular, the credit spreads of different maturity can move in opposite directions. However, for long term maturities, the correlation between the credit spreads and the interest rate is always negative. These theoretical results not only have important implications in risk management, but also build a bridge between the current credit risk models and the literature that studies the comovement of the equity and riskless bond prices (e.g., Barsky (1989) and Abel (1988)).

(4) *The effect of inflation on credit risk.* Corporate debts are usually in nominal terms and the bondholders are subject to inflation risk. Given the importance of inflation on fixed-income securities, however, there has rarely been any work, either theoretical or empirical, that study the role of inflation on credit risk. It is well known that in a simple capital structure such as that of Merton (1974), the risky debt can be expressed as follows:

\[ D_t = G_t - P_t, \]

where \( D_t \) is the credit-risky debt, \( G_t \) is the riskless debt, and \( P_t \) is a put option on the firm asset value with the exercise price as the face value of debt. This paper shows that the put option behaves like an exchange option between asset value, which is subject to both production and inflation risk, and the face value of debt, which is subject to inflation risk. A higher money supply will always reduce the credit spread because a larger nominal asset value reduces default risk. The relationship between expected inflation and the credit spread, however, is trickier. When expected inflation changes due to monetary
shocks (i.e., either money growth rate or volatility changes), the expected inflation and credit spread will move in opposite directions because a higher money growth will boost up expected inflation but reduce credit risk. However, when the expected inflation changes in response to a change in the production growth rate, the expected inflation and credit spreads can move in the same direction. The intuition is that a higher production growth can reduce both inflation and credit risk.

This work extends the money-in-the-utility models, including Danthine and Donaldson (1986), Stulz (1986), and Bakshi and Chen (1996), by adding a capital structure. In addition, the exchange economy implies that the exogenous production is independent of capital structure. This is in the spirit of Modigliani and Miller (1958) and consistent with a similar assumption in Merton (1970). We argue that in reality firms do not change their debt structure frequently. Most of the variation in the debt/equity ratio comes from the endogenous market price change. Therefore, by holding the debt structure constant, the model is still close to the reality and can shed important insights into the study of credit risk.

At this time we are aware of the work by Chang and Sundaresan (1999) who model a Cox-Ingersoll-Ross production economy (1985) with both production and borrowing technologies and the investor’s optimal choice of consumption and default will determine the prices of financial assets. Without analytical solutions, their simulation results show a counter-cyclical equity premium and a positive correlation between equity premium and credit spread. This paper adopts a pure-exchange economy (e.g., Lucas (1978) and Bakshi and Chen (1996)) that, even though can not cover the insightful relationship between consumption and default in Chang and Sundaresan (1999), can alternatively obtain analytical solutions and largely facilitate our study of market risk and default risk.

The rest of the paper proceeds as follows: In Section I the equilibrium properties for the economy are obtained. Sections II and III deal with the financial securities in
real and nominal terms respectively and in both cases market risk and credit risk are analyzed. A concluding remark is provided in Section IV.

II. A general equilibrium model

A. The set up of the economy

The setup follows closely Bakshi and Chen (1996) except that we introduce a capital structure. The investors' optimal portfolio choice is first established in discrete time and equilibrium prices are derived in the continuous time limit. Imagine an economy with an exogenous production technology where the output $Y_t$, in units of a completely perishable single good, follows:

$$\frac{\Delta Y_t}{Y_t} = \mu_{Y_t} \Delta t + \sigma_{Y_t} \varepsilon_{Y_t} \sqrt{\Delta t},$$

(4)

where $\mu_{Y_t}$ and $\sigma_{Y_t}$ are the instantaneous conditional mean and standard deviation of the output growth per unit time respectively which can be stochastic but observable at time $t$. $\{\varepsilon_{Y_t}, t = 0, \Delta t, 2 \Delta t, ...\}$ is an i.i.d. standard normal process that is defined on the probability space $\{-, P, F (F_t, t = 0, \Delta t, ...)\}$. In addition, the money supply of the economy follows

$$\frac{\Delta M_t}{M_t} = \mu_{M_t} \Delta t + \sigma_{M_t} \nu_{M_t} \sqrt{\Delta t},$$

(5)

where $\nu_{M_t}$ is an i.i.d. standard normal process that is defined on the probability space $\{-, P, F (F_t, t = 0, \Delta t, ...)\}$. The information structure of the economy is revealed through the filtration $F_t$ on $\varepsilon_{Y_t}$ and $\nu_{M_t}$.

The economy is populated with many identical investors who will live infinitely and consume the non-storable output in each period. Following Danthine and Donaldson (1986), Stulz (1986), and Bakshi and Chen (1996), we assume that the investors derive utility from both consumption and holding real cash balance because money facilitates transactions. As Feenstra (1986) shows, this is equivalent to using money in the liquidity
costs in the budget constraint. The investors seek to maximize their lifetime expected utility

\[ \sup_{c_t, M^d_t} \sum_{t=0}^{\infty} E_0 \left\{ e^{-\beta t} u \left( c_t, \frac{M^d_t}{P_t} \right) \Delta t \right\}, \]

where \( c_t \) is the consumption flow in \([t, t+\Delta t)\), \( M^d_t \) is the nominal money demand in \((t-\Delta t, t]\), and \( P_t \) is the price level of the consumption good. \( m_t = \frac{M^d_t}{P_t} \) is by definition the real money balance. We further assume that the utility function is twice continuously differentiable and concave with respect to consumption and the real money balance.

**B. The equilibrium**

There exists one share of equity and corporate debt respectively in net positive supply in the economy that, in combination, claim the output from the exogenous production. The exchange economy framework implies that the exogenous production is independent of capital structure. This help us separate the valuation problem into two steps. In the first step the representative agent chooses to hold the asset to maximize his expected utility. This step enables us to calculate the asset value. In the second step, the equity and risky debt values are calculated as contingent claims on the firm asset. Such a setting will not be able to determine the optimal capital structure but only correctly price the equity and debt given any capital structure. The flow chart of the paper is in Figure 1.

Insert Figure 1 here

Formally, let us denote by \( F_t \) the nominal price of the asset and by \( W_t \) the agent's wealth. In addition to the firm asset, there is one riskless real bond that pays in units of the consumption good and its instantaneous return is \( r_t \). There is also a riskless nominal bond that will pay in cash and its instantaneous return is denoted as \( R_t \). There are \( N \)
financial securities whose cum-dividend nominal prices are $P_{it}, i = 1, ..., N$. At each
time $t$ we use $Q_{it}, i = F, 1, 2, ..., N$, to denote the number of holdings of the asset and
the financial securities respectively. We use $\zeta_{rt}$ to denote the amount the investor holds
in the real bond in units of the consumption good, and $\zeta_{Rt}$ the dollar value the investor
holds in the nominal bond. The following accounting constraint must hold:

$$W_t = M^d_t + (F_t + P^c_t Y_t \Delta t) Q_{Ft} + \sum_{i=1}^{N} P_{it} Q_{it} + \zeta_{rt} + \sum_{i=1}^{N} P_{it} Q_{it} = P^c_t c_t \Delta t + M^d_{t+\Delta t} + F_t Q_{F_{t+\Delta t}} + P^c_t \frac{\zeta_{rt+\Delta t}}{1 + r_t \Delta t}$$

$$+ \frac{\zeta_{R_{t+\Delta t}}}{1 + r_t \Delta t} + \sum_{i=1}^{N} P_{it} Q_{it+\Delta t},$$

(7)

where the first equality is what the current portfolio is worth and the second is how
the investor will rebalance the portfolio. The maximized indirect utility function should
satisfy the Bellman equation:

$$V(M^d_t, Q_{Ft}, \zeta_{rt}, \zeta_{Rt}, Q_{it}, i = 1, ..., N)$$

$$= \sup_{Q_{Ft+\Delta t}, \zeta_{rt+\Delta t}, \zeta_{Rt+\Delta t}, M^d_{t+\Delta t}, Q_{it+\Delta t}, i = 1, ..., N} E_t \{ u \left( \frac{M^d_t}{P^c_t} \right)$$

$$- e^{-\rho \Delta t} V(M^d_{t+\Delta t}, Q_{F_{t+\Delta t}}, \zeta_{rt+\Delta t}, \zeta_{R_{t+\Delta t}}, Q_{it+\Delta t}, i = 1, ..., N) \}.$$

(8)

The equilibrium of the economy is defined as such that the representative agent
will maximize his lifetime expected utility subject to the budget constraint and market
clearing. In particular, $M^d_t = M^d_t, Q_{Ft} = 1, \zeta_{rt} = 0, \zeta_{Rt} = 0$, and $Q_{it} = 0, i = 1, ..., N$.

**Theorem 1** In the continuous time limit, equilibrium interest rates and the prices are
calculated as follows:

(1) The real interest rate is

$$r_t = \rho - \frac{Y_t u_{11}}{u_1} \frac{1}{dt} E_t \left\{ \frac{dY_t}{Y_t} \right\} - \frac{1}{2} \frac{Y_t^2 u_{11}}{u_1} \frac{1}{dt} var_t \left\{ \frac{dY_t}{Y_t} \right\}$$

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The nominal interest rate is

\[ R_t = \frac{u_2(c_t, m_t)}{u_1(c_t, m_t)}. \]  

(3) The real value of the firm is

\[ f_t = E_t \int_t^\infty e^{-\rho(s-t)} \frac{u_1(c_s, m_s)}{u_1(c_t, m_t)} Y_s ds. \]  

(4) The price of the consumption good can be calculated through

\[ \frac{1}{P^c_t} = E_t \int_t^\infty e^{-\rho(s-t)} \frac{u_1(c_s, m_s)}{u_1(c_t, m_t)} \frac{1}{P^c_s} ds. \]


The following theorem describes how the risk premium of any financial instrument in equilibrium should behave:

**Theorem 2** (1) In the continuous time limit, the instantaneous risk premium of any cum-dividend real price over the real interest rate is

\[ E_t \left( \frac{dp_{i,t}}{p_{i,t}} - r_t dt \right) = \frac{u_{11}(c_t, m_t)}{p_{i,t}u_1(c_t, m_t)} Cov[Y_t, dp_{i,t}] \]

\[ - \frac{u_{12}(c_t, m_t)}{p_{i,t}u_1(c_t, m_t)} Cov[dm_t, dp_{i,t}] \].

(2) In the continuous time limit, the instantaneous risk premium of any cum-dividend nominal price over the nominal interest rate is

\[ E_t \left( \frac{dP_{i,t}}{P_{i,t}} - R_t dt \right) = \frac{u_{11}(c_t, m_t)}{P_{i,t}u_1(c_t, m_t)} Cov[Y_t, dP_{i,t}] \]

\[ - \frac{u_{12}(c_t, m_t)}{P_{i,t}u_1(c_t, m_t)} Cov[dm_t, dP_{i,t}] \]

\[ + Cov \left[ \frac{dP^c_t}{P^c_t}, \frac{dP_{i,t}}{P_{i,t}} \right]. \]
Proof: see Appendix A.

Because of the concavity of the utility function, the coefficients on the right hand side of both equations are non-negative. The first equation states that a financial instrument will have a positive real risk premium if its real price is positively correlated with the production and real money supply. In the second equation, a financial instrument will have a positive nominal risk premium if its nominal price is positively correlated with the production, real money supply, and the inflation process.

We assume, for the rest of the paper, that the production and money supply follow their continuous time versions, i.e.,

\[
\frac{dY_t}{Y_t} = \mu_{Y,t} dt + \sigma_{Y,t} dZ_{Y,t}, \quad (15)
\]
\[
\frac{dM_t^\gamma}{M_t^\gamma} = \mu_{M,t} dt + \sigma_{M,t} dZ_{M,t}. \quad (16)
\]

where \(Z_{Y,t}\) and \(Z_{M,t}\) are Brownian motion on the probability space \([-\infty, \mathcal{F}(F_t, t \in [0, \infty))]\), and \(E_t (dZ_{Y,t} dZ_{M,t}) = \varphi_{Y,t,M}\). The representative agent has the following utility function

\[
u (c_t, m_t) = \varphi \frac{c_t^\gamma - 1}{\gamma} + (1 - \varphi) \frac{m_t^\beta - 1}{\beta}. \quad (17)
\]

When \(\beta = 0\) it reduces to

\[
u (c_t, m_t) = \varphi \frac{c_t^\gamma - 1}{\gamma} + (1 - \varphi) \ln (m_t). \quad (18)
\]

which we will use for tractability considerations.\(^4\) We further assume that \(\mu_{M,t}, \sigma_Y, \) and \(\sigma_M\) as constants but \(\mu_{Y,t}\) can be time-varying. The equilibrium interest rates and the consumption prices are calculated as follows:

**Theorem 3** (1) The *real interest rate* is

\[
r_t = \rho + (1 - \gamma)\mu_{Y,t} - \frac{(2 - \gamma)(1 - \gamma)}{2}\sigma_Y^2. \quad (19)
\]

\(^4\)When both \(\gamma = \beta = 0\), the investor will have a log utility function as in Stulz (1985) and Bakshi and Chen (1996).
(2) The nominal interest rate is

\[ R_t = \rho + \mu_M - \sigma_M^2; \quad (20) \]

(3) The consumption price level is

\[ P_t^c = \frac{\phi}{1 - \phi} (\rho + \mu_M - \sigma_M^2) Y_{t-1}^t M_t; \quad (21) \]

Proof: see Appendix B.

Due to the additivity of the utility function, monetary policy does not affect the prices of financial securities that have claims in units of the consumption good. This enables us to study market risk and credit risk in real terms and nominal terms respectively. In particular, the same results in the real terms will hold for those economies without money. To avoid confusion, for the rest of the paper, we use lower cases to represent real financial prices, and upper cases to represent nominal prices.

III. Market risk and credit risk in real terms

In this section all the financial securities have claims in units of the consumption good. As we will show shortly, capital structure irrelevance in the continuous time implies a dynamic capital structure. In particular, because production affects the equity value and the debt value disproportionately, the leverage ratio will be counter-cyclical for given outstanding debt amount. This produces a counter-cyclical equity premium which is positively correlated with the credit spread. In addition, the stochastic leverage ratio also implies that the stock beta is very volatile. This makes the traditional two-pass regression (Fama and Macbeth (1973)) biased. On the other hand, any variable that is counter-cyclical might have power in explaining the systematic risk because it is correlated with the stock beta.

We also find that credit spreads react differently to interest rate shocks, depending on investors' risk aversion and the debt maturity. Under some cases, credit spreads
with different maturities can move in different directions. This finding has important implication for the risk management of corporate bond portfolios, or for the hedging of credit derivatives.

A. Case 1: Constant interest rate

The real interest rate in Theorem 3 is a standard result that is consistent with Cox, Ingersoll, and Ross (1985), Breeden (1986), and Barsky (1989). A higher $\rho$ implies that the investors are reluctant to save wealth for future consumption, forcing a higher interest rate to attract saving. With $(1-\gamma) > 0$, the interest rate is positively driven by the production growth rate $\mu_{Y,t}$ because a higher $\mu_{Y,t}$ increases the expected future consumption and reduces marginal rate of substitution. Investors thus require a higher interest rate for saving. Finally, the last term captures the precautionary saving. A higher variance of the production process will push the investors to buy riskless bonds, and thus drives the interest rate down.

In this subsection we assume that $\mu_{Y,t} = \mu_Y$ is a constant, which implies a constant interest rate. The real firm value can be calculated from Theorem 2 as

$$f_t = \frac{Y_t}{r^*}, \quad (22)$$

where

$$r^* = \rho - \gamma \mu_Y + \gamma (1 - \gamma) \frac{\sigma_Y^2}{2} \quad (23)$$

The equity and risky debt are contingent claims on the asset. To simplify the analysis, we assume a very simple capital structure, where the risky debt is a discount bond with maturity $T$ and face value $k$ in units of consumption good. At maturity if $f_T > k$, the firm can issue new equity worth $k$ to pay debt holders and the old equity value will decrease by $k$. On the other hand, if $f_T < k$, the firm is bankrupt and the debt holders take over the firm, holding equity worth $f_T$. Even though the representative agent
owns both equity and debt, this does not prevent each claim being correctly priced and transferred under the above terms.

The real equity value $s_t$ is a summation of two parts: the present value of the production during the period $(t, T]$ and a call option on the asset:

$$s_t = E_t \int_t^T e^{-r(s-t)} \left( \frac{Y_s}{Y_t} \right)^{r-1} Y_s ds + E_t \left[ e^{-r(T-t)} \left( \frac{Y_T}{Y_t} \right)^{r-1} \max[f_T - k, 0] \right].$$

(24)

In addition, the risky debt value can be calculated as the price of an instrument that pays

$$k - \max[k - f_T, 0]$$
at time $T$.

**Theorem 4** (1) The equilibrium real equity value is

$$s_t = f_t (1 - e^{-r(T-t)}) + f_t e^{-r(T-t)} N(d_1^*) - ke^{-r(T-t)} N(d_2^*)$$

(25)

$$= f_t (1 - e^{-r(T-t)}) + c_t,$$

where

$$c_t = f_t e^{-r(T-t)} N(d_1^*) - ke^{-r(T-t)} N(d_2^*),$$

$$d_1^* = \frac{\ln \left( \frac{f_T}{k} \right) + (r - r^* + \sigma_Y^2/2)(T-t)}{\sigma_Y \sqrt{T-t}},$$

$$d_2^* = \frac{\ln \left( \frac{f_T}{k} \right) + (r - r^* - \sigma_Y^2/2)(T-t)}{\sigma_Y \sqrt{T-t}}.$$

(2) The equilibrium price of the risky debt is

$$d_t = ke^{-r(T-t)} + f_t e^{-r(T-t)} N(-d_1^*) - ke^{-r(T-t)} N(-d_2^*)$$

(26)

$$= g_t - p_t.$$
where

\[ g_t = ke^{-r(T-t)} , \]
\[ p_t = ke^{-r(T-t)}N(-d_2^*) - f_t e^{-r(T-t)}N(-d_1^*) . \]

Proof: this is a special case of the proof in Appendix G.

\( c_t \) is a standard European call option on the asset with a dividend payout rate \( r^* \) because the asset value is a perpetuity with discount rate \( r^* \). The risky bond is equal to the riskless bond value \( g_t \) minus a put option \( p_t \) that represents the expected bankruptcy cost. It can be easily checked that

\[ s_t + d_t = f_t . \]

The yield spread \( y_t \) is calculated as

\[
y_t = \frac{\ln \left( \frac{d_2^*}{d_1^*} \right)}{T-t} - r
\]

\[
= \frac{\ln \left( \frac{kN(d_2^*) - f_t e^{-r(T-t)}N(-d_1^*)}{kN(d_2^*)} \right)}{T-t} .
\]

\( kN(d_2^*) \) is the expected future value of a bond that pays its face value at maturity if there is no default and zero otherwise. \( f_t e^{-r(T-t)}N(-d_1^*) \) is the expected future value of a security that pays the asset value if default happens and zero otherwise. So the denominator in the logarithm is the expected future value of the risky bond at maturity in the risk-neutral world.

A.1 Dividend-price ratio, asset premium, equity premium, and credit spread

We are now ready to provide theoretical explanations to some well-known empirical regularities. Let's start with dividend-price ratio. The dividend/asset price ratio, which is
the dividend/unlevered equity ratio in the traditional models, is a constant, \( r^* \). However, the dividend/equity price ratio, \( \delta_t \), calculated as

\[
\delta_t = \frac{Y_t}{s_t} = \frac{1}{1 - e^{-r^*(T-t)} \cdot \frac{1}{T} \times \sigma_t^2}
\]

is negatively driven by production (i.e., \( \frac{\partial \delta_t}{\partial Y_t} < 0 \)). This counter-cyclical pattern of the dividend/equity ratio has been well documented empirically (e.g., Fama and French (1989)). Therefore we have just shown that adding capital structure to the traditional models can help explain this empirical phenomenon. In addition, because of the monotonic relationship between the dividend/equity price ratio and the production, the current production level can be inverted from \( \delta_t \). Let us denote the inverse function as

\[
Y_t = I(\delta_t).
\]

The expected future production at future time \( \tau \), where \( t < \tau < T \), can be calculated as

\[
E_t [Y_\tau] = Y_t e^{\mu(\tau-t) + \frac{1}{2} \sigma_\tau^2(\tau-t)} = I(\delta_t) e^{\mu(\tau-t) + \frac{1}{2} \sigma_\tau^2(\tau-t)}.
\]

In other words, the dividend/equity price ratio can be used to predict future dividend earning. In particular, a higher \( \delta_t \) predicts a lower future dividend earning, consistent with Fama and French (1995). Note this predictability comes from the non-linearity of the equity price in production. To the contrary, the asset price is a linear function of the production, causing a constant dividend/asset price ratio without any power of predictability.

We next use Theorem 2 to study real risk premia and their correlations. The annualized asset risk premium, \( s_t' \), is:

\[
s_t' = E_t \left( \frac{df_t}{f_t} - r_t dt \right) \frac{1}{dt} = (1 - \gamma) \sigma_t^2.
\]
The constant asset premium is strictly positive due to the positive correlation between the production and the asset price. Intuitively, it is increasing in both the relative risk aversion parameter $1 - \gamma$ and the production volatility. The equity risk premium $S_t^P$ can be calculated from Theorem 2 as follows:

Theorem 5 The annualized equity risk premium is:

$$s_t^P = \left\{ 1 + \frac{ke^{r(T-t)} N(d_2^*)}{s_t} \right\} (1 - \gamma) \sigma^2\gamma = \beta_{E,t} \times s_t^f > 0,$$

where

$$\beta_{E,t} = 1 + \frac{ke^{r(T-t)} N(d_2^*)}{s_t}.$$

The equity premium is stochastic and strictly positive. In addition, it has several very important and appealing properties. First, the equity risk premium is always higher than the asset premium due to the leverage. The underlying rationale is that the proportional change of a call option price is always higher than that of its underlying asset. In addition, the equity premium is a function of both the level and maturity of debt, which is intuitive. In fact this model gives equity premium a term structure that is shown in Figure 2. While the asset premium is fixed at 4 percent, the equity premium can be as high as 20 percent and is a decreasing function of the maturity of debt. When debt maturity increases the call option feature will be subdued and the default risk decreases, in which case the equity premium converges to the asset premium. This is consistent with Chang and Sundaresan (1999) who show that the two premia converge when default risk disappears.

---

5While the equity premium is always higher than the asset premium, the monotonic pattern shown in Figure 2 is specific to the simple capital structure we assumed.
Second, it can be shown that
\[
\frac{\partial \beta_{E, t}}{\partial Y_t} = \frac{\partial \beta_{E, t}}{\partial Y_t} (1 - \gamma) \sigma_Y^2 \\
= \frac{1}{\lambda_t} (1 - \gamma) \sigma_Y^2 \ln(k e^{-\lambda T - t}) \Pi_t.
\]
(32)

where
\[
\Pi_t = \left\{ N'(d_2) \left[ \frac{1}{\sigma_Y \sqrt{T - t}} - N(d_2) \right] s_t - k e^{-\lambda(T - t)} N(d_2)^2 \right\}.
\]
(33)

For the most common case where \(f_t e^{-r(T-t)} > k e^{-r(T-t)}\), the sign of \(\frac{\partial \beta_{E, t}}{\partial Y_t}\) is negative for a wide range of reasonable parameters.\(^6\) Intuitively, a lower (higher) production will reduce (increase) the equity value and increase (decrease) the leverage ratio, causing a higher (lower) equity premium. The equity premium is thus counter-cyclical purely due to the leverage. To fully understand this result, note the equity premium can be rewritten in the traditional cost of equity format:
\[
s^2_t = (1 + L_t) \times s_t^f - L_t \times s_t^d,
\]
(35)

where \(L_t = \frac{d_t}{s_t}\) is the debt/equity ratio and
\[
s_t^d = \left\{ \left[ \frac{k e^{-r(T-t)} N(d_2)}{d_2} \right] \times s_t^f \right\}.
\]
(36)
is the risk premium on corporate bond. Because \(\frac{\partial \beta_{E, t}}{\partial Y_t} < 0\), the leverage ratio is counter-cyclical and it is the factor driving the variation of the leverage ratio is the dominant part of the variation of the counter-cyclical equity premium. However, while the equity premium is increasing in the leverage ratio for most of the cases, it can decrease with leverage ratio when the leverage ratio is very high. This is because a very risky debt behaves like the firm asset itself, and the return on equity will return back to the level of the return on firm asset. This can be clearly shown in Figure 3. For small production level (i.e., very high leverage ratio), the equity premium actually increases in production.

\(^6\) Helwege and Turner [32] imply that \(f_t e^{-r(T-t)} > k e^{-r(T-t)}\) is generally satisfied in reality. Parameters are reasonable if their combination makes \(r > 0\) and \(r - r^* > 0\). This implies that the interest rate is positive and the asset value grows at a positive rate in the risk-neutral world.
On the other hand, for sensible production level (i.e., leverage ratio), the equity premium is decreasing in production.

Insert Figure 3 here.

We have obtained the results with the relative risk aversion parameter $1 - \gamma$ held constant. When the investors are more risk-averse, the counter-cyclical behavior is more pronounced and the pattern remains the same.

This paper thus produces an equity premium whose stochastic and counter-cyclical feature is primarily caused by the leverage effect. This is distinctly different from those papers, including Barberis, Huang, and Santos (1998) and Campbell and Cochrane (1999), that model the counter-cyclical pattern.

We now probe the correlation between equity premium and credit spread. It can be shown that

$$
\frac{\partial y_t}{\partial Y_t} = -\frac{1}{T-t} \times \frac{1}{\mathcal{N}(d_1)\mathcal{N}(-d_1)} + \frac{1}{Y_t} < 0.
$$

(37)

In other words, the credit spread is negatively driven by production. This finding is hardly surprising considering that the credit spread is basically the yield version of a put option on the asset that represents the expected bankruptcy cost. A lower production will increase the value of the put and the credit spread. This produces a counter-cycle pattern for credit spread.

More important, the above analysis implies that the equity premium and credit spread are, in general, positively correlated due to the stochastic leverage ratio. Because the equity premium is mainly driven by a call option feature and the credit spread is driven by a put option, their correlation can be understood as a put-call relationship. Intuitively, the credit spread not only reveals business condition, as usually interpreted in empirical studies, but also should be tightly related to the equity premium because of this relationship. In addition, the equity premium and credit spreads are positively
correlated in a non-linear way, inconsistent with the linear regression practice in many empirical papers.

Finally, the stochastic leverage ratio also shed light on why the book-to-market ratio and the equity size might work in explaining the variation of equity returns. The firm asset value is in fact the market portfolio, i.e., the whole portfolio that is in positive net supply in the economy. From Theorem 2, it can be shown that the real risk premium of any security, \( \lambda_{i,t} \), can be written as

\[
\lambda_{i,t} = \beta_{i,t} \Psi_{i,t},
\]

where \( \Psi_{i,t} \) is the market portfolio premium, which is the firm asset premium in this case. Heuristically, the variation of the risk premium thus comes from the variation in \( \beta_{i,t} \), \( \Psi_{i,t} \), and their covariance. In our model, the market portfolio premium is a constant, which means that the market portfolio has no power in explaining the equity premium. Therefore, time series regression of the equity premium on the market portfolio will not yield correct update of the true beta. On the other hand, the equity beta \( \beta_{E,t} \), as we showed earlier on, is stochastic and counter-cyclical due to the endogenous leverage ratio. Any variable that is counter-cyclical can have explanatory power of the true beta. These variables include the leverage ratio, equity size, dividend/equity price ratio, credit spread, and possibly the book-market ratio.

Fama and French (1992, 1993, 1995, 1996) provide convincing empirical evidence that the book-market ratio and the equity size are significant to explain cross-section of equity returns. However, the fundamental question left is, as they state (1993), "how does profitability, or any other fundamental, produce common variation in returns associated with size and BE/ME that is not picked up by the market return?" Our analysis above provides a very plausible answer to their question. This is also consistent with Hecht (2000) who shows that the book-market ratio and the size lose much of the explanatory power at the firm level, suggesting that capital structure plays a key role for these factors.
A.2 Comovement of asset, interest rate, and credit spread

In this subsection we study how credit spreads react to interest rate shocks. To achieve this, we need to look at how interest rate and firm asset value are correlated, which can be summarized as follows:

\[
\begin{align*}
\frac{\partial r}{\partial \mu_Y} &> 0, \quad \left\{ \begin{array}{ll}
\frac{\partial \mu_Y}{\partial \mu_Y} > 0 & \text{if } \gamma > 0 \\
\frac{\partial \mu_Y}{\partial \mu_Y} < 0 & \text{if } \gamma < 0 
\end{array} \right. \\
\frac{\partial r}{\partial \sigma_Y} &< 0, \quad \left\{ \begin{array}{ll}
\frac{\partial \sigma_Y}{\partial \sigma_Y} < 0 & \text{if } \gamma > 0 \\
\frac{\partial \sigma_Y}{\partial \sigma_Y} > 0 & \text{if } \gamma < 0.
\end{array} \right.
\]

Note we implicitly assume that only the change of \( \mu_Y \) or \( \sigma_Y \) will cause the interest rate to change. While a higher production growth rate \( \mu_Y \) always props up the interest rate, it decreases the asset value if \( \gamma > 0 \) and increases the asset value if \( \gamma < 0 \), with a dividing point at \( \gamma = 0 \), in which case the investors have a log utility function. This pattern is caused by the tradeoff between the growth rate and the sensitivity to the marginal rate of substitution (Barsky (1989)). Similarly, a change of the volatility can either encourage or depress asset prices depending on the sign of \( \gamma^7 \). These results are fully consistent with Barsky (1989) and Abel (1988) who study the comovement of the riskless bond and unlevered equity.\(^8\)

To summarize, the interest rate and asset value can change in the same or opposite direction, depending on the risk aversion level of the investors. We might guess that similar results should hold for the interest rate and credit spread because the asset drives the default risk. This conjecture is proven in the following theorem:

**Theorem 6** The interest rate and credit spread will move in opposite directions for long maturity bonds. For short maturity, it depends on the sign of \( \gamma \) and what causes interest

\(^7\)For a detailed explanation of these standard results, see Barsky [4].

\(^8\)We can also show that the comovement of the equity price and interest rates is a function of debt structure, which extends Barsky's [4] framework.
rate to change. If $\gamma > 0$, the interest rate and credit spread move negatively. If $\gamma < 0$, for short maturity the change of $\mu_Y$ will move them in the same direction, and the effect of $\sigma_Y$ on credit spread is uncertain. To summarize,

$$\frac{\partial r}{\partial \mu_Y} > 0, \frac{\partial r}{\partial \sigma_Y} < 0,$$

$$\begin{cases} \frac{\partial \mu}{\partial \mu_Y} < 0, \frac{\partial \mu}{\partial \sigma_Y} > 0, & \text{if } \gamma > 0, \\ \frac{\partial \mu}{\partial \mu_Y} > 0, \frac{\partial \mu}{\partial \sigma_Y} \geq 0 \text{ for short maturity}, \\ \frac{\partial \mu}{\partial \mu_Y} < 0, \frac{\partial \mu}{\partial \sigma_Y} > 0 \text{ for long maturity} \end{cases}$$

Proof: See Appendix C.

Therefore the comovement of the interest rate and the credit spread depends on not only the risk aversion but also the debt maturity. Let us borrow the standard risk-neutral pricing language to explain the issue. From the equity price formula we know that the asset price grows at a rate

$$r - r^* = \mu_Y - (1 - \gamma)\sigma_Y^2$$

in the risk-neutral world. Therefore the asset grows faster if the production growth rate $\mu_Y$ is higher or the volatility $\sigma_Y$ is lower, which means the interest rate is higher. Thus even though a higher interest rate might reduce the current asset value, the asset also grows faster and will dominate if the debt maturity is long enough. Consequently the long term default risk will be lower and the credit spread and interest rate move in opposite directions, which is consistent with Longstaff and Schwartz (1995). For short maturities the comovement between interest rates and the credit spread depends on the relationship between interest rate and asset value, which can be positive or negative as we just showed.

Interestingly, credit spreads for different maturities can change in different directions. In particular, it is shown in Appendix C that the sign of $\frac{\partial r}{\partial \mu_Y}$ depends on
If $\gamma < 0$, there is a break-even maturity $\frac{T}{\gamma^2}$ at which the credit spread does not change with respect to changes in $\mu_Y$. For shorter maturities the spread goes up and for longer maturities the spread goes down.

The above theorem builds a bridge between the credit risk literature, including Longstaff and Schwartz (1995), and the asset pricing models that study the comovement between the interest rate and the equity price (e.g., Barsky (1989)). While the former is more consistent with bonds with long maturities, the latter matters more for short maturities. These results are not only interesting, but also have important implications for the risk management of corporate bond portfolios and credit derivatives.

**B. Case 2: Stochastic interest rate**

In this section we let the real interest rate be stochastic, and investigate whether our earlier results still hold. We still assume that $\sigma_Y$ is a constant but $\mu_{Y,t}$ obeys the following process:

$$\frac{d\mu_{Y,t}}{\mu_{Y,t}} = \kappa (\alpha - \mu_{Y,t}) dt + \sigma_{\mu} dZ_{\mu,t};$$

where $\kappa$, $\alpha$, $\sigma_{\mu}$, and $E_t (dZ_{Y,t} dZ_{\mu,t}) = \delta_{Y,\mu}$ are constants. The interest rate is still the same as in Theorem 2. Ito's lemma shows that

$$dr_t = \kappa (\bar{r} - r_t) dt + \sigma_{\mu} dZ_{\mu,t};$$

where

$$\bar{r} = \rho + (1 - \gamma) \alpha - \frac{(2 - \gamma)(1 - \gamma)}{2} \sigma_Y^2.$$

This is a Vasicek type (1977) interest rate model with mean reversion where the real interest rate is driven solely by the stochastic production growth rate. Similar modeling of the interest rate can be found in Goldstein and Zapatero (1996) and Yan (1999).
Theorem 7 (1) The equilibrium real asset price is
\[ f_i = Y_i \int_t^\infty \exp \left( -(s - t) \left( \rho - \gamma \alpha + \gamma(1 - \gamma) \frac{\sigma_Y^2}{2} \right) \right) \times \exp \left( \gamma \left( \mu_{Y_t} - \alpha \right) \Gamma(t, s) + \gamma^2 \sigma_Y \sigma_\mu \varphi_{Y_t} \int_t^s \Gamma(\tau, s) d\tau + \frac{1}{2} \gamma^2 \sigma_\mu^2 \int_t^s \Gamma(\tau, s)^2 d\tau \right) ds, \]
where
\[ \Gamma(t, s) = \frac{(1 - e^{-\kappa(s-t)})}{\kappa}. \]

(2) When \( \gamma = 0 \),
\[ f_i = \frac{Y_i}{\rho} \]
Proof: see Appendix E.

In the constant interest rate case of last section, the asset price is
\[ f_t = Y_i \int_t^\infty \exp \left( -(s - t) \left( \rho - \gamma \alpha + \gamma(1 - \gamma) \frac{\sigma_Y^2}{2} \right) \right) ds. \]
Comparing this with the above theorem, it is clear that the production growth rate's mean reversion, volatility, and covariance with the production all play a role in determining the asset price. When \( \gamma = 0 \), the utility function is reduced to a log utility, in which case the decreasing marginal utility (which tends to reduce the asset demand) exactly offsets the precautionary saving (which tends to increase the asset demand). This makes the asset price neutral to \( \mu_{Y_t} \) and \( \sigma_Y \).

Theorem 8 The price of a riskless discount bond in this economy is
\[ g(t, T) = A(t, T) e^{-\Gamma(t, T) r}, \]
where
\[ A(t, T) = \exp \{ -\frac{1}{2} (T - t) \} - \Gamma(t, T) \}
\[ + \frac{1}{2} (1 - \gamma)^2 \sigma_\mu^2 \int_t^T \Gamma(t, T)^2 d\tau + (1 - \gamma)^2 \varphi_{Y_t} \sigma_Y \sigma_\mu \int_t^T \Gamma(t, T) d\tau \}, \]
Proof: see Appendix F.
To obtain analytical solutions we assume log utility, i.e., $\gamma = 0$, in which case the short rate of interest is just $r_t = \rho + \mu_t - \sigma^2_t$.

**Theorem 9** (1) The equilibrium equity price is

$$s_t = f_t \left(1 - e^{-\sigma(T-t)}\right) + f_t e^{-\sigma(T-t)} N(d_1) - kg(t,T)N(d_2),$$

where

$$d_1 = \ln \left(\frac{T}{t}\right) - \rho(T-t) - \ln g(t,T) + \frac{1}{2} V(t,T)^2,$$

$$d_2 = \ln \left(\frac{T}{t}\right) - \rho(T-t) - \ln g(t,T) - \frac{1}{2} V(t,T)^2,$$

$$V(t,T) = \sqrt{\sigma^2_V(s-t) + 2\sigma_V \sigma g \int_t^T \Gamma(\tau,s)d\tau + \sigma^2 \int_t^T \Gamma(\tau,s)^2 d\tau}.$$

(2) The equilibrium risky bond price is

$$d(t,T) = g(t,T) + f_t e^{-\sigma(T-t)} N(-d_1) - kg(t,T)(-d_2).$$

**Proof:** see Appendix G.

The format of the option value above is very similar to the case with a constant interest rate. With a constant interest rate, $-\ln g(t,T)$ will be reduced to $r(T-t)$ and $V(t,T) = \sqrt{\sigma^2_V(T-t)}$. The credit spread is calculated as

$$y_t = -\frac{\ln \left(\frac{T}{t}\right)}{T-t} + \frac{\ln \left(\frac{g(t)}{g(t,T)}\right)}{T-t},$$

$$= -\frac{1}{T-t} \left\{ \ln \left[kN(d_2) + e^{-\sigma(T-t)} \frac{1}{g(t,T)} f_t N(-d_1) \right] \right\},$$

which is stochastically driven by the production and its growth rate.

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B.1 Risk premia and "flight to quality"

It is important to check whether the results from the constant interest rate case still hold. The covariance between the credit spread and interest rate is

\[
\text{Cov}[dy_t, dr_t] = \frac{1}{T-t} \frac{1}{\mathcal{N}(\mu, \sigma^2)} \left( \varphi \gamma \sigma \sigma_f + \Gamma(t, T) \sigma^2_f \right). \tag{46}
\]

The sign of this instantaneous covariance is not clear and depends on the correlation between the production and its growth rate, $\varphi \gamma$. It is arguably true that during recessionary periods both the production level and its growth rate are low, which implies a positive correlation, i.e., $\varphi \gamma > 0$. In this case the credit spread and the interest rate are negatively correlated for all maturities.

Note we have assumed $\gamma = 0$ to obtain analytical solution, in which case the asset value is independent of the interest rate. We thus have lost the rich short term scenario for the correlation between interest rates and the credit spread obtained before. On the other hand, the mean-reverting pattern of interest rate does not change our earlier results. This is because

\[
E_t [r_s | r_t] = (1 - e^{-\kappa(s-t)}) \bar{r} + e^{-\kappa(s-t)} r_t.
\]

In other words, the interest rate is expected to converge to its long term mean in the limit. A higher initial interest rate always implies a higher expected interest rate, thus a higher growth rate of production in our case.

What will happen if $\gamma \neq 0$? From Theorem 7 we know $\frac{\partial \bar{r}}{\partial \mu_{Y,t}} > 0$ if $\gamma > 0$ and $\frac{\partial \bar{r}}{\partial \mu_{Y,t}} < 0$ if $\gamma < 0$. This is the same as in the constant interest rate case. Because the mean reversion does not change the long term pattern, we can easily see that the results we obtained in the constant interest rate case should hold. In particular, even though we do not have analytical solutions, there should be some break-even maturity that makes the credit spread immune to interest rate shocks.
The rest of this section assumes $\varphi_{Y,\mu} > 0$ and $\gamma = 0$. The asset premium is still $(1 - \gamma) \sigma_Y^2$ (with $\gamma = 0$). The equity premium will be different:

**Theorem 10** The annualized equity risk premium is

\[ s_t^p = \left\{ 1 + \frac{k g(t, T) N(d_2)}{s_t} \right\} (1 - \gamma) \sigma_Y^2 
+ \frac{k N(d_2) g(t, T) \Gamma(t, T) \sigma_Y \sigma_{\mu} \varphi_{Y,\mu}}{s_t}. \]  

(47)

Compared to the constant interest rate case, the second term is new in the equity premium formula, which accounts for the interest rate risk. In other words, with $\varphi_{Y,\mu} > 0$, equity premium will be even higher than that of the constant interest rate case. To study how the equity risk premium is stochastically driven by the production and its growth rate, we obtain the following partial derivatives

\[ \frac{\partial s_t^p}{\partial Y_t} = \frac{1}{2} (1 - \gamma) \sigma_Y^2 k e^{-r(T-t)} \Pi_t, \]  

(48)

\[ \frac{\partial s_t^p}{\partial \mu_{Y,t}} = \frac{(1 - \gamma) \Gamma(t, T) \sigma_Y^2 g(t, T) k}{s_t^2} \Pi_t 
+ \frac{\Gamma(t, T)^2 \sigma_Y \sigma_{\mu} \varphi_{Y,\mu} g(t, T) k}{s_t^2} \Pi_t, \]  

(49)

where

\[ \Pi_t \equiv \left[ N'(d_2) \right. \frac{V(t, T)}{N(d_2)} \left. - N(d_2) \right] s_t - k g(t, T) N(d_2)^2. \]

The signs of $\frac{\partial s_t^p}{\partial Y_t}$ and $\frac{\partial s_t^p}{\partial \mu_{Y,t}}$ depend on $\Pi_t$. Note $\Pi_t$ is exactly the same as in the constant interest rate case except the riskless bond price and the volatility are replaced by the stochastic part. It can be shown that when $f_t > k$ then $\Pi_t$ is in general negative for reasonable parameters, which implies that the equity premium is counter-cyclical. In addition, because $\frac{\partial \Pi_t}{\partial \mu_{Y,t}} = \frac{\partial \Pi_t}{\partial \mu_t} < 0$ and $\frac{\partial \Pi_t}{\partial t} < 0$, the credit spread is positively correlated to the equity premium. The model thus implies an equity premium that is stochastic, counter-cyclical, and positively correlated with the credit spread even when the interest rate is stochastic.
We next look at the correlation between the real interest rate and the equity price. It is easily shown that
\[
\frac{\partial s_t}{\partial r_t} = \frac{\partial s_t}{\partial \mu_Y} = kN(d_2) g(t, T) \Gamma(t, T) > 0.
\]

The equity price is thus positively correlated with the interest rate. This is consistent with the "flight to quality" phenomenon. When the state of the economy sours, \( \mu_Y \) decreases. The lower demand for the equity and higher demand for the riskless bond will cause a positive correlation between equity price and the interest rate. While this relationship has been discussed extensively (e.g., Barsky (1989)), this paper enriches the understanding by showing that both the equity premium and the credit spread will increase when "flight to quality" happens.

IV. Market risk and credit risk in nominal terms

In this section we study market risk and credit risk in nominal terms. We assume \( \mu_Y \) is a constant. From Theorem 3, Ito's lemma tells that the consumption price process is
\[
\frac{dP_t^c}{P_t^c} = \left( \mu_M - (1 - \gamma)\mu_Y + \frac{(2 - \gamma)(1 - \gamma)}{2} \sigma_Y^2 - (1 - \gamma)\sigma_Y\sigma_M \varphi_{Y,M} \right) dt + \sigma_M dZ_{Mt} - (1 - \gamma)\sigma_Y dZ_{Yt},
\]
and the expected instantaneous inflation, defined as \( E_t \left[ \frac{dY_t}{Y_t} \right] / dt \), is simply the drift rate:
\[
\pi_t = \mu_M - (1 - \gamma)\mu_Y + \frac{(2 - \gamma)(1 - \gamma)}{2} \sigma_Y^2 - (1 - \gamma)\sigma_Y\sigma_M \varphi_{Y,M}
= R_t - r_t + \sigma_M^2 - (1 - \gamma)\sigma_Y\sigma_M \varphi_{Y,M}.
\]

Ample empirical evidence (e.g., Friedman and Schwartz (1982) and Christiano (1991)) shows that monetary policy is procyclical (i.e., \( \varphi_{Y,M} > 0 \)). Note the nominal rate is independent of the production process. This is because whenever interest
rate changes in response to the production change, the expected inflation rate will decrease by exactly the same amount, leaving the nominal rate intact. In addition, Fisher separation does not hold in this setting because

$$R_t - \pi_t = \pi_t + (1 - \gamma)\sigma_Y \sigma_M \varphi_{Y,M} - \sigma_M^2.$$ (52)

We now calculate the nominal credit-risky bond price and the nominal credit spread. The risky debt $D_t$ is equal to the riskless nominal bond minus a put option. That is,

$$D_t = B_t - P_t,$$ (53)

where $P_t$ can be calculated using the following formula:

$$\frac{P_t}{P_t} = e^{-r(T-t)}E_t \left\{ \frac{u_1(\sigma_T, m_T)}{u_1(c_t, m_t)} \frac{1}{P_T} \max (K - f_TP_T^e, 0) \right\}. \quad (54)$$

**Theorem 11** (1) The value of the risky debt is

$$D_t = B_t - P_t,$$ (55)

where

$$B_t = Ke^{(\sigma + \mu_M - \sigma_M)(T-t)}$$

$$= Ke^{-R(T-t)},$$

$$P_t = Ke^{-R(T-t)}N(-d_2) - e^{-r(T-t)}P_T^e f_t N(-d_1)$$

$$d_1 = \frac{\ln (\frac{P_T^e}{K}) + (R - r^* + \frac{1}{2}\nu^2)(T - t)}{\nu\sqrt{T - t}},$$

$$d_2 = d_1 - \nu\sqrt{T - t},$$

$$\nu^2 = \sigma_M^2 - 2\gamma \varphi_{Y,M}\sigma_M\sigma_Y + \gamma^2 \sigma_Y^2. $$

and $\varphi$ is the correlation between $dz_{Y,t}$ and $dz_{M,t}$.

(2) The value of equity is

$$S_t = f_tP_T^e \left( 1 - e^{-r(T-t)} \right) + f_tP_T^e e^{-r(T-t)} N(d_1) - Ke^{-R(T-t)} N(d_2). \quad (56)$$
**Proof:** See Appendix H.

Corporate bond holders effectively have sold a put option to the equity holders. The equity holders have the right to sell the asset to the bond holders at the market price, which is subject to both production and inflation risk, or at the debt face value, which is subject to inflation risk. This suggests that the put option is in fact an exchange option which represents the expected bankruptcy cost the bond holders undertake.

A. Risk premia, inflation, and credit spread

This part studies how risk premia and their relationship behave in an economy with inflation. From the pricing formula we observe that the production and money supply stochastically drive the equity and risky bond only through their effect on the nominal asset price, which is

\[
F_t = f_t \times P^e_t
= \frac{\phi R}{r^*(1 - \phi)} Y_t^\gamma M_t.
\]  

(57)

The nominal asset price is positively driven by money supply. The effect of production on the nominal asset price, however, depends on the sign of \(\gamma\). This is because while a higher production boosts the real asset price, it also increases the demand for money and makes the unit price of consumption good less valuable. Which effect dominates depends on the magnitude of \(\gamma\). If \(\gamma > 0\), then the nominal asset price is increasing in \(Y_t\). If \(\gamma < 0\), nominal price is decreasing in \(Y_t\). When \(\gamma = 0\) the investor has a log utility as in Stulz (1986) or Bakshi and Chen (1996), and the nominal asset is independent of production shocks because the two effects offset each other.

Following Theorem 3, the annualized nominal asset premium is

\[
\sigma^2_M + \gamma \sigma_M \sigma_Y \gamma Y_M.
\]

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which depends on the sign of \( \gamma \) because, while investors require compensation for money risk, the effect of production on nominal price is determined by \( \gamma \). Empirical evidence implies that the covariance between production and the money supply is either positive or negative but sufficiently small to make the asset premium strictly positive. Similarly, the nominal equity premium is

\[
S_t^p = \left( 1 + \frac{K e^{-r(T-t)} N(d_2)}{S_t} \right) \left( \sigma_M^2 + \gamma \sigma_M \sigma_Y \phi_{Y,M} \right),
\]

which is strictly higher than asset premium due to the call feature.

Both the equity premium and credit spread are affected by production or money only through the nominal asset price. For example,

\[
\frac{\partial S_t^p}{\partial Y_t} = \frac{\partial S_t^p}{\partial F_t} \frac{\partial F_t}{\partial Y_t}.
\]

The \( \frac{\partial S_t^p}{\partial Y_t} \) part is the same as in the economy without money, which implies that the result depends on \( \frac{\partial F_t}{\partial Y_t} \). The same applies to \( \frac{\partial S_t^p}{\partial M_t} \), \( \frac{\partial F_t}{\partial Y_t} \), and \( \frac{\partial S_t^p}{\partial M_t} \). We can immediately conclude that equity premium and credit spread are positively correlated, independent of the sign of \( \gamma \). The change of money supply will cause a counter-cyclical equity premium and credit spread. The effect of production, on the other hand, depends on the sign of \( \gamma \). A positive \( \gamma \) causes a counter-cycle equity premium and credit spread, and a negative \( \gamma \) causes a procyclical pattern.

We next study how expected inflation and credit spread are related. The credit spread of corporate debt over the nominal rate is simply

\[
y_t = \frac{\ln \left( \frac{R}{2} \right)}{T-t} - R = \ln \left\{ N \left( d_2 \right) + F_t e^{(R-r)(T-t)} N \left( -d_1 / R \right) \right\} / T-t
\]

It can be shown that

\[
\frac{\partial y_t}{\partial \mu_M} = -\frac{1}{T-t} \left[ \frac{1}{R} + (T-t) \right] \frac{1}{\frac{N(d_2)}{N(-d_2)}} + 1 < 0,
\]
A higher money growth or lower money volatility will increase expected inflation but reduce credit spread, which implies that credit spread and expected inflation move in opposite directions due to monetary shocks. This prediction can explain the generous mortgage lending conditions in the 1970s when inflation was very high. The default risk was really in the short term and in the long term, due to inflation, the nominal income of the borrowers would grow large enough to pay down the loans. By contrast, mortgage borrowers got much less generous offer in the 1990s when inflation was very low.

On the other hand,

\[
\frac{\partial \pi_t}{\partial \mu_Y} < 0, \quad \frac{\partial y_t}{\partial \mu_Y} < 0 \text{ if } \gamma > 0, \quad \frac{\partial y_t}{\partial \mu_Y} > 0 \text{ if } \gamma < 0.
\]

Therefore when the shock is from production growth rate, the relationship between expected inflation and credit spread is not clear. When \( \gamma < 0 \) they are negatively related but when \( \gamma > 0 \) they are positively related. This is because, when \( \gamma > 0 \), a higher production growth reduces both expected inflation and credit risk.

To summarize, we find that the nominal equity premium is strictly higher than the asset premium, and positively correlated with the credit spread. While money supply produces a counter-cyclical equity premium, the effect of production depends on investors' risk aversion (\( \gamma < 0 \) or \( \gamma > 0 \)). In addition, monetary shocks (\( \mu_M \) or \( \sigma_M \)) cause a negative comovement of the credit spread and expected inflation. However, the change of production growth will move the credit spread and expected inflation in the same direction if \( \gamma > 0 \).

Finally, the credit spread over real interest rate \( y_t \), is

\[
y_t = \pi_t - \sigma_M^2 + (1 - \gamma)\sigma_Y\sigma_M\varphi_{\gamma,M} + y_t
\]
\[ = (R_t - r_t) + y_t. \]

We thus have effectively separated the nominal credit spread over the real interest rate into two components, the first solely due to inflation risk and the second due to the effect of production and inflation on credit risk.

V. Concluding remarks

In this paper we start from exogenous assumptions about production and money supply. Homogenous investors maximize their expected utility by holding the firm asset and real money balance. Equity and corporate debt are treated as contingent claims on the firm asset. A critical implication of the exchange economy is that the production is not affected by the firm’s capital structure. We contribute to the literature by building a bridge between the current equilibrium asset pricing models wherein the capital structure or the default risk is usually not considered, and the credit risk models which are usually of partial equilibrium in nature.

Such an approach helps to shed new light on three important issues. First, we propose a concept of dynamic capital structure irrelevance. That is, Modigliani and Miller’s (1958) capital structure irrelevance holds, but, given the current outstanding amount of debt, the capital structure changes endogenously because the production affects the value of equity and debt disproportionately. This will produce a counter-cyclical pattern for both the leverage ratio and the equity premium. Because the credit spread is also negatively driven by production, this induces a positive correlation between the equity premium and credit spread, consistent with empirical evidence. By adding a capital structure, we not only provide a new way to capture the counter-cyclical equity premium, but also are able to study the relationship between equity risk and credit risk.

The dynamic capital structure also causes trouble to empirical studies. The stock beta is very volatile and it captures the counter-cyclical pattern of the capital structure
that is not going to be explained by the market portfolio. This implies that the usual two-pass approach will yield a biased estimate of systematic risk. On the other hand, any variable that is counter-cyclical has hope to pick up the variation of systematic risk due to capital structure. These variables include equity value (i.e., size), leverage ratio, dividend yield, credit spread, and possibly, book-to-market ratio.

Second, we find that credit spreads can react differently to interest rate shocks, depending on investors' risk aversion and the maturity of debt. Under some circumstances, credit spreads for different maturities can change in opposite directions. However, for relatively long maturity, the credit spread and the interest rate are always negatively correlated. This has important implication for the risk management of corporate bond portfolios or credit derivatives.

Third, we provide one of the first models to ever study the role of inflation on credit risk. We find that, while a higher money supply reduces credit risk, the relationship between the expected inflation and the credit spread is complicated. When the expected inflation changes due to the monetary shocks, it will move with the credit spread in opposite directions. However, when the expected inflation changes due to a shock from the production growth rate, it can move with the credit spread in the same direction. This is because a higher production growth can reduce both expected inflation and default risk.

Some limitations and possible extensions are listed here for further research. First, while our two-step pricing method can value equity and corporate debt for any given capital structure, the issue of optimal capital structure might be handled as Leland (1994) and Leland and Toft (1996) do by adding the bankruptcy cost and tax benefits. Second, the simple debt structure can be extended to accommodate more flexibility and default boundaries. Any debt can be treated as a contingent claim on the asset value, as generalized in Leland (1994) and Leland and Toft (1996). The techniques that are derived in Leland (1994), Leland and Toft (1996), Longstaff and Schwartz (1995), and
Collin-Dufresne and Goldstein (2001) can be readily borrowed for the risky bond pricing. Third, the issue of default correlation can be tackled by modeling multiple firms in the economy.
A. Appendix

In this appendix we calculate the risk premium. The following Euler equations can be obtained from the optimization problem:

\[
\begin{align*}
\frac{u_1(c_t, m_t)}{u_1(c_t, m_t)} &= e^{-\rho \Delta t} E_t \left\{ u_1(c_{t+\Delta t}, m_{t+\Delta t}) + u_2(c_{t+\Delta t}, m_{t+\Delta t}) \Delta t \right\} \left( \frac{P_c}{P_c^{t+\Delta t}} \right), \quad (62) \\
\frac{u_1(c_t, m_t)}{u_1(c_t, m_t)} &= e^{-\rho \Delta t} E_t \left\{ u_1(c_{t+\Delta t}, m_{t+\Delta t}) \left( \frac{f_{t+\Delta t} + Y_{t+\Delta t} \Delta t}{f_t} \right) \right\}, \quad (63) \\
\frac{u_1(c_t, m_t)}{u_1(c_t, m_t)} &= e^{-\rho \Delta t} E_t \left\{ u_1(c_{t+\Delta t}, m_{t+\Delta t}) \left( 1 + r_t \Delta t \right) \right\}, \quad (64) \\
\frac{u_1(c_t, m_t)}{u_1(c_t, m_t)} &= e^{-\rho \Delta t} E_t \left\{ u_1(c_{t+\Delta t}, m_{t+\Delta t}) \left( 1 + R_t \Delta t \right) \left( \frac{P_c}{P_c^{t+\Delta t}} \right) \right\}, \quad (65) \\
\frac{u_1(c_t, m_t)}{u_1(c_t, m_t)} &= e^{-\rho \Delta t} E_t \left\{ u_1(c_{t+\Delta t}, m_{t+\Delta t}) \frac{P_{i+\Delta t}}{P_{i_t}} \right\}, \quad i = 1, ..., N, \quad (66)
\end{align*}
\]

where \( f_t = \frac{R_t}{P_t} \) is the real asset price, and \( p_{i_t} = \frac{P_{i_t}}{P_t} \), \( i = 1, ..., N \) are the real prices of the financial securities. Note the following transversality conditions must be satisfied:

\[
\begin{align*}
&\frac{u_1(c_T, m_T)}{u_1(c_t, m_t)} \frac{P_{i_t}}{P_T} \hspace{1cm} \rightarrow 0, \quad \text{as} \quad T \rightarrow \infty, \quad (67) \\
&\frac{u_1(c_T, m_T)}{u_1(c_t, m_t)} \frac{1}{P_T} \hspace{1cm} \rightarrow 0, \quad \text{as} \quad T \rightarrow \infty. \quad (68)
\end{align*}
\]

Subtracting the fifth Euler equation by the third equation yields:

\[
E_t \left\{ \frac{u_1(c_{t+\Delta t}, m_{t+\Delta t})}{u_1(c_t, m_t)} \left( \frac{\Delta P_{i_t}}{P_{i_t}} - r_t \Delta t \right) \right\} = 0. \quad (69)
\]

We take Taylor expansion on \( u_1(c_{t+\Delta t}, m_{t+\Delta t}) \) and in the limit obtain

\[
E_t \left( \frac{dP_{i_t}}{P_{i_t}} - r_t dt \right) = - \frac{u_{11}(c_t, m_t)}{P_{i_t} u_1(c_t, m_t)} Cov \left[ dY_t, dP_{i_t} \right] - \frac{u_{12}(c_t, m_t)}{P_{i_t} u_1(c_t, m_t)} Cov \left[ dm_t, dP_{i_t} \right]. \quad (70)
\]

Similarly subtracting the fifth Euler equation by the third equation yields
\[ E_t \left\{ \frac{u_1(c_{t+\Delta t}, m_{t+\Delta t})}{u_1(c_t, m_t)} \left( \frac{P^c_t}{P_{t+\Delta t}^c} \right) \left( \frac{\Delta P_{t,t}}{P_{t,t}} - R_t \Delta t \right) \right\} = 0. \]  

(71)

This can be rewritten as

\[ E_t \left( \frac{\Delta P_{t,t}}{P_{t,t}} - R_t \Delta t \right) E_t \left( \frac{u_1(c_{t+\Delta t}, m_{t+\Delta t})}{u_1(c_t, m_t)} \left( \frac{P^c_t}{P_{t+\Delta t}^c} \right) \right) = -C_{\sigma u} \left\{ \frac{u_1(c_{t+\Delta t}, m_{t+\Delta t})}{u_1(c_t, m_t)} \left( \frac{P^c_t}{P_{t+\Delta t}^c} \right) \right\} . \]  

(72)

Take Taylor expansion on \( u_1(c_{t+\Delta t}, m_{t+\Delta t}) \) and \( \frac{1}{P_{t+\Delta t}} \), rearrange, and in the continuous time limit we obtain the nominal spread equation.

QED.

**B. Appendix**

In this appendix we prove Theorem 3. The real and nominal interest rate can be calculate directly from Theorem 1. For the consumption price:

\[ \frac{1}{P_t^c} = E_t \int_t^\infty e^{-\rho(s-t)} \frac{w_2(c_s, m_s)}{u_1(c_t, m_t)} \frac{1}{P_s^c} ds \]

\[ = \frac{1-\phi}{\phi Y_t^{\gamma-1}} E_t \int_t^\infty e^{-\rho(s-t)} M_s^{\beta-1} (P_s^c)^{-\beta} ds. \]  

(73)

To solve the above equation analytically, assume \( \beta = 0 \), in which case

\[ \frac{1}{P_t^c} = \frac{1-\phi}{\phi Y_t^{\gamma-1}} \int_t^\infty e^{-\rho(s-t)} E_t \left( M_s^{-1} \right) ds \]

\[ = \frac{1-\phi}{\phi} \left[ \frac{1}{\rho + \mu_M - \sigma_M^2} \right] Y_t^{1-\gamma} \frac{1}{M_t}. \]  

(74)

QED.
C. Appendix

In this appendix we derive Theorem 6. The credit spread is

\[ y_t = \frac{\ln \left( \frac{K}{X_t} \right)}{T - t}, \]  

(75)

where

\[ X_t = K N(d_2) + \frac{F_t}{e^{r(T-t)}} e^{r(T-t)} N(-d_1^*). \]

So

\[ \text{Sign} \left( \frac{\partial y_t}{\partial x} \right) = -\text{Sign} \left( \frac{\partial X_t}{\partial x} \right). \]

We obtain

\[ \frac{\partial X_t}{\partial \mu} = e^{r(T-t)} N(-d_1^*) \frac{F_t}{e^{-r(T-t)}} (T-t) + \frac{\gamma}{r^*}. \]  

(76)

So if \( \gamma > 0 \) then \( \frac{\partial X_t}{\partial \mu} > 0 \) and \( \frac{\partial X_t}{\partial \mu} < 0 \). If \( \gamma < 0 \), then for short maturity \( \frac{\partial X_t}{\partial \mu} < 0 \) and 
\( \frac{\partial X_t}{\partial \mu} > 0 \). For large maturity \( \frac{\partial X_t}{\partial \mu} > 0 \) and \( \frac{\partial X_t}{\partial \mu} < 0 \).

\[ \frac{\partial X_t}{\partial \sigma} = -KN'(d_2^*) \sqrt{T-t} \]

\[ + e^{r(T-t)} N(-d_1^*) W_t \left[ (T-t)[-(1-\gamma)] - \frac{1}{r^*} \gamma(1-\gamma) \sigma \right]. \]  

(77)

So if \( \gamma > 0 \) then \( \frac{\partial X_t}{\partial \sigma} < 0 \) and \( \frac{\partial X_t}{\partial \mu} > 0 \). If \( \gamma < 0 \), then for short maturity \( \frac{\partial X_t}{\partial \sigma} > 0 \) and 
\( \frac{\partial X_t}{\partial \sigma} < 0 \) if \( F_t > K \), and the sign is not clear if \( F_t < K \). For large maturity \( \frac{\partial X_t}{\partial \mu} < 0 \) and 
\( \frac{\partial X_t}{\partial \mu} > 0 \).

D. Appendix

In this appendix we calculate the following moments for later use. If we have the 
following processes:

\[ dX_t = \mu_t dt + \sigma_1 dZ_{1,t}, \]  

(78)

\[ d\mu_t = \kappa(\alpha - \mu_t) dt + \sigma_2 dZ_{2,t}, \]  

(79)

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and $E_t [dZ_1, dZ_2] = \varphi_{1,2}$, then

$$
\mu_s = \alpha + e^{-\kappa(s-t)}(\mu_t - \alpha) + e^{-\kappa s} \sigma_2 \int_t^s e^{r\tau} dZ_\tau. \tag{80}
$$

$$
X_s = X_t + \int_t^s \mu_\tau d\tau + \int_t^s \sigma_1 dZ_\tau. \tag{81}
$$

This gives

$$
E [\mu_s | F_t] = \alpha + e^{-\kappa(s-t)}(\mu_t - \alpha), \tag{82}
$$

$$
VAR[\mu_s | F_t] = \frac{\sigma_2^2}{2\kappa} \left( 1 - e^{-2\kappa(s-t)} \right), \tag{83}
$$

$$
E [X_s | F_t] = X_t + \alpha(s-t) + \frac{(\mu_t - \alpha)}{\kappa} \left[ 1 - e^{-\kappa(s-t)} \right], \tag{84}
$$

$$
VAR[X_s | F_t] = \sigma_1^2(s-t) + 2\varphi_{1,2}\sigma_1\sigma_2 \int_t^s \frac{(1 - e^{-\kappa(s-\tau)})}{\kappa} d\tau \tag{85}
$$

$$
+ \sigma_2^2 \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right)^2 d\tau. \tag{86}
$$

$$
COV[X_s, \mu_s | F_t] = \varphi_{1,2} \sigma_1 \sigma_2 \Gamma(t, s) + \frac{\sigma_2^2}{\kappa} \int_t^s (1 - e^{-\kappa(s-\tau)}) e^{-\kappa(s-\tau)} d\tau, \tag{87}
$$

where

$$
\Gamma(t, s) = \frac{(1 - e^{-\kappa(s-t)})}{\kappa}. \tag{88}
$$

E. Appendix

In this appendix we derive the real asset price when the interest rate is stochastic.

$$
Y_t = \int_t^\infty e^{-\rho(s-t)} E_t \left[ \left( \frac{Y_s}{Y_t} \right)^\gamma \right] ds
$$

$$
= Y_t \int_t^\infty e^{-\rho(s-t)} e^{-\gamma \ln(Y_t)} E_t (e^{\gamma \ln(Y_s)}) ds. \tag{89}
$$

Let $X_t = \ln(Y_t)$. Then $X_t$ and $\mu_{Y_t}$ are joint normal and from the previous appendix

$$
E_t[X_s] = \ln(Y_t) + \left( \alpha - \frac{\sigma_Y^2}{2} \right) (T - t) + (\mu_{Y_t} - \alpha) B(t, s) \tag{90}
$$
\[ V_{ar_t}[X_s] = \sigma_Y^2 (s-t) + 2\varphi_Y \sigma_Y \sigma_\mu \int_t^s \Gamma(\tau, s) d\tau + \sigma_\mu^2 \int_t^s \Gamma(\tau, s)^2 d\tau. \] (91)

So

\[ f_t = Y_t \int_t^\infty e^{-\rho(s-t)} e^{-\gamma \ln(Y_t) + \gamma E[X_s] + \frac{1}{2} \gamma^2 V_{ar}[X_s]} ds \]

\[ = Y_t \int_t^\infty \exp \left(- (s-t) \left( \rho - \gamma \alpha + \gamma (1-\gamma) \frac{\sigma_Y^2}{2} \right) \right) \]

\[ \times \exp \left( \gamma (\mu_{Y,t} - \alpha) \Gamma(t, s) + \gamma^2 \sigma_Y \sigma_\mu \int_t^s \Gamma(\tau, s) d\tau + \frac{1}{2} \gamma^2 \sigma_\mu^2 \int_t^s \Gamma(\tau, s)^2 d\tau \right) \] (92)

QED.

F. Appendix

In this appendix we calculate the price of the riskless bond.

\[ G(t, T) = e^{-\rho(T-t)} E_t \left[ \left( \frac{Y_T}{Y_t} \right)^{\tau-1} \right] \]

\[ = e^{-\rho(T-t)} Y_t^{1-\gamma} E_t \left[ e^{(\gamma-1) \ln Y_t} \right] \]

\[ = e^{-\rho(T-t)} Y_t^{1-\gamma} e^{(\gamma-1) E[X_T] + \frac{1}{2} (1-\gamma)^2 V_{ar}[X_T]}, \] (93)

where \( X_T \) is as in Appendix D. Using Appendix D leads to Theorem 8. QED.

G. Appendix

In this appendix we calculate the equity and risky bond price when interest rate is stochastic.

\[ s_t = E_t \int_t^T e^{-\rho(s-t)} \left( \frac{Y_s}{Y_t} \right)^{\gamma-1} Y_s d\tau + e^{-\rho(T-t)} E_t \left[ \left( \frac{f_T}{f_t} \right)^{-1} \max \left( f_T - k, 0 \right) \right] \]

\[ = f_t \left( 1 - e^{-\rho(T-t)} \right) + e^{-\rho(T-t)} f_t E_t \left[ \max \left[ 1 - \frac{k}{f_T}, 0 \right] \right]. \] (94)
If we denote $X_T \equiv \ln (f_T)$, then from Appendix D we know $X_T$ is normally distributed with

$$
E[X_T | F_t] = \ln f_t + (\alpha - \frac{\sigma^2}{2})(T-t) + (\mu_t - \alpha)\Gamma(t, T),
$$

(95)

$$
VAR[X_T | F_t] = \sigma_Y^2(T-t) + 2\varphi_{Y, t}\sigma_Y\sigma_\mu \int_t^T \Gamma(t, \tau)d\tau
+ \sigma_\mu^2 \int_t^T \Gamma(t, \tau)^2d\tau.
$$

(96)

So

$$
s_t = f_t (1 - e^{-\rho(T-t)}) + e^{-\rho(T-t)} f_t \mathbb{E}_t \left[ \max \left[ 1 - \frac{k}{f_T}, 0 \right] \right]
= f_t (1 - e^{-\rho(T-t)}) + e^{-\rho(T-t)} f_t \mathbb{E}_t \left[ \max \left[ 1 - ke^{-X_T}, 0 \right] \right]
= f_t (1 - e^{-\rho(T-t)}) + e^{-\rho(T-t)} \int_{\ln K}^{\infty} (1 - ke^{-X_T}) \frac{1}{\sqrt{2\pi\text{VAR}(X_T)}} e^{-\frac{(X_T - \mathbb{E}(X_T))^2}{2\text{VAR}(X_T)}} dX_T
= f_t (1 - e^{-\rho(T-t)}) + f_t e^{-\rho(T-t)} N (d_1) - kg(t, T) N (d_2),
$$

(97)

where

$$
d_1 = \frac{\ln (\frac{f_t}{K}) - \rho(T-t) - \ln g(t, T) + \frac{1}{2}V(t, T)^2}{V(t, T)},
$$

(98)

$$
d_2 = \frac{\ln (\frac{f_t}{K}) - \rho(T-t) - \ln g(t, T) - \frac{1}{2}V(t, T)^2}{V(t, T)},
$$

(99)

$$
V(t, s) = \sqrt{\sigma_Y^2(s-t) + 2\varphi_{Y, t}\sigma_Y\sigma_\mu \int_t^s \Gamma(t, \tau)d\tau + \sigma_\mu^2 \int_t^s \Gamma(t, \tau)^2d\tau}.
$$

(100)

Following the same procedure will give us the equilibrium risky bond price:

$$
d(t, T) = g(t, T) + f_t e^{-\rho(T-t)} N (-d_1) - kg(t, T) (-d_2).
$$

(101)

QED.

**H. Appendix**

In this appendix we prove Theorem 11. The value of the risky debt is calculated as

$$
\frac{D_t}{P_t^c} = \mathbb{E}_t \left\{ \frac{u_1(c_t, m_s)}{u_1(c_t, m_s)} \frac{1}{P_T} [K - \max (K - f_T P_T, 0)] \right\}.
$$

(102)
So

\[
D_t = P_t^P K E_t \left\{ \frac{u_1(c_s, m_s)}{u_1(c_t, m_t)} \frac{1}{P_t^P} \right\} - P_t^P \left\{ \frac{u_1(c_s, m_s)}{u_1(c_t, m_t)} \frac{1}{P_t^P} \max(K - f_T P_t^P, 0) \right\}
\]

= \text{B}(t, T) - P(t, T), \quad (103)

where \(B(t, T)\) is the nominal debt value and \(P(t, T)\) is the nominal price of a put on the nominal asset value with exercise price \(K\).

So

\[
D_t = Ke^{-r(T-t)} - e^{-\alpha(T-t)} K M_t E_t \{\max(x_1T - x_2T, 0)\}, \quad (104)
\]

where \(x_1 = \frac{B(t, T)}{M_T}, x_2 = \frac{\partial\Phi}{1-\Phi} \frac{1}{r} (Y_T)^\gamma\). Ito’s lemma shows that

\[
\frac{dx_1}{x_{1t}} = (-\mu_M + \sigma_M^2) dt - \sigma_M dZ_{M,t} \tag{105}
\]

\[
\frac{dx_2}{x_{2t}} = \left( \gamma \mu_Y + \frac{1}{2} \gamma (\gamma - 1) \sigma_Y^2 \right) dt + \gamma \sigma_Y dZ_{Y,t}. \tag{106}
\]

\(E_t \{\max(x_1T - x_2T, 0)\}\) is thus the expected future payoff of an exchange option, with both \(x_1T\) and \(x_2T\) following geometric Brownian motion. By double integration it can be shown that (see, for example, Zhang (1998), Page 366)

\[
E_t \{\max(x_1T - x_2T, 0)\} = x_1 e^{(-\mu_M + \sigma_M^2)(T-t)} N(d_1) - x_2 e^{(\gamma \mu_Y + \frac{1}{2} \gamma (\gamma - 1) \sigma_Y^2)(T-t)} N(d_2)
\]

\[
= x_1 e^{-(r-\rho)(T-t)} N(d_1) - x_2 e^{-(r-\rho)(T-t)} N(d_2), \tag{107}
\]

where

\[
d_1 = \frac{\ln \left( \frac{x_1}{x_2} \right) + (R - r^* + \frac{1}{2} \nu^2)(T-t)}{\nu \sqrt{T-t}},
\]

\[
d_2 = d_1 + \nu \sqrt{T-t},
\]

\[
\nu^2 = \sigma_M^2 - 2 \gamma \rho \sigma_M \sigma_Y + \gamma^2 \sigma_Y^2.
\]

Substitute this back to the above equation and we obtain Theorem 11. The equity can be derived in the same way. QED.

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Production  

Money supply  

Investors make decision  

asset  

equity  

debt  

Interest rate  

Equity premium  

Credit spread  

inflation  

Note: The dotted lines imply correlation.

Figure 1: Flow chart of the paper.
Figure 2: Term structure of the equity premium

Parameter values are: $Y_t = 1, \mu = 0.03, \sigma = 0.1, \rho = 0.04, \frac{F e^{-\gamma (T-t)}}{K e^{-\mu (T-t)}} = 1.25, \gamma = -3$.

While the asset premium is 4 percent, the equity premium is a function of the time to maturity of debt.
Figure 3: Equity premium as a function of production

Parameter values are: $Y_t = 1, \mu = 0.03, \sigma = 0.1, \rho = 0.04, \frac{\delta-r}{\rho} = 1.25, \gamma = -3$.

While the asset premium is 4 percent, the equity premium is a function of the production. When production is higher than 1.552, which corresponds to an equity/asset ratio of 22 percent, the equity is a negative function of production.
Illiquidity and expected return of Treasury Securities

Long Chen 9

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ABSTRACT

We develop one of the first theoretical models to study how the illiquidity of Treasury securities affects their return. The tradeoff among price, bid-ask spread, brokerage fee, maturity, and expected investment horizon is presented. We are able to show that (1) The (ask and bid) price is a decreasing and convex function of the bid-ask spread. Alternatively, the expected return is an increasing and concave function of the spread; (2) An investor with longer expected holding period will require a lower return from a given discount bond than an investor with shorter expected holding period does; (3) The clientele effect: an investor with longer expected holding period is more willing to hold bonds with larger bid-ask spread, holding everything else constant; and (4) The yield spread of bonds with different bid-ask spreads is a decreasing and convex function of the time to maturity. In particular, the yield spread is a linear function of the reciprocal of the time to maturity. These results are supported by empirical work for fixed income securities and are consistent with similar theoretical and empirical results for common stocks.
I. Introduction to the second essay

Although the relationship between the liquidity of common stocks and their returns has been studied extensively in the past decade, similar theoretical analysis for fixed income securities has rarely, if ever, been done. This paper intends to narrow this gap by studying how illiquidity affects the price and return of fixed income securities.

In their ground-breaking equilibrium model, Amihud and Mendelson (1986) show that the return of common stocks is a decreasing and concave function of the bid-ask spreads, and the portfolios that are expected to be held for a longer time will have higher bid-ask spreads. They further provide empirical evidence of the negative and concave return-spread relationship. One major assumption in the paper is that the stock prices are constant over time. It is thus yet to see whether the results can be applied to fixed income securities, where the prices are clearly a function of time. In addition, no closed form solution is obtained.

Kane (1994) adds particular assumptions to Amihud and Mendelson’s model (1986) to obtain a closed form solution. In a Markovian steady state equilibrium model, Aiyaguri and Gertler (1991) show a negative return-spread relationship. Again, one of their major assumptions is that the asset prices are constant over time. Vayanos (1999) uses an overlapping general equilibrium model to readdress the issue. He shows that, when investors can adjust their expected holding horizon, the relationship between transaction costs and price is not clear anymore. Yu (1998) relaxes some assumptions in previous studies, including constant prices and no short selling, and obtains similar return-spread relationship.

On the empirical side, Amihud and Mendelson’s (1986) proposition that the bid-ask spread will increase with the expected holding period is verified by Atkins and Dyl (1997), controlling firm size and return volatilities. In particular, the expected holding period is approximated by the stock shares outstanding divided by the trading volume.
In addition, they find that the causality between the bid-ask spread and the expected holding period is mutual.

Amihud and Mendelson (1991) show that the negative and concave relationship between return and spread also holds for US treasury securities, though no theoretical model is attempted. Similar result is empirically supported again by Elton and Green (1998). Interestingly, Amihud and Mendelson (1991) find that the yield spread of bonds with different bid-ask spreads is a decreasing and convex function of time to maturity. Kamara (1994) further investigate the determinants of the liquidity spreads between Treasury securities.

To summarize, the trade-off between transaction costs and expected return is well studied, both theoretically and empirically, for equities. Similar relationship is also widely tested empirically for Treasury bonds. However, the same theoretical attempt has rarely been made for Treasury bonds. Because fixed income securities are very different from equities, it is important to develop a theoretical model to investigate (1) whether similar results in Amihud and Mendelson (1986) hold for Treasury bonds; and (2) what are the features specific to fixed income securities.

The study on the illiquidity of fixed income securities is not only interesting for intellectual reasons. It is common for many central banks to routinely update their bond holding to the most liquid benchmark. A thorough understanding of the liquidity spreads for debts with different maturities and ages will help the central banks to control debt service costs. In addition, the liquidity spreads can be explored as arbitrage opportunities. According to the Economist, one of the major strategies of Long Term Capital Management was to short on-the-run US Treasury bonds and buy off-the-run Treasury bonds and bet that the return would converge in the short run.\textsuperscript{10}

\textsuperscript{10}For example, in US treasury market. 5 year note is issued quarterly in the second month of each quarter. Once UST 4.25 11/03 was launched in November, it replaced UST 5.25 8/03 as the on-the-run 5 year benchmark (also called hot run or current). The latter immediately became off-the-run 5 year benchmark. Usually the return of on-the-run treasury securities will be lower than that of off-the-run treasury securities.
In this paper we focus on the liquidity spread between two discount Treasury bonds, identical in every aspect except for bid-ask spreads. In fact, it is not clear that bid-ask spreads should be priced. One example can be seen in Figure 4, where one of the bonds has higher bid-ask spread than the other. Because there are more participants on both the buy and sell sides for the more liquid bond, its proportional bid-ask spread is smaller. However, if the mid-points of the bid and ask prices are the same for both bonds, then their expected return should still be the same. The bid-ask spread does not appear to be priced. We show in the model that, so long as investors have positive probability to sell the bonds before maturity, the mid-prices of the two bonds will not be the same (i.e., expected transaction costs will be priced).

Insert Figure 4 here.

We develop a simple model where the investors face the trade-off among price, bid-ask spread, brokerage fee, maturity, and expected investment horizon. We are able to show that (1) The (ask and bid) price is a decreasing and convex function of the bid-ask spread. Alternatively, the expected return is an increasing and concave function of the spread; (2) An investor with a longer expected holding period will require a lower return from a given discount bond than an investor with a shorter expected holding period does; (3) The clientele effect: an investor with longer expected holding period is more willing to hold bonds with larger bid-ask spread, holding everything else constant; and (4) The yield spread of bonds with different bid-ask spreads is a decreasing and convex function of the time to maturity. In particular, the yield spread is a linear function of the reciprocal of the time to maturity. Such a pattern is observed by Amihud and Mendelson (1991), who find that the correlation between the yield spread and the reciprocal of the time to maturity is as high as 0.97. But no theoretical reason is provided in their paper. In our model we prove the proposition with an elegant expression.

The rest of the paper is organized as follows: Section I derives the bond prices in
equilibrium. Section II proves the propositions we have shown. A conclusion is provided in Section III.

II. The bond prices in equilibrium

In this section we derive the bond prices in equilibrium. This is achieved in three steps. First, we study how investors choose among discount bonds with the same maturity but different relative spreads (defined as the ratio of the bid-ask spread to the ask price). Second, we obtain how investors choose among discount bonds with different relative spreads and maturity. Finally, the first two steps help us to develop the equilibrium market prices.

A. Trade-off between bonds with the same maturity

We assume there are \( n + 1 \) competitively traded discount bonds, \( P_0, P_1, P_2, \ldots, P_n \), with the same face value \( $1 \) and time to maturity \( T \). The ask prices are \( a_0, a_1, \ldots, a_n \) respectively, which are functions of time. The relative bid-ask spread of bond \( P_i \) is \( \alpha_i \) and its bid price is \( b_i = a_i \times (1 - \alpha_i) \). Bond \( P_0 \) is perfectly liquid and has zero relative spread, i.e., \( \alpha_0 = 0 \). In addition, we assume that \( a_i = k_{i,j} \times a_j \), where \( k_{i,j} \) is a constant. This implies that the bond prices may change with time but they will change in a parallel fashion.

Now there is an investor considering buying one of the discount bonds at current time \( 0 \) on the market. He knows that at some known time \( t (t < T) \) in the future there is probability \( p \) he will continue to hold the bond until maturity and with probability \( 1 - p \) he will sell the bond and use the cash elsewhere. Such an assumption is reasonable as the expected investment horizon is usually shorter than the maturity of bonds. The uncertainty is introduced to capture the fact that the investor is not quite sure about the investment horizon. We also assume that all investors are risk-neutral and only care
about the expected payoff of their investment.

**Proposition 12** For an investor with holding probability \( p \), he will price bond \( i \) as follows:

\[
\alpha_i = \frac{\alpha_0}{1 - \alpha_i} \times \frac{k_{i,0} - p}{(1 - p)}.
\]  

(108)

or alternatively,

\[
k_{i,0} = \frac{1}{1 + \alpha_i (\frac{1}{p} - 1)}.
\]  

(109)

**Proof:** See Appendix A.

The left hand side of Equation (109) is the relative price, i.e., the price ratio of the bond with spread \( \alpha_i \) to the bond with zero spread. A higher bid-ask spread (i.e., a more illiquid bond) will yield a lower price. In addition, \( p \) can be regarded as the variable that represents the expected holding period. Intuitively, an investor with a longer expected holding period will be more tolerant to potential transaction costs and is willing to pay a relatively higher price. The equation describes how the investor evaluates different discount bonds, all identical except different relative spread \( \alpha_i \), relative to the perfectly liquid bond price. It is a trade-off between the potential transaction cost \( (\alpha_i) \), and the expected holding period \( (p) \). Note the prices in Proposition 1 are the ones that the particular investor is willing to pay for the bonds. They are not the market prices. As we will show later, it is the marginal investors that will determine the market prices.

One major insight in Vayanos (1998) is that when the transaction cost increases, the investors can extend the expected holding period to offset the potential transaction cost. This flavor is also clear in Equation (109). When the potential transaction cost \( \alpha_i \) increases, the price of the security should decrease, holding everything else constant. However, if the expected holding period is a function of \( \alpha_i \), the investor could adjust his expected investment horizon by increasing \( p \). These two effects work in opposite directions. The net effect on the required bond price depends on which effect will dominate.
Proposition 1 hinges on two important assumptions on the investor’s investment pattern. First, the investor knows that at time \( t \) he needs to make the decision, but for now he only knows the probability with which he will hold the bond at time \( t \). This combination of fixed decision time and randomness in the decision itself is a simple proxy to the random holding period. It would be desirable if we could let decision time \( t \) follow some distribution. However, this would let the expected future value be dependent on the path of the ask price throughout time. Unless we assume a very particular functional form of the ask price as a function of time, the question is difficult to solve. The same problem exists for most of the previous models for equities, where the asset prices are assumed to be constants over time instead.\(^{11}\)

Second, the investor only needs cash on two time points. This assumption is not as restrictive as it appears. In fact, any investor, no matter how many time points he want to choose, can be divided into the two-point-type investors we just described. For example, suppose there is an investor who wants to invest in Treasury bonds at current time \( t_0 \). He knows that with probability \( p_1 \) he needs cash at time \( t_1 \), with probability \( p_2 \) he needs cash at \( t_2 \), and with probability \( p_3 \) he needs cash at \( t_3 \). The investor can be separated into two mini-investors: the first investor with probability \( \frac{p_1}{p_1 + p_2} \) will cash out at \( t_1 \) and with probability \( \frac{p_2}{p_1 + p_2} \) will cash out at \( t_2 \); The second investor with probability \( \frac{p_3}{p_2 + p_3} \) will cash out at \( t_2 \) and with probability \( \frac{p_2}{p_2 + p_3} \) will cash out at \( t_3 \). Thus the original investor can be treated as a group of two-point-type mini investors and Equation (109) holds for these mini investors.

As we will show shortly, for any bond with fixed maturity, it is the marginal investor

\(^{11}\)For example, in Aminihud and Mendelson (1986), investor \( i \) maximizes the following expected payoff:

\[
\mathbb{E}_{T_i} \left\{ \int_{t_0}^{T_i} e^{-\rho u} \left[ \sum_{j=0}^{N} x_{ij} d_j \right] du \right\} + \mathbb{E}_{T_i} e^{-\rho T_i} \sum_{j=0}^{N} x_{ij} V_j (1 - S_j).
\]

where \( T_i \) is the expected holding period which follows exponential distribution; \( \rho \) is the discount rate; \( x_{ij} \) is the units of security \( j \) held by investor \( i \); \( d_j \) is the dividend payment per instant time; \( V_j \) is the ask price; and \( S_j \) is the relative bid-ask spread. In this case the expected holding period is random, but the ask price and dividend payment are constant over time. However, in the case of fixed income securities, the ask price will surely be a function of time and thus a function of \( T_i \). Unless we specify the functional form, the expression is not solvable.
who has the shortest maturity will determine the price. Because two-point-type investors tend to have shorter maturity, they will be the dominant type and we can omit investors with more sophisticated investment patterns. From now on, we assume that the world is populated only with two-point-type investors.

B. The trade-off between bonds with different maturities

One special feature of most of the fixed income securities is the finite maturity. An investor can avoid the potential costs of bid-ask spreads by buying bonds with short maturities, and reinvest the proceeds again when the bonds are retired. By doing this he has to pay additional brokerage fee (i.e., commission). Alternatively, he can reduce the number of transactions by buying bonds with long maturity, and only sell the bonds when cash is needed. Therefore, by choosing bonds with different maturity, an investor faces the trade-off between two types of transaction costs.

From now on we assume that the term structure in the economy is deterministic. The only uncertainty in the economy is the investors' investment horizon. This will make the trade-off between different maturity as the one between two types of transaction costs. Denote $a(0, t_i, \alpha_j)$ as the ask price of a bond with bid-ask spread $\alpha_j$ at time 0 that matures at $t_i$. By definition its bid price is $b(0, t_i, \alpha_j) = a(0, t_i, \alpha_j) \times (1 - \alpha_j)$.

An investor is considering buying one of the discount bonds at current time 0 on the market. He knows that he will need the money with probability $1 - p$ at some known time $t_i$ and with probability $p$ at a later time $t_k$ ($t_i < t_k$). This implies that his expected holding period is within $[t_i, t_k]$. There is a constant brokerage fee $B$ per trade whenever a transaction is made. The investor thus faces the following trade-off:

Strategy #3: Buy bond maturing at $t_i$ and with probability $p$ he will reinvest the money between $[t_i, t_k]$.

Strategy #4: Buy bond maturing at $t_k$ and with probability $1 - p$ he will cash out
If Strategy \#3 is adopted, a short term bond will be purchased at time zero. If Strategy \#4 is adopted, a long term bond will be purchased. In addition to the two strategies, the investor could buy bonds maturing before \( t_t \) and roll over repetitively. It is clear that Strategy \#3 will dominate any strategy of this sort because it requires rolling over at most once and thus saves the brokerage cost. The investor could also buy bonds maturing after \( t_k \). With probability \( 1 - p \) he will cash out at \( t_t \) and with probability \( p \) he will cash out at \( t_k \). This kind of strategy is dominated by Strategy \#4 because with probability 1 the former will incur bid-ask spread cost while the later will incur bid-ask spread only with probability \( 1 - p \). Therefore we concentrate on the trade-off between Strategy \#3 and \#4.

**Proposition 13** The investor will be indifferent to the following equation:

\[
1 - p + p \times (1 - B) \times \frac{1}{k_{m,0}} = \frac{1}{1 + \alpha_j(\frac{1}{p} - 1)} - B(1 - p),
\]

where \( m \) is the bond that the investor buys at \( t_t \) following Strategy \#3 (i.e., rolling over strategy), and \( j \) is the bond that the investor buys at time 0 following Strategy \#3.

*Proof: See Appendix B.*

The left hand side of Equation (110) is the expected payoff of Strategy \#3 at \( t_t \). Let's assume that the investor has initial fund worth \( a(0, t_t, \alpha_j) \), where bond \( j \) is the one that the investor intends to buy with Strategy \#3. Intuitively, if the investor buys the bond, with probability \( 1 - p \) the bond is worth \$1 \) to him at \( t_t \). With probability \( p \), however, he has to reinvest the money into bond \( m \), which matures at \( t_k \). He can only spend \( 1 - B \) on the bond due to the brokerage cost. He can thus buy \( (1 - B) \times \frac{1}{a(t_k, t_k, a_m)} \) that is worth \( (1 - B) \times \frac{a(t_k, t_k, a_m)}{a(t_k, t_k, a_m)} = (1 - B) \times \frac{1}{k_{m,0}} \) to him. Similarly, the right hand side of Equation (110) is the expected payoff of Strategy \#4 at \( t_t \). With the initial fund worth \( a(0, t_t, \alpha_j) \), he will buy a bond that matures at \( t_k \). The value of this strategy grows at.
the risk-free interest rate. So its value at \( t_i \) is 
\[ \frac{a_{(0,t_i)}}{a(0,t_i,0)} = \frac{1}{1+\alpha/(p-1)} \]
With probability \( 1-p \), however, the brokerage cost of \( B \) will incur.

**Proposition 14** The expected payoff of the rolling over strategy is decreasing in the
rolling over cost (brokerage fee) and the expected holding period, and increasing in bid-ask spread.

**Proof:**

Take the first derivative of the left hand side of Equation (110) with respect to \( B \), \( p \),
and \( \alpha_m \) respectively. The first two are negative and the last is positive. QED.

Intuitively, the cost of the rolling over strategy is increasing in the brokerage fee
because it is the major transaction cost for this strategy. In addition, we have used \( p \) as
the proxy to the expected holding period. The longer the expected holding period, the
more likely that the investor needs to roll over, and thus more like the brokerage cost
will incur. However, bid-ask spread actually benefits the rolling over strategy, which
always involves holding the bonds until maturity. As bonds with larger bid-ask spreads
tend to be cheaper, the expected payoff the strategy is increasing in the bond's bid-ask
spread.\(^{12} \) On the other hand, the expected payoff of Strategy \#4 is increasing in the
expected holding period but decreasing in the bid-ask spread. Obviously, investors with
longer expected holding period get benefits by purchasing the bond with long maturity.
In addition, bid-ask spread is the major burden for Strategy \#4.

To summarize, Equation (110) incorporates the trade-off between two bonds with
different maturity. It provides the whole picture of interplay among expected return,
bid-ask spread, brokerage cost, maturity, and expected holding period.

**C. Prices in equilibrium**

\(^{12}\)Because bond \( m \) and \( j \) are the ones that the investor uses in Strategy \#3, this implies that they
are the bonds with largest bid-ask spreads.
In Equation (110) the investor is indifferent between the two strategies. The left hand side, which implies the purchase of short term bond, is monotonically increasing in \( p \). The right hand side, which implies the choice of long term bond, is monotonically decreasing in \( p \). Therefore for given bid-ask spreads, there is a unique holding probability \( p^* \) to make the two strategies indifferent. All investors with \( p \in [0, p^*) \) will choose the rolling over strategy and buy bond \( a(0, t_i, \alpha_i) \) at time zero. All investors with \( p \in (p^*, 1] \) will choose to hold the long term bond \( a(0, t_k, \alpha_k) \). And investors with \( p^* \) will be indifferent between the two strategies.

The same trade-off applies to all investors choosing between \( t_i \) and \( t_k, i = 1, 2, \ldots , k-1 \). Because Equation (110) always holds for \( p^* \) for given bid-ask spreads, it turns out that the same \( p^* \) will hold for all \( i = 1, 2, \ldots , k-1 \). So all investors with \( p \in (p^*, 1] \) will choose bonds with relatively longer maturity. Let us call them the first class of investors. Similarly, all investors with \( p \in [0, p^*) \) will buy bonds with relatively shorter maturity. We call them the second class of investors. Because this class of investors will hold the bond until maturity (Strategy \#3), they can be treated the same as the first class of investors with \( p = 1 \).

We conclude that the demand for any bond comes from two classes of investors: the first class with expected holding period shorter than the bond maturity (Strategy \#4) and the second class with expected holding period longer than the bond maturity (Strategy \#3), while the second class can be treated the same as the first class with \( p = 1 \). Therefore for each bond the marginal investor is the one with \( p^* \), and Equation (109) tells us that he is willing to pay the lowest price among the investors.

The supply of the bonds comes from the investors who need cash. The dealers are in the middle and charge bid-ask spreads. We assume that the supply of bonds is competitive and can be described by a flat supply curve. This assumption is reasonable given the fact that in our model only investors needing cash will sell the bonds before maturity. We also implicitly assume that the supply of bonds is enough to meet the
As the dealers can not distinguish the investors' expected investment horizon, this implies a standard downward sloping demand curve and a flat supply curve. The equilibrium market price is determined by the marginal investors with \( p^* \), who will evaluate the bonds according to Equation (100). Investors with \( p \neq p^* \) will enjoy consumer surplus. In particular, investors with \( p \in [0, p^*) \) and \( p = 1 \) are the ones with the largest consumer surplus. The same pricing rule applies for bonds with all maturities \( t_1 < t_2 < ... < t_i < ... < t_N \), and with all bid-ask spreads. What we have just described is shown in Figure 5.

The equilibrium we get is similar to the case for equities in the sense that it is the marginal investors with the shortest investment period that determine the price of the securities. However, in our model the investors are segmented into bond markets with different maturity. This is different from the equity market, where there is only one maturity (i.e., infinity). In addition, note even though the bid-ask spreads are priced into the relative prices, the brokerage fee is not because investors that employ the rolling over strategy are not the marginal investors.

### III. Properties of the equilibrium market price

As the equilibrium market price of a bond is determined by the marginal investor with the holding probability \( p^* \), we can learn the properties of the market price by analyzing the choice of the marginal investor.

**Proposition 15** The price of a discount bond (both ask price and bid price) is a decreasing function of the bid-ask spread. Alternatively, the rate of return of a discount bond is an increasing function of the bid-ask spread.
Proof. See Appendix C.

Put in words, the price of a more liquid bond will be higher. Alternatively, because of the inverse relationship between return and price, the expected return of the more liquid bond will be lower than that of the less liquid bond. The intuition is very clear: the investor requires a higher expected return to compensate for the loss in liquidity. This result is supported by empirical evidence from Anil Hud and Mendelson (1991) and Elton and Green (1998).

Because a coupon bond is a linear combination of discount bonds, the same relationship should also apply to coupon bonds. That is, for the investor, the expected return of a more liquid coupon-bearing bond will be lower than that of a less liquid coupon-bearing bond, assuming everything else the same.

A concern that arises from Proposition 4 is the possible existence of arbitrage opportunity. Theoretically the following arbitrage portfolio is possible: short the liquid bond, invest the proceeds in the illiquid bond, and hold the portfolio until maturity. Such a portfolio will guarantee profit at zero initial cost. However this is not available in reality. The short selling of bonds can be either directly done among the dealers or through a reverse repurchase agreement on the repo market in practice. In both cases the following three disadvantages will affect the performance of the portfolio:

(1) The short selling proceeds can not be used. In a repo agreement, the proceeds are lent out to the owner of the bonds. If the shorting is done between dealers, the proceeds are in general deposited in the hands of the dealer who lends out the bonds. Thus the proceeds from shorting can not be used to finance the purchase of the cheaper bonds. As a result, a large amount of initial investment is required in this "arbitrage portfolio".

(2) There is transaction cost in short selling. In the case of repo market, the bond that needs to be borrowed (i.e., the bond with higher price) is usually on special and the
overnight interest earned is lower than normal. This is additional cost for the shorting side. Similarly, in a normal short selling contract, in addition to the shorting proceeds, the shorting party has to put up more margin in the bond lender’s hand.

(3) The shorting position has to be rolled over. The repo market becomes very thin when the maturity of the contract increases. Similarly, it is very hard to find a bond lender who does not call the bonds for a relatively long time.

The above discussion suggests that even if an arbitrage portfolio is possible, the additional cost attached might very likely wipe out any yield difference. In fact, the only arbitrage available is "speculative arbitrage", i.e., rolling over the short position and hoping the yield of the two bonds will converge quickly. Again, the risk involved in keeping the position might more than wipe out any profit from the yield spread. This explains the existence of discount bonds with the same face value and maturity, but different bid-ask spreads and yields, as empirically investigated by Amihud and Mendelson (1991) and Kamara (1994).

Treating $p$ as the proxy to expected holding period, we immediately get the following proposition:

**Proposition 16** Assume the term structure is deterministic. An investor with longer expected holding period will require a lower return from a given discount bond than the investor with shorter expected holding period does.

*Proof: see Appendix C.*

**Proposition 17** (Clientele effect). Assume the term structure is deterministic. An investor with longer expected holding period is more willing to buy bonds with larger bid-ask spread than an investor with shorter expected holding period does, holding everything else constant.

*Proof: see appendix C.*
The above two propositions are very intuitive. The potential transaction cost caused by bid-ask spread is amortized over the investor's holding period. An investor with longer expected holding period will thus require less compensation for the loss of liquidity than an investor with shorter expected holding period does (Proposition 5). This will make bonds with higher spread relatively more attractive for the investor with longer expected holding period. As a result, the bid-ask spread is positively correlated with the expected holding period (Proposition 6). As shown in Figure 5, investors with holding probability higher than the marginal investor will enjoy consumer surplus. Amihud and Mendelson (1986) call it clientele effect when it holds for equities, which is empirically supported by Arkins and Dyl's (1997). Our model predicts that it should also hold for treasury bonds. In particular, for each maturity, if we approximate the expected holding period \( H \) as

\[
H = \frac{\text{outstanding issues}}{\text{trading volume}},
\]

then the clientele effect predicts that the relative bid-ask spread is increasing in \( H \).

The lesson is that an investor should choose more illiquid bonds if he knows that with a high probability he will hold the bond for a long time. The extreme case happens when he knows the exact investment horizon, in which case \( p^* = 1 \). He should go to the market and buy the cheapest bond (i.e., the bond with the largest bid-ask spread) with the corresponding maturity.

**Proposition 18** The ask price and the bid price are convex functions of the bid-ask spread.

*Proof: see appendix D.*

Because of the inverse relationship between bond price and return, it can be shown that the expected return is a concave function of bid-ask spread, a proposition empirically supported by Amihud and Mendelson (1991). While the proof of the proposition
is similar to Yu (1998) for equities, our paper distinguishes from his in two major aspects: First, an assumption of infinite time horizon of securities and finite birth-death of investors is needed in Yu (1998). Such a model is not suitable to securities with finite maturity. Second, his proof depends crucially on the positive dividend payoff before the security is sold. Again, this excludes discount bonds, where no cash flow will incur after the purchase until the bond is sold or matures.

Proposition 4 indicates that a higher expected return is required by an investor for bonds with larger bid-ask spread to compensate for the potential loss in illiquidity. The convexity of bond price in relative spread comes from the assumption that when the investor decides not to sell the bond before maturity, discount bonds with the same maturity have the same value to him, independent of the bid-ask spreads. When the bid-ask spread increases, the investor requires higher return, but only part of the increase in the spread is regarded as potential cost and discounted in the price. Therefore the required return will not increase as fast as the bid-ask spread. This suggests a concave relationship between required return and the bid-ask spread.

**Proposition 19** The yield spread is a decreasing and convex function of the time to maturity. In particular, the yield spread is a linear function of the reciprocal of the time to maturity. That is,

\[ y_i - y_0 = \frac{1}{T} \ln \left[ 1 + \alpha_i \left( \frac{1}{p} - 1 \right) \right], \quad (111) \]

where \( y_i \) is the yield of the bond \( i \) with relative spread \( \alpha_i \), \( y_0 \) is the yield of the perfectly liquid bond with the same maturity as bond \( i \), \( T \) is the maturity of the two bonds, and \( p \) is the probability of holding the bond until maturity for the marginal investor.

**Proof:** see Appendix E.

When the holding period is not certain, an investor faces the trade-off between repeatedly buying shorter term securities with cumulative brokerage costs and buying bonds with longer maturity. However, an investor with longer expected holding period
will surely hold securities with corresponding longer maturity to minimize transaction costs. If the potential transaction cost due to bid-ask spread is amortized on the time to maturity period, we obtain that the yield spread is a decreasing and convex function of the time to maturity. The result that the yield spread is a linear function of the reciprocal of time to maturity comes from the assumption that the price ratio of two bonds with the same maturity but different bid-ask spread will remain constant through time.

Proposition 8 confirms with Amihud and Mendelson’s (1991) empirical work. They show that the yield spread is a decreasing and convex function of the time to maturity. They further observe that the yield spread seems a linear function of the reciprocal of the time to maturity. The correlation between the yield spread and the inverse of the time to maturity is as high as 0.97. Without any theoretical proof to their conjecture, they run a regression of yield spread on the inverse of the time to maturity and a coupon variable and find significant relationships. Here we provide a theoretical proof supporting their regression.

IV. Concluding remarks

Although the trade-off between illiquidity and expected return for equities has been studied extensively in the past decade, similar theoretical analysis for fixed income securities has rarely, if ever, been done. We developed a simple model to narrow this gap by studying how illiquidity can affect the return of fixed income securities.

First we develop the criteria investors use to choose bonds with different liquidity but the same maturity, and then with different maturity. With the additional assumption of the market supply and demand structure, we are able to obtain the market prices in equilibrium. An important finding is that the relative bid-ask spread instead of the commission cost will be priced in equilibrium. This is because the marginal investors who
determine the market prices usually do not use a reinvestment strategy (i.e., buying short term bonds and rolling over, which avoids bid-ask spread but has to pay commission).

We clearly show the trade-off among price, bid-ask spread, brokerage fee, maturity, and expected investment horizon. We find that the equilibrium prices are determined by some marginal investor with unique holding probability $p^*$, regardless of the maturity. We then study the properties of the market price determined by the marginal investor. These proportions are intuitive and consistent with the theoretical and empirical evidence regarding the liquidity-return relationships in the equity literature. In addition, we are able to incorporate the important properties specific for fixed income securities, such as maturity, rolling over costs, and term structure in equilibrium.
References


A. Appendix

This appendix proves Proposition 1. Consider the following two strategies for the investor:

Strategy #1: Buy one unit of $P_2$ at time 0.

Strategy #2: Buy $k$ unit of $P_1$ at time 0.

If the investor buys one unit of $P_2$, the initial cost is $a_2$. At the known future time $t < T$, his expected payoff is

$$p \times PV + (1 - p) \times a_2 \times (1 - \alpha_2)$$  \hspace{1cm} (112)

where PV is the future value of a discount bond at time $t$ paying $\$1$ at time $T$. Alternatively, the investor can buy $k$ units of $P_1$, with the same initial cost $k \times a_1 = a_2$. At the future time $t < T$, his expected payoff is

$$k \times p \times PV + k \times (1 - p) \times a_1 \times (1 - \alpha_1)$$  \hspace{1cm} (113)

To make the investor indifferent, the expected payoff of the two strategies must be equal:

$$p \times PV + (1 - p) \times a_2 \times (1 - \alpha_2) = k \times p \times PV + k \times (1 - p) \times a_1 \times (1 - \alpha_1).$$

This provides us an indifference price relationship between $a_1$ and $a_2$:

$$a_2 = \frac{1 - \alpha_1}{1 - \alpha_2} \times k \times a_1 + \frac{p \times PV}{(1 - p) \times (1 - \alpha_2)} \times (k - 1)$$  \hspace{1cm} (114)

The above equation should also hold for $P_0$, with $\alpha_0 = 0$:

$$a_1 = \frac{1}{1 - \alpha_1} \times k \times a_0 + \frac{p \times PV}{(1 - p) \times (1 - \alpha_1)} \times (k - 1)$$  \hspace{1cm} (115)

We also assume that the bond $P_0$, because of its low exposure to transaction cost, has been chosen to fit the term structure curve, and its yield will be used as the appropriate
discount rate for the treasury bond with the same maturity. This suggests that \( a_0 = PV \) at any time. Thus we can further simplify the above equation as:

\[
a_1 = \frac{1}{1 - \alpha_1} \times k_{1,0} \times a_0 + \frac{p \times a_0}{(1 - p) \times (1 - \alpha_1)} \times (k_{1,0} - 1)
\]

(116)

\[
a_1 = \frac{a_0}{1 - \alpha_1} \times \frac{k_{1,0} - p}{(1 - p)}
\]

(117)

where \( k_{i,j} \) is the ask price ratio of bonds \( P_i \) to \( P_j \). Similarly, we can obtain the relationship between \( a_i \) and \( a_0 \):

\[
a_i = \frac{a_0}{1 - \alpha_i} \times \frac{k_{i,0} - p}{(1 - p)}.
\]

(118)

Substituting \( a_i = k_{i,0} \times a_0 \) into the equation, after rearranging, we immediately get

\[
k_{i,0} = \frac{p}{1 - (1 - \alpha_i)(1 - p)} = \frac{1}{1 + \alpha_i \left( \frac{1}{p} - 1 \right)}
\]

(119)

QED.

B. Appendix

This appendix proves Proposition 2. The investor faces the following trade-off:

Strategy #3: Buy bond maturing at \( t_i \) and with probability \( p \) he will reinvest the money between \([t_i, t_k]\).

Strategy #4: Buy bond maturing at \( t_k \) and with probability \( 1 - p \) he will cash out at \( t_i \).

If the investor adopts strategy #3, let us assume he buys one unit of bond \( j \) maturing at \( t_i \). The initial cost is \( a(0, t_i, \alpha_j) \). The expected value at \( t_i \) is\(^{13}\)

\[
V_0(t_i) = (1 - p) \times 1 + p \times \frac{1 - B}{a(t_i, t_k, \alpha_m)} \times a(t_i, t_k, 0)
\]

\(^{13}\)With probability \( 1 - p \) he will get $1 and with probability \( p \) he will use the $1 to reinvest, in which case he will lose brokerage fee \( B \). But he can buy \( \frac{1}{a(t_i, t_k, \alpha_m)} \) units of bonds with relative spread \( \alpha_m \) at that time, which is worth \( \frac{1}{a(t_i, t_k, \alpha_m)} a(t_i, t_k, 0) \) for him.

82
\[ = 1 - p - p \times (1 - B) \times \frac{1}{k_{m,0}} \]  
(120)

where \( \alpha_m \) is the bid-ask spread of the bond \( a(t_i, t_k, \alpha_m) \).

What is the expected value for Strategy #4 at \( t_i \)? As in Strategy #3, the investor has initial money of \( a(0, t_i, \alpha_j) \). He buys \( x \) unit of bond \( a(0, t_k, \alpha_0) \), where \( x = \frac{a(0, t_i, \alpha_j)}{a(0, t_i, \alpha_j)} \), and \( \alpha_i \) is the bid-ask spread of this bond. So the two strategies have the same initial costs. The expected value of this investment at \( t_i \) is

\[
(x \times a(t_i, t_k, \alpha_i) \times (1 - \alpha_i) - B) \times (1 - p) + x \times PV \times p 
\]

\[
= x \times a(t_i, t_k, \alpha_0) \times (k_{i,0} - p) + x \times p \times a(t_i, t_k, \alpha_0) - B(1 - p) 
\]

\[
= x \times a(t_i, t_k, \alpha_0) \times k_{i,0} - B(1 - p) 
\]

\[
= x \times a(t_i, t_k, \alpha_1) - B(1 - p) 
\]  
(121)

where to obtain the first equality we have made use of Equation (108) and to obtain the last equality we have used the definition of \( k_{i,0} \). The equation can be further rewritten as

\[
\frac{a(0, t_i, \alpha_j) \times a(t_i, t_k, \alpha_i)}{a(0, t_k, \alpha_0)} - B(1 - p) 
\]

\[
= a(0, t_i, \alpha_j) \times \frac{a(0, t_k, \alpha_0)}{a(0, t_k, \alpha_0)} \times \frac{a(t_i, t_k, \alpha_i)}{a(t_i, t_k, \alpha_0)} \times \frac{a(t_i, t_k, \alpha_0)}{a(0, t_k, \alpha_0)} - B(1 - p) 
\]

\[
= a(0, t_i, \alpha_j) \times \frac{k_{i,0}}{k_{i,0}} \times \frac{1}{1} \times \frac{a(t_i, t_k, \alpha_0)}{a(0, t_k, \alpha_0)} - B(1 - p) 
\]

\[
= a(0, t_i, \alpha_j) \times \frac{a(t_i, t_k, \alpha_0)}{a(0, t_k, \alpha_0)} - B(1 - p) 
\]

\[
= a(0, t_i, \alpha_j) \times \frac{a(0, t_i, \alpha_0) \times a(t_i, t_k, \alpha_0)}{a(0, t_i, \alpha_0)} - B(1 - p) 
\]

\[
= \frac{a(0, t_i, \alpha_j)}{a(0, t_i, \alpha_0)} - B(1 - p) 
\]

\[
= k_{i,0} - B(1 - p) 
\]

\[
= \frac{1}{1 + \alpha_j \left( \frac{1}{p} - 1 \right)} - B(1 - p) 
\]  
(122)

where to obtain the fifth equality we have used the basic property of forward rate with deterministic interest rate.
To let the investor be indifferent between the two strategies, the following has to hold:

\[
\frac{1}{1 + \alpha_j \left( \frac{1}{p} - 1 \right)} - B(1 - p) = 1 - p + p \times (1 - B) \times \frac{1}{k_{m,0}}. \tag{123}
\]

QED.

C. Appendix

This appendix proves Proposition 4, 5, and 6. Equation (109) indicates that \( k_{i,0} < 1 \). This implies Proposition 4. In addition, \( \frac{\partial k_{i,0}}{\partial p} > 0 \). In other words, an investor with longer expected holding period is willing to pay a higher price for a given bond. Alternatively, be required a lower return. This proves Proposition 5. Finally, holding price constant, it is clear that a higher expected holding period is accompanied by a higher bid-ask spread, which is Proposition 6.

QED.

D. Appendix

In this appendix we prove Proposition 7.

Consider three discount bonds, \( P_1 \), \( P_2 \), and \( P_3 \), with the same maturity and face value, but \( \alpha_1 < \alpha_2 < \alpha_3 \). In other words, \( P_3 \) is a more illiquid bond than \( P_1 \) or \( P_2 \). We observe the following relationship:

\[
\alpha_2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1} \times \alpha_1 + \frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1} \times \alpha_3 \tag{124}
\]

Therefore, if we regard the ask price as a function of the bid-ask spread, let

\[
a^* = \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1} \times \alpha_1 + \left(1 - \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}\right) \times \alpha_3 \tag{125}
\]
$a^* > a_2$ would suggest that the ask price is a convex function of the bid-ask spread. Now define $a_2 = k^* \times a^*$. Suppose the previous investor is considering the following two strategies:

Strategy #5: Buy one unit of $P_2$.

Strategy #6: Buy $k^* \times \frac{a_3 - a_2}{a_3 - a_1}$ unit of $P_1$ and $k^* \times (1 - \frac{a_3 - a_2}{a_3 - a_1})$ unit of $P_3$.

For Strategy #5, the initial cost is $a_2$, and the expected payoff at time $t$ is

$$P \times PV + (1 - P) \times a_2 \times (1 - \alpha_2).$$

For Strategy #6, the initial cost is

$$a^* \times \frac{a_3 - a_2}{a_3 - a_1} \times a_1 + k^* \times (1 - \frac{a_3 - a_2}{a_3 - a_1}) \times a_3 = k^* \times a^* = a_2.$$

So the two strategies have the same initial costs. The expected payoff of Strategy #6 at time $t$ is

$$k^* \times \frac{a_3 - a_2}{a_3 - a_1} \times [P \times PV + (1 - P) \times a_1 \times (1 - \alpha_1)] + k^* \times (1 - \frac{a_3 - a_2}{a_3 - a_1}) \times [P \times PV + (1 - P) \times a_3 \times (1 - \alpha_3)].$$

$$= k^* \times P \times PV + k^* \times (1 - P) \left\{ \frac{a_3 - a_2}{a_3 - a_1} \times a_1 \times (1 - \alpha_1) + (1 - \frac{a_3 - a_2}{a_3 - a_1}) \times a_3 \times (1 - \alpha_3) \right\}.$$

To be indifferent, the two expected payoffs should be equalized. This gives rise to the following equation:

$$a_2 = \frac{p \times (k^* - 1)}{(1 - p)(1 - \alpha_2)} \times PV \tag{126}$$

$$+ k^* \times \frac{1}{1 - \alpha_2} \times \left[ \frac{a_3 - a_2}{a_3 - a_1} \times a_1 \times (1 - \alpha_1) + \frac{a_2 - \alpha_1}{a_3 - a_1} \times a_3 \times (1 - \alpha_3) \right].$$

The first item on the right hand side is positive if $k^* > 1$. In the second item,

$$\left[ \frac{a_3 - a_2}{a_3 - a_1} \times a_1 \times (1 - \alpha_1) + \frac{a_2 - \alpha_1}{a_3 - a_1} \times a_3 \times (1 - \alpha_3) \right] \tag{127}$$
So the second item is simplified as

\[ k^* \times a^* + k^* \times \frac{(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)(a_1 - a_3)}{(\alpha_3 - \alpha_1)(1 - \alpha_2)} \]

\[ > k^* \times a^* = a_2 \]

This contradicts the fact that \( k^* \times a^* = a_2 \). Therefore, \( k^* \) has to be less than 1.

So \( a_2 < a^* \). That is, the ask price is a convex function of the bid-ask spread. In the same fashion, we can show that the bid price is also a convex function of the bid-ask spread.\(^{14} \) QED.

E. Appendix

In this appendix we prove Proposition 8.

Equation (108) is

\[ a_i = \frac{1}{1 + a_i(\frac{1}{p} - 1)} \times a_0. \]

By definition, the yield \( y_i \) of a discount bond \( P_i \) is expressed as

\[ e^{T \times y_i} = \frac{1}{a_i} \quad (128) \]

This implies

\[ y_i = \frac{1}{T} \ln(\frac{1}{a_i}) \quad (129) \]

\(^{14}\)This proposition can actually be easily proven by taking the second derivative of Equation (109), but the current way of proof provides more intuition.
So the yield differential is expressed as

\[
y_t - y_0 = \frac{1}{T} \left[ \ln \left( \frac{1}{a_t} \right) - \ln \left( \frac{1}{a_0} \right) \right]
\]

(130)

\[
= \frac{1}{T} \ln \left[ \frac{a_0}{a_t} \right]
= \frac{1}{T} \ln \left[ 1 + \alpha_t \left( \frac{1}{p} - 1 \right) \right]
\]

The item inside the logarithm operator is a constant in the model. So the yield spread is a linear function of the reciprocal of time to maturity. Finally, it is trivial to show that the yield spread is a decreasing and convex function of the time to maturity.

QED.
Two bonds with different spreads but the same mean

Figure 4: Why should the bid-ask spread be priced?

The figure describes one possible scenario: if the two bonds have the same mean which is used to calculate return, then it seems that the bid-ask spread is not priced.
The horizontal axis is the quantity of investors, and the vertical is dollar. The supply curve is assumed to be flat. The demand curve is downward sloping. The marginal investor with $p^*$ will determine the market price.
An Indirect Estimate of Transaction Costs for Corporate Bonds

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An Indirect Estimate of Transaction Costs for Corporate Bonds

Abstract

This paper deals with the estimation of transaction costs for corporate bonds. We make three contributions to the literature. First, methodologically, similar to the treatment of equity by Lesmond, Ogden and Trzcinka (1999), we propose a limited dependent variable model for bonds. With this method, we are able to include thinly traded bonds, which are usually discarded in previous studies. The estimated median round-trip transaction cost is 0.59%, much higher than those in previous studies. However, when using only the more liquid half of the data, the median is 0.23%, very close to 0.26%, the estimate obtained by Schultz (2001). Second, this paper enriches the understanding of the determinants of transaction costs and liquidity. In particular, transaction costs are found to be positively related to volatility and duration, and negatively related to firm size, bond size and trading frequency. Firm size has a bigger impact on bond liquidity than bond size. In addition, contrary to previous studies, we find that, for bonds with moderate age, their liquidity can increase with age. Third, we investigate how illiquidity affects bond yield. We find that for the median total spread of 1.58%, 1.43% is due to credit risk and 0.15% due to illiquidity. For individual bonds, depending on their rating and trading frequency, the liquidity spread could be even higher relative to the total yield spread.
I. Introduction to the third essay

While an accurate estimate of transaction costs for corporate bonds is crucial for studying investment strategies and understanding market liquidity, such an estimate is lacking in the current empirical studies. This largely owes to the lack of transparency in the corporate bond market. Most of the bonds are traded infrequently over the counter, and only bid prices, instead of bid/ask prices, are quoted. In addition, during the life of a particular bond, its trading will decline as it is increasingly held by institutional investors for long term investments. How can we obtain, in spite of all these obstacles, an intuitive and comprehensive measure of the transaction costs? What is the expected transaction cost for a typical investor trading corporate bonds? What are the determinants of the costs? Is illiquidity priced into the bond yields? These are the major issues we intend to address in this paper.

At least two different approaches have been employed to estimate the transaction costs for corporate bonds. The first is to estimate the effective spread as the difference between the weighted average of the buy and sell prices of the same bond on the same day. This group of work includes Chakravarty and Sarkar (1999), Hong and Warga (2000), and Kalimipalli and Warga (2000). Working with transaction data for both the dealer-market bonds and the exchange-based (NYSE’s Automated Bond System, or ABS in short) bonds, Hong and Warga (2000) find that per $100 par value, the average spreads range from 13 cents to 20 cents. In addition, bond age and maturity both contribute positively to the estimated spreads. On the other hand, larger outstanding amount and trading volume (NYSE’s ABS) result in lower spreads. Furthermore, the effective spreads have declined over time. Using the same method, Chakravarty and Sarkar (1999) show that the effective spreads for the municipal, corporate, and government bond markets are 22 cents, 21 cents and 11 cents, respectively. They investigate similar determinants of the effective spreads as in Hong and Warga (2000). Again, they find evidence of increased liquidity over time. Working with the most liquid bonds on the NYSE’s ABS,
Kalimipalli and Warga (2000) argue that, among other things, the volatility and observed spread are positively related.

Despite the interesting findings, the effective-spread method has two serious limitations. To be included in the sample, the selected bond must have at least one trade at the buy price and another at the sell price within the same day. This implies that only liquid bonds are used and the estimated spreads are most likely biased downward. In addition, because the trades do not occur at the same time within a day, the reliability of the estimated spread is subject to doubt. This drawback is partly shown by the low explanatory power in their regressions of the estimated spreads on independent variables (e.g., the R-square is less than 2 percent in Hong and Warga (2000) and less than 2.39 percent in Chakravarty and Sarkar (1999)).

Schultz (2001) provides the second method by regressing the difference between the trade price and the estimated bid quote on a dummy variable that is equal to one for buys and zero for sells. The slope coefficient is a proxy for the round trip transaction costs, which he finds to be about 0.26%. In addition, trading cost is lower for larger trades. Smaller institutions pay higher transaction costs and smaller dealers charge higher spreads. There is no evidence that lower-graded bonds cost more to trade. The study is limited by the fact that only monthly quotes are available and the bid quotes must be inferred. To infer daily quotes, Schultz (2001) assumes that the price changes in corporate bonds are proportional to price changes in treasury bonds. Such a method can only apply to high quality bonds. In addition, only bonds in the Lehman Brothers Fixed Income Index are used, which excludes bonds that are not actively traded.

This paper proposes a third method, the so-called Limited Dependent Variable (LDV hereafter) technique, to estimate the transaction costs for corporate bonds. The premise is that, while the true value of the bond is driven by many stochastic factors, a transaction will occur only if the information value of the informed marginal trader exceeds the total transaction costs. This implies that there exists a transaction cost threshold
for each bond, which is equivalent to the minimum information value for a trade. We will observe zero returns (corresponding to no trade) within the threshold and non-zero returns that are outside of the transaction cost thresholds. This leads to the natural use of the LDV model originated by Tobin (1958) and Rosett (1959). We use the maximum likelihood method to jointly estimate the true return generating function (i.e., the systematic risk) and the upper and lower transaction thresholds.

This method is also a natural extension of Glosten and Milgrom (1985), who illustrate that trades will happen when the information value exceeds the transaction costs defined by the bid-ask spreads. It is well known that comprehensive transaction costs include not only the bid-ask spreads, but also the commission as well as the expected time to execute the transaction. The estimated thresholds from the LDV model appear to capture all these effects and stand out as a good candidate for the measure of comprehensive transaction costs. Lesmond, Ogden and Trzcinka (1999) use the LDV model to estimate the transaction costs for equities. As they show, the estimated transaction costs are indeed highly correlated with other commonly used estimators such as the bid-ask quotes plus a constant (e.g., Stoll and Whaley (1983)), but their estimate is considered to be more accurate.

The paper contributes to the literature in three important aspects. First, methodologically, to our knowledge, we provide the first LDV model to estimate the transaction costs for corporate bonds. This method only needs the time series data of corporate bonds that update the prices whenever a trade occurs. As such, it can include bonds with very infrequent trading, which are usually discarded in previous studies. As liquidity is the major subject of research, the method presents an important improvement by including illiquid bonds. Our median estimate of the round-trip transaction costs is 0.59%, much higher than those in the previous studies. However, when only the more liquid half of the data are included, the median is 0.23%, very close to 0.26% obtained by Schultz (2001).
Second, we enrich the understanding of the determinants for transaction costs and bond liquidity. The estimated transaction costs are found to be positively related to volatility and duration, and negatively related to trading frequency. In addition, bonds with a higher rating react more to interest rate shocks and less to the stock market shocks. The opposite is true for low quality bonds. Moreover, contrary to Hong and Warga (2000), aging does not contribute monotonically to higher transaction costs. In fact, for bonds that are relatively young (e.g., less than 10 years old), liquidity actually increases with the age. However, when the bonds are old (e.g., more than 13 years old), their liquidity can drop quickly and they gradually leave the market.

It is well known that the average transaction cost for equity is negatively related to the firm's capitalization (Demsetz (1968), Benston and Hagerman (1974), Copeland and Galai (1983), Stoll and Whaley (1983), Roll (1984), and Lesmond, Ogden and Trzcinka (1999)). Similarly, the transaction cost for bonds is negatively related to bond size (e.g., Hong and Warga, 2000). The unaddressed question is, which factor, the firm size or the bond size, is more important for the bond's liquidity? We show that the firm size dominates the bond size in determining the liquidity of bonds. This is consistent with the hypothesis that bonds issued by larger firms have better publicity and are better distributed, resulting in a higher trading activity and liquidity. In fact, the non-callable bonds in our sample are much more liquid than the callable bonds even though their bond sizes are very similar, primarily because larger firms tend to issue non-callable bonds and smaller firms tend to issue callable bonds.

Third, theoretically, Amihud and Mendelson (1986) and Chen (2001) show that investors require a higher return for securities with higher transaction costs. While previous studies estimate transaction costs for corporate bonds, they stop short of investigating whether illiquidity is priced in bond yields. We observe that bonds with a smaller outstanding amount tend to be the ones with lower rating quality, and exhibit a higher spread over riskless bonds. The issues of credit risk and illiquidity are thus
intertwined. It is very important to know how each part contributes to the total spread. This knowledge will help to predict the evolution of a particular bond's spread over time because credit risk and illiquidity have different dynamics. In addition, in the credit risk literature, default probability is usually recovered from the total spread and used to evaluate other credit-sensitive instruments. Clearly, the liquidity spread part in the total spread should have been excluded because it does not reveal the default probability of the firm. We show that both credit rating and trading frequency significantly contribute to the total spreads. In particular, for a median yield spread of 1.58%, about 1.43% is credit spread and 0.15% is liquidity spread. In other words, about 9.5% of the total spread is due to illiquidity. Holding the credit spread at 1.43%, the liquidity spread can be as high as 0.54%.

The rest of the paper is organized as follows. Section I outlines the LDV model adapted to bonds. Section II describes the data. Section III discusses the empirical results. Section IV concludes. Tables are collected in the appendix.

II. The Return Generating Model

Asset pricing models predict that the return of corporate bonds is driven instantaneously by stochastic factors. However, despite the frequent variation of macro variables such as the interest rate, we do not observe frequent trading of corporate bonds. To reconcile this, following Lesmond, Ogóe and Trzcinka (1999), we assume that the observed returns generated from the model are constrained by the existence of transaction costs. Specifically, the bond will be traded by the marginal informed investor only when the potential move in price more than offsets the transaction costs; otherwise the observed bond price remains unchanged although the true price has changed. To facilitate exposition, we employ the following notation:

\[ P_t : \text{observed bond price at time } t, \]
$P_t^*$ : true bond price at time $t$,

$R_t$ : observed log return at time $t$, i.e., $\ln(P_t) - \ln(P_{t-1})$,

$R_t^*$ : the true counterpart of $R_t$, i.e., $\ln(P_t^*) - \ln(P_{t-1}^*)$.

$D_t$ : Macaulay duration of the bond,

$r_{st,t}$ : short-term interest rate at time $t$ (approximated by the 3-month T-bill rate),

$r_{lt,t}$ : long-term interest rate at time $t$ (approximated by the 10-year T-note yield),

$R_{sp,t}$ : log return on the S&P500 index at time $t$.

It is well known that a corporate bond is a hybrid of equity and fixed income securities. As such, we assume that the return of the corporate bond is generated by two systematic risk factors, the first being the interest rate and the second the S&P 500 index. All other effects are assumed to be nonsystematic and are captured by a random error term. The precise LDV model is specified as follows:

$$R_t^* = \beta_1(DR_t) + \beta_2(SP_t) + \varepsilon_t \quad (131)$$

where

$$R_t = R_t^* - \alpha_1 \quad \text{if} \quad R_t^* < \alpha_1,$$

$$R_t = 0 \quad \text{if} \quad \alpha_1 < R_t^* < \alpha_2,$$

$$R_t = R_t^* - \alpha_2 \quad \text{if} \quad R_t^* > \alpha_2.$$

In the above, $DR_t = D_t \times \Delta r_{lt,t}$, $SP_t = D_t \times R_{sp,t}$, and $\varepsilon_t$ is assumed to be a normally distributed variable with a mean of zero and stochastic standard deviation of $\sigma_t = D_t \times \sigma_{LT} \times \sigma$, where $\sigma$ is a constant. Let us first explain the term, $DR_t = D_t \times \Delta r_{lt,t}$. When $D_t$ is the Macaulay duration and $\Delta r_{lt,t}$ is the change in long term interest rate, $DR_t$ represents the percentage change in the riskless bond price with the duration similar to that of the corporate bond. The sign of the coefficient $\beta_1$ is expected to be negative given the negative relationship between bond prices and interest rates.
The term \( SP_t = Dt \times R_{pt} \) captures the bond return sensitivity to the stock market. Many authors (e.g., Cornell and Green (1991)) have documented that high-grade bonds are more sensitive to interest rate changes and less sensitive to changes in stock prices, and that the opposite holds true for low-grade bonds. Therefore it is hoped that the inclusion of the term \( SP_t \) will improve the general fitting of the bond returns. Notice that Cornell and Green (1991) simply use the S&P500 index return in their regressions without the duration adjustment. This may be justified in that the duration of a bond fund is arguably very stable. However, when we characterize return relations over time for individual bonds, it is important to recognize the diminishing return variations. Authors such as Boquist, Racette and Schlarbaum (1975) and Jarrow (1978) have shown theoretically that a bond's beta should be proportional to its duration, a result on which we base our adjustment. In addition, a higher market return implies a higher return for stocks because most of the stocks have positive correlation with the market portfolio (i.e., systematic risk). Controlling the effect of interest rate, this reduces the default risk and increases the bond price. Therefore, \( \beta_2 \) is expected to be positive. This is consistent with Kwan (1996) who shows a positive relationship between corporate bond returns and stock returns. However, \( \beta_2 \) is expected to matter more for high-yield bonds and less for investment grade bonds.

The standard deviation of bond returns is modelled as \( \sigma_t = Dt \times \sigma_{a,t} \times \sigma \). The duration adjustment is both intuitive and consistent with bond pricing theories (e.g. the standard deviation should approach zero when the bond approaches its maturity). The interest rate adjustment factor \( r_{a,t}^{1.5} \) is based on the empirical behavior for interest rates and the fact that bond return standard deviations tend to be proportional to that of interest rates. Chan, Karolyi, Longstaff and Sanders (1992) show that, among many alternatives, \( r_{a,t}^\beta \times \sigma \) (where \( \beta = 1.5 \)) best describes the interest rate diffusion coefficient, or the standard deviation.

The threshold for trades on negative information is \( \alpha_1 \) and that on positive informa-
tion is $\alpha_2$. When the true return is within the bounds, or when $\alpha_1 < R_t^* < \alpha_2$, there will be no trade and the observed return is zero. Naturally, $\alpha_2 - \alpha_1$ serves as an indirect measure of round-trip transaction costs for a marginal investor.

Because we use daily data, Equation (131) implies that the true bond return (not the observed) is updated daily through the two factors. However, the observed bond price will not change until a new trade occurs. As a result, the bond return for a trading day immediately following a period of no trade is actually the cumulative return over the whole period. But in Equation (131) the interest rate change and market index return are actually daily, which could lead to a data mis-match problem. To avoid this problem, we modify Equation (131) as

$$\sum_{i=0}^{n-1} R_{t-i}^* = \beta_1 \sum_{i=0}^{n-1} (D R_{t-i}) + \beta_2 \sum_{i=0}^{n-1} (S P_t) + \epsilon_t,' \quad (132)$$

where $\sum_{i=0}^{n-1} R_{t-i}^* = \ln(P_t^*) - \ln(P_{t-n}^*)$, $n$ is the number of consecutive days in which no trade occurs, and $\epsilon_t'$ is assumed to be a normally distributed variable with a mean of zero and stochastic standard deviation of $\sigma_t = \sigma \sqrt{\sum_{i=0}^{n-1} (D_t \times \sigma_{i,t'})^2}$. Thus for a cumulative return of $n$ days, the interest rate, market index return, and the cumulative error term for the same period will be used, adjusted by the duration. This adjustment is in the same spirit as that by Schultz (2001), who effectively assumes a one-factor return generating function and adjusts the error term accordingly using monthly data.

The above setup naturally lends itself to maximum likelihood estimations. Similar to Lesmond, Ogden and Trzcinka (1999), we can write down the resulting likelihood function as
\[ L = \prod_{t \in 1} \frac{1}{\sigma_t} \phi \left[ \frac{R_{t+1} - \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right] \]
\[ \times \prod_{t \in 0} \left[ \Phi \left( \frac{\alpha_1 + \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right) - \Phi \left( \frac{\alpha_1 - \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right) \right] \]
\[ \times \prod_{t \in 2} \frac{1}{\sigma_t} \phi \left[ \frac{R_{t+1} + \alpha_1 + \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right], \quad \text{(133)} \]

where \( \phi \) and \( \Phi \) stand for the standard normal density function and the cumulative distribution function respectively. \(-1, -0, -2 \) stand for, respectively, the negative, zero and positive return regions. \(^{16}\) The log-likelihood function can be written as

\[ \ln L = -\frac{1}{2} \sum_{t \in 1} \ln(2\pi \sigma_t^2) - \sum_{t \in 1} \frac{(R_{t+1} - \beta_1(DR_t) - \beta_2(SP_t))^2}{2\sigma_t^2} \]
\[ + \sum_{t \in 0} \ln \left[ \Phi \left( \frac{\alpha_1 + \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right) - \Phi \left( \frac{\alpha_1 - \beta_1(DR_t) - \beta_2(SP_t)}{\sigma_t} \right) \right] \]
\[ -\frac{1}{2} \sum_{t \in 2} \ln(2\pi \sigma_t^2) - \sum_{t \in 2} \frac{(R_{t+1} + \alpha_1 + \beta_1(DR_t) - \beta_2(SP_t))^2}{2\sigma_t^2}. \quad \text{(134)} \]

The function is to be maximized using \( R_t, DR_t, SP_t, D_t, \) and \( r_t \) as inputs. The parameter vector is \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma) \). Our immediate interest is in \( \alpha_1 \) and \( \alpha_2 \). As discussed by Lesmond, Ogden and Trzcinka (1999), although it is intuitive to think that positive (negative) price movements tend to be associated with a buying (selling) initiative in a trade, the model does not require that all positive (negative) price movements are buyer (seller) initiated. All we need to say is that a trade will occur only if the price movement is big enough to offset the transaction costs associated with the trade.

\(^{16}\) For those days when a trade occurs immediately following a period of no trade, the relevant quantities in Equation (133) are to be replaced by items in Equation (132).
III. The Data

The data cover more than 700 corporate bonds from U.S. issuers that are traded in the U.S. market. They are obtained from Interactive Data Company (IDC) through Datastream. IDC is a leading supplier of global securities pricing and covers daily evaluation of more than 2.5 million active fixed income issues. It collects corporate bond quotes from primary dealers. Most of the bonds can be traced back to 1990 only, and as a result, we choose December 31, 1989 as the sample starting point. We have deleted bonds with obvious erroneous prices and bonds that are never traded.

This leaves us with 701 bond series in total, out of which 119 belong to finance companies and the rest belong to industrial companies. A prominent feature of this daily data is that the quoted prices do not change until a trade is initiated. Similar to the treatment of equity in Lesmond, Ogden and Trzcinka (1999), we separate the data into years. That is, we treat the data for one bond within one particular year as a separate time series. This will allow us to detect the potential change in liquidity over time. It also enables us to investigate how liquidity is related to such quantities as amount outstanding and firm's capitalization. 17

Other summary information is reported in Exhibit 1. Note that the "average proportion of no trade" is calculated as the ratio of the number of days in which the bond is not traded over the total number of trading days within the bond year. The "average combined rating" is calculated in the following way. First, we assign a numerical value to each rating class from Standard and Poor’s and Moody’s. For Standard and Poor’s, the numerical values of 1, 2, 3, 4, .... are assigned to ratings AAA+, AAA, AAA−, AA+, .... and for Moody’s, the numerical values of 1, 2, 3, 4, .... are assigned to ratings Aaa1, Aaa2, Aaa3, Aa1........ and so on. Second, for bonds rated by both companies, we

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17 The information on the number of bonds outstanding became available only toward the end of the sample period. We are therefore forced to assume that the outstanding amount remains the same for the 10 year period we investigate. We do find that the outstanding amount for most bonds is not different from the bond issue size, making our assumption reasonable.
take the simple arithmetic average of the two numerical ratings; for bonds rated by only one company we simply use that company's rating. The resulting numerical rating is called the combined rating and will be used throughout the study.

**Exhibit 1. Summary Statistics for Bonds**

<table>
<thead>
<tr>
<th>Metric</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average maturity (years)</td>
<td>13.975</td>
</tr>
<tr>
<td>Average duration (years)</td>
<td>6.6185</td>
</tr>
<tr>
<td>Average issue size ($ million)</td>
<td>424.47</td>
</tr>
<tr>
<td>Average amount outstanding ($ million)</td>
<td>413.08</td>
</tr>
<tr>
<td>Average proportion of trade (%)</td>
<td>42.94</td>
</tr>
<tr>
<td>Total number of bond years</td>
<td>5947</td>
</tr>
<tr>
<td>Number of issuing firms</td>
<td>340</td>
</tr>
<tr>
<td>Number of callable bond years</td>
<td>3897</td>
</tr>
<tr>
<td>Number of non-callable bond years</td>
<td>2050</td>
</tr>
<tr>
<td>Median combined rating (S&amp;P or Moody's)</td>
<td>8.8045 (A-/A or A2/A3)</td>
</tr>
<tr>
<td>Number of rated bond years</td>
<td>4580</td>
</tr>
</tbody>
</table>

**IV. Empirical Results and Analysis**

**A. Trading frequency and other variables**

In the LDV model, infrequent trading data reveal important information regarding a bond's liquidity. As we will show shortly, liquidity directly affects the magnitude of the estimates for transaction costs. In this subsection we investigate the pattern of trading frequency and its relation to other variables, including maturity, duration, bond age, amount outstanding, callability, yield to maturity, bond rating, and yield spread. Here, a bond's yield spread refers to the difference between the bond’s yield to maturity and its riskfree counterpart.

The statistics are reported in Table I. In Panel A, a total of 5947 bond years are separated into 10 equal deciles according to the size of trading proportions. We first
notice that bonds in the first three deciles are either not traded at all, or very thinly traded, even though they still have more than 6 years of maturity. This is a group characterized by considerably higher ages than any other decile, and we will call them the senior group hereinafter. Presumably they are held by institutional investors and have left the market. Note that the senior group has the largest yield to maturity and yield spread. In addition, the senior group consists mostly of callable bonds. For the other seven deciles, bonds with a longer maturity tend to be traded less frequently and non-callable bonds tend to be traded more frequently. Trading frequency is found to be positively related to bond size, and negatively related to yield to maturity and yield spread. Finally, outside of the senior group, there is no clear relationship between bond age and trading frequency. In fact, it appears that older bonds are traded relatively more frequently. Note that the difference in age among the groups is relatively small. Therefore, aging, as a factor, does not always contribute to less liquidity. For a relatively younger bond, its liquidity can actually increase over time. This is either because the bond is getting more known, or simply because the liquidity for corporate bonds has increased within the sample period. Nonetheless, when the bond grows much older (e.g., 13 years old), the pattern reverses: liquidity decreases dramatically and the bond gradually leaves the market, as seen in the senior group.

Obviously we can not learn much from a bond that is already out of the market. Therefore in Panel B we only include bond years with at least 5 percent of trading within a year, and regroup them into 10 deciles according to trading frequency. In Panel C we further restrict to bond years with rating information. The patterns observed in Panel A also prevail in Panels B and C. In addition, there is a negative relationship between trading frequency and rating quality. Moreover, we observe that bonds with a smaller outstanding amount are traded less frequently, and earn a higher yield to maturity. This seems consistent with the intuition that investors require a higher return

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18 The variable "callability" is equal to zero if the bond is callable and one if it is not. The reported figure in each decile can be understood as the median proportion of non-callable bond years.
for lower liquidity. However, the less liquid bonds are also the ones with relatively lower rating quality. It is thus not clear whether the higher return is due to bond's credit risk or illiquidity. We will investigate this issue later.

To see if there are discernible differences between callable and non-callable bonds, we re-group bonds according to callability and report the results in Table II. It is seen that the two types of bonds share some similar traits. For example, older bonds are traded relatively more frequently, and a higher trading frequency is related to a lower yield to maturity and yield spread. On the other hand, the two types of bonds also exhibit many different properties. To start with, the positive relationship between trading frequency and bond size is much stronger for callable bonds than for non-callable bonds. In addition, the relationship between maturity and trading frequency is positive for callable bonds, and negative for non-callable bonds. Moreover, callable bonds have a much longer maturity and age than non-callable bonds. Finally, callable bonds command a much higher yield than non-callable bonds (98 basis points more). This difference in yields could be due to three reasons. First, callability, as an option for the issuer, presents risk to the bond holder who will demand a higher yield to compensate for this risk. Second, it is seen that the callable bonds have marginally lower rating quality than non-callable bonds. Third, non-callable bonds are traded much more frequently than callable bonds, and hence may have a lower liquidity premium. Incidentally, note that the bond size difference between the two groups is very small. In fact, as we will show later, it is the firm size that drives the liquidity difference since larger firms tend to issue non-callable bonds.

B. Estimated transaction costs

We only estimate bond years with at least 12 trades a year (roughly equivalent to 5 percent trading proportion). For each of the 4186 bond years, we fit the return generating process in (131) by maximizing the log likelihood function in (134). We exclude 76 bond
years for which the estimations either do not converge, or produce an unreasonably high volatility estimate (higher than 80%) that we consider as bad convergence. This leaves us with 4110 bond years. Table III provides summary information for the estimated parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \text{and } \sigma$.

Except for $\beta_2$, all the parameters are estimated with at least 5 percent significance. $\alpha_1$ (the lower threshold) is negative and $\alpha_2$ (the upper threshold) is positive as they should be. In addition, the median estimate of $\alpha_1$ is greater than that of $\alpha_2$ in magnitude. This implies that more trades occur at the buy side, consistent with Schultz (2001). The average round-trip transaction cost is 1.90% and the median counterpart is 0.59%. Therefore the round-trip transaction costs are not normally distributed and are affected by some large estimates. We shall from now on use the median as the reliable measure for future discussions.

The median estimate of 0.59% (or 59 cents per 100 dollars) is much larger than those obtained by Hong and Wagar (2000), Chakravarty and Sarkar (1999), and Schultz (2001). The highest estimate they obtain is 26 cents on $100 par. However, as we mentioned earlier, the estimates from the previous studies tend to be biased downward. If we only include the more liquid half of the bond years, the median transaction cost is 0.23%, very close to that in other studies. Therefore our results can be considered as more reliable and realistic in that they reflect a wider spectrum of bond markets liquidity. In fact, as Kalimipalli and Warga (2000) show, even for the most liquid bonds traded on the New York Stock Exchange's ABS, half of the spread quotes are larger than 0.5%.

The interest rate sensitivity parameter $\beta_1$ is estimated with a median of -0.4583 at 5% significance. The negative sign is what we would expect in that interest rate increases reduce bond prices. The market index sensitivity parameter $\beta_2$ is estimated with a median of 0.0016, which is not statistically significant. This is not surprising in that most of the bonds in the sample are investment grade bonds, which tend to respond
less to stock market movements. Finally, the estimate for the volatility parameter \( \sigma \) is highly significant.

Again, we observe an apparent difference between callable and non-callable bond years. While the mean estimates of the transaction costs are relatively close (1.82\% vs. 2.03\%), the median estimates for callable bonds are much higher (0.70\% vs. 0.41\%). In addition, callable bonds respond relatively less to interest rate changes but more to stock market movements, and they tend to have a higher volatility. In a nutshell, compared with non-callable bonds, callable bonds in our sample behave more like lower rating bonds with less liquidity.

Taken together, the return generating process in (131) or the LDV model seems to be well specified. All parameters save the stock market sensitivity parameter are estimated with statistically significant values. In what follows we will perform further analysis of the estimation results.

C. Transaction costs and their determinants

C.1 Relating transaction cost estimates to other variables

To see how the estimated transaction costs are related to other variables, we group bond years into deciles by the round-trip transaction cost estimate, \( \alpha_2 - \alpha_1 \), and then report in Table IV the median value for related variables in each decile (except for callability and rating score, where the mean is reported). It is seen in Panel A that the estimated transaction costs are related negatively to trading frequency and positively to volatility. Bond years with a higher trading frequency will have lower transaction costs and a lower volatility. The early results sorted by trading frequency are thus expected to hold here. In particular, bond years with higher transaction costs also have a longer maturity, a longer duration, and a smaller outstanding amount. They have a higher yield to maturity and a bigger yield spread. In addition, they are more likely to be callable bonds. Note that, for moderately old bonds, aging does not contribute to less
liquidity. This is contrary to the results in Hong and Warga (2000). Furthermore, less liquid bonds react less to interest rate shocks. There is no clear evidence that they react more to stock market shocks because $\beta_2$ is estimated insignificantly.

Panel B reports the results for bond years with rating information. It is seen that the overall patterns are similar to those in Panel A. In addition, lower grade bonds tend to be less liquid. Again, credit risk and illiquidity are intertwined and compensated by a bigger yield spread.

We also examine the results by grouping bonds into callable and non-callable bonds. It turns out that the overall results for all bonds also hold for each type of bonds, with only two exceptions. First, duration and maturity seem to decrease with transaction costs for callable bonds, whereas there are no discernible patterns for non-callable bonds. This is consistent with the findings in Table II. Second, as observed before, callable bonds exhibit properties similar to those of less liquid, lower grade bonds.

C.2 Relating all variables to each other

Before we proceed with further analysis, it is useful to quantify the inter-relationship among all the variables we have been examining so far. To this end, we calculate a correlation matrix for the variables in question and present it in Table V. We briefly discuss the pair-wise correlations for each variable in the order they appear in the correlation matrix. First, as we have already observed, trading frequency is found to be positively correlated with bond size, bond age, non-callability, and rating quality, and negatively correlated with yield to maturity and yield spread. The correlation between trading frequency and maturity or duration is not very strong. Second, yield to maturity is negatively correlated with age, non-callability, and rating quality, but positively correlated with maturity, amount outstanding, yield spread, and rating. The correlations to yield spread and rating are sizable at 0.92 and 0.34, indicating that, other things being equal, a higher yield to maturity is primarily due to a higher credit risk. Third, maturity is
positively related to duration as expected and negatively related to yield spread, amount outstanding, age, non-callability, and rating. The correlations involving the outstanding amount and rating imply that bonds with a bigger circulation and a better quality tend to have shorter maturities. In addition, longer maturity bonds tend to be callable bonds. Fourth, duration has essentially the same relationship with other variables as maturity does except for the outstanding amount. Fifth, yield spread does not appear to be related to the outstanding amount, although it is negatively correlated with age and non-callability, and positively correlated with rating. Sixth, bonds with a bigger circulation tend to be younger, callable, and have a higher rating. Seventh, regarding age, older bonds tend to be callable and have a higher rating. Finally, there is no discernible relationship between callability and rating.

C.3 Regressions of transaction costs on their determinants

We run simple OLS regression of the following form:

\[ y_i = a_0 + a_1x_{1i} + a_2x_{2i} + a_3x_{3i} + a_4x_{4i} + a_5x_{5i} + a_6x_{6i} + \epsilon_i \]  

(135)

where index \( i \) indicates bond year \( i \), \( y_i = \alpha_{2i} - \alpha_{Hi} \) (estimated transaction cost band), \( x_{1i} \) = trading proportion, \( x_{2i} = \sigma_i \) (estimated volatility), \( x_{3i} \) = logarithm of dollar amount outstanding, \( x_{4i} = D_i \) (duration), \( x_{5i} = \text{age} \), \( x_{6i} = \text{rating score} \), and \( \epsilon_i \) = error term. We run five step-wise regressions, the first with only two independent variables, \( x_{1i} \) and \( x_{2i} \), the second with an additional independent variable, \( x_{3i} \), and so on. The results are summarized in Table VI.

We first discuss Panel A, where all bond years are included. As expected, transaction costs decrease in trading frequency and amount outstanding, two of the most important liquidity measures. Contrary to Crabbe and Turner (1995), we find that larger bond size is related to a better liquidity and lower transaction costs. In addition, the transaction

\[ \text{It is indeed possible for bond size to be negatively correlated with maturity but positively correlated with duration if the coupon rate is not evenly distributed cross-sectionally.} \]
costs also increase in the estimated volatility and duration. Many studies (e.g., Hong and Warga, 2000) explain the positive relationship between the transaction costs and duration by assuming that bond volatility increases with duration, and thus duration is a good proxy for the volatility. In this paper we are able to estimate the true volatility separately, and establish that duration and volatility are both related to transaction costs as two separate variables.

Note the volatility we recovered is the "true volatility" instead of observed volatility. Because of low trading frequency, low liquidity bonds tend to have low observed volatility. It appears that a lower bond volatility is associated with higher bond return (e.g., Cornell and Green, 1991). In this paper, we are able to separate the effect of transaction thresholds from the true market movements. As a result, the estimated volatility is positively related to transaction costs, and positively related to bond yield. These results are consistent with Kalimipalli and Warga (2000) for NYSE's ABS data. They are also in line with the empirical evidence in the exchange rate market (e.g., Bollerslev and Melvin, 1994).

Transaction costs are found to be negatively related to the age of bonds. As we discussed earlier, the liquidity of bonds seems to improve with age until the bonds are sufficiently old, at which time the liquidity will dry out quickly. In addition, transaction costs are positively related to rating quality.

When the set of explanatory variables increases from 2 to 6 in the five regressions, the regression coefficients are all stable and significant at 5% level (except for one coefficient in the third regression). The adjusted R-square increases from 0.629 to 0.686. The R-square itself is very impressive, and much higher than those in Hong and Warga (2000, less than 0.02) and Chakravarty and Sarkar (1999, less than 0.024).

In Panels B and C we re-run the regressions for callable and non-callable bonds separately. In both cases, trading frequency, volatility, and duration are all significant and stable as in the overall bond years case. However, for callable bonds, bond size is
estimated with the right sign, but not significant until in the full (fifth) regression. For non-callable bonds, bond size is never significantly estimated, while age has a positive slope and rating has a negative slope. Therefore, it appears that when all bonds are considered (Panel A), bond size is significantly and negatively related to transaction costs.

D. Impact of firm size vs. bond size

We have shown that bond liquidity is affected by its outstanding amount. On the other hand, it is well known that there is a negative relationship between the equity transaction costs and the firm capitalization size (Demsetz (1968), Beuston and Hagerman (1974), Copeland and Galai (1983), Stoll and Whaley (1983), Roll (1984), and Lesmond, Ogden and Trzcinka (1999)). For corporate bonds, the unaddressed question is: which factor is more important for bond liquidity, the firm size or the bond size? Intuitively, we could conjecture that bond issues by larger firms tend to be well marketed and hence widely traded afterwards.

To investigate this issue, we match bond year data with last year’s firm capitalization size, collected from CRSP. We then repeat the analysis for Table IV with an additional item: log of firm size. The results are reported in Table VII. In addition to the similar patterns observed in Table IV, the firm size is found to be negatively related to transaction costs. It appears that larger firms are associated with larger bond issues, a lower yield to maturity, and a smaller yield spread.

Again, to pinpoint the precise relationship between transaction costs and related variables for this sub-sample, we run regressions similar to that in (135):

\[ y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i} + \beta_7 x_{7i} + \epsilon_i \]  

where index \( i \) indicates bond \( i \), \( y_i = \alpha_2 t - \alpha_1 t \) (estimated transaction cost band), \( x_{1i} \) = trading frequency, \( x_{2i} = \sigma_i \) (estimated volatility), \( x_{3i} = \log \) logarithm of bond size, \( x_{4i} = \log \)
of firm size, \( x_D = D_i \) (duration), \( x_a = \text{age} \), \( x_R = \text{rating score} \), and \( \epsilon_i = \text{error term} \). The results are summarized in Table VIII.

As before, transaction costs are negatively related to trading frequency and positively related to bond volatility. However, when we add both the bond size and firm size into the regression, the slope is positive for the former and negative for the latter. This relationship is significant for the overall bond years case and for the full (fifth) regression of the callable bond case, but not significant for the non-callable bond case. In addition, the positive slope for bond size and negative slope for firm size are obtained for all nine regressions, although they are not always statistically significant. Furthermore, as in Table VI, for both the overall data and the callable bond data, we obtain negative slopes for the age and rating score. For non-callable bonds, neither is statistically significant.

Therefore, when both bond size and firm size are considered, transaction costs are negatively driven by firm size and positively driven by bond size. This supports the hypothesis that, compared to bond size, firm size is a bigger driving force for liquidity.

At this point we can understand why callable bonds are less liquid. In Table VII, the difference of the median firm sizes between non-callable bonds and callable bonds is 1.2. Taking the exponential, this implies that the firms that issue non-callable bonds have capitalization three times that of the firms issuing callable bonds. Therefore larger firms tend to issue non-callable bonds that are traded more frequently. It is the firm size difference that drives the liquidity difference between the two groups.

E. How much of the yield spread is liquidity spread?

In the credit risk literature the yield spread of corporate bonds over riskless bonds has usually been treated as if it is solely due to credit risk. However, as we already show in this paper, less liquid bonds tend to be the ones with lower rating quality, and have higher yield spreads. The effects of credit risk and illiquidity are thus intertwined. It is very important to know how each part contributes to the total spread. This knowledge
will help to predict the evolution of a particular bond's spread over time since credit risk and illiquidity tend to have different dynamics. In addition, in the credit risk literature, default probability is usually recovered from the total spread and used to evaluate other credit-sensitive instruments. Clearly, the liquidity spread in the total spread should be excluded because it does not reveal the default probability of the firm.

To formulate the framework, we assume that, cross-sectionally, rating score summarizes the credit risk. In addition, duration is assumed to be an important factor for bond returns. As such, credit spread is expected to be determined by these two factors. We further assume that trading frequency is a good proxy for liquidity risk. Therefore, we run the following regression:

\[ y_i = c_0 + c_1 x_{1i} + c_2 x_{2i} + c_3 x_{3i} + \varepsilon_i \]  (137)

where index \( i \) indicates bond year \( i \), \( y_i = \) yield spread, \( x_{1i} = \) rating score of the bond year, \( x_{2i} = \) duration, \( x_{3i} = \) trading frequency, and \( \varepsilon_i = \) error term. The results are summarized in Table IX.

For the cases of all bonds and callable bonds, all three regression coefficients are significant at 1 percent significance level, and the coefficients for rating score and trading frequency in the two regressions are very similar. For non-callable bonds, the coefficient estimates for trading frequency and duration are not significant. This is consistent with our earlier findings that liquidity seems less of a factor for non-callable bonds.

To appreciate the results, we assume that the regression for the overall data is correctly specified and estimated. That is,

\[ E[y_i] = 0.0044 + 0.0019 \times x_{1i} - 0.0054 \times x_{2i} - 0.0003 \times x_{3i}. \]  (138)

From Table IV, the median duration, rating score, and trading frequency are 7.1271, 8.9, and 0.7241 respectively. If the bond is perfectly liquid (i.e., trading proportion = 1), the credit spread will be 1.43\%. With the trading proportion of 0.7241, the total
yield spread will be 1.58%. The 0.15% difference is due to the illiquidity of the bond and it is \( \frac{0.15\%}{1.58\%} = 9.49\% \) of the yield spread. Imagine the trade frequency reducing to 0%. The liquidity spread will be 0.54%, which is 27.41% of the total spread.

F. Rating versus return sensitivity to interest rates and market index

Intuitively, risky bonds would respond more to general stock market movements than to interest rate changes, and safer bonds would behave in the opposite way. Insofar as rating is a composite measure of a bond’s riskiness, we would expect some linkage to exist between a bond’s rating and its interest rate and market index sensitivity parameters, \( \beta_1 \) and \( \beta_2 \). In addition, it is interesting to know if a higher interest rate sensitivity is also accompanied by a higher market index sensitivity. To this end, we run the following regression for the rated bonds:

\[
s_i = d_0 + d_1 \text{(rating)} + \epsilon_i
\]  

(139)

where index \( i \) indicates bond year \( i \) and \( s_i = |\beta_{2i}| / (|\beta_{1i}| + |\beta_{2i}|) \), the relative importance of the market index sensitivity with respect to the total sensitivity. The intercept \( d_0 \) and the coefficient \( d_1 \) are estimated as 0.0313 and 0.0029 with corresponding t-values of 6.0278 and 5.2658. The adjusted R-square is 0.01. Recalling that a bigger number is assigned to a lower (risky) rating, we have essentially confirmed the intuitive prediction: the riskier the bond, the bigger impact will the general stock market movements have on bond returns.

V. Summary and Conclusions

This paper deals with the estimation of transaction costs for corporate bonds. We advance the literature on three fronts. First, methodologically, similar to the treatment
of equity by Lesmund, Ogden and Trzcinka (1999), we propose the limited dependent variable (LDV) method for bonds. This method allows us to include thinly traded bonds in the estimation, which are usually discarded in previous studies. Using a sample of daily price quotes for 701 bonds, we estimate the round-trip median transaction costs to be 0.59%, much larger than those obtained in previous studies (e.g., Hong and Wargar 2000. Chakravarty and Sarkar, 1999, and Schultz, 2001). However, when only the more liquid half of data are used, the median estimate is 0.23%, very close to that obtained in Schultz (2001).

Second, we enrich the understanding of the determinants of transaction costs and liquidity. We find that the estimated transaction costs are positively related to bond volatility and duration, and negatively related to bond size, firm size, and trading frequency. In addition, bonds with higher rating quality tend to respond more to interest rate shocks and less to stock market shocks. The opposite is true for lower grade bonds.

Another important finding is that, firm size dominates bond size as the determining factor for liquidity. Because larger firms tend to issue non-callable bonds, this implies that non-callable bonds are more liquid than callable bonds and have a lower yield to maturity. Moreover, contrary to Hong and Wargar (2000), we find that aging does not always contribute to less liquidity. In fact, for bonds with a moderate age (i.e. less than 13 years old), their liquidity can increase with age.

Third, we break down the total yield spread into two components due to credit risk and illiquidity, respectively. We are able to quantify the individual contributions of the two factors. For example, for a median yield spread of 1.58% in our sample, 1.43% is due to credit risk and 0.15% due to liquidity risk. For individual bonds, depending on their trading frequency and rating, the liquidity spread could be even higher relative to the total yield spread.
References


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<th>8</th>
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<td>0.00</td>
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<td>2.03</td>
<td>2.40</td>
<td>1.97</td>
<td>1.76</td>
<td>1.39</td>
<td>1.24</td>
<td>1.18</td>
<td>1.06</td>
<td>1.06</td>
<td>1.65</td>
</tr>
</tbody>
</table>

Panel B: bonds with at least 5% trading proportion

| number of bonds | 418 | 418 | 418 | 418 | 418 | 418 | 418 | 418 | 418 | 418 | 418 |
| trading proportion (%) | 0.12 | 0.22 | 0.26 | 0.49 | 0.63 | 0.75 | 0.81 | 0.86 | 0.91 | 0.93 | 0.73 |
| maturity (years) | 11.46 | 13.28 | 12.84 | 10.32 | 10.99 | 11.49 | 11.46 | 11.54 | 11.98 | 14.25 | 11.34 |
| duration (year) | 6.35 | 6.88 | 6.75 | 6.41 | 6.52 | 6.79 | 7.16 | 7.43 | 4.91 | 8.61 | 6.73 |
| age (year) | 6.00 | 5.00 | 7.00 | 5.00 | 5.00 | 10.00 | 8.00 | 4.00 | 6.00 | 7.00 | 6.00 |
| log of amount outstanding | 11.31 | 11.71 | 11.51 | 11.92 | 11.74 | 11.92 | 12.18 | 12.32 | 11.92 | 11.95 | 11.92 |
| callability | 0.55 | 0.45 | 0.29 | 0.38 | 0.31 | 0.22 | 0.26 | 0.45 | 0.75 | 0.86 | 0.42 |
| yield to maturity (%) | 8.90 | 4.37 | 8.13 | 9.11 | 7.96 | 8.19 | 8.37 | 7.75 | 6.84 | 7.34 | 7.96 |
| yield spread (%) | 2.03 | 1.66 | 1.66 | 1.42 | 1.20 | 1.21 | 1.17 | 1.06 | 1.10 | 1.51 | 1.40 |

Panel C: rated bonds with at least 5% trading proportion

| number of bonds | 344 | 344 | 344 | 344 | 344 | 344 | 344 | 344 | 344 | 344 | 344 |
| trading proportion (%) | 0.23 | 0.25 | 0.27 | 0.51 | 0.65 | 0.62 | 0.87 | 0.91 | 0.94 | 0.72 |
| duration (year) | 7.75 | 7.18 | 7.47 | 6.72 | 0.83 | 7.41 | 7.82 | 7.24 | 5.39 | 8.44 | 7.11 |
| age (year) | 6.00 | 5.00 | 7.00 | 5.00 | 6.00 | 5.00 | 13.50 | 5.00 | 5.00 | 7.00 | 7.00 |
| log of amount outstanding | 11.51 | 11.72 | 11.74 | 11.92 | 11.92 | 11.92 | 12.21 | 12.32 | 11.92 | 12.07 | 11.92 |
| callability | 0.27 | 0.48 | 0.42 | 0.41 | 0.20 | 0.23 | 0.28 | 0.55 | 0.74 | 0.64 | 0.44 |
| yield to maturity (%) | 5.59 | 8.33 | 8.00 | 9.41 | 7.94 | 8.03 | 8.09 | 7.49 | 6.80 | 7.46 | 7.86 |
| yield spread (%) | 1.92 | 1.56 | 1.51 | 1.20 | 1.07 | 1.06 | 0.96 | 0.98 | 1.14 | 1.55 | 1.29 |

The bond years are divided into 10 deciles, by the ranking of trading frequency. Panel A includes all bond years. Panel B includes bond years with at least 5% trading frequency. Panel C includes bond years with at least 5% trading frequency and rating scores. Callability and bond rating are reported by the mean , and the rest are by the median. For callability, 1 stands for non-callable and 0 stands for callable.

Table I: Trading frequency and other characteristics
Both panels include all bond years with at least 5% trading frequency.

Table II: Trading frequency and other characteristics - callable vs. noncallable
### Estimated parameters from the LDV model

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<td>0.000008</td>
<td>0.010124</td>
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<td><strong>Panel C:</strong> non-callable bonds with 5% trading</td>
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<td>0.020252</td>
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<td>0.001066</td>
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<tr>
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<td>0.000004</td>
<td>0.009592</td>
<td>0.000677</td>
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</table>

Both panels include all bond years with at least 5% trading frequency.

Table III: Estimated parameters from LDV model
### Estimated Transaction Costs and Related Variables

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<tr>
<th>decile</th>
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<th>9</th>
<th>10</th>
<th>total</th>
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</thead>
<tbody>
<tr>
<td>Panel A: All bond years</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>number of bond years</td>
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<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
<td>411</td>
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<tr>
<td>round-trip transaction costs (%)</td>
<td>0.05</td>
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<td>0.25</td>
<td>0.25</td>
<td>0.47</td>
<td>0.78</td>
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<td>2.31</td>
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<td>0.91</td>
<td>0.86</td>
<td>0.81</td>
<td>0.78</td>
<td>0.68</td>
<td>0.51</td>
<td>0.39</td>
<td>0.25</td>
<td>0.20</td>
<td>0.71</td>
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<td>-0.32</td>
<td>-0.31</td>
<td>-0.27</td>
<td>-0.28</td>
<td>-0.32</td>
<td>-0.41</td>
<td>-0.46</td>
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<tr>
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<td>0.25</td>
<td>0.17</td>
<td>0.27</td>
<td>0.29</td>
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<td>0.28</td>
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<td>0.03</td>
<td>0.06</td>
<td>0.08</td>
<td>0.10</td>
<td>0.12</td>
<td>0.12</td>
<td>0.17</td>
<td>0.22</td>
<td>0.34</td>
<td>0.10</td>
</tr>
<tr>
<td>maturity (years)</td>
<td>4.52</td>
<td>13.86</td>
<td>10.66</td>
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<td>10.77</td>
<td>10.82</td>
<td>11.70</td>
<td>12.03</td>
<td>12.11</td>
<td>17.82</td>
<td>11.45</td>
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<td>duration (years)</td>
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<td>5.38</td>
<td>6.98</td>
<td>7.58</td>
<td>6.73</td>
<td>6.13</td>
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<td>6.68</td>
<td>6.48</td>
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<td>6.76</td>
</tr>
<tr>
<td>age (years)</td>
<td>7.00</td>
<td>7.00</td>
<td>6.00</td>
<td>7.00</td>
<td>12.00</td>
<td>8.00</td>
<td>6.00</td>
<td>6.00</td>
<td>6.00</td>
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<tr>
<td>log of amount outstanding</td>
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<td>12.21</td>
<td>12.21</td>
<td>12.21</td>
<td>11.92</td>
<td>11.90</td>
<td>11.76</td>
<td>11.61</td>
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<td>11.51</td>
<td>11.92</td>
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<tr>
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<td>0.63</td>
<td>0.51</td>
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<td>0.26</td>
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<td>0.25</td>
<td>0.39</td>
<td>0.59</td>
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<tr>
<td>yield to maturity (%)</td>
<td>7.06</td>
<td>7.41</td>
<td>7.52</td>
<td>8.06</td>
<td>8.03</td>
<td>8.21</td>
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<td>8.46</td>
<td>9.27</td>
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<td>1.57</td>
<td>1.15</td>
<td>1.15</td>
<td>1.26</td>
<td>1.28</td>
<td>1.28</td>
<td>1.53</td>
<td>1.71</td>
<td>1.57</td>
<td>1.28</td>
</tr>
</tbody>
</table>

| Panel B: bond years with credit rating |
| number of bond years | 339 | 339 | 339 | 339 | 339 | 339 | 339 | 339 | 339 | 343 | 389 |
| round-trip transaction costs (%) | 0.05 | 0.11 | 0.21 | 0.31 | 0.44 | 0.72 | 1.26 | 2.24 | 3.99 | 7.99 | 0.55 |
| trading proportion | 0.91 | 0.92 | 0.87 | 0.62 | 0.78 | 0.70 | 0.55 | 0.40 | 0.24 | 0.72 | 0.72 |
| Beta1 | -0.85 | -0.85 | -0.72 | -0.44 | -0.32 | -0.34 | -0.29 | -0.28 | -0.28 | -0.35 | -0.50 |
| beta2*100 | 0.05 | 0.25 | 0.21 | 0.24 | 0.26 | 0.15 | 0.06 | 0.03 | 0.15 | 0.14 | 0.15 |
| sigma | 0.02 | 0.03 | 0.06 | 0.08 | 0.09 | 0.11 | 0.12 | 0.15 | 0.21 | 0.33 | 0.10 |
| duration (years) | 2.74 | 5.45 | 6.95 | 7.54 | 6.66 | 7.17 | 6.86 | 6.85 | 8.35 | 7.13 | 7.13 |
| age (years) | 7.00 | 7.00 | 6.00 | 7.00 | 11.00 | 8.00 | 7.00 | 7.00 | 6.00 | 5.00 | 7.00 |
| log of amount outstanding | 11.92 | 12.21 | 12.21 | 12.21 | 11.92 | 11.92 | 11.74 | 11.74 | 11.59 | 11.74 | 11.92 |
| callability | 0.65 | 0.65 | 0.56 | 0.31 | 0.26 | 0.27 | 0.37 | 0.37 | 0.42 | 0.53 | 0.44 |
| yield to maturity (%) | 7.02 | 7.43 | 7.40 | 7.06 | 7.01 | 7.98 | 8.26 | 8.35 | 8.33 | 8.33 | 7.36 |
| yield spread (%) | 1.07 | 1.58 | 1.98 | 2.50 | 2.03 | 1.10 | 1.69 | 1.43 | 1.60 | 1.50 | 1.28 |
| rating score | 8.10 | 8.41 | 8.52 | 8.91 | 8.77 | 9.05 | 9.05 | 9.44 | 9.62 | 9.11 | 8.90 |

This table reports the estimated transaction costs and other variables, grouped into deciles by the size of round-trip transaction costs. All figures are medians of the decile, except for callability and rating score, where mean of the deciles is reported.

Table IV: Estimated transaction costs and related variables
### Table V: Correlation matrix for relevant variables

<table>
<thead>
<tr>
<th></th>
<th>TF</th>
<th>Yi</th>
<th>Ma</th>
<th>Du</th>
<th>S</th>
<th>$</th>
<th>Age</th>
<th>C</th>
<th>RS</th>
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</thead>
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<td>Yi</td>
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<td>0.92</td>
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<td>-0.17</td>
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<td></td>
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<tr>
<td>$</td>
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<td>Age</td>
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<td>-0.27</td>
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<td>-0.10</td>
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<td>-0.06</td>
<td>-0.13</td>
<td>0.00</td>
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</table>

TF = trading frequency, Yi = yield to maturity, Ma = maturity, Du = duration, S = credit spread, $ = log of amount outstanding, Age = age of the bond year, C = callability, RS = rating score.
### Regression of transaction costs

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<th>a9</th>
<th>a1</th>
<th>a2</th>
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<th>a4</th>
<th>a5</th>
<th>a6</th>
<th>Adjusted R-square</th>
</tr>
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<tbody>
<tr>
<td><strong>Panel A: all bonds</strong></td>
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<td></td>
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<td>0.049</td>
<td>0.097</td>
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<td>0.0012</td>
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<td>0.0022</td>
<td>0.0002</td>
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<td>0.0002</td>
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<td>Regression 5</td>
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<td>0.0002</td>
<td>13.1524</td>
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<td></td>
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<tr>
<td><strong>Panel C: all non-callable bonds</strong></td>
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<td></td>
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<td>0.002</td>
<td>0.0001</td>
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<tr>
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<td>0.0001</td>
<td>0.0002</td>
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</table>

The cross-sectional regressions take the following form:

\[ y_i = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \alpha_3 x_{i3} + \alpha_4 x_{i4} + \alpha_5 x_{i5} + \alpha_6 x_{i6} + \epsilon_i \]

where index \( i \) = bond year \( i \), and \( y_i = \alpha_2 - \alpha_3 \) (estimated transaction cost band), \( x_{i1} = \) trading frequency, \( x_{i2} = \sigma_i \) (estimated volatility), \( x_{i3} = \) duration, \( x_{i4} = \log \) of bond size, \( x_{i5} = \) age, \( x_{i6} = \) rating score, \( \epsilon_i = \) error term.

For each regression, the first row contains the parameter estimates, and second row contains corresponding t-values. The last column reports the adjusted R-square.

Table VI: Regression of transaction costs

122
### Transaction costs and related variables (with firm size)

<table>
<thead>
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<th>2</th>
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<th>4</th>
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<th>9</th>
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<th>total</th>
</tr>
</thead>
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<td>152</td>
<td>152</td>
<td>152</td>
<td>152</td>
<td>152</td>
<td>152</td>
<td>152</td>
<td>152</td>
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<td>152</td>
</tr>
<tr>
<td>round-trip transaction costs (%)</td>
<td>0.16</td>
<td>0.13</td>
<td>0.23</td>
<td>0.26</td>
<td>0.59</td>
<td>1.07</td>
<td>1.41</td>
<td>2.83</td>
<td>4.06</td>
<td>9.27</td>
<td>8.55</td>
</tr>
<tr>
<td>trading proportion</td>
<td>0.92</td>
<td>0.91</td>
<td>0.85</td>
<td>0.81</td>
<td>0.75</td>
<td>0.60</td>
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<td>-0.86</td>
<td>-0.68</td>
<td>-0.44</td>
<td>-0.23</td>
<td>-0.23</td>
<td>-0.23</td>
<td>-0.23</td>
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<td>-0.23</td>
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</tr>
<tr>
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<td>-0.16</td>
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<td>-0.45</td>
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<td>0.03</td>
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<td>0.13</td>
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<td>0.18</td>
<td>0.20</td>
<td>0.24</td>
<td>0.11</td>
</tr>
<tr>
<td>maturity (years)</td>
<td>5.90</td>
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<td>10.01</td>
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<td>9.98</td>
<td>11.13</td>
<td>10.92</td>
<td>16.80</td>
<td>15.01</td>
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</tr>
<tr>
<td>duration (years)</td>
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<td>4.88</td>
<td>5.04</td>
<td>3.56</td>
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<td>3.50</td>
<td>5.95</td>
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<tr>
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<td>5.00</td>
<td>5.00</td>
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<td>5.00</td>
<td>5.00</td>
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</tr>
<tr>
<td>log of amount outstanding</td>
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<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
<td>12.43</td>
</tr>
<tr>
<td>yield to maturity (%)</td>
<td>7.19</td>
<td>7.27</td>
<td>7.43</td>
<td>8.54</td>
<td>8.78</td>
<td>8.16</td>
<td>8.37</td>
<td>8.55</td>
<td>8.60</td>
<td>9.44</td>
<td>8.89</td>
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<tr>
<td>credit spread (%)</td>
<td>1.15</td>
<td>1.37</td>
<td>1.08</td>
<td>1.72</td>
<td>1.77</td>
<td>1.49</td>
<td>1.63</td>
<td>1.33</td>
<td>1.71</td>
<td>1.29</td>
<td>1.44</td>
</tr>
<tr>
<td>log of firm size</td>
<td>16.83</td>
<td>17.01</td>
<td>16.38</td>
<td>15.67</td>
<td>15.49</td>
<td>15.45</td>
<td>15.06</td>
<td>15.05</td>
<td>14.72</td>
<td>15.02</td>
<td>15.57</td>
</tr>
</tbody>
</table>

#### Panel A: bonds that have firm size data

| number of bond years | 76 | 76 | 78 | 78 | 78 | 78 | 78 | 78 | 78 | 78 | 78 |
| round-trip transaction costs (%) | 0.07 | 0.24 | 0.26 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| trading proportion | 0.92 | 0.84 | 0.81 | 0.76 | 0.63 | 0.49 | 0.33 | 0.28 | 0.22 | 0.16 | 0.55 |
| beta | -0.4 | -0.36 | -0.26 | -0.16 | -0.16 | -0.28 | -0.28 | -0.28 | -0.28 | -0.28 | -0.28 |
| beta2*100 | 0.11 | 0.37 | 0.29 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 |
| sigma | 0.02 | 0.05 | 0.10 | 0.12 | 0.14 | 0.12 | 0.13 | 0.19 | 0.20 | 0.22 | 0.28 |
| duration (years) | 9.43 | 7.57 | 6.51 | 5.82 | 5.03 | 5.86 | 6.61 | 7.21 | 4.5 | 7.92 | 6.70 |
| age (years) | 8.00 | 5.00 | 5.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| log of amount outstanding | 12.43 | 12.61 | 12.43 | 11.92 | 11.92 | 11.87 | 11.91 | 11.59 | 11.51 | 11.21 | 11.92 |
| yield to maturity (%) | 7.08 | 8.12 | 8.44 | 8.44 | 8.91 | 8.50 | 8.51 | 8.95 | 9.24 | 9.40 | 5.89 |
| credit spread (%) | 2.20 | 1.58 | 2.47 | 2.36 | 2.29 | 2.58 | 1.90 | 2.18 | 2.11 | 2.13 | 2.19 |

#### Panel B: callable bonds that have firm size data

| number of bond years | 76 | 76 | 74 | 74 | 74 | 74 | 74 | 74 | 74 | 74 | 74 |
| round-trip transaction costs (%) | 0.05 | 0.09 | 0.15 | 0.22 | 0.24 | 0.34 | 0.75 | 1.52 | 2.49 | 5.12 | 0.51 |
| trading proportion | 0.91 | 0.92 | 0.90 | 0.88 | 0.82 | 0.72 | 0.53 | 0.46 | 0.28 | 0.21 | 0.78 |
| duration (years) | -9.2 | -1.0 | -0.6 | -0.6 | -0.26 | -0.54 | -0.49 | -0.43 | -0.26 | -0.26 | -0.77 |
| age (years) | 3.64 | 7.09 | 6.94 | 6.04 | 6.13 | 5.93 | 6.85 | 6.90 | 7.66 | 9.94 | 6.76 |
| yield to maturity (%) | 6.72 | 7.08 | 6.73 | 6.97 | 7.25 | 7.37 | 7.62 | 7.68 | 6.60 | 6.85 | 7.45 |
| yield spread (%) | 1.01 | 1.23 | 0.90 | 0.80 | 0.95 | 0.81 | 0.96 | 1.04 | 1.26 | 1.29 | 1.95 |
| log of firm size | 16.29 | 17.25 | 17.01 | 16.88 | 16.27 | 15.88 | 15.65 | 15.47 | 15.21 | 15.61 | 16.06 |

This table replicates Table 4 with bonds that have firm size data.

Table VII: Transaction costs and related variables

123
### Regression of transaction costs on related variables (with firm size)

<table>
<thead>
<tr>
<th></th>
<th>b0</th>
<th>b1</th>
<th>b2</th>
<th>b3</th>
<th>b4</th>
<th>b5</th>
<th>b6</th>
<th>b7</th>
<th>Adjusted R-square</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: all bonds with firm size data</strong></td>
<td></td>
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<tr>
<td>Regression 1</td>
<td>0.0339</td>
<td>-0.0051</td>
<td>0.1024</td>
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<td></td>
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<td>0.6287</td>
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<td></td>
<td>21.5773</td>
<td>-25.5674</td>
<td>24.2896</td>
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<td></td>
</tr>
<tr>
<td>Regression 2</td>
<td>0.0164</td>
<td>-0.0028</td>
<td>0.1113</td>
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<tr>
<td></td>
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<td>-20.3859</td>
<td>28.1608</td>
<td>15.9357</td>
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<td></td>
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<td></td>
<td></td>
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<tr>
<td>Regression 3</td>
<td>-0.0095</td>
<td>-0.0050</td>
<td>0.1158</td>
<td>0.0024</td>
<td>0.0023</td>
<td>-0.0005</td>
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<td></td>
<td>0.6922</td>
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<td></td>
<td>-1.3793</td>
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<td>29.3774</td>
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<td>7.1409</td>
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<td>Regression 4</td>
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<td>-0.0047</td>
<td>0.1151</td>
<td>0.0023</td>
<td>0.0022</td>
<td>-0.0005</td>
<td>-0.0002</td>
<td></td>
<td>0.6474</td>
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<tr>
<td>Regression 5</td>
<td>0.0088</td>
<td>-0.0051</td>
<td>0.1175</td>
<td>0.0021</td>
<td>0.0016</td>
<td>-0.0006</td>
<td>-0.0002</td>
<td>-0.0004</td>
<td>0.6927</td>
</tr>
</tbody>
</table>

**Panel B: all callable bonds with firm size data**

| Regression 1 | 0.0135  | -0.0054  | 0.0877  |          |          |          |          |          | 0.6028          |
|              | 13.0675 | -20.6278 | 18.4655 |          |          |          |          |          |                  |
| Regression 2 | 0.0132  | -0.0054  | 0.0843  | 0.0035  |          |          |          |          | 0.6962          |
|              | 4.8961  | -23.2776 | 21.8458 | 12.5231 |          |          |          |          |                  |
| Regression 3 | 0.0078  | -0.0059  | 0.1054  | 0.0034  | 0.0004  | -0.0001 |          |          | 0.6559          |
|              | 0.9727  | -24.9108 | 21.5211 | 15.4760 | 1.0265  | -0.199  |          |          |                  |
| Regression 4 | 0.0127  | -0.0059  | 0.1053  | 0.0032  | 0.0006  | -0.0004 | -0.0003 |          | 0.6499          |
|              | 1.5368  | -25.0125 | 21.0192 | 12.2518 | 1.4856  | -0.6771 | -3.0121 |          |                  |
| Regression 5 | 0.0341  | -0.0058  | 0.1019  | 0.0027  | 0.0004  | -0.0031 | -0.0004 | -0.0002 | 0.6958          |
|              | 3.4571  | -19.1922 | 18.4522 | 9.4466  | 0.6338  | -2.3484 | -3.5155 | -1.5981 |                  |

**Panel C: all non-callable bonds with firm size data**

| Regression 1 | 0.0048  | -0.0018  | 0.1112  |          |          |          |          |          | 0.676           |
|              | 14.5851 | -17.3455 | 17.0324 |          |          |          |          |          |                  |
| Regression 2 | 0.0254  | -0.0047  | 0.1178  | 0.0021  |          |          |          |          | 0.6995          |
|              | 6.3457  | -15.4328 | 18.4964 | 7.7249  |          |          |          |          |                  |
| Regression 3 | -0.0142 | -0.0022  | 0.1201  | 0.0017  | 0.0001  | -0.0003 |          |          | 0.7093          |
|              | -1.5819 | -16.3985 | 19.1271 | 10.1287 | 5.7129  | -1.4504 |          |          |                  |
| Regression 4 | -0.0172 | -0.0023  | 0.1207  | 0.0017  | 0.0002  | -0.0003 | 0.0001  |          | 0.7105          |
|              | -1.2844 | -16.4421 | 19.9393 | 6.2417  | 5.2965  | -1.2124 | 1.2408  |          |                  |
| Regression 5 | -0.025 | -0.0014  | 0.1243  | 0.0018  | 0.0005  | -0.0002 | 0.0001  | 0.0002  | 0.7103          |
|              | -1.5953 | -13.9576 | 13.1273 | 5.8849  | 3.9240  | -0.9669 | 1.2823  | 0.6002          |

The cross-sectional regressions take the following form:

\[ y_i = b_0 + b_1 x_{1i} + b_2 x_{2i} + b_3 x_{3i} + b_4 x_{4i} + b_5 x_{5i} + b_6 x_{6i} + b_7 x_{7i} + \epsilon_i, \]

where index \( i = \) bond year \( i, \) and \( y_i = \alpha_{2i} - \alpha_{1i} \) (estimated transaction cost band), \( x_{1i} = \) trading frequency, \( x_{2i} = \sigma_i \) (estimated volatility), \( x_{3i} = \) duration, \( x_{4i} = \) log of bond size, \( x_{5i} = \) log of firm size, \( x_{6i} = \) age, \( x_{7i} = \) rating score, \( \epsilon_i = \) error term. For each regression, the first row contains the parameter estimates, and second row contains corresponding t-values. The last column reports the adjusted R-square.

Table VIII: Regression of transaction costs on related variables (with firm size)
The cross-sectional regressions take the following form:

\[ y_i = c_0 + c_1 x_{1i} + c_2 x_{2i} + c_3 x_{3i} + \epsilon_i, \]

where index \( i \) = bond year \( i \), and \( y_i \) = yield spread, \( x_{1i} \) = rating score, \( x_{2i} \) = trading frequency, \( x_{3i} \) = duration, \( \epsilon_i \) = error term. For each regression, the first row contains the parameter estimates, and second row contains corresponding \( t \)-values. The last column reports the adjusted R-square.

### Table IX: Regression of yield spread on related variables

<table>
<thead>
<tr>
<th>Panel A: all bonds with rating score</th>
<th>Panel B: callable bonds with rating score</th>
<th>Panel C: non-callable bonds with rating score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression 1:</td>
<td></td>
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</tr>
<tr>
<td>( c_0 )</td>
<td>0.0044</td>
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<tr>
<td>( c_1 )</td>
<td>0.0019</td>
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<td>( c_2 )</td>
<td>-0.0054</td>
<td>-0.0005</td>
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<tr>
<td>( c_3 )</td>
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<td>0.0001</td>
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<tr>
<td>adjusted R-square</td>
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<tr>
<td>( t )-value</td>
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<tr>
<td>Regression 2:</td>
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<tr>
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<tr>
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<td>( t )-value</td>
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