Nonlinear State-Estimation for Spacecraft Attitude Determination

by

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A thesis submitted in conformity with the requirements for the degree of Master of Applied Science
Graduate Department of Aerospace Science and Engineering
University of Toronto

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Abstract

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Spacecraft requirements are changing. Smaller, lower-cost spacecrafts are more desirable, while the desired performance is increasing. The effect of using cheaper hardware may be offset by the use of more sophisticated controller and estimation schemes.

This thesis compares the Extended Kalman Filter (EKF) with the Nonlinear Predictive Filter (NPF) and the extended $H_{\infty}$ filter (EHF). The EHF is a linearized $H_{\infty}$ filter, completely analogous to the EKF, which is a linearized Kalman filter.

Under certain circumstances, the NPF outperforms the EKF. However, the requirements for these circumstances conflict with filter robustness with respect to initial estimate error. This problem is solved by applying the NPF differently, but comes at the cost of robustness with respect to plant error. The EHF performs poorly in all aspects.

The EKF has the least sensitivity to sensor noise level and is the estimation scheme of choice.
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Chapter 1

Introduction

State estimation is an important function within an engineering system. A large number of systems, including spacecraft dynamics, can be described by a set of first order ordinary differential equations

\[ \dot{x} = f(x, t). \]

Knowledge of the states, \( x \), gives the complete information of the system behaviour, and is necessary for the control purpose. However, in many instances it is too expensive or downright impossible to measure a system's states directly (consider the attitude parameterizations, which are mathematical creations useful for describing the spacecraft attitude, but not physically measurable). In these cases, a state estimation scheme is needed to calculate the states from the available measurements. Even in cases where the states are measured directly, state estimation schemes can be useful to minimize the effect of sensor noise.

The Extended Kalman Filter (EKF) has become the standard state estimation scheme for nonlinear systems. The EKF is based upon the linearized dynamics of the system it is estimating. It has the advantage that it is computationally inexpensive, and fairly robust with respect to model errors. However, the disadvantage is that when used for severely nonlinear systems (the EKF being a linearized technique), the EKF may perform poorly since the nonlinearities are not fully accounted for. Large initial estimation errors may also cause the EKF to diverge since its inherent stability is only guaranteed locally.

With the advent of the commercial space industry comes the need for smaller, cheaper spacecrafts, while the performance requirements are becoming more stringent (for
example, a space telescope is required to accurately track a star. The degradation in performance due to the use of cheaper hardware may be offset via the use of more sophisticated controller and estimation schemes.

Traditionally, the EKF has been the estimation scheme used for spacecraft attitude determination. The past decade has seen much research done in the field of nonlinear state estimation. However, few of these schemes have been applied to the spacecraft attitude determination problem.

The remainder of this chapter surveys recently developed approaches to the general nonlinear filtering problem and different approaches that have been attempted for the spacecraft attitude determination problem. Finally, the scope of work in this thesis is addressed.

1.1 Nonlinear Filtering

Since the spacecraft attitude dynamics are continuous in time, but the measurements are at discrete instances only, we only concern ourselves with the continuous-discrete filter (i.e. continuous plant, discrete measurements). The spacecraft dynamics can be discretized and hence the discrete-discrete filter is also useful, but to keep the study to a manageable size, it is skipped.

The approaches taken to solving the filtering problem can be divided into two general categories: the stochastic approach, and the deterministic approach. In the stochastic approach, the state variables are assumed to be random, and satisfy the n-dimensional stochastic differential equation

$$dx_t = f(x_t, t)dt + G(x_t, t)d\beta_t$$

where $x_t \in \mathbb{R}^n$ is the random state-variable, $f(x_t, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $G(x_t, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times q}$ is a function determining how the disturbance $\beta$ is added to the dynamics, and $\beta_t \in \mathbb{R}^q$ is Brownian motion with covariance $Q(t)dt$. The m-dimensional measurement equation is assumed to be of the form

$$y_k = h(x_{tk}, t) + v_k$$

where $y_k \in \mathbb{R}^m$ is the measurement, $h(x_{tk}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ and $v_k$ is a white Gaussian sequence with covariance $R_k$. 

In the deterministic case, the plant is assumed to have the form

$$\dot{x} = f(x,t) + G(x,t)w(t)$$

where $x \in \mathbb{R}^n$ is the state-variable and $w(t) \in \mathbb{R}^w$ is a deterministic disturbance. The measurement equation has the same form as for the stochastic case, but the noise has no assumed statistical properties.

1.1.1 Stochastic Setting

We define the set of measurements up to and including time $t$ as $Y_t = \{y_k : t_k \leq t\}$. From probability theory, the density function $p(x,t)$ of the random variable $x$ has the property that the mean value of $x$ (the estimate of $x$) is given by

$$\hat{x} = \int x p(x,t)dx.$$

The filtering problem therefore becomes one of finding an equation to propagate the conditional density function $p(x,t|Y_{t_k})$. The exact solution to this problem is well-known.

Between observations $t_k$ and $t_{k+1}$, the conditional density diffuses according to Kolmogorov's forward equation (also known as the Fokker-Planck equation)

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial (pf_i)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 [p(GQG^T)_{ij}]}{\partial x_i \partial x_j}.$$

At observations, the conditional density is updated using Baye’s formula

$$p(x,t_{k+1}|Y_{t_{k+1}}) = \frac{p(y_{k+1}|x)p(x,t_{k+1}|Y_{t_k})}{\int p(y_{k+1}|\xi)p(\xi,t_{k+1}|Y_{t_k})d\xi}$$

where

$$p(y_{k+1}|x) = \exp\{-\frac{1}{2}[y_k - h(x,t_k)]^T R_k^{-1}[y_k - h(x,t_k)]\}/\sqrt{(2\pi)^m|R_k|}.$$

Clearly, there are problems with the implementation of this filter. The first is that in general, closed-form solutions to Kolmogorov’s equation do not exist. Secondly, the integrals in Baye’s formula and in the calculation of the state estimates are over infinite domains, and are computationally expensive.

The research in the stochastic domain focuses on solving Kolmogorov’s equation. Many approaches are available, and they are all necessarily approximate. Most papers suggest using series solutions for the conditional density function, such as the application
of Galerkin's method\textsuperscript{1,14,23}. They present different methods of choosing the basis functions. Application of this kind of method results in Kolmogorov's equation being reduced to a set of first order linear time-varying ordinary differential equations. Beard et al.\textsuperscript{1} have called this approach the Nonlinear Projection Filter. Challa and Bar-Shalom\textsuperscript{2} suggest an explicit finite difference method, however this is not the main contribution of their work. Their main contribution is a systematic way of truncating the domain over which the integrals are calculated, using Chebychev's inequality theorem.

A point of interest is that for linear systems, the exact nonlinear filter reduces to the standard Kalman filter\textsuperscript{11}. This suggests that finding a good approximation to the exact filter will result in improvements over the extended Kalman filter.

1.1.2 Deterministic Setting

While there are a number of different approaches to nonlinear filtering in the deterministic setting, most approaches focus on minimizing a ratio of norms of the form

\[ J = \frac{||\text{estimation error}||^2}{||\text{plant and sensor noise}||^2}. \]

In the linear case, this approach is called the $H_\infty$ filter, based on the fact that it is the $H_\infty$ induced norm that is being minimized. In the nonlinear case, it has been analogously termed the nonlinear $H_\infty$ filter, even though there is no $H_\infty$ norm to speak of.

There are many variants on the way the nonlinear $H_\infty$ filtering problem is defined\textsuperscript{10,16,17,18,19,20}. The drawback of much of this work is that they only give sufficient conditions for a filter to exist, but leave the user with the onus of finding such a filter. Also, in most cases, the form of the filter is chosen a-priori, so there is no guarantee that the resulting filter will be globally optimal.

1.1.3 Nonlinear Predictive Filter

A radically different approach to the nonlinear filtering problem is the Nonlinear Predictive Filter (NPF), derived by Crassidis and Markley\textsuperscript{4}. The details of this filter can be found in Chapter 3.

The advantage of this filter is that no assumption is made about the plant disturbances, and it is estimated as part of the filtering process. This suggests that the filter has very good robustness properties.
1.2 Spacecraft Attitude Determination

There are many approaches to the spacecraft attitude determination problem. Many of them do not use dynamic filters\textsuperscript{24}. These require the presence of at least two vector measurements, and they run into problems when the vector measurements become coaligned. Moreover, these methods can result in poor angular rate estimates. For this reason, dynamic filters are more desirable.

As already mentioned, the EKF has often been used. Lefferts et al. provide a thorough description of this\textsuperscript{13}. Crassidis and Markley have applied the NPF to the spacecraft attitude determination problem and have shown it to be more robust than the EKF when only one vector measurement is available\textsuperscript{5}. They have not compared it with more measurements available. Markley et al. have applied nonlinear $H_{\infty}$ filtering to the attitude determination problem\textsuperscript{16}. However, they assume a specific form of filter, and so it is not necessarily globally optimal. They also use continuous-time measurements.

1.3 This Thesis

In this thesis, three filters are compared, the extended Kalman filter, the Nonlinear Predictive Filter and the extended $H_{\infty}$ filter (EHF). The EHF is a linearized $H_{\infty}$ filter, completely analogous to the EKF, where instead of applying the Kalman filter to the system's linearized dynamics, a linear $H_{\infty}$ filter is applied. This is to obtain a comparison between the stochastic and deterministic approaches. The NPF was chosen, since it has the potential for a considerable amount of robustness with respect to modelling error.

The filter performances are compared with respect to four criteria: performance with respect to model error, performance with respect to sensor misalignment, performance with respect to initial estimate error and performance with respect to sensor noise level. The effect of each is examined separately.

The spacecraft under consideration is unforced (no control input). The justification for this is that the control input is generally additive in both the spacecraft dynamics and the filter dynamics. Hence, the control input cancels out in the estimation error dynamics.

The main contributions of this thesis are:

- a consistent framework is established for comparing the performances of different filters in the context of spacecraft attitude determination,
- the existing NPF is extended to time-varying systems,
- the EHF is derived and spacecraft attitude determination filter implementations for the NPF and EHF are established.

It is also shown that the EKF is the superior technique for spacecraft attitude determination of the different techniques attempted in this thesis, thus presenting the need for further investigation into more sophisticated techniques such as the Nonlinear Projection Filter and the Nonlinear $H_\infty$ Filter.
Chapter 2

Spacecraft Attitude Dynamics

2.1 Preliminaries

The notation used is the vectrix notation coined by Hughes\(^9\). The following reference frames are used in this chapter:

- \( F_I \) Geocentric inertial frame
- \( F_b \) Body-fixed spacecraft frame
- \( F_o \) Perifocal coordinate system frame

Vernal equinox is taken to occur at time \( t = 0 \). Consequently, the sun’s direction is given by

\[
\mathbf{s} = \mathbf{F}_I^T s_I
\]  

(2.1)

where \( s_I = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \).

Generally speaking, the last subscript on a column vector denotes the frame of reference it is expressed in (unless it is otherwise defined). Given that the simulations are for two orbits (~202 minutes), which is very small compared to the earth’s orbital period about the sun, the sun-pointing vector is taken to be fixed.

The right ascension of the Greenwich meridian \( \alpha_G(t) \) is taken to satisfy

\[
\alpha_G(0) = 0.
\]  

(2.2)
The operator \((\cdot)^x\) is the cross-product operator on a 3-dimensional column vector and is defined as
\[
\begin{bmatrix}
  a_1 & a_2 & a_3
\end{bmatrix}^T \overset{\times}{=} \begin{bmatrix}
  0 & -a_3 & a_2 \\
  a_3 & 0 & -a_1 \\
  -a_2 & a_1 & 0
\end{bmatrix}.
\]

A convex region is defined as being a region such that a straight line joining any two points in that region is also contained in that region. Mathematically this is stated as follows:

**Definition** A region \(S \in \mathbb{R}^3\) is said to be convex if for every point \(x, y \in S\) and every real number \(\theta, 0 < \theta < 1\), the point \(\theta x + (1 - \theta) y \in S\) also.

### 2.2 Orbital Dynamics Model

The orbital dynamics are taken to be Keplerian. This is a highly simplified treatment of them. However, the focus of this thesis is spacecraft attitude determination, not orbit determination. In essence, we are assuming that the spacecraft orbital dynamics can be perfectly modeled. While the solution to the unperturbed two-body problem is well-known, it is summarized here for completeness.

The orbit is described by six orbital elements which are fixed for a Keplerian orbit:

- \(a\) semi-major axis
- \(e\) eccentricity
- \(i\) inclination
- \(\Omega\) right ascension of the ascending node
- \(\omega\) argument of perigee
- \(t_0\) time of periapsis (= 0)

Given these, the spacecraft position and velocity are readily calculated for any time \(t\).

We begin by solving Kepler’s equation for the eccentric anomaly, \(E\), given by
\[
g(E) \overset{\triangle}{=} E - e \sin E - \sqrt{\frac{\mu}{a^3}} (t - t_0) = 0 \quad (2.3)
\]

This is solved using a Newton-Raphson method iteratively which involves consecutively evaluating
\[
E_{k+1} = E_k - \frac{g(E_k)}{g'(E_k)}, \quad k \in \{0, 1, 2, \ldots\}
\]
Section 2.2. Orbital Dynamics Model

until either \(|E_{k+1} - E_k| < \delta\) or \(|g(E_k)| < \epsilon\) where \(0 < \delta \ll 1\) and \(0 < \epsilon \ll 1\) are prescribed tolerances. The derivative of \(g(E)\) is given by \(g'(E) = 1 - e \cos E\). As initial condition we use \(E_0 = \sqrt{\mu/\mathbf{a}}(t - t_0)\), and \(\mu\) is the geocentric gravitational parameter.

Having calculated \(E\), the true anomaly, \(\theta\) is then calculated from

\[
\theta = 2 \arctan \left( \frac{1 + e}{1 - e} \tan \left( \frac{E}{2} \right) \right).
\]  

(2.4)

Given the true anomaly, the orbital radius is calculated

\[
r = \frac{a(1 - e^2)}{1 + e \cos \theta}.
\]  

(2.5)

The spacecraft speed is then calculated from the vis-viva equation

\[
v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)}.
\]  

(2.6)

With \(r\) and \(\theta\) in hand, the spacecraft position and velocity are readily calculated in inertial coordinates as follows:

\[
\mathbf{r} = \mathbf{F}_I^T \mathbf{R}_I = \mathbf{F}_I^T \mathbf{C}_{fo} \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ 0 \end{bmatrix}
\]  

(2.7)

and

\[
\mathbf{v} = \mathbf{F}_I^T \mathbf{v}_I = \mathbf{F}_I^T \mathbf{C}_{fo} \begin{bmatrix} -\sin \theta \\ \cos \theta + e \end{bmatrix} \sqrt{\frac{\mu}{a(1 - e^2)}}
\]  

(2.8)

where 

\[
\mathbf{C}_{fo} = \mathbf{C}_{3I}^T = [C_3(\omega)C_1(i)C_3(\Omega)]^T
\]

\[
= \begin{bmatrix}
c_{\Omega}c_{\omega} - s_\omega c_{\Omega}s_{\Omega} & -c_{\Omega}s_{\omega} - c_{\omega}c_{\Omega}s_{\Omega} & s_{\Omega}s_{\Omega} \\
s_{\Omega}c_{\omega} + s_{\omega}c_{\Omega}s_{\Omega} & -s_{\Omega}s_{\omega} + c_{\omega}c_{\Omega}s_{\Omega} & -s_{\Omega}s_{\Omega} \\
s_{\omega}c_{\omega} & s_{\omega}c_{\omega} & c_{\omega}
\end{bmatrix}.
\]

Equations (2.7) and (2.8) completely describe the spacecraft orbital motion, and contain information required by subsequent calculations of the earth’s magnetic field and external torques on the spacecraft.

\footnote{We use the notation \(s_a = \sin a\), \(c_a = \cos a\) and \(t_a = \tan a\).}
2.3 Attitude Dynamics Model

The spacecraft rotational motion is fully described by the absolute angular velocity and a 3-dimensional attitude parameterization. The absolute angular velocity is denoted by $\mathbf{\omega} = \mathbf{F}_b^T \omega$. The spacecraft attitude is parameterized by a 3-2-1 Euler sequence denoted by $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$, which has a corresponding rotation matrix

$$C_b = [C_1(\theta_1)C_2(\theta_2)C_3(\theta_3)]^T.$$  \hspace{1cm} (2.9)

The attitude kinematics, which couple $\mathbf{\omega}$ and $\theta$ are given by

$$\dot{\theta} = S^{-1}(\theta)\mathbf{\omega}$$ \hspace{1cm} (2.10)

where

$$S^{-1}(\theta) = \begin{bmatrix} 1 & s_\theta \cdot s_\phi & c_\theta \cdot s_\phi \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta / c_\phi & c_\theta / c_\phi \end{bmatrix}.$$ \hspace{1cm}

The spacecraft dynamics expressed in the spacecraft body frame, $F_b$, are simply Euler’s equations with disturbance terms and are given by

$$I\ddot{\mathbf{\omega}} + \mathbf{\omega} \times I\mathbf{\omega} = G_{gb} + G_{mb} + G_{ab} + G_{sb}$$ \hspace{1cm} (2.11)

where $I$ is the spacecraft moment of inertia matrix expressed in $F_b$. $G_{gb}$, $G_{mb}$, $G_{ab}$ and $G_{sb}$ are the gravity-gradient, geomagnetic, aerodynamic and solar-pressure torques respectively. Note that there are no control torques - the spacecraft under consideration is unforced. The gravity-gradient and geomagnetic torques are given by

$$G_{gb} = \frac{3\mu}{r^5} R_b^x I R_b, \quad R_b = C_b R_f$$ \hspace{1cm} (2.12)

and

$$G_{mb} = m_b^x B_b, \quad B_b = C_b B_f$$ \hspace{1cm} (2.13)

where $R_b$, $m_b$ and $B_f$ are the spacecraft position, spacecraft residual magnetic dipole moment and earth’s magnetic field respectively. The aerodynamic and solar-pressure torques are dealt with later in this chapter.

Equations (2.10) and (2.11) together completely determine the attitude motion of the spacecraft. It will prove useful for filter design to write these in the first order form

$$\dot{x} = f(x, t).$$ \hspace{1cm} (2.14)
This is accomplished by defining the state vector to be $x^T \triangleq \begin{bmatrix} \omega^T & \theta^T \end{bmatrix}$ and defining

$$f(x, t) \triangleq \begin{bmatrix} I^{-1}(-\omega^T \omega + G_{gb} + G_{mb} + G_{ab} + G_{ab}) \\ S^{-1}(\theta) \omega \end{bmatrix}.$$ 

The geomagnetic, aerodynamic and solar-pressure torques are treated as disturbance torques, and as such are included in the simulation of spacecraft motion, but not in the plant model used for filter design. The justifications for this are two-folded. Firstly, the gravity-gradient is the only torque that can be modelled accurately while maintaining simplicity in the mathematics (the form of the geomagnetic torque is simple also, but unlike the inertia matrix $I$, the spacecraft residual magnetic dipole $m_b$ is not accurately known, nor is it necessarily constant). Secondly, as shown in figures 2.1-2.4, for the scenario under consideration, the gravity-gradient torque is of order $10^{-5}$ Nm, where as the geomagnetic and solar-pressure torques are of order $10^{-6}$ Nm and the aerodynamic torque is of order $10^{-7}$ Nm. Hence, the main driving torque for the attitude dynamics is the gravity-gradient torque, and thus, the filter model does capture most of the spacecraft attitude dynamics. Thus, the nominal plant used for filter design is given by

$$\dot{x} = f(x, t) \quad (2.15)$$

where

$$f(x, t) \triangleq \begin{bmatrix} f_\omega \\ f_\theta \end{bmatrix}$$

with

$$f_\omega = I^{-1}(-\omega^T \omega + G_{gb})$$

$$f_\theta = S^{-1}(\theta) \omega$$

### 2.4 Geomagnetic Field Model

To maintain simplicity, the geomagnetic field model adopted is a dipole approximation. In inertial coordinates it is given by

$$\mathbf{B}_I = \begin{bmatrix} (B_r c_\delta + B_\theta s_\delta)c_\alpha - B_\phi s_\alpha \\ (B_r c_\delta + B_\theta s_\delta)s_\alpha + B_\phi c_\alpha \\ B_r s_\delta - B_\theta c_\delta \end{bmatrix} \quad (2.16)$$
where \( \alpha \) and \( \delta \) are that spacecraft right ascension and declination respectively. They are given by \(^2\)

\[
\alpha = \arctan \left( \frac{[R_f]_2}{[R_f]_1} \right)
\]

(2.17)

and \(^3\)

\[
\delta = \arctan \left( \frac{[R_f]_3}{\sqrt{[R_f]_1^2 + [R_f]_2^2}} \right)
\]

(2.18)

\{B_r, B_\theta, B_\phi\} are the geomagnetic field components expressed in a spherical coordinate frame fixed to the rotating earth. They are given by

\[
B_r = 2 \left( \frac{a_e}{r} \right)^3 \left[ g^0_c \theta_m + (g^1_c \phi_m + h^1_\phi s \phi_m) s \theta_m \right]
\]

\[
B_\theta = \left( \frac{a_e}{r} \right)^3 \left[ g^0_s \theta_m - (g^1_s \phi_m + h^1_\phi s \phi_m) c \theta_m \right]
\]

\[
B_\phi = \left( \frac{a_e}{r} \right)^3 \left[ g^1_s \phi_m - h^1_\phi c \phi_m \right]
\]

where \( a_e \) is the radius of the earth, \( \theta_m = 90^\circ - \delta \) is the spacecraft co-elevation, \( \phi_m = \alpha - \alpha_G \) is the spacecraft east-latitude and \( g^0_s, g^1_s \) and \( h^1_\phi \) are geomagnetic field coefficients.

Given the assumption (2.2), the right ascension of the Greenwich meridian is given by

\[
\alpha_G(t) = \alpha_G t
\]

(2.19)

where \( \alpha_G = 360.9856469 \) deg/day.

\section*{2.5 Aerodynamic Torque}

To avoid unnecessarily complex calculations (aerodynamic torque is an unmodelled disturbance torque anyway), some simplifying assumptions are made.

**Assumptions:**

1. The momentum of molecules arriving at the spacecraft surface is totally lost to the surface.
2. The motion of the earth's atmosphere is negligible compared to the spacecraft speed.
3. The velocity of points on the spacecraft due to its rotational motion is negligible compared to the spacecraft orbital speed.

\(^2\)The signs of \{R_f\}_1 and \{R_f\}_2 determine which quadrant \( \alpha \) lies in.

\(^3\)-\( \frac{\pi}{2} \leq \delta \leq \frac{\pi}{2} \).
4. The spacecraft body is convex.

Having made these assumptions, the aerodynamic torque is readily found to be

\[ \mathbf{G}_a = \mathbf{c}_{pa} \times \mathbf{F}_a \]

where

\[
\begin{align*}
\mathbf{c}_{pa} &= \frac{1}{A_{pa}} \int \int_{A_{wa}} \hat{\rho} \frac{\hat{n} \cdot \mathbf{v}}{v} \, dA \\
&= \text{centre of pressure}
\end{align*}
\]

\[
\begin{align*}
\mathbf{F}_a &= -\rho_a v A_{pa} \mathbf{v} \\
&= \text{net aerodynamic force}
\end{align*}
\]

\[
A_{pa} = \int \int_{A_{wa}} \frac{\hat{n} \cdot \mathbf{v}}{v} \, dA
\]

= projected area

where \( \rho_a \) is the atmospheric density, \( \hat{n} \) is the unit outward pointing normal for the spacecraft surface, \( \hat{\rho} \) is the vector locating a point on the spacecraft surface from the spacecraft center of mass, and \( A_{wa} \) is that part of the spacecraft surface that is wetted, which is defined as being all points on the surface that satisfy \( \hat{n} \cdot \mathbf{v} \geq 0 \).

To simplify calculations, the projected area is eliminated by defining

\[
\begin{align*}
\mathbf{c}_{pa} &= \int \int_{A_{wa}} \hat{\rho} \hat{n} \cdot \mathbf{v} \, dA \\
\mathbf{F}_a &= -\rho_a \mathbf{v}.
\end{align*}
\]

Then, the aerodynamic torque is given by

\[ \mathbf{G}_a = \mathbf{c}_{pa} \times \mathbf{F}_a, \quad (2.21) \]

consequently, in \( \mathbf{F}_b \) we have

\[ \mathbf{G}_{ab} = \mathbf{c}_{pab} \times \mathbf{F}_{ab} = -\rho_b \mathbf{c}_{pab} \mathbf{C}_{bl} \mathbf{v}_l, \quad (2.22) \]

with

\[ \mathbf{c}_{pab} = \int \int_{A_{wa}} \rho_b \hat{n}_b^T \mathbf{C}_{bl} \mathbf{v}_l \, dA. \quad (2.23) \]

For simplicity, the atmospheric density is assumed to be constant throughout the orbit.
2.6 Solar-Pressure Torque

As with the aerodynamic torque, some simplifying assumptions are in order.

Assumptions:
1. The spacecraft absorbs all incident solar radiation.
2. The spacecraft body is convex.

It is in this domain that the necessity of the second assumption is best illustrated. The second assumption guarantees that no part of the spacecraft "throws a shadow" on another part. This is what allows the definition of the wetted surface to be all points on the surface satisfying \( \hat{\mathbf{u}} \cdot \hat{\mathbf{s}} \geq 0 \) (\( \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} \geq 0 \) in the case of aerodynamic torque), where \( \hat{\mathbf{s}} \) is the unit vector pointing in the sun's direction.

With these assumptions, the solar pressure torque takes a form very similar to the aerodynamic torque,

\[
\mathbf{G}_s = \hat{\mathbf{e}}_{ps} \times \mathbf{F}_s
\]

(2.24)

where

\[
\hat{\mathbf{e}}_{ps} = \frac{1}{A_{ps}} \int \int_{A_{ws}} \hat{\rho} \hat{\mathbf{u}} \cdot \hat{\mathbf{s}} \, dA
\]

\( \hat{\mathbf{e}}_{ps} \) = centre of pressure

\[
\mathbf{F}_s = -p A_{ps} \hat{\mathbf{s}}
\]

\( \mathbf{F}_s \) = net solar pressure force

\[
A_{ps} = \int \int_{A_{ws}} \hat{\mathbf{u}} \cdot \hat{\mathbf{s}} \, dA
\]

\( A_{ps} \) = projected area

where \( p \) is the solar pressure. Again, the projected area can be eliminated by defining

\[
\hat{\mathbf{e}}_{ps} = \int \int_{A_{ws}} \hat{\rho} \hat{\mathbf{u}} \cdot \hat{\mathbf{s}} \, dA
\]

\[
\mathbf{F}_s = -p \hat{\mathbf{s}}.
\]

Then,

\[
\mathbf{G}_s = \hat{\mathbf{e}}_{ps} \times \mathbf{F}_s,
\]

(2.25)

consequently, in \( F_b \) we have

\[
\mathbf{G}_{sb} = \hat{\mathbf{e}}_{psb} \times \mathbf{F}_{sb} = -p \hat{\mathbf{c}}_{psb}^x C_{bl} \hat{\mathbf{s}}_l,
\]

(2.26)
Section 2.7. Calculation of $\hat{c}_{p-b}$

with

$$\hat{c}_{p-b} = \int \int_{A_{w}} \rho_b \hat{n}_b^T C_{bf} s_f dA.$$  \hspace{1cm} (2.27)

2.7 Calculation of $\hat{c}_{p-b}$

The integrals for the “centre of pressure” for the aerodynamic and solar-pressure torques are identical in form,

$$\hat{c}_{p-b} = \int \int_{A_{w}} \rho_b \xi dA$$  \hspace{1cm} (2.28)

where

$$\xi = \begin{cases} \hat{n}_b^T C_{bf} v_f & \text{for aerodynamic torque} \\ \hat{n}_b^T C_{bf} s_f & \text{for solar-pressure torque} \end{cases}.$$

The “wetted” area, $A_{w}$, is defined as being those points on the spacecraft surface, $A$, for which $\xi \geq 0$.

In their complete generality, the integrals in (2.28) are difficult to evaluate. To circumvent this problem, we assume that the spacecraft surface is made up of $m$ flat surfaces. To formulate the solution to (2.28) we begin by considering the $n$th surface, $A_n$ ($n \in \{1, \ldots, m\}$).

The outward pointing unit normal to the $n$th surface is denoted $\hat{n}_{nb}$, and is constant over the surface $A_n$. For this surface, we also have

$$\xi_n = \begin{cases} \hat{n}_{nb}^T C_{bf} v_f & \text{for aerodynamic torque} \\ \hat{n}_{nb}^T C_{bf} s_f & \text{for solar-pressure torque} \end{cases}.$$

Consequently, surface $A_n$ is said to be “wetted” if $\xi_n \geq 0$. Note that $\xi_n$ is constant over $A_n$, so that

$$\int \int_{A_{w}} \rho_b \xi_n dA = \int \int_{A_n} \rho_b dA \xi_n.$$  \hspace{1cm} (2.29)

Defining the centre of area of surface $A_n$ to be

$$\rho_{cn} \triangleq \frac{1}{A_n} \int \int_{A_n} \rho_b dA,$$

the integral (2.29) becomes

$$\int \int_{A_{w}} \rho_b dA \xi_n = \xi_n A_n \rho_{cn}.$$  \hspace{1cm} (2.30)

Revisiting integral (2.28), we see that

$$\hat{c}_{p-b} = \sum_n \left\{ \int \int_{A_n} \rho_b \xi_n dA \right\}, \hspace{0.5cm} n \in \{(1, \ldots, m) | \xi_n \geq 0\}$$  \hspace{1cm} (2.31)
Substituting (2.30) into (2.31) gives the desired result for \( \dot{\epsilon}_{p\beta} \)

\[
\dot{\epsilon}_{p\beta} = \sum_{n} \{ \xi_n A_n \rho_{cn} \}, \quad n \in \{(1, \ldots, m) | \xi_n \geq 0 \}.
\]  

(2.32)

\section*{2.8 Eclipse Model}

The spacecraft is determined to be in eclipse if both of the following conditions are satisfied:

\[
R_T^T s_f < 0
\]

and

\[
| R_f - (R_T^T s_f)s_f | < a_e.
\]
Section 2.8. Eclipse Model

Figure 2.1: Gravity-Gradient Torque

Figure 2.2: Geomagnetic Torque
Chapter 2. Spacecraft Attitude Dynamics

Figure 2.3: Solar-Pressure Torque

Figure 2.4: Aerodynamic Torque
Chapter 3

Filter Mathematics

This chapter contains derivations of the respective filter algorithms. Even though some of them exist in the literature\textsuperscript{3,4,11}, they are included here since they are instructive and provide insight into the respective filter mechanisms.

3.1 Notational Preliminaries

Given variables $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$, and a function $\phi(\alpha, \beta) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q$, the Jacobian of $\phi$ with respect to $\alpha$ is a $q \times n$ matrix whose $ij$th term is defined by

$$
\left\{ \frac{\partial \phi(\alpha, \beta)}{\partial \alpha^j} \right\}_{ij} = \frac{\partial \phi_i(\alpha, \beta)}{\partial \alpha_j}.
$$

The notation $a(b|c)$ means the value of $a$ at time $b$ given all measurements up to and including time $c$.

The notation $x \sim N(a, B)$ means that $x$ is a random variable with a Gaussian probability distribution and its mean and variance are given by

$$
E\{x\} = a
$$
$$
E\{xx^T\} = B
$$

where $E\{\cdot\}$ denotes the expectation operator.
3.2 The Extended Kalman Filter

We use the Kalman Filter for discrete linear systems as the starting point. This is not derived, but presented as a theorem without proof. The proof can be found in Chui and Chen.

Consider the discrete dynamical system

\[ x_{k+1} = \Phi(k+1,k)x_k + \Gamma(k)w_k \]
\[ y_k = H(k)x_k + v_k \]  

(3.1)

where \( x_k \in \mathbb{R}^n \) is the system state, \( y_k \in \mathbb{R}^m \) is the system measurement, \( \Phi(k+1,k) \in \mathbb{R}^{n \times n} \) is the state transition matrix, \( \Gamma(k) \in \mathbb{R}^{n \times r} \) and \( H(k) \in \mathbb{R}^{m \times n} \) are discrete-time matrix functions, and \( w_k \in \mathbb{R}^r \) and \( v_k \in \mathbb{R}^m \) are Gaussian white-noise sequences satisfying

\[ E\{w_k\} = 0, \quad E\{w_k w_k^T\} = Q(k)\delta_{kl} \]
\[ E\{u_k\} = 0, \quad E\{u_k u_k^T\} = R(k)\delta_{kl} \]

where \( Q(k) \in \mathbb{R}^{r \times r} \) and \( R(k) \in \mathbb{R}^{m \times m} \) are symmetric positive-definite covariance matrices, and \( \delta_{kl} \) is the discrete Dirac function satisfying

\[ \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \]

The discrete-time Kalman filter is stated as follows:

**Theorem** (Kalman-Bucy filter for discrete systems)\(^{11}\)

The optimal (minimum variance) filter for the discrete system (3.1) consists of difference equations for the conditional mean and covariance matrix. Between observations,

\[ \hat{x}(k+1|k) = \Phi(k+1,k)\hat{x}(k|k) \]
\[ P(k+1|k) = \Phi(k+1,k)P(k|k)\Phi^T(k+1,k) + \Gamma(k)Q(k)\Gamma^T(k) \]

At observations,

\[ \hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1)(y_{k+1} - H(k+1)\hat{x}(k+1|k)) \]
\[ P(k+1|k+1) = [I - K(k+1)H(k+1)]P(k+1|k)[I - K(k+1)H(k+1)]^T + K(k+1)R(k+1)K^T(k+1) \]
Section 3.2. The Extended Kalman Filter

where

\[ K(k+1) = P(k+1|k)H^T(k+1)[H(k+1)P(k+1|k)H^T(k+1) + R(k+1)]^{-1} \]

is the Kalman gain.

The following derivation of the Extended Kalman Filter is directly from Jazwinski\textsuperscript{11}. Consider the following nonlinear stochastic system:

\[
\dot{x} = f(x, t) + G(t)w(t), \quad x(t_0) \sim N(\bar{x}(t_0), P(t_0)) \\
y(t_k) = h(x(t_k), t_k) + u_k
\]

where \( w(t) \) is a zero-mean, white Gaussian noise process with

\[ E\{ww^T\} = Q(t)\delta(t - \tau) \]

where \( \delta(t - \tau) \) is the Dirac-delta function. Now, we generate a reference deterministic trajectory \( \bar{x}(t) \), with initial condition \( \bar{x}(t_0) \), satisfying

\[ \dot{\bar{x}} = f(\bar{x}, t), \quad t \geq t_0 \]

Next we define the perturbation from the reference trajectory to be

\[ \delta x = x - \bar{x} \]

for which we obtain a new stochastic system

\[ \delta \dot{x} = f(x, t) - f(\bar{x}, t) + G(t)w(t), \quad \delta x(t_0) \sim N(\bar{x}(t_0) - \bar{x}(t_0), P(t_0)) \]

Assuming \( \|\delta x\| \) is small, we may make the following approximation

\[ f(x, t) - f(\bar{x}, t) \approx F(\bar{x}, t)\delta x \]

where

\[ F(\bar{x}, t) = \frac{\partial f}{\partial x}\bigg|_{\bar{x}, t} \quad (3.3) \]

is the Jacobian matrix with respect to \( x \) evaluated along the reference trajectory. Thus, we have a linear equation approximating the perturbation dynamics

\[ \delta \dot{x} \approx F(\bar{x}, t)\delta x + G(t)w(t), \quad \delta x(t_0) \sim N(\bar{x}(t_0) - \bar{x}(t_0), P(t_0)). \quad (3.4) \]
To implement the Kalman filter in the theorem, we need a discrete equation. Let $\Phi(t_{k+1}, t_k)$ be the state-transition matrix corresponding to (3.4), and let $t_k, \ k = 0, 1, 2, \ldots$ be a discrete-time system with $t_{k+1} - t_k = T_s, \ \forall k \in [0, 1, \ldots, \infty)$ being the sample period\(^1\). Then, the discrete form of (3.4) is

$$\delta x(t_{k+1}) = \Phi(t_{k+1}, t_k)\delta x(t_k) + w_k, \quad \delta x(t_0) \sim N(\bar{x}(t_0) - \bar{x}(t_0), P(t_0))$$

(3.5)

where $w_k$ is a zero-mean white Gaussian sequence $w_k \sim N(0, Q(k))$, where

$$Q(k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)G(\tau)Q(\tau)G^T(\tau)\Phi^T(t_{k+1}, \tau) \, d\tau.$$ 

For a sufficiently small sample period, the state-transition matrix may be approximated by

$$\Phi(t_{k+1}, t_k) \approx 1 + F(\bar{x}(t_k), t_k)T_s.$$  

(3.6)

Now, we deal with the measurement equation. Analogous to the reference trajectory, we define a reference measurement

$$\bar{y}(t_k) \triangleq h(\bar{x}(t_k), t_k)$$

(3.7)

and the measurement perturbation as

$$\delta y(t_k) \triangleq y(t_k) - \bar{y}(t_k).$$

(3.8)

For sufficiently small $\|\delta y(t_k)\|$, we may approximate (3.8) with the linear equation

$$\delta y(t_k) \approx H(\bar{x}(t_k), t_k)\delta x(t_k) + v_k$$

(3.9)

where

$$H(\bar{x}(t_k), t_k) = \frac{\partial h}{\partial x} \bigg|_{\bar{x}(t_k), t_k}.$$ 

Thus, we now have a discrete-time linear system for the perturbation dynamics

$$\begin{align*}
\delta x(t_{k+1}) &= \Phi(t_{k+1}, t_k)\delta x(t_k) + w_k, \quad \delta x(t_0) \sim N(\bar{x}(t_0) - \bar{x}(t_0), P(t_0)) \\
\delta y(t_k) &= H(\bar{x}(t_k), t_k)\delta x(t_k) + v_k
\end{align*}$$

(3.10)

Treating $\delta y(t_k)$ as a measurement, we may apply the Kalman filter to estimate the perturbation. In view of the definition for $\delta x$, we define the state estimate to be

$$\dot{x} \triangleq \dot{x} + \delta \dot{x}.$$  

(3.11)

\(^1\)It is not necessary to have a fixed sample period, but we do have one in this thesis.
Now, all that remains is to choose initial conditions for the reference trajectory. A good choice for the initial condition is

$$\bar{x}(t_0) = \hat{x}(t_0),$$

and as a consequence of this we have that

$$\delta \bar{x}(t_0|t_0) = 0.$$ 

Since the trajectory $\bar{x}(t)$ does not include the effects of $w(t)$, even for a good choice of $\bar{x}(t_0)$, the trajectory may diverge significantly enough from the true trajectory $x(t)$ that the linearization is no longer valid. To circumvent this problem, we redefine the reference trajectory at each sampling instant after the measurement has been processed, with new initial condition

$$\bar{x}(t_k) = \hat{x}(t_k|t_k).$$

In doing so, we have that

$$\delta \bar{x}(t_k|t_k) = 0 \quad \forall k \in [0, 1, \ldots, \infty).$$

We are now in a position to apply the discrete-time Kalman filter.

**Step 1** Before measurement.

$$\delta \bar{x}(t_{k+1}|t_k) = \Phi(t_{k+1}, t_k)\delta \bar{x}(t_k) = 0.$$ 

$$P(t_{k+1}|t_k) = \Phi(t_{k+1}, t_k)P(t_k|t_k)\Phi^T(t_{k+1}, t_k) + Q(k) \quad (3.12)$$ 

**Step 2** Processing measurement,

$$\delta \bar{x}(t_{k+1}|t_{k+1}) = \delta \bar{x}(t_{k+1}|t_k) + K(k + 1)[\delta y(t_{k+1}) - H(\bar{x}(t_{k+1}), t_{k+1})\delta \bar{x}(t_{k+1})]$$

which leads to

$$\delta \bar{x}(t_{k+1}|t_{k+1}) = K(k + 1)\delta y(t_{k+1}) \quad (3.13)$$

By (3.11), we have that $\hat{x}(t_{k+1}|t_k) = \bar{x}(t_{k+1})$, and substituting (3.8) into (3.13) we have that

$$\hat{x}(t_{k+1}|t_{k+1}) = \hat{x}(t_{k+1}|t_k) + K(k + 1)[y(t_k) - h(\bar{x}(t_k), t_k)]. \quad (3.14)$$
The propagation of the covariance matrix remains the same

\[ P(t_{k+1}|t_k) = [1 - K(k+1)H(\hat{z}(t_{k+1}|t_k), t_{k+1})]P(t_{k+1}|t_k)\begin{bmatrix} 1 - K(k+1)H(\hat{z}(t_{k+1}|t_k) \end{bmatrix}^T + K(k+1)R(k+1)K^T(k+1) \]

and the Kalman gain matrix is

\[ K(k+1) = P(t_{k+1}|t_k)H^T(\hat{z}(t_{k+1}|t_k), t_{k+1}) \times [H(\hat{z}(t_{k+1}|t_k), t_{k+1})P(t_{k+1}|t_k)H^T(\hat{z}(t_{k+1}|t_k), t_{k+1}) + R(k+1)]^{-1}. \]

To summarize the Extended Kalman Filter algorithm, we have:

1. Store \( \hat{x}(t_k|t_k) \), \( P(t_k|t_k) \).
2. Evaluate
   \[ \hat{x}(t_{k+1}|t_k) = \hat{x}(t_k|t_k) + \int_{t_k}^{t_{k+1}} f(\hat{x}(t|t_k), t) dt. \]
3. Calculate \( F(\hat{x}(t_k|t_k)) \), \( \Phi(t_{k+1}, t_k) \) and \( H(\hat{x}(t_{k+1}|t_k), t_{k+1}) \) from (3.3), (3.6) and (3.9) respectively.
4. Calculate \( P(t_{k+1}|t_k) \) and \( K(k+1) \) from (3.12) and (3.16) respectively.
5. Calculate new state-estimate, \( \hat{x}(t_{k+1}|t_{k+1}) \) from (3.14).
6. Calculate new covariance matrix, \( P(t_{k+1}|t_{k+1}) \) from (3.15).
7. Store \( \hat{x}(t_{k+1}|t_{k+1}) \), \( P(t_{k+1}|t_{k+1}) \).
8. Set \( k \) to \( k + 1 \).
9. Return to 1.

### 3.3 The Nonlinear Predictive Filter

The Nonlinear Predictive Filter is developed by Crassidis and Markley, and is presented in full in their paper\(^4\). The filter they present is for time-invariant systems. The filter for the more general case of time-varying systems is developed here.

In the nonlinear predictive filter, the state estimates are obtained by propagating an equation of the plant dynamics, which are assumed to be of the form

\[ \dot{\hat{x}} = f(\hat{x}, t) + G(\hat{x})d(t) \quad t_k \leq t \leq t_{k+1} \]

(3.17)
3.3. The Nonlinear Predictive Filter

where \( \hat{z} \in \mathbb{R}^n \) is the state estimate, \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is sufficiently differentiable and contains the modelled (nominal) plant dynamics, \( t_k \) is a sampling instant, \( d(t_k) \in \mathbb{R}^q \) is a to be determined model error vector and \( G(\hat{z}) : \mathbb{R}^n \to \mathbb{R}^{n \times q} \) is the model error distribution matrix which determines how the model error is introduced to the plant dynamics.

State-observable discrete measurements are assumed to be of the form

\[
y(t_k) = h(z(t_k), t_k) + v_k
\]

where \( y(t_k) \in \mathbb{R}^m \) is the measurement at time \( t_k \), \( h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m \) is sufficiently differentiable, \( z(t) \in \mathbb{R}^n \) is the true state and \( v_k \in \mathbb{R}^m \) is the measurement noise, which is assumed to be a zero-mean Gaussian white noise sequence\(^2\) satisfying

\[
E\{v_k\} = 0 \quad E\{v_k v_k^T\} = R \delta_{kl}
\]

where \( R \in \mathbb{R}^{m \times m} \) is a positive-definite covariance matrix.

Corresponding to (3.18), the measurement estimate is defined as

\[
\hat{y} \triangleq h(\hat{z}, t)
\]

where \( \hat{y} \in \mathbb{R}^m \) is the measurement estimate.

Expanding the output estimate in a Taylor series from one sampling instant to the next gives

\[
\hat{y}(t_{k+1}) \approx \hat{y}(t_k) + z(\hat{z}(t_k), t_k, T_s) + \Lambda(T_s) S(\hat{z}(t_k), t_k) d(t_k)
\]

where \( T_s \triangleq t_{k+1} - t_k \) is the sampling period and

\[
z(\hat{z}(t_k), t_k, T_s) = \text{col} \left\{ \sum_{k=1}^{n} \frac{T_s^k}{k!} L_f^k(h_i) \right\}
\]

\[
\Lambda(T_s) = \text{diag} \left\{ \frac{T_s^p}{p!} \right\}
\]

\[
S(\hat{z}(t_k), t_k) = \text{col} \left\{ L_{g_i} [L_f^{p-1}(h_i)] \right\}
\]

and \( p_i \) is the lowest order of the derivatives of \( h_i(\hat{z}(t), t) \) in which any component of \( d(t_k) \) appears. \( L_f^k(h_i) \) is a \( k \)th order Lie derivative defined by

\[
L_f^0(h_i) = h_i(\hat{z}(t), t), \quad k = 0
\]

\[
L_f^k(h_i) = \frac{\partial L_f^{k-1}(h_i)}{\partial \hat{z}^T} f(\hat{z}, t) + \frac{\partial L_f^{k-1}(h_i)}{\partial t}, \quad k \geq 1.
\]

\(^2\)This assumption is not necessary, but is made so that a positive definite matrix \( R \) reflecting the noise intensity is available.
The Lie derivative with respect to $L_y$, is given by

$$L_y[L_y^{-1}(h_i)] = \frac{\partial L_y^{-1}(h_i)}{\partial x^T} G(\dot{x}).$$

The model error vector $d(t_k)$ is found at each sampling instant to minimize the cost functional

$$J(d(t_k)) = \frac{1}{2}[y(t_{k+1}) - \hat{y}(t_{k+1})]^T R^{-1} [y(t_{k+1}) - \hat{y}(t_{k+1})] + \frac{1}{2} d(t_k)^T W d(t_k) \tag{3.24}$$

where $W \in \mathbb{R}^{q \times q}$ is a positive semi-definite weighting matrix.

Substituting (3.20) into (3.24) and minimizing gives the model error vector $d(t_k)$

$$d(t_k) = -[S(\dot{x}(t_k), t_k) \Lambda(T_s)^T R^{-1} \Lambda(T_s) S(\dot{x}(t_k), t_k) + W]^{-1} S(\dot{x}(t_k), t_k)^T \Lambda(T_s)^T R^{-1} [z(\dot{x}(t_k), t_k, T_s) - y(t_{k+1}) + \hat{y}(t_k)]. \tag{3.25}$$

In summary then, the Nonlinear Predictive Filter algorithm is:

1. Calculate $\Lambda(T_s)$ from (3.22).
2. Store $\dot{x}(t_k)$, $\hat{y}(t_k)$ and $y(t_{k+1})$.
3. Calculate $z(\dot{x}(t_k), t_k, T_s)$ and $S(\dot{x}(t_k), t_k)$ from (3.21) and (3.23).
4. Calculate $d(t_k)$ from (3.25).
5. Integrate (3.17) from $t_k$ to $t_{k+1}$ to get $\dot{x}(t_{k+1})$.
6. Calculate $\hat{y}(t_{k+1})$ from (3.19).
7. Set $k$ to $k + 1$.
8. Return to 2.

### 3.4 The Extended $H_\infty$ Filter

We use as starting point an a-posteriori sub-optimal $H_\infty$ filter for linear discrete systems as derived by Hassibi et al. As with the extended Kalman filter, the linear filter is not derived, but is stated as a theorem without proof. The proof can be found in the papers of Hassibi et al.

Consider the discrete dynamical system given by (3.1). This time, we make no assumptions.
Section 3.4. The Extended $H_\infty$ Filter

about the statistics of $w_k$ and $v_k$. Instead we assume that $\{w_k\} \in h_2$ and $\{v_k\} \in h_2$. Then, the sub-optimal a-posteriori $H_\infty$ problem can be stated as follows:

Given measurements up to and including time $n$, find a filter that achieves

$$\sup_{x_0, w \in h_2, v \in h_2} \frac{\sum_{k=0}^n [\hat{x}(t_k|t_k) - x(t_k)]^T [\hat{x}(t_k|t_k) - x(t_k)]}{\Pi_0^{-1} [\hat{x}(t_0|t_0) - x(t_0)] + \sum_{k=0}^n w_k^T w_k + \sum_{k=0}^n v_k^T v_k} < \gamma^2$$

(3.26)

where $\Pi_0 = \Pi_0' > 0$ and $\gamma > 0$.

The $H_\infty$ filter is stated as follows

**Theorem (An $H_\infty$ a-posteriori filter)**

Given $\gamma > 0$, if the $[\Phi(k+1, k) \quad \Gamma(k)]$ have full rank, then a filter that achieves (3.26) exists iff

$$P(k)^{-1} + H(k)^T H(k) - \gamma^2 I > 0, \quad k = 0, 1, \ldots, n$$

where $P(0) = \Pi_0$ and $P(k)$ satisfies the Riccati recursion

$$P(k+1) = \Phi(k+1, k)P(k)\Phi^T(k+1, k) + \Gamma(k)\Gamma^T(k)$$

$$- \Phi(k+1, k)P(k) \left[ H^T(k) \quad 1 \right] R_e^{-1}(k) \left[ \begin{array}{c} H(k) \\ 1 \end{array} \right] P(k)\Phi^T(k+1, k)$$

with

$$Re(k) = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} H(k) \\ 1 \end{bmatrix} P(k) \begin{bmatrix} H^T(k) & 1 \end{bmatrix}$$

A possible $H_\infty$ filter is given by

$$\hat{x}(k+1|k+1) = \Phi(k+1, k)\hat{x}(k|k) + K(k+1)[y(k+1) - H(k+1)\Phi(k+1, k)\hat{x}(k|k)]$$

and

$$K(k+1) = P(k+1)H^T(k+1)[1 + H(k+1)P(k+1)H^T(k+1)]^{-1}$$

where $K(k+1)$ is the $H_\infty$ gain.

From this point onwards, the derivation is exactly the same as the Extended Kalman Filter, except that the discret-time plant noise is given by

$$w_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)\Gamma(\tau)w(\tau) \, d\tau.$$
In summary, the Extended $H_\infty$ Filter algorithm is as follows:

1. Store $\hat{x}(t_k|t_k)$, $P(k)$.
2. Evaluate
   \[ \dot{x}(t_{k+1}|t_k) = \dot{x}(t_k|t_k) + \int_{t_k}^{t_{k+1}} f(\dot{x}(t|t_k), t) \, dt. \]
3. Calculate $F(\dot{x}(t_k|t_k), t_k)$, $H(\dot{x}(t_k|t_k), t_k)$ and $H(\dot{x}(t_{k+1}|t_k), t_{k+1})$ from (3.3), and (3.9) respectively.
4. Calculate the state-transition matrix $\Phi(t_{k+1}, t_k)$ from (3.6).
5. Evaluate
   \[ P(k+1) = \Phi(t_{k+1}, t_k)P(k)\Phi^T(t_{k+1}, t_k) + 1 - \Phi(t_{k+1}, t_k)P(k) \times \left[ \begin{array}{c} H^T(\dot{x}(t_k|t_k), t_k) 1 \\ 1 1 \end{array} \right] R_e^{-1}(k) \left[ \begin{array}{c} H(\dot{x}(t_k|t_k), t_k) \\ 1 \end{array} \right] P(k)\Phi^T(t_{k+1}, t_k) \]

where
   \[ R_e(k) = \left[ \begin{array}{ccc} 1 + H(\dot{x}(t_k|t_k), t_k)P(k)H^T(\dot{x}(t_k|t_k), t_k) & H(\dot{x}(t_k|t_k), t_k)P(k) \\ P(k)H^T(\dot{x}(t_k|t_k), t_k) & P(k) - \gamma^2 1 \end{array} \right]. \]
6. Evaluate
   \[ K(k+1) = P(k+1)H^T(\dot{x}(t_{k+1}|t_k), t_{k+1}) \times [1 + H(\dot{x}(t_{k+1}|t_k), t_{k+1})P(k+1)H^T(\dot{x}(t_{k+1}|t_k), t_{k+1})]^{-1}. \]
7. Calculate the new state estimate from
   \[ \hat{x}(t_{k+1}|t_{k+1}) = \hat{x}(t_{k+1}|t_k) + K(k+1)[y(t_k) - h(\hat{x}(t_k), t_k)]. \]
8. Store $\hat{x}(t_{k+1}|t_{k+1})$, $P(k+1)$.
9. Set $k$ to $k+1$.
10. Return to 1.
Chapter 4

Filter Implementation

This chapter contains the information necessary to implement the filters for spacecraft attitude determination.

4.1 Measurement Equations

Two measurement scenarios are investigated. In the first scenario, measurements of the spacecraft angular rate, earth’s magnetic field and sun’s position vector are available. The measurement equation for this case is given by

\[
y_k = \begin{bmatrix} h_\omega \\ h_m \\ h_s \end{bmatrix} + \begin{bmatrix} v_{wk} \\ v_{mk} \\ v_{sk} \end{bmatrix}
\]

where \( y_k \in \mathbb{R}^3 \) is the measurement, \( h_\omega = \omega(t_k), h_m = C_h(t_k)B(t_k), h_s = C_s(t_k)s_l \), and \( v_{wk}, v_{mk} \) and \( v_{sk} \) are zero-mean additive noises with discrete-time covariances

\[
E\{v_{wk}v_{wk}^T\} = r_w1\delta_{kl} \\
E\{v_{mk}v_{mk}^T\} = r_m1\delta_{kl} \\
E\{v_{sk}v_{sk}^T\} = r_s1\delta_{kl}
\]

The sun measurements are obviously not available during eclipse. During eclipse, the measurement equation becomes

\[
y_k = \begin{bmatrix} h_\omega \\ h_m \end{bmatrix} + \begin{bmatrix} v_{wk} \\ v_{mk} \end{bmatrix}
\]

(4.2)
where \( y_k \in \mathbb{R}^6 \) is the measurement.

In the second scenario, the rate measurements are removed, and it is assumed that the sun's measurements are available at all times (i.e., no eclipse). This is to maintain two vector measurements at all times. The measurement equation in this case is

\[
y_k = \begin{bmatrix} h_m \\ h_s \end{bmatrix} + \begin{bmatrix} v_{mk} \\ v_{sk} \end{bmatrix}
\]

(4.3)

where \( y_k \in \mathbb{R}^6 \) is the measurement.

### 4.2 Extended Kalman Filter and Extended \( H_{\infty} \) Filter

The filter implementations are based on (2.15) for the plant. The plant jacobian is given by

\[
F(x, t) = \begin{bmatrix} \frac{\partial f_\omega}{\partial \omega^T} & \frac{\partial f_\theta}{\partial \theta^T} \\ \frac{\partial f_\omega}{\partial \theta^T} & \frac{\partial f_\theta}{\partial \theta^T} \end{bmatrix}
\]

(4.4)

Evaluating the partial derivatives we have

\[
\frac{\partial f_\omega}{\partial \omega^T} = I^{-1}[(I\omega)^T - \omega^T I]
\]

\[
\frac{\partial f_\omega}{\partial \theta^T} = \frac{3\mu}{r^3} I^{-1}[R_6^T I - (I R_6)^T] \begin{bmatrix} \frac{\partial C_m}{\partial \theta_1} R_1 & \frac{\partial C_m}{\partial \theta_2} R_2 & \frac{\partial C_m}{\partial \theta_3} R_3 \end{bmatrix}
\]

\[
\frac{\partial f_\theta}{\partial \omega^T} = S^{-1}(\theta)
\]

\[
\frac{\partial f_\theta}{\partial \theta^T} = \begin{bmatrix} \frac{\partial S^{-1}(\theta)}{\partial \theta_1} \omega & \frac{\partial S^{-1}(\theta)}{\partial \theta_2} \omega & \frac{\partial S^{-1}(\theta)}{\partial \theta_3} \omega \end{bmatrix}
\]

In measurement scenario one, the measurement jacobian is given by

\[
H(x, t) = \begin{bmatrix} \frac{\partial h_m}{\partial \omega^T} & \frac{\partial h_m}{\partial \theta^T} \\ \frac{\partial h_s}{\partial \omega^T} & \frac{\partial h_s}{\partial \theta^T} \\ \frac{\partial h_m}{\partial \omega^T} & \frac{\partial h_m}{\partial \theta^T} \end{bmatrix}
\]

(4.5)

and during eclipse by

\[
H(x, t) = \begin{bmatrix} \frac{\partial h_m}{\partial \omega^T} & \frac{\partial h_m}{\partial \theta^T} \\ \frac{\partial h_s}{\partial \omega^T} & \frac{\partial h_s}{\partial \theta^T} \\ \frac{\partial h_m}{\partial \omega^T} & \frac{\partial h_m}{\partial \theta^T} \end{bmatrix}
\]

(4.6)
In measurement scenario two, the measurement Jacobian is given by

\[ H(x, t) = \begin{bmatrix} \frac{\partial h_m}{\partial \omega^T} & \frac{\partial h_m}{\partial \theta^T} \\ \frac{\partial h_s}{\partial \omega^T} & \frac{\partial h_s}{\partial \theta^T} \end{bmatrix} . \]  

(4.7)

The non-zero partial derivatives are given by

\[ \frac{\partial h_s}{\partial \theta^T} = \begin{bmatrix} \frac{\partial C_{h|m}B_I}{\partial \theta_1} & \frac{\partial C_{h|m}B_I}{\partial \theta_2} & \frac{\partial C_{h|m}B_I}{\partial \theta_3} \\ \frac{\partial C_{h|m}S_I}{\partial \theta_1} & \frac{\partial C_{h|m}S_I}{\partial \theta_2} & \frac{\partial C_{h|m}S_I}{\partial \theta_3} \end{bmatrix} . \]

For both the Extended Kalman Filter and the Extended $H_{\infty}$ Filter, we take $P(0) = 0.011$. For the Extended Kalman Filter, the disturbance is assumed to affect the spacecraft dynamics only, and so, the disturbance covariance matrix is taken to be of the form

\[ Q(k) = \text{block diag}(q1, 0), \quad k = 0, 1, 2, \ldots \]  

(4.8)

In scenario one, the sensor noise covariance matrix is given by

\[ R(k) = \text{block diag}(r_\omega 1, r_m 1, r_s 1), \]  

(4.9)

and during eclipse by

\[ R(k) = \text{block diag}(r_\omega 1, r_m 1). \]  

(4.10)

In scenario two, the sensor noise covariance matrix is given by

\[ R(k) = \text{block diag}(r_m 1, r_s 1). \]  

(4.11)

### 4.3 The Nonlinear Predictive Filter

The equation for the attitude kinematics is exact, and hence, the model error need only be added to the spacecraft dynamics equation ($d(t_k) \in \mathbb{R}^3$). In the first scenario it is added as a disturbance torque, and in this case the filter equation becomes

\[ \dot{x} = f(\hat{x}, t) + G(\hat{x})d(t) \quad t_k \leq t \leq t_{k+1} \]  

(4.12)

where

\[ G = \begin{bmatrix} I^{-1} \\ 0 \end{bmatrix} . \]
In the second scenario (no rate sensor), the model error is added directly to the filter dynamics, and hence in this case

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ 

### 4.3.1 Angular Rate Measurements

The angular rate measurements depend directly on $\omega$ (obviously) and hence, the order of $p_i$ in this case is one. Thus,

$$z_\omega = T_s L_f^1(h_\omega),$$  \hspace{1cm} (4.13)

$$\Lambda_\omega = T_s 1,$$  \hspace{1cm} (4.14)

### 4.3.2 Magnetometer and Sun-sensor Measurements

The magnetometer and sun-sensor measurements do not depend explicitly on $\omega$, and hence, the lowest order of derivative of those in which any component of $d(t_k)$ appears is $p_i = 2$. Thus,

$$z_m = T_s L_f^1(h_m) + \frac{T_s^2}{2} L_f^2(h_m),$$  \hspace{1cm} (4.15)

$$\Lambda_m = \frac{T_s^2}{2} 1,$$  \hspace{1cm} (4.16)

and

$$z_s = T_s L_s^1(h_s) + \frac{T_s^2}{2} L_s^2(h_s),$$  \hspace{1cm} (4.17)

$$\Lambda_s = \frac{T_s^2}{2} 1.$$  \hspace{1cm} (4.18)

In scenario one,

$$z(\hat{x}(t), t, T_s) = \begin{bmatrix} z_\omega \\ z_m \\ z_s \end{bmatrix}, \quad \Lambda(T_s) = \begin{bmatrix} \Lambda_\omega & 0 & 0 \\ 0 & \Lambda_m & 0 \\ 0 & 0 & \Lambda_s \end{bmatrix},$$  \hspace{1cm} (4.19)

and during eclipse

$$z(\hat{x}(t), t, T_s) = \begin{bmatrix} z_\omega \\ z_m \end{bmatrix}, \quad \Lambda(T_s) = \begin{bmatrix} \Lambda_\omega & 0 \\ 0 & \Lambda_m \end{bmatrix}.$$  \hspace{1cm} (4.20)
In scenario two,
\[
\begin{pmatrix}
  z_m \\
  z_s
\end{pmatrix}, \quad
\Lambda(T_s) = \begin{bmatrix}
  \Lambda_m & 0 \\
  0 & \Lambda_s
\end{bmatrix}.
\] (4.21)

### 4.3.3 Lie Derivatives

The first-order Lie derivatives are given by
\[
\begin{align*}
L_1^1(h_\omega) &= \Gamma^{-1}[ -\omega^* I\omega + G_g ], \\
L_1^1(h_m) &= \frac{\partial (C_{bl}(\hat{\theta})B_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta})\omega + \left\{ C_{bl} \frac{dB_I}{dt} \right\}, \\
L_1^1(h_s) &= \frac{\partial (C_{bl}(\hat{\theta})s_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta})\omega.
\end{align*}
\]

\[
\begin{align*}
L_2^1(h_m) &= \frac{\partial (C_{bl}(\hat{\theta})B_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta}) \Gamma^{-1}[ -\omega^* I\omega + G_g ] \\
&\quad + \frac{\partial}{\partial \hat{\theta}} \left[ \frac{\partial (C_{bl}(\hat{\theta})B_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta})\omega \right] S^{-1}(\hat{\theta})\omega \\
&\quad + \left\{ 2 \frac{\partial}{\partial \hat{\theta}} \left[ C_{bl}(\theta) \frac{dB_I}{dt} \right] S^{-1}(\theta)\omega + C_{bl}(\theta) \frac{d^2B_I}{dt^2} \right\}, \\
L_2^1(h_s) &= \frac{\partial (C_{bl}(\hat{\theta})s_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta}) \Gamma^{-1}[ -\omega^* I\omega + G_g ] \\
&\quad + \frac{\partial}{\partial \hat{\theta}} \left[ \frac{\partial (C_{bl}(\hat{\theta})s_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta})\omega \right] S^{-1}(\hat{\theta})\omega.
\end{align*}
\]

Note that the terms in braces \{ \} account for the time dependence of the magnetic field measurements. It's effect may be negligible.

The Lie derivatives with respect to \( L_{g1} \) in scenario one are given by
\[
\begin{align*}
L_{g\omega}[L_f^0(h_\omega)] &= \Gamma^{-1}, \\
L_{gm}[L_f^1(h_m)] &= \frac{\partial (C_{bl}(\hat{\theta})B_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta}) \Gamma^{-1}, \\
L_{gs}[L_f^1(h_s)] &= \frac{\partial (C_{bl}(\hat{\theta})s_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta}) \Gamma^{-1},
\end{align*}
\]

and in scenario two,
\[
L_{gm}[L_f^1(h_m)] = \frac{\partial (C_{bl}(\hat{\theta})B_I)}{\partial \hat{\theta}} S^{-1}(\hat{\theta}),
\]
\[ L_{gs}[L^1_t(h_s)] = \frac{\partial (C_{bl}(\hat{\theta})^T)}{\partial \hat{\theta}^T} S^{-1}(\hat{\theta}). \]

### 4.3.4 The Modified Nonlinear Predictive Filter

What follows is a modified implementation of the Nonlinear Predictive Filter when rate sensors are present (scenario 1). In this case, model error is added to the attitude kinematics equation as well as to the attitude dynamics \((d(t_k) \in \mathbb{R}^6)\). The model error is added directly to both the attitude dynamics and the attitude kinematics, and hence.

\[
G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

In this case, \(p_i = 1\) for all measurements. Thus,

\[
z_{\omega} = T_z L^1_t(h_{\omega}), \quad \Lambda_{\omega} = T_z 1, \quad (4.22)
\]

\[
z_m = T_z L^1_t(h_m), \quad \Lambda_m = T_z 1, \quad (4.24)
\]

\[
z_s = T_z L^1_t(h_s), \quad \Lambda_s = T_z 1. \quad (4.26)
\]

The Lie derivatives with respect to \(L_{gi}\) are given by

\[
L_{gw}[L^0_t(h_{\omega})] = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

\[
L_{gm}[L^1_t(h_m)] = \begin{bmatrix} 0 & \frac{\partial (C_{bl}B_l)}{\partial \theta^T} \end{bmatrix},
\]

\[
L_{gs}[L^1_t(h_s)] = \begin{bmatrix} 0 & \frac{\partial (C_{bl}s_l)}{\partial \theta^T} \end{bmatrix}.
\]

### 4.3.5 Relevant Matrices

The following matrices are needed in the calculation of the Lie derivatives:

\[
\frac{\partial (C_{bl}(\theta)x)}{\partial \theta^T} = \left[ \frac{\partial C_{bl}(\theta)}{\partial \theta_1} x \quad \frac{\partial C_{bl}(\theta)}{\partial \theta_2} x \quad \frac{\partial C_{bl}(\theta)}{\partial \theta_3} x \right].
\]

where \(x \in \mathbb{R}^3\) does not depend on \(\theta\).

\[
\frac{\partial}{\partial \theta^T} \left[ \frac{\partial (C_{bl}(\theta)x)}{\partial \theta^T} S^{-1}(\theta) \omega \right] = \text{row} \left( \frac{\partial}{\partial \theta_i} \left[ \frac{\partial (C_{bl}(\theta)x)}{\partial \theta^T} S^{-1}(\theta) \omega \right] \right). \quad i = 1, 2, 3.
\]
Section 4.3. The Nonlinear Predictive Filter

where

\[
\frac{\partial}{\partial \theta_i} \left[ \frac{\partial (C_{Bf}(\theta)) x}{\partial \theta^T} S^{-1}(\theta) \omega \right] = \left[ \frac{\partial^2 (C_{Bf}(\theta)) x}{\partial \theta_1 \partial \theta_1} \frac{\partial^2 (C_{Bf}(\theta)) x}{\partial \theta_1 \partial \theta_2} \frac{\partial^2 (C_{Bf}(\theta)) x}{\partial \theta_1 \partial \theta_3} \right] \\
+ \frac{\partial (C_{Bf}(\theta)) x}{\partial \theta^T} \frac{\partial S^{-1}(\theta)}{\partial \theta_i} \omega, \quad i = 1, 2, 3.
\]

4.3.6 Time Derivatives of \( B_f(t) \)

Generally, the time derivatives \( \frac{dB_f}{dt} \) and \( \frac{\partial B_f}{\partial t^2} \) are difficult or impossible to compute analytically, especially given the empirical nature of real geomagnetic field models. A method by which they may be computed is by using a backward \( n \)-point Lagrangian approximation. The \( n \in \mathbb{N} \) may be chosen as small or as large as desired to maintain accuracy, as well as satisfy filter hardware memory limitations. The mathematical details of this approximation may be found in Lomax et al.\textsuperscript{12} In the simulations performed in this thesis, the rate of change of the geomagnetic field as seen by the spacecraft was very slow, and was found to be negligible.
Chapter 5

Simulation Descriptions

5.1 Orbital Dynamics

The spacecraft orbit is assumed to be Keplerian, and the same for all simulations. The orbital elements are given in table 5.1.

<table>
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<th>a (km)</th>
<th>e</th>
<th>i (deg)</th>
<th>Ω (deg)</th>
<th>ω (deg)</th>
<th>t₀ (s)</th>
</tr>
</thead>
<tbody>
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<td>7171.2</td>
<td>0</td>
<td>94.6</td>
<td>157.5</td>
<td>180</td>
<td>0</td>
</tr>
</tbody>
</table>

5.2 Space Environment

The radius of the earth is taken to be \( a_e = 6371.2 \text{ km} \). The geomagnetic field constants are; \( g_0 = -29682 \text{ nT} \), \( g_1 = -1789 \text{ nT} \) and \( h_1 = 5310 \text{ nT} \). The atmospheric density on orbit is \( \rho_a = 4.6 \times 10^{-14} \text{ kg/m}^3 \) and the solar pressure is \( p = 4.5 \times 10^{-6} \text{ N/m}^2 \).

5.3 Spacecraft Characteristics

The spacecraft inertia matrix is given by \( \mathbf{I} = \text{diag}([10 \ 12 \ 2]) \text{ kg\cdot m}^2 \), and the spacecraft residual magnetic dipole moment is \( \mathbf{m}_b = [0.1 \ 0.1 \ 0.1]^T \text{ A\cdot m}^2 \).

The spacecraft is assumed to consist of a central hub, and two sails (e.g. solar panels). The effect of the hub on the aerodynamic and solar pressure torques is ignored.
Thus, four surfaces are needed and are defined by areas; \( A_1 = A_2 = A_3 = A_4 = 0.5 \text{ m}^2 \), centroids;

\[
\begin{align*}
\mathbf{\rho}_1 = \mathbf{\rho}_2 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \\
\mathbf{\rho}_3 = \mathbf{\rho}_4 &= \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T,
\end{align*}
\]

and surface normals;

\[
\begin{align*}
\mathbf{\hat{n}}_1 &= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^T, \\
\mathbf{\hat{n}}_2 &= \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}^T, \\
\mathbf{\hat{n}}_3 &= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}^T
\end{align*}
\]

and

\[
\mathbf{\hat{n}}_4 = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^T.
\]

5.4 Numerical Integration and Sampling Rate

The spacecraft states are propagated using a fourth-order Runge-Kutta procedure with the stepsize being the sampling period, \( T_s = 0.1 \text{s} \).

5.5 Spacecraft Initial Conditions

The spacecraft initial conditions are

\[
\mathbf{x}(0) = [0.005 \text{rad/s} \ 0.005 \text{rad/s} \ 0.005 \text{rad/s} \ 10^6 \ 10^6 \ 10^6]^T.
\]

5.6 Sensor Noise and Sensor Suites

The sensor noise components are generated using the equation

\[
v_i = \sqrt{12\sigma_i} \left[ \sigma_i - \frac{1}{2} \right]
\]

where \( \sigma_i \) is a random variable between 0 and 1 generated using the Fortran intrinsic function \text{RAND}. The values of the sensor noise covariances are given in table 5.2. These covariances are approximately one-hundredth of what could be expected in a practical situation. This
is to allow the examination of other aspects of the filter performance such as performance with respect to plant error, measurement error and initial estimate error, without clouding these results due to the sensor noise. The effect of sensor-noise level is examined separately.

Table 5.2: Sensor Noise Covariances

<table>
<thead>
<tr>
<th>$\sqrt{\sigma_O}$ (deg/s)</th>
<th>$\sqrt{\sigma_m}$ (T)</th>
<th>$\sqrt{\sigma_s}$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00016</td>
<td>$2 \times 10^{-3}$</td>
<td>0.0014</td>
</tr>
</tbody>
</table>

5.7 Performance Measures

The filter performance is measured by calculating the root mean square (RMS) of the angular rate and Euler angle errors. These are given by

$$E_{rms_w} = \left( \frac{\sum_{k=1}^{n} e_{\omega_k} e_{\omega_k}}{n} \right)^{\frac{1}{2}}$$

(5.2)

$$E_{rms_\theta} = \left( \frac{\sum_{k=1}^{n} e_{\theta_k} e_{\theta_k}}{n} \right)^{\frac{1}{2}}$$

(5.3)

where $e_{\omega_k} = \omega(t_k) - \dot{\omega}(t_k)$ and $e_{\theta_k} = \theta(t_k) - \dot{\theta}(t_k)$ are the estimation errors in the angular rates and Euler angles respectively, and $n$ is the total number of estimate updates during the simulation.

5.8 Plant Error

Model error in the plant is introduced by errors in the spacecraft inertia matrix. The true spacecraft inertia matrix remains fixed throughout all simulations, but the filter knowledge of it changes from simulation to simulation. The error in the inertia matrix is introduced in two ways simultaneously. There is an error in the value of the principal inertias, and there is an offset of the principal axes. The filter principal inertias are calculated from

$$\{I_{pf}\}_{i,i} = \{I_{ps}\}_{i,i}[1 + 0.02e_p], \ i = 1, 2, 3$$
where \( \{I_{pe}\}_{i,i} \) and \( \{I_{pe}\}_{i,i} \) are the true and filter principal inertias respectively. The angular offset of the principal axes is given by a 3–2–1 Euler sequence,

\[
\theta_f = e_p [3^\circ, 3^\circ, 3^\circ]^T
\]

From this the filter inertia matrix is calculated:

\[
I_e = C^T(\theta_f)I_{pe}C(\theta_f)
\]

where \( C(\theta) \) is the rotation matrix corresponding to \( \theta_f \). It is clear then that \( e_p \) is a measure of the plant error.

### 5.9 Measurement Error

Model error in the measurements is introduced in two ways simultaneously. The filter knowledge of the geomagnetic field is not exact, and the sensors are offset. The geomagnetic field (as far as the filter is aware) is calculated using the geomagnetic field constants given by

\[
\begin{align*}
g_{le}^0 &= g_i^0 (1 + 0.01e_m), \\
g_{le}^1 &= g_i^1 (1 + 0.01e_m), \\
h_{le}^1 &= h_i^1 (1 + 0.01e_m).
\end{align*}
\]

The earth's radius (as far as the filter is aware) is calculated from:

\[
a_{ee} = a_e (1 + 0.01e_m)
\]

The angular offsets of the sensors are given by 3-2-1 Euler sequences

\[
\begin{align*}
\theta_w &= e_m [1^\circ - 1^\circ - 1^\circ]^T, \\
\theta_m &= e_m [-1^\circ 1^\circ 1^\circ]^T, \\
\theta_s &= e_m [1^\circ 1^\circ - 1^\circ]^T,
\end{align*}
\]

for the rate-sensor, magnetometer, and sun-sensor respectively. Thus, the actual measurements are given by:

\[
y_k = \begin{bmatrix} C(\theta_w)\omega(t_k) \\ C(\theta_m)C_{Bf}(\theta(t_k))B_f(t_k) \\ C(\theta_s)C_{Bf}(\theta(t_k))s_f(t_k) \end{bmatrix} + \begin{bmatrix} v_{\omega k} \\ v_{mk} \\ v_{sk} \end{bmatrix}
\]
Section 5.10. Initial Estimate Error

where $C(\theta_i)$ is the rotation matrix corresponding to $\theta_i$. Thus, $e_m$ is a measure of the measurement error.

5.10 Initial Estimate Error

The initial condition of the state estimates is given by

$$\dot{x}(0) = x(0) + e_i \bar{e}$$

where

$$\bar{e} = [0.005\text{rad/s} \ 0.005\text{rad/s} \ 0.005\text{rad/s} \ 5^\circ \ 5^\circ \ 5^\circ]^T.$$ 

Hence, $e_i$ is a measure of the initial estimate error.

5.11 Filter Tuning

For the Extended Kalman Filter, the discrete-time plant noise covariance matrix, $Q$, is assumed to be of the form

$$Q = \begin{bmatrix} qI & 0 \\ 0 & 0 \end{bmatrix}.$$ 

For the Nonlinear Predictive Filter, the model error weighting matrix, $W$, is assumed to be of the form

$$W = wI.$$ 

In tuning the filters, $e_p$, $e_m$ and $e_i$ were set to zero (the filter is designed for the nominal plant), and $q$ and $w$ were sought to minimize $E_{rms_c}$ and $E_{rms_a}$. The minimum for both did not always occur for the same value of $q$ or $w$. and a compromise between the two was achieved instead by the use of engineering judgement.

In scenario 1, the values chosen are, $q = 1 \times 10^{-12} \text{ s}^{-4}$ for the EKF, $w = 1 \times 10^4 \text{ kg}^2\text{m}^4\text{s}^{-4}$ for the NPF, $w = 1 \times 10^8 \text{ kg}^2\text{m}^4$ for the MNPF and $\gamma = 14.2$ for the EHF. In scenario 2 the values are are, $q = 5 \times 10^{-12} \text{ s}^{-4}$ for the EKF and $w = 5 \times 10^6 \text{ s}^{-4}$ for the NPF.
Table 5.3: Simulation Sets

<table>
<thead>
<tr>
<th>Sim. set</th>
<th>( e_p ) range</th>
<th>( e_m ) range</th>
<th>( e_i ) range</th>
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</thead>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
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<td>-6.6</td>
<td>0</td>
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<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-18.16</td>
</tr>
</tbody>
</table>

5.12 Simulations Performed

With the filters tuned for the nominal plant, the filters were perturbed to examine performance with respect to plant error, measurement error, and sensitivity to initial estimate error. The simulation sets are summarised in Table 5.3.

Simulation set 1 examines the filter sensitivity to plant error, simulation set 2 examines the filter sensitivity to measurement error, and simulation set 3 examines the filter sensitivity to initial estimate error. All simulation sets were run for both sensor suite scenarios. Simulations were also performed for the nominal plant \( (e_p = e_m = e_i = 0) \) where the sensor noise levels in Table 5.2 were multiplied by 10 and 100. The filters were redesigned for the new sensor noise levels.
Chapter 6

Results

6.1 Initial Estimate Error

6.1.1 Scenario 1

Figure 6.1 shows that with rate sensors present, all filters perform essentially the same for the angular rate estimates (when they do converge). This means that the angular rate estimates achieve a similar rate of convergence to the true angular rates for all filters. Examining figures B.8, B.16, B.24 and B.32, shows this to indeed be the case, and that the rate of convergence is very rapid. This is not too surprising, since with the angular rates being measured directly, their estimates can be expected to converge rapidly. Figure 6.1 shows that the EKF has the most rapid convergence of all the filters with respect to the angular rates. Figure 6.2 on the other hand, tells an entirely different story for the Euler angle estimates. The NPF and EHF Euler angle RMS estimation errors increase rapidly as the initial estimate error increases. The cause of this, as verified by figures B.16 and B.32, is that the rate of convergence of the Euler angle estimates of both of these filters is very slow. The EKF Euler angle estimates, on the other hand, converge very rapidly (see figure B.8).

The slow rate of convergence of the NPF and EHF Euler angle estimates is a reason for their small ranges of convergence. As shown in table A.3, the EHF had the smallest range of initial error over which the filter converged, closely followed by the NPF. The EKF has a surprisingly large range over which convergence occurs, given that the EKF is a linearized technique.

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A possible explanation for the behaviour of the NPF with rate measurements, is that unlike the EKF, the Euler angle estimates are not updated directly. They are updated indirectly by integration of (3.17). The angular rates are however affected directly by the model error, $d(t_k)$. With rate-sensors present, the angular rates are included in the cost function (3.24) as well as the Euler angle related measurements (magnetometer and sun-sensor), and the filter attempts to track the angular rates at the expense of the Euler angles. Since the Euler angles are integrals of the rates, the initial condition of the Euler angles greatly affects the resulting Euler-angle trajectory. If the NPF puts most emphasis on tracking the angular rates, then it does not allow the Euler angles to converge. To circumvent this problem, we add an artificial model error to the attitude kinematics equation. The resulting filter is what we have called the Modified Nonlinear Predictive Filter (MNPF). In this implementation, both the rates and the Euler angles are directly influenced by the model error, and hence, the rates can not be tracked at the expense of the Euler angles. The results in Table A.3 show that this has the desired effect. The MNPF has the largest range over which convergence occurs of all the filters. However, Figure 6.2 shows that the convergence of the MNPF is significantly slower than the EKF. Figure B.24 verifies this.
Figure 6.1: Angular Rate Performance with respect to $e_i$ (Scenario 1)

Figure 6.2: Euler Angle Performance with respect to $e_i$ (Scenario 1)
6.1.2 Scenario 2

Table A.6 shows that the NPF has a larger range of initial estimate errors for which convergence occurs than the EKF. The EHF did not converge at all. Figures 6.3 and 6.4 show that the EKF converges much quicker in both the angular rates and the Euler angles than the NPF does. Comparing Figures 6.3 and 6.1, we see that when the rate-sensors are removed, the EKF angular rate estimates converge much more slowly. The same holds for the NPF.
Section 6.1. Initial Estimate Error

Figure 6.3: Angular Rate Performance with respect to $e_i$ (Scenario 2)

Figure 6.4: Euler Angle Performance with respect to $e_i$ (Scenario 2)
6.2 Plant Error

6.2.1 Scenario 1

Figures 6.5 and 6.6 show that the NPF outperforms all other filters with respect to plant error. In fact, the flat line corresponding to the NPF shows that when rate-sensors are present, the NPF is completely insensitive to plant error. All other filters perform progressively worse as the plant error increases. In terms of the angular rate estimates, figure 6.5 shows that the EKF is the most sensitive to plant error, and the MNPF and EHF have similar sensitivities to each other, with the EHF being slightly less sensitive. Examining figure 6.6 shows that with regard to the Euler angle estimates, the EKF is the least sensitive (not including the NPF), followed by the MNPF. The EHF performs the worst.

It may seem at first surprising that while the NPF is completely insensitive to plant error, the MNPF is not. It must be remembered, however, that the plant dynamics being propagated in the NPF are of a natural form, whereas those in the MNPF are not. The attitude kinematics equation in the NPF is exact, where as in the MNPF, model error is added. A noticeable effect of this is that under nominal conditions \((e_p = 0, e_m = 0\) and \(e_i = 0\)), the Euler angle estimates of the MNPF are less accurate than those of the NPF. A surprising result is that the angular rate estimates of the MNPF are more accurate than those of the NPF.
Section 6.2. Plant Error

Figure 6.5: Angular Rate Performance with respect to $e_p$ (Scenario 1)

Figure 6.6: Euler Angle Performance with respect to $e_p$ (Scenario 1)
6.2.2 Scenario 2

Figure 6.7 shows that when the rate-sensors are removed, the NPF angular rate estimates are still essentially insensitive to plant error. However, figure 6.8 shows that the NPF Euler angle estimates become more sensitive to plant error than the EKF Euler angle estimates. Comparing figures 6.8 and 6.6, we see that surprisingly, without rate-sensors, the EKF Euler angle estimates are less sensitive to plant error, than when rate-sensors are present.
Figure 6.7: Angular Rate Performance with respect to $e_p$ (Scenario 2)

Figure 6.8: Euler Angle Performance with respect to $e_p$ (Scenario 2)
6.3 Measurement Error

6.3.1 Scenario 1

Table A.2 shows that the NPF has the largest range of measurement error over which the state estimates converge. The EHF has the smallest range. Figure 6.9 shows that all filters have the same sensitivity to measurement error with respect to the angular rate estimates. Figure 6.10 shows that the NPF is the least sensitive to measurement error with respect to the Euler angle estimates, and the MNPF is the most sensitive.
Section 6.3. Measurement Error

Figure 6.9: Angular Rate Performance with respect to $e_m$ (Scenario 1)

Figure 6.10: Euler Angle Performance with respect to $e_m$ (Scenario 1)
6.3.2 Scenario 2

Figures 6.11 and 6.12 show that when the rate-sensors are removed, the EKF is significantly more robust in the angular rate estimates with respect to measurement error than the NPF, but both perform exactly the same in the Euler angle estimates.
Section 6.3. Measurement Error

Figure 6.11: Angular Rate Performance with respect to $e_m$ (Scenario 2)

Figure 6.12: Euler Angle Performance with respect to $e_m$ (Scenario 2)
6.4 Sensor Noise Level

6.4.1 Scenario 1

Figures 6.13 and 6.14 show that the EKF is the least sensitive to sensor noise level, followed by the MNPF. The EHF is the most sensitive.

6.4.2 Scenario 2

Figures 6.15 and 6.16 show that the EKF is much less sensitive to sensor noise level than the NPF.
**Section 6.4. Sensor Noise Level**

**Figure 6.13:** Angular Rate Performance with respect to sensor noise level (Scenario 1)

**Figure 6.14:** Euler Angle Performance with respect to sensor noise level (Scenario 1)
Chapter 6. Results

Figure 6.15: Angular Rate Performance with respect to sensor noise level (Scenario 2)

Figure 6.16: Euler Angle Performance with respect to sensor noise level (Scenario 2)
6.5 General Comments

Comparing Figures 6.5-6.8 to Figures 6.9-6.12, we see that all filters are much more sensitive to measurement error than to plant error. Hence, much emphasis should be made on accurate sensor placement. An accurate model for the spacecraft dynamics is less important. Figures B.5, B.13, B.21, B.29, B.37 and B.45 show that for even small measurement errors, the state estimates become significantly bad.
Chapter 7

Conclusions and Recommendations

In this work, three state-estimation techniques have been compared: the EKF, the NPF and the EHF. This comparison is in the context of spacecraft attitude determination.

The EHF did not perform well in any of the situations, and is definitely the worst of the filters.

With rate-sensors present, the NPF outperformed the EKF with respect to plant and measurement error. However, the NPF has a very small range of convergence. This is due to the way the filter is formulated, with the model error \( d(t_k) \) being added only to the spacecraft dynamics and not to the attitude kinematics, leading to a conflicting requirement between tracking the rates, and tracking the attitude parameterization related measurements (magnetometer and sun-sensor). This results in a very slow rate of convergence and for even small initial errors, divergence of the state estimates. The problem with convergence can be overcome by reformulating the NPF by artificially adding model error to the attitude kinematics as well as adding it to the attitude dynamics. This has the desired effect of increasing the range of convergence, which is the largest of all of the filters, as well as significantly decreasing the filter sensitivity to sensor noise level (though not as good as the EKF). However, it comes at the cost of poorer performance with respect to model and measurement error, which are precisely where the NPF outperform the EKF.

Without rate-sensors, the EKF outperforms the NPF in all aspects other than angular rate estimate sensitivity to plant error and Euler angle sensitivity to measurement error. The NPF has a slightly larger range of convergence.

Given that the EKF has the least sensitivity to sensor noise level (both with and without rate-sensors), and that the EKF performs well with respect to plant error,
measurement error and initial estimate error, it is the estimation scheme of choice of all filters investigated in this thesis.

Because the Nonlinear Projection Filter\textsuperscript{1} approximates the nonlinear generalization of the Kalman Filter, it is recommended that this filter be the focus of a future study. It could be that the EHF performed poorly since it is not suitable as a linearized technique. This does not rule out investigation of its nonlinear counterpart. For this reason, it is suggested that the nonlinear H\textsubscript{\infty} filter also be the subject of a future study. To exploit the different benefits of each filter, the subject of filter scheduling between the different filters could be examined also.
Chapter 8

References


Appendix A

Simulation Results

Note: DNC = Did Not Converge

A.1 Scenario 1 (With Rate-Sensors)
Appendix A. Simulation Results

Table A.1: Filter Performance with respect to $e_p$ (Scenario 1)

<table>
<thead>
<tr>
<th>$e_p$</th>
<th>$\text{Erms}_{\omega}$ (deg/s)</th>
<th>$\text{Erms}_{\alpha}$ (deg)</th>
</tr>
</thead>
<tbody>
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<td>EKF</td>
<td>NPF</td>
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<td>.13408e-3</td>
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Table A.2: Filter Performance with respect to $e_m$ (Scenario 1)

<table>
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<th>$\text{Erms}_{\alpha}$ (deg)</th>
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</thead>
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Table A.3: Filter Performance with respect to $e_i$ (Scenario 1)

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A.2 Scenario 2 (Without Rate-Sensors)

Table A.4: Filter Performance with respect to $e_p$ (Scenario 2)

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Table A.5: Filter Performance with respect to $e_m$ (Scenario 2)

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<th>$e_m$</th>
<th>$E_{rms_w}$ (deg/s)</th>
<th>$E_{rms_a}$ (deg)</th>
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<td>.35836</td>
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<td>.55041e-1</td>
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<td>.10032</td>
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Table A.6: Filter Performance with respect to $e_i$ (Scenario 2)

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<th>$E_{rms_a}$ (deg)</th>
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<td>NPF</td>
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Appendix B

Simulation Plots

B.1 Scenario 1 (With Rate-Sensors)

B.1.1 Extended Kalman Filter
Figure B.1: EKF State Estimates. $e_p = 0, e_m = 0, e_i = 0$
Section B.1. Scenario 1 [With Rate-Sensors]

Figure B.2: EKF Estimation Errors. $e_p = 0$, $e_m = 0$, $e_i = 0$
Figure B.3: EKF State Estimates, $e_p = -14$, $e_m = 0$, $e_i = 0$
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Figure B.7: EKF State Estimates. $e_p = 0$, $e_m = 0$, $e_i = -16$
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Section B.1. Scenario 1 (With Rate-Sensors)

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Section B.1. Scenario 1 (With Rate-Sensors)

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Section B.1. Scenario 1 (With Rate-Sensors)

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Section B.1. Scenario 1 (With Rate-Sensors)

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Section B.2. Scenario 2 (Without Rate-Sensors)

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Section B.2. Scenario 2 (Without Rate-Sensors)

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