Resolving Multiplicities in the Tensor Product of Irreducible Representations of Semisimple Lie Algebras

by

David John Brooke

A thesis submitted in conformity with the requirements for the degree of Ph.D.
Graduate Department of Mathematics
University of Toronto

© Copyright by David John Brooke (2008)
Resolving multiplicities in the tensor product of irreducible representations of semisimple Lie algebras

Doctoral thesis 2008
by David John Brooke
Department of Mathematics
University of Toronto

ABSTRACT: When the tensor product of two irreducible representations contains multiple copies of some of its irreducible constituents, there is a problem of choosing specific copies: resolving the multiplicity. This is typically accomplished by some ad hoc method chosen primarily for convenience in labelling and calculations. This thesis addresses the possibility of making choices according to other criteria. One possible criterion is to choose copies for which the Clebsch-Gordan coefficients have a simple form. A method fulfilling this is introduced for the tensor product of three irreps of $su(2)$. This method is then extended to the tensor product of two irreps of $su(3)$. In both cases the method is shown to construct a full nested sequence of basis independent highest weight subspaces. Another possible criterion is to make choices which are intrinsic, independent of all choices of bases. This is investigated in the final part of the thesis with a basis independent method that applies to the tensor product of finite dimensional irreps of any semisimple Lie algebra over $\mathbb{C}$. 
Acknowledgements

I would like to thank Prof. Joe Repka for guiding and supporting me with such confidence and care throughout the last five years. I much appreciate the many hours he has put into thinking about my thesis work as well as the many hours of unfruitful labour that he has often guarded me from with his insightful and careful advice. My wife and I are also very grateful for Prof. Repka’s generous hospitality on numerous occasions. I would also like to thank Prof. David Rowe and all who attended our weekly mathematical physics seminar for grounding my work with physical interpretations and practical concerns. My thanks also go to Sandeep Bhargava, Maria Wesslen and Lindsay Shorser for their friendship and for many interesting and helpful mathematical discussions over the last five years. During the work of this thesis I have been very grateful for the generous financial support of the Canadian Rhodes Scholars Foundation and the University of Toronto. Finally, and most importantly, I would like to thank my wife, Louise, who was willing to move to another country for my studies and who has been a constant loving support and encouragement in all that I have done.
Contents

1 An Introduction
   1.1 The general problem ............................................. 1
   1.2 Some general representation theory ............................. 2

2 Decomposing highest weight spaces in the tensor product of three irreps
   of $su(2)$
   2.1 A brief overview .................................................. 4
   2.2 An introduction to the tensor product of three copies of $su(2)$ .... 5
   2.3 Clebsch-Gordan Coefficients ..................................... 7
      2.3.1 Possible Clebsch-Gordan coefficients ....................... 7
      2.3.2 An example ................................................... 9
      2.3.3 The One Dimensional case .................................. 10
   2.4 A condition for RSR-generating vectors ......................... 11
      2.4.1 Examining the C.G. coefficients ............................ 11
      2.4.2 A C.G. coefficient that determines the RSR-generating property ... 12
      2.4.3 What is the C.G. coefficient of $|\frac{p-2N}{2}\rangle|\frac{q-2N}{2}\rangle|\frac{r-2N}{2}\rangle$? ...................... 14
   2.5 More on determining RSR-generating highest weight vectors ....... 14
      2.5.1 Examining the highest weight vectors ...................... 14
      2.5.2 On levels determining the RSR-generating property for highest weight
            vectors ....................................................... 16
      2.5.3 Classifying the RSR-generating highest weight vectors .......... 19
   2.6 Constructing orthogonal RSR-generating highest weight vectors .... 20
      2.6.1 Further properties of C.G. coefficients on a given level .......... 20
      2.6.2 A method for constructing orthogonal RSR-generating highest weight
            vectors ....................................................... 21
List of Tables

2.1 A table of general C.G. coefficients determined by C.G. coefficients in the $\frac{p+q+r-2}{2}$ weight space. ........................................ 9
2.2 A table of the levels of the C.G. coefficients of highest weight $\frac{p+q+r-4}{2}$. .... 15
2.3 An example of degenerate C.G. coefficients of highest weight $\frac{p+q+r-4}{2}$ when $|r-4\rangle$ does not exist. ........................................ 26
3.1 A table listing the action of $T_+$ and $U_+$ on elements of $U$. ....................... 41
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>The weight diagram of the irrep $(2, 1)$</td>
<td>36</td>
</tr>
<tr>
<td>3.2</td>
<td>The vectors $U_+, T_+$, $\lambda_1$ and $\lambda_2$.</td>
<td>37</td>
</tr>
<tr>
<td>3.3</td>
<td>The weight diagram of the irrep $(2, 1)$ with some bases of weight spaces labeled.</td>
<td>37</td>
</tr>
<tr>
<td>3.4</td>
<td>The weight diagrams of $(1, 1) \otimes (2, 1)$.</td>
<td>40</td>
</tr>
<tr>
<td>3.5</td>
<td>The HWCD of $(2, 1)$ in $(1, 1) \otimes (2, 1)$</td>
<td>40</td>
</tr>
<tr>
<td>3.6</td>
<td>A general nondegenerate HWCD.</td>
<td>43</td>
</tr>
<tr>
<td>3.7</td>
<td>The positions of $R^<em>_1$ and $L^</em>_2$ on the first irrep in a non-degenerate HWCD.</td>
<td>45</td>
</tr>
<tr>
<td>3.8</td>
<td>The dimension of the weight spaces in the non-degenerate HWCD tensor product.</td>
<td>45</td>
</tr>
<tr>
<td>3.9</td>
<td>The HWCD for a non-degenerate two-dimensional highest weight space.</td>
<td>47</td>
</tr>
<tr>
<td>3.10</td>
<td>The HWCD of a six-dimensional weight space with two dimensional highest weight subspace.</td>
<td>52</td>
</tr>
<tr>
<td>3.11</td>
<td>A general degenerate HWCD.</td>
<td>55</td>
</tr>
<tr>
<td>3.12</td>
<td>The HWCD for $(1, 2) \otimes (2, 2)$.</td>
<td>62</td>
</tr>
<tr>
<td>4.1</td>
<td>A diagram of the chain of relations that enable $L^<em>_1$ to be written as a linear combination of the $R^</em>_1$.</td>
<td>73</td>
</tr>
<tr>
<td>4.2</td>
<td>A diagram of the chain of relations that enable $L^<em>_2$ to be written as a linear combination of the $R^</em>_1$.</td>
<td>73</td>
</tr>
<tr>
<td>4.3</td>
<td>A diagram of the chain of relations that enable $L^<em>_1$ to be written as a linear combination of the $R^</em>_1$.</td>
<td>74</td>
</tr>
<tr>
<td>4.4</td>
<td>The HWCD for a non-degenerate two-dimensional highest weight space.</td>
<td>74</td>
</tr>
<tr>
<td>4.5</td>
<td>The HWCD for $(1, 2) \otimes (2, 2)$.</td>
<td>76</td>
</tr>
<tr>
<td>4.6</td>
<td>The HWCD of $(2, 1) \otimes (2, 2)$ for $(1, 2)$.</td>
<td>78</td>
</tr>
</tbody>
</table>
Chapter 1

An Introduction

1.1 The general problem

Weyl’s theorem (see Humphreys [3]) states that representations of semisimple Lie algebras over \( \mathbb{C} \) are completely reducible. So the tensor product of \( m \) irreducible representations, itself being a representation, can be written as a direct sum of irreducible representations.

\[
U_1 \otimes U_2 \otimes \ldots U_m = W_1 \oplus W_2 \ldots W_k \tag{1.1}
\]

Now equation (1.1) implies that by using a change of basis transformation, it is possible to convert the tensor product basis into a basis for the direct sum decomposition. The change of basis coordinates are called Clebsch-Gordan coefficients (hereafter C.G. coefficients). The question of decomposing the tensor product of two irreducible representations of a semisimple Lie algebra into the direct sum decomposition is much researched, see for example de Graaf [2]. In some cases, such as \( su(3) \), the decompositions are well known and there are several algorithms for finding them, see for example Pfeifer [5] for the theory of Young tableaux.

In the cases where irreducible representations (hereafter irreps) occur more than once as a direct summand, very little has been done by way of general attempts to distinguish particular copies. Rather it is left to the reader to choose copies, usually according to the conveniences of their own work. The underlying opinion being that whilst some choices are better than others in specific cases, all repeated copies are essentially the same.

This thesis maintains that certain decompositions are better than others and aims to begin an investigation of the possibilities for decomposing irreps that occur with multiplicity
in such tensor products. In chapter 2, a method is proposed that constructs orthogonal irreps with C.G. coefficients that can be expressed in a simple form in the tensor product of three irreps of $su(2)$. The method is shown to produce a full nested sequence of basis independent highest weight spaces. The method also has the attractive property that it can be applied to either the highest weight space or the lowest weight space with the same results. Then, in chapter 3, this method is shown to generalise to decompose multi-dimensional highest weight spaces in the tensor product of two irreps of $su(3)$; once again producing a full nested sequence of basis independent highest weight spaces. Finally chapter 4 examines a method that is basis independent and applies to the tensor product of finite dimensional irreducible representations of any semisimple Lie algebra over $\mathbb{C}$.

One of the first questions to raise is how to decide whether a particular decomposition is special or desirable. This work has been guided under the assumption that it is desirable to have orthogonal irreps (with easily describable Clebsch-Gordan coefficients) or irreps that are basis independent. Note that it is impossible to have both these conditions simultaneously, as a change of basis alters the dot product, and hence affects the orthogonality condition.

It should also be noted that an irrep is determined by its highest weight vector. So to specify a decomposition of irreps of a particular highest weight it suffices to choose a basis of highest weight vectors. Hence the problem may be re-stated as that of finding a well defined basis (up to scalar multiplication) for any multi-dimensional highest weight space; indeed this is the way the results in this thesis are formulated.

Finally note that it has been necessary to define some similar concepts in the different settings described above. Due to the similarities it has often been convenient to use the same word in analogous circumstances. As such, definitions are occasionally repeated with appropriate changes and must always be read in the context of the chapter in which they are set.

1.2 Some general representation theory

Much of the background theory on the representations of semisimple Lie algebras that is used in this thesis, can be found in Humphreys’ book on the topic [3]. This section briefly covers a few useful results with the notation and theory all being taken from Humphreys [3]. Recall that finite dimensional modules of finite dimensional semisimple Lie algebras decompose under the action of the Cartan subalgebra, H (the maximal toral subalgebra).
Let $V$ be such a module.

$$V = \bigoplus V_{\lambda} \quad \text{with} \ \lambda \in H^* \ \text{and} \ V_{\lambda} = \{v \in V | h.v = \lambda(h)v \ \forall h \in H\} \quad (1.2)$$

If $V_{\lambda} \neq 0$ then $V_{\lambda}$ is said to be a weight space, and $\lambda$ a weight of $V$. Recall that in the special case of the adjoint action decomposing the Lie algebra itself, the term roots and root spaces are used instead. When the elements in a particular root space, $x \in L_\alpha$, act on elements $v \in V_{\lambda}$ the resulting vector is in $V_{\lambda + \alpha}$. Let $h \in H$; then equation (1.3) establishes this result.

$$h.x.v = x.h.v + [h,x].v = (\lambda(h) + \alpha(h))x.v \quad (1.3)$$

This may be written as $L_\alpha : V_{\lambda} \to V_{\lambda + \alpha}$. Also recall that a finite dimensional irrep is determined by its highest weight space (this weight space is killed under the action of all the positive roots) and that all the weights are integer combinations of a finite basis of fundamental weights. Also important for this thesis is the easy result that the weight of the tensor product of two elements is the sum of the weights of the respective parts.
Chapter 2

Decomposing highest weight spaces in the tensor product of three irreps of $su(2)$

2.1 A brief overview

This chapter examines the tensor product of three irreps of $su(2)$. This seems to be the logical place to begin enquiries into tensor product decompositions as it is often possible to make calculations explicitly. Recall that there are no multiplicities in the direct sum decomposition of two irreducible representations of $su(2)$. The work in this chapter is concerned with unitary irreps, unless otherwise stated. After introducing the necessary notation, this chapter begins with an examination of the possibilities for C.G. coefficients in the unitary basis and notes that, at best, it might be hoped that all the C.G. coefficients can be square roots of rational numbers. The one dimensional case is reviewed as an example that such a possibility does indeed occur when no choices are involved. Then a particular C.G. coefficient is identified that determines whether all the C.G. coefficients of a particular irrep are indeed square roots of rational numbers. This is followed by some more practical conditions that determine whether all the C.G. coefficients of a particular irrep are square roots of rational numbers. Then the highest weight vectors of such irreps are classified. A method is then introduced that constructs a basis for the highest weight space of orthogonal vectors, that give rise to irreps with C.G. coefficients that are square roots of rational numbers. This method is first proved for the non-degenerate case, then immediately adapted to apply to any highest
weight space in the tensor product of three irreps of $su(2)$. Intrinsic to this method is
the construction of a full nested sequence of basis independent highest weight subspaces;
and after some discussion it is established that the method will construct orthogonal irreps
regardless of the bases of the irreps forming the tensor product. Finally the method is applied
to lowest weight spaces and it is proved that it gives rise to the same irreps as when applied
to the corresponding highest weight spaces.

2.2 An introduction to the tensor product of three
copies of $su(2)$

Recall that $su(2)$ is the 3-dimensional real Lie algebra of skew-Hermitian $2 \times 2$-matrices with
vanishing trace. Recall also that the irreps of $su(2)$ are well known and may be put into
correspondence with the non-negative half-integers. Unless otherwise stated, it is assumed
that the irreps are unitary. Let $|p, a\rangle$ denote the vectors of a (unitary) irrep of $su(2)$ where
$p, a \in \mathbb{Z}$, and $p$ is the highest weight of the irrep in which the vector is contained and $a$
is the weight of the vector itself. Recall that $-p \leq a \leq p$ and $a \equiv p (mod 2)$. Where the
highest weight is clear $|p, a\rangle$ may be written more concisely as $|a\rangle$. For example $|p - 2, 2\rangle$
will denote $|p, p - 2\rangle$. Let $T_+$ and $T_- \in su(2)$ be the usual raising and lowering operators on the
irreps of $su(2)$. In Pfeifer [5] the action of $T_+$ and $T_-$ in a unitary irrep is derived and may
be expressed as follows:

\[
T_+ |\frac{p}{2}, \frac{p - 2a}{2}\rangle = \sqrt{a(p - a + 1)} |\frac{p}{2}, \frac{p - 2(a - 1)}{2}\rangle
\]

\[
T_- |\frac{p}{2}, \frac{p - 2a}{2}\rangle = \sqrt{(a + 1)(p - a)} |\frac{p}{2}, \frac{p - 2(a + 1)}{2}\rangle
\]

In this context the numbers $\sqrt{a(p - a + 1)}$ and $\sqrt{(a + 1)(p - a)}$ shall be referred to
as raising and lowering coefficients respectively. Now let $P, Q$ and $R$ be the irreducible
representations with highest weights $\frac{p}{2}$, $\frac{q}{2}$ and $\frac{r}{2}$ respectively. Consider the tensor product
$P \otimes Q \otimes R$. Then since each of $P, Q$ and $R$ have a unique (up to scalar multiplication) highest
weight vector, the unique (up to scalar multiplication) highest weight vector of $P \otimes Q \otimes R$ is
given by $|\frac{p}{2}\rangle \otimes |\frac{q}{2}\rangle \otimes |\frac{r}{2}\rangle$. In vector notation, where it is clear, the tensor product sign will be
omitted; for example this highest weight will be written as $|\frac{p}{2}, \frac{q}{2}, \frac{r}{2}\rangle$. Recall that choosing a
highest weight vector determines an irrep (simply lower using $T_+$ to get the whole irrep). So it is clear that $P \otimes Q \otimes R$ has a unique irrep of highest weight $\frac{p+q+r}{2}$. It will be convenient to put a lexicographic ordering on the tensor product and when listing vectors in the tensor product always list the lexicographically greatest first. So $v_0 = |\frac{a_0}{2}\rangle |\frac{b_0}{2}\rangle |\frac{c_0}{2}\rangle$ is compared to $v_1 = |\frac{a_1}{2}\rangle |\frac{b_1}{2}\rangle |\frac{c_1}{2}\rangle$ by first considering if $a_0$ is greater than or less than $a_1$. If either of these is true, then the inequality for the $v_i$ is the same as for the $a_i$. If $a_0 = a_1$ then consider the $b_i$ and so on. It will also be convenient to assume throughout that $p \geq q \geq r$.

Now consider the weight space of $P \otimes Q \otimes R$ of weight $\frac{p+q+r-2}{2}$. If $p,q,r \geq 1$, then this weight space is spanned by $\{|\frac{a^3}{2}\rangle |\frac{b^2}{2}\rangle, |\frac{a^2}{2}\rangle |\frac{b^2}{2}\rangle |\frac{c}{2}\rangle, |\frac{a^2}{2}\rangle |\frac{b}{2}\rangle |\frac{c}{2}\rangle\}$. Let $v$ be an arbitrary vector in this weight space.

$$v = \alpha_1|\frac{p^3}{2}\rangle |\frac{q^2}{2}\rangle |\frac{r-2}{2}\rangle + \alpha_2|\frac{p^2}{2}\rangle |\frac{q^2}{2}\rangle |\frac{r}{2}\rangle + \alpha_3|\frac{p^2}{2}\rangle |\frac{q}{2}\rangle |\frac{r}{2}\rangle$$

Consider the action of $T_+$ on $v$.

$$T_+v = (\sqrt{r}\alpha_1 + \sqrt{q}\alpha_2 + \sqrt{p}\alpha_3)|\frac{p^3}{2}\rangle |\frac{q}{2}\rangle |\frac{r}{2}\rangle$$

(2.1)

For this to be a highest weight vector the action of $T_+$ must annihilate $v$. That is, if $v$ is to be a highest weight vector, $T_+v = 0$. Setting equation 2.1 to zero gives $\sqrt{r}\alpha_1 + \sqrt{q}\alpha_2 + \sqrt{p}\alpha_3 = 0$. This condition for a vector to be a highest weight vector may be rewritten as $\alpha_3 = \frac{-1}{\sqrt{p}}(\sqrt{r}\alpha_1 + \sqrt{q}\alpha_2)$. So this three dimensional weight space contains a two dimensional highest weight space spanned by $\{|\frac{a^3}{2}\rangle |\frac{b^2}{2}\rangle |\frac{c}{2}\rangle - \sqrt{q}|\frac{a^2}{2}\rangle |\frac{b^2}{2}\rangle |\frac{c}{2}\rangle, |\frac{a^2}{2}\rangle |\frac{b^2}{2}\rangle |\frac{c}{2}\rangle - \sqrt{p}|\frac{a^2}{2}\rangle |\frac{b}{2}\rangle |\frac{c}{2}\rangle\}$. The third dimension is accounted for in the irrep with highest weight $|\frac{a^2}{2}\rangle |\frac{b^2}{2}\rangle |\frac{c}{2}\rangle$.

**Definition 2.2.1** In an $n$-dimensional highest weight space the $n$ lexicographically greatest tensor product vectors will be called basic vectors. The C.G. coefficients attached to the basic vectors will be called basic C.G. coefficients (or just 'basic coefficients').

In the above example, the highest weight space is 2-dimensional and $\alpha_1$ and $\alpha_2$ are the basic C.G. coefficients.

**Lemma 2.2.2** Let $p,q,r \geq n$, then the weight space of $P \otimes Q \otimes R$ of weight $\frac{n+q+r-2(n-1)}{2}$ has a highest weight subspace of dimension $n$.

**Proof:** A counting argument gives the dimension of the weight space as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and inductively the first $n-1$ terms in the sum are accounted for by irreps of higher weight.
For the exposition that follows let $p, q$ and $r$ be greater than $n$. This ensures that all the vectors in the weight space of the highest weight actually exist and results such as lemma 2.2.2 above hold. There is a discussion in section 2.8.1 on page 24 that generalises to the degenerate situation. However, the following lemma is stated in such a way that it holds in general.

**Lemma 2.2.3** Let $p \geq q, r$. In an $n$-dimensional highest weight space the basic C.G. coefficients are precisely the coefficients that are attached to vectors of the form $|\frac{p}{2}\rangle \times \langle -| \times \langle -|$.

When $p, q$ and $r \geq n$, lemma 2.2.3 is immediate as such vectors are exactly the $n$ lexicographically greatest. In general the result relies on the assumption that $p \geq q, r$. Lemma 2.2.3 is stated here in generality, but the proof is postponed until the degenerate case is discussed on page 24. Lemma 2.2.3 provides a useful alternative way of thinking about and recognising the basic C.G.

### 2.3 Clebsch-Gordan Coefficients

#### 2.3.1 Possible Clebsch-Gordan coefficients

Recall that the Clebsch-Gordan coefficients of a tensor product of irreps are the coefficients of the basis transformation that takes the tensor product basis to that of a direct sum decomposition. Once the highest weights of all the irreps in the direct sum are determined, so are all the C.G. coefficients. This is because the lowering operators may be applied to the highest weight vectors to determine the whole of each irrep. The purpose of the work that follows is to choose highest weights that give rise to C.G. coefficients that can be expressed in a simple form. Before doing so, it is worthwhile examining just how good or bad the C.G. coefficients might be with any given choice of highest weight vector.

Let $\alpha_1, \alpha_2, ..., \alpha_{\frac{\left( n+1 \right)}{2}}$ be the coefficients attached to the (lexicographically ordered) tensor product basis elements in the weight space of $P \otimes Q \otimes R$ of weight $\frac{p+q+r-2n}{2}$. Then the $\alpha_i$ are themselves C.G. coefficients and, under certain relations, form a highest weight space of weight $\frac{p+q+r-2n}{2}$. Further all the C.G. coefficients for an irrep with highest weight $\frac{p+q+r-2n}{2}$ are found by lowering a vector from this highest weight space. More precisely, the C.G.
coefficients are of the form \( \sum_{i=1}^{n(n+1)/2} b_i c_i \alpha_i \) where the \( c_i \) are products of square roots of lowering coefficients with factors of the form \( \sqrt{(l+1)(p-l)}, \sqrt{(m+1)(q-m)} \) and \( \sqrt{(n+1)(r-n)} \) with \( l, m, n \in \mathbb{N} \) and the \( b_i \) are binomial coefficients. Further note that for \( k > n \), each \( \alpha_k \) is a linear combination of basic C.G. coefficients; specifically \( \alpha_k = \sum_{i=1}^{n} b^k_i c^k_i \alpha_i \) where \( b^k_i \) is a binomial coefficient and \( c^k_i \) is a quotient of products of raising coefficients (or lowering coefficients). (More rigorous details concerning the dependence of coefficients in the highest weight space are established in section 2.5.1.) Thus it is evident that every C.G. coefficient, \( \Psi \), may be written as a linear combination of basic coefficients in the following form:

\[
\Psi = \sum_{i=1}^{n(n+1)/2} b_i c_i \alpha_i = \sum_{i=1}^{n} C_i \alpha_i \quad \text{where} \quad C_i = b_i c_i + \sum_{j=n+1}^{n(n+1)/2} b^j_i c^j_i \quad (2.2)
\]

The basic coefficients determine an irrep and thus also determine all the C.G. coefficients for that irrep as outlined in equation (2.2). The aim is to choose the basic coefficients, \( \alpha_i \), in such a way that all the C.G. coefficients for the irrep are as simple as possible. It turns out that the \( \alpha_i \) may be chosen so that each of the \( C_i \alpha_i \) in equation (2.2) contain the same product of square roots. A sensible condition to insist upon is that every C.G. coefficient has the form \( Y(p,q,r)Z(p,q,r) \) where \( Y(p,q,r) \) is a quotient of products of raising coefficients and \( Z(p,q,r) \) is an integer polynomial in \( p, q \) and \( r \). This is clearly equivalent to insisting that the square of each C.G. coefficient is a rational number.

**Definition 2.3.1** A C.G. coefficient that is the square root of a rational number is said to be RSR (for rational square root).

**Definition 2.3.2** A highest weight vector is said to be RSR-generating if, for all values of \( p, q \) and \( r \in \mathbb{Z} \), every C.G. coefficient in the corresponding irrep is RSR.

The notation \( \alpha_i \in \sqrt{\mathbb{Q}} \) will be used when \( \alpha_i \) is RSR. Similarly if a highest weight vector, \( v \), is RSR-generating, the notation \( v \in \sqrt{\mathbb{Q}} \) will be used. Note that definition 2.3.2 excludes vectors that might have all entries as square roots of rational numbers in specific cases where the values of \( p, q \) and \( r \) are such that the relevant raising operators are rational themselves. Insisting all C.G. coefficients are RSR seems to be the strongest possible condition attainable for all C.G. coefficients simultaneously. That it is indeed attainable will be demonstrated explicitly, after some basic theory is developed.
2.3.2 An example

In this subsection the weight space of $P \otimes Q \otimes R$ of weight $\frac{p+q+r-2}{2}$ is considered in detail. As mentioned in section 2.2, if $p, q, r \geq 1$, then this weight space is spanned by $V = \{|\frac{p}{2}\rangle |\frac{q}{2}\rangle |\frac{r-2}{2}\rangle, |\frac{p-2}{2}\rangle |\frac{q}{2}\rangle |\frac{r}{2}\rangle, |\frac{p-2}{2}\rangle |\frac{q-2}{2}\rangle |\frac{r}{2}\rangle\}$. (Recall also that this space contains a two dimensional highest weight space.)

<table>
<thead>
<tr>
<th>Element of tensor product basis</th>
<th>General C.G. coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\frac{p}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-2}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-2}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-2}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-2}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-4}{2}\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\frac{p-4}{2}\rangle</td>
</tr>
</tbody>
</table>

Table 2.1: A table of general C.G. coefficients determined by C.G. coefficients in the $\frac{p+q+r-2}{2}$ weight space.

Let $\alpha_1, \alpha_2$ and $\alpha_3$ be the (lexicographically ordered) coefficients of a vector in the span of $V$. Table 2.1 records the C.G. coefficients of a few selected elements in the tensor product basis that result from the lowering of an arbitrary vector of weight $\frac{p+q+r-2}{2}$. Note that insisting the vector of weight $\frac{p+q+r-2}{2}$ is a highest weight vector requires $\alpha_3 = \frac{-1}{\sqrt{q}}(\sqrt{r}\alpha_1 + \sqrt{q}\alpha_2)$.

For example, consider the C.G. coefficient of $|\frac{p}{2}\rangle |\frac{q-2}{2}\rangle |\frac{r-2}{2}\rangle$ as given in table 2.1; it is $\sqrt{q}\alpha_1 + \sqrt{r}\alpha_2$. So choosing $\alpha_1 = \sqrt{q}$ and $\alpha_2 = \sqrt{r}$ gives the C.G. coefficient as $q + r$. This is certainly the square root of a rational number and following the notation in the previous section we have $Y(p, q, r) = 1$ and $Z(p, q, r) = q + r$. Alternatively choosing $\alpha_1 = \sqrt{r}$ and $\alpha_2 = \sqrt{q}$ would give this C.G. coefficient as $2\sqrt{q}\sqrt{r}$. Again this is the square root of a rational number, and in the notation of the previous section $Y(p, q, r) = \sqrt{q}\sqrt{r}$ and $Z(p, q, r) = 2$. However there are many choices of $\alpha_1$ and $\alpha_2$ that would not make this particular C.G. coefficient the square root of a rational number. For example, $\alpha_1 = 1$ and $\alpha_2 = 1$, gives $\sqrt{q} + \sqrt{r}$ which is not the square root of a rational number for all $q$ and all $r$ and so is not in the required form. This example is concerned with how the choice of the basic coefficients
affect one C.G. coefficient. The more general aim is to try to choose basic coefficients so that all C.G. coefficients are RSR simultaneously. Insisting that the C.G. coefficients are all square roots of rational numbers is easily satisfied by choosing all bar one of the \( \alpha_i \) to be zero. However this particularly straightforward choice does not give orthogonal vectors, which is another of the required properties.

With table 2.1 in this section, it will be useful to clarify two definitions now.

**Definition 2.3.3** Fix some set, \( S \), of C.G. coefficients that determine an irrep. A general C.G. coefficient, \( \Psi \), with respect to \( S \), is a C.G. coefficient described as a linear combination of C.G. coefficients in \( S \). (i.e. \( \Psi = \sum_{i=1}^{n} c_i(p,q,r)\alpha_i \).)

**Definition 2.3.4** A partial sum of a general C.G. coefficient, \( \Psi \), is a sum of any of the \( c_i(p,q,r)\alpha_i \) that occur in \( \Psi \), as in definition 2.3.3.

So for example, in table 2.1, \( S \) has been chosen to be \( \{\alpha_1, \alpha_2, \alpha_3\} \). The general C.G. coefficient of \(|p-\frac{4}{2}\rangle|q-\frac{2}{2}\rangle|r-\frac{2}{2}\rangle\), with respect to \( S \), is \( 3\sqrt{2p-2}(\sqrt{pq}\alpha_1 + \sqrt{pr}\alpha_2 + 2\sqrt{qr}\alpha_3) \). Further \( 3\sqrt{2p-2}(\sqrt{pr}\alpha_2 + 2\sqrt{qr}\alpha_3) \) is a partial sum of this coefficient, as is \( 6\sqrt{2p-2}\sqrt{qr}\alpha_3 \).

If no set \( S \) is mentioned, it will be assumed that \( S \) is the set of basic C.G. coefficients.

2.3.3 The One Dimensional case

The intention is to establish a method that will decompose a multi-dimensional highest weight space into highest weight vectors that will give rise to orthogonal irreps with RSR C.G. coefficients. To reinforce that these are sensible conditions to require, this section demonstrates that all one dimensional highest weight spaces give rise to irreps that have RSR C.G. coefficients.

**Lemma 2.3.5** Every one dimensional highest weight space gives rise to an irrep with RSR C.G. coefficients.

**Proof:** Consider the irrep with highest weight \( \frac{p+q+r}{2} \). The weight space of dimension \( \frac{p+q+r}{2} \) is one dimensional and is the entire highest weight space. It is spanned by a single element of the tensor product basis and therefore all C.G. coefficients are simply multiples of this coefficient by lowering coefficients and binomial coefficients. So choosing this highest C.G. coefficient to be RSR forces all C.G. coefficients to be RSR.
A one dimensional highest weight space may also occur when $p, q$ and/or $r$ are less than $n$. In this case the weight space will be more than one dimensional. However there will be a single basic C.G. coefficient in the highest weight space and then again all other coefficients are multiples of this coefficient by lowering coefficients and binomial coefficients. (Note that a rigorous proof relies on proposition 2.5.4 and the discussion in section 2.8.1) So choosing this single basic C.G. coefficient to be RSR gives all C.G. coefficients as RSR.

2.4 A condition for RSR-generating vectors

2.4.1 Examining the C.G. coefficients

It will be convenient to define $P_a = \sqrt{a(p-a+1)}$. Equivalently, $P_a$ is the raising coefficient given when $T_+$ is applied to $|p, \frac{p-2a}{2}\rangle$ in the unitary basis. Let $a \geq b$ and denote by $P_a^b$ the product of raising coefficients that occur from raising the vector $|\frac{p-2a}{2}\rangle$ $b-a$ times to a scalar multiple of $|\frac{p-2b}{2}\rangle$, so $P_a^b = \sqrt{a(p-a+1)} \sqrt{(a-1)(p-a+2)} \ldots \sqrt{(b+1)(p-b)}$. $Q_a^b$ and $R_a^b$ are defined similarly in terms of $q$ and $r$. Of course, since only unitary representations are currently being considered, these coefficients may also be regarded as lowering coefficients and the notation may be used without ambiguity for lowering coefficients.

Given that the C.G. coefficients are found by lowering the highest weight vectors, it is sensible to examine more closely the general C.G. coefficients with respect to the C.G. coefficients attached to vectors in the highest weight space: $\alpha_1, \ldots, \alpha_{n(n+1)/2}$ where the $\alpha_i$ are lexicographically ordered as described in section 2.2. It will be convenient to write $\alpha_{a,b}$ for the C.G. coefficient that is attached to the tensor product basis element of the form $|\frac{p-2a}{2}\rangle |\frac{q-2b}{2}\rangle |\frac{r-2c}{2}\rangle$. So for example, $\alpha_1 = \alpha_{0,0}$ and $\alpha_2 = \alpha_{0,1}$.

Notation 2.4.1 Let $\Psi_{k,l,m}$ denote the general C.G. coefficient with respect to the set $\{\alpha_{a,b}\}$, of C.G. coefficients attached to vectors of the form $|\frac{p-2k}{2}\rangle |\frac{q-2l}{2}\rangle |\frac{r-2m}{2}\rangle$. Also let $\alpha_{a,b}^{k,l,m}$ denote the coefficient of $\alpha_{a,b}$ in $\Psi_{k,l,m}$.

Note that $\alpha_{a,b}^{k,l,m} \neq 0$ only when $k \geq a, l \geq b$ and $m \geq (n - a - b - 1)$ since the vector is obtained by lowering from the highest weight vector. Then a combinatorial argument gives that
\[ \alpha_{a,b}^{k,l,m} = B_{a,b}^{k,l,m} P_k Q_l^{a} R_m^{n-a-b-1} \]

where \( B_{a,b}^{k,l,m} = \binom{k+l+m+n}{k-a} \binom{l+m+a+1-n}{l-b} \) is the number of distinct ways in which we may lower from \( \binom{p-2a}{q-2b} \binom{r-2(n-a-b-1)}{2} \) to \( \binom{p-2k}{q-2l} \binom{r-2m}{2} \) and \( P_k Q_l^{a} R_m^{n-a-b-1} \) are the appropriate lowering coefficients. Using this notation it is now possible to establish the following result about determining the RSR-generating property.

### 2.4.2 A C.G. coefficient that determines the RSR-generating property

**Lemma 2.4.2** Let \( p, q \) and \( r \geq N \). Consider an irreducible subrepresentation of \( P \otimes Q \otimes R \) with highest weight \( \binom{p+q+r-2N}{2} \). Let the C.G. coefficient of \( \binom{p-2N}{q-2N} \binom{r-2N}{2} \) in this irrep be a rational square root. Suppose also that every partial sum (with respect to the C.G. coefficients \( \alpha_{a,b} \)) of this general C.G. coefficient is also a rational square root. Then all the C.G. coefficients for the irrep are rational square roots.

Note that in the usual notation, the highest weight space of weight \( \binom{p+q+r-2N}{2} \) has dimension \( n \). So \( n = N + 1 \). Also note that requiring every partial sum of the C.G. coefficient to be \( \in \sqrt{Q} \) is to avoid awkward numbers that cancel in this particular coefficient, but then appear in other C.G. coefficients. In the example in section 2.3.2, taking \( \alpha_1 = \sqrt{r} (p - q) \pi, \alpha_2 = -\sqrt{q} (r - q) \pi \) and \( \alpha_3 = -\sqrt{p} (r - q) \pi \) with \( n = 1 \) gives that the coefficient of \( \binom{p-2}{q-2} \binom{r-2}{2} \) is zero, which is certainly in \( \sqrt{Q} \). But with these choices of \( \alpha_i \), \( i = 1, 2, 3 \) the coefficient of \( \binom{p-2}{q-2} \binom{r-2}{2} \) is \( \sqrt{p} \sqrt{q} (p + q) \pi \) which is not in \( \sqrt{Q} \). However this example does not satisfy the partial sum condition imposed in Lemma 2.4.2. The partial sum condition is a minor issue in terms of identification. In particular, if all terms in the sum (before any cancellation of terms) have the same factors up to integer polynomials (in \( p, q \) and \( r \)), then every partial sum is also in \( \sqrt{Q} \). Prior to the proof it will be convenient to introduce a definition that will be useful in structuring the proof of lemma 2.4.2

**Definition 2.4.3** A vector \( v_0 = \binom{a_0}{q_0} \binom{b_0}{b_0} \binom{c_0}{2} \) is said to be totally greater than \( v_1 = \binom{a_1}{q_1} \binom{b_1}{b_1} \binom{c_1}{2} \) if \( a_0 \geq a_1, b_0 \geq b_1 \) and \( c_0 \geq c_1 \). In such a case, \( v_1 \) is also said to be totally less than \( v_0 \).

**Proof:** The proof of lemma 2.4.2 is in three parts. Recall notation 2.4.1: that \( \Psi_{k,l,m} \) is the C.G. coefficient of \( \binom{p-2k}{q-2l} \binom{r-2m}{2} \) in the irrep.
Part 1 If the C.G. coefficient of $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ is in $\sqrt{Q}$, then the C.G. coefficient of any vector totally less than $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ is in $\sqrt{Q}$.

Proof of part 1: Let $k,l$ and $m$ be such that $k,l,m \geq N$. Denote the coefficient of $\alpha_1$ in $\Psi_{k,l,m}$ by $\alpha_{1}^{k,l,m}$. Then by Eq (2.3) $\alpha_{1}^{k,l,m} = P_0^0 Q_l^0 R_m (\frac{k+l+m-N}{m-N})^{(k+l)}$ and $\alpha_{1}^{N,N,N} = P_N^0 Q_N^0 R_N (\frac{k+l+m-N}{m-N})^{(k+l)}$. So $\alpha_{1}^{k,l,m} = P_k^N Q_l^N R_m (\frac{k+l+m-N}{m-N})^{(k+l)} \alpha_{1}^{N,N,N}$. A similar argument gives the general case:

$$\alpha_{i}^{k,l,m} = P_k^N Q_l^N R_m B_i \alpha_{i}^{N,N,N}$$

where $B_i$ is a quotient of the appropriate binomial coefficients. This gives:

$$\Psi_{k,l,m} = \sum_i \alpha_{i}^{k,l,m} \alpha_i = P_k^N Q_l^N R_m \sum_i B_i \alpha_{i}^{N,N,N} \alpha_i$$

(2.4)

By assumption, $\sum_i \alpha_{i}^{N,N,N} \alpha_i$ is RSR and therefore so is the final expression in equation (2.4).

Part 2 If the C.G. coefficient of $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ and all its partial sums are RSR, then the C.G. coefficient of any vector totally greater than $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ is RSR.

Proof of part 2: Using an argument similar to that in the proof of part (1), it is evident that the desired C.G. coefficient is actually a partial sum of the C.G. coefficient for $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ divided by the appropriate product of square roots. By assumption, all such partial sums are also RSR. Note: The C.G. coefficient examined here is not a true partial sum of the $\alpha_i$ as they appear in $\Psi_{N,N,N}$ in the sense of definition 2.3.4. This is because it does not contain the same binomial coefficients. However, changes to the binomial coefficients do not affect the RSR property for a partial sum.

Part 3 If the C.G. coefficient of $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ and all its partial sums are RSR, then every C.G. coefficient is RSR.

Proof of part 3: The vector $|\frac{p-2k}{2}|\frac{q-2l}{2}|\frac{r-2m}{2}\rangle$ is totally less than some vector which is totally greater than $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$. Combining the proofs of (a) and (b) gives the required result.

$\blacksquare$
2.4.3 What is the C.G. coefficient of $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$?

Now using the theory and notation of section 2.4.1 it has been established that the coefficient of $\alpha_{a,b}$ in $\Psi_{N,N,N}$ is $B_{a,b}^{N,N,N}P_{N}^{a}Q_{N}^{b}R_{N}^{N-a-b}$ where the latter three factors are the products of the lowering coefficients that occur when lowering from $|\frac{p-2a}{2}|\frac{q-2b}{2}|\frac{r-2(N-a-b)}{2}\rangle$ to $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$ and $B_{a,b}^{N,N,N}(:=B_{a,b}) = \binom{2N}{N-a}\binom{2N}{N-b}$ is the number of distinct ways in which the lowering can be performed.

**Proposition 2.4.4** The C.G. coefficient of $|\frac{p-2N}{2}|\frac{q-2N}{2}|\frac{r-2N}{2}\rangle$, denoted $\Psi_{N,N,N}$, can be written explicitly as follows:

$$\Psi_{N,N,N} = B_{0,0}P_{N}^{0}Q_{N}^{0}R_{N}^{0}\alpha_{0,0} + B_{0,1}P_{N}^{0}Q_{N}^{1}R_{N}^{1}\alpha_{0,1} + \cdots + B_{0,N}P_{N}^{0}R_{N}^{N}\alpha_{0,N} +$$

$$B_{1,0}P_{N}^{1}Q_{N}^{0}R_{N}^{1}\alpha_{1,1} + \cdots + B_{N,0}Q_{N}^{N}R_{N}^{0}\alpha_{N,0}$$

So it is evident that choosing $\alpha_{a,b}$ to be $P_{a}^{0}Q_{b}^{0}R_{N-a-b}^{0}S$, where $S$ is any rational number, results in all summands having the same rational square root factors; this ensures that this C.G. coefficient is RSR and that all its partial sums are also RSR. So lemma 2.4.2 gives that all C.G. coefficients are RSR. The results established in the next section are analogous but are concerned with conditions on highest weight spaces that ensure that particular highest weight vectors are RSR-generating.

2.5 More on determining RSR-generating highest weight vectors

2.5.1 Examining the highest weight vectors

The fact that there is a single C.G. coefficient that determines the RSR property is perhaps less surprising in the light of some equivalent conditions that ensure a highest weight vector is RSR-generating. Such conditions will be established in this section. The hypothesis that $p, q$ and $r$ are sufficiently large ($\geq n$) for all necessary highest weight vectors to exist will continue to apply.

Consider a highest weight vector $(\alpha_{1}, \ldots \alpha_{n}, \alpha_{n+1}, \ldots \alpha_{n(n+1)})$ in an $n$-dimensional highest weight space, where the $\alpha_{i}$ correspond to the tensor product basis with the lexicographic
ordering (given in section 2.2). Then the basic $\alpha_1, \ldots, \alpha_n$ determine a highest weight vector and each $\alpha_k$ ($k = n + 1, \ldots, \frac{n(n+1)}{2}$) can be written as a linear combination of the basic $\alpha_i$ with $i = 1, \ldots, n$.

**Definition 2.5.1** Fix a weight space. For $1 \leq k \leq \frac{n(n+1)}{2}$, $\alpha_k$ is said to be of level $a$ if $\alpha_k$ is a (non-trivial) linear combination of exactly $a+1$ of the basic highest weight C.G. coefficients $\alpha_1, \ldots, \alpha_n$.

The concept of level is well defined as the $\alpha_1, \ldots, \alpha_n$ are fixed and the linear combinations uniquely determined. Table 2.2 records the levels of the C.G. coefficients of a highest weight vector $\frac{p+q+r-4}{2}$.

<table>
<thead>
<tr>
<th>Element of tensor product basis</th>
<th>C.G. coefficient</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vert \frac{1}{2} \rangle \vert \frac{3}{2} \rangle \vert \frac{p-2}{2} \rangle$</td>
<td>$\alpha_1$</td>
<td>0</td>
</tr>
<tr>
<td>$\vert \frac{p}{2} \rangle \vert \frac{q-2}{2} \rangle \vert \frac{r-2}{2} \rangle$</td>
<td>$\alpha_2$</td>
<td>0</td>
</tr>
<tr>
<td>$\vert \frac{p}{2} \rangle \vert \frac{q-4}{2} \rangle \vert \frac{r}{2} \rangle$</td>
<td>$\alpha_3$</td>
<td>0</td>
</tr>
<tr>
<td>$\vert \frac{p-2}{2} \rangle \vert \frac{q}{2} \rangle \vert \frac{r-2}{2} \rangle$</td>
<td>$\alpha_4 = \frac{1}{\sqrt{p}}(\sqrt{2r} - 2\alpha_1 + \sqrt{q}\alpha_2)$</td>
<td>1</td>
</tr>
<tr>
<td>$\vert \frac{p-2}{2} \rangle \vert \frac{q-2}{2} \rangle \vert \frac{r}{2} \rangle$</td>
<td>$\alpha_5 = \frac{1}{\sqrt{q}}(\sqrt{p}\alpha_2 + \sqrt{2q} - 2\alpha_3)$</td>
<td>1</td>
</tr>
<tr>
<td>$\vert \frac{p-4}{2} \rangle \vert \frac{q-2}{2} \rangle \vert \frac{r}{2} \rangle$</td>
<td>$\alpha_6 = \frac{1}{\sqrt{p}\sqrt{q}}(2\sqrt{r}\sqrt{2p} - 2\alpha_1 + 2\sqrt{q}\sqrt{2q} - 2\alpha_3)$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2.2: A table of the levels of the C.G. coefficients of highest weight $\frac{p+q+r-4}{2}$.

The next step is to examine the properties of the coefficients in given levels. First consider the coefficient of $\vert \frac{p-2k}{2} \rangle \vert \frac{q-2l}{2} \rangle \vert \frac{r-2(n-k-l-1)}{2} \rangle$. It is equal to a linear combination of the coefficients of $\vert \frac{p-2(k-1)}{2} \rangle \vert \frac{q-2(l+1)}{2} \rangle \vert \frac{r-2(n-k-l)}{2} \rangle$ and $\vert \frac{p-2(k-1)}{2} \rangle \vert \frac{q-2l}{2} \rangle \vert \frac{r-2(n-k-l-1)}{2} \rangle$. This is because these are the only three vectors that raise to $\vert \frac{p-2(k-1)}{2} \rangle \vert \frac{q-2l}{2} \rangle \vert \frac{r-2(n-k-l-1)}{2} \rangle$. So when constructing the highest weight space, a linear combination of these three coefficients is set to zero.

So it is clear that the level 1 coefficients are $\alpha_{n+1+k}$ with $k = 0, \ldots, n - 2$. These are precisely the coefficients attached to the tensor product basis elements $\vert \frac{p-2}{2} \rangle \vert \frac{q-2k}{2} \rangle \vert \frac{r-2(n-k)}{2} \rangle$. Moreover $\alpha_{n+1+k}$ is a linear combination of $\alpha_{k+1}$ and $\alpha_{k+2}$ for $k = 0, \ldots n - 2$. Using similar reasoning it is evident that in general, the level $a$ coefficients are precisely those attached to the tensor product basis elements of the form $\vert \frac{p-2a}{2} \rangle \vert \frac{q-2k}{2} \rangle \vert \frac{r-2(n-a-k-1)}{2} \rangle$ for
\( k = 0, 1, 2, \ldots, n - a - 1 \). By a simple induction, it is easy to establish that the coefficient attached to this particular tensor product basis element is a (non-trivial) linear combination of \( \alpha_{1+k}, \alpha_{2+k}, \ldots, \alpha_{1+k+a} \). This reasoning establishes the following two results.

**Lemma 2.5.2** There are exactly \( n - a \) coefficients of level \( a \) for \( a = 0, 1, \ldots, n - 1 \).

**Lemma 2.5.3** Fix \( a \in \{0, 1, \ldots, n - 1\} \). For each \( z = 1, \ldots, n - a \), there is exactly one level a coefficient that is a non-trivial linear combination of \( \alpha_z, \alpha_{z+1}, \ldots, \alpha_{z+a} \) with \( z = 1, \ldots, n - a \).

**Proposition 2.5.4** Consider the \( n \)-dimensional highest weight space inside the \( \frac{n(n+1)}{2} \)-dimensional weight space of weight \( \frac{p+q+r-2(n-1)}{2} \). Let \( \alpha_i \) be attached to \( |\frac{p}{2}|^{q-2(i-1)}|\frac{r-2(n-i)}{2} \rangle \) and let \( \alpha_{a,b} \) be attached to \( |\frac{p-2a}{2}|^{q-2b} |\frac{r-2(n-a-b-1)}{2} \rangle \). Then

\[
\alpha_{a,b} = \sum_{i=1}^{n} d_i \alpha_i \quad \text{where } d_i = \begin{cases} 0 & \text{if } i < b + 1 \text{ or } a + b + 1 < i \\
\frac{(-1)^a}{P_a} \binom{a}{b-i+1} Q_{r-1}^{\alpha-a-b-1} & \text{otherwise}
\end{cases}
\]

and \( \binom{a}{b-i+1} \) is the number of distinct ways to lower \( |\frac{q-2(i-1)}{2}|^{r-2(n-i)} \) to \( |\frac{q-2b}{2}|^{r-2(n-a-b-1)} \).

**Proof:** The result will be proved by induction on the levels. The coefficient \( \alpha_{a,b} \) is attached to \( |\frac{p-2a}{2}|^{q-2b} |\frac{r-2(n-a-b-1)}{2} \rangle \) and so is on level \( a - 1 \). The highest weight equations give rise to equations of the following form:

\[
\sqrt{a(p - a + 1)} \alpha_{a,b} + \sqrt{(b + 1)(q - b)} \alpha_{a-1,b+1} + \sqrt{(n - a - b)(r - n + a + b + 1)} \alpha_{a-1,b} = 0
\]

And \( \alpha_{a-1,b+1} \) and \( \alpha_{a-1,b} \) are both on level \( a - 2 \) and known by inductive hypothesis.

**2.5.2 On levels determining the RSR-generating property for highest weight vectors**

Proposition 2.5.4 gives the C.G. coefficients in the highest weight space in terms of the basic C.G. coefficients. Using this result and the discussion on levels in the previous section it is now possible to examine how the coefficients on a given level determine whether a vector is RSR-generating.
Lemma 2.5.5  Consider an irreducible subspace of $P \otimes Q \otimes R$ with highest weight $\frac{p+q+r-2(n-1)}{2}$. Let the C.G. coefficient of $|\frac{p-2(n-1)}{2}\rangle |\frac{q}{2}\rangle |\frac{r}{2}\rangle$ be a rational square root (i.e. the unique level $n-1$ C.G. coefficient). Suppose also that every partial sum of this general C.G. coefficient (with respect to the basic C.G. coefficients) is also a rational square root. Then all the C.G. coefficients for the irrep are rational square roots.

The proof of this result is entirely analogous to Lemma 2.4.2. Lemma 2.4.2 considered a weight space and using the $su(2)$ actions determined a single C.G. coefficient that indicated whether all C.G. coefficients were RSR. The proof of lemma 2.5.5 begins by solving for the highest weight subspace of the weight space of weight $\frac{p+q+r-2(n-1)}{2}$ (the results of which are summarised in proposition 2.5.4) and then using the $su(2)$ action in exactly the same way to determine a similar single C.G. coefficient that indicates whether all C.G. coefficients are RSR. Alternatively, lemma 2.5.5 may be proved as a special case of the following more general result. Suppose $\alpha_1, \ldots, \alpha_n$ determined a highest weight space and hence an irrep. In the irrep, suppose also that the C.G. coefficient of some tensor product basis element $v$, say $\alpha_v$, is a linear combination of $\alpha_1, \ldots, \alpha_n$ in which each $\alpha_i$ with $i = 1, \ldots, n$ has a non-zero coefficient. Then $\alpha_v$ (and its partial sums) determine the RSR-property for the whole irrep. Once again, the general proof follows exactly the proof of lemma 2.4.2.

Corollary 2.5.6  If the expression (and its partial sums) given in equation (2.5) is RSR, then so are all the C.G. coefficients of the corresponding irrep in $P \otimes Q \otimes R$ of highest weight $\frac{p+q+r-2(n-1)}{2}$.

\[
\left(\frac{-1}{P_{n-1}^0}\right)^{n-1}R_{n-1}^0\alpha_1 + \binom{n-1}{1}R_{n-2}^0Q_1\alpha_2 + \binom{n-1}{2}R_{n-3}^0Q_2\alpha_3 + \cdots + Q_{n-1}^0\alpha_n \tag{2.5}
\]

Proof: Equation (2.5) is the general C.G. coefficient attached to $|\frac{p-2(n-1)}{2}\rangle |\frac{q}{2}\rangle |\frac{r}{2}\rangle$ (i.e., the unique level $n-1$ C.G. coefficient). The result now follows from lemma 2.5.5.

So choosing $\alpha_i = R_{n-1}^{i-1} Q_{n-1}^{i-1} S_i$ for $i = 2, \ldots, n-1$ and choosing $\alpha_1 = Q_{n-1}^0 S_1$ and $\alpha_n = R_{n-1}^0 S_n$ where $S_j$ is any rational number, gives that equation (2.5) and all its partial sums are RSR and so by Lemma 2.5.5 such a highest weight vector is RSR-generating. To use this result as a general method to prove that a highest weight vector is RSR-generating
requires many calculations. The next result, and in particular its corollary, will be of more practical use in determining the RSR-generating property for highest weight vectors and is concerned with examining the C.G. coefficients on other levels.

**Proposition 2.5.7** Fix $a \in \{1, \ldots, n-1\}$. Suppose that all level $a$ C.G. coefficients are RSR and all partial sums in each level $a$ coefficient are RSR. Then all C.G. coefficients are RSR.

**Proof:** The proof will proceed by induction. It will establish that if the conditions of the proposition hold at level $a$ then they hold at level $a+1$. Iterating this result gives that if the conditions hold at any non-zero level, then they hold at level $n-1$. The result then follows by Lemma 2.5.5.

Suppose $\alpha_1, \ldots, \alpha_n$ are chosen so that all level $a$ coefficients are RSR and all partial sums of the level $a$ coefficients are also RSR. Then the level $a+1$ coefficients are linear combinations of the level $a$ coefficients (up to a factor of $\sqrt{(a+1)(p-a)}$, but this does not affect the RSR property as it occurs everywhere and so will be ignored in this proof hereafter). Now suppose $\Psi^a_i$ and $\Psi^a_{i+1}$ are consecutive C.G. coefficients on level $a$. That is, $\Psi^a_j$ is a non-trivial sum of $\alpha_k$ where $k = j, \ldots, j+a$ for $j = i, i+1$. Recall that such $\Psi^a_j$ for $j = 1, \ldots, n-a$ are all the level $a$ coefficients. Now each level $a+1$ coefficient is a linear combination of the form $\Psi^{a+1}_i = \gamma \Psi^a_i + \delta \Psi^a_{i+1}$ for some lowering coefficients $\gamma$ and $\delta$. Now consider the coefficient of $\alpha_k$ in $\Psi^{a+1}_i$; denoted by $\alpha^{i,a+1}_k$. Then $\alpha^{i,a+1}_k = \gamma \alpha^{i,a}_k + \delta \alpha^{i+1,a}_k$ where $\alpha^{j,a}_k$ is the coefficient of $\alpha_k$ in $\Psi^a_j$ for $j = i, i+1$. But $\gamma \alpha^{i,a}_k$ and $\delta \alpha^{i+1,a}_k$ contain exactly the same lowering coefficients and so differ only by a binomial factor. Moreover, for $j = i, i+1$, $\alpha^{j,a}_k \alpha_k$ and $\alpha^{j,a}_{k+1} \alpha_{k+1}$ share the same irrational factors by hypothesis. Therefore $\left(\gamma \alpha^{i,a}_k + \delta \alpha^{i+1,a}_k\right) \alpha_k + \left(\gamma \alpha^{i,a}_{k+1} + \delta \alpha^{i+1,a}_{k+1}\right) \alpha_{k+1}$ is RSR because each of the summands contains the same irrational factors. Thus all partial sums of two consecutive elements of the level $a+1$ coefficients are RSR, and by transitivity of addition of the RSR property, the required result is established.

A very useful corollary follows immediately from proposition 2.5.7. This corollary will be the most practical way to check whether all C.G. coefficients are RSR.

**Corollary 2.5.8** All C.G. coefficients are RSR if all level 1 and level 0 coefficients are RSR.

**Proof:** Proposition 2.5.7 gives that if all level 1 C.G. coefficients are RSR and the partial sums of all level 1 C.G. coefficients are RSR, then all C.G. coefficients are RSR. But the
partial sums of the level 1 C.G. coefficients are just level 0 coefficients multiplied by the square root of a rational number.

Proposition 2.5.4 may be used to give explicitly the C.G. coefficients on a specified level and then proposition 2.5.7 details how to use these C.G. coefficients to establish conditions for RSR-generating highest weight vectors. In general, it is of course easiest to use the level 1 and level 0 C.G. coefficients. In the next section the RSR-generating highest weight vectors are classified from exactly these conditions.

2.5.3 Classifying the RSR-generating highest weight vectors

To find RSR-generating highest weight vectors, corollary 2.5.8 states that it suffices to consider the level 0 and level 1 C.G. coefficients. The level 1 coefficients are precisely:

\[
\frac{-1}{\sqrt{p}}(R_{n-i}\alpha_i + Q_i\alpha_{i+1}) \quad \text{for } i = 1, \ldots, n-1
\]  

(2.6)

So for a highest weight vector to be RSR-generating it is necessary and sufficient that the basic coefficients are RSR (i.e. the level 0 coefficients) and that \(\alpha_i\) differs from \(\alpha_{i+1}\) by a factor of \(\frac{Q_i}{R_{n-i}}\) (up to rational multiples). As usual let \(\alpha_1, \alpha_2, \ldots, \alpha_n\) be the basic C.G. coefficients. It will be convenient to rewrite these \(\alpha_i\) in terms of more convenient \(\beta_i\).

\[
\alpha_1 = \frac{\sqrt{p}\sqrt{2p-2} \ldots \sqrt{(n-1)p-(n-1)(n-2)}}{\sqrt{q}\sqrt{2q-2} \ldots \sqrt{(n-1)q-(n-1)(n-2)}} \beta_1 = \frac{P_{n-1}^0}{R_{n-1}^0} \beta_1
\]

\[
\alpha_2 = \frac{\sqrt{p}\sqrt{2p-2} \ldots \sqrt{(n-1)p-(n-1)(n-2)}}{\sqrt{q}\sqrt{2q-2} \ldots \sqrt{(n-2)q-(n-2)(n-3)}} \beta_2 = \frac{P_{n-2}^0}{R_{n-2}^0} \beta_2
\]

\[
\vdots
\]

\[
\alpha_n = \frac{\sqrt{p}\sqrt{2p-2} \ldots \sqrt{(n-1)p-(n-1)(n-2)}}{\sqrt{q}\sqrt{2q-2} \ldots \sqrt{(n-1)q-(n-1)(n-2)}} \beta_n = \frac{P_{n-1}^0}{Q_{n-1}^0} \beta_n
\]  

(2.7)

If \(\beta_i \in \mathbb{Q}\), then the \(\alpha_i\) determine a RSR-generating highest weight vector. This is most easily seen by checking the conditions in Eq (2.6). Moreover these conditions suffice to ensure that these are all possible RSR-generating highest weight vectors up to multiplication by a rational square root.

**Proposition 2.5.9** Equation (2.7) describes all possible RSR-generating highest weight vectors.
Proof: Now $\alpha_1$ is a C.G. coefficient, so choose $\alpha_1 = \mu \in \sqrt{Q}$. Now $\sqrt{(n-1)(r-n+2)}\alpha_1 + \sqrt{q}\alpha_2$ is also a C.G. coefficient. This coefficient being in $\sqrt{Q}$ ensures

$$\alpha_2 = \frac{\sqrt{(n-1)(r-n+2)}}{\sqrt{q}} \mu b_1 \quad \text{for some } b_1 \in Q$$

Continuing to use the level 1 conditions, produces the $n$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ which can be put into the required form above by multiplying the entire vector by the scalar $\frac{P_0^0}{R_{n-1}^0}$.

2.6 Constructing orthogonal RSR-generating highest weight vectors

2.6.1 Further properties of C.G. coefficients on a given level

This section introduces a method for constructing orthogonal RSR-generating highest weight vectors in a highest weight space of any dimension. It begins with a few lemmas that will be needed to prove that the construction does indeed give highest weight vectors with the required properties.

Lemma 2.6.1 Setting all level $a$ coefficients to zero gives rise to $n - a$ linearly independent equations in the basic C.G. coefficients.

Proof: Let $\alpha_k, \ldots, \alpha_{k+n-a-1}$ be the (ordered) C.G. coefficients on level $a$. Then lemma 2.5.3 gives that $\alpha_{k+j-1}$ is a (non-zero) linear combination of $\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+a}$. So consider

$$0 = a_k \alpha_k + a_{k+1} \alpha_{k+1} + \cdots + a_{k+n-a-1} \alpha_{k+n-a-1}$$

Then $a_k = 0$ because $\alpha_k$ contains the only instance of $\alpha_1$. But now $a_{k+1} = 0$ because the only occurrences of $\alpha_2$ are in $\alpha_k$ and $\alpha_{k+1}$. And continuing to argue iteratively establishes that $a_j = 0$ for all $j = k, \ldots, k + n - a - 1$.

Lemma 2.6.2 For a particular irrep, if all the C.G. coefficients of level $a$ are zero, then all the coefficients on levels longer than $a$ are zero. (i.e. all coefficients on level $b$ with $b \geq a$ are zero.)

20
Proof: This is because the coefficients on a given level are linear combinations of the coefficients on the previous shorter level. Arguing inductively gives the result.

Lemma 2.6.3 The highest weight vectors for which all the level $a$ coefficients are 0 form a subspace of the highest weight space of dimension $a$.

Proof: It is clearly a subspace as the zero coefficients remain zero under scalar multiplication and addition. The condition that the level $a$ coefficients are all zero gives $n - a$ independent homogeneous equations in $n$ variables. So gives rise to an $a$ dimensional subspace of the $n$ dimensional highest weight space.

2.6.2 A method for constructing orthogonal RSR-generating highest weight vectors.

The method

Step 1: Choose the first highest weight vector to be the vector such that all the level 1 coefficients are zero. By lemma 2.6.3 this determines a highest weight vector (up to scalar multiplication).

Step 2: Now consider the space spanned by the highest weight vectors where all the level 2 coefficients are zero. By lemma 2.6.3, this is a 2-dimensional space that contains the first vector. So choosing a second vector that is in this space and orthogonal to the first, again determines a highest weight vector (up to scalar multiplication).

Step k: Consider the space spanned by the highest weight vectors for which all the level $k$ coefficients are zero. Again by lemma 2.6.3, this is a $k$-dimensional vector space that contains the $k - 1$ orthogonal vectors that have already been constructed inductively. So it is possible to choose the $k^{th}$ vector to be in this space and orthogonal to the first $k - 1$ vectors.

Step n: At the final step $(n-1)$ orthogonal vectors have been constructed in an $n$-dimensional highest weight space. So it is possible to choose the final vector to be orthogonal to the initial $(n - 1)$ vectors.
This construction clearly gives a set of (well-defined up to scalar multiplication) orthogonal highest weight vectors that span the entire highest weight space. It turns out that these vectors are also RSR-generating highest weight vectors.

**Proposition 2.6.4** The construction described on page 21 gives rise to orthogonal RSR-generating highest weight vectors that span the highest weight space.

**Proof:** Orthogonality is given by construction, and spanning is a result of having the same number of orthogonal vectors as the dimension of the space containing them. Consider the highest weight vectors given by \((\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n(n+1)})\) inside an \(n\) dimensional highest weight space. Now set all level 1 coefficients to zero: i.e. \(\alpha_k = 0\) for \(k \geq n + 1\). Consider the equations in (2.6.2) that are satisfied by the coefficients of every highest weight vectors:

\[
\sqrt{p} \alpha_{n+1} + R_{n-1} \alpha_t + Q_{t} \alpha_{t+1} = 0 \quad \text{for} \ t = 1, \ldots, n-1.
\]

Since \(\alpha_{n+1} = 0\) for \(t = 1, \ldots, \frac{n(n+1)}{2}\) the equations (2.6.2) become \(\alpha_t = \frac{Q_t}{R_{n-1}} \alpha_{t+1}\) and in terms of the \(\beta_i\) this gives \(\beta_t = -\beta_{t+1}\). This determines a 1-dimensional space and hence (up to scalar multiplication) a highest weight vector. It is easy to see that this is an RSR-generating vector given the complete description of RSR vectors in proposition 2.5.9. In the notation of that section, this first vector given by the method on page 21 may be written as \(\vec{\beta} = (1, -1, 1, \ldots, (-1)^{n+1})\).

Now assume inductively that this method has given us \(l\) orthogonal RSR-generating vectors with each \(\beta^i_j \in \mathbb{Q}\), where \(\beta^i_j\) is the \(j^{th}\) \(\beta\)-entry in the \(i^{th}\) vector.

Now set all level \(l + 1\) coefficients to zero. Each coefficient of level \(l + 1\) gives rise to a rational linear combination of the \(\beta_i\) equalling zero. We have \(n - (l + 1)\) such equations and they are all linearly independent by lemma 2.6.1. So we have an \(l + 1\) dimensional space which contains \(l\) other orthogonal (and hence linearly independent) vectors.

It is now possible to choose the \((l + 1)^{th}\) vector by insisting it is orthogonal to the other \(l\). Let \(v_1, \ldots, v_l\) be the RSR-generating vectors that have already been constructed and let \(v_{l+1}(\beta_1, \ldots, \beta_n)\) be the vector with level \(l+1\) coefficients zero. Now each \(v_i\) is RSR-generating, so it is of the form \((\frac{P_0}{Q_{n-1}}, \frac{P_0}{Q_{n-2}}, \beta_1, \ldots, \frac{P_0}{Q_{n-1}}, \beta_n)\) with the \(\beta^i_j \in \mathbb{Q}\). Take \(v_{l+1}(\beta_1, \ldots, \beta_n)\) to be of the form \((\frac{P_0}{Q_{n-1}}, \beta_1^{l+1}, \frac{P_0}{Q_{n-2}}, \beta_2^{l+1}, \ldots, \frac{P_0}{Q_{n-1}}, \beta_n^{l+1})\). Then considering the \(\beta^{l+1}_j\) as unknowns, \(v_i \cdot v_{l+1}(\beta_1^{l+1}, \ldots, \beta_n^{l+1}) = 0\) for \(i = 1, \ldots, l\) is a system of \(l\) homogenous linear equations in the
...\(\beta_{i}^{j+1}\)'s with rational coefficients. Furthermore each of the \(\beta_{j}^{l+1}\) with \(j > l + 1\) can be written as a rational linear combination of \(\beta_{i}^{l+1}\) with \(i = 1, \ldots, l + 1\), so the system of \(l\) homogenous equations can be written in \(l + 1\) unknowns with rational coefficients. So there is a rational solution.

\[\square\]

**Corollary 2.6.5** The method described on page 21 allows for the construction of orthogonal RSR-generating highest weight vectors of unit length, that span the highest weight space.

**Proof:** Proposition 2.6.4 establishes much of this result; all that remains is to make the vectors unit in length. The method leaves a choice for the magnitude of the vectors, as the method determines the vector only up to scalar multiplication. Suppose the method on 21 has been applied to produce RSR highest weight vectors of arbitrary length. Now the magnitude of a vector is the square root of the sum of the squares of the entries. So an RSR-generating vector always has magnitude in \(\sqrt{Q}\). Therefore it is possible to unitise the RSR vectors given by the method on page 21 and preserve the RSR-generating property and orthogonality.

\[\square\]

### 2.7 Some results

Applying the algorithm on page 21 to the (non-degenerate) 2-dimensional highest weight space gives the following highest weight vectors.

\[
v_1 = \begin{pmatrix} \sqrt{q} \\ -\sqrt{r} \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \sqrt{r} \\ \sqrt{q} \\ -\frac{(r+q)}{\sqrt{p}} \end{pmatrix}
\]

Applying the algorithm to the (non-degenerate) 3-dimensional highest weight space gives the following highest weight vectors.
\[ v_1 = \begin{pmatrix} \sqrt{2q - 2} \sqrt{q} \\ -\sqrt{2q - 2} \sqrt{r} \\ \sqrt{2r - 2} \sqrt{q} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\sqrt{q} \sqrt{2r - 2} \\ r - q \\ \sqrt{2q - 2} \sqrt{r} \\ \frac{\sqrt{q}}{\sqrt{p}} (r + q - 2) \\ -\frac{\sqrt{r}}{\sqrt{p}} (r + q - 2) \\ 0 \end{pmatrix} \]

\[ v_3 = \begin{pmatrix} \sqrt{2r - 2} \sqrt{r} \\ 2\sqrt{r} \sqrt{q} \\ -\sqrt{q} \sqrt{2r} \\ -\sqrt{q} 2(q + r - 1) \\ -\sqrt{r} 2(q + r - 1) \\ \frac{1}{\sqrt{p} \sqrt{2p - 2}} 2(q + r)(q + r - 1) \end{pmatrix} \]

### 2.8 The degenerate cases

#### 2.8.1 Examining the general degenerate case

Recall that \( p \geq q, r \). Consider the level zero C.G. coefficients attached to the tensor product basis elements \( |\frac{p}{2}|^{q-2(i-1)} |\frac{r}{2}|^{r-2(n-i)} \) for \( i = 1, \ldots n \). Previously, it was assumed that all such vectors exist and so, in such a case, the highest weight space is \( n \)-dimensional. It will be convenient to continue to assign the C.G. coefficients (denoted \( \alpha \)) to the same tensor product basis elements as in the non-degenerate case; but write of a C.G. coefficient not existing, if the corresponding tensor product basis element does not exist.

Now if \( q \) is small, some of the vectors \( |\frac{q}{2}|^{q-2(i-1)} |\frac{r}{2}|^{r-2(n-i)} \) might not exist. So there is some \( j \) such the tensor product basis element attached to \( \alpha_i \) for \( i = 1, 2, \ldots, j - 1 \) does not exist. Similarly for small \( r \), some \( k \) is found such that the tensor product basis element attached to \( \alpha_i = j + k + 1, j + k + 2, \ldots, n \) does not exist.

So the basic C.G. coefficients that exist are \( \alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+k-1}, \alpha_{j+k} \). Importantly, note that all the level 0 C.G. coefficients that exist are consecutive. It is the case that the level 1 C.G. coefficients that depend on two of these existing level 0 coefficients, are still level 1 C.G. coefficients (i.e. are a non-trivial linear combination of two existing level 0 coefficients). This is because the coefficient attached to \( |\frac{p}{2}|^{p-2} |\frac{q}{2}|^{q-2} |\frac{r}{2}|^{r-2(n-l-2)} \) raises to form an equation with
the level 0 coefficients attached to $|\frac{p}{2}\rangle|\frac{q-2(l+1)}{2}\rangle|\frac{r-2(n-m-l-1)}{2}\rangle$ and $|\frac{p}{2}\rangle|\frac{q-2l}{2}\rangle|\frac{r-2(n-l-1)}{2}\rangle$. And because $p \geq q, r$ the first of these tensor product basis elements exists if the second and third exist. There are three possibilities for the other level 1 coefficients: they are either attached to tensor product basis elements that don’t exist or are $\alpha_{n+j-1}$ which is a multiple of $\alpha_j$ or are $\alpha_{n+j+k}$ which is a multiple of $\alpha_{j+k}$. So this gives $k$ level 1 coefficients that are still a linear combination of two existing level 0 coefficients. It is possible to continue this reasoning which at the general stage considers the relation satisfied by the following three tensor product basis elements.

$$|\frac{p-2m}{2}\rangle|\frac{q-2l}{2}\rangle|\frac{r-2(n-m-l-1)}{2}\rangle,$$

$$|\frac{p-2(m-1)}{2}\rangle|\frac{q-2(l+1)}{2}\rangle|\frac{r-2(n-m-l-1)}{2}\rangle,$$

$$|\frac{p-2(m-1)}{2}\rangle|\frac{q-2l}{2}\rangle|\frac{r-2(n-m-l)}{2}\rangle.$$

The existence of the latter two ensures the existence of the first. In particular if the C.G. coefficients attached to the latter two are level $m - 1$ then the C.G. coefficient of the first is of level $m$.

Importantly, this discussion shows by construction that the level 0 coefficients that exist are still independent and that they determine the highest weight space. Note that it has also been shown that the dimension of the highest weight space is given by the number of tensor product basis elements of the form $|\frac{p}{2}\rangle|\rangle|\rangle$ (i.e., this proves lemma 2.2.3 on page 7). This is because this is exactly the number of level 0 coefficients. Note that this result is only true under the condition that $p \geq q, r$.

The next definition generalises the concept of level as first introduced in definition 2.5.1

**Definition 2.8.1** A C.G. coefficient is said to be of level $a$ if it is the sum of $a + 1$ basic C.G. coefficients and is attached to a tensor product basis element of the form $|\frac{p}{2}\rangle|\rangle|\rangle$.

**Theorem 2.8.2** In an $n$-dimensional highest weight space, the method given on page 21 gives rise to $n$ orthogonal RSR-generating vectors.

Proof: All the results for the non-degenerate case hold in the degenerate case with this new definition of level and appropriate generalisations to the actual raising and lowering coefficients (which are still square roots of rational numbers). So the proof of this result is exactly the same as the proof of proposition 2.6.4.

$\blacksquare$
Note that applying the algorithm to an n-dimensional degenerate case does not embed nicely into the n-dimensional nondegenerate case. This is because when the dot product is used to find orthogonal vectors, the resulting equations in the degenerate case involve extra terms given from coefficients that are not defined as being in any level. This is illustrated more clearly in the following example.

2.8.2 A degenerate example

This subsection will consider in detail the example from page 15 of irreps with highest weight \( \frac{p+q+r-4}{2} \), although now with the assumption that \( |\frac{r-4}{2}\rangle \) does not exist (i.e. \( r = 0 \) or \( 1 \)). Table 2.3 gives the tensor product elements and the C.G. coefficients that ensure a given vector is a highest weight vector.

<table>
<thead>
<tr>
<th>Element of tensor product basis</th>
<th>C.G. coefficient</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\frac{p}{2}\rangle</td>
<td>\frac{q}{2}\rangle</td>
</tr>
<tr>
<td>(</td>
<td>\frac{p-2}{2}\rangle</td>
<td>\frac{q-2}{2}\rangle</td>
</tr>
<tr>
<td>(</td>
<td>\frac{r-2}{2}\rangle</td>
<td>\frac{q-4}{2}\rangle</td>
</tr>
<tr>
<td>(</td>
<td>\frac{p-2}{2}\rangle</td>
<td>\frac{q}{2}\rangle</td>
</tr>
<tr>
<td>(</td>
<td>\frac{p-4}{2}\rangle</td>
<td>\frac{q-2}{2}\rangle</td>
</tr>
<tr>
<td>(</td>
<td>\frac{p-4}{2}\rangle</td>
<td>\frac{q-2}{2}\rangle</td>
</tr>
</tbody>
</table>

Table 2.3: An example of degenerate C.G. coefficients of highest weight \( \frac{p+q+r-4}{2} \) when \( |\frac{r-4}{2}\rangle \) does not exist.

Note that in the degenerate case it is not sensible to generalise the definition of level a highest weight C.G. coefficients as given in definition 2.5.1 to include all C.G. coefficients that are a (non-trivial) linear combination of \( a + 1 \) basic C.G. coefficients. In such a case, \( \alpha_5 \) and \( \alpha_6 \) in table 2.3 would both be level 1. But setting both \( \alpha_5 \) and \( \alpha_6 \) to zero gives only the zero solution and so construction of orthogonal highest weight vectors would not work with such a definition.

Note also that if \( \alpha_6 \) did not exist because \( |\frac{p-4}{2}\rangle \) did not exist, then \( \alpha_1 \) and \( \alpha_3 \) would not exist either because neither \( |\frac{q-4}{2}\rangle \) nor \( |\frac{r-4}{2}\rangle \) would exist due to the assumption that \( p \geq q, r \). Applying the method on page 21 to the example in table 2.3 gives the following two RSR-generating orthogonal vectors.
$$v_1 = \begin{pmatrix} \sqrt{2q - 2} \\ -\sqrt{r} \\ \frac{\sqrt{2q - 2}}{\sqrt{p}} \\ 0 \\ \frac{\sqrt{2q - 2}}{\sqrt{p}\sqrt{2q - 2}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\sqrt{r}(p + q - 1)(q - p) \\ \sqrt{2q - 2}((p + q)(p - 1) + qr) \\ -\sqrt{\frac{r}{p}}(p + q - 1)(q - p) \\ -\frac{1}{\sqrt{p}}(r(p + q - 1)(p - q) + (2q - 2)((p + q)(p - 1) + qr)) \\ \frac{\sqrt{r}}{\sqrt{2p - 2}}(2pr(p - q) + (2q - 2)((p + q + r)(p - 1) + r)) \end{pmatrix}$$

Note that these vectors have more complicated C.G. coefficients than the results for the general 3-dimensional case on page 23 because much of the symmetry is broken without the existence of $|\frac{r-4}{2}\rangle$ (i.e., without $\alpha_1$).

2.9 Using an arbitrary basis

2.9.1 A brief overview

The results so far have only been concerned with irreps with respect to the unitary basis. However the method outlined on page 21 can be used with other bases. The method will give the correct number of orthogonal highest weight vectors no matter what the raising or lowering coefficients in the irreps are. This will be discussed in more detail in this section. It is worth noting that the RSR-generating condition will hold exactly if all the raising and lowering coefficients are square roots of rational numbers. In particular, the method on page 21 would produce RSR-generating highest weight vectors for two irreps with easy-to-use raising and lowering operators: irreps with raising coefficients all 1, and equally irreps with the lowering coefficients all 1.

2.9.2 An example with an arbitrary basis

Recall that the unitary basis acts as follows on an irrep of highest weight $\frac{p}{2}$:

$$T_+|\frac{p-2}{2}\rangle = \sqrt{p}|\frac{p}{2}\rangle$$
$$T_-|\frac{p}{2}\rangle = \sqrt{p}|\frac{p-2}{2}\rangle$$

Any change of basis must preserve the weight spaces and so the only change of basis that is possible is a rescaling of the vectors. Let $|\frac{p-2}{2}\rangle_{\nu_p} = \nu_p^a|\frac{p-2}{2}\rangle$ where $\nu_p^a$ is some constant
and $|p^2\rangle$ is the usual vector in the unitary irrep. Note that it is possible to rescale all the vectors simultaneously and so, without loss of generality, it is convenient to impose the condition that $|p^2\rangle = |\frac{p}{2}\rangle_{\nu_p}$ (i.e. $\nu_p^0 = 1$). In the following exposition the superscript $a$ will be dropped from $\nu_a^p$ as the only $a$ being considered in the example is $a = 1$.

In this rescaled basis the action of $T_\pm$ is as follows:

$$T_+|\frac{p^2}{2}\rangle_{\nu_p} = \nu_p \sqrt{p}\frac{p}{2}\rangle_{\nu_p}$$
$$T_-|\frac{p^2}{2}\rangle_{\nu_p} = \frac{\sqrt{p}}{\nu_p}|\frac{p^2}{2}\rangle_{\nu_p}$$

The basis may be changed in a similar way for irreps $Q$ and $R$ and the same notation will be adopted as above, with $p$ replaced appropriately by $q$ or $r$.

Recall that if $p, q, r \geq 1$ then $P \otimes Q \otimes R$ contains a non-degenerate highest weight space of weight $\frac{p+q+r-2}{2}$. With an arbitrary basis for $P$, $Q$ and $R$ this weight space is spanned by the following three tensor product basis elements:

$$|\frac{p^2}{2}\rangle_{\nu_p}|\frac{q^2}{2}\rangle_{\nu_q}|\frac{r^2}{2}\rangle_{\nu_r}$$

Consider an arbitrary vector, $v$, in this weight space.

$$v = \alpha_1|\frac{p^2}{2}\rangle_{\nu_p}|\frac{q^2}{2}\rangle_{\nu_q}|\frac{r^2}{2}\rangle_{\nu_r} + \alpha_2|\frac{p^2}{2}\rangle_{\nu_p}|\frac{q^2}{2}\rangle_{\nu_q}|\frac{r^2}{2}\rangle_{\nu_r} + \alpha_3|\frac{p^2}{2}\rangle_{\nu_p}|\frac{q^2}{2}\rangle_{\nu_q}|\frac{r^2}{2}\rangle_{\nu_r}$$

The action of $T_+$ on $v$ is as follows:

$$T_+ v = (\nu_r \sqrt{r}\alpha_1 + \nu_q \sqrt{q}\alpha_2 + \nu_p \sqrt{p}\alpha_3)|\frac{p^2}{2}\rangle_{\nu_p}|\frac{q^2}{2}\rangle_{\nu_q}|\frac{r^2}{2}\rangle_{\nu_r}$$

For $v$ to be a highest weight vector it is required that $T_+ v = 0$. This implies that $\nu_r \sqrt{r}\alpha_1 + \nu_q \sqrt{q}\alpha_2 + \nu_p \sqrt{p}\alpha_3 = 0$. Rearranging this equation gives the following condition on $\alpha_3$ to ensure that the vector $v$ is a highest weight vector:

$$\alpha_3 = \frac{-1}{\nu_p \sqrt{p}}(\nu_r \sqrt{r}\alpha_1 + \nu_q \sqrt{q}\alpha_2)$$

Now following the method on page 21, the first vector in the highest weight space is found by setting the $\alpha_3$ coefficient to zero; this results in $0 = \nu_r \sqrt{r}\alpha_1 + \nu_q \sqrt{q}\alpha_2$. So it is possible to choose the first highest weight vector, $v_1$, to be given by $\alpha_1 = \nu_q \sqrt{q}$, $\alpha_2 = -\nu_r \sqrt{r}$ and $\alpha_3 = 0$. The second vector, $v_2$, is found by choosing it to be orthogonal to the first.
As such the following equation is derived: $\nu q \sqrt{q} \alpha_1 - \nu r \sqrt{r} \alpha_2 = 0$. Solving this equation gives (up to scalar multiplication) $\alpha_1 = \nu r \sqrt{r}$, $\alpha_2 = \nu q \sqrt{q}$ and $\alpha_3 = \frac{1}{\nu q \sqrt{p}} (\nu^2 r + \nu q^2 q)$. So $v_1$ and $v_2$ decompose the highest weight space according to the method established on page 21. Changing the bases back to unitary bases will enable a comparison of this decomposition with respect to arbitrary bases with the results established in section 2.7 for which the calculation and statement of results are with respect to unitary bases.

Let $v'_1 = \frac{1}{\nu q} v_1$, then $v'_1 = (\sqrt{q}, -\sqrt{r}, 0)$ with respect to the unitary basis. This is exactly the first vector given by applying the method to vectors with respect to the unitary basis and summarised in section 2.7. So, in the two dimensional non-degenerate case, applying the method to irreps with respect to any basis results in a well-defined highest weight vector (up to scalar multiplication). Now the second vector is given as follows:

$$v_2 = \nu r \sqrt{r} | \frac{p}{2} \rangle_{\nu r} | \frac{q}{2} \rangle_{\nu q} | \frac{r - 2}{2} \rangle_{\nu r} + \nu q \sqrt{q} | \frac{p}{2} \rangle_{\nu p} | \frac{q - 2}{2} \rangle_{\nu q} | \frac{r}{2} \rangle_{\nu r} + \frac{-1}{\nu \sqrt{p}} (\nu^2 r + \nu q^2 q) | \frac{p - 2}{2} \rangle_{\nu r} | \frac{q}{2} \rangle_{\nu q} | \frac{r}{2} \rangle_{\nu r}$$

Writing the co-ordinates of $v_2$ with respect to the unitary basis gives:

$$v_2 = (\nu r \sqrt{r}, \nu q \sqrt{q}, \frac{-1}{\sqrt{p}} (\nu^2 r + \nu q^2 q))$$

Whilst this has some similarity to the second vector in section 2.7, it is not a scalar multiple. It is of course a highest weight vector and so in the span of the two vectors given by applying the method on page 21 to the unitary basis. It is not surprising that $v_2$ is not a scalar multiple of the second vector in section 2.7 as the method used constructs orthogonal highest weight vectors and unfortunately there are no orthogonal vectors that are basis independent. This is because changing the basis alters the dot product equations.
that ensure orthogonality.

2.9.3 Orthogonality and basis independence

Since it is impossible to construct orthogonal vectors that are basis independent it is necessary to choose which property is more important. Given that explicit calculations are usually done in a specific basis, it is plausible to give priority to the orthogonality property. However it is prudent to consider how close the vectors given by the method on page 21 are to being basis independent. This subsection will establish that the method is as good as possible and will explicitly demonstrate a nested sequence of vector spaces that are basis independent and spanned by the appropriate constructed highest weight vectors.

**Lemma 2.9.1** Choose any bases for \( P, Q \) and \( R \). Then the method described in section 2.6.2 constructs \( n \) orthogonal (well-defined up to scalar multiplication) highest weight vectors.

**Proof:** Since a change of basis involves multiplying vectors in the various irreps by scalars, the concept of levels remains well defined and the necessary properties also remain intact. So that setting all the coefficients of a certain level to zero gives a space of the correct dimension.

Since the method in section 2.6.2 gives orthogonal highest weight vectors, it does not give rise to the same vectors if a different basis is chosen at the beginning. However, there are several vector spaces that are basis invariant.

**Lemma 2.9.2** Choose any bases for \( P, Q \) and \( R \). In a highest weight space of dimension \( n \), setting the coefficients of level \( a \) to zero gives rise to a well defined vector space of dimension \( a \).

**Proof:** Setting the level \( a \) coefficients to zero certainly gives rise to an \( a \) dimensional highest weight space by lemma 2.6.3. It is basis independent as a change of basis does not affect zero coefficients and because the concept of level is basis independent. So setting the level \( a \) coefficients to zero for two sets of irreps with different bases gives rise to two vector spaces of the same dimension each containing the other.

Now finally, using lemma 2.9.2, it is clear that whilst the method cannot establish orthogonality and basis independence, it does establish orthogonality and a nested sequence of vector spaces that are basis independent.
Lemma 2.9.3 Choose any bases for $P$, $Q$ and $R$ and consider a highest weight space of dimension $n$. Then the method outlined in section 2.6.2 establishes $n$ orthogonal highest weight vectors $v_1, v_2, \ldots, v_n$. Further it also establishes a nested sequence of basis independent vector spaces, $V_1 \subset V_2 \subset \ldots V_{n-1} \subset V_n$ such that $v_i \in V_i$ and $V_i$ is of dimension $i$.

Proof: The $V_i$ for $i = 1, \ldots, n - 1$ are precisely the subspaces of the highest weight space given by setting the coefficients of level $i$ to zero. $V_n$ is simply the whole highest weight space. These subspaces are basis independent as explained in lemma 2.9.2 and the nesting and dimensions follow from the theory as outlined in section 2.5.1.

Corollary 2.9.4 Choose any bases for $P, Q$ and $R$. Then the first vector constructed by setting the level 1 coefficients to zero is basis independent (up to scalar multiplication).

Proof: The subspace $V_1$ is basis independent and one dimensional and $v_1 \in V_1$.

2.10 Lowest weight vectors

The method on page 21 decomposed the tensor product of three irreps of $su(2)$ by decomposing the highest weight space. It would make equal sense to apply the method on page 21 to a multi-dimensional lowest weight space. The theory outlined in section 2.5.1 would hold in its entirety with the definition of basic vectors now being taken to mean vectors of the form $|\mathbb{2}^p\rangle|\mathbb{2}^{a}\rangle|\mathbb{2}^{b}\rangle$. Basic coefficients determine the whole lowest weight space and moreover the level $k$ coefficients are those attached to vectors of the form $|\mathbb{2}^{p+2k}\rangle|\mathbb{2}^{a}\rangle|\mathbb{2}^{b}\rangle$. In this section, it will be shown that applying the method on page 21 to a highest weight space of weight $\lambda$ in $P \otimes Q \otimes R$, results in exactly the same irreps as applying the method to a lowest weight space of weight $-\lambda$.

2.10.1 Background results for $P \otimes Q$

Consider the tensor product of two unitary irreps of $su(2); P \otimes Q$. As in section 2.2 let $P$ be of highest weight $\frac{p}{2}$ and let $|\mathbb{2}^{p}\rangle|\mathbb{2}^{a}\rangle$ denote the vector in $P$ of weight $\frac{p}{2}$. Similar notation is adopted for $Q$ and $R$. Let $J_j$ denote the irrep with highest weight $\frac{j}{2}$ in the direct sum.
decomposition of $P \otimes Q$. Denote by $\Psi_{jm}$ the vector of weight $\frac{m}{2}$ in $J_j$. Then $\Psi_{jm}$ can be expressed as a linear combination of appropriate tensor product basis elements as in equation (2.8)

$$\Psi_{jm} = \sum_{a,b} (pa, qb|jm)(\frac{p}{2}, \frac{a}{2})|\frac{q}{2}, \frac{b}{2}\rangle$$

(2.8)

In equation (2.8) $(pa, qb|jm)$ denotes exactly the appropriate C.G. coefficient. In such a case Repka and Rowe [6] give the relation between certain C.G. coefficients described in equation (2.9).

$$\quad (pa, qb|jm) = (-1)^{\frac{p+q-j}{2}}(p - a, q - b|j - m)$$

(2.9)

2.10.2 The lowest weight space in the unitary case

Now consider $P \otimes Q \otimes R$. First consider $P \otimes Q$; which has a direct sum decomposition with particular irreps occurring with multiplicity at most one. Then for each irrep in $P \otimes Q$, tensoring it with $R$ again gives a direct sum decomposition with particular irreps occurring with multiplicity at most one. Note that a highest weight space in $P \otimes Q \otimes R$ can have multiplicity greater than one, but the irreps are now identified because of the order in which the tensor product is taken.

Define by $\Psi_{J_j;JM}$ the vector of weight $\frac{M}{2}$ in the irrep of highest weight $\frac{J}{2}$ in $J_j \otimes R$. Then equation (2.10) is established immediately and can also be found in Repka and Rowe [6].

$$\Psi_{J_j;JM} = \sum_{a,b,c} (pa, qb|jk)(jk, rc|JM)(\frac{p}{2}, \frac{a}{2})|\frac{q}{2}, \frac{b}{2}\rangle|\frac{r}{2}, \frac{c}{2}\rangle$$

(2.10)

**Proposition 2.10.1** Let $J$ be a unitary irreducible representation of highest weight $\frac{J}{2}$ in $P \otimes Q \otimes R$. Then for $J$ the C.G. coefficient of $|\frac{p}{2}, \frac{a}{2}\rangle|\frac{q}{2}, \frac{b}{2}\rangle|\frac{r}{2}, \frac{c}{2}\rangle$ is equal to $(-1)^{\frac{p+q+r-J}{2}}$ times the C.G. coefficient of $|\frac{p}{2}, -\frac{a}{2}\rangle|\frac{q}{2}, -\frac{b}{2}\rangle|\frac{r}{2}, -\frac{c}{2}\rangle$.

**Proof:** Consider the C.G. coefficients of $J_j \otimes R$ with respect to the tensor product basis of $P\otimes Q \otimes R$. By equation (2.10) the CG coefficient of $|\frac{p}{2}, \frac{a}{2}\rangle|\frac{q}{2}, \frac{b}{2}\rangle|\frac{r}{2}, \frac{c}{2}\rangle$ is $(pa, qb|jk)(jk, rc|JM)$. Similarly the C.G. coefficient of $|\frac{p}{2}, -\frac{a}{2}\rangle|\frac{q}{2}, -\frac{b}{2}\rangle|\frac{r}{2}, -\frac{c}{2}\rangle$ is $(p - a, q - b|j - k)(j - k, r - c|J - M)$. Moreover by equation (2.9) it is known that $(pa, qb|jk) = (-1)^{\frac{p+q-j}{2}}(p - a, q - b|j - k)$ and $(jk, rc|JM) = (-1)^{\frac{j+J-M}{2}}(j - k, r - c|J - M)$. Hence $(pa, qb|jk)(jk, rc|JM) =$
This is the required result for these particular irreps. However, any unitary irrep of highest weight \( \frac{1}{2} \) in \( P \otimes Q \otimes R \), can be written as a linear combination of these particular irreps and so the result holds in general with respect to the unitary basis.

**Proposition 2.10.2** Applying the method on page 21 to the lowest weight space gives exactly the same irreps as applying this same method to the highest weight space.

**Proof:** Proposition 2.10.1 establishes this result immediately if \( P, Q \) and \( R \) are unitary irreps; setting the level 1 C.G. coefficients to zero in the highest weight space determines an irrep in which all level 1 C.G. coefficients in the lowest weight space are zero. Thus this irrep is the same irrep that is determined by applying the method to the lowest weight space. Iterating this argument, establishes the result for all irreps in the unitary basis produced by the method on page 21. Note that in a arbitrary basis, proposition 2.10.1 does not hold in its entirety. However, in an arbitrary basis it is still the case the the C.G. coefficient of \( |p_2, a_2 \rangle |q_2, b_2 \rangle |r_2, c_2 \rangle \) is zero if and only if the C.G. coefficient of \( |p_2, -a_2 \rangle |q_2, -b_2 \rangle |r_2, -c_2 \rangle \) is zero. This suffices to prove the proposition holds in an arbitrary basis.

**2.11 Summary**

This chapter has examined, fairly comprehensively, the tensor product of three irreps of \( su(2) \). A method was established on page 21 that decomposed any non-degenerate highest weight space in the tensor product of three unitary irreps of \( su(2) \). The decomposition gave orthogonal highest weight vectors that were RSR-generating. The degenerate case was examined in section 2.8.1 and the method extended easily to this case with an appropriate generalisation of the concept of level in definition 2.8.1. The properties of orthogonality and RSR-generating continued to hold for the highest weight vectors established in the degenerate case. This was followed by a brief discussion on basis independence and the highest weight vectors were viewed as being contained in a full nested sequence of basis independent vector spaces. In particular, this established that the first vector produced by the method on page 21 is basis independent and that the method produces orthogonal irreps when \( P, Q \) and \( R \) are given with respect to any bases (although the RSR-generating property fails to hold). Finally
the method was applied to lowest weight spaces and proved to give the same decomposition as when applied to the corresponding highest weight spaces (for irreps $P, Q$ and $R$ given with respect to any bases).
Chapter 3

Decomposing highest weight spaces in the tensor product of two irreps of $su(3)$

3.1 A brief overview

Having studied $su(2)$ in detail, the tensor product of two irreps of $su(3)$ is now examined. The fact that $su(3)$ contains copies of $su(2)$ is exploited, and a method of decomposition is established that is surprisingly analogous to the method in the last chapter. The irreps of $su(3)$ are more complicated than those of $su(2)$ and this chapter begins by establishing the raising actions need to calculate the highest weight spaces in the tensor product. These actions are then used to determine relations between various C.G. coefficients of importance. When certain of these C.G. coefficients are set to zero they determine basis independent highest weight subspaces of each dimension. In an analogous manner to the previous chapter, this full nested sequence of basis independent highest weight subspaces determines a decomposition in the non-degenerate case. The RSR property is examined and suggested by an example in a particular basis, but no general result is established. The degenerate case is then examined and the method is shown to extend easily; once again giving rise to a well-defined full nested sequence of basis independent highest weight subspaces. The chapter ends by mentioning an equivalent method that uses the action of a different copy of $su(2) \subset su(3)$ to provide a different decomposition of the tensor product.
3.2 The finite dimensional representations of $su(3)$

The Lie algebra $su(3)$ consists of $3 \times 3$ skew-hermitian matrices with vanishing trace. It also contains three important copies of $su(2)$. Denote by $T_3, T_+, T_- \in su(3)$ a basis for one such copy of $su(2)$ satisfying $[T_3, T_\pm] = \pm T_\pm$ and $[T_+, T_-] = 2T_3$. This particular copy of $su(2)$ will be denoted by $su(2)_T$. Similar relations are satisfied by elements $U_*$ and $V_*$ with $* \in \{+,-,3\}$ and these copies of $su(2)$ will be denoted by $su(2)_U$ and $su(2)_V$ respectively. Then these three copies of $su(2)$ interact non-trivially; in particular $[U_\pm, T_\pm] = \mp V_\pm$ and $[U_\pm, T_+] = 0$. Recall that the finite dimensional representations of $su(3)$ can be realised as hexagons (or triangles in the degenerate case) in a triangular weight lattice. The nodes of the lattice are associated with weight spaces $V_\lambda$ in the usual way (see Humphreys [3]). The finite dimensional irreps are defined by their highest weight, which may be written as a linear combination of elements from the basis of fundamental weights $\lambda_1$ and $\lambda_2$. Figure 3.1 is the weight diagram for the irrep of $su(3)$ with highest weight $2\lambda_1 + \lambda_2$, which will often be denoted by $(2,1)$. In general the irrep with highest weight $a_1\lambda_1 + a_2\lambda_2$ will be denoted by $(a_1, a_2)$.

![Weight Diagram](image)

Figure 3.1: The weight diagram of the irrep $(2,1)$.

Recall also that the dimension of the weight spaces increase by one with each hexagonal shell towards the center; beginning with one dimensional weight spaces in the outer shell. The dimension stabilises once a triangular shaped shell is reached. The dimension of a particular weight space will be illustrated by the appropriate number of circles. For example in figure 3.1, the weight spaces in the outer shell are one dimensional and those in the inner (triangular) shell are two dimensional.

The actions of $U_+$ and $T_+$ on the weight lattice are in the directions indicated on figure 3.2. It is possible to decompose the multi-dimensional weight spaces according to the action
Figure 3.2: The vectors $U_+, T_+, \lambda_1$ and $\lambda_2$.

of $su(2)_T$ or $su(2)_U$ or $su(2)_V$. However it is impossible to decompose an arbitrary irrep according to any two of these actions simultaneously. It will be convenient to fix a basis. The work that follows will use a $su(2)_T$-basis and the nodes/vectors of an irrep will be labeled $|b, c, d\rangle$. Here the $b$ denotes the horizontal row in which the nodes/vector occurs; numbering the top row as 1 and counting down from there. The second number, $c$, denotes the highest weight of the $su(2)_T$-irrep in which this vector lies. Finally the third number, $d$, denotes the $T_3$ weight of the vector itself.

Figure 3.3: The weight diagram of the irrep $(2, 1)$ with some of the weight vectors labeled accordingly.

For convenience, when listing vectors, the vectors will always be ordered lexicographically. The vector with the smallest $b$ (i.e. the top most row) will be first. If several vectors are in the same row then, of these, the vector with largest $c$ will be placed first and if the vectors are all in the same $su(2)_T$-irrep then the vector with the highest $T_3$-weight will be listed first. Figure 3.3 shows the irrep $(2, 1)$ with some of the weight vectors labeled accordingly. When listing tensor products of two vectors, the order is determined by the first vector in the tensor product and only if those two vectors are the same is the second vector considered.
### 3.3 $su(3)$ actions: $U_+$ and $V_+$ actions on a $T_+$ basis

Before trying to decompose multi-dimensional highest weight spaces in the tensor product, it is first necessary to calculate the highest weight spaces. To do this, the action of $T_+$ and $U_+$ on an arbitrary basis vector, $|a,b,c⟩$, must be known. In general the action of $U$ and $V$ on such a vector will result in a (non-trivial) linear combination of at least two other basis vectors. Before establishing these actions in detail, it will be convenient to choose the $su(2)_T$ action to be the unitary action. This choice is due to the fact that the unitary basis is the most commonly used and because trial and error with several other bases did not seem to simplify the problem of decomposing multi-dimensional highest weight spaces.

**Lemma 3.3.1** The action of $T_+$ on an arbitrary basis vector $|b, p, \frac{p-2a}{2}⟩$ with $b, p \in \mathbb{N}$ is as follows:

$$T_+|b, p, \frac{p-2a}{2}⟩ = \sqrt{a(p-a+1)}|b, p, \frac{p+2-2a}{2}⟩$$

**Proof:** The more familiar form for the unitary $su(2)$ has raising action:

$$T_+|j, m⟩ = \sqrt{j(j+1) - m(m+1)}|j, m+1⟩$$

This can be found, with derivation, in Pfeifer [5]. Using the notation established above the raising coefficient becomes:

$$\sqrt{\frac{p(p+2)}{2} - \frac{p-2a(p-2a)}{2}} = \sqrt{\frac{4ap-2a(2a-2)}{4}} = \sqrt{a(p-a+1)}$$

**Lemma 3.3.2** The $U_+$ and $V_+$ action on the $su(2)_T$-basis is described as follows.

\begin{align*}
\gamma_b \delta_k U_+|b, \frac{k}{2}, \frac{k-2a}{2}⟩ &= \sqrt{\frac{a+1}{k+1}}|b+1, \frac{k+1}{2}, \frac{k-(2a+1)}{2}⟩ + \sqrt{\frac{k-a}{(k+1)}}|b+1, \frac{k-1}{2}, \frac{k-(2a+1)}{2}⟩ \\
\gamma_b \delta_k V_+|b, \frac{k}{2}, \frac{k-2a}{2}⟩ &= \frac{\sqrt{k+1-a}}{\sqrt{k+1}}|b+1, \frac{k+1}{2}, \frac{k+1-2a}{2}⟩ - \frac{\sqrt{a}}{\sqrt{k+1}}|b+1, \frac{k-1}{2}, \frac{k+1-2a}{2}⟩
\end{align*}

(3.1) (3.2)
where $\gamma_b$ may be chosen freely for each row $b$ and $\delta_k = \frac{\sqrt{k'}}{\sqrt{k+1}}$ where $k'$ is the highest $T_3$-weight of the largest $su(2)_T$-irrep in row $b$.

Given two $su(2)_T$-irreps in the same row of a weight diagram of an $su(3)$ irrep, how $V_+$ acts on the highest weight of one of the $su(2)_T$-irreps determines how $V_+$ acts on the highest weight of the other $su(2)_T$ irrep. This is the reason for the occurrence of the factor $\delta_k'$ and is due to the fact that $U_+$ and $V_+$ satisfy equation (3.3).

$$[U_+, V_+] = U_+V_+ - V_+U_+ = 0 \quad (3.3)$$

This is stated precisely in lemma 3.3.3. The proof of lemma 3.3.2 and lemma 3.3.3 may be found in the appendix on page 82. The result is established by induction and the exposition not particularly enlightening.

**Lemma 3.3.3** Suppose that $V_+|b, \frac{k}{2}, \frac{k}{2}\rangle = \gamma_b\delta_k|b + 1, \frac{k+1}{2}, \frac{k+1}{2}\rangle$. Then:

$$V_+|b, \frac{k-2n}{2}, \frac{k-2n}{2}\rangle = \frac{\sqrt{k+1}}{\sqrt{k+1-2n}}\gamma_b\delta_k|b + 1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2}\rangle$$

### 3.4 An example of a multi-dimensional highest weight space

Having explicitly established the actions of $T_+$ and $U_+$ it is now possible to find the highest weight spaces inside the tensor product of two arbitrary finite dimensional irreps of $su(3)$. This section is devoted to an explicit example. When considering highest weight spaces, it is often useful to visualise the weight lattices of the contributing irreps. In particular, it is useful to be able to see exactly the weight spaces of each irrep that contribute to the weight space in which a highest weight space is contained. Consider $(1, 1) \otimes (2, 1)$ (i.e. the tensor product of irreps with highest weight $\lambda_1 + \lambda_2$ and $2\lambda_1 + \lambda_2$).

Using Young tableaux (for details see Pfeifer [5]) it is possible to find the following direct sum decomposition for this tensor product.

$$(1, 1) \otimes (2, 1) = (3, 2) \oplus (1, 3) \oplus (4, 0) \oplus 2(2, 1) \oplus (0, 2) \oplus (1, 0)$$
Note that there are two irreps of highest weight $2\lambda_1 + \lambda_2$ and hence there is a two dimensional highest weight space of this weight. The weight space of weight $2\lambda_1 + \lambda_2$ in the tensor product is six dimensional. It is spanned by the following set, $\mathcal{U}$, of six elements of the tensor product basis.

$$\mathcal{U} = \{ |1, \frac{1}{2}, \frac{1}{2}\rangle \otimes |2, \frac{3}{2}, \frac{1}{2}\rangle, |1, \frac{1}{2}, \frac{1}{2}\rangle \otimes |2, \frac{1}{2}, \frac{1}{2}\rangle, |1, \frac{1}{2}, -\frac{1}{2}\rangle \otimes |2, \frac{3}{2}, \frac{3}{2}\rangle,$$

$$|2, 1, 1\rangle \otimes |1, 1, 0\rangle, |2, 1, 0\rangle \otimes |1, 1, 1\rangle, |2, 0, 0\rangle \otimes |1, 1, 1\rangle \}$$

The elements occurring in $\mathcal{U}$ can be shown on a weight diagram as depicted in figure 3.5. A diagram of the form depicted in figure 3.5 shall be referred to as a highest weight contributing diagram (hereafter HWCD). The diagram in figure 3.5 actually represents many more elements than occur in $\mathcal{U}$ unless it is understood that a HWCD "depicts" the tensor product basis elements of a given fixed weight. In the case of figure 3.5 that fixed weight is
2\lambda_1 + \lambda_2$. In a general HWCD the fixed weight may always be found as the weight of the tensor product basis element that is formed from the tensor product of the highest weight from the (partial) weight diagram on the left and the lowest weight from the (partial) weight diagram on the right (or vice versa).

In this example, the intention is to find a two-dimensional subspace of \( \text{span}\{U\} \) that is killed by the action of \( T_+ \) and \( U_+ \) (and also \( V_+ \) but the raising actions of \( T_+ \) and \( U_+ \) suffice since \( V_+ = [T_+, U_+] \)). Note that the remaining four dimensions are accounted for in other irreps in the direct sum decomposition, namely \((1, 3), (4, 0)\) and two in \((3, 2)\). To find the required highest weight space, begin by taking an arbitrary vector, \( v \), composed of a linear combination of the basis elements in \( U \). It will be convenient to denote these arbitrary coefficients by \( \alpha_i \) with \( i = 1, \ldots, 6 \), where the coefficients inherit the ordering of the tensor product basis element to which they are attached (see section 3.2). To find the highest weight space apply \( T_+ \) to \( v \) and set the result to zero; solving for the coefficients. Then do the same with \( U_+ \). The relevant information is compiled in table 3.1.

<table>
<thead>
<tr>
<th>Element of ( U )</th>
<th>After applying ( T_+ )</th>
<th>After applying ( U_+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1</td>
<td>1, \frac{1}{2}, \frac{1}{2}\rangle \otimes</td>
<td>2, \frac{3}{2}, \frac{1}{2}\rangle )</td>
</tr>
<tr>
<td>( \alpha_2</td>
<td>1, \frac{1}{2}, \frac{1}{2}\rangle \otimes</td>
<td>2, \frac{1}{2}, \frac{1}{2}\rangle )</td>
</tr>
<tr>
<td>( \alpha_3</td>
<td>1, \frac{1}{2}, \frac{1}{2}\rangle \otimes</td>
<td>2, \frac{3}{2}, \frac{1}{2}\rangle )</td>
</tr>
<tr>
<td>( \alpha_4</td>
<td>2, 1, 1\rangle \otimes</td>
<td>1, 1, 0\rangle )</td>
</tr>
<tr>
<td>( \alpha_5</td>
<td>2, 1, 0\rangle \otimes</td>
<td>1, 1, 1\rangle )</td>
</tr>
<tr>
<td>( \alpha_6</td>
<td>2, 0, 0\rangle \otimes</td>
<td>1, 1, 1\rangle )</td>
</tr>
</tbody>
</table>

Table 3.1: A table listing the action of \( T_+ \) and \( U_+ \) on elements of \( U \).

The arbitrary vector, \( v \), is the sum of entries in the first column. Totalling the second column and the third column and setting each to zero gives the following four equations, relating the coefficients of \( v \).

\[
\sqrt{3}\alpha_1 + \alpha_3 = 0 \\
\sqrt{2}\alpha_4 + \sqrt{2}\alpha_5 = 0 \\
\frac{\sqrt{2}}{\sqrt{3}}\alpha_1 + \frac{2}{\sqrt{3}}\alpha_2 + \alpha_4 = 0 \\
\alpha_3 + \frac{1}{\sqrt{2}}\alpha_5 + \frac{3}{\sqrt{2}}\alpha_6 = 0
\]
Solving these equations in terms of $\alpha_1$ and $\alpha_2$ gives:

\[
\begin{align*}
\alpha_3 &= -\sqrt{3} \alpha_1 \\
\alpha_4 &= -\frac{\sqrt{3}}{3} \alpha_1 - \frac{2}{\sqrt{3}} \alpha_2 \\
\alpha_5 &= \frac{\sqrt{2}}{\sqrt{3}} \alpha_1 + \frac{2}{\sqrt{3}} \alpha_2 \\
\alpha_6 &= \frac{2\sqrt{7}}{3\sqrt{3}} \alpha_1 - \frac{2}{3\sqrt{3}} \alpha_2
\end{align*}
\]

This is clearly a two dimensional highest weight space as every coefficient depends on $\alpha_1$ and $\alpha_2$. Moreover any two linearly independent choices of $(\alpha_1, \alpha_2)$ determine two distinct highest weight vectors and thus two distinct irreps in the direct sum. The intention of this thesis is to construct a method that will choose such coefficients in a well-defined and sensible way.

### 3.5 More general highest weight spaces in a tensor product of irreps of $su(3)$

Fortunately, when determining a highest weight space it is not necessary to work out explicitly all the coefficients of an arbitrary highest weight space, as this can be a very long process. The purpose of this section is to introduce and label some of the important coefficients and show how some of these coefficients are related to each other. This will provide enough information to decompose the highest weight spaces in a method analogous to that discussed for the tensor product of three copies of $su(2)$ (see section 2.6.2).

First it will be useful to consider a HWCD diagram that is more general than the example in figure 3.5. Note that figure 3.6 could be denoted by $P_{HWCD} \otimes R_{HWCD}$ and that $R_{HWCD}$ is the same as $P_{HWCD}$ with $p$ replaced by $r$. Recall that figure 3.6 does not intend to represent all the elements than can be formed from the tensor product of vectors in the various weight spaces depicted. Rather the HWCD represents all the tensor product basis elements of a fixed weight. The specified weight is always the weight of the tensor product basis element formed by taking the tensor product of the highest vector in the first irrep (namely $|1, \frac{p}{2}, \frac{p}{2}\rangle$) with the vector of lowest weight that occurs in $R_{HWCD}$. In figure 3.6, $n$ denotes the number of rows, and it is assumed that each row contains a larger $su(2)_T$ irrep than any of the irreps in the previous row. It will also be convenient to denote by $x + n - 1$ the number of columns in $P_{HWCD}$ (in the $U_{\pm}$ direction; see figure 3.2) and assume $p \geq x + n - 1$. This last condition
ensures that the top row of $P_{\text{HWCD}}$ is the full length of the whole of $P_{\text{HWCD}}$. Similarly, assume $r \geq x + n - 1$. Such a HWCD will be called nondegenerate. Note that such a HWCD gives rise to an $n$ dimensional highest weight space.

Having introduced the HWCD in figure 3.6 to assist with analyzing the highest weight spaces, it is now appropriate to consider how certain coefficients in a highest weight vector are related.

**Lemma 3.5.1** There are exactly (up to scalar multiplication) two tensor product basis elements that raise to the same vector, $|p_i\rangle \otimes |r_i\rangle$, under the action of $T_+$. These are exactly $(T_-|p_i\rangle) \otimes |r_i\rangle$ and $|p_i\rangle \otimes (T_-|r_i\rangle)$.

**Proof:** The proof is immediate (with the explicit action given in lemma 3.3.1) as these are clearly the only two tensor product basis elements that raise under the action of $T_+$ to the required vector.

Using the appropriate definition of basic C.G. coefficients (definition 4.3.1), Lemma 3.5.1 means that in the highest weight space all C.G. coefficients (given as linear combinations of the basic C.G. coefficients) of vectors from the same horizontal $su(2)_T$-irrep are scalar multiples of each other. That is to say the C.G. coefficients (given as linear combinations of the basic C.G. coefficients) of vectors of the form \{\[b, \frac{p_i}{2}, \frac{k}{2}\] $\otimes |b', \frac{r_i}{2}, \frac{l-k}{2}\) : fixed $p_i, r_i$ and $l$ and appropriate $k}\} are all scalar multiples of each other.

Figure 3.6: A general nondegenerate HWCD.
Lemma 3.5.2 There are (if they exist) four tensor product basis elements that raise to $|b, \frac{p_i}{2}, \frac{k}{2} \rangle \otimes |b', \frac{r_i}{2}, \frac{l}{2} \rangle$ under the action of $U_+$. These are:

- $|b + 1, \frac{p_i + 1}{2}, \frac{k + 1}{2} \rangle \otimes |n - b, \frac{r_i}{2}, \frac{l}{2} \rangle$
- $|b + 1, \frac{p_i - 1}{2}, \frac{k + 1}{2} \rangle \otimes |n - b, \frac{r_i}{2}, \frac{l}{2} \rangle$
- $|b, \frac{p_i}{2}, \frac{k}{2} \rangle \otimes |n - b + 1, \frac{r_i + 1}{2}, \frac{l + 1}{2} \rangle$
- $|b, \frac{p_i}{2}, \frac{k}{2} \rangle \otimes |n - b + 1, \frac{r_i - 1}{2}, \frac{l + 1}{2} \rangle$

Proof: Again the proof is immediate by considering all possibilities as given by the raising action of $U_+$ described in lemma 3.3.2.

These relations relate the coefficients on adjacent rows. Notice that if $p_i$ is the largest irrep in row $b$ and $k = p_i$ then $|b + 1, \frac{p_i + 1}{2}, \frac{k + 1}{2} \rangle$ does not exist and so the coefficient of the first tensor product basis in the list in lemma 3.5.2 is a linear combination of the two coefficients of the latter two tensor product basis elements. Similarly, if $r_i$ is the largest irrep in row $b'$ and $l = r_i$ then $|n - b + 1, \frac{r_i - 1}{2}, \frac{l + 1}{2} \rangle$ does not exist.

Lemmas 3.5.1 and 3.5.2 give all the types of relation needed to find a highest weight space and will be fundamental in constructing a method to decompose arbitrary highest weight spaces. However it is first necessary to describe and label some of the important C.G. coefficients in the highest weight space. In a highest weight space, consider the tensor product basis elements of the appropriate weight with $R_{\text{HWCD}}$ component equal to $|k, \frac{r + k - 1}{2}, \frac{r + k - 1}{2} \rangle$.

With the usual ordering let $L^j_k$ be the $j^{th}$ such tensor product basis element (with the usual ordering outlined in section 3.2). Similarly let $R^j_k$ be the coefficients of the $j^{th}$ tensor product basis element of the appropriate weight with $P_{\text{HWCD}}$ component equal to $|k, \frac{r + k - 1}{2}, \frac{r + k - 1}{2} \rangle$.

(Explicitly, $R^j_k$ is attached to the tensor product basis element: $|k, \frac{r + k - 1}{2}, \frac{r + k - 1}{2} \rangle \otimes |n + 1 - k, \frac{r + n - k - 2(j - 1)}{2}, \frac{r + n - k - 2(j - 2)}{2} \rangle$.) Consider the highest weight contributing diagram and look at $P_{\text{HWCD}}$ (i.e., the left hand partial weight diagram); then the coefficients $L$ are attached to tensor product elements that occur down the left hand side of $P_{\text{HWCD}}$ and the coefficients $R$ occur on the right hand side. Pictorially, these coefficients are attached to tensor product elements that occur in the positions in $P_{\text{HWCD}}$ as indicated in figure 3.7. Note that $*$ has been used to simplify notation and indicates any appropriate number.

So $R^j_1$ for $j = 1, \ldots, n$ are attached to tensor product elements that contain the highest weight of $P$. With the usual ordering (and notation from section 3.2) $R^j_1 = \alpha_j$. Continuing in the non-degenerate case, fix $t$, then there are exactly $n + 1 - t$ elements with coefficients
The next result relates these $\mathcal{R}$- and $\mathcal{L}$-coefficients explicitly. The precise statement is most easily written with the use of notation for the raising coefficients with respect to the action of $U_+$ and notation for the product of coefficients from multiple actions of $T_+$. In lemma 3.3.2 it was shown that the action of $U_+$ on an arbitrary vector $|b, \frac{c}{2}, \frac{d}{2}\rangle$ had the following form:

$$U_+|b, \frac{c}{2}, \frac{d}{2}\rangle = *^-|b - 1, \frac{c - 1}{2}, \frac{d - 1}{2}\rangle + *^+|b - 1, \frac{c + 1}{2}, \frac{d - 1}{2}\rangle$$

The coefficients $*^+$ and $*^-$ may be calculated for any irrep using the result in lemma...
3.3.2. However for the purposes of this section it will be convenient to denote \( *^- = U_{[b,c,d]}^- \) and \( *^+ = U_{[b,c,d]}^+ \). Also it will be convenient to denote by \( \{ a_1, b, c_2, \ldots, c_n \} \) the product of \( su(2) \) raising coefficients from the action of \( T_3 \) on vectors with \( T_3 \) weight \( b, b+2, \ldots, c_n \) in an \( su(2)_T \) irrep of highest weight \( a \).

**Lemma 3.5.3** The coefficients \( L_k^* \) and \( R_k^* \) are related in the following fashion:

- \( R_k^j \) is a linear combination of \( R_{k-1}^j \) and \( R_{k-1}^{j+1} \) for \( k = 2, \ldots, n \)
- \( L_k^j \) is a linear combination of \( L_{k-1}^{j+1} \) and \( L_{k-1}^{j-1} \) for \( k = 2, \ldots, n \)
- \( L_{n+1-k}^1 \) is a linear multiple of \( R_k^1 \) for \( k = 1, \ldots, n \)

These relationships may be stated precisely as follows:

\[
R_k^j = -\frac{1}{U_{[k,p+k-1,p+k-1]}^-} \left( U_{[n-k+2,r+n-k+1-2(j-1),r-n-k+1-2(x-2)]}^- R_{k-1}^j \right. \\
+ U_{[n-k+2,r+n-k+1-2j,r-n-k+1-2(x-2)]}^+ R_{k-1}^{j+1} \left. \right) 
\]  
(3.4)

\[
L_k^j = -\frac{1}{U_{[n+1-k,p+n-k-2(j-1),p-n-k-2(x-2)]}^+} \left( U_{[n+1-k,p+n-k-2(j-2),p-n-k-2(x-2)]}^+ L_{k-1}^{j+1} \right. \\
+ U_{[k,r+k-1,r+k-1]}^- L_{k-1}^{j-1} \left. \right) 
\]  
(3.5)

\[
L_{n+1-k}^1 = \left\{ \frac{p+k-1}{2}, \frac{p+k-1}{2}, \frac{p+k-3-2(x+n)}{2} \right\} R_k^1 
\]  
(3.6)

**Proof:** The vectors/coefficients that appear in the relations are given by lemma 3.5.2 and the explicit raising coefficients may be calculated from lemma 3.3.2.

\[ \blacksquare \]
3.6 A more general example of a 2-dimensional highest weight space

Consider the tensor product $P \otimes R$ with a 2-dimensional highest weight space with highest weight contributing diagram as given in figure 3.9.

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array}
\]

\[
|1, \frac{p}{2}, \frac{r}{2}\rangle
\]

\[
\otimes
\]

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array}
\]

\[
|2, \frac{p+1}{2}, \frac{r+1}{2}\rangle
\]

Figure 3.9: The HWCD for a non-degenerate two-dimensional highest weight space.

The 2-dimensional highest weight space given by the HWCD in figure 3.9 sits inside a $(4x + 2)$-dimensional weight space. Let $\alpha_i$ be the coefficient of the $i^{th}$ vector in this ambient weight space (as usual with respect to the ordering established in section 3.2). Then comparing the $\alpha_i$ with the notation established on page 45 and depicted in figure 3.7, establishes that:

\[
\begin{align*}
\alpha_{2x+1} &= L_2^1 & \alpha_1 &= R_1^1 \\
\alpha_{4x+1} &= L_1^1 & \alpha_2 &= R_1^2 \\
\alpha_{4x+2} &= L_1^2 & \alpha_{2x+2} &= R_2^1
\end{align*}
\]

Recall that it is possible to use the actions of $T_+$ and $U_+$ to find the highest weight space by raising an arbitrary vector of the correct weight and setting the result to zero (see page 40 for full details of the case $(1, 1) \otimes (2, 1)$). The $T_+$ action gives rise to equations (3.7) and (3.8).
\[ \alpha_1 = \frac{-\sqrt{p}}{\sqrt{x(r + 2 - x)}} \]
\[ \alpha_3 = \frac{\sqrt{p} \sqrt{2(p - 1)}}{\sqrt{x(r + 2 - x)} \sqrt{(x - 1)(r + 3 - x)}} \alpha_5 = \ldots \]
\[ \ldots = (-1)^x \{ \frac{p}{2}, \frac{p - 2x}{2} \} \} \alpha_{2x+1} \] (3.7)
\[ \alpha_2 = \frac{-\sqrt{p}}{\sqrt{(x - 1)(r + 1 - x)}} \]
\[ \alpha_4 = \frac{\sqrt{p} \sqrt{2(p - 1)}}{\sqrt{(x - 1)(r + 1 - x)} \sqrt{(x - 2)(r + 2 - x)}} \alpha_6 = \ldots \]
\[ \ldots = (-1)^{x+1} \{ \frac{p}{2}, \frac{p - 2(x - 1)}{2} \} \} \alpha_{2x} \] (3.8)

The \( T_+ \) action also gives rise to similar relations between \( \alpha_{4x+1} \) and \( \alpha_{2x+2} \) and also between \( \alpha_{4x+2} \) and \( \alpha_{2x+4} \). There are also many relations given by the action of \( U_+ \) and in general these tend to be more complicated than the relations that arise from the action of \( T_+ \). However in this particular case there are only two relations from the action of \( U_+ \) that are needed to explicitly give all the coefficients of the highest weight space in terms of the coefficients \( \alpha_1 \) and \( \alpha_2 \). The relations from the action of \( U_+ \) that are used in this respect are given in equation (3.9) and equation (3.10). Recall from section 3.3 that the \( \gamma_{s,s} \) depend on the \( su(3) \) irrep and are determined if the basis of the irrep is determined and otherwise may all be chosen to be equal to 1.

\[ \gamma_{2,p+1} \frac{\sqrt{p + 1}}{\sqrt{p + 2}} \alpha_{2x+2} + \gamma_{2,r+1} \frac{\sqrt{r + 1 - x}}{\sqrt{r + 2}} \alpha_1 + \gamma_{2,r+1} \frac{\sqrt{r + 2} \sqrt{x}}{r} \alpha_2 = 0 \] (3.9)

\[ \gamma_{2,r+1} \frac{\sqrt{r + 1}}{\sqrt{r + 2}} \alpha_{2x+1} + \gamma_{2,p+1} \frac{\sqrt{p + 1 - x}}{\sqrt{p + 2}} \alpha_{4x+1} + \gamma_{2,p+1} \frac{\sqrt{x} \sqrt{p + 2}}{p} \alpha_{4x+2} = 0 \] (3.10)

These equations relate the coefficients attached to tensor product basis elements on different rows. In particular equation (3.9) relates the \( R \) coefficients to each other and equation (3.10) relates the \( L \) coefficients to each other. It is equation (3.9) that will be used to decompose this highest weight space in the next section.
3.7 A method for decomposing highest weight spaces

3.7.1 Some background results

In this section a method is introduced for decomposing multi-dimensional highest weight spaces in the non-degenerate case. It is very similar to the method outlined for the tensor product of three copies of $su(2)$ in section 2.6.2. Here the $R^*_k$ satisfy similar conditions to the $\alpha_*$ of level $k$ and indeed form an analogous structure. Moreover lemmas 3.7.1, 3.7.2 and 3.7.3 establish relationships between the $R^*_k$ that mirror the results concerning the coefficients on level $k$ in lemmas 2.6.1, 2.6.2, and 2.6.3 on page 20.

**Lemma 3.7.1** In the highest weight space, $R^*_k$ is a linear combination of $R^*_1, R^*_1+1, \ldots, R^*_1+k−1$.

**Proof:** Lemma 3.7.1 is established by an easy induction using equation (3.4) in lemma 3.5.3 which gives that $R^*_k$ is a linear combination of $R^*_k$ and $R^*_k+1$.

**Lemma 3.7.2** Fix $k \in \{1, 2, \ldots, n\}$. The $n+1−k$ coefficients of the form $R^*_k$ are linearly independent.

**Proof:** The $R^*_1$ are independent and lemma 3.7.1 gives that $R^*_k$ are of the correct form for lemma 2.6.1 to apply.

**Lemma 3.7.3** Fix $k \in \{1, 2, \ldots, n\}$. If $R^*_j = 0$ for all $j$, then $R^*_k+t = 0$ for $t = 1, \ldots, n−k$.

**Proof:** This is immediate from lemma 3.7.1 since it is possible to write any $R^*_k+t$ as a linear combination of coefficients of the form $R^*_k$.

**Lemma 3.7.4** Fix $k$. The highest weight vectors with $R^*_k = 0$ for all $j$, form a vector space of dimension $k−1$.

**Proof:** This is clearly a vector space as such vectors are closed under scalar multiplication and addition and the space contains the zero vector. In a highest weight space of dimension $n$, setting $n+1−k$ linearly independent coefficients to zero results in a $k−1$ dimensional vector space.
Lemmas 3.7.1, 3.7.2 and 3.7.3 provide the essential results required to proceed with a method for decomposing a non-degenerate highest weight space, in analogy to the method described for the tensor product of three copies of $su(2)$ on page 21.

### 3.7.2 The Method

**The Method**

Step 1: Choose the first highest weight vector to be the vector such that for $j = 1, \ldots, n - 1$ the coefficients of the form $R^j_2$ are zero. By lemma 3.7.4 this determines a highest weight vector (up to scalar multiplication).

Step 2: Now consider the space spanned by the highest weight vectors for which $R^j_3$ ($j = 1, \ldots, n - 2$) are zero. By lemma 3.7.4, this is a 2-dimensional space that contains the first vector. So choosing a second vector that is in this space and orthogonal to the first again determines a highest weight vector (up to scalar multiplication).

Step $k$: Consider the space spanned by the highest weight vectors for which $R^j_{k+1}$ ($j = 1, \ldots, n + 1 - k$) are zero. Again by lemma 3.7.4, this is a $k$-dimensional vector space that contains the $k - 1$ orthogonal vectors already constructed. So it is possible to choose the $k^{th}$ vector to be in this vector space and orthogonal to the first $k - 1$ vectors.

Step $n$: At the final step, the method has constructed $(n - 1)$ orthogonal vectors in an $n$-dimensional highest weight space and the final vector may be chosen to be orthogonal to all the rest.

**Proposition 3.7.5** The method described in this section, 3.7.2, gives rise to a set of orthogonal highest weight vectors that span the entire highest weight space.

**Proof:** Lemma 3.7.4 ensures that step 1 determines a highest weight vector (up to scalar multiplication). At step 2, lemma 3.7.4 ensures the space spanned by the highest weight vectors for which $R^j_3$ ($j = 1, \ldots, n - 2$) are zero, is a 2-dimensional space that contains the first vector. In general, lemma 3.7.4 ensures that the space spanned by the highest weight vectors for which $R^j_{k+1}$ ($j = 1, \ldots, n + 1 - k$) are zero, is a $k$-dimensional space that contains the first $k - 1$ vectors. Therefore, the final $n^{th}$ vector can be chosen to be orthogonal to all the rest.
vectors for which $R^j_{k+1}$ ($j = 1, \ldots, n+1-k$) are zero is a $k$-dimensional vector space that contains the $k-1$ orthogonal vectors that have already constructed.

3.7.3 An example

Recall, in section 3.4, the example of $(1,1) \otimes (2,1)$ which contained the irrep $(2,1)$ with multiplicity 2. In this example there is a six dimensional highest weight space of weight $2\lambda_1 + \lambda_2$ with the coefficients of the tensor product basis denoted by $\alpha_1, \alpha_2, \ldots, \alpha_6$. In the notation of this section, $R^1_1 = \alpha_1, R^2_1 = \alpha_2$ and $R^4_2 = \alpha_4$. Recall that it was established that the 2-dimensional highest weight coefficients were related as follows:

\[
\begin{align*}
\alpha_1 & = -\sqrt{3} \alpha_1 \\
\alpha_2 & = -\frac{\sqrt{2}}{\sqrt{3}} \alpha_1 + \frac{2}{\sqrt{3}} \alpha_2 \\
\alpha_3 & = \frac{\sqrt{2}}{\sqrt{3}} \alpha_1 + \frac{2}{\sqrt{3}} \alpha_2 \\
\alpha_4 & = \frac{2\sqrt{2}}{3\sqrt{3}} \alpha_1 - \frac{2}{3\sqrt{3}} \alpha_2 \\
\alpha_5 & = \frac{2\sqrt{2}}{3\sqrt{3}} \alpha_1 + \frac{2}{3\sqrt{3}} \alpha_2 \\
\alpha_6 & = -\sqrt{3} \alpha_1 - \sqrt{3} \alpha_2
\end{align*}
\]

So following the method described in section 3.7.2 the first vector is constructed by setting $\alpha_4 = 0$. This gives rise to the highest weight vector $(-\sqrt{6}, \sqrt{3}, 3\sqrt{2}, 0, 0, -2)$. The second highest weight vector is found by setting to zero the dot product of the first vector with an arbitrary highest weight vector. This gives the second highest weight vector as $(3\sqrt{6}, 48\sqrt{3}, -9\sqrt{2}, -90, 90, -28)$.

3.7.4 The RSR property

As already noted, there are many similarities between the method described on page 21 for decomposing highest weight spaces in the tensor product of three irreps of $su(2)$ and this method for decomposing highest weight spaces in the tensor product of two irreps of $su(3)$. In the first case it was proved that the method gave rise to irreps with C.G. coefficients that were all square roots of rational numbers (hereafter RSR, for rational square root). It is natural to ask whether this is the case with the decomposition of the tensor product of irreps of $su(3)$ as described on page 50. This is much harder than in the $su(2)$ setting and this section discusses only some very elementary results.
Proposition 3.7.6  The first highest weight vector found using the method described on page 50 has all coefficients in $\sqrt{\mathbb{Q}}$.

Proof: It is evident from the relations in lemma 3.5.3 that setting all $R_i^2 = 0$ for $i = 1, \ldots, n - 1$ ensures that each $L$-coefficient is a scalar multiple of $R_1^1$ by a factor in $\sqrt{\mathbb{Q}}$. But the $L$ and $R$ coefficients are related to all the other coefficients by scalar multiplication by factors in $\sqrt{\mathbb{Q}}$. (Note that the $R_j^1$ coefficients are scalar multiples of the $L_j^n$ coefficients and so all $R$-coefficients are also scalar multiples of $R_1^1$ or zero). Hence all coefficients in the highest weight vector are in $\sqrt{\mathbb{Q}}$, given that $R_1^1$ is chosen to be a rational square root.

Note that proposition 3.7.6 mentions nothing of the RSR-generating property. That is, the result proves that the coefficients of the highest weight vector are rational square roots, but does not establish anything about the C.G. coefficients of vectors of lower weight in the same irrep. It is also hard to make general statements about the vectors that are constructed from orthogonality conditions. However, the next subsection includes the details of one example to demonstrate that, in general, it is hopeful that the C.G. coefficients of the highest weight vector may be in $\sqrt{\mathbb{Q}}$.

3.7.5  An example of the RSR property

Consider the two dimensional highest weight space in $P \otimes R$ with the HWCD as given in figure 3.10.

\[
\begin{array}{ccc}
\bullet & \bullet & |1, \frac{p}{2}, \frac{p}{2}\rangle \\
\otimes & \bullet & |1, \frac{r}{2}, \frac{r}{2}\rangle \\
\bullet & \bullet & |2, \frac{p+1}{2}, \frac{p+1}{2}\rangle \\
\bullet & \bullet & |2, \frac{r+1}{2}, \frac{r+1}{2}\rangle
\end{array}
\]

Figure 3.10: The HWCD of a six-dimensional weight space with two dimensional highest weight subspace.

Solving for the highest weight space gives that the six coefficients of the tensor product basis with the usual lexicographic ordering satisfy the following set of equations. The first two equations are found by applying $T_+$ to an arbitrary highest weight vector.
\[ \sqrt{p}\alpha_3 + \sqrt{r+1}\alpha_1 = 0 \]
\[ \sqrt{r}\alpha_4 + \sqrt{p+1}\alpha_5 = 0 \]

The next two equations are found by applying \( U_+ \) to an arbitrary highest weight vector.

\[ \gamma_p \sqrt{p+1}\alpha_4 + \gamma_r \sqrt{r+2}\alpha_1 + \gamma_r \frac{\sqrt{r+2}}{r}\alpha_2 = 0 \]
\[ \gamma_r \sqrt{r+2}\sqrt{r+2}\alpha_3 + \gamma_p \sqrt{p+1}\alpha_5 + \gamma_p \frac{\sqrt{p+2}}{p}\alpha_6 = 0 \]

Rearranging these four equations gives the C.G. coefficients of the 2-dimensional highest weight space in terms of the first two C.G. coefficients.

\[ \alpha_1 \]
\[ \alpha_2 \]
\[ \alpha_3 = -\frac{\sqrt{r+1}}{\sqrt{p}}\alpha_1 \]
\[ \alpha_4 = \frac{\gamma_r \sqrt{r+2}}{\gamma_p \sqrt{p+1}} \left( \frac{\sqrt{r}}{\sqrt{r+2}}\alpha_1 + \frac{\sqrt{r+2}}{r}\alpha_2 \right) \]
\[ \alpha_5 = \frac{\gamma_r \sqrt{r+2}}{\gamma_p (p+1)} \left( \frac{\sqrt{r}}{\sqrt{r+2}}\alpha_1 + \frac{\sqrt{r+2}}{r}\alpha_2 \right) \]
\[ \alpha_6 = \frac{-\gamma_p p}{\gamma_p \sqrt{p+2}} \left( \frac{-\gamma_r (p+r+1)}{(p+1)\sqrt{p}}\alpha_1 + \frac{\sqrt{p+2}}{(p+1)\sqrt{r}}\alpha_2 \right) \]

To find the first highest weight vector, the method on page 50 sets \( \alpha_4 = 0 \). This results in the following relation:

\[ \frac{\sqrt{r}}{\sqrt{r+2}}\alpha_1 = -\frac{\sqrt{r+2}}{r}\alpha_2 \quad (3.11) \]

So the first highest weight vector is given (up to scalar multiplication) by setting \( \alpha_1 = -(r+2) \) and \( \alpha_2 = r^{\frac{3}{2}} \). This gives the following C.G. coefficients for the first highest weight vector:

\[ \alpha_1 = -(r+2) \]
\[ \alpha_2 = r^{\frac{3}{2}} \]
\[ \alpha_3 = \frac{(r+2)\sqrt{r+1}}{\sqrt{p}} \]
\[ \alpha_4 = 0 \]
\[ \alpha_5 = 0 \]
\[ \alpha_6 = -\gamma_p \sqrt{p} \sqrt{r+2} \sqrt{r+2} \frac{(r+1)}{\gamma_p \sqrt{p+2}} \]
The coefficients of this highest weight vector are clearly in $\sqrt{\mathbb{Q}}$, as proposition 3.7.6 already proved. Note that $\alpha_6$ is a linear combination of $\alpha_3$ and $\alpha_5$. In the above highest weight vector $\alpha_5 = 0$ and so $\alpha_6$ can be viewed as a $\sqrt{\mathbb{Q}}$ multiple of $\alpha_3$ which is in turn a linear multiple of $\alpha_1$. Hence the example here does not contradict the theory outlined in the proof of proposition 3.7.6.

So the first vector is as required. To find the second vector we must take the dot product of the above vector with an arbitrary highest weight vector and set the result to zero. Doing just this gives rise to equation (3.12).

\[-(r + 2)\alpha_1 + r^2 \alpha_2 - \frac{(r + 1)(r + 2)}{p} \alpha_1 \]
\[+ \frac{\gamma_2^2 p^3 \sqrt{r + 2}(r + 1)}{\gamma_2^2 (p + 2)} \left( \frac{-(p + r + 1)}{(p + 1) \sqrt{r + 2}} \alpha_1 + \frac{\sqrt{p} \sqrt{r + 2}}{(p + 1) \sqrt{r}} \alpha_2 \right) = 0 \]

(3.12)

Rearranging equation (3.12) gives equation (3.13).

\[\sqrt{r}(p+r+1)\left(\gamma_2^2(p+2)(p+1)(r+2)+\gamma_2^2 p^2 (r+1)\right) \alpha_1 = p\left(\gamma_2^2(p+2)(p+1)r^2+\gamma_2^2(r+2)(r+1)p^2\right) \alpha_2 \]

(3.13)

Equation (3.13) is rather long, but the only concern is whether the highest weight C.G. coefficients are in $\sqrt{\mathbb{Q}}$. Define $k$ as follows:

\[k := \frac{p\left(\gamma_2^2(p+2)(p+1)r^2+\gamma_2^2(r+2)(r+1)p^2\right)}{r(p+r+1)\left(\gamma_2^2(p+2)(p+1)(r+2)+\gamma_2^2 p^2 (r+1)\right)} \]

Then $\alpha_1 = \sqrt{r}k\alpha_2$ where $k \in \mathbb{Q}$. It is possible to write all the C.G. coefficients of the highest weight vector in terms of $\alpha_1$ as follows:
\[\begin{align*}
\alpha_1 &= 1 \\
\alpha_2 &= \frac{1}{k\sqrt{r}} \alpha_1 \\
\alpha_3 &= \frac{-\sqrt{r+1}}{\sqrt{p}} \alpha_1 \\
\alpha_4 &= \frac{-\gamma_r \sqrt{p+2(kr^2+r^2)}}{\gamma_p \sqrt{p+2kr} \alpha_1} \\
\alpha_5 &= \frac{-\gamma_r \sqrt{p+2(kr^2+r^2)}}{\gamma_p (p+1) \sqrt{p+2kr} \alpha_1} \\
\alpha_6 &= \frac{\gamma_p (p+r+1) kr - p(r+2)}{\gamma_p (p+1) \sqrt{p+2kr}} \alpha_1
\end{align*}\]

So it is clear that in this example (for any value of \(p\) and \(r\)) the C.G. coefficients of the highest weight space are in \(\sqrt{Q}\) whenever \(\alpha_1 \in \sqrt{Q}\). Hopefully this example also highlights how messy the problem is in general.

### 3.8 The degenerate case

#### 3.8.1 A brief overview

![Diagram](image.png)

Figure 3.11: A general degenerate HWCD.
Consider the HWCD of $P \otimes R$, where $P = (p,q)$ and $R = (r,s)$. If $p, q, r$ or $s$ are small enough it is possible for the outline of $P_{HWCD}$ and $R_{HWCD}$ to be (distinct) irregular hexagons. Figure 3.11 depicts a general outline. Let $N$ denote the number of rows in the HWCD and let $n$ continue to be the dimension of the highest weight space. In particular, $N \geq n$. In $P = (p,q)$, let $N_r = N - q - 1$; $N_r$ is labeled in figure 3.11 and can be interpreted as the number of rows in the HWCD below the longest row in the irrep. Note that it is not the case that every degenerate case gives rise to a multi-dimensional highest weight space. However every multi-dimensional highest weight space has a HWCD and all cases are included in the diagram in figure 3.11.

It is now necessary to generalise the notation of section 3.5. In the degenerate case the definition of $R$-vectors is extended to mean tensor product basis vectors of the form $|\ast, c, c\rangle \otimes |\ast\rangle$ where $c$ is the highest weight of the largest $T_{\pm}$-irrep in row $\ast$. We call C.G. coefficients attached to such vectors $R$-coefficients and label by $R^j_i$ the $j^{th}$ $R$-coefficient (with the usual ordering given in section 3.2) that occurs in the $i^{th}$ row. Note that it is possible for $P$ and $R$ to combine in such a way that there are no non-zero $R$-vectors on a particular row in a degenerate HWCD. Similarly the definition of $L$ vectors is extended to mean the $R$ vectors in $R \otimes P$. Note that pictorially, the $R$ vectors occur on the right-hand side of the degenerate $P_{HWCD}$ and the $L$ vectors on the left-hand side of the degenerate $P_{HWCD}$. There are several results that need to be established before a generalisation of the method in section 3.7.2 on page 50 to the general degenerate case can be introduced.

**Lemma 3.8.1** Given a weight space of dimension $t$ in an $su(3)$ irrep, all adjacent weight spaces in the irrep have dimension equal to either $t - 1, t$ or $t + 1$.

**Proof:** Section 3.2 gives that the weight spaces increase in dimension by one with each smaller consecutive concentric hexagonal shell and that the dimension of the weight spaces stabilise once the shells become triangular in shape. The actions of $T, U$ and $V$ only pass to consecutive (or remain in the same) hexagonal (or triangular) shell.

**Lemma 3.8.2** Suppose an irrep has $N$ rows (with $N = N_r + q + 1$). Then the $U_+$ and $V_+$ actions on a vector in a weight space of dimension $t$ in the bottom $N_r$ rows take the vector into a weight space of dimension $t$ or $t + 1$. Also the $U_+$ and $V_+$ actions on a vector in a weight space of dimension $t$ in the top $q + 1$ rows take the vector into a weight space of dimension $t$ or $t - 1$.  

56
Proof: In the bottom \( N \) rows, the actions of \( U_+ \) and \( V_+ \) either map a vector to another vector in the same hexagonal (or triangular) shell or map it to a vector in the adjacent smaller shell. Similarly, in the top \( q + 1 \) rows the actions of \( U_+ \) and \( V_+ \) map any vector to another vector in the same hexagonal shell or map it to a vector in the adjacent larger shell. Note that lemma 3.8.2 includes the top row of the irrep in the statement and it holds trivially true as the 1-dimensional weight spaces are annihilated under the action of \( U_+ \) and \( V_+ \) and thus these actions take such vectors into a 0-dimensional weight space.

Further note that if the action of \( U_+ \) is repeatedly applied to a vector in a weight space of dimension \( t \) then one of the following occurs:

**Either** The dimensions of the subsequent containing weight spaces increase by 1 with each application of \( U_+ \), then stabilise, and then decrease in increments of 1 until they reach 0.

**Or** The dimensions of the subsequent containing weight spaces remain the same with each application of \( U_+ \) and then decrease in increments of 1 until they reach 0.

**Or** The dimensions of the subsequent containing weight spaces decrease in increments of 1 with each application of \( U_+ \) until they reach 0.

Exactly the same statement holds with \( V_+ \) replacing \( U_+ \). It is the dimensions of the containing weight spaces under repeated application of \( U_+ \) followed by repeated application of \( V_+ \) that are of interest for generalising the method of section 3.7.2. By lemma 3.8.2, it is not possible for the containing weight spaces to decrease under the action of \( U_+ \) and then increase under subsequent applications of \( V_+ \).

Consider \( \mathcal{R}_1^t \). Suppose there are \( k \) such coefficients in the first row. Note that \( N \geq k \geq n \). These \( k \) coefficients determine the whole highest weight vector by lemma 4.3.2 and it is sensible to ask how these \( k \) coefficients are related to the \( \mathcal{R} \)-coefficients of the form \( \mathcal{R}_2^t \) (i.e. the \( \mathcal{R} \)-coefficients on the next row down). Notice that \( \mathcal{R}_1^t \) is attached to the tensor product basis element \(|1, \frac{p}{2}, \frac{p}{2}\rangle \otimes |N, \frac{r_N}{2}, \frac{r_N-2a}{2}\rangle \) where \( \frac{r_N}{2} \) is the highest weight of the largest \( su(2)_T \)-irrep on row \( N \) of the irrep \( R \). Under the action of \( U_+ \) this raises to a linear combination containing a multiple of \(|1, \frac{p}{2}, \frac{p}{2}\rangle \otimes |N-1, \frac{r_N-1}{2}, \frac{r_N-1-2a}{2}\rangle \). The other two vectors that also raise under the action of \( U_+ \) to linear combinations containing scalar multiples of this vector.
are $|1, \frac{p}{2}, \frac{p}{2} \rangle \otimes |N, \frac{rN-2}{2}, \frac{rN-2a}{2} \rangle$ and $|2, \frac{p+1}{2}, \frac{p+1}{2} \rangle \otimes |N-1, \frac{rN-1}{2}, \frac{rN-1-2a}{2} \rangle$. And so in a highest weight space, $R^1_1, R^2_1$ and $R^1_2$ are linearly dependent. Continuing this process of examining the relationships between the $R$-coefficients by applying $U_+$ or $V_+$ to the attached tensor product basis vectors yields all the necessary results. Now fix $i$ and let there be $j$ $R$-coefficients on row $i$: $R^1_i, \ldots, R^j_i$. Suppose such coefficients are attached to tensor product basis elements $|p \rangle \otimes |r \rangle$. Suppose also that row $i+1$ of $P$ contains a $su(2)_T$-irrep of dimension one larger than the largest such on row $i$. Then there are three possibilities to consider depending on whether the action of $U_+$ sends the $|r \rangle$ to a weight space in $R$ of dimension one greater than, the same as or one less than, the dimension of the weight space in $R$ containing $|r \rangle$.

**Case I:** Suppose the action of $U_+$ on the weight space in $R$ containing $|r \rangle$ sends these vectors to a weight space of greater dimension. Then the $R$-coefficients on row $i$ and $i+1$ are related as follows:

- $R^1_{i+1}$ is a scalar multiple of $R^1_i$
- $R^k_{i+1}$ is a linear combination of $R^{k-1}_i$ and $R^k_i$ for $k = 2, \ldots, j$
- $R^j_{i+1}$ is a scalar multiple of $R^j_i$

**Case II:** Suppose the action of $U_+$ on the weight space in $R$ containing $|r \rangle$ sends these vectors to a weight space of equal (non-zero) dimension. Then the $R$-coefficients on row $i$ and row $i+1$ are related as follows:

- $R^k_{i+1}$ is a linear combination of $R^k_i$ and $R^{k+1}_i$ for $k = 1, \ldots, j - 1$
- $R^j_{i+1}$ is a scalar multiple of $R^j_i$

**Case III:** Suppose the action of $U_+$ on the weight space in $R$ containing $|r \rangle$ sends these vectors to a weight space of lesser (non-zero) dimension. Then the $R$-coefficients on row $i$ and row $i+1$ are related as follows:

- $R^k_{i+1}$ is a linear combination of $R^k_i$ and $R^{k+1}_i$ for $k = 1, \ldots, j - 1$
These cases are all established by similar reasoning to the case of $R_1^1$, $R_2^2$ and $R_1^1$; this involves repeated application of lemma 3.5.2 which indicates which C.G. coefficients are related. Note that the non-zero condition is unimportant in case III, since the dimension of adjacent weight spaces decreases by one; the important result being that the dimension of decreasing weight spaces passes through 1 at some point. There are equivalent possibilities to consider in the case that it is the action of $V_\pm$ that relates the $R$ coefficients on adjacent rows. The full result is included here as some of the indices do change. Such a case is established under the assumption that row $i$ of $P$ contains a $su(2)_T$-irrep of dimension one larger than the largest such on row $i + 1$ of $P$. Then, as with the action of $U_+$, there are three possibilities to consider depending on whether the action of $V_+$ sends the $|r_i\rangle$ to a weight space in $R$ of dimension one greater than, the same as or one less than, the dimension of the weight space in $R$ containing the $|r_i\rangle$.

**Case I:** Suppose the action of $V_+$ on the weight space in $R$ containing $|r_i\rangle$ sends such vectors to a weight space of greater dimension. Then the $R$-coefficients on rows $i$ and $i + 1$ are related as follows:

- $R_{i+1}^1$ is a scalar multiple of $R_i^1$
- $R_{i+1}^k$ is a linear combination of $R_i^{k-1}$ and $R_i^k$ for $k = 2, \ldots, j$
- $R_{i+1}^j$ is a scalar multiple of $R_i^j$

**Case II:** Suppose the action of $V_+$ on the weight space in $R$ containing $|r_i\rangle$ sends such vectors to a weight space of equal dimension. Then the $R$-coefficients on rows $i$ and $i + 1$ are related as follows:

- $R_{i+1}^1$ is a scalar multiple of $R_i^1$
- $R_{i+1}^k$ is a linear combination of $R_i^{k-1}$ and $R_i^k$ for $k = 2, \ldots, j$

**Case III:** Suppose the action of $V_+$ on the weight space in $R$ containing $|r_i\rangle$ sends such vectors to a weight space of lesser dimension. Then the $R$-coefficients on rows $i$ and $i + 1$ are related as follows:

- $R_{i+1}^k$ is a linear combination of $R_i^k$ and $R_i^{k+1}$ for $k = 1, \ldots, j - 1$
3.8.2 The method in the degenerate case

Having established how the $\mathcal{R}$-coefficients on adjacent rows are related, it is now possible to generalise the method of page 50 to the degenerate case. But first some notation and lemmas, that will aid the proof of such a method, are introduced. Fix $i$ and let $m_i = |\{\mathcal{R}_i^k\}|$ where $|$ denotes the cardinality of the set. Still with $i$ fixed, let $V_i$ be the vector subspace of the highest weight vector space containing only highest weight vectors for which $\mathcal{R}_i^k = 0$ for each $k$.

Lemma 3.8.3 Suppose $1 + m_i = m_{i+1}$, then $V_i = V_{i+1}$.

Proof: Suppose that $\mathcal{R}_i^1, \ldots, \mathcal{R}_i^{m_i}$ are all zero. Then by lemma 3.5.3, $\mathcal{R}_{i+1}^1$ are all linear combinations of the $\mathcal{R}_i^1$, which are all zero. So $V_i \subseteq V_{i+1}$. Conversely suppose the $\mathcal{R}_{i+1}^1$ are all zero. Then by examining the case I’s in section 3.8.1, $\mathcal{R}_i^1$ is a linear multiple of $\mathcal{R}_{i+1}^1$ and so is zero. But again by the case I’s in section 3.8.1 $\mathcal{R}_i^2$ is a linear combination of $\mathcal{R}_{i+1}^2$ and $\mathcal{R}_i^1$; both of which are zero. Continuing iteratively establishes that $V_{i+1} \subseteq V_i$. Both inclusions give the required result.

Lemma 3.8.4 Suppose $m_i = m_{i+1}$, then $V_i = V_{i+1}$.

Proof: This result is established in exactly the same way as lemma 3.8.3. The proof is included here anyway with the appropriate changes to the indices. Suppose that $\mathcal{R}_i^1, \ldots, \mathcal{R}_i^{m_i}$ are all zero. Then by lemma 3.5.3, $\mathcal{R}_{i+1}^1$ are all linear combinations of the $\mathcal{R}_i^1$, which are all zero. So $V_i \subseteq V_{i+1}$. Conversely suppose the $\mathcal{R}_{i+1}^1$ are all zero. Then by the case II’s in section 3.8.1, $\mathcal{R}_i^{m_i}$ is a linear multiple of $\mathcal{R}_{i+1}^{m_i}$ and so is zero. But again by the case II’s in section 3.8.1, $\mathcal{R}_i^{m_i-1}$ is a linear combination of $\mathcal{R}_{i+1}^{m_i-1}$ and $\mathcal{R}_i^{m_i}$; both of which are zero. Continuing iteratively establishes that $V_{i+1} \subseteq V_i$. Both inclusions give the required result.

Lemma 3.8.5 Suppose $m_i - 1 = m_{i+1}$ then $V_i \subseteq V_{i+1}$. Further $V_i = V_{i+1}$ or $1 + \dim V_i = \dim V_{i+1}$

Proof: As in lemma 3.8.3 and 3.8.4, $V_i \subseteq V_{i+1}$ because $\mathcal{R}_i^1$ all being zero implies that $\mathcal{R}_{i+1}^1$ are all zero. Now if $V_i \neq V_{i+1}$ then we can extend a basis for $V_i$ to a basis for $V_{i+1}$ by
including a vector with $R_1^i \neq 0$ and the other $R_k^i$ determined by the relations obtained from setting $R_{i+1}^i = 0$.

Lemma 3.8.6 It is the case that $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_N$. Further $V_1 = 0$, $\dim V_N = n - 1$ or $n$ and either $\dim V_i = \dim V_{i+1}$ or $\dim V_i + 1 = \dim V_{i+1}$.

Proof: The nesting of the $V_i$ is clear from lemmas 3.8.3, 3.8.4 and 3.8.5. The only vector for which all $R_1^*$ are zero is the zero vector, because these coefficients determine the whole highest weight space by lemma 4.3.2. So $V_1 = 0$. Finally $V_N$ is determined by setting the single coefficient $R_N^1$ to zero. If this coefficient (and vector) is non-zero then $V_N$ is $(n - 1)$-dimensional and if it is necessarily zero (or does not exist) in the highest weight space then $V_N$ is the whole highest weight space.

It is now possible to state a generalisation of the method for decomposing highest weight spaces given on page 50 to the degenerate cases.

The method

Step 1: Now $V_1 = 0$ by lemma 3.8.6. By lemmas 3.8.3, 3.8.4 and 3.8.5, $V_2$ is either equal to $V_1$ or is a 1-dimensional space (containing $V_1$). If $V_2$ is 1-dimensional, choose a vector in $V_2$. If $V_2 = 0$ consider $V_3$ and so on. It must be the case that there exists a $V_1'$ that is 1-dimensional where $V_1' = V_m$ for some value of $m$. This is because $V_N$ is $n$ or $n - 1$ dimensional and the nested vector spaces increase by at most one dimension at a time. Choose any non-zero $v_1 \in V_1'$.

Step 2: Let $V_2'$ be two dimensional where $V_2' = V_l$ for some $l$. Such a $V_2'$ must exist by lemma 3.8.6 as the nested sequence of vector spaces increases by at most one dimension each time. Note also that $V_1' \subset V_2'$. Now choose the second highest weight vector, $v_2$, to be a vector in $V_2'$ that is orthogonal to $V_1'$. This is well-defined up to scalar multiplication.

Step k: Let $V_k'$ be $k$ dimensional where $V_k' = V_l$ for some $l$. Again the existence of such a $k$ is guaranteed by lemma 3.8.6. Note that $V_{k-1}' \subset V_k'$. Now choose the $k^{th}$ highest
weight vector to be a vector in $V'_k$ that is orthogonal to $V'_{k-1}$. This is well-defined up to scalar multiplication.

**Step n:** At the final stage we have constructed $V'_{n-1}$ and may choose the final vector to be in the $n$-dimensional highest weight space and orthogonal to $V'_{n-1}$. This is well-defined up to scalar multiplication.

**Proposition 3.8.7** The method described above gives rise to orthogonal highest weight vectors (defined up to scalar multiplication).

**Proof:** The method itself describes the results that it relies upon. The only outstanding issue is whether $V'_k$ is well-defined. This is indeed the case, because although there may be several values of $i$ for which $V_i$ is of the correct dimension, lemma 3.8.6 with lemma 3.8.2 show that these vector spaces are all the same.

3.8.3 An example

Consider $M(1,2) \otimes M(2,2)$. It contains two copies of the irrep $M(1,2)$. The two dimensional highest weight space of weight $\lambda_1 + 2\lambda_2$ has HWCD as given in figure 3.12.

![Figure 3.12: The HWCD for (1,2) \otimes (2,2).](image_url)

The space of weight $\lambda_1 + 2\lambda_2$ is 17-dimensional. Let $\alpha_i$ with $i = 1, \ldots, 17$ be the C.G. coefficients with the usual ordering as given in section 3.2. Then in the notation established on page 56, it is the case that $R_1^1 = \alpha_1, R_1^2 = \alpha_2, R_1^3 = \alpha_3, R_2^1 = \alpha_6, R_2^2 = \alpha_7, R_3^1 = \alpha_{13}$. By applying the actions of $T_+$ and $U_+$ it is possible to establish the two dimensional highest weight space. Since $\alpha_1 = 0$ in the highest weight space we will record the coefficients in terms of $\alpha_2$ and $\alpha_3$.  

62
\[\begin{align*}
\alpha_1 &= 0, & \alpha_2, & \alpha_3, \\
\alpha_4 &= 0, & \alpha_5 &= -\sqrt{2} \alpha_2, & \alpha_6 &= \frac{-5}{3\sqrt{2}} \alpha_2, \\
\alpha_7 &= -\frac{5}{2} \left(\frac{1}{3} \alpha_2 + \alpha_3\right), & \alpha_8 &= \frac{5}{3} \alpha_2, & \alpha_9 &= \frac{5}{2\sqrt{2}} \left(\frac{1}{3} \alpha_2 + \alpha_3\right), \\
\alpha_{10} &= 0, & \alpha_{11} &= \frac{-5}{6\sqrt{2}} (\alpha_2 - \alpha_3), & \alpha_{12} &= \frac{-5}{\sqrt{6}} \alpha_2, \\
\alpha_{13} &= \frac{5}{\sqrt{2}} (\alpha_2 + 2\alpha_3), & \alpha_{14} &= \frac{-5}{3} (\alpha_2 + 2\alpha_3), & \alpha_{15} &= \frac{5}{6\sqrt{2}} (\alpha_2 - \alpha_3), \\
\alpha_{16} &= \frac{5}{3\sqrt{2}} (\alpha_2 + 2\alpha_3), & \alpha_{17} &= \frac{-5}{6} (-\alpha_2 + \alpha_3)
\end{align*}\]

The procedure for the degenerate case outlined on page 61 sets \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). This gives rise to the zero vector, so no vectors are determined. Moving onto the next level the method sets \(\alpha_6 = \alpha_7 = 0\) (i.e., \(\mathcal{R}_2^1\) and \(\mathcal{R}_2^2\) are set to zero). This still gives rise to only the zero vector and so determines no vector. Finally setting \(\alpha_{13} = 0\) gives rise to the relation that \(\alpha_2 = -2\alpha_3\) and this determines the following highest weight vector:

\[v_1 = (0, 2, -1, 0, -2\sqrt{2}, -\frac{5\sqrt{2}}{3}, \frac{5}{6}, \frac{10}{3}, -\frac{5}{6\sqrt{2}}, 0, -\frac{5}{2\sqrt{2}}, -\frac{5\sqrt{2}}{\sqrt{3}}, 0, 0, -\frac{5}{2\sqrt{2}}, 0, 2)\]

The second and final vector is now prescribed as a highest weight vector orthogonal to \(v_1\). Setting to zero the dot product of an arbitrary highest weight vector with \(v_1\) gives rise to the condition \(\alpha_3 = 3\frac{22}{199} \alpha_2\). It is not illuminating to write out \(v_2\), but it is worth noting that in this example, all the coefficients continue to be rational square roots.

### 3.9 Orthogonality and basis independence

Much of the discussion on basis independence given in section 2.9.1 in the context of the tensor product of three copies of \(su(2)\) applies in this case. Given section 2.9.1, this section is straightforward and the notation of the last section lends itself towards the results compiled here; however the results are worth stating explicitly. Since the method for decomposing multi-dimensional highest weight spaces in section 3.8.2 gives orthogonal highest weight vectors, it will not give rise to the same vectors if a different basis is chosen at the beginning. However there does exist a nested sequence of vector spaces that are basis invariant.

**Lemma 3.9.1** Choose any bases for \(P\) and \(R\) and consider a highest weight space of dimension \(n\) in \(P \otimes R\). Then the method outlined in section 3.8.2 establishes \(n\) orthogonal highest
weight vectors \( v_1, v_2, \ldots, v_n \). Further, it also establishes a nested sequence of basis independent vector spaces, \( V'_1 \subset V'_2 \subset \ldots \subset V'_{n-1} \subset V'_n \) such that \( v_i \in V'_i \) and \( V'_i \) is \( i \) dimensional.

**Proof:** Since a change of basis preserves weight spaces in both irreps, the vector spaces \( V'_i \) remain unchanged and the necessary properties also remain intact. So the method given in section 3.8.2 gives vector spaces of the correct dimension that are basis independent.

\[ \blacksquare \]

**Corollary 3.9.2** Choose any bases for \( P \) and \( R \). Then the first vector constructed using the method in section 3.8.2, is basis independent (up to scalar multiplication).

**Proof:** The subspace \( V'_1 \) is basis independent and one dimensional and \( v_1 \in V'_1 \).

\[ \blacksquare \]

### 3.10 Uniqueness of the method

The method given in section 3.8.2 decomposes a highest weight space in the tensor product of \( P \otimes R \) by considering the coefficients attached to vectors on the right hand side of \( P_{HWCD} \). Examining the relationship between these coefficients gives rise to the method. It would have been possible to consider exactly the same construction, but using the coefficients attached to vectors on the top row of \( P_{HWCD} \). Examining the relationship between these coefficients would also give rise to a similar method for decomposing a multi-dimensional highest weight space. This is a distinct method in the sense that it would give rise to different highest weight vectors and even a different sequence of nested vector spaces. It would seem sensible to use an \( su(2)_U \)-basis whilst performing this alternative method. Indeed doing so, it is evident that both methods are essentially the same (i.e. applying the method outlined in section 3.8.2 to the irrep \((p,q)\) gives the same nested sequence of vector spaces up to a transformation of the Weyl group as applying the method along the top row of \( P_{HWCD} \) to the irrep \((q,p)\)). This is unavoidable due to the symmetry of \( su(3) \) and in particular due to the fact that the highest weight vector of an irrep of \( su(3) \) is also the highest weight vector of two ‘edge’ \( su(2) \)-irreps.
3.11 Summary

This chapter has extended the method for decomposing multi-dimensional highest weight spaces in the triple tensor product of irreps of $su(2)$ given in chapter 2, to the tensor product of two irreps of $su(3)$. The method used was surprisingly analogous with the concept of level (definition 2.8.1) being fulfilled by the C.G. coefficients of the form $\mathcal{R}_k^*$. The properties of orthogonality and a basis invariant nested sequence of vector spaces were exhibited. The RSR-property was not proved, but suggested by an general example. Finally the method was extended to apply to the degenerate case. It is hoped that this method of decomposition may extend to the tensor product of irreps of $su(4)$ and eventually all $su(n)$. Such a generalisation would use an ‘edge’ $su(2)$-irrep of $su(n)$, and as described in the last section it might be expected that $su(n)$ would have $n - 1$ equivalent but different decompositions.
Chapter 4

A basis independent approach

4.1 A brief overview

In the work done so far, highest weight spaces have been decomposed into orthogonal vectors that are contained in a sequence of basis-independent nested vector spaces. In this chapter the property of orthogonality is abandoned for a method that gives rise to basis-independent vectors. In fact the method, that will be established in the next few sections, produces highest weight vectors that are basis independent and the basic coefficients of the vectors are seen to be ‘orthogonal’. This new method is introduced by considering the example of $(1,1) \otimes (2,1)$. Then the general method, that applies to the irreps of any semisimple Lie algebra over $\mathbb{C}$, is explained and some basic results discussed. This is followed by a more detailed investigation of the method applied to the tensor product of two irreps of $su(3)$. Results for the non-degenerate two-dimensional highest weight spaces are exhibited in this case. Finally some of the difficulties of extending the method to the degenerate cases are discussed with examples.

4.2 An example of a different approach on $(1,1) \otimes (2,1)$

Recall that in section 3.4 on page 39 the two dimensional highest weight space was explicitly calculated for the irrep $(2,1)$ in $(1,1) \otimes (2,1)$. Recall that the weight space of weight $2\lambda_1 + \lambda_2$ in the tensor product is six dimensional. Furthermore, following the notation of section 3.4, it is spanned by the following set, $U$, of six elements of the tensor product basis.
\[ \mathcal{U} = \{ |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |2, -\frac{1}{2} \rangle, |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |2, \frac{1}{2}, \frac{1}{2} \rangle, |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |2, \frac{3}{2}, \frac{3}{2} \rangle, |2, 1, 1 \rangle \otimes |1, 1, 0 \rangle, |2, 1, 0 \rangle \otimes |1, 1, 0 \rangle, |2, 0, 0 \rangle \otimes |1, 1, 1 \rangle \} \]

It was established that the six dimensional weight space contained a two dimensional highest weight space whose C.G. coefficients satisfied the following relations, (using the usual ordering and notation from section 3.2).

\[
\begin{align*}
\alpha_1 & = -\sqrt{3} \alpha_3 \\
\alpha_2 & = -\frac{\sqrt{2}}{\sqrt{3}} \alpha_4 - \frac{2}{\sqrt{3}} \alpha_5 \\
\alpha_3 & = \frac{\sqrt{2}}{\sqrt{3}} \alpha_4 + \frac{2}{\sqrt{3}} \alpha_5 \\
\alpha_4 & = \frac{2\sqrt{2}}{3\sqrt{3}} \alpha_6 - \frac{2}{3\sqrt{3}} \alpha_6
\end{align*}
\]

Given this two dimensional highest weight space, any two linearly independent choices of \((\alpha_1, \alpha_2)\) determine two distinct highest weight vectors and thus two distinct copies of \((2, 1)\) in the direct sum. As previously, the aim is to develop a method that will determine these values. Notice that \(\alpha_1\) and \(\alpha_2\) are the coefficients of (all) the elements of \(\mathcal{U}\) which contain the highest weight of the irrep \((1, 1)\). Similarly, \(\alpha_5\) and \(\alpha_6\), which are (all) the coefficients of the elements in \(\mathcal{U}\) which contain the highest weight of the irrep \((2, 1)\), determine \(\alpha_i\) for \(i = 1, \ldots, 6\). So any two linearly independent choices of \((\alpha_5, \alpha_6)\) also give rise to two distinct highest weight vectors. Note that this is not true for any two arbitrary \(\alpha_i, \alpha_j\). Both \{\(\alpha_1, \alpha_3\)\} and \{\(\alpha_4, \alpha_5\)\} fail to determine an entire highest weight vector.

Consider the abstract 2-dimensional linear map, denoted by \(A\), that maps \(\alpha_1\) to \(\alpha_5\) and which also maps \(\alpha_2\) to \(\alpha_6\). That is, \(A\) is the linear map from the coefficients of elements of \(\mathcal{U}\) which contain the highest weight of the first irrep to the coefficients of elements in \(\mathcal{U}\) which contain the highest weight of the second irrep. Explicitly:
\[ A(\alpha_1) = \frac{\sqrt{2}}{\sqrt{3}} \alpha_1 + \frac{2}{\sqrt{3}} \alpha_2 \quad (= \alpha_5) \]
\[ A(\alpha_2) = \frac{2\sqrt{2}}{3\sqrt{3}} \alpha_1 - \frac{2}{3\sqrt{3}} \alpha_2 \quad (= \alpha_6) \]

As a matrix with respect to the basis \( \{\alpha_1, \alpha_2\} \), \( A \) may be written as the matrix in equation (4.1).

\[
A = \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & \frac{2\sqrt{2}}{3\sqrt{3}} \\
\frac{2}{\sqrt{3}} & -\frac{2}{3\sqrt{3}}
\end{pmatrix}
\tag{4.1}
\]

One possibility is to hope that \( A \) has distinct real eigenvalues; this will ensure that the associated eigenvectors are uniquely determined (up to scalar multiplication). For an eigenvector, \( (x, y)^T \), put \( x \) as the value of \( \alpha_1 \) and \( y \) as the value of \( \alpha_2 \) to determine a highest weight vector (and thus a particular copy of the desired irrep).

In this example, \( \det(xI - A) = x^2 + \frac{2 - 3\sqrt{2}}{3\sqrt{3}} x - \frac{2\sqrt{2}}{3} \) and using the quadratic formula this characteristic polynomial has roots:

\[
x = \frac{3\sqrt{2} - 2}{6\sqrt{3}} \pm \sqrt{\frac{22 + 60\sqrt{2}}{108}}
\]

This is already starting to get complicated and does not lead to easy-to-use Clebsch-Gordan coefficients. However, when \( A \) was being defined as the linear map for which \( A(\alpha_1) = \alpha_5 \) and \( A(\alpha_2) = \alpha_6 \), it would have made equal sense to have defined \( A \) by insisting \( A(\alpha_1) = \alpha_6 \) and \( A(\alpha_2) = \alpha_5 \). The reason for the initial choice was merely the lexicographic ordering; a different choice (and as the dimension of the highest weight space increases there will be many different choices) would have led to a different map, \( A' \), with different eigenvalues and different eigenvectors. To solve this ambiguity, consider instead the map \( A^T A \), where \( A^T \) is the transpose of \( A \). Again, the aim is to find the eigenvectors of \( A^T A \), which are uniquely determined (up to scalar multiplication) if the roots of the characteristic polynomial are distinct. Notice that since \( A^T A \) is a real symmetric matrix the eigenvalues are real and there is an orthogonal set of eigenvectors (although this does not mean that the corresponding highest weight vectors are orthogonal).

In the example, with \( A \) as in equation (4.1), the matrix \( A^T A \) actually turns out to be diagonal.
Thus the eigenvectors \((1, 0)^T\) and \((0, 1)^T\) of \(A^T A\) give rise to the following highest weight vectors.

\[
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (3\sqrt{3}, 0, -9, -3\sqrt{2}, 3\sqrt{2}, 2\sqrt{2})
\]

\[
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (0, 3\sqrt{3}, 0, -6, 6, -2)
\]

The next section will examine the general case, as well as prove some necessary results for the construction to work. After this the two-dimensional non-degenerate case will be calculated explicitly, and it will be evident that \(A^T A\) is not usually diagonal.

### 4.3 The general case

#### 4.3.1 Some background results

**Definition 4.3.1** Let \(|P\rangle\) be the highest weight vector for an irrep, \(P\), of some semisimple Lie algebra, \(L\). Also let \(|R\rangle\) be the highest weight vector for an irrep, \(R\), of \(L\). Consider \(P \otimes R\). In a highest weight space, \(\Lambda\), of weight \(\lambda\) the vectors \(|P\rangle \otimes |\rangle\) will be called \(P\)-basic vectors and the corresponding Clebsch-Gordan coefficient will be called a \(P\)-basic coefficient. Similarly let vectors of the form \(\langle \rangle \otimes |R\rangle\) be called \(R\)-basic vectors and the corresponding C.G. coefficients will be called \(R\)-basic coefficients.

Given the detailed example outlined in section 4.2, it may well be clear that the intention is to construct the linear map from the \(P\)-basic coefficients to the \(R\)-basic coefficients. The matrix of such a map will be multiplied by its own transpose and the task is to find the eigenvectors of this matrix (which are uniquely determined up to scalar multiplication if the eigenvalues are distinct). These eigenvectors will thus determine highest weight vectors and hence also well-defined copies of irreps of a particular highest weight. Before beginning to do this it is necessary to prove that the basic coefficients of the correct weight do indeed determine the entire highest weight space. This is indeed the case. Moreover it is the case for any semisimple Lie algebra; thus the following lemma is stated in full generality.
Lemma 4.3.2 Let $P$ and $R$ be two finite dimensional irreps of a semisimple Lie algebra, $L$. Let $|P\rangle$ and $|R\rangle$ denote the highest weight vectors of $P$ and $R$ respectively. Consider $P \otimes R$. Each highest weight space is determined by the $P$-basic coefficients.

Proof: Let $T^1_+, T^2_+, \ldots, T^n_+$ be a basis of raising operators for $L$. Suppose that the $P$-basic coefficients are all zero. Let $V_\lambda$ be a weight space of $P$ such that all vectors of the tensor product basis with higher $P$-component have only zero coefficients. Let $|v_1\rangle, \ldots, |v_k\rangle$ be a basis for $V_\lambda$ and let $|p_i\rangle$ be of weight $\lambda + \lambda_i$. Then the only tensor product basis elements that raise to $|p_i\rangle \otimes |r\rangle$ under the action of $T^i_+$ are of the form $|v_i\rangle \otimes |r\rangle$ or $|p_i\rangle \otimes |*\rangle$ where $*$ is some element of $R$. Now the coefficient of any vector with $|p_i\rangle$ in the first component is zero by hypothesis. Consequently, the action of $T^i_+$ forces some linear combination of the coefficients of $|v_i\rangle$ to zero. Moreover, the only solution for the system of homogeneous equations given by the action of $T^i_+$ for $i = 1, \ldots, n$ is the zero solution. To show this, let $w \in V_\lambda$ have coefficients that satisfy such a system of equations. Then $T^i_+ w = 0$ for $i = 1, \ldots, n$. But $w$ is not a highest weight vector for $P$ and so $w = 0$. This suffices to establish the proposition.

Corollary 4.3.3 The $R$-basic coefficients also determine the highest weight space.

Proof: This is proved in exactly the same way as lemma 4.3.2 or can be considered exactly lemma 4.3.2 given the symmetry on the tensor product.

Definition 4.3.4 A highest weight space of weight $\lambda$ and of dimension $n$ in $P \otimes R$ is said to be non-degenerate (for the purposes of this chapter) if there are exactly $n$ $P$-basic vectors of weight $\lambda$ and exactly $n$ $R$-basic vectors of weight $\lambda$.

Now let a non-degenerate highest weight space have weight $\lambda$ and dimension $n$. Write $\alpha_i$ ($i = 1, \ldots, n$) for the $P$ basic coefficients and $\beta_i$ ($i = 1, \ldots, n$) for the $R$-basic coefficients. Then by lemma 4.3.2 and corollary 4.3.3 both $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ determine highest weight vectors. So it is possible to define a linear map:

$$A(\alpha_i) = \beta_i = \sum_{j=1}^n a_j \alpha_j \quad (4.2)$$

The map $A$ is not well defined (unless an ordering is fixed) and so instead it is preferable to consider the well defined map $A^T A$ where $A^T$ is the transpose of $A$. (Suppose two different
maps $A$ and $B$ say, are defined as above on the same coefficients with different orderings. The bases may be ordered so that $A$ is the identity matrix and $A^T A = I$. Then $B$ would be a permutation matrix and as such $B^T B = I$ also. So $A^T A = B^T B$ is well defined). Note that it would make equal sense to use $A A^T$ as a well defined map with eigenvectors that determine highest weight vectors. Unfortunately $A A^T$ and $A^T A$ give rise to different sets of highest weight vectors and so it is necessary to choose one of these maps.

Now $A^T A$ is a real symmetric matrix and so has $n$ real orthogonal eigenvectors. Each eigenvector determines the $P$-basic coefficients and so determines a highest weight vector. Thus if the eigenvalues are distinct, then the eigenvectors provide a well-defined (up to scalar multiplication) way to decompose the highest weight space. Note that the eigenvectors being orthogonal means that the $P$-basic coefficients are orthogonal and not, unfortunately, the highest weight vectors themselves. The next three lemmas are straightforward observations that apply to the finite dimensional irreps of any semisimple Lie algebra over $\mathbb{C}$.

**Lemma 4.3.5** $A$ and $A^T A$ are invertible. In particular, 0 is not an eigenvalue of $A$ or $A^T A$.

**Proof:** $A$ is a linear map from an $n$-dimensional space onto an $n$-dimensional space and must therefore be invertible. So $A^T A$ is also invertible as the product of invertible matrices.

**Lemma 4.3.6** Let $A$ be constructed as above for a non-degenerate highest weight space of weight $\lambda$ in $P \otimes R$. Let $B$ be constructed as above (using the same ordering on the tensor product basis as $A$) for a non-degenerate highest weight space of weight $\lambda$ in $R \otimes P$. Then $A$ has distinct eigenvalues if and only if $B$ had distinct eigenvalues.

**Proof:** Using the notation in equation 4.2 it is possible to explicitly construct the inverse of $A$ as $A^{-1}(\beta_i) = \alpha_i$. Thus if the same ordering has been used $B = A^{-1}$ and $B$ has reciprocal eigenvalues to those of $A$.

**Corollary 4.3.7** If $P = R$, then $A^2 = I_n$. Moreover, $A$ is diagonalisable.

**Proof:** If $P = R$, then by lemma 4.3.6, $A = A^{-1}$ and so $A^2 = I_n$. Thus $A$ satisfies a minimum polynomial with linear factors and is therefore diagonalisable.
4.4 An algorithm for constructing $A$ for a non-degenerate $su(3)$ highest weight space

The discussion in the last section described a very different way to select highest weight vectors in a multi-dimensional highest weight space in a tensor product of two irreps of (the same) semisimple Lie algebra. This section considers in more detail the specific case of $su(3)$. Before trying to find the eigenvectors of $A^T A$, $A$ must first be calculated. Fortunately it is possible to write the $R$-basic coefficients as a linear combination of $P$-basic coefficients without calculating all the coefficients in the highest weight space. This section will outline an algorithm for obtaining $A$. It seems that the algorithm outlined is the most efficient such algorithm as it uses all the relations given in lemma 3.5.2 that contain three coefficients (and none of the relations that contain four coefficients).

The notation established in section 3.5 will continue to be used. In this notation the algorithm aims to write $L_j$ for each $j = 1, \ldots, n$ as a linear combination of $R_{j1}$ with $j = 1, \ldots, n$. Consider $L_1$. This may be written as a scalar multiple of $R_{1n}$ by lemma 3.5.3 and in turn $R_{1n}$ can be written as a linear combination of $R_{1j}$ with $j = 1, \ldots, n$ by repeated application of lemma 3.5.3. Pictorially, it is possible to visualise these connecting relations as moving through the highest weight contributing diagram following the arrows in figure 4.1. The coefficients that are related 'along' the rows are related by scalar multiplication, by lemma 3.5.1. A coefficient attached to a vector on the right hand side of $P_{HWCD}$ is equal to a linear combination of two coefficients in the above row by lemma 3.5.3.

Now that the coefficient $L_1$ is in the required form, it is possible to consider $L_2$. By lemma 3.5.3 it is a linear combination of $L_1$ and $L_2$. Given that $L_1$ is known we need only find $L_2$. But by lemma 3.5.3, $L_2$ is a scalar multiple of $R_{1n-1}$ which in turn can be written as a linear combination of $R_{1j}$ with $j = 1, \ldots, n - 1$ again by lemma 3.5.3. This process is similarly demonstrated in figure 4.2. It might be helpful to imagine finding $L_2$ by superimposing figure 4.2 onto figure 4.1.

This process is iterated so that when $L_t$ is being considered, $L_1$ are already known for $j = 1, \ldots, t - 1$. Further, the construction of $L_1^m$ has also yielded all $L_y$ such that $y + z = m$. By plowing through the relations given in lemma 3.5.3 once more, it is evident that $L_1$ is a linear combination of $L_1^{t-1}$ and $L_2^{t-1}$. The first of these is known iteratively and, by lemma 3.5.3, the second is a linear combination of $L_2^{t-2}$ and $L_3^{t-1}$. Once again the first of these is known inductively and the second may be rewritten as a linear combination of $L_3^{t-2}$ and
\[ \sum_{j=1}^{n} a_j^1 R_j^1 \]
\[ \sum_{j=1}^{n-1} a_j^2 R_j^2 \]
\[ a_1^{n-1} R_{n-1}^1 + a_2^{n-1} R_{n-1}^2 \]
\[ L_1 \]
\[ \times \]
\[ R_{HWCD} \]
\[ b_1^{n-2} R_{n-2}^1 + b_2^{n-2} R_{n-2}^2 \]
\[ L_2 \]
\[ \times \]
\[ R_{HWCD} \]

Figure 4.1: A diagram of the chain of relations that enable \( L_1 \) to be written as a linear combination of the \( R_1^* \).

Figure 4.2: A diagram of the chain of relations that enable \( L_2 \) to be written as a linear combination of the \( R_1^* \).
This process continues until $L_t^1$ may be written as a linear combination of $L^*_1$ that have already been constructed and $L_t^1$. This final coefficient is a scalar multiple of $R_{n+1-t}^1$, which in turn may be written as a linear combination of $R_t^1$ in the usual manner. This step of the process is illustrated in figure 4.3. The proof of this algorithm is implicit in the structure and relies only on sorting through the relations in lemma 3.5.3.

\[
\sum_{j=1}^{n+1-t} c_j^1 R_j^1 \\
\sum_{j=1}^{n-t} c_j^2 R_j^2 \\
\vdots \\
\bigotimes_{i=1}^{(n+1-t)C} R_{\text{HWCD}} \\
\bigoplus_{i=1}^{(n-t)C} R_t^1 \bigoplus_{i=1}^{(n+1)C} R_{n+1-t}^1 \\
L_t^1 \\
\vdots
\]

Figure 4.3: A diagram of the chain of relations that enable $L_t^1$ to be written as a linear combination of the $R_t^1$.

### 4.5 A 2-dimensional non-degenerate case for $su(3)$

\[
\begin{array}{ccccccccc}
\bigotimes & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & |1, \frac{p}{2}, \frac{p}{2}\rangle \\
\bigotimes & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & |2, \frac{p+1}{2}, \frac{p+1}{2}\rangle \\
\bigotimes & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & |1, \frac{r_1}{2}, \frac{r_1}{2}\rangle \\
\bigotimes & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & |2, \frac{r_1+1}{2}, \frac{r_1+1}{2}\rangle
\end{array}
\]

Figure 4.4: The HWCD for a non-degenerate two-dimensional highest weight space.
Recall that the HWCD in figure 4.4 on page 74 was examined in section 3.6 and all the relations between coefficients were stated. Continuing to solve the relations in section 3.6 to find $L_1^1 (= \alpha_{4x+1})$ and $L_1^2 (= \alpha_{4x+2})$ in terms of $R_1^1 (= \alpha_1)$ and $R_1^2 (= \alpha_2)$ gives the following equations:

\[
L_1^1 = (-1)^{n+1} \frac{(r, \frac{r-2}{2}, \frac{r-2x}{2})}{(p+1, \frac{p-1}{2}, \frac{p+1-2x}{2})} \frac{\sqrt{p+2}}{\sqrt{p+1}} \frac{\gamma_{r+1}}{\gamma_{p+1}} \left( \frac{\sqrt{r+1-n}}{\sqrt{r+2}} R_1^1 + \frac{\sqrt{r+2} \sqrt{x}}{r} R_1^2 \right) \quad (4.3)
\]

\[
L_1^2 = (-1)^{x} \frac{(r, \frac{r-2}{2}, \frac{r-2x}{2})}{(p+1, \frac{p-1}{2}, \frac{p+1-2x}{2})} \frac{\gamma_{r+1}}{\gamma_{p+1}} \left( \frac{p}{\sqrt{x} \sqrt{p+2}} \left( \frac{\sqrt{p+1-x} \sqrt{r+1-x}}{\sqrt{p+1} \sqrt{r+2}} - \frac{(r+1) \sqrt{p+1}}{\sqrt{r+2} \sqrt{r+2-x} \sqrt{p+2-x}} \right) R_1^1 + \frac{p \sqrt{p+1-n} \sqrt{r+2}}{r \sqrt{p+1} \sqrt{p+2}} R_1^2 \right) \quad (4.4)
\]

Given that scalar multiplication does not affect the eigenvectors of a matrix, nor whether the eigenvalues are distinct, it will be convenient to divide the above equations by suitable numbers to get slightly simpler matrices.

Given these matrices, the method for constructing highest weight vectors produces immediate
results. When \( r \neq 2\sqrt{p + 1 - x} \), then \( A^T A \) is a symmetric \( 2 \times 2 \) matrix. Thus \( A^T A \), because it is not a scalar multiple of the identity, has distinct eigenvalues and the method described on page 71 applies. However, in general, this method does not give rise to C.G. coefficients that are easy to use or describe.

### 4.6 The degenerate case

Whilst this eigenvalue method seems to generalise easily to other semisimple Lie algebras, it is not so easy to see a well-defined way to generalise it to the degenerate cases. The purpose of this section is illustrate by examples some of the difficulties of generalising this method to the degenerate cases in the special case of \( su(3) \).

**Example 1:** Consider \((1,2)\) as a direct summand of \((1,2) \otimes (2,2)\). This is examined in detail on page 62. Recall that it has HWCD as shown in figure 4.5.

![Figure 4.5: The HWCD for \((1,2) \otimes (2,2)\).](image)

Notice that the following \( R \)-coefficients are attached to the following tensor product basis elements.

\[
\begin{align*}
R_1^1 & \quad |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |3, 2, 0 \rangle \\
R_1^2 & \quad |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |3, 1, 0 \rangle \\
R_1^3 & \quad |1, \frac{1}{2}, \frac{1}{2} \rangle \otimes |3, 0, 0 \rangle \\
R_2^1 & \quad |2, 1, 1 \rangle \otimes |2, \frac{1}{2}, \frac{1}{2} \rangle \\
R_2^2 & \quad |2, 1, 1 \rangle \otimes |2, \frac{1}{2}, -\frac{1}{2} \rangle \\
R_3^1 & \quad |3, \frac{3}{2}, \frac{3}{2} \rangle \otimes |1, 1, -1 \rangle
\end{align*}
\]

There are two well-defined \( R \)-basic coefficients that determine the two dimensional highest weight space; \( \alpha_{16} \) attached to \(|3, \frac{3}{2}, -\frac{1}{2} \rangle \otimes |1, 1, 1 \rangle \) and \( \alpha_{17} \) attached to \(|3, \frac{1}{2}, -\frac{1}{2} \rangle \otimes |1, 1, 1 \rangle \). It
is necessary to find two other coefficients that determine the whole weight space and it seems sensible to try and look amongst the $R$-coefficients. In the non-degenerate case it is the $P$-basic coefficients that are used; in this case there are three such coefficients. However in the highest weight space $R_1^1$ is necessarily 0. This is because $\alpha_1 = R_1^1$ is a linear multiple of $\alpha_4$, since these are the only two vectors that raise under the action of $T_+$ to $|1, \frac{1}{2}, \frac{1}{2}\rangle \otimes |3, 2, 1\rangle$. But $\alpha_4 = 0$ since it is the only vector that raises under the action of $T_+$ to $|1, \frac{1}{2}, \frac{1}{2}\rangle \otimes |3, 2, 2\rangle$. Moreover, $R_1^2$ and $R_1^3$ are independent as they determine the whole two dimensional highest weight space by lemma 4.3.2 (since $R_1^1 = 0$). The calculations on page 62 give:

$$\begin{align*}
\alpha_{16} &= \frac{5}{3\sqrt{2}} (\alpha_2 + \alpha_3) \\
\alpha_{17} &= \frac{5}{6} (\alpha_2 - \alpha_3)
\end{align*}$$

The linear transformation defined by $A(\alpha_2) = \alpha_{16}$ and $A(\alpha_3) = \alpha_{17}$ may be written as the matrix:

$$A = \begin{pmatrix}
\frac{5}{3\sqrt{2}} & \frac{5}{6} \\
\frac{5\sqrt{2}}{3} & -\frac{5}{6}
\end{pmatrix}$$

This gives the following well-defined real symmetric matrix.

$$A^T A = \begin{pmatrix}
125 & -25 \\
-25 & 25
\end{pmatrix} \begin{pmatrix}
18 & 18\sqrt{2} \\
18\sqrt{2} & 25
\end{pmatrix}$$

A quick calculation gives eigenvectors $(1, 2\sqrt{2} - 3)^T$ and $(1, 2\sqrt{2} + 3)^T$. Note that these C.G. coefficients are not rational square roots. In general it is clear that the existence of a surplus of basic vectors is not problematic when there are only $n$ non-zero basic coefficients in the highest weight space. Lemma 4.6.1 expands on the example above.

**Lemma 4.6.1** If $(n - 1, q) \otimes (r, s)$ has an $n$-dimensional highest weight space, then there are exactly $n$ non-zero $(n - 1, q)$-basic coefficients.

**Proof:** Since the $(n - 1, q)$-basic coefficients determine the $n$-dimensional highest weight space by lemma 4.3.2 at least $n$ of them must be non-zero. Suppose there are $N$ $(n - 1, q)$-basic coefficients. This means that in $(r, s)$ there are $N$ vectors of the appropriate weight space. Label these vectors $|k, a \frac{1}{2}, b\rangle, |k, a - 2 \frac{1}{2}, b\rangle, \ldots, |k, a - 2(N - 1) \frac{1}{2}, b\rangle$.

Now $|1, \frac{n-1}{2}, -\frac{n+1}{2}\rangle \otimes |k, a \frac{1}{2}, \frac{b+2n}{2}\rangle$ raises under the action of $T_+$ to a scalar multiple of $|1, \frac{n-1}{2}, -\frac{n+1}{2}\rangle \otimes |k, a \frac{1}{2}, \frac{b+2n}{2}\rangle$. Moreover it is the only vector to do so and must therefore
be zero in a highest weight space. Iterating the action of $T_+$ gives that the coefficient of $|1, \frac{n-1}{2}, \frac{n-1}{2}\rangle \otimes |k, \frac{a}{2}, \frac{b}{2}\rangle$ is zero. This argument can be repeated with all the vectors of the form $|1, \frac{n-1}{2}, \frac{n-1}{2}\rangle \otimes |k, \frac{a-2k}{2}, \frac{b}{2}\rangle$ for $k = 0, \ldots, 2n + 1$. In other words, the only $(n-1,q)$-basic vectors with non-zero coefficients are $|1, \frac{n-1}{2}, \frac{n-1}{2}\rangle \otimes |k, \frac{a-2(N+1-2j)}{2}, \frac{b}{2}\rangle$ with $j = 1, \ldots, n$.

So there are some cases where the method outlined on page 71 extends with ease. However there are many cases where it does not; one of which is considered next.

**Example 2:** Consider $(2, 1) \otimes (2, 2)$. It contains two copies of $(2, 1)$ for which it has the HWCD as given in figure 4.6

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure46.png}
\caption{The HWCD of $(2, 1) \otimes (2, 2)$ for $(1, 2)$.}
\end{figure}

Notice that the following $R$-coefficients are attached to the following tensor product basis elements:

\begin{align*}
R_1^1 & \quad |1, 1, 1\rangle \otimes |3, 2, 0\rangle \\
R_2^1 & \quad |1, 1, 1\rangle \otimes |3, 1, 0\rangle \\
R_3^1 & \quad |1, 1, 1\rangle \otimes |3, 0, 0\rangle \\
R_1^2 & \quad |2, \frac{3}{2}, \frac{3}{2}\rangle \otimes |2, \frac{3}{2}, \frac{1}{2}\rangle \\
R_2^2 & \quad |2, \frac{3}{2}, \frac{3}{2}\rangle \otimes |2, \frac{1}{2}, \frac{1}{2}\rangle \\
R_3^2 & \quad |3, 1, 1\rangle \otimes |1, 1, 1\rangle
\end{align*}

There are two well-defined $R$-basic coefficients that determine the two dimensional highest weight space; $\alpha_{16}$ attached to $|3, 1, 0\rangle \otimes |1, 1, 1\rangle$ and $\alpha_{17}$ attached to $|3, 0, 0\rangle \otimes |1, 1, 1\rangle$. Now it is necessary to find two other coefficients that determine the whole weight space and it is sensible to try to look amongst the $R$-coefficients. In the non-degenerate case it is the $P$-basic coefficients that are used; but here there are three non-zero such coefficients.
\( \alpha_1 + 5\alpha_2 + 10\alpha_3 = 0 \) is the only condition. How can we choose two of these or combine these three coefficients into two numbers? Alternatively consider the \( \mathcal{R} \)-coefficients that occur on the second row. Unfortunately, \( \mathcal{R}_2^1 \) and \( \mathcal{R}_2^2 \) are linear multiples of each other, and thus do not determine the entire highest weight space. It is evident that they are multiples of each other because under the action of \( U_+ \), these are the only two vectors that raise to \( |2, \frac{3}{2}, \frac{3}{2} \rangle \otimes |1, 1, -1 \rangle \). So there is no obvious choice from amongst the \( \mathcal{R} \)-coefficients for two well-defined coefficients that determine the highest weight space.

As mentioned in section 3.10, it is possible to decompose the highest weight space by the action of \( su(2)_T \) action of along the top row in a \( su(2)_U \)-basis (rather than the action of \( su(2)_U \) in a \( su(2)_T \)-basis). So instead of using the \( \mathcal{R} \)-coefficients that are attached to vectors in the tensor product on the right hand side of \( P_{HWCD} \), the tensor product would be decomposed using analogous \( T \)-coefficients attached to vectors situated along the top row of \( P_{HWCD} \). It may be the case that some method exists that uses either the \( \mathcal{R} \) or \( T \)-coefficients. Consider the example \((2, 1) \otimes (2, 2)\) where there is no set of two linearly independent \( \mathcal{R} \)-coefficients in the same row, there are however two linearly independent \( T \)-coefficients in the same \( U_\pm \)-column. An alternative approach would be to search for a basis such that all non-independent basic coefficients are zero.

### 4.7 Summary

This chapter has introduced a basis independent method for decomposing multi-dimensional highest weight spaces in the tensor product of two irreps of any semisimple Lie algebra. The method was constructed explicitly in the 2-dimensional non-degenerate cases for \( su(3) \) and a basis independent decomposition could be established. It would be interesting to establish similar such results for highest weight spaces of greater dimension. This chapter leaves many unanswered questions. Firstly what else can be established about the method in the case of irreps of \( su(3) \)? In particular might a specific basis enable the method to generalise to the degenerate case or give rise to simpler C.G. coefficients? What happens when this method is applied to the tensor product of irreps of a different semisimple Lie algebra?
Chapter 5

Summary and further work

5.1 Summary

The purpose of this thesis is to consider methods for decomposing multiple occurrences of irreps that appear in the tensor product of irreps of semisimple finite dimensional Lie algebras. The first case studied in depth was the tensor product of three irreps of $su(2)$. In such a case a method was presented that would ensure the resulting irreps were orthogonal and, in the unitary case, had C.G. coefficients that were rational square roots. The first vector produced by the method was basis independent and subsequent vectors were contained in a nested sequence of basis-independent vector spaces. The method also had the attractive property of applying equally well to the lowest weight space and still producing exactly the same irreps as when applied to the highest weight space. The method established the ‘nicest’ results when applied to a non-degenerate highest weight space, but the above statements were shown to hold equally well in the degenerate case.

The method from the first part is extended and the $su(2)$ action of $su(2)_U$ is exploited to decompose the highest weight space into orthogonal highest weight vectors. Unfortunately, due to the fact that $su(3)$ contains two ‘edge’ copies of $su(2)$, there are two possible decompositions that are not the same. The property of basis-independent vector spaces occurs again and it seems likely that, in the unitary case, the C.G. coefficients of the irreps in the decomposition are all rational square roots.

Finally, a rather different method is examined that constructs highest weight vectors that are basis independent, but not orthogonal, for any semisimple Lie algebra. The non-degenerate two-dimensional case was established for the tensor product of two irreps of $su(3)$. 

80
However no general results were proved for highest weight spaces of greater dimension, nor was it entirely clear how this method might extend to the degenerate case.

It is hoped that whilst there may be other possible decomposition than those investigated here, (and even ‘better’ decompositions for whatever reason) the work of this thesis demonstrates that some decompositions are better than others and that there is much work left to do on this topic.

5.2 Further work

There are many loose ends to tie up and this project seems to leave with more questions than it began (at least more precise ones!). It would be good to establish the result that the method for decomposing the tensor product of $su(3)$ described on page 50 does indeed always give irreps with C.G. coefficients that are the square roots of rational numbers. It would also be good to prove (or disprove) the result that this method gives rise to exactly the same irreps when applied to the lowest weight. Furthermore there are many leftover questions from the eigenvector method described on page 71. The two biggest questions being how to generalise the method to the degenerate case, as well as proving that the eigenvalues are necessarily distinct. There is also always the over-hanging question of other possible decompositions and how one might show a particular decomposition is the most desirable.

After this it seems sensible to begin looking at $su(4)$ and eventually $su(n)$. It is hoped that the initial method described on page 50 will generalise to $su(n)$. This seems reasonable as it is shown in this thesis how to extend the method from the tensor product of three copies of $su(2)$ to the tensor product of two copies of $su(3)$. A similar generalisation should work, although as with $su(3)$ having two ‘equivalent’ decompositions, $su(n)$ can be expected to have $n - 1$ ‘equivalent’ decompositions, each one corresponding to a different edge of the weight diagram that meets at the highest weight vertex. It would also be of interest to look at decomposing the tensor product of an arbitrary number of irreps of $su(2)$, using the decomposition of the tensor product of three irreps of $su(2)$ in chapter 2 to establish a result inductively.
Appendix A

Establishing the action of $U_+$ and $V_+$ on an irrep of $su(3)$.

Lemma A.0.1 The $U_+$ and $V_+$ action on the $T_3$-basis is described as follows.

$$
\gamma_b \delta_k U_+ |b, \frac{k}{2}, \frac{k-2a}{2}\rangle = \sqrt{\frac{a+1}{k+1}} |b+1, \frac{k+1}{2}, \frac{k-(2a+1)}{2}\rangle + \sqrt{\frac{k-a}{(k+1)}} |b+1, \frac{k-1}{2}, \frac{k-(2a+1)}{2}\rangle
$$

(A.1)

$$
\gamma_b \delta_k V_+ |b, \frac{k}{2}, \frac{k-2a}{2}\rangle = \sqrt{\frac{k+1-a}{\sqrt{k+1}}} |b+1, \frac{k+1}{2}, \frac{k+1-2a}{2}\rangle - \frac{\sqrt{a}}{\sqrt{k+1}} |b+1, \frac{k-1}{2}, \frac{k+1-2a}{2}\rangle
$$

(A.2)

where $\gamma_b$ may be chosen freely for each row $b$ and $\delta_k = \frac{\sqrt{k+1}}{\sqrt{k+1}'}$ where $k'$ is the highest $T_3$-weight of the largest $su(2)_T$-irrep in row $b$.

Given two $su(2)_T$-irreps in the same row of a weight diagram of an $su(3)$ irrep, how $V_+$ acts on the highest weight of one of the $su(2)_T$-irreps determines how $V_+$ acts on the highest weight of the other $su(2)_T$ irrep. This is the reason for the occurrence of the factor $\delta_k$ and is due to the fact that $U_+$ and $V_+$ satisfy equation (A.3).

$$[U_+, V_+] = U_+ V_+ - V_+ U_+ = 0$$

(A.3)

Proof of lemma A.0.1: First note that due to the universal property of the tensor product, the action of $U_+$ on $|b, \frac{k}{2}, \frac{l}{2}\rangle$ may be viewed as the tensor product of two $su(2)$-modules and
written $U_+ \otimes |b, \frac{k}{2}, \frac{1}{2}\rangle$. Likewise, a similar statement holds for $V_+$.

The base case for the action of $U_+$: The proof of lemma A.0.1 will proceed by induction; beginning with the base case. Choose

$$V_+ \otimes |b, \frac{k}{2}, \frac{k-2}{2}\rangle = |b + 1, \frac{k+1}{2}, \frac{k-1}{2}\rangle \quad (A.4)$$

Note that this expression is determined only up to scalar multiplication, and for convenience the scalar multiple is chosen to be one. If a different scalar multiple is desired in the above expression (e.g., if the entire irrep is to be unitary), all of the following formulas are multiplied by this same constant. This is taken into account in lemma A.0.1 with the constant $\gamma_b$.

Applying $T_-$ to both sides of equation (A.4) gives equation (A.5).

$$U_+ \otimes |b, \frac{k}{2}, \frac{k-2}{2}\rangle + \sqrt{k}V_+ \otimes |b, \frac{k}{2}, \frac{k-2}{2}\rangle = \sqrt{k+1} |b + 1, \frac{k+1}{2}, \frac{k-1}{2}\rangle \quad (A.5)$$

Now the vector on which $T_+$ vanishes is:

$$\sqrt{\frac{k}{k+1}} U_+ \otimes |b, \frac{k}{2}, \frac{k}{2}\rangle - \sqrt{\frac{1}{k+1}} V_+ \otimes |b, \frac{k}{2}, \frac{k-2}{2}\rangle = |b + 1, \frac{k-1}{2}, \frac{k-1}{2}\rangle \quad (A.6)$$

Simply multiplying both sides of equation (A.6) by $\sqrt{k+1}$ gives equation (A.7).

$$\sqrt{k}U_+ \otimes |b, \frac{k}{2}, \frac{k}{2}\rangle - V_+ \otimes |b, \frac{k}{2}, \frac{k-2}{2}\rangle = \sqrt{k+1} |b + 1, \frac{k-1}{2}, \frac{k-1}{2}\rangle \quad (A.7)$$

Adding $\sqrt{k}$ times equation (A.7) to equation (A.5) gives equation (A.8).

$$(k+1)U_+ \otimes |b, \frac{k}{2}, \frac{k}{2}\rangle = \sqrt{k+1} |a, \frac{k+1}{2}, \frac{k-1}{2}\rangle + \sqrt{k(k+1)} |b + 1, \frac{k-1}{2}, \frac{k-1}{2}\rangle \quad (A.8)$$

Dividing equation (A.8) by $k + 1$ gives equation (A.9).

$$U_+ \otimes |b, \frac{k}{2}, \frac{k}{2}\rangle = \sqrt{\frac{1}{k+1}} |a, \frac{k+1}{2}, \frac{k-1}{2}\rangle + \sqrt{\frac{k}{(k+1)}} |b + 1, \frac{k-1}{2}, \frac{k-1}{2}\rangle \quad (A.9)$$

This establishes the base case and it remains to prove the inductive step.
The inductive step for the action of $U_+$: The inductive hypothesis is given in equation (A.10).

$$U_+ \otimes |b, \frac{k-k-n}{2}\rangle = \sqrt{\frac{k-n+1}{k+1}}|b+1, \frac{k-1}{2}, \frac{k-(n+1)}{2}\rangle + \sqrt{\frac{k-n}{(k+1)}}|b+1, \frac{k-1}{2}, \frac{k-1}{2}\rangle$$

(A.10)

In order to get $U_+ \otimes |b, \frac{k-n-2}{2}\rangle$ from equation (A.10), $T_-$ is applied. This results in equation (A.11).

$$\sqrt{\frac{k}{2}-\frac{k-n}{2}}U_+ \otimes |b, \frac{k-n}{2}\rangle =$$

$$\sqrt{\frac{k+1}{k+1}}\sqrt{\frac{k+1}{2}}\frac{k+3}{2} - \frac{k-(n+1)}{2}\frac{k-(n+3)}{2}|b+1, \frac{k+1}{2}, \frac{k-(n+3)}{2}\rangle +$$

$$\sqrt{\frac{k-n}{(k+1)}}\sqrt{\frac{k-1}{2}}\frac{k+1}{2} - \frac{k-(n+1)}{2}\frac{k-(n+3)}{2}|b+1, \frac{k-1}{2}, \frac{k-(n+3)}{2}\rangle$$

(A.11)

Simplifying the coefficients of equation (A.11) gives equation (A.12).

$$\sqrt{(-\frac{n}{2}+k)(\frac{n}{2}+1)}U_+ \otimes |b, \frac{k-n-2}{2}\rangle =$$

$$\sqrt{\frac{k-n}{2}}\sqrt{\frac{n+2}{2}}(-\frac{n}{2}+k)|b+1, \frac{k+1}{2}, \frac{k-(n+3)}{2}\rangle +$$

$$\sqrt{\frac{k-n}{(k+1)}}\sqrt{\frac{n+1}{2}}(k-\frac{n}{2}-1)|b+1, \frac{k-1}{2}, \frac{k-(n+3)}{2}\rangle$$

(A.12)

And dividing equation(A.12) by $\sqrt{(-\frac{n}{2}+k)(\frac{n}{2}+1)}$ gives the required result.
\[ U_+ \otimes |b, \frac{k - n - 2}{2}\rangle = \sqrt{\frac{n + 2}{k + 1}} |b + 1, \frac{k + 1}{2}, \frac{k - (n + 3)}{2}\rangle + \sqrt{\frac{k - \frac{n}{2} - 1}{(k + 1)}} |b + 1, \frac{k - 1}{2}, \frac{k - (n + 3)}{2}\rangle \]  

(A.13)

This establishes the required result for the action of \( U_+ \). The action of \( U_+ \) will now be used in order to establish the action of \( V_+ \). The \( V_+ \) action will be proved by induction.

**The base case for the action of \( V_+ \):** The base case follows from combining (A.5) and (A.7) to make \( V_+ \otimes |b, \frac{k - 2}{2}\rangle \) the subject of the equation. This gives equation (A.14).

\[ V_+ \otimes |b, \frac{k - 2}{2}\rangle = \sqrt{\frac{k}{k + 1}} |b + 1, \frac{k + 1}{2}, \frac{k - 1}{2}\rangle - \frac{1}{\sqrt{k + 1}} |b + 1, \frac{k - 1}{2}, \frac{k - 1}{2}\rangle \]  

(A.14)

**The inductive step for the action of \( V_+ \):** The inductive step requires the inductive hypothesis given in equation (A.15).

\[ V_+ \otimes |b, \frac{k - n}{2}\rangle = \frac{\sqrt{k + 1} - \frac{n}{2}}{\sqrt{k + 1}} |b + 1, \frac{k + 1}{2}, \frac{k - (n - 1)}{2}\rangle - \frac{\sqrt{\frac{n}{2}}}{\sqrt{k + 1}} |b + 1, \frac{k - 1}{2}, \frac{k - (n - 1)}{2}\rangle \]  

(A.15)

Applying \( T_+ \) to equation (A.15) gives equation (A.16).

\[ U_+ \otimes |b, \frac{k - n}{2}\rangle + \sqrt{\frac{k k + 2}{2}} - \frac{k - n k - (n + 2)}{2} V_+ \otimes |b, \frac{k - (n + 2)}{2}\rangle = \]

\[ \frac{\sqrt{k + 1} - \frac{n}{2}}{\sqrt{k + 1}} \sqrt{\frac{k + 1}{2} \frac{k + 3}{2} - \frac{k - (n - 1) k - (n + 1)}{2}} |b + 1, \frac{k + 1}{2}, \frac{k - (n + 1)}{2}\rangle \]

\[ - \frac{\sqrt{\frac{n}{2}}}{\sqrt{k + 1}} \sqrt{\frac{k - k + 1}{2} - \frac{k - (n - 1) k - (n + 1)}{2}} |b + 1, \frac{k - 1}{2}, \frac{k - (n + 1)}{2}\rangle \]  

(A.16)

By using the \( U_+ \) action given in equation (A.13), and factoring the coefficients, equation
(A.16) may be rewritten as equation (A.17). Note that both sides have been multiplied by √k+1 to make the expression more palatable.

\[
\sqrt{\left(\frac{n}{2} + 1\right)}\left(\frac{-n}{2} + k\right)(k+1)V_+ \otimes |b, \frac{k - (n + 2)}{2}\rangle = \]

\[
\sqrt{\frac{n}{2} + 1}\left(\frac{-n}{2} + k\right)|b + 1, \frac{k + 1}{2}, \frac{k - (n + 1)}{2}\rangle - \left(\frac{n}{2} + 1\right)\sqrt{\frac{-n}{2}}|b + 1, \frac{k - 1}{2}, \frac{k - (n + 1)}{2}\rangle
\]

(A.17)

Now it is clear that this is exactly the desired expression for \( V_+ \otimes |b, \frac{k - (n + 2)}{2}\rangle \).

\[
V_+|b, \frac{k - (n + 2)}{2}\rangle = \sqrt{\frac{-n + k}{k + 1}}|b + 1, \frac{k + 1}{2}, \frac{k - (n + 1)}{2}\rangle - \sqrt{\frac{n}{2} + 1}|b + 1, \frac{k - 1}{2}, \frac{k - (n + 1)}{2}\rangle
\]

(A.18)

This establishes how \( U_+ \) and \( V_+ \) act on an \( su(2)_T \)-irrep in \( su(3) \) given it is known (or chosen) how \( V_+ \) acts on the highest weight \( |b, \frac{k}{2}, \frac{k}{2}\rangle \). However, because of the relation \( 0 = U_+ V_+ - V_+ U_+ \), given two \( su(2)_T \)-irreps in the same row of a weight diagram of an \( su(3) \) irrep, how \( V_+ \) acts on the highest weight of one of the \( su(2)_T \)-irreps determines how \( V_+ \) acts on the highest weight of the other \( su(2)_T \) irrep. This is stated precisely in lemma A.0.2, which together with the above working establish lemma A.0.1.

\[\boxed{\text{Lemma A.0.2}}\]

\[\text{Suppose that } V_+|b, \frac{k}{2}, \frac{k}{2}\rangle = \gamma_b \delta_k |b + 1, \frac{k + 1}{2}, \frac{k + 1}{2}\rangle. \text{ Then:} \]

\[
V_+|b, \frac{k - 2n}{2}, \frac{k - 2n}{2}\rangle = \frac{\sqrt{k + 1}}{\sqrt{k + 1 - 2n}}\gamma_b \delta_k |b + 1, \frac{k + 1 - 2n}{2}, \frac{k + 1 - 2n}{2}\rangle
\]

Proof: Again the result is established by induction and the base case is examined first.

The base case:

\[
0 = \left(U_+ V_+ - V_+ U_+\right)|b - 1, \frac{k - 1}{2}, \frac{k - 1}{2}\rangle
\]

\[
= \gamma_{b-1} \delta_{k-1} U_+|b, \frac{k}{2}, \frac{k}{2}\rangle - \gamma_{b-1} \delta_{k-1} V_+ \left(\frac{1}{\sqrt{k}}|b, \frac{k}{2}, \frac{k - 2}{2}\rangle + \frac{\sqrt{k - 1}}{\sqrt{k}}|b, \frac{k - 2}{2}, \frac{k - 2}{2}\rangle\right)
\]

(A.19)

Given that the factors of \( \gamma_{b-1} \) and \( \delta_k \) occur in each part of equation (A.19), it may be factored out. Continuing to rearrange equation (A.19) gives equation (A.20).
\[
0 = \gamma_b \delta_k \left( \frac{1}{\sqrt{k+1}} |b + 1, \frac{k+1}{2}, \frac{k-1}{2} \rangle + \frac{\sqrt{k}}{\sqrt{k+1}} |b + 1, \frac{k-1}{2}, \frac{k-1}{2} \rangle \right) \\
-\gamma_b \delta_k \left( \frac{1}{\sqrt{k+1}} |b + 1, \frac{k+1}{2}, \frac{k-1}{2} \rangle - \frac{1}{\sqrt{k}\sqrt{k+1}} |b + 1, \frac{k-1}{2}, \frac{k-1}{2} \rangle \right) \\
-\gamma_b \delta_{k-2} \frac{\sqrt{k-1}}{\sqrt{k}} |b + 1, \frac{k-1}{2}, \frac{k-1}{2} \rangle
\]

(A.20)

So the coefficient of \(|b + 1, \frac{k-1}{2}, \frac{k-1}{2} \rangle\) is zero and it may be rewritten as in equation (A.21).

\[
\gamma_b \delta_k \left( \frac{\sqrt{k}}{\sqrt{k+1}} + \frac{1}{\sqrt{k}\sqrt{k+1}} \right) = \frac{\sqrt{k-1}}{\sqrt{k}} \gamma_b \delta_k - 2 \quad (A.21)
\]

Equation (A.21) simplifies to give:

\[
\gamma_b \delta_k - 2 = \frac{\sqrt{k+1}}{\sqrt{k-1}} \gamma_b \delta_k
\]

This establishes the base case. The inductive step is identical and is included for the sake of completeness.

**The inductive step:** The inductive hypothesis for \(n - 1\) is given in equation (A.22).

\[
0 = \left( U_+ V_- V_+ U_+ \right) |b-1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2} \rangle \\
= \gamma_{b-1} \delta_{k+1-2n} U_+ |b, \frac{k+2-2n}{2}, \frac{k+2-2n}{2} \rangle \\
-\gamma_{b-1} \delta_{k+1-2n} V_+ \left( \frac{1}{\sqrt{k+2-2n}} |b, \frac{k+2-2n}{2}, \frac{k-2n}{2} \rangle + \frac{\sqrt{k+1-2n}}{\sqrt{k+2-2n}} |b, \frac{k-2n}{2}, \frac{k-2n}{2} \rangle \right)
\]

(A.22)

Given that the factor of \(\gamma_{b-1} \delta_{k+1-2n}\) occurs in each expression it may be factored out and ignored. Continuing to manipulate equation (A.22) gives equation (A.23).
\[
0 = \gamma_b \delta_{k+2-2n} \left( \frac{1}{\sqrt{k+3-2n}} |b+1, \frac{k+3-2n}{2}, \frac{k+1-2n}{2} \right) + \frac{\sqrt{k+2-2n}}{\sqrt{k+3-2n}} |b+1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2} \right) \]

\[
-\gamma_b \delta_{k+2-2n} \left( \frac{1}{\sqrt{k+3-2n}} |b+1, \frac{k+3-2n}{2}, \frac{k+1-2n}{2} \right) - \frac{1}{\sqrt{k+2-2n} \sqrt{k+3-2n}} |b+1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2} \right) \]

\[
-\gamma_b \delta_{k-2n} \frac{\sqrt{k+1-2n}}{\sqrt{k+2-2n}} |b+1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2} \right)\]

(A.23)

So the coefficient of \(|b+1, \frac{k+1-2n}{2}, \frac{k+1-2n}{2}\) is zero and it may be rewritten as equation (A.24)

\[
\gamma_b \delta_{k+2-2n} \left( \frac{\sqrt{k+2-2n}}{\sqrt{k+3-2n}} + \frac{1}{\sqrt{k+2-2n} \sqrt{k+3-2n}} \right) = \frac{\sqrt{k+1-2n}}{\sqrt{k+2-2n}} \gamma_b \delta_{k-2n} \quad (A.24)
\]

And equation (A.24) simplifies to give equation (A.25).

\[
\delta_{k-2n} = \frac{\sqrt{k+3-2n}}{\sqrt{k+1-2n}} \delta_{k+2-2n} \quad (A.25)
\]

And the inductive hypothesis gives that \(\delta_{k+2-2n} = \frac{\sqrt{k+1}}{\sqrt{k+3-2n}} \delta_k\). So it is now possible to conclude with the required equation (A.26).

\[
\delta_{k-2n} = \frac{\sqrt{k+3-2n}}{\sqrt{k+1-2n}} \frac{\sqrt{k+1}}{\sqrt{k+3-2n}} \delta_k = \frac{\sqrt{k+1}}{\sqrt{k+1-2n}} \delta_k \quad (A.26)
\]


