Cost Based Design Optimization of a Laminated Plate

by

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A thesis submitted in conformity with the requirements
for the degree of Masters of Applied Science
Graduate Department of Institute for Aerospace Studies
University of Toronto

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Abstract

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2008

The focus of this thesis is to introduce a proof-of-concept illustrating the integration of cost and performance as primary design drivers for structural design. Of particular interest to the aerospace industry is laminated structural design, therefore an example problem of a laminated plate structure is selected. This problem presents two primary challenges: linking cost in as general a way as possible to the design variables, and dealing with the many discrete design options available in a laminated plate. The first issue is examined by using a theoretical cost model for advanced composite fabrication and using a Direct Operating Cost model. The second issue is addressed using a gradient based optimization algorithm and a Discrete Material Optimization (DMO) method, which is typically used in topology optimization of structures.
Acknowledgements

I would like to thank my adviser, Professor Jorn Hansen, for his direction, support, insight and time. I would also like to acknowledge my peers, particularly Graeme Kennedy, who provided a great deal of patient direction throughout my time here, and James Forbes, who taught me the value of hard work and to strive for excellence. Finally, I wish to acknowledge the love and support of my mother, father and sister, family, and Amie Baker, all of whom provided me the confidence and support I needed to succeed.
## Contents

1 **Introduction**  
   1.1 Background ........................................ 1  
   1.2 Motivation .......................................... 3  
   1.3 Objectives .......................................... 4  

2 **Engineering Design Optimization**  
   2.1 Background ........................................ 6  
   2.1.1 Integrated Design ...................................... 7  
   2.1.2 Cost Based Optimization Approaches ...................... 7  
   2.1.3 Discrete Option Optimization Approaches .................. 8  
   2.2 Optimization Problem Statement ........................... 9  
   2.3 Optimization Algorithm Selection ......................... 9  
   2.4 SIMP Method .......................................... 10  
   2.5 Multi-Material SIMP formulation ........................... 12  
   2.6 DMO Convergence ...................................... 16  
   2.7 SIMP Sensitivity Calculation .............................. 18  
      2.7.1 Original SIMP Sensitivity ............................ 19  
      2.7.2 Multi-Material SIMP Sensitivity ....................... 20  
      2.7.3 Multi-Material SIMP with Normalization Sensitivity ... 22  

3 **Engineering Cost Modelling**  
   3.1 Cost Modelling in Design ............................... 24  
      3.1.1 Recurring and Non-Recurring Costs .................... 25  
      3.1.2 Direct and Indirect Costs ............................ 25  
      3.1.3 Life Cycle Costs .................................... 25  
   3.2 Cost Estimation Methods ................................ 25  
      3.2.1 Conventional Methods ................................ 26  
      3.2.2 Advanced Methods ................................... 27
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.3 Uncertainty in Costing</td>
<td>28</td>
</tr>
<tr>
<td>3.3 Cost Integration</td>
<td>28</td>
</tr>
<tr>
<td>3.4 Selected Cost Estimation Method - Gutowski Process Based Cost Model</td>
<td>29</td>
</tr>
<tr>
<td>3.4.1 Gutowski Theoretical Cost Model - Size Effects</td>
<td>29</td>
</tr>
<tr>
<td>3.4.2 Gutowski Theoretical Cost Model - Complexity Effects</td>
<td>32</td>
</tr>
<tr>
<td>3.4.3 Gutowski Theoretical Cost Model - Cost Driver</td>
<td>33</td>
</tr>
<tr>
<td>3.4.4 Cost Model Implementation</td>
<td>34</td>
</tr>
<tr>
<td>3.5 Material Costs</td>
<td>42</td>
</tr>
<tr>
<td>3.6 Cost and Weight Sensitivity Calculation</td>
<td>43</td>
</tr>
<tr>
<td>3.7 Direct Operating Costs</td>
<td>43</td>
</tr>
<tr>
<td>3.7.1 Stiffened Plate Design DOC Integration</td>
<td>44</td>
</tr>
<tr>
<td>3.7.2 Generic Aircraft Assumptions</td>
<td>48</td>
</tr>
<tr>
<td>3.7.3 Aircraft Performance Calculations</td>
<td>49</td>
</tr>
<tr>
<td>3.7.4 Crew Costs</td>
<td>52</td>
</tr>
<tr>
<td>3.7.5 Loan Servicing</td>
<td>52</td>
</tr>
<tr>
<td>3.7.6 Depreciation</td>
<td>53</td>
</tr>
<tr>
<td>3.7.7 Insurance</td>
<td>53</td>
</tr>
<tr>
<td>3.7.8 Mission Servicing</td>
<td>54</td>
</tr>
<tr>
<td>3.7.9 Fuel Costs</td>
<td>54</td>
</tr>
<tr>
<td>3.7.10 Engine Maintenance</td>
<td>55</td>
</tr>
<tr>
<td>3.7.11 Airframe Maintenance</td>
<td>56</td>
</tr>
<tr>
<td>3.7.12 Landing Fees</td>
<td>57</td>
</tr>
<tr>
<td>3.8 Direct Operating Cost Sensitivity Calculation</td>
<td>58</td>
</tr>
<tr>
<td>4 Structural Analysis</td>
<td>61</td>
</tr>
<tr>
<td>4.1 Reissner-Mindlin Plate Model</td>
<td>61</td>
</tr>
<tr>
<td>4.1.1 Constitutive Matrix calculation example</td>
<td>62</td>
</tr>
<tr>
<td>4.2 Failure Calculation</td>
<td>66</td>
</tr>
<tr>
<td>4.2.1 Failure Stress to Strain Conversion</td>
<td>66</td>
</tr>
<tr>
<td>4.2.2 Mid-Surface Strain and Curvature Calculation</td>
<td>68</td>
</tr>
<tr>
<td>4.2.3 Local Boundary Strain Calculation</td>
<td>68</td>
</tr>
<tr>
<td>4.2.4 Rotate to local material axis</td>
<td>69</td>
</tr>
<tr>
<td>4.2.5 Failure Check</td>
<td>70</td>
</tr>
<tr>
<td>4.2.6 Exceedance Amalgamation</td>
<td>72</td>
</tr>
<tr>
<td>4.2.7 Highest Exceedance Selection</td>
<td>74</td>
</tr>
<tr>
<td>4.2.8 Failure Load Aggregation</td>
<td>75</td>
</tr>
</tbody>
</table>
## Contents

4.2.9 Secondary KS requirement .................................................. 76
4.3 Sensitivity Calculation .......................................................... 78

5 Results and Discussion ........................................................... 81

5.1 Sample Problem ............................................................... 81
5.1.1 Geometric Details .......................................................... 81
5.1.2 Finite Element Mesh Details .............................................. 82
5.1.3 Materials ................................................................. 82
5.1.4 Loading and Boundary Conditions ..................................... 83
5.2 Optimization Function ......................................................... 85
5.3 Reference Plates .............................................................. 87
5.4 Optimization Results ........................................................... 89
5.4.1 4 Ply Optimization .......................................................... 89
5.4.2 8 Ply Optimization .......................................................... 92
5.4.3 Fuel Cost Effects ............................................................ 93
5.4.4 Labour Cost Effects ........................................................ 96
5.4.5 Stiffener Height Effects .................................................... 100
5.4.6 Plate Size Effects .......................................................... 103
5.4.7 Stiffener Spacing Effects .................................................. 106
5.4.8 DOC Model Plate Cost and Mass Effects ......................... 108
5.5 Conclusions ................................................................. 110
5.6 Future Work ................................................................. 111

References .................................................................. 113

Appendix .................................................................. 116

A DOC Sensitivity Calculation Constants ................................ 117
A.1 Discussion .................................................................. 117
A.2 Fuel ........................................................................ 117
A.3 Airframe Maintenance ................................................... 118
A.4 Landing .................................................................. 120

B H Convergence ................................................................. 122

C Mass over Area Proof ......................................................... 125
### List of Tables

3.1 Manufacturing Time Constants .................................................. 35
3.2 A/C Cost Assumptions .......................................................... 47
3.3 A/C Performance and Weight Assumptions ................................. 48
3.4 AC Engine Assumptions .......................................................... 49
3.5 DOC Crew Cost Assumptions ..................................................... 52
3.6 DOC Loan Assumptions .......................................................... 52
3.7 DOC Depreciation Assumptions ............................................... 53
3.8 DOC Mission Servicing Assumptions ........................................ 54
3.9 Jet Fuel Cost variability over 2007-2008 ................................. 55
3.10 DOC Maintenance Assumptions ............................................. 56
3.11 Selected Airport Landing Fees (2007) ...................................... 58

4.1 Selected Material Failure Loads ............................................... 67
4.2 Selected Material Failure Strains ............................................ 72

5.1 Plate geometries ....................................................................... 81
5.2 Selected Material Failure Properties ........................................ 83
5.3 Imposed Load Cases ................................................................ 83
5.4 Initial Cases ............................................................................ 88
5.5 Load Case Percentage Change - 4 Ply Optimum vs. Reference Plates ................................................................. 91
5.6 DOC Percentage Change - 4 Ply Optimum vs. Reference Plates ................................................................. 91
5.7 Load Case Percentage Change - 8 Ply Optimum vs. Reference Plates ................................................................. 93
5.8 DOC Percentage Change - 8 Ply Optimum vs. Reference Plates ................................................................. 93
5.9 Plate Attribute Percentage Change - fuel at year 2000 cost, Optimum vs. Reference Plates ................................................................. 95
5.10 DOC Percentage Change - 4 Ply, fuel at year 2000 cost, Optimum vs. Reference Plates ................................................................. 95
5.11 DOC Percentage Change - 4 Ply, labour costs multiplied by 3.5, Optimum (Gr/Ep (90)4 skin, Aluminum (0)4 st), vs. Reference Plates ................................................................. 99
List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Basic SIMP Concept</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>Multi-Material SIMP; 2 materials, with p = 1, 3, 5, 7</td>
<td>14</td>
</tr>
<tr>
<td>2.3</td>
<td>Normalized Multi-Material SIMP; 2 materials, with p = 1, 3, 5, 7</td>
<td>15</td>
</tr>
<tr>
<td>2.4</td>
<td>DMO Convergence tolerance at $\varepsilon = 0.95, 0.975, 0.995$ and SIMP penalty = 1</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>DMO Convergence tolerance at $\varepsilon = 0.95, 0.975, 0.995$ and SIMP penalty = 5</td>
<td>18</td>
</tr>
<tr>
<td>3.1</td>
<td>Time estimation comparison, conservative vs. Mabson approximation, hand layup of 3\textdegree{} composite tape</td>
<td>31</td>
</tr>
<tr>
<td>4.1</td>
<td>2 Ply element failure criteria, showing cusp at transition between failure from lower to upper ply</td>
<td>77</td>
</tr>
<tr>
<td>4.2</td>
<td>2 Ply element failure criteria, showing cusp at transition between failure from lower to upper ply, corrected with KS function</td>
<td>78</td>
</tr>
<tr>
<td>5.1</td>
<td>Finite Element vs. Design Variable Discretization</td>
<td>82</td>
</tr>
<tr>
<td>5.2</td>
<td>Load Cases 0 through 3</td>
<td>84</td>
</tr>
<tr>
<td>5.3</td>
<td>Load Case 1: Ultimate Positive Pressure</td>
<td>85</td>
</tr>
<tr>
<td>5.4</td>
<td>Example Plate Boundary Conditions</td>
<td>86</td>
</tr>
<tr>
<td>5.5</td>
<td>Data Flow Structure</td>
<td>87</td>
</tr>
<tr>
<td>5.6</td>
<td>4 Ply Optimum Ply Orientations</td>
<td>89</td>
</tr>
<tr>
<td>5.7</td>
<td>4 Ply Optimum vs. Reference Plate Comparisons</td>
<td>90</td>
</tr>
<tr>
<td>5.8</td>
<td>8 Ply Optimum vs. Reference Plate Comparisons</td>
<td>92</td>
</tr>
<tr>
<td>5.9</td>
<td>4 Ply Optimum Ply Orientations - Year 2000 Fuel Costs</td>
<td>94</td>
</tr>
<tr>
<td>5.10</td>
<td>DOC Cost Comparison of Year 2000 Fuel Cost optimum plate vs. Year 2008 Fuel Cost optimum plate</td>
<td>96</td>
</tr>
<tr>
<td>5.11</td>
<td>DOC Percentage Breakdown Comparison</td>
<td>97</td>
</tr>
<tr>
<td>5.12</td>
<td>DOC Fuel Cost Sensitivity $\frac{\partial (DOC)}{\partial M}$ with respect to Individual Plate Mass</td>
<td>97</td>
</tr>
<tr>
<td>5.13</td>
<td>Fabrication Cost Variation Study</td>
<td>98</td>
</tr>
</tbody>
</table>
5.14 Airframe and engine maintenance and servicing cost sensitivity to labour costs with changing plate mass .................................................. 100
5.15 DOC Comparison of viable designs with 4 plies and varying stiffener heights . . . 101
5.16 Moment resultant \( \kappa_y \) values for \((0\ 90)_4\) cross-ply vs. \((90)_4\) skin and \((0)_4\) stiffener optimum at stiffener heights of 60 and 27 mm .......................................... 102
5.17 Size Variation Effects ............................................................................. 104
5.18 Stiffener Spacing Variation Effects ............................................................. 107
5.19 DOC compared to fuselage mass and cost ................................................... 109
5.20 DOC compared to fuselage mass and cost ................................................... 109
B.1 Finite Element Meshes used for convergence study ..................................... 123
Chapter 1

Introduction

1.1 Background

Engineering is traditionally a field in which known scientific principles are rationally applied in some fashion to obtain a desired result. Since the discipline is an application of science, it lies at the intersection between abstract theories and principles established by careful experimentation and theory, and between tribal knowledge and experiential trades of fashioning a physical object. Engineers are aware that their final products will be either a physical manifestation, or will produce, in some way, some physical result in the real world, and are always aiming to satisfy some perceived need. This means that engineers must make rational decisions about the design of objects in the physical world, and "If in other sciences we should arrive at certainty without doubt and truth without error, it behooves us to place the foundations of knowledge in mathematics"[1]. Since an engineer’s work is directly linked to the physical world, it is immediately constrained to operate within the confines of the defined natural laws of physics. Additionally, it is also constrained to abide by the man-made laws of society, policy and economics. This means that although an engineer would prefer to solve the entire design problem using rational mathematics, other less quantifiable external factors, such as time to completion and resource allocation, that may be out of their direct control, may also play a pivotal role. The engineering challenge, therefore, is to balance competing requirements, not just from the technical standpoint, typically a complex and challenging area in and of itself, but also to balance economic, time and resource advantages with a view to producing a finished design that performs as required, within a reasonable amount of time, and with the fewest possible resources.

The typical way in which this is accomplished is to pursue an iterative design path, in which the design progresses through a series of well-defined process steps, running from an establishment of a specific need, through conceptual, preliminary and detail design, and ending in a final
design. This initial design is then passed through further phases of analysis, such as production design, manufacturing, and distribution. The requirements of these further steps may, and frequently do, require reiteration of the design to meet performance, manufacturability, economic or any other critical considerations. However, in a free-market competitive environment, there is a constant incentive to achieve a finished design with as few iterative repetitions as possible, while including such varied metrics as performance, functionality, cost, manufacturability, and product life-cycle in the design phase. The task of ‘closing the loop’ in terms of reducing several disparate, iterative design and post-design stages into a few large cross-coupled steps begins to touch upon the relatively new area of integrated design.

Integrated design entails considering the multiple design steps as a series of interdependent stages with each having influence on the other. Rather than so-called ‘over the wall’ design, in which completed designs are foisted upon the remaining production, manufacturing, distribution and operating engineers with minimal feedback, integrated design seeks to involve these future steps as much as possible and as early as possible in the design process. Unfortunately, the ability of any skilled designer to assimilate and incorporate the myriad of various disciplines into every design decision within a reasonable time span and with little revision would be a taxing task. Fortunately, a rigorous mathematical solution can be applied to those factors which are numerically quantifiable in the form of numerical optimization. The rigorous definition of optimization involves maximizing or minimizing a given function. Typically, engineering requirements dictate that this be done within some set of constraints. Obviously, selection of the appropriate objective function and constraint parameters is vital for the optimization to proceed to the optimal solution. Additional complexity is added when many coupled, non-linear variables are considered. For this reason, and since most designs are dominated by more than one specific engineering discipline, the evolving field of multi-disciplinary optimization presents a framework within which many of these disciplines can be incorporated simultaneously. Since the time required to set up a more complex analysis must be less than or equal to the economic benefit of doing so, such complex optimization schemes are typically performed only in those fields where exercises in precise analysis will yield a potential gain at least equal to the additional cost of performing the task. Typically, this occurs in fields where weight must be reduced to the bare minimum without compromising safety and performance. Therefore, outside of a few niche applications, the majority of these advanced analyses are only widely applied in the aerospace field.

Within the specific discipline of structural optimization, the traditional optimization objective is to minimize weight or compliance. However, recent developments in the marketplace have shown a new focus on designing from the outset for profitability; therefore, there has been
a shift in thinking away from performance to cost as the ultimate arbiter of design. In the aerospace context, airlines are increasingly focused towards reducing the direct operating cost of an airliner \[2\]. Such costs include all factors that contribute to the daily operating expense of an aircraft, including, but not limited to, the initial capital cost of acquiring the airframe, fuel consumption rate, maintenance concerns and crew costs. Traditional structural design is focused on reducing design weight as a way to improve the cost of a component. However, in light of the fact that this requirement will only reduce the fuel consumption rate, and indirectly may contribute to increased manufacture, material and non-recurring capital costs, this may in fact, lead to a more expensive, and hence to a less competitive product. Since up to 80\% of a component's final cost may be fixed during the early design phase \[3\], it is of interest to implement cost factors when the design is still in this phase. Unfortunately, since the detailed design is still unrefined at this point, modeling the cost of the finished product at this stage is extremely difficult. For this reason, weight remains the primary analogue to cost in aerospace structural design. Despite this, there is great interest in including cost in the preliminary design phase, while recognizing that some performance metrics must still be included to ensure that a proper trade-off between cost and performance is met.

Within the context of the above discussion, several areas of interest appear: (a) numerical optimization, (b) preliminary cost estimation and (c) structural analysis all with a view to develop some adequate fairly detailed preliminary design. Using a numerical optimization scheme and incorporating in the objective function both performance and cost metrics, a preliminary design framework for a representative aerospace structure would be an ideal testing ground for determining the viability of including cost early in the design process. The concept of including both cost and performance metrics in the same objective function in the context of a structural design is not new; it has been proposed in a non-aerospace context by Iqbal[4].

1.2 Motivation

With a view to extending the cost-based integrated design scheme proposed by Iqbal [4], the motivation of this thesis is to pursue a cost-based integrated design of a representative typical aerospace structure. The structure selected is a laminated stiffened plate of simple geometry. This structure is numerically subjected to multiple representative load cases, is structurally analyzed using a traditional finite element method and is ‘designed’ using a numerical optimizer, which is allowed to freely select from a chosen list of various material options for each ply for each discretized element of the plate. The analysis of this structure provides insight into the methodologies of cost estimation, material selection, failure determination, and numerical
optimization. Additionally, an investigation of the effect of varying performance and cost metrics is conducted, with a view to explore the design space. Of particular interest is the effect of non-performance related factors, such as labour and material cost on the preliminary design of the structure. In order to facilitate analysis of the plate, costing methodologies, discrete material selection, failure calculation as well as optimization considerations are addressed. These provide additional insight into the cost-based integrated design optimization of a stiffened laminated plate.

1.3 Objectives

In order to model the entire potential costs of even a simple representative structure (such as a stiffened laminated plate) would be unrealistic within the scope of a thesis, and therefore, the only cost considerations that are addressed are the fabrication and material costs and the associated effects on Direct Operating Cost of a representative aircraft. Additionally, the costs are restricted to factors only resulting from a structural engineering standpoint; therefore, the effect of aircraft performance outside of the direct effect of weight is not considered. The selection of a gradient based optimizer implies the requirement of a sensitivity calculation; therefore, optimization sensitivities are calculated using a combination of analytic, complex step, and adjoint methods, where each method is appropriate. Within these confines the objectives of the study are:

- Develop a numerical model which can accurately simulate the loading and boundary conditions on a stiffened-laminated plate, with the intention of conducting a design optimization.

- Study and incorporate discrete material selection considerations from both an objective and failure constraint point-of-view.

- Study and incorporate cost-modelling methods for engineering design into the numerical model from an objective standpoint.

- Investigate and determine the utility of including Direct Operating Cost considerations in the preliminary design phase.

Within the scope of this thesis is the consideration of multiple potential materials for each laminated ply using a Discrete Material Optimization (DMO) method, following the work of Stegmann et al. [5], the inclusion of fabrication costs based on a fabrication costing model by Gutowski et al. [6], the inclusion of material costs and the incorporation of aircraft direct operating cost, similar to the approach proposed by Curran et al. [7], and following established IATA
and NASA Direct Operating Cost reports [8][9]. The above factors are all considered within a numerical optimization context. Additionally, a novel approach for determining failure in a DMO-based structure is presented, and incorporated in the optimization as a design constraint.
Chapter 2

Engineering Design Optimization

2.1 Background

Almost as soon as it was recognized that data could be fitted to a curve, it was realized that engineering design decisions, especially trade-offs, could be facilitated using a mathematical process which became known as optimization. In this process, a defined design space is established, and a mathematical construct of the competing mathematically abstracted requirements is then constructed within it. Using techniques of calculus, an optimal choice of parameters can be made whereby the parameters minimize or maximize the objective function such that the parameters can be said to be optimum, within given bounds and satisfying any imposed constraints. A limitation exists; however, in that the this method only works if the function is differentiable and continuous. Originally, the field was known as linear programming and was primarily focused on linear problems that had the virtue of being fundamentally solvable using the linear simplex solution method.

The linear simplex method is very powerful, and is extremely useful in operations research and industrial engineering applications, where man-made systems tend to be linear in scope. However, challenging engineering design problems typically are related to such natural phenomena, as viscous fluid flow and plastic deformation. Although simple linear models can be and are applied as much as possible in analysis, at some point the required accuracy for a given application cannot be satisfied with a linear model, at which time a more complex non-linear model must be introduced. It must be recognized that linear models do not necessarily lead to a purely linear optimization. If non-linearity is incorporated in the design space, this leads to a whole new set of challenges in optimization. Additionally, there may be serious problems incorporating discrete design variables, as well as the difficulty of handling any constraints that may arise. Constraints may, themselves, be non-linear.

In terms of non-linear optimization approaches, there are two main branches: gradient and
non-gradient based algorithms. There are families of algorithms belonging to each branch, and it is outside the scope of this thesis to detail the particulars of each algorithm; however, it is informative to assess selections by others to gain some insight into the reasons behind the choice of a particular optimization algorithm over other potential contenders. In general terms, non-gradient based algorithms can be said to be more accommodating of discrete design variables, whereas gradient based algorithms can handle many more design variables and typically converge faster.

2.1.1 Integrated Design

One of the primary areas of interest which is further enabled by rigorous mathematical optimization is the area of integrated design. Integrated design was discussed briefly in Section 1.1 but essentially is defined as an integration of the various steps in the design cycle. Some articles have been published with respect to the various aspects of integrated design. Although these aspects are not always specifically applied to an aerospace context, they are nonetheless valuable in providing insight into concurrent engineering processes. Various reviews involving life cycle analysis (LCA), design for manufacture and assembly (DFMA), and concurrent engineering (CE) have been published, such as the article by Singh [10], which reports an analytical multi-objective framework for the concurrent design of processes and products. In Singh’s work, the focus is on a joint consideration of process selection and product tolerances. A generic framework for integrated product and process development (IPPD) is proposed by Yan [11], with the focus being on stages in the life cycle of the product. This is combined with an exhaustive search to ‘optimize’ the design for given objectives throughout the product life-cycle. Design for manufacture and assembly techniques, are addressed by Boothroyd [12], who identifies that complexity and number of assembly steps are also extremely important in a manufacturability context. Of these few examples, Yan specifically integrates process path optimization, which allows many potential configurations to be considered at any stage of the lifecycle that is capable of being quantified and modelled. However, this example involves an exhaustive search of the entire process path tree which is only feasible when the number of potential design variables is low or when the product model is extremely simplified.

2.1.2 Cost Based Optimization Approaches

It is informative to observe the choices made by previous authors when considering an integrated design optimization approach where cost is included in the analysis. Although the field is small, there are a few examples of various optimization approaches. Castagne et al [13], rely on a generalised reduced gradient solver which is included in the Excel software package. This
approach is also followed by Curran et al [7], and allows a user-friendly and rapidly customizable approach to the optimization. The Sequential Quadratic Programming (SQP) algorithm is used by Peoples [14] to perform a value-based multi-disciplinary optimization, whereas Crossley et al [15] use CONMIN, a legacy Method of Feasible Directions (MFD) optimizer, to perform a multi-objective optimization of a commercial transport aircraft for cost and weight. Although not using cost in the objective function, a cost constrained optimization is performed by Rais-Rohani [16], and in that case an SQP algorithm is employed. Iqbal [4] uses a Method of Moving Asymptotes (MMA) for a cost-based integrated design optimization. There is considerable industrial and academic collaboration involved in the COSTADE and MADESmart models [17], which contain an Improving Hit and Run (IHR) optimization algorithm; the article also indicates that some preliminary work with neural networks and response surfaces is performed as well. Of particular interest is a process-based manufacturing cost model which Bao [18] uses to perform a multi-disciplinary design optimization of a wing using an SQP algorithm.

As can be seen from the above examples, there are no established, generally accepted optimization algorithms for these problems. The majority of the above authors used gradient-based algorithms, of which the SQP, MFD and MMA algorithms are all a form. The only non-gradient based method discussed above is the Improving Hit and Run algorithm, further details of which were not given in the literature of interest [17].

2.1.3 Discrete Option Optimization Approaches

Solving a discrete optimization problem presents a particular challenge due to the discontinuous nature of discrete problems. Several different methodologies are used. Typically, the advantage given by non-gradient based approaches is balanced by their relatively slow rate of convergence and inability to handle large numbers of design variables. A common non-gradient approach is to apply a genetic algorithm, as Rahul et al [19], who use a modified genetic algorithm to optimize FRP composites and allow the algorithm to select specific ply angles for each lamina of a composite. This problem has relatively few design variables, yet with 8 parallel processors takes approximately 20,000 seconds to converge to the optimal solution. On the other end of the spectrum, Stegmann et al [5] use a penalized interpolation scheme, typically used in topological optimization, and the gradient-based MMA algorithm to optimize a composite fiber layup problem with many design variables. They achieved convergence with 3840 design variables in approximately 1800 seconds, although it is important to note that the convergence criteria were relatively relaxed. Further details of this convergence method is discussed in Section 2.6.
2.2 Optimization Problem Statement

An optimization problem is posed in a rigorous, mathematical form, which presents a clear, concise synopsis of the objective and constraint functions, as well as what is to be optimized and whether it should be maximized or minimized. The standard generalized optimization is:

\[
\text{Find } X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ which minimizes } f(X) \\
\text{subject to:}
\]

\[g_j(X) \leq 0 \quad j = 1, 2, \ldots, M\]

\[h_k(X) = 0 \quad k = 1, 2, \ldots, L\]  \hspace{1cm} (2.1)

Where:

- \(X\) is an \(n\) dimensional vector of design variables
- \(f(X)\) is the objective function
- \(g_j(X)\) is an \(m\) dimensional vector of inequality constraint functions
- \(h_k(X)\) is an \(l\) dimensional vector of equality constraint functions

The number of the design variables affects the dimensionality of the design space. When these problems possess only one or two design variables, then, besides being easy to solve, the design space and objective value can be plotted in a visually realizable way and the optimum verified visually. For problems encountered in typical engineering applications the number of design variables may vary widely. Note that the constraint functions may be functions as much or more complex than the objective function itself. The constraint functions are grouped as inequality and equality constraints because these are usually handled differently by the optimization algorithm. Upper and lower limits on the design variables may be imposed either directly in some algorithms, or indirectly in others by creating additional constraints as required.

2.3 Optimization Algorithm Selection

In the context of this thesis, it is known that discrete material choices are desired and that there are many design variables owing to the finite element discretization. Additionally, the analysis
is fairly computationally intensive, requiring a finite element solution; therefore, an optimization method which requires as few function evaluations as possible is desirable. Unfortunately, non-gradient based algorithms cannot typically converge without performing many function evaluations and are typically inefficient with large numbers of design variables. Therefore, an SQP gradient based optimization algorithm is used to solve the current problem. This will require the calculation of objective and constraint function gradients to determine sensitivities with respect to the design variables. The issue of handling multiple discrete potential material options is resolved by employing the SIMP method.

### 2.4 SIMP Method

The Solid Isotropic Micro-structures with Penalization, or SIMP method, is a penalized interpolation scheme that is applied to allow discrete event problems to be modelled as continuous and differentiable problems; such conditions are required for gradient based optimization. The SIMP method is used primarily in problems of topology optimization and was originally proposed by Bendsoe [20] to determine the presence or absence of material within a set domain. The initial problem of the presence or absence of material is posed as:

\[ C(x) = xC \quad x \in (0, 1) \]  

(2.2)

where:

- \( x \) is the variable controlling the presence or absence of material. \( X \) may also be considered as a normalizing value corresponding to the variable \( C \)

- \( C \) is some constant of interest which is being used in the optimization, such as elastic modulus

The SIMP method reformulates the above to be:

\[ C(x) = x^pC \quad 0 \leq x \leq 1 \]  

(2.3)

Here \( p \) is a penalty parameter that affects the discrete control variable \( x \) by penalizing all numbers that are not at the endpoints; the endpoints represent the discrete values considered in the original problem statement. The effect of the penalty parameter on the SIMP interpolation is illustrated in Figure 2.1. The effect of the above interpolation allows a previously discrete function to be treated as a continuous, differentiable function. This is critical for gradient based optimizers where the objective function must be defined as continuous and differentiable in order to provide valid gradient information. A discrete optimization problem has the form:

\[ \min f(x) \quad x \in 0, 1 \]
Chapter 2. Engineering Design Optimization

Figure 2.1: Basic SIMP Concept

\[ g_j(X) \leq 0 \quad j = 1, 2, \ldots, m \]
\[ h_k(X) = 0 \quad k = 1, 2, \ldots, l \]

(2.4)

where:

- \( X = (x_0, x_1, \ldots, x_n)^T \) is a vector of design variables.
- \( f(X) \) is the objective function.
- \( g_j(X) \) is a vector of \( m \) inequality constraint equations.
- \( h_k(X) \) is a vector of \( l \) inequality constraint equations.

By including the SIMP penalty parameter, the problem can be replaced with an optimization problem of the form:

\[ \min f(X) \quad x \in R \quad 0 \leq x \leq 1 \]
\[ g_j(X) \leq 0 \quad j = 1, 2, \ldots, m \]
\[ h_k(X) = 0 \quad k = 1, 2, \ldots, l \]

(2.5)

where:

- \( X = (x_0^p, x_1^p, \ldots, x_n^p)^T \) is the new vector of design variables.
This change allows a potential solution using gradient based optimizers when the problem is convex. Note that the value of $x$ is now able to range anywhere from 0 to 1 in the real number set. If the optimal solution is found to lie somewhere between the discrete choices, the penalty parameter can be successively increased until the solution is forced towards one of the discrete values.

### 2.5 Multi-Material SIMP formulation

The SIMP concept can be extended to address more complex discrete problems, rather than simply the presence or absence of a given material. It has been extended by Stegmann et al to include a multi-material weighting capability [5]. The multi-material model proposed is an extension of equation 2.3 with an added term which ensures that an increase in the direct weighting term also reduces the likelihood of other weights. The formula, as applied to three materials, takes the form:

$$C^e = \frac{(x_0^e)^p[1 - (x_1^e)^p][1 - (x_2^e)^p]}{w_0} C_0 + \frac{(x_1^e)^p[1 - (x_0^e)^p][1 - (x_2^e)^p]}{w_1} C_1$$

$$+ \frac{(x_2^e)^p[1 - (x_0^e)^p][1 - (x_1^e)^p]}{w_2} C_2$$

(2.6)

where:

- $e$ denotes an element of interest
- $p$ is the SIMP penalty parameter
- $C$ is some constant value of the materials in question, such as stiffness or mass

The generalized form of the above equation is:

$$C^e = \sum_{i=1}^{n^e} \left[ (x_i^e)^p \prod_{j=1}^{n^e} \left[ 1 - (x_{j \neq i}^e)^p \right] \right] C_i$$

(2.7)

where:

- $e$ denotes each element
- $n^e$ is the number of design variables per element
- $i, j$ are the particular number identifying the design variables per element
- $p$ is the SIMP penalty parameter
$x$ is the design weight

$C$ is a constant value of the material in question

Note that equation [2.7] can be applied to multiple layers in a single element by replacing the $e$ index in equation [2.7] with $l$, which denotes a particular layer, and repeating the above calculation for each successive layer in a given element, establishing the element properties as required, and then repeating the entire sequence for each element. As discovered by Stegmann et al. [5], in general, the weights, $w_i(x_i, p)$, which are functions of the design variables $x_i$, may not sum to one, and this implies that when $C$ is some physical parameter, such as mass, unrealistic intermediate results may occur. This is best illustrated by a brief example. Consider a two material element with the following characteristics:

- $x_i \in \mathbb{R}$
- $0 \leq x_0 \leq 1$
- $x_1 = x_0 - 1$
- $C_0 = 10$
- $C_1 = 15$
- $p = 1, 3, 5, 7$

As in Figure [2.2], when $x_0 = 0$, the sum of the constant terms is 15. At this point, the total $C$ value is equal to $C_1 = 15$, and at the other end, when $x_0 = 1$, then the total $C$ value is at 10, equal to the $C_0 = 10$ value only. However, during the interpolation, when $x_0$ is not 0 or 1, the formulation results in the interim interpolations providing a total $C$ value that may be less than either of the two discrete values. This is a non-physical result, and furthermore, any optimizer which is required to minimize the problem as posed will stall in the local optimum between the desired discrete values. As evidenced in Figure [2.2], this cannot be resolved by incrementing the penalty parameter. The formulation is therefore adjusted to normalize the weight values $w_i$ by the sum of the weights calculated previously. The resulting formulation requires successive calculation over a given element or layer:

\[
\text{Step 1} : \quad S_w = \sum_{k=1}^{n_e} \left[ (x_k^e)^p \prod_{j=1}^{n_k} [1 - (x_j^e \neq k)^p] \right] \frac{w_k}{\sum_{k=1}^{n_e} w_k}
\]

\[
\text{Step 2} : \quad C^e = \sum_{i=1}^{n_e} \left( \frac{1}{S_w} \right) \left[ (x_i^e)^p \prod_{j=1}^{n_e} [1 - (x_j^e \neq i)^p] \right] C_i \quad (2.8)
\]
where:

- $k, i, j$ denote the design variables for a given element $e$, and are equal to one another
- $x$ is a design variable allowed to vary between 0 and 1
- $n$ is the number of design variables for this particular element
- $p$ is the SIMP penalty parameter
- $C$ is the material property of interest

Note that equation 2.8 requires that the sum of the SIMP weights $\sum_{k=1}^{n^e} w_k$ be calculated for each element first, following which each weight $w_i$ is divided by this summation during a normalization step. This ensures that the sum of the weights $\sum_{i=1}^{n^e} w_i$ over each element will be unity. As before, the above equation can be used for a layer rather than an element, with the only additional effect being that this 2 step calculation must be performed for each layer in each element. The effect of the additional normalization step on the sample two-material problem is displayed in Figure 2.3, which shows that for all values of $x_0$ and all penalties, the value of the interpolation function lies between the constants $C_0$ and $C_1$ in the vertical axis, which were equal to 10 and 15, respectively. It can also be seen that the interpolation has changed significantly, and that as the weights approach a discrete data point the interpolation
curve curvature is greatly reduced. This creates a problem of reduced convergence rate as the weights approach the desired discrete data points, and is an undesirable side-effect of using the normalization equation 2.8. Unfortunately, this form is required because all the values upon which the SIMP weights are applied have a physical interpretation, and therefore require that the sum of the weights be unity at all times, in order to ensure physically plausible designs during the optimization process. A significant drawback to using the normalizing equation 2.8 form, exists. If the discrete values between which the interpolation is being used are equal, then, assuming no other influencing factors, the design space will be flat, and the optimizer will stall. This problem is not resolved by increasing the penalty parameter, as the intention of the normalization form is essentially to ensure that all designs in the space remain feasible at all times, and if the two designs are equivalent, the only feasible design point would be that value. In the example illustrated in Figure 2.3, it would be as if the two constants $C_0$ and $C_1$ were both 10. At this point, the only valid interpolation form with the normalization equation would be a straight line. This brings about competing requirements that dictate that the design must be physical during intermediate steps and yet must converge to a discrete point. Obviously, if the designs being considered are functionally equivalent, one cannot physically realizable designs throughout the optimization and promote convergence to discrete design points, given equations 2.7 and 2.8. To avoid this fact, at least one of the design factors to be considered (among cost, mass and stiffness) will need to be interpolated using the non-physical but penalizable equation.
which will ensure convergence to the discrete design points of interest. The normalization equation also experiences reduced convergence with increased penalization. To address the reduced convergence rate of this formulation, a fairly weak convergence tolerance criterion is used. The convergence criterion is used by Stegmann et al. [5], and is referred to as DMO (Discrete Material Optimization) convergence.

2.6 DMO Convergence

As discussed in Section 2.5, the SIMP formulation is modified and normalized to ensure physical results during intermediate points in the optimization sequence. A side effect of the normalization is that the rate of convergence is decreased as the design variables approach the desired discrete values. To counteract this, a fairly lenient convergence criterion is used. For each layer of each element, the equation below is applied:

\[
(w_l^e)^c \geq \varepsilon \sqrt{((w_0^e)^c)^2 + ((w_1^e)^c)^2 + \ldots + ((w_n^e)^c)^2} \tag{2.9}
\]

where:

\( l \) is the layer of interest in element \( e \)

\( w \) is the normalized material weight, which is a function of the design variables, as calculated using equation 2.8

\( n \) is the number of design variables associated with a particular layer in a particular element

\( \varepsilon \) is a tolerance level, which is recommended by Stegmann et al. [5] to vary between 95-99.5%

If the inequality in equation 2.9 is satisfied for a particular layer in a particular element, it is flagged as converged. The final convergence number is then simply the ratio of converged elements to the total number of elements, as in equation 2.10:

\[
h_\varepsilon = \frac{N_c^l}{N^l} \tag{2.10}
\]

where:

\( h_\varepsilon \) is the total ratio of converged layers in all elements, \( N_c^l \), to total layers in all elements, \( N^l \)

The relative leniency of this convergence criteria is illustrated in Figure 2.4. The figure depicts the variation of two weighting parameters \( w \) with respect to the value of the design variable \( x_0 \). Also plotted are the threshold DMO convergence values at various tolerances. The point at which a layer is said to be converged for each particular tolerance is highlighted by the vertical line, which indicates the point at which the DMO convergence is below the weight value plotted.
on the y-axis, and above which the layer of interest would be flagged as converged. In the given example, the SIMP penalty parameter is 1, and it can be seen that with the DMO tolerances set to 0.95, 0.975 and 0.995, the DMO convergence numbers, as indicated on the y-axis, are 0.75, 0.82 and 0.91, respectively. The effect of an increased SIMP penalty number on these convergence numbers is small, as in Figure 2.5. When the penalty is incremented to 5, although the shape of the SIMP interpolations and DMO convergence curve are altered, the DMO convergence numbers for the same three tolerance values are 0.75, 0.82 and 0.91, as before. Of particular interest is the fact that the penalization still has an effect, because the design variable values, $x$, at which the weights, $w$, are calculated to have the 'converged' values, are much more relaxed when the penalty is increased, as can be clearly noted when comparing the x-intercepts of the 'converged' values for the differing SIMP penalties. With a penalty of 1, the required $x$ values for convergence are 0.64, 0.68 and 0.76, for tolerances of 0.95, 0.975 and 0.995, respectively, whereas with a penalty of 5, the required $x$ values for convergence are 0.55, 0.57 and 0.61, respectively. This relaxation effect which occurs with increasing penalization serves to counteract the flatter interpolation curve which results with higher penalization numbers, as was shown in Figure 2.3.
2.7 SIMP Sensitivity Calculation

In order to perform gradient-based optimization, it is necessary to determine the gradients of the objective and constraint functions. The gradients may be calculated numerically, but with the SIMP formulations described in sections 2.4 and 2.5, it is possible to derive the derivatives analytically. Analytic derivations are desirable, since this is the fastest computational way in which to arrive at the required gradient information. In a generalized matrix form, the SIMP weighting scheme for a single value of interest may be expressed as:

\[
\begin{bmatrix}
    w_0 & 0 & 0 & 0 \\
    0 & w_1 & 0 & 0 \\
    0 & 0 & \ddots & 0 \\
    0 & 0 & 0 & w_i
\end{bmatrix}
\begin{bmatrix}
    C_0 \\
    C_1 \\
    \vdots \\
    C_i
\end{bmatrix}
\]

where:

- \( w \) is the SIMP weight, and is a function of the design variables \( x \)
- \( i \) corresponds to the total number of constants \( C \) to be weighted

Although the constants may be any parameter of interest, in this problem the constant is either a fabrication cost, a material weight or a material stiffness. SIMP weights \( (w) \) are calculated
with equation [2.3], [2.7] or [2.8]. Gradients are calculated by taking the partial derivative of the objective function with respect to each of the design variables. The gradients are expressed as:

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial w_j}{\partial x_i} \quad i = 1, 2, \ldots, n \quad (2.11)$$

The difference between the gradient calculation for each of the different SIMP weighting equations is best illustrated by an example.

**Example Problem**

Consider a sample problem discretized into two separate elements. Each element may be composed of three distinct materials. There will then be 6 weighting terms, represented by the vector \([x_0, x_1, x_2, x_3, x_4, x_5]\). The six potential material masses are represented by the vector \([C_0, C_1, C_2, C_3, C_4, C_5]\), where \(C_0 = C_3\), \(C_1 = C_4\), \(C_2 = C_5\), due to the fact that three distinct material weights are being considered repeatedly. Applying a SIMP equation will transform the design variable weighting terms, \(x\), into functions of both the design variables \(x\) and the penalty \(p\), although \(p\) is not a design variable and is not allowed to be altered by the optimizer during an optimization cycle. When considering the three gradient calculation methods outlined below, note that when considering laminated ply elements, the calculations may be applied to a single ply of a single element.

### 2.7.1 Original SIMP Sensitivity

The original SIMP form is given in equation [2.3]. The analytic derivative equation of the original SIMP form, given in equation [2.3] is:

$$f(x) = x^p \quad \frac{\partial f}{\partial x} = px^{p-1} \quad (2.12)$$

Note that an implication of the above equation is that any design variable that is used to determine a SIMP weight for a specific element is not used again. This is important because it allows the gradient of each element to be computed independently. Using the sample problem described above in section [2.7], the matrix form of this particular method for the first element is expressed as:

$$C^0 = \begin{bmatrix} w_0(x_0) & w_1(x_1) & w_2(x_2) \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$
This equation may be recast in terms of the design variables only, where the effect of each design variable is isolated for simplicity.

\[
C^0 = \begin{bmatrix}
w_0 \quad w_1 \quad w_2 \\
x_0^p \quad x_1^p \quad x_2^p
\end{bmatrix}
\begin{bmatrix}
C_0 \\
C_1 \\
C_2
\end{bmatrix}
\]

Given that the weights above are only functions of a single design variable, the combined gradient corresponding to the matrix above is:

\[
\left[ \frac{\partial f}{\partial x_{0,1,2}} \right]^T = \left[ \frac{\partial w_0}{\partial x_0} \frac{\partial w_0}{\partial x_0} \frac{\partial w_2}{\partial x_2} \right] + \left[ \frac{\partial w_0}{\partial x_1} \frac{\partial w_1}{\partial x_1} \frac{\partial w_2}{\partial x_2} \right] + \left[ \frac{\partial w_0}{\partial x_2} \frac{\partial w_1}{\partial x_2} \frac{\partial w_2}{\partial x_2} \right] \begin{bmatrix}
C_0 \\
C_1 \\
C_2
\end{bmatrix}
\]

\[
\left[ \frac{\partial f}{\partial x_{0,1,2}} \right]^T = \left[ \begin{bmatrix}
px_0^{p-1} & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & px_1^{p-1} & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & px_2^{p-1}
\end{bmatrix} \right] \begin{bmatrix}
C_0 \\
C_1 \\
C_2
\end{bmatrix}
\]

(2.13)

The gradient vector for all three design variables for the first element of the example problem would then be:

\[
\left[ \frac{\partial f}{\partial x_{0,1,2}} \right]^T = \begin{bmatrix}
px_0^{p-1}C_0 & px_1^{p-1}C_1 & px_2^{p-1}C_2
\end{bmatrix}
\]

2.7.2 Multi-Material SIMP Sensitivity

The multi material SIMP form is given in equation 2.7. The generalized analytic gradient for this equation is:

\[
f(x_i) = \sum_{i=1}^{n} [(x_i)^p \prod_{j=1, j \neq i}^{n} (1 - (x_j)^p)] C_i
\]

\[
\frac{\partial f}{\partial x_i} = px_i^{p-1} \left\{ \sum_{j=1, j \neq i}^{n} (1 - x_j^p) C_i - \sum_{j=1, j \neq i}^{n} x_j^p \prod_{k=1, k \neq j}^{n-1} (1 - x_k^p) C_j \right\}
\]

(2.14)

Note that as in the original SIMP equation, an implication of the above equation is that any design variable which is used to determine SIMP weights for a specific element is not used again. However, unlike the original SIMP equation, design variables are used in the calculation for each SIMP weight in an element. Although this format allows gradients to be computed independently for each element, the gradient calculation for each element is now more involved. Considering the six design variable, two element sample problem discussed in section 2.4, and
using equation 2.7 the matrix form of this particular method for the first element that is a function of three design variables, may be expressed in SIMP weights as:

\[
C^0 = \begin{bmatrix}
  w_0(x_0, x_1, x_2) & w_1(x_0, x_1, x_2) & w_2(x_0, x_1, x_2)
\end{bmatrix} \begin{bmatrix}
  C_0 \\
  C_1 \\
  C_2
\end{bmatrix}
\]

For calculation purposes the weighting equation may also be recast in terms of the design variables, where each column in the design variable matrix corresponds to a SIMP weight \( w \).

\[
C^0 = \begin{bmatrix}
  (w_0)_{(0)}(1 - (x_1)^p)(1 - (x_2)^p) & (w_1)_{(0)}(1 - (x_0)^p)(1 - (x_2)^p) & (w_1)_{(2)}(1 - (x_0)^p)(1 - (x_1)^p)
\end{bmatrix} \begin{bmatrix}
  C_0 \\
  C_1 \\
  C_2
\end{bmatrix}
\]

If evaluated, equation 2.15 above would provide a solution equivalent to equation 2.6. Since it is apparent that the SIMP weights for this particular element are now functions of each related design variable, the gradient is also more complex. Gradients for the objective with respect to each design variable \( x_0, x_1 \) and \( x_2 \) are shown below.

\[
\frac{\partial f}{\partial x_{0,1,2}} = \begin{bmatrix}
  \frac{\partial w_0}{\partial x_0} \\
  \frac{\partial w_0}{\partial x_1} \\
  \frac{\partial w_0}{\partial x_2}
\end{bmatrix}^T + \begin{bmatrix}
  \frac{\partial w_1}{\partial x_0} \\
  \frac{\partial w_1}{\partial x_1} \\
  \frac{\partial w_1}{\partial x_2}
\end{bmatrix}^T + \begin{bmatrix}
  \frac{\partial w_2}{\partial x_0} \\
  \frac{\partial w_2}{\partial x_1} \\
  \frac{\partial w_2}{\partial x_2}
\end{bmatrix}^T \begin{bmatrix}
  C_0 \\
  C_1 \\
  C_2
\end{bmatrix}
\]

\[
\frac{\partial f}{\partial x_{0,1,2}} = \begin{bmatrix}
  px_0^{p-1}(1 - x_1^p)(1 - x_2^p) \\
  -px_0^{p-1}x_1^p(1 - x_2^p) \\
  -px_0^{p-1}x_2^p(1 - x_1^p)
\end{bmatrix}^T + \begin{bmatrix}
  -px_1^{p-1}x_0^p(1 - x_2^p) \\
  px_1^{p-1}(1 - x_0^p)(1 - x_2^p) \\
  -px_1^{p-1}x_2^p(1 - x_0^p)
\end{bmatrix}^T
\]

\[
+ \begin{bmatrix}
  -px_2^{p-1}x_0^p(1 - x_1^p) \\
  -px_2^{p-1}x_1^p(1 - x_2^p) \\
  px_2^{p-1}(1 - x_0^p)(1 - x_1^p)
\end{bmatrix}^T \begin{bmatrix}
  C_0 \\
  C_1 \\
  C_2
\end{bmatrix}
\]

(2.16)

When examining the above matrix, it is apparent that the partial derivative of the design variable corresponding to that weight in particular is treated differently than other terms (e.g. \( \frac{\partial w_i}{\partial x_i} \) as compared to \( \frac{\partial w_i}{\partial x_{j \neq i}} \)). The full derivative, as shown in equation 2.14 may be divided into components satisfying each of these cases. The derivative of the first term, denoted as \( A \) in equation 2.14 corresponds to the first half of equation 2.14 and the derivative of the second term, denoted as \( B \) in equation 2.14 correspond to the second half. The general analytic derivative
of each of the terms is:

\[
A : \quad \frac{\partial w_i}{\partial x_i} = px_i^{p-1} \prod_{j=1, j \neq i}^n (1 - x_j^p)C_i \tag{2.17}
\]

\[
B : \quad \frac{\partial w_i}{\partial x_{j \neq i}} = px_i^{p-1} x_j \prod_{k=1, k \neq j}^{n-1} (1 - x_k^p)C_j \tag{2.18}
\]

where:

\( p \) is the penalty

\( n \) is the number of design variables in this particular optimization

\( x \) is a design variable

The gradient with respect to all three design variables for the first element of the example problem in section 2.7 would be:

\[
\begin{bmatrix}
px_0^{p-1}(1 - x_1^p)(1 - x_2^p) - px_0^{p-1}x_1(1 - x_2^p) - px_0^{p-1}x_2(1 - x_1^p) \\
px_1^{p-1}(1 - x_0^p)(1 - x_2^p) - px_1^{p-1}x_0(1 - x_2^p) - px_1^{p-1}x_2(1 - x_0^p) \\
px_2^{p-1}(1 - x_0^p)(1 - x_1^p) - px_2^{p-1}x_0(1 - x_1^p) - px_2^{p-1}x_1(1 - x_0^p)
\end{bmatrix} C_0
\]

\[
\begin{bmatrix}
px_0^{p-1}(1 - x_1^p)(1 - x_2^p) - px_0^{p-1}x_1(1 - x_2^p) - px_0^{p-1}x_2(1 - x_1^p) \\
px_1^{p-1}(1 - x_0^p)(1 - x_2^p) - px_1^{p-1}x_0(1 - x_2^p) - px_1^{p-1}x_2(1 - x_0^p) \\
px_2^{p-1}(1 - x_0^p)(1 - x_1^p) - px_2^{p-1}x_0(1 - x_1^p) - px_2^{p-1}x_1(1 - x_0^p)
\end{bmatrix} C_1
\]

\[
\begin{bmatrix}
px_0^{p-1}(1 - x_1^p)(1 - x_2^p) - px_0^{p-1}x_1(1 - x_2^p) - px_0^{p-1}x_2(1 - x_1^p) \\
px_1^{p-1}(1 - x_0^p)(1 - x_2^p) - px_1^{p-1}x_0(1 - x_2^p) - px_1^{p-1}x_2(1 - x_0^p) \\
px_2^{p-1}(1 - x_0^p)(1 - x_1^p) - px_2^{p-1}x_0(1 - x_1^p) - px_2^{p-1}x_1(1 - x_0^p)
\end{bmatrix} C_2
\]

### 2.7.3 Multi-Material SIMP with Normalization Sensitivity

The multi-material SIMP equation, 2.7, can also be normalized, as discussed in Section 2.5. The normalized SIMP equation is equation 2.8 and is reproduced below.

**Step 1**: \( S_w = \sum_{k=1}^{n_e} [(x_k^e)^p \prod_{j=1}^{n} (1 - (x_j^e)^p)] \)

**Step 2**: \( C_e = \sum_{q=1}^{n_e} \left( \frac{1}{S_w} \right) \left[ (x_q^e)^p \prod_{j=1}^{n} (1 - (x_j^e)^p) \right] C_q \)

Collapsing the above form in terms of weights results in equation 2.19 below.

\[
C_e = \sum_{q=1}^{n_e} \left( \frac{w_q}{S_w} \right) C_q \tag{2.19}
\]
Taking the analytic derivative of this equation, using the quotient rule results in:

\[
\frac{\partial \xi_q}{\partial x_i} = \left[ \frac{\partial w_j}{\partial x_i} \right] - \frac{w_j \sum_{k=1}^{n^e} \frac{\partial w_k}{\partial x_i}}{(S_w)^2} C_j \quad i, j, q = 1, 2, \ldots, n^e
\]  

(2.20)

where:

\[ i, j \] are indices indicating the particular design variable and weight being considered

\[ n^e \] is the number of design variables and hence weights in the element of interest

Using the two element six design variable example, and equation 2.16, the normalized SIMP gradient calculation for the three design variables related to element one would have the algebraic form:

\[
\frac{\partial f}{\partial x_{0,1,2}} = \left[ \begin{array}{c} \frac{\partial [x_0 w_0]}{\partial x_0} \\ \frac{\partial [x_1 w_0]}{\partial x_0} \\ \frac{\partial [x_2 w_0]}{\partial x_0} \\ \frac{\partial [x_0 w_1]}{\partial x_0} \\ \frac{\partial [x_1 w_1]}{\partial x_0} \\ \frac{\partial [x_2 w_1]}{\partial x_0} \end{array} \right]^T + \left[ \begin{array}{c} \frac{\partial [x_0 w_2]}{\partial x_0} \\ \frac{\partial [x_1 w_2]}{\partial x_0} \\ \frac{\partial [x_2 w_2]}{\partial x_0} \end{array} \right]^T + \left[ \begin{array}{c} \frac{\partial [x_0 w_0]}{\partial x_1} \\ \frac{\partial [x_1 w_0]}{\partial x_1} \\ \frac{\partial [x_2 w_0]}{\partial x_1} \\ \frac{\partial [x_0 w_1]}{\partial x_1} \\ \frac{\partial [x_1 w_1]}{\partial x_1} \\ \frac{\partial [x_2 w_1]}{\partial x_1} \end{array} \right]^T + \left[ \begin{array}{c} \frac{\partial [x_0 w_2]}{\partial x_2} \\ \frac{\partial [x_1 w_2]}{\partial x_2} \\ \frac{\partial [x_2 w_2]}{\partial x_2} \end{array} \right]^T \left[ \begin{array}{c} C_0 \\ C_1 \\ C_2 \end{array} \right]
\]  

(2.21)
Chapter 3

Engineering Cost Modelling

3.1 Cost Modelling in Design

Although the field of Multi-Disciplinary Optimization is relatively new, it is accepted that an important sub-discipline to consider, along with other more traditional disciplines, is cost. Since most engineering designs are undertaken in an environment where money is limited, it is important that the design satisfy the need while remaining economic. Unfortunately, the majority of the overall cost to produce a given design is usually fixed after the preliminary configuration has been decided [3]. The additional difficulty is that it is extremely difficult to estimate the cost of a part, and even defining how costs are to be accounted when a part is in daily production is very difficult. This is complicated by the inter-related nature between individual recurring costs, non-recurring costs, infrastructure and deciding where to draw the line in terms of accounting for all cost factors. Clearly, from a preliminary design standpoint, considering all potential options and their associated costs in the context of a single part would likely be intractable and provide very little insight. Therefore, preliminary design cost estimation methodologies necessarily simplify the assumptions surrounding costs. An example of this would be to exclude the non-recurring costs, such as re-usable tooling, associated with a particular design, and only consider recurring costs, such as material and labour. Cost estimation has traditionally been consigned to the financial field; however, it is now being treated seriously from an engineering perspective. As discussed above, considering the total cost picture during preliminary design would be intractable, therefore, it is important to focus upon certain areas of interest when considering cost for a preliminary design. It is also important to determine the various methodologies used to account for costs of a given design, and establish whether they should be applied within the objective of this thesis.
3.1.1 Recurring and Non-Recurring Costs

Formally, recurring costs are costs which occur during the production process of a given design, and account for consumables, labour, overhead and other consumables which are expended on a per product basis. Non-recurring costs are costs which are associated with the initial infrastructure setup, such as tooling, engineering design and production system setup. Recurring costs are proportional to the number of parts made, but when discussing mass-produced parts, it is typical also to assume that recurring costs decrease during the production run, as a learning curve effect and subsequent efficiency increases may occur. A pertinent optimization study of learning curve effects on the recurring costs of composite laminated structures is the subject of a paper by Apostolopoulos [21].

3.1.2 Direct and Indirect Costs

Direct costs are those which are easily discretized into clear sub-units, and can be clearly accounted for in the production of a product, or delivery of a service. An example of direct costs would be the time required to assemble a component, or the purchasing costs of a given material. Indirect costs are costs which are still associated with the production of a part or delivery of a service, but are not easily discretized into a per unit basis, and must be allocated to the entire process. Some examples of indirect costs would be the cost of utilities, building maintenance and employee amenities. Indirect costs are usually referred to as ‘overhead’.

3.1.3 Life Cycle Costs

In the context of a delivered system, it is common to perform a life-cycle cost analysis. Life cycle costs are accounted for differently by different authors, but generally, attempt to take into account the cost of a given system or part from the moment of design until eventual disposal or recycling. An excellent review of life cycle cost analysis has been performed by Asiedu and Gu [22]. Interest in life cycle costing is increasing due to environmental stewardship concerns. A subset of life cycle costing which is of particular interest to the airline industry is Direct Operating Cost (DOC), which will be addressed further in section 3.7 of this document.

3.2 Cost Estimation Methods

A comprehensive review of aerospace engineering cost modeling has been compiled by Curran [23] and covers a wide variety of cost modeling methodology, including direct, indirect and life cycle costs, activity based costing (ABC), analogous, parametric, bottom-up, feature based, fuzzy logic, neural networks, uncertainty and genetic causal modeling. The two significant
conclusions of the review are that there is no single established theoretical approach to cost modeling, and that traditional design to meet performance and technical specifications will, in the future, be supplemented with cost estimation tools to help designers. Of the many varied methodologies listed by Curran in the review, particular interest lies in the cost estimating methods, which are broken down by Curran into conventional and advanced methods.

### 3.2.1 Conventional Methods

Conventional methods are broken down into analoguous or case-based reasoning methods, parametric methods, or activity based costing and bottom up methods.

**Analogous / Case-Based Reasoning**

The analogical method is also known as the case-based method. The method consists of a method whereby costs are estimated based on commonality with existing designs. This procedure requires a great deal of data about previous such designs, such that a conclusion can be drawn with regards to the new design. This also implies that the new design must share significant commonality with previous designs, or this method will not provide an accurate means of cost estimation. A comparison between the case based reasoning and parametric cost based methods was studied by Duverlie [24], where it is shown that to employ either method requires significant in-depth knowledge of the manufacturing processes such that accurate quantification of costs can occur. The case-based method was also studied by Rehman [3], where cost estimation was automated by linking design to production knowledge.

**Parametric Methods**

The parametric method is employed by estimating the cost of the design based on a few specific features, which are observed to change as cost changes, from which cost estimation relationships (CERs) can be developed. These are incorporated into an algorithm for cost estimation. These relationships are general; however, they are usually only applied to a specific design type with a specific production method. The parametric method is also employed by Watson [25] to estimate costing of machined outsourced parts for an aerospace supply chain. A study of costing tools for decision making within integrated aerospace design, with particular emphasis on the effect on cost of relaxing design tolerances, has been conducted by Curran [26]. Kassapoglou [27] [28] implemented a combined weight and cost minimization scheme related to the optimization of helicopter fuselage frames, considering two metallic and two composite fabrication schemes, and including manufacturing constraints, while using parametric equations for cost estimation. As with Curran [26], the parametric equations were obtained using empirical proprietary data.
sourced from actual production floor statistics.

**Activity Based Costing**

The analytical method, or activity based costing, is also known as the bottom up approach, and is the most time-consuming and typically most accurate step. It requires a decomposition of the product design into elementary steps, and then a full accounting of the time, materials and processes applied allow direct dollar values to be inferred. Specific processes are broken down and published in manuals for reference, an example being the Advanced Composites Cost Estimating Manual (ACCEM)\[29\]. An example of a simplified activity based costing model is that proposed by Gutowski et al \[6\][30][31][32]. Bao \[18\] demonstrates that assembly and manufacturing costs are more significant than material costs when applying the process-based manufacturing cost models of Gutowski et al into a traditional performance based MDO problem.

### 3.2.2 Advanced Methods

Advanced methods addressed are: Feature Based Modelling, Neural Networks and Fuzzy Logic.

**Feature Based Modelling**

Feature based modelling is based upon the concept that cost functions can be associated with certain design features, and that linked together, these features will contribute to the global cost estimation. Curran \[23\] proposes a genetic-causal approach which employs this aspect of feature based modelling.

**Neural Networks**

Neural networks are heuristic computer algorithms which seek to ‘learn’ relationships between variables by using historic data and ‘training’ the network using some concepts from biology. These networks can detect relationships between variables which are not intuitively evident to a human observer; however, to learn properly, these networks need large historical databases as precedents, and the new product to be analyzed must be fairly similar to those contained in the databases.

**Fuzzy Logic**

Fuzzy logic addresses the uncertainty in costing by using fuzzy sets. Fuzzy sets establish a relationship between the design variables, constraints, objectives and the design space, using a complex set of rules to determine the relationships. These rules are a sequence of ‘if-then’ rules
which establish different relationships between the variables based upon previous decisions. As with neural networks, these are not yet widely understood or deployed.

### 3.2.3 Uncertainty in Costing

Costing with uncertainty is not a costing methodology of itself, but is an advanced way of modifying any of the other methods listed. Although there are significant differences in the models described above, only the advanced and underutilized neural network and fuzzy logic methods explicitly take into account non-deterministic costing. Since uncertainty exists at all points in the life cycle of a system or part, and the degree of variability may change based on outside factors, to accurately calculate the cost of a given system or part with great precision is unrealistic. The inclusion of stochastic modelling adds a degree of complexity to the models, but may be necessary to provide a more realistic assessment of the potential costs associated with a particular design. Scanlan et. al [33] propose an approach applying statistical modeling to cost in the preliminary design phase, with a reduction in uncertainty and gradual refinement of cost estimates as the design definition matures, although some uncertainty is posited throughout. Rais-Rohani [16] applied probabilistic methods to the optimization of a spar model, while incorporating reliability, manufacturability and cost constraints, and the interaction between these factors was investigated. Uncertainty is also addressed by Frangopol [34], in which he reviews the life-cycle reliability based optimization field. Of particular interest is his highlighting of the large degree of uncertainty associated specifically with composite structures. Peoples [14] presents a value-based MDO with business-risk uncertainty for aircraft design, in which stochastic models represent uncertainty in the marketplace.

### 3.3 Cost Integration

Deployment of the cost models into design optimization usually takes one of two forms, typically described as Design For Cost (DFC) or Design To Cost (DTC), where the former refers to a design process where cost is taken into account, typically using some kind of weighting variable, and the latter case typically sets a specified cost target. Unfortunately, several authors believe that imposing a fixed cost in an optimal design context leads to an underperforming design, which is still not guaranteed to actually cost what the cost estimate indicates [7]. This has led to the implementation of a cost-based integrated design optimization scheme, as proposed by Iqbal [4], and this scheme has been shown to give a good trade-off between both traditional structural performance requirements and cost considerations. This scheme is known as design for integrated cost and performance (DICP), because it contains both performance and cost metrics within the objective function. Conversely, if the cost is directly linked to the performance, then
cost may be minimized alone, and this would achieve a similar result to DICP.

3.4 Selected Cost Estimation Method - Gutowski Process Based Cost Model

The cost model which is used combines some of the simplistic attributes of an analytical method with some of the detailed accuracy of an activity based model, while remaining general enough to be utilized in a preliminary design situation. The theoretical cost model developed by Gutowski et al. [6],[30],[31],[32] and studied by several graduate students at MIT, has yielded a fairly simple mathematical form which should be generalizable to both additive (composite) and subtractive (machining) processes. Through several experimental studies, the model showed a strong correlation with experimental results. This model is also being used by Boeing to make cost estimates, as described by Neoh [31]. The additional benefit of this model is that it provides a preliminary tool, but also allows for the subsequent refinement of fundamental timing estimates based on real-world data, which allows the model to be refined further, as time and resources permit. Note that in the instance of this thesis, the time constant data used was that provided in Neoh’s thesis.

3.4.1 Gutowski Theoretical Cost Model - Size Effects

The theoretical cost model developed by Gutowski et al is designed around the concept that the detailed process models used in bottom-up accounting provide the most accurate results, but are too detailed for uncharacterized processes. The approach, therefore, is to develop a simple dynamic model for primitive steps, to simplify and sum each of these steps, and then develop some form of complexity theory to address part complexity. Considering all processes as a combination of primitive steps allows a relationship between one or two variables to be established with respect to time. Well characterized primitive steps are tabulated in time standards, as in the Means Man Hour Standards book used in the construction industry [35], or in the ‘AM Cost Estimator’ [36] with regards to machining processes. The Gutowski model assumes that these primitive tasks may be modelled as first-order systems in velocity response, since the majority of these processes move over a region of space, and the time to move will depend upon the distance to be covered. Since most processes are curvilinear in nature, a generalized relationship between the time required and the length (L) of the action in space can be drawn, as:

\[ t_L = f(L) \]
By summing these operations through the number of actions for each length required, the total time for a given operation can be determined. The sum would typically correspond to an area or volume in space, upon which the process was applied. From that, the typical first order equation can be derived. It is of the form

\[
\frac{d\lambda}{dt} = v_0 \left(1 - e^{-\frac{t}{\tau}}\right)
\]

(3.2)

where:

- \( v \) has the dimensions \( \frac{\lambda}{\text{time}} \)
- \( \lambda \) is the appropriate variable for the task, typically a length, but area, volume or weight may also be applied

Integrating the equation above with respect to time \( t \), from 0 to \( t \) leads to the relationship:

\[
\lambda = v_0 [t - \tau (1 - e^{-\frac{t}{\tau}})]
\]

(3.3)

This equation cannot be explicitly inverted for time, but since time is of interest, it can be approximated using:

\[
t \approx \sqrt{\frac{2\tau L}{v_0}} \quad t < \tau \tag{3.4}
\]

\[
t \approx \tau + \frac{L}{v_0} \quad t > \tau \tag{3.5}
\]

Note that in the above equations \( \lambda \) has been replaced with \( L \) as a standard length scale. Note also that these are meant to approximate the different segments of a standard first order equation, with the first term approximating the non-linear ‘ramp up’ phase, and the second term approximating the linear ‘steady-state’ phase. An approximation for the entire range has also been suggested as:

\[
t = \sqrt{\frac{L^2}{v_0^2} + \frac{2\tau L}{v_0}}
\]

(3.6)

This equation, referred to as the Mabson approximation, approximates both segments of the curve. In terms of a conservative estimate, equation 3.5 provides an excellent approximation for long times, and is conservative for short times. A comparison between these time estimation schemes for a 3 inch composite tape hand lay up, with \( \tau = 1.1146 \) minutes, being the ‘rise-time’, and \( v_0 = 4.6355 \, \text{m/min} \), being the steady state velocity, as provided by Neoh, is demonstrated by plotting the time estimates for various tape lengths in Figure 3.1. Note that for all lengths, the conservative estimate gives more time for the procedure than the Mabson approximation, although in the limit these two would converge. Additionally, the conservative estimate necessarily neglects the non-linear portion which begins at zero and ends at the steady-state velocity value.
Figure 3.1: Time estimation comparison, conservative vs. Mabson approximation, hand layup of 3" composite tape

This portion may be more accurately estimated by the Mabson equation; however, Gutowski suggests the conservative form in his paper, [6], because the potential overestimate shown in the figure when \( L = 0 \) is misleading, due to the fact that for most manufacturing processes there are additional time delays. If the development of equation 3.5 is continued, where the process is repeated for \( n \) strips of width \( w \) to create an area \( A \), then a linear sum in the plane would be:

\[
\sum_{1}^{n} t = n\tau + \frac{A}{wv_0} \tag{3.7}
\]

Summing \( N \) individual layers of thickness \( h \),

\[
t = \sum_{1}^{N} \sum_{1}^{n} t = nN\tau + \frac{V}{whv_0} \tag{3.8}
\]

Finally, including delay and setup times, equation 3.8 will be:

\[
t = Setups + N_L(Delays + \tau) + \frac{V}{whv_0} \tag{3.9}
\]

where \( N_L = nN \). This same methodology can be applied to the entire range approximation, as in equation 3.6 and would then yield:

\[
t = \frac{V}{hwv_0} \sqrt{\frac{1 + 2(Delays + \tau)N_Lhwv_0}{V}} + Setups \tag{3.10}
\]
Although equation 3.10 gives a more general relationship between size and process time, including the non-linearity at the start of a given process, it was found in general that the effect of this non-linearity was negligible, and Gutowski suggests that equation 3.9 be used for process estimation. Excellent correlation between various different processes employing different lengths, areas and volumes have been observed, and are illustrated extensively in references [6], [30] and [31].

Finally, although the above discussion relates to a single step of a manufacturing process, it has been found that often one step, such as layup, dominates all the other steps, and that a fair estimation of a total process time for a preliminary design may be calculated based on consideration of only one or two process steps. This simplification was employed by Bao [18] when he performed an MDO study to include process-based manufacturing cost models into a traditional performance-focused optimization.

### 3.4.2 Gutowski Theoretical Cost Model - Complexity Effects

The above model works well for extremely simple parts, built up in a linear manner, but for any more complex fiber layup, additional factors need to be taken into account. Ply orientation effects affect the pieces per volume, or $N_L/V$, and a general formula related to this ratio was developed to deal with ply orientations in a given rectangular beam of composite as:

$$\frac{N_L}{V} = \frac{\overline{D}H}{wh}$$

(3.11)

where:

- $H$ is the part thickness
- $w$ is the strip width
- $h$ is the strip thickness
- $\overline{D}$ is the pseudo-width of the part, based upon a critical angle $\psi_{crit}$

The pseudo-width of a rectangular part, used to determine how many strips of tape are required if tape is being used, uses two equations dependent upon a critical angle $\psi_{crit}$. $\psi$ is the angle of the applied fiber angle measured relative to the length axis of the part. The critical angle is calculated as:

$$\psi_{crit} = \tan\left(\frac{W}{L}\right)$$

(3.12)

where:

- $W$ is the part width
Chapter 3. Engineering Cost Modelling

$L$ is the part length

The pseudo-width is calculated as:

\[
D = \left( L + \frac{W}{\tan \psi} \right) \sin \psi \quad \text{if } \psi > \psi_{\text{crit}}
\]

\[
D = \cos \psi (W + L \tan \psi) \quad \text{if } \psi < \psi_{\text{crit}}
\]

where:

$\psi$ is the angle of the applied fiber angle measured relative to the length axis of the part

Ply drop-off effects can be taken into account as well, but these were not included in the implemented model, therefore the equations are not included here. For simple bend curvature of the type which may potentially be encountered in the stringers, it was found that lay-up times varied linearly with cumulative bend angle. $\tau$ would be modified to include the linear relationship of this angle, to become

\[
\tau = \tau_0 + \sum_{j=1}^{N_e} b_j \theta_j
\]

where:

$b_j$ is a proportionality constant with units of $\frac{1}{\text{rad}}$

$\theta_j$ is a summation through $j$ of the enclosed angles

Enclosed angle is calculated as:

\[
\theta_n = \pi - 2 \cos^{-1}\left[ \sin \psi \cos \left( \frac{\alpha}{2} \right) \right]
\]

where:

$\alpha$ is the surface bend angle

$\psi$ is the angle of the fiber measured from the bend axis (i.e. $\psi = 90$ is perpendicular to the bend)

A full explanation of these formulas may be found in Neoh [31].

3.4.3 Gutowski Theoretical Cost Model - Cost Driver

As discussed in section [3.4.1] research showed that in most manufacturing processes, a few process steps dominate the others. This ‘vital few and trivial many’ rule implies that a few or even a single process will be the primary cost driver, and that if this were to be scaled to the
appropriate level, an approximation of the total process time could be calculated. This allows general process steps to be estimated, using the equation:

\[
    t_{total} = m(t_{most\text{dominating}})
\]

where \( m \) is

\[
    m = 1 + \frac{t_{2nd\text{most\text{dominating}}}}{t_{most\text{dominating}}} + \sum \frac{t_{avg\text{other steps}}}{t_{most\text{dominating}}}
\]

This scaling methodology allows time estimates to be made for processes which are not fully characterized, as would be the case in preliminary design. As long as the few ‘vital’ steps in the process are correctly identified, and the scaling relationship established, total process time can be estimated. Process time can then be multiplied by an appropriate factor to give a dollar value if required, or left as a time value.

3.4.4 Cost Model Implementation

The cost modeling method outlined above is the generalized method. It is employable for any type of fabrication step in which a linear, areal or volumetric operation is to be employed. Specific fabrication steps are selected for the sample problem to illustrate the versatility of the model. Additional steps or different processes altogether can be accommodated within the framework, if necessary. Since most aerospace stiffened plate structures consist primarily of aerospace grade aluminum, laminated composites, or some combination of the two, these two materials and their unique fabrication processes are considered. The model is posed in such a way that any particular layer may be composed of one of several optional materials; therefore, it is possible that the preliminary design for a multi-ply structure may consist of alternating layers of composite and metallic materials. This form of composite structure already exists in industry, in the form of GLARE (Glass Laminate Aluminum Reinforced). GLARE is utilized by Airbus Industries, and consists of alternating layers of thin aluminum sheet 0.016 in thick, interspersed with glass reinforced epoxy, which are bonded together [37]. The intention of the model as implemented is to facilitate preliminary design optimization and not precisely accurate costing; therefore, detailed processing steps such as transportation, placement, removal and positioning steps are omitted from the model. These steps can be modelled, and specific information about their details exist in the thesis by Neoh [31], but their contribution is not large when compared to major dominating steps. Processing steps which would be common to both materials are omitted, as they provide no means of differentiation for the optimizer. Although the processing steps may be quite varied, they are all calculated using a linear, areal or volumetric variation of equation 3.6 or, in the case of no geometric dependence, a combination of setup and delay constants. Process calculations implemented in the model that are unique to aluminum
Fabrication Process Constants

<table>
<thead>
<tr>
<th>Material</th>
<th>Process</th>
<th>Geometry</th>
<th>$v_0$</th>
<th>$m^n_{\text{min}}$</th>
<th>$\tau$ (min)</th>
<th>Setup</th>
<th>Delay</th>
<th>Eqn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>Mask Part</td>
<td>Area</td>
<td>0.09032</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td></td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>Chemimill</td>
<td>Thickness</td>
<td>$2.032E-5$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td>3.6</td>
</tr>
<tr>
<td>Graphite-Epoxy</td>
<td>Hand Layup</td>
<td>Area</td>
<td>0.353</td>
<td>1.146</td>
<td>2.5</td>
<td>1</td>
<td></td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>Automatic Trim</td>
<td>Perimeter</td>
<td>1.494</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td></td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>Debulk</td>
<td>Area</td>
<td>0.06451</td>
<td>2.1</td>
<td>5</td>
<td>0.083</td>
<td></td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>Remaining Steps</td>
<td>Area</td>
<td>1.102</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>3.9</td>
</tr>
</tbody>
</table>

Table 3.1: Manufacturing Time Constants

sheet costing include masking the sheet in preparation for chemimilling, and chemimilling the sheet itself to the desired thickness. The chemimilling step acts on the area of the part, but the time required is a function of the thickness of the initial sheet and thickness of the final sheet, requiring the linear version of the fabrication time equations. Process calculations implemented in the model that are unique to laminated composite costing include hand tape layup, automatic trimming, debulking, and remaining smaller steps, where the remaining steps include vacuum bagging, damming and dam removal. In the case of the layup step, equations 3.13 and 3.14 are used to determine the number of strips required, based on the desired layup angle. The selected processes and their salient constants are summarized in Table 3.1.

Note that the number of people involved in each process, the crew size, is 1 for each of the above cases, but the capacity to change the number of involved personnel in a fabrication step is included in the model. These processes provide time estimates, based on part geometry, which can be converted to cost by multiplying by the labour cost per hour. As illustrated by equations 3.6 and 3.9 and by Figure 3.1 there is a time benefit for one continuous operation over a given geometry, rather than repeated operations. Since the sample problem is discretized both for the structural solution and for design variable selection purposes, it is important that this effect be included in the model. For this reason, the fabrication time for a given part of the structure, such as the plate or a single stiffener, is estimated as if that entire structure were to be fabricated using that single material. Once this calculation has been performed, the final fabrication time is then divided by the number of discrete elements contained in that structure, and then multiplied by the appropriate labour time and SIMP weight to determine the cost contribution of that particular material for that particular element. This exercise may be repeated for multiple layers. The fabrication cost calculation corresponds to the $C_i$ term discussed in equations 2.3, 2.7 and 2.8. Equation 2.4 is now reproduced below, with the weight
terms collapsed and the cost term expanded.

\[ Cost^e = \sum_{i=1}^{n_e} \left[ \sum_{j=1}^{n_e} (x_{ej}^e)^p \prod_{j=1}^{n_e} \left[ 1 - (x_{ej}^e)^p \right] \right] C_i \]

\[ Cost^e = \sum_{i=1}^{n_e} \left( \sum_{j=1}^{m} \left( t_j^i (L, W, H) \right) \right) \]

\[ (3.19) \]

where:

\( n \) is equal to the number of materials to be considered in the element of interest \( e \), that is also equal to the number of design variables \( x \)

\( i \) is an index indicating the current material

\( k \) is the number of discretized elements comprising the current structure

\( l \) is the labour cost per hour, including labour overhead

\( m \) is the number of fabrication processes for the current of material of interest \( i \)

\( j \) is an index indicating the current process being considered

\( t \) is the time required for a particular process, which is a function of \( L, W, H \), the geometric features of the current structure

Note that the above calculation may be duplicated for each ply in a given structure. Additionally, a special case exists in terms of the thickness term, denoted in equation (3.19) as \( H \), which is applied only over the layer of interest, whereas the other geometry terms \( L \) and \( W \) are taken to be over the entire structure of interest. The implementation of this method is best illustrated by an example.

**Cost Calculation Example**

The example problem consists of the following constant parameters, listed below:

- Dimensions: 1 x 0.5 x (2 x 0.000125 m)
- Number of Discretized Elements: 2 x 1
- Material Options: 0 = Aluminum, 1 = Graphite/Epoxy at 0 degrees, 2 = Graphite/Epoxy at 45 degrees.
• Manufacturing Constants as in Table 3.1

• Graphite / Epoxy tape width = 0.0762 m (3"

• Labour Cost = \$35.09/hr as indicated by Statistics Canada [38]

• Design Variable vector = [(x^0_0), (x^0_1), (x^0_2), (x^1_0), (x^1_1), (x^1_2), (x^2_0), (x^2_1), (x^2_2), (x^3_0), (x^3_1), (x^3_2)]
  = [0.4, 0.1, 0.5, 0.3, 0.3, 1, 0, 0, 0.1, 0.1, 0.9]

Note that the design vector above uses the notation

\((x^e_l)_m\)

where:

e is the element number

l is the layer number

m is the material number

Note also that the sum of the SIMP weights in this case are normalized to 1, meaning that the weights on the three materials comprising each layer of each element sum to 1. Following the cost procedure indicated above, the cost to manufacture a ply of dimensions 1 x 0.5 x 0.000125 m in Aluminum, using the manufacturing processes in the order listed in Table 3.1 can be calculated. The first time calculation is the masking step, which is an area-dependent step. Recall that these time calculations are applied to the entire area of the plate in question.

\[
t^0_0 = \sqrt{\frac{A^2}{v_0^2} + \frac{2\tau A}{v_0} + setup + delays}
\]

\[
t^0_0 = \sqrt{\frac{(0.5)^2}{0.09032^2} + \frac{2(0)(0.5)}{0.09032} + 3 + 0}
\]

\[
t^0_0 = 8.5357
\]

The second manufacturing step considered for aluminum would be chemimilling the sheet to the required thickness. Given that a standard 35 gage aluminum sheet is 0.1422 mm thick, and the ply thicknesses are required to be 0.125 mm thick, the chemimilling step will need to remove
0.0172 mm of material.

\[ t_0^1 = \sqrt{\frac{t^2}{v_0^2} + \frac{2\tau}{v_0}} + \text{setup + delays} \]

\[ t_0^1 = \sqrt{\frac{(1.72E-5)^2}{(2.032E-5)^2} + \frac{2(0)(1.72E-5)}{(2.032E-5)^2}} + 1 + 0 \]

\[ t_0^1 = 1.8464 \]

It is important to note that sheet placement / alignment and transportation times are not included, because it is assumed these will be similar for both aluminum and Graphite / Epoxy sheets, and that these will provide no differentiation between the two. Further it is assumed that fastening costs between plies of varying materials follows an identical process irrespective of material, and is not time consuming, relative to the steps indicated. The fabrication time assumptions above are intended to provide a first order approximation for preliminary design. Further details can be included into the framework to achieve desired accuracy.

Continuing with the example, the Graphite Epoxy fabrication costs will be identical in all respects except for the fibre placement, and therefore, in the interest of brevity, all remaining steps will be calculated only once, with the exception of the ply layup steps. Checking the critical angle for this part:

\[ \Psi_{\text{crit}} = \tan\left(\frac{W}{L}\right) \]

\[ \Psi_{\text{crit}} = \tan\left(0.5\right) \]

\[ \Psi_{\text{crit}} = 26.565^\circ \]

\(\overline{D}\) for the 45 degree case is:

\[ \overline{D}_{45} = (L + \frac{W}{\tan\psi})\sin\psi \]

\[ \overline{D}_{45} = (1 + \frac{0.5}{\tan(45)})\sin(45) \]

\[ \overline{D}_{45} = 1.0606 \]

Using this to determine how many strips of the composite will be deposited:

\[ (N_L)_{45} = \frac{\overline{D}_{45}}{w} \]

\[ (N_L)_{45} = \frac{1.0606}{0.0762} \]

\[ (N_L)_{45} = 6.5616 \]
$\overline{D}$ for the 0 degree case is:

\[
\overline{D}_0 = \cos \psi (W + L \tan \psi) \\
\overline{D}_0 = \cos(0)(0.5 + 1.0 \tan(0)) \\
\overline{D}_0 = 0.5
\]

Using this to determine how many strips of the composite will be deposited:

\[
(N_L)_0 = \frac{\overline{D}}{\text{stripwidth}} \\
(N_L)_0 = \frac{0.5}{0.0762} \\
(N_L)_0 = 13.9194
\]

A modified Mabson equation is used for the composite ply layup. This form takes into account the repetitions of the ply strips without calculating precise areas covered in each strip layup:

\[
t_1^0 = \sqrt{\frac{a^2}{v_0^2} + \frac{2(\tau + \text{delays})AN_L}{v_0}} + \text{setup} \\
t_1^0 = \sqrt{\frac{(0.5)^2}{0.353^2} + \frac{2(1.146 + 1)(0.5)(6.5616)}{0.353}} + 2.5 \\
t_1^0 = 8.9706
\]

The time required for the 45 degree layup is:

\[
t_2^0 = \sqrt{\frac{a^2}{v_0^2} + \frac{2(\tau + \text{delays})AN_L}{v_0}} + \text{setup} \\
t_2^0 = \sqrt{\frac{(0.5)^2}{0.353^2} + \frac{2(1.146 + 1)(0.5)(13.9194)}{0.353}} + 2.5 \\
t_2^0 = 11.8073
\]

The remaining calculations are applied to the part once it has been completed; that is, once all plies are laid down. Therefore, these are calculated individually for each ply for computational purposes, and then divided by the number of plies. The automatic trim calculation is applied to the perimeter of the plate, and is shown below:

\[
t_1^1 = t_2^1 = \sqrt{\frac{p^2}{v_0^2} + \frac{2\tau_P}{v_0}} + \text{setup + delays} \\
t_1^1 = t_2^1 = \sqrt{\frac{(3)^2}{1.494^2} + \frac{2(1)(3)}{1.494}} + 2.5 + 1 \\
t_1^1 = t_2^1 = 9.8369
\]
The debulking calculation is an area related calculation:

\[
\begin{align*}
\tau_1^2 &= \tau_2^2 = \sqrt{\frac{A^2}{v_0^2} + \frac{2\tau A}{v_0} + \text{setup} + \text{delays}} \\
\tau_1^2 &= \tau_2^2 = \sqrt{\frac{(0.5)^2}{6.451E-2^2} + \frac{2(2.1)(0.5)}{6.451E-2} + 5 + 8.3E-2} \\
\tau_1^2 &= \tau_2^2 = 14.706
\end{align*}
\]

Remaining process steps include rapid operations associated with composite manufacture, such as debagging. These operations are area dependent, and use the simple form, because the Mabson equation is not conservative enough in this instance.

\[
\begin{align*}
\tau_1^3 &= \tau_2^3 = \frac{A}{v_0} + \tau + \text{setup} + \text{delays} \\
\tau_1^3 &= \tau_2^3 = \frac{0.5}{1.102} \\
\tau_1^3 &= \tau_2^3 = 0.4537
\end{align*}
\]

Although each of the above remaining process steps are calculated to only take half of a minute, there are estimated to be 23 such steps in a composite fabrication application of this type [30], therefore the total time from this equation is given as:

\[
\begin{align*}
\tau_1^3 &= \tau_2^3 = (0.4537)(23) \\
\tau_1^3 &= \tau_2^3 = 10.4356
\end{align*}
\]

Using the above information, the fabrication time estimate for a ply of aluminum is:

\[
\begin{align*}
t_0 &= t_0^0 + t_0^1 \\
t_0 &= 8.5357 + 1.8464 \\
t_0 &= 10.3821
\end{align*}
\]

The fabrication time estimate for a ply of 0 degree composite, taking into account the fact that the trim, debulk and remaining process steps will only be performed once over the entire part thickness, is:

\[
\begin{align*}
t_1 &= t_1^0 + \frac{t_1^1 + t_1^2 + t_1^3}{n} \\
t_1 &= 8.9706 + \frac{9.8369 + 14.706 + 10.4356}{2} \\
t_1 &= 26.4599
\end{align*}
\]

The fabrication time estimate for a ply of 45 degree composite is:

\[
\begin{align*}
t_2 &= t_2^0 + \frac{t_2^1 + t_2^2 + t_2^3}{n} \\
t_2 &= 11.8073 + \frac{9.8369 + 14.706 + 10.4356}{2} \\
t_2 &= 29.2966
\end{align*}
\]
By multiplying all of the fabrication time values by their appropriate SIMP weight, dividing for
the level of discretization and then multiplying by labour costs, a cost estimate for the plate is
developed. Note that the actual numerical solutions are abbreviated.

\[
F_{\text{abcost}} = \frac{1}{k} \left[ (t_0^0, t_0^1), (t_0^0, t_1^0), (t_0^0, t_1^1), (t_0^1, t_0^0), (t_0^1, t_1^0), (t_0^1, t_1^0), (t_1^1, t_0^0), (t_1^1, t_0^1), (t_1^1, t_1^0) \right]
\]

\[
F_{\text{abcost}} = \frac{1}{2} \left( \frac{35.09}{60} \right) \left[ 10.3, 26.5, 29.3, 10.3, 26.5, 29.3, 10.3, 26.5, 29.3, 10.3, 26.5, 29.3 \right]
\]

The above solution procedure provides a fabrication time estimate for the plate, taking into
account discretization, while remaining capable of incorporating detailed fabrication informa-
tion. Additionally, since the first order time estimation model is dependent upon a steady state
velocity, the effect of using the entire area to estimate the time ensures that any activities which
can leverage the geometry to their advantage (as the 0 versus 45 degree example above), will be
incorporated and considered in the optimization.
3.5 Material Costs

Including material feedstock costs into the model requires a determination of the relationship between material geometry, material density and material feedstock costs. Material costs can be calculated using the same methodology as the fabrication costs. By altering equation 3.19, the material cost calculation is:

\[
\text{Cost}^e = \sum_{i=1}^{n^e} \left( \sum_{j=1}^{n^e} \left[ (x^e_i)^p \prod_{j \neq i} \left[ 1 - (x^e_j)^p \right] \right] \right) M_i
\]

\[
\text{Cost}^e = \sum_{i=1}^{n^e} \left( \frac{1}{kU_i} (c)(V)((\rho_i)) \right) M_i
\]

\[(3.20)\]

where:

- \(n\) is equal to the number of materials to be considered in the element of interest \(e\), which is also equal to the number of design variables \(x\) related to that element
- \(i\) is an index indicating the current material
- \(k\) is the number of discretized elements comprising the current structure
- \(U\) is the material utilization number, that indicates the amount of waste material
- \(c\) is the the material cost per kg
- \(V\) is the volume of the current part
- \(\rho\) is the material density

This material cost value can be added to the fabrication cost calculated in Section 3.4 to develop a more accurate cost estimation, which includes the mass of the material. Material utilization values can either be estimated or found in the literature, such as the article by Bader [39]. All other constant values are self-explanatory, or are identical to those discussed in section 3.4.4. Equation 3.20 can be calculated in a similar fashion to equation 3.19. Note that this equation can be modified by removing the material cost per kg constant, designated as \(c\) in the equation, resulting in an estimation of the material weight, \(m\). Overall material weight is an important independent parameter that is used in the Direct Operating Cost comparison.
3.6 Cost and Weight Sensitivity Calculation

Due to the fact that fabrication and material cost values are to supply objective values for an optimization problem, sensitivity values with respect to the design variables are desired. Fortunately, the design variables are entirely contained within the SIMP equations, and do not affect the fabrication or material cost equations directly. Therefore, full objective sensitivities can be calculated using the techniques of SIMP function differentiation discussed in Section 2.7, and the fabrication cost, material weight or material cost can all be treated as constants in the context of the partial derivative with respect to the design variable.

3.7 Direct Operating Costs

In order to truly evaluate the effect of a preliminary design in an aerospace context, it is necessary to attach some value to reducing material weight. In the literature this is typically done by attaching a constant dollar value to material mass, and then minimizing cost [7]. However, some literature has pointed to the inclusion of aircraft Direct Operating Cost (DOC) in the analysis [13], since it can incorporate much more detail than a clear linear relationship which doesn’t capture such nuances as the relationship between aircraft size and material weight, maintenance, mission or variable fuel costs. Some literature has included the evaluation of a theoretical aircraft sub-system solely in terms of DOC [40]. A Direct Operating Cost model, including all of these factors can be used to provide additional detail with respect to a preliminary part design, and serves to broaden the context within which it is considered. This model also adds a few layers of complexity, and necessarily requires assumptions on the part of the aircraft dimensions, mission, fuel burn and performance characteristics. However, it is felt that by including such a model ‘on top’ of the existing costing methodology, additional valuable insight can be gained, such as the relationship between increasing fuel cost and preliminary material design. Additionally, it is recognized that the selection of a very expensive yet very light material may or may not lower an aircraft DOC, dependent upon the relative weighting of the acquisition cost and associated financing details when compared to mission length, fuel burn and fuel costs. Using the methods indicated in the ATA DOC calculation method [8], and three NASA contractor reports, [9], [41], [42], a generalized DOC model can be assumed. DOC costs can be broken down into several sub areas, each of which will be briefly discussed in the context of the applied model. Note that in the interest of brevity, some details will be omitted. The sub-areas of the DOC operating model are:

- Crew
- Loan Servicing
Chapter 3. Engineering Cost Modelling

- Depreciation
- Insurance
- Mission Services
- Fuel
- Engine Maintenance
- Airframe Maintenance
- Landing Fees

The preliminary design of the stiffened plate is salient to the above areas in terms of affecting the airframe cost of the aircraft, and also in terms of affecting the fuselage weight of the aircraft. When considering all of the above areas, it is apparent that many of the mission, aircraft weight, aircraft equipment and financial costs will be affected by assumptions about the factors affecting them. Therefore, the intention is to select a generic aircraft of similar features to a real aircraft in a given family size, for comparison purposes and to validate the model, and then use the given assumptions in the optimization to minimize the DOC by changing the design weights. Each of the sub-areas above will be addressed by listing the assumptions made for each and by indicating whether the sub-area is affected by airframe cost, fuselage weight, or both. Each sub-area calculation is calibrated to provide the DOC for a year of operation on the assigned route.

3.7.1 Stiffened Plate Design DOC Integration

Changes in the design variables affecting the stiffened plate design will affect the DOC by changing the acquisition cost of the airframe by some amount, and by changing the mass of the aircraft by some amount. These changes will in turn be related to the sub-areas of the DOC operating model. To break down the effect on acquisition cost and airframe weight, the plate is assumed to compose some amount of the airframe. The fuselage is selected to be composed of some number of these stiffened plates. Again, it is important to note that this assumption is intended to provide an approximation of the effect of changing materials on DOC, not to provide a high-fidelity DOC estimate. Further refinement of the structural model would be required, but the existing framework contains the versatility to do so, if desired. The most important assumption then becomes how much of the total airframe cost and weight the fuselage comprises. The assumption is made that 90% of the surface area of the fuselage of the aircraft is composed of the stiffened plates. Determining the number of plates which compose this surface provides
a factor by which individual cost and weight estimates can be scaled and input into the DOC model. The relevant surface area calculation is:

\[ n = \frac{(0.9)(FSA)}{A} \]

where

- \( n \) is the number of plates required to ‘fill in’ the fuselage
- \( FSA \) is the fuselage surface area, as indicated in Table 3.3
- \( A \) is an individual plate area, as defined by the user during the initial geometry set up

The \( n \) value above is then applied to an individual plate cost and weight estimate, and the final numbers input into the DOC estimation model.

**DOC Weight Breakdown**

In terms of airframe weight, the aircraft empty weight less the weight of the engines \((ACEW - LE)\) is the value upon which some initial fuselage guess percentage should be imposed. Using historical aircraft airframe weight breakdowns, the fuselage is estimated to comprise 22\% of the \((ACEW - LE)\) \[43\]. The remaining airframe weight is assumed to remain unchanged, yet is retained for proper performance calculation. The remaining immutable weight value is now referred to as the aircraft empty weight less engines less fuselage \((ACEW - LE - LF)\). From this base weight, all other weights can be calculated. In this way, the only variable is the airframe fuselage weight. All other values are assumed, although it is important to note that all remaining weight values are derived from existing publicly available airframe information, and are included in Table 3.3. Upon calculation of the fuselage weight \( m \), all other weights can be derived. The overall empty weight \((AC_{OEW})\) would then be:

\[ AC_{OEW} = AC_{EW - LE - LF} + EW + m \]

where \( m \) is the fuselage weight estimate, given by the equation:

\[ m = n(m_{plate}) \]

where the \( m_{plate} \) value is calculated as discussed in equation 3.20 in section 3.5. The maximum take off weight \((AC_{MTOW})\) that is used for range calculation, including the aircraft maximum fuel and maximum payload weight \((AC_{MF - MP})\) would be:

\[ AC_{MTOW} = AC_{OEW}(m) + AC_{MF - MP} \quad (3.21) \]

where:
$AC_{MF_{-}MP}$ is the maximum fuel and payload mass, given in Table 3.3.

Another value of importance is the aircraft no fuel weight, or $(AC_{NFW})$. This value is calculated using the equation below:

\[
(AC_{NFW}) = AC_{MTOW} - (AC_{fuelcap})(\rho_{fuel})
\]

where:

$AC_{fuelcap}$ is the fuel capacity of the aircraft, given in Table 3.3

$\rho_{fuel}$ is the fuel density, which is given in Table 3.3

**DOC Cost Breakdown**

To estimate the airframe cost requires some empirical relationship between cost and airframe weight and also a way of including the calculated plate cost. Using the empirical relationship on P.17 of reference [42], the value for the cost of the airframe without the fuselage is calculated:

\[
AC_{C-LE-LF} = \left(\frac{AC_{EW-LE-LF} \times 2.20462}{1000}\right)^{0.7475}(1.3255E + 6)(i_{93})
\]

where

$AC_{EW-LE-LF}$ is the aircraft empty weight less engines and fuselage

$i_{93}$ is an inflation factor to take into account the fact that this formula was derived in 1993

The inflation factor is used to ensure dollar values are roughly calibrated to current (2006) prices; however, it is important to note that core inflation does not necessarily accurately reflect the inflation pertaining to specific hardware. The above equation can also be used to calculate the airframe weight with the fuselage. Assuming that 50% of the fuselage cost is assembly, and that 22% of the airframe weight is due to the fuselage, taken to be an average of the various existing aircraft types listed in reference [43], the overall aircraft price is:

\[
AC_{PRICE} = AC_{C-LE-LF} + (ENG_n)(ENG_{COST}) + AC_{C-LE}(\frac{FA}{MIT}) + AC_{FUSE-COST}
\]

where:

$AC_{C-LE-LF}$ is the airframe cost without engines or fuselage taken into account, as calculated in equation 3.22.

$ENG_{COST}$ is the cost of an engine, an assumed value

$ENG_n$ is the number of engines, which is assumed and listed in Table 3.4.
$AC_{C-LE}$ is the cost of the airframe without engines

$F$ is a % factor indicating how much of the fuselage cost is due to fuselage assembly costs which are not determined in the stiffened plate model

$A$ is the initial amount of airframe weight which is due to the fuselage

$M$ is a manufacturing support cost factor

$I$ is a miscellaneous manufacturing support cost factor

In order to determine the actual dollar value to purchase an airframe, a spares burden for both airframe and engines is usually assumed as well. The final actual dollar value spent would then be:

$$AC_{PURCHASE} = AC_{PRICE} + (AC_{C-LE})(AF_{SPARES}) + [(ENG_{COST})(ENG_n)](ENG_{SPARES})$$

where

$AF_{SPARES}$ is a % factor used to estimate the spare burden based on the airframe less engine cost

$ENG_{SPARES}$ is a constant estimated based on the initial engine costs

Constant values used in the equations above were extracted from references [9], [41], [42], [44] and [43] and, where not otherwise indicated are listed in Table 3.2. The inflation factor is calculated by inflating OECD core inflation values over the time period.

<table>
<thead>
<tr>
<th>A/C Cost Constant Factors</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation 1993-2006</td>
<td>$i_{93}$</td>
<td>1.39216</td>
</tr>
<tr>
<td>Airframe Spare Burden</td>
<td>$AF_{SPARES}$</td>
<td>0.06%</td>
</tr>
<tr>
<td>Fuselage Assembly Cost (%)</td>
<td>$F$</td>
<td>50%</td>
</tr>
<tr>
<td>Fuselage % of Airframe</td>
<td>$A$</td>
<td>22%</td>
</tr>
<tr>
<td>Manufacturing Support Factor</td>
<td>$M$</td>
<td>1.05</td>
</tr>
<tr>
<td>Miscellaneous Manufacturing Costs</td>
<td>$I$</td>
<td>1.05</td>
</tr>
<tr>
<td>Engine Cost</td>
<td>$ENG_{COST}$</td>
<td>$5700000$</td>
</tr>
<tr>
<td>Engine Spare Burden</td>
<td>$ENG_{SPARES}$</td>
<td>$6600000$ /engine</td>
</tr>
</tbody>
</table>

Table 3.2: A/C Cost Assumptions
3.7.2 Generic Aircraft Assumptions

The generic aircraft is assumed to be a large single-aisle trans-continental ranged aircraft, similar in size and configuration to an existing modern single-aisle aircraft such as the Boeing 737-800 or Airbus A320. The mission is assumed to remain constant during the life cycle of the aircraft, and consists of a mission distance of 2500 km, which takes approximately 3 hours in an aircraft of this type. The missions per year, using the NASA CR-145190 [9] empirical correlation given on P.6, would then be 963, or approximately 2.5 per day. Aircraft performance and weight assumptions are tabulated below. Most of these assumptions are made based upon data freely available for the A320 and 737-800. Indeed, most of the assumed weights and performance numbers are rounded approximations resting in between these aircraft, where possible. Wing surface area is estimated using the 737-800 as a baseline. Fuselage surface area is estimated by using 90% of the surface area a fuselage tube cylinder of 4 m diameter and 40 m length, approximately similar to that of the 737-800 or A320. All other values are closely approximated based upon 737-800 and/or A320 values. Engine performance assumptions are tabulated in

<table>
<thead>
<tr>
<th>A/C Performance and Weight Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Wing Surface Area</td>
</tr>
<tr>
<td>Fuselage Surface Area</td>
</tr>
<tr>
<td>Aspect Ratio</td>
</tr>
<tr>
<td>Total Wetted Area</td>
</tr>
<tr>
<td>Skin Friction Coefficient</td>
</tr>
<tr>
<td>Span Efficiency Factor</td>
</tr>
<tr>
<td>Fuel Capacity</td>
</tr>
<tr>
<td>Seats</td>
</tr>
<tr>
<td>Load Factor</td>
</tr>
<tr>
<td>Service Altitude</td>
</tr>
<tr>
<td>Mission Distance</td>
</tr>
<tr>
<td>Climb / Descend Time</td>
</tr>
<tr>
<td>Max Landing Weight</td>
</tr>
<tr>
<td>Max Payload</td>
</tr>
<tr>
<td>Max Fuel and Payload</td>
</tr>
<tr>
<td>Jet Fuel Density</td>
</tr>
</tbody>
</table>

Table 3.3: A/C Performance and Weight Assumptions
Table 3.4. Engine spare burden was derived based upon data supplied in References [9] and [22].

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Engines</td>
<td>ENG_n</td>
<td>2</td>
</tr>
<tr>
<td>Engine SFC Cruise</td>
<td>SFC</td>
<td>0.62 lb/hr/lb</td>
</tr>
<tr>
<td>Engine Weight</td>
<td>(EN_w)</td>
<td>2318.18 kg</td>
</tr>
<tr>
<td>Engine Cruise Thrust</td>
<td>EN_CT</td>
<td>2363 kgf</td>
</tr>
<tr>
<td>Combustor Exit Temperature</td>
<td>(CET)</td>
<td>1673 K</td>
</tr>
</tbody>
</table>

Table 3.4: AC Engine Assumptions

[41]. All other engine values were estimated based upon the CFM56, an engine that is used as a powerplant in both the 737 and A320.

3.7.3 Aircraft Performance Calculations

Aircraft performance parameters such as range, missions per year, cruise time, cruise speed and fuel consumption are calculated in order to determine operating parameters that affect fuel burn, mission servicing, maintenance and other sub-areas in the DOC framework.

Range Calculation

The range calculation of the aircraft itself is not of interest; however, it is used as a cross-validation against realistic figures for similar aircraft such as the 737-800 and A320, and also to calculate the fuel efficiency during cruise, which is used to determine the fuel consumption over one year of operation. The jet propelled aircraft range equation is a modified Breguet equation, given as:

\[
    r = \frac{2}{c_t} \sqrt{\frac{2}{\rho_\infty S} \frac{C_L^{\frac{1}{2}}}{C_D}} (W_0^{\frac{1}{2}} - W_1^{\frac{1}{2}}) \quad (3.23)
\]

where

- \( c_t \) is the thrust specific fuel consumption
- \( \rho_\infty \) is the air density at the service ceiling of 11582.4 m, taken to be a constant 0.3331 kg/m\(^3\)
- \( \frac{C_L^{\frac{1}{2}}}{C_D} \) is a measure of the lift over drag ratio
- \( W_0^{\frac{1}{2}} \) is the starting aircraft weight
- \( W_1^{\frac{1}{2}} \) is the ending aircraft weight
Note that the above equation is calibrated for imperial units of feet, and must be multiplied by the appropriate factors to obtain a range in the desired units. Assuming that the aircraft is flying at the most fuel efficient velocity for a jet aircraft [45], that given by \( V(\frac{C_L}{C_D})_{max} \), then the maximum value of \( \frac{C_L}{C_D} \) is given by the equation:

\[
\frac{C_L}{C_D}_{max} = \frac{3}{4} \left( \frac{1}{3KC_{D,0}} \right)^{\frac{3}{2}}
\]

where \( K \) is a series of drag factors corresponding to skin friction, pressure and wave drag, and

\[
K = k_1 + k_2 + k_3
\]

\[
k_3 = \frac{1}{\pi(e)(AR)}
\]

\[
k_2 = 0
\]

\[
k_1 = \frac{1}{3}k_3
\]

where \( e \) is the span efficiency factor and \( AR \) the aspect ratio. The value of \( C_{D,0} \) is given by:

\[
C_{D,0} = \frac{S_{wet}}{S}(C_{f_e})
\]

where

\[
\frac{S_{wet}}{S} \text{ is the ratio of the wetted surface area } S_{wet} \text{ of the aircraft to the wetted planform area of the wing } S
\]

\((C_{f_e}) \text{ is the skin friction coefficient}

The weights in equation 3.23 are the \( AC_{MTOW} \) and the \( AC_{NFW} \), the Maximum Take Off Weight and the No Fuel Weight, respectively. All constants and equations in this section were extracted from Anderson [45] and used where appropriate.

**Cruise Speed**

The cruise speed is a required value to determine the fuel efficiency of the aircraft. As discussed in the range paragraph above, the range is at a maximum when the aircraft is flying at \( V(\frac{C_L}{C_D})_{max} \), which is defined as:

\[
V(\frac{C_L}{C_D})_{max} = \left( \frac{2}{\rho_\infty} \sqrt{\frac{3K}{C_{D,0}}} \frac{W}{S} \right)^{\frac{3}{2}}
\]

where \( \rho_\infty \), \( K \) and \( C_{D,0} \) are as defined in equation 3.23 and \( \frac{W}{S} \) is the wing loading, which is given by the weight of the aircraft \( W \) over the wing planform surface area \( S \). Note that this number
will change as the weight of the aircraft changes, due to the reduction in fuel mass. To avoid unnecessary complexity, the weight is assumed to be fixed at a known, conservative, $AC_{MTOW}$ in order to calculate a fixed wing loading. The $AC_{MTOW}$ used in the wing loading calculation is 78,000 kg. Since the geometry of the wing is fixed, the result is a fixed value for $V\left(\frac{C_L}{C_D}\right)_{max}$, which is calculated to be 260 m/s, or Mach 0.884 at the cruise altitude listed in Table 3.3.

**Mission Calculations**

The assumption is that the aircraft flies an identical length route throughout its service. Mission length can be estimated by dividing the mission length by the cruise speed, and adding a factor to take into account climbing and descending, as required. The calculation is:

$$t = \frac{d}{V\left(\frac{C_L}{C_D}\right)_{max}} + CDT$$

where $d$ is the mission distance listed in Table 3.3 $V\left(\frac{C_L}{C_D}\right)_{max}$ was calculated in equation 3.25, and $CDT$ is the constant assumed climbing and descending time, assumed to be 20 minutes total, as in Table 3.3. The total time resulting from the calculation is approximately 3 hours. Using an empirical mission per year correlation given on P.6 of ref [9], the missions per year are calculated as:

$$mis_{year} = \frac{3205.0}{t + 0.327}$$

The missions per year in this case are calculated to be 963.

**Fuel Consumption**

Fuel consumption is calculated using the maximum range determined from equation 3.23. The range equation assumes the maximum range given complete consumption of the entire fuel mass. This would never occur in a real flight, but by using the $AC_{MTOW}$ and the subsequent aircraft no fuel weight $AC_{NFW}$, a cruise fuel efficiency number can be calculated. Although the cruise speed and missions per year are fixed by assumptions, the range calculation is variable, because it is based upon an aircraft $AC_{MTOW}$ and $AC_{NFW}$ which are subject to a variation in mass based upon the weight of the fuselage, which itself is a function of the weight of the stiffened plate design. This variability in the range equation translates into a variability in fuel consumption, which affects the DOC fuel cost calculation, and penalizes heavier design by a higher fuel burn DOC. The fuel consumed in a year is calculated as below:

$$fuel_{year} = mis_{year} \left(\frac{f_{cap}}{r}\right) (d)$$
3.7.4 Crew Costs

Crew cost assumptions are given in Table 3.5. Yearly crew costs are calculated using the equation

\[ Crew = [(C_{FLIGHT})(S_{FLIGHT}) + (C_{CABIN})(S_{CABIN})](O) \]

Note that there is no relationship between aircraft weight or acquisition cost and crew costs. These values are computed for model validation and to properly compare to the overall DOC, thereby ensuring that the percentage effects of those areas affected by the design variables are properly represented.

<table>
<thead>
<tr>
<th>Crew Assumptions</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flight Crew</td>
<td>(C_{FLIGHT})</td>
<td>2</td>
</tr>
<tr>
<td>Flight Crew Average Salary</td>
<td>(S_{FLIGHT})</td>
<td>$140,380.00</td>
</tr>
<tr>
<td>Cabin Crew</td>
<td>(C_{CABIN})</td>
<td>4</td>
</tr>
<tr>
<td>Cabin Crew Average Salary</td>
<td>(S_{CABIN})</td>
<td>$65,000</td>
</tr>
<tr>
<td>Crew Overhead Burden</td>
<td>(O)</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Table 3.5: DOC Crew Cost Assumptions

3.7.5 Loan Servicing

It is assumed that the aircraft is purchased with a loan. This loan is to be repaid over a financing period equal to the career service life of the aircraft with the carrier. Assumptions related to this calculation are shown in Table 3.6. The loan calculation is affected by the above assumptions and the aircraft acquisition cost. The loan payment calculation is given by:

\[ Loan(c) = \left( \frac{AC_{PURCHASE}(c)}{(\frac{L}{P})(1 + \frac{L}{P})^{PS}} \right)(P) \]
As evidenced by the equation, the design variables are linked to the loan via the aircraft purchase price, which is in turn linked to the plate fabrication costs, \( c \).

### 3.7.6 Depreciation

Depreciation costs of the aircraft are also taken into account, since the intrinsic value of the aircraft is reduced the longer it is in service. Assumptions related to the depreciation cost are listed in Table 3.7. The depreciation calculation is:

\[
Depreciation(c) = (AC_{PURCHASE}(c))((1 - D)^{A - 1} - (1 - D)^A)
\]

The depreciation DOC value is a function of the plate cost, linked through the plate cost and the aircraft age, which is assumed to be 5 years.

### 3.7.7 Insurance

The insurance rate is based on an assumed yearly hull-loss insurance rate of 1%. It is calculated by multiplying the current aircraft book value by the insurance rate, as shown in the equation below.

\[
Insurance(c) = (AC_{PURCHASE}(c)(1 - D)^{A - 1})(I)
\]

Where

- \( I \) is the insurance rate
- \( D \) is the depreciation rate
- \( A \) is the aircraft current age

These values are supplied in Table 3.7.
3.7.8 Mission Servicing

Mission servicing costs include cleaning the aircraft, restocking galleys, fueling and any other short term daily tasks completed while the aircraft is on the ground during regular service, excluding the cost of fuel and maintenance. Mission servicing costs are calculated using an empirical correlation on P.164 of reference [9]. The correlation is:

\[
\text{mis}_{\text{service}} = ((0.02) (AC_{\text{seats}})(l)(o) + (0.002)(AC_{\text{seats}})(i_{76}))(mis_{\text{year}})
\]

Assumptions related to the above calculation are tabulated in Table 3.8. Note that the inflation factor is included to account for the fact that the empirical correlation was developed in 1976. Since the missions per year are a function of the mission distance and cruise velocities, and equation 3.27, the mission servicing costs are not actually a function of either the fuselage cost or weight. As in the case of mission crew costs, these values are computed for model validation and to properly scale the overall DOC, thereby ensuring that the percentage effects of those areas affected by the design variables are properly represented.

3.7.9 Fuel Costs

Fuel costs are an important factor, since they are directly linked to the weight of the airframe through the fuel consumption, range and weight calculations equations, 3.28, 3.23 and 3.21, respectively. The fuel cost calculation is simply a multiplication of the \( fuel_{\text{year}} \) value arrived at in equation 3.28 by the cost of fuel.

\[
\text{fuel}(m) = fuel_{\text{year}}(m)(mis_{\text{year}})
\]

Current fuel costs at the time of writing have increased by over 100% in one year, therefore this number is highly volatile and subject to change. As an example, sample aviation Jet Fuel weekly prices as quoted by the International Air Transport Association (IATA) in U.S. dollars for three dates within a year of writing this document are as shown in the Table below: The yearly fuel cost calculation acts as a significant penalty to extra mass resulting from the stiffened plate optimization.

<table>
<thead>
<tr>
<th>Mission Servicing Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Inflation 1976-2006</td>
</tr>
<tr>
<td>Mission Service Labour</td>
</tr>
<tr>
<td>Mission Service Overhead</td>
</tr>
</tbody>
</table>

Table 3.8: DOC Mission Servicing Assumptions
IA T A Jet Fuel Prices

<table>
<thead>
<tr>
<th>Date</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 August 2007</td>
<td>0.5416 $/L</td>
</tr>
<tr>
<td>10 May 2008</td>
<td>0.9006 $/L</td>
</tr>
<tr>
<td>7 July 2008</td>
<td>1.0810 $/L</td>
</tr>
</tbody>
</table>

Table 3.9: Jet Fuel Cost variability over 2007-2008

3.7.10 Engine Maintenance

Engine maintenance costs are calculated using empirical correlations based on past maintenance data. The mean time between breakdown in hours is calculated using the equation:

$$MTBR = \frac{3604}{e^{0.000324(CET)}}(t)^{0.28}$$

where:

- $CET$ is the combustor exit temperature, given in Table 3.4
- $t$ is the individual mission time, given in section 3.7.3

Labour costs per man hour per engine are calculated using the equation:

$$MHPE = (0.0440 + 0.143t + \left(\frac{(1936 + 0.705(EN_w))}{MTBR}\right)t)(l_{red})$$

where:

- $(EN_w)$ is the engine weight given in Table 3.4
- $t$ is the individual mission time, calculated in equation 3.26
- $MTBR$ is calculated above
- $(l_{red})$ is a labour reduction factor, assuming that maintenance labour has reduced by 1% per year since this correlation formula was developed in 1976.

The engine labour costs per year are calculated using the equation:

$$EN_{lpyear} = (MHPE)(n)(O)(mis_{year})(l)$$

where:

- $n$ is the number of engines
- $MHPE$ is calculated above
mis\_year is calculated in section 3.27.

Remaining constants are given in Table 3.10. Engine material costs per year are estimated using the equation:

\[
EN_{\text{mpyear}} = [0.326 + 0.829t + 0.0906(EN_{\text{cost}}MTBR)l(n)(mis\_year)]
\]

where \(n\) is the number of engines, \(EN_{\text{cost}}\) is the cost of an individual engine as indicated in Table 3.4 and all other values are as indicated. Total engine maintenance costs per year are calculated by adding together the labour cost and material costs, as below:

\[
EN_{\text{MAINT}} = EN_{\text{lpyear}} + EN_{\text{mpyear}}
\]

Engine maintenance is based on the assumptions and equations listed, and since the number of missions per year is considered to be constant for this analysis, the engine maintenance calculation is not affected by plate mass or cost. As in the case of mission servicing and crew costs, engine maintenance is included in order to provide validation and proper scaling of the areas affected by the design variables.

### 3.7.11 Airframe Maintenance

Airframe maintenance is calculated using empirical formulas for labour and material costs, as in the engine maintenance cost calculation. Yearly labour costs are calculated by determining the labour per flight and the labour per flight hour.

\[
MHPF = 0.05 \left( \frac{(AC_{OEW})2.20462}{1000} \right) + 6 - \left( \frac{630}{((AC_{OEW})2.20462)} \right) + 120
\]

where \(AC_{OEW}(m)\) is a function of the plate weight variable \(m_{\text{plate}}\). Labour per flight hour is calculated as:

\[
MHPFH = 0.59MHPF
\]

Finally, the total yearly labour cost is calculated using:

\[
AF_{\text{lpyear}}(m) = \left( ((MHPFH + (MHPF)(t))\left(l_{\text{red}}\right))\left(\sqrt{\text{mach}}\right)\right)(l)(O)
\]
where:

$t$ is an individual mission length

$mis_{year}$ are the missions per year calculated in equation 3.27

$l_{red}$ is a labour reduction rate to take into account an assumed reduction in maintenance time from the time of the formula, as indicated in Table 3.10

$mach$ is the cruising speed mach number, which is given in section 3.7.3 as 0.88

Note that labour costs are a function of airframe mass, and therefore a function of the stiffened plate weight.

Maintenance material costs per year are calculated as:

$$AF_{mp_{year}}(c) = (\frac{(3.08)(AC_{price}(c) - EN_{cost})}{1E + 6})(t) + (\frac{(6.24)(AC_{price}(c) - EN_{cost})}{1E + 6})(mis_{year})$$

where $AC_{price}$ is the aircraft price without spares taken into account. Note that the parts per year cost is a function of the aircraft price, which implies that it is a function of the stiffened plate cost.

Total maintenance costs are calculated to be:

$$AFM_{year}(m, c) = AF_{lp_{year}}(m) + AF_{mp_{year}}(c)$$

Airframe maintenance is therefore a function of both plate weight and plate cost, and will be affected by changes in both of these plate design variables.

### 3.7.12 Landing Fees

Landing fees vary drastically from one airport to another; however, since the mission length is fixed, two Canadian airports in this distance range are selected. The airports selected were Ottawa and Regina. Landing fees are typically a function of aircraft landing weight and number of passengers. A Table containing four representative airports and their associated fees is included in Table 3.11. In the aforementioned Table, note that $(LF)$, refers to the aircraft load factor, or seats utilized on average. This value is assumed to be 0.8, as indicated in Table 3.3. The DOC of landing is calculated by assuming that each mission lands at one or the other of Ottawa or Regina airports, and then using the calculated aircraft $AC_{CURTOW}$. Parking fees are not included. Current aircraft takeoff weight is required to determine mission landing weight, which is used in landing fee calculations above. Current take-off weight is calculated with the equation:

$$AC_{CURTOW} = AC_{OEW} + \left(\frac{f_{cap}}{r}\right)(d)(\rho_{fuel}) + m_{rfuel} + 102.27(AC_{seats})(LF)$$

(3.29)
Table 3.11: Selected Airport Landing Fees (2007)

<table>
<thead>
<tr>
<th>Airport</th>
<th>Flat Rate (FR) ($)</th>
<th>Seat Multiplier (SM) ($/seat)</th>
<th>Ldg Multiplier (LM) ($/1000kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>London (LHR)</td>
<td>500</td>
<td>15(LF)</td>
<td>0</td>
</tr>
<tr>
<td>Toronto (YYZ)</td>
<td>0</td>
<td>9.86(LF)</td>
<td>35.09</td>
</tr>
<tr>
<td>Ottawa (YOW)</td>
<td>499.44</td>
<td>1.44</td>
<td>6.05</td>
</tr>
<tr>
<td>Regina (YQR)</td>
<td>0</td>
<td>1.44</td>
<td>5.35</td>
</tr>
</tbody>
</table>

where \( \left( \frac{f_{\text{cap}}}{r} \right) d \) is a calculation of the fuel consumption during a mission based upon range \( r \) and mission distance \( d \). The passenger load allowance is given by \( 102.27(AC_{\text{seats}})(LF) \), which calculates the total number of passengers on the flight by multiplying the load factor \( LF \) by the number of seats on the aircraft and then applying approximately 100 kg per passenger, based on empirical data. The mission reserve fuel mass \( m_{r,fuel} \) is determined by calculating the amount of fuel mass required to traverse 1852 km based on the aircraft cruise fuel consumption (ATA Domestic Rules, p.58 of NASA-CR145190 [9]). The aircraft current mission landing weight can be calculated by subtracting the mission fuel consumption from the equation 3.29:

\[
AC_{\text{CURLW}} = AC_{\text{OEW}} + m_{r,fuel} + 102.27(AC_{\text{seats}})(LF)
\]  

(3.30)

The \( AC_{\text{CURLW}} \) number is used in all landing fee calculations. The assumption is that the aircraft carries passengers and the mission fuel plus reserve only, no additional cargo, and that it is fueled to this same level each flight. This ignores any operational reasons for varying fuel carried during each flight for simplicity. The actual landing fee calculation is:

\[
LFee(m) = FR + (AC_{\text{seats}})(SM) + AC_{\text{CURLW}}(LM)
\]

Landing fee DOC is a function of aircraft weight, and therefore will be affected by the plate weight in the optimization process.

### 3.8 Direct Operating Cost Sensitivity Calculation

Because the overall objective of the optimization is to minimize direct operating cost, the sensitivity of the DOC to plate mass and plate cost is required. Computationally, the fastest way to calculate sensitivities is to derive them analytically, and then program the analytic solution into the computer to be calculated as required. This requires that each of the sub-disciplines discussed in section 3.7 above be differentiated with respect to cost and mass, and then the
chain rule invoked to determine their sensitivity to the design variables. The general sensitivity equation is:

\[
\frac{dO}{dx} = \left[ \frac{\partial O}{\partial m} \right] \left[ \frac{\partial m}{\partial x} \right] + \left[ \frac{\partial O}{\partial c} \right] \left[ \frac{\partial c}{\partial x} \right]
\]  

(3.31)

where:

\( O \) indicates the Direct Operating Cost

\( x \) denotes the design variables

\( m \) is the plate mass

\( c \) is the plate fabrication cost

In terms of sensitivity calculation, both the derivative of mass with respect to the design variables, \( \frac{\partial m}{\partial x} \), and cost with respect to the design variables, \( \frac{\partial c}{\partial x} \), are calculated using the techniques discussed in section 2.7. The derivatives of each DOC sub discipline with respect to cost and mass are derived using analytical differentiation techniques. The relevant sub-disciplines which are partially differentiated with respect to mass are fuel, maintenance and landing costs. Sub-disciplines partially differentiated with respect to plate cost are depreciation, loan, maintenance and insurance. Exhaustive differentiation details are not included below. Each of the derivatives has a number of local constants which affect the analytic value, but in some cases are quite detailed. These constants are included, but details of their make-up is excluded here for brevity. Further details of the make-up of the appropriate constants may be found in Appendix A. Derivatives of each of the relevant disciplines with respect to mass are listed below:

\[
\frac{\partial \text{fuel}}{\partial m} = (\text{C}1_{0}\frac{1}{\sqrt{\text{C}6+m-\sqrt{\text{C}7+m}}})^2(0.5)(\frac{1}{\sqrt{\text{C}6+m-\sqrt{\text{C}7+m}}}) \\
\frac{\partial \text{maint}}{\partial m} = \text{C}2_{2} + \text{C}2_{6} + \text{C}2_{4}\text{C}1_{11}(\text{C}11(m) + \text{C}1_{8})^2 + \text{C}2_{5}\text{C}11(\text{C}11(m) + \text{C}1_{8})^{-2} \\
\frac{\partial \text{landing}}{\partial m} = \text{C}2_{0} - \frac{(0.5)[\text{C}2_{2}(\frac{1}{\sqrt{m+C_{11}}}) - (\frac{1}{\sqrt{m+C_{12}}})]}{(\sqrt{m+C_{11}}\sqrt{m+C_{12}})^2} \\
\]

Derivatives of each of the relevant disciplines with respect to cost are listed below:

\[
\frac{\partial \text{dep}}{\partial c} = (1-D)^{A-1} - (1-D) \\
\frac{\partial \text{loan}}{\partial c} = 2(\frac{L_P}{P}(1 + \frac{L_P}{P})PS)^P(1 + \frac{L_P}{P})PS - 1 \\
\frac{\partial \text{maint}}{\partial c} = \frac{(\text{mis}_{year})}{1E + 6}(3.08t + 6.24) \\
\frac{\partial \text{ins}}{\partial c} = I(1-D)^{A-1} \\
\]
The DOC sensitivity calculation can then be modified by the appropriate partials.

\[
\frac{dO}{dx} = \left[ \frac{\partial O}{\partial m} \frac{\partial m}{\partial x} \right] + \left[ \frac{\partial O}{\partial c} \frac{\partial c}{\partial x} \right]
\]

\[
\frac{dO}{dx} = \left[ \frac{\partial \text{fuel}}{\partial m} \frac{\partial m}{\partial x} + \frac{\partial \text{maint}}{\partial m} \frac{\partial m}{\partial x} + \frac{\partial \text{landing}}{\partial m} \frac{\partial m}{\partial x} \right] + \left[ \frac{\partial \text{dep}}{\partial c} + \frac{\partial \text{loan}}{\partial c} + \frac{\partial \text{maint}}{\partial c} + \frac{\partial \text{ins}}{\partial c} \right] \frac{\partial c}{\partial x}
\]

(3.32)

This calculation can be used to determine the gradient for the entire field of design variables, since there is no direct link to the sensitivities of the DOC values to the design variables directly, the sensitivity of these to mass and cost can be calculated once and thereafter remain constant, acting as multiplication factors for the design variable sensitivity to cost and mass.
Chapter 4

Structural Analysis

4.1 Reissner-Mindlin Plate Model

Structural analysis of the stiffened plate is required to ensure that under given loading conditions, the optimized design does not exceed some specified failure criteria. In order to calculate the stresses and strains inside the stiffened plate, the finite-element method is used. Details of the finite element method are outside the scope of this document; however, a brief explanation of the Reissner-Mindlin plate model is salient, in that it is directly related to the calculation of the failure criteria which bear upon the optimization setup. This formulation also assumes a laminated ply, where mid-ply strains and curvatures are integrated over each composite layer to obtain the stress tensor. The matrix representation is given below:

\[
\sigma = [K][\epsilon]
\]

(4.1)

where:

\[
\sigma = \begin{bmatrix} N \\ M \\ Q \end{bmatrix}, \quad K = \begin{bmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & \bar{A} \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon^o \\ \bar{\kappa} \\ \gamma^o \end{bmatrix}
\]

In the expanded form, the matrices are:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy} \\
Q_x \\
Q_y
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\
A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} & 0 & 0 \\
A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} & 0 & 0 \\
B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} & 0 & 0 \\
B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} & 0 & 0 \\
B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{44} & \bar{A}_{45} \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{54} & \bar{A}_{55}
\end{bmatrix}
\begin{bmatrix}
\tilde{\epsilon}_x \\
\tilde{\epsilon}_y \\
\tilde{\gamma}_{xy} \\
\tilde{\kappa}_x \\
\tilde{\kappa}_y \\
\tilde{\kappa}_{xy} \\
(\gamma)_{xz} \\
(\gamma)_{yz}
\end{bmatrix}
\]
The A-matrix is a value relating the stress resultant to tensile strain resultants, the B matrix is a coupling matrix linking the stress resultants to the curvatures and the moments to the mid-surface strain. The B-matrix indicates the degree to which the anisotropy of the material causes membrane-strain-moment coupling. The D matrix relates the curvature to moments directly, and the A bar matrix relates out-of-plane shears to the transverse shears. The above A,B,D and A bar matrices are calculated according to the formulas:

\[
A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1}); i, j = 1, 2, 6 
\]

\[
B_{ij} = \left(\frac{1}{2}\right) \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2); i, j = 1, 2, 6 
\]

\[
D_{ij} = \left(\frac{1}{3}\right) \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3); i, j = 1, 2, 6 
\]

\[
\bar{A}_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2); i, j = 4, 5 
\]

where:

- \( \bar{Q}_{ij} \) is the particular material lamina constitutive matrix, transformed to some structure reference system.

- \( k \) is the number of laminates in the plate

- \( h \) is the distance from the boundary of a given ply to the centreline

Additionally, note that the matrix \( \bar{Q}_{ij} \) is modified from a simple material \( Q_{ij} \) matrix rotated to the correct axis, since the additional aspect of the SIMP weights need to be taken into account. Therefore, a full accounting of the material properties can be summarized as an exercise in determining a SIMP weighted \( Q_{ij} \) matrix for each layer, which are then summed over each ply making up the plate element, and contributing to the A,B,D,\( \bar{A} \) as shown in equations 4.2, 4.3, 4.4 and 4.5 above.

### 4.1.1 Constitutive Matrix calculation example

As an illustrative example showing the calculation of the appropriate \( \bar{Q}_{ij} \) matrix, consider a SIMP-weighted plate element, consisting of 2 plies, with three material options: aluminum, and Graphite/Epoxy Hercules - IM7/8551 oriented at 0 and 45 degrees, relative to the material axis. The relevant Aluminum material properties are:
$E = 70.3$ GPa  \\
$G = 26.4$ GPa  \\
$\nu = 0.33$

and the relevant Graphite/Epoxy properties are:

$E_{11} = 164.0$ GPa  \\
$E_{22} = 8.30$ GPa  \\
$G_{12} = G_{26} = 2.10$ GPa  \\
$G_{16} = 1.20$ GPa  \\
$\nu_{12} = 0.34$

The current SIMP weights $w_i^j$ for each material in each ply are assumed to be evenly distributed, that is to say, that:

$w_0^0 = w_1^0 = w_2^0 = 0.33$  \\
$w_1^1 = w_2^1 = 0.33$

In order to calculate the constitutive matrix for layer 0, the individual constitutive matrices of the above materials need to be established. The relevant $Q_{ij}$ matrices are established based on the plane stress assumption, and are calculated using the following formulas:

\[
Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \\
Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \\
Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} \\
Q_{44} = G_{23} \\
Q_{55} = G_{13} \\
Q_{66} = G_{12}
\]  

Therefore, using the above equations and assumptions, the $Q_{ij}$ matrices for the materials in question are:

Aluminum:
Since the aluminum is assumed to be isotropic, any transformation of the above aluminum constitutive matrix is irrelevant. The Graphite/Epoxy ply which is oriented at 0 degrees to the material axis is already correctly aligned with the material axis, and also does not need to be rotated. However, one option of Graphite/Epoxy is set at 45 degrees, and therefore, a transformation matrix is required to rotate the Graphite/Epoxy constitutive matrix from 45 degrees back to 0 to align with the material axis. Due to the independent nature of the forces in-plane versus those out of plane, two separate transformations can be used, one on the in-plane forces and shears, 1,2 and 6, and another for the out of plane shears, 4 and 5. The transformation matrix for a 2-D in plane rotation is:

\[
T = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\]

Additionally, to take into account the $\frac{1}{2}$ factor present in strain conversion between engineering strain and tensor strain, an additional Reuter matrix is used to multiply the shear strains by 2, where appropriate.

\[
R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\]

The final $\bar{Q}$ matrix for 45 degrees therefore is calculated by using the equation:

\[
\bar{Q} = T^{-1}QRTR^{-1}
\]
The transformation matrix for the two out of plane shear directions, 4 and 5, is given as:

\[ T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

After the transformations, the Q matrix for the Graphite/Epoxy rotated from 45 degrees is:

Graphite/Epoxy at 45 degrees:

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{14} & Q_{15} & Q_{16} \\
Q_{21} & Q_{22} & Q_{24} & Q_{25} & Q_{26} \\
Q_{41} & Q_{42} & Q_{44} & Q_{45} & Q_{46} \\
Q_{51} & Q_{52} & Q_{54} & Q_{55} & Q_{56} \\
Q_{61} & Q_{62} & Q_{64} & Q_{65} & Q_{66}
\end{bmatrix} = \begin{bmatrix}
46.85 & 42.65 & 0 & 0 & 39.15 \\
42.65 & 46.85 & 0 & 0 & 39.15 \\
0 & 0 & 1.20 & 0 & 0 \\
0 & 0 & 0 & 1.20 & 0 \\
39.15 & 39.15 & 0 & 0 & 41.91
\end{bmatrix}
\]

GPa

The combined Q matrix for a single ply then would be SIMP weighted simply by summing the appropriate, properly transformed constitutive matrices and multiplying them by the salient SIMP weight. The formula for a single ply would be:

\[
\sum_{i=1}^{N} w_i Q_i
\]  

(4.8)

where \( N \) is the number of material options for that particular ply. Continuing with the example from above, the final constitutive matrix for a given ply would be given as:

\[
Q_i = (Q_{Al})(w_{Al}) + (Q_{GrEp0})(w_{GrEp0}) + (Q_{GrEp45})(w_{GrEp45})
\]  

(4.9)

Completing the above equation with numbers:

\[
\begin{bmatrix}
96.90 & 23.84 & 0 & 0 & 13.05 \\
23.84 & 44.70 & 0 & 0 & 13.05 \\
0 & 0 & 9.69 & 0 & 0 \\
0 & 0 & 0 & 9.69 & 0 \\
13.05 & 13.05 & 0 & 0 & 23.48
\end{bmatrix} = \begin{bmatrix}
78.89 & 26.03 & 0 & 0 & 0 \\
26.03 & 78.89 & 0 & 0 & 0 \\
0 & 0 & 26.43 & 0 & 0 \\
0 & 0 & 0 & 26.43 & 0 \\
0 & 0 & 0 & 0 & 26.43
\end{bmatrix}
\]

(0.33)+

\[
\begin{bmatrix}
164.97 & 2.84 & 0 & 0 & 0 \\
2.84 & 8.35 & 0 & 0 & 0 \\
0 & 0 & 1.20 & 0 & 0 \\
0 & 0 & 0 & 1.20 & 0 \\
0 & 0 & 0 & 0 & 2.10
\end{bmatrix} + \begin{bmatrix}
46.85 & 42.65 & 0 & 0 & 39.15 \\
42.65 & 46.85 & 0 & 0 & 39.15 \\
0 & 0 & 1.20 & 0 & 0 \\
0 & 0 & 0 & 1.20 & 0 \\
39.15 & 39.15 & 0 & 0 & 41.91
\end{bmatrix}
\]

(0.33)

Note that all of the above numbers have been rounded to the nearest significant digit, and so may not precisely meet a cursory mathematical examination.
4.2 Failure Calculation

In order to establish whether a SIMP weighted laminated plate element has failed, an existing failure criteria is used, and modified to take into account the weighting factor associated with the SIMP method. The failure criteria used is a strain failure criteria, which is predicated on the fact that the constitutive matrix of the material is known, and that the material fibre and matrix failure properties in tension, compression and shear have been clearly established. If dealing with an homogenous isotropic material, such as aluminum, the yield, compressive and shear failure yield stresses can be used. Once the element stiffnesses are assembled according to the SIMP weights and constitutive matrices defined elsewhere, loads and boundary conditions applied, and a structural solve taken place, the failure of a given plate element is calculated using the following steps:

1. Convert material failure stresses to material failure strains
2. Get local middle-surface strains and curvatures from local displacements
3. Convert middle-surface strains and curvatures to local boundary strains, based on distance from the centreline
4. Average boundary strains to get single strain values for each ply
5. For each ply, for each material option, rotate local ply strains to ply axis
6. For each ply and each material rotated strain, divide the local ply strain by the appropriate failure strain of the given material to determine how close that particular strain is to exceeding the failure criteria for that strain, material and ply.
7. Sum all of the strain exceedances calculated for each of the materials in a given ply, after having multiplied them by their appropriate SIMP weights.
8. Determine the highest exceedance, then invert it and output that number as the multiple of the current load required to fail the part.
9. Use the multiple of the current load required to fail in a KS function which encompasses each gauss point in each element

4.2.1 Failure Stress to Strain Conversion

This step need only be performed once, at the beginning of the analysis. Conversely, the failure strains could be calculated ‘manually’, outside of this analysis entirely, but since it is customary to give material failure details in terms of stress, a quick explanation of this calculation is made
here. For a given material, such as Hercules IM7/8551 Graphite/Epoxy or Aluminum the failure stresses are given in Table 4.1, where X, Y and S symbolize failure in the fiber, failure in the matrix, and a failure in shear, respectively. The prime terms refer to failure under compression.

In the case of a homogeneous material, such as aluminum, the X and Y numbers are taken to be equal, although in reality there are processing variations which make even homogeneous materials such as aluminum slightly anisotropic. These failure stresses can be converted to failure strains by multiplying by the inverse of the constitutive matrix using the basic stress strain relation \( \sigma = K \epsilon \). Adding in the known material properties of the composite in the table, the result is:

\[
\epsilon = \sigma K^{-1}
\]  

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} = 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{21} & Q_{22} & Q_{26} \\
Q_{61} & Q_{62} & Q_{66}
\end{bmatrix}^{-1}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.0145 \\
0.0038 \\
0.0871
\end{bmatrix} = 
\begin{bmatrix}
6.10 & -2.07 & 0 \\
-2.07 & 120.48 & 0 \\
0 & 0 & 0.00047
\end{bmatrix}
\begin{bmatrix}
2410 \\
73 \\
183
\end{bmatrix}
\]

Note that the above numerical example shows the conversion of the failure stresses in tension only. A similar exercise would need to be undertaken for calculation of failure strains in compression. Note also that the above failure strains define a strain failure surface which assumes that all strains are imposed when the material is aligned with the uniaxial fiber direction along the 1 direction, and orthogonal to the 2 direction. This necessitates strain transformation for any strains which are not aligned in such fashion.
4.2.2 Mid-Surface Strain and Curvature Calculation

This step is combined with the Finite Element analysis, and consists of extracting the appropriate middle surface strains and curvatures from the finite element analysis at a given integration point. The extracted strain vector has the form:

\[
\begin{bmatrix}
\epsilon^o_x \\
\epsilon^o_y \\
\gamma^o_{yz} \\
\gamma^o_{xz} \\
\gamma^o_{xy} \\
\tilde{\kappa}_x \\
\tilde{\kappa}_y \\
\tilde{\kappa}_{xy}
\end{bmatrix}
\]

4.2.3 Local Boundary Strain Calculation

Since the problem is a laminated plate, the middle surface strains and curvatures need to be converted into local average ply strains, based on the equation:

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
\epsilon^o_x + \frac{(h_k+h_{k-1})}{2}\tilde{\kappa}_x \\
\epsilon^o_y + \frac{(h_k+h_{k-1})}{2}\tilde{\kappa}_y \\
\gamma^o_{yz} \\
\gamma^o_{xz} \\
\epsilon^o_{xy} + \frac{(h_k+h_{k-1})}{2}\tilde{\kappa}_{xy}
\end{bmatrix}
\]

Where:

\(h_k\) is a measure of the distance from the upper boundary of the ply of interest to the centerline

\(\epsilon^o_x, \epsilon^o_y, \epsilon^o_{xy}\) are mid-surface strains

\(\tilde{\kappa}_x, \tilde{\kappa}_y, \tilde{\kappa}_{xy}\) are the plane curvatures

The above equation will give average strains in the plane, including out of plane shear, based on the middle surface strains and curvature, for each of the plies.

Example Problem: Local Strain Conversion

As part of an illustrative example, an arbitrary strain state is imposed and calculated on an arbitrary 2 ply laminate, with the following assumptions:

- Ply thicknesses 0.125 mm, giving the values for the boundaries of these plies from the centerline as:

\[
\begin{bmatrix}
h_0 \\
h_1 \\
h_2
\end{bmatrix}
= 
\begin{bmatrix}
-0.000125 \\
0 \\
0.000125
\end{bmatrix} \text{ m}
\]

- Potential materials for each ply: Aluminum, Graphite/Epoxy at 0 degrees, Graphite/Epoxy at 45 degrees

- Current SIMP weights: 0.33 for all materials in both plies
• Arbitrary strain distribution as described

The arbitrary strain state is given as:

\[
\begin{bmatrix}
\epsilon_o^x \\
\epsilon_o^y \\
\gamma_o^{xy} \\
\tilde{\kappa}_x \\
\tilde{\kappa}_y \\
\tilde{\kappa}_{xy} \\
\gamma_o^{yz} \\
\gamma_o^{xz} \\
\gamma_o^{yx}
\end{bmatrix} =
\begin{bmatrix}
0.1 \\
-0.1 \\
0 \\
0.1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

This arbitrary strain state enforces a positive displacement in the 1 direction, and equally sized negative displacement in the 2 direction, as well as a curvature parallel to the x axis. When this strain state is converted into the strains for ply 0, the result is:

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\epsilon_o^x + \frac{(h_k+h_{k-1})}{2} \tilde{\kappa}_x \\
\epsilon_o^y + \frac{(h_k+h_{k-1})}{2} \tilde{\kappa}_y \\
\gamma_o^{yz} \\
\gamma_o^{xz} \\
\epsilon_o^{yx} + \frac{(h_k+h_{k-1})}{2} \tilde{\kappa}_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.09999375 \\
-0.1 \\
0 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
0.1 + \frac{(-0.000125+0.0)}{2} \\
-0.1 + \frac{(-0.000125+0.0)}{2} \\
0 \\
0 \\
0 + \frac{(-0.000125+0.0)}{2}
\end{bmatrix}
\]

This example illustrates the fact that the ply thicknesses are only relevant when middle surface curvature is present. The \(\epsilon_y\) term remained unaffected by \(\tilde{\kappa}_x\), and retained the original strain, whereas \(\epsilon_x\) was relieved slightly. For ply 1, the opposite would occur and the expectation would be that a higher strain would result.

### 4.2.4 Rotate to local material axis

Since the definition of a SIMP element implies that each ply may potentially consist of several material options. To properly evaluate failure the local ply strains are rotated into the particular material axis, so that they may be evaluated against the strain surface. For any isotropic materials or composites already at 0 degrees, this step is ignored. Otherwise, the strains as calculated above are transformed using the appropriate angle transformation matrix.
Example: Strain Rotation

Rotating the strains to the local axis is only necessary for those materials which are anisotropic and which are not aligned with the material axis. Since, in the example, the only material meeting these criteria is the Graphite/Epoxy at 45 degrees, that is the only transformation that must be applied to the strains. The transformation is accomplished by using an amalgamated 2D rotation matrix, which rotates the two out of plane shears simultaneously, as shown in the equation below:

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
45
= \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 0 & 0 & 2 \cos \theta \sin \theta \\
\sin^2 \theta & \cos^2 \theta & 0 & 0 & -2 \cos \theta \sin \theta \\
0 & 0 & \cos \theta & -\sin \theta & 0 \\
0 & 0 & \sin \theta & \cos \theta & 0 \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & 0 & 0 & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
_{0}
\]

\[
\begin{bmatrix}
\cos^2(45) & \sin^2(45) & 0 & 0 & 2 \cos(45) \sin(45) \\
\sin^2(45) & \cos^2(45) & 0 & 0 & -2 \cos(45) \sin(45) \\
0 & 0 & \cos(45) & -\sin(45) & 0 \\
0 & 0 & \sin(45) & \cos(45) & 0 \\
-\sin(45) \cos(45) & \sin(45) \cos(45) & 0 & 0 & \cos^2(45) - \sin^2(45)
\end{bmatrix}
_{0}
\begin{bmatrix}
0.09999375 \\
0 \\
-0.1 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
-3.125E-6 \\
-3.125E-6 \\
0 \\
0 \\
-0.199994
\end{bmatrix}
_{45}
\]

4.2.5 Failure Check

After the strain vectors have been transformed where appropriate, each material which may contribute to a ply is checked against its material failure criteria, by dividing the appropriate strain by the appropriate failure strain. This also includes checking whether the strain of interest is in compression or tension, and dividing it by the appropriate failure strain. A vector of strain exceedances for each potential material of each ply would then result. The calculation shown below assumes that all strains are positive (i.e. in tension), but note that this is not necessarily always the case, and if not, the appropriate compressive failure criteria would be
checked.

\[
\begin{bmatrix}
\epsilon_{11m}^m \\
\epsilon_{22m}^m \\
\epsilon_{26m}^m \\
\epsilon_{12m}^m \\
\end{bmatrix}
= 
\begin{bmatrix}
(\lambda_{11})_m \\
(\lambda_{22})_m \\
(\lambda_{26})_m \\
(\lambda_{12})_m \\
\end{bmatrix}
\]

(4.11)

The result will be a vector of non-dimensionless exceedances, designated as \(\lambda\), for each material \(m\) which indicate the relative ‘degree of failure’ of a particular material strain, evaluated against its appropriate failure criteria.

**Example: Exceedence Calculation**

For the lower layer ply, the strain states imposed on each of the materials is:

\[
\begin{bmatrix}
0.09999375 \\
-0.1 \\
0 \\
0 \\
0 \\
\end{bmatrix}_{Aluminum}
= 
\begin{bmatrix}
0.09999375 \\
-0.1 \\
0 \\
0 \\
0 \\
\end{bmatrix}_{GrEp(0)}
= 
\begin{bmatrix}
-3.125E-6 \\
-3.125E-6 \\
0 \\
0 \\
-0.199994 \\
\end{bmatrix}_{GrEp(45)}
\]

Note that these strains are particular to the lower layer ply, and are different from the ply above the mid-surface by virtue of the imposed curvature. Further, the initially established SIMP weights are:

\[
w_{Aluminum} = 0.33 \quad w_{GrEp(0)} = 0.33 \quad w_{GrEp(45)} = 0.33
\]

Given the information above, the strain failure criteria for the materials in question are as indicated below in Table 4.2.
Selected Material Failure Strains

<table>
<thead>
<tr>
<th>Failure Mode</th>
<th>GrEp</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.0145433</td>
<td>0.0039457</td>
</tr>
<tr>
<td>Y</td>
<td>0.0037990</td>
<td>0.0039457</td>
</tr>
<tr>
<td>S</td>
<td>0.0871429</td>
<td>0.0065081</td>
</tr>
<tr>
<td>X'</td>
<td>-0.0059826</td>
<td>-0.0039457</td>
</tr>
<tr>
<td>Y'</td>
<td>-0.0186874</td>
<td>-0.0039457</td>
</tr>
<tr>
<td>S'</td>
<td>-0.0871429</td>
<td>-0.0065081</td>
</tr>
</tbody>
</table>

Table 4.2: Selected Material Failure Strains

Therefore, calculating the exceedances as per equation 4.11 above:

\[
\begin{pmatrix}
0.09999375 \\
0.0039457
\end{pmatrix}
\begin{pmatrix}
25.3427 \\
-0.1
\end{pmatrix}
+ 
\begin{pmatrix}
0.0039457 \\
-0.0145433
\end{pmatrix}
\begin{pmatrix}
6.87557 \\
-3.125E-6
\end{pmatrix}
= 
\begin{pmatrix}
0.000522346
\end{pmatrix}
\]

\[
\begin{pmatrix}
-0.1 \\
0.0065081
\end{pmatrix}
\begin{pmatrix}
25.3443 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
5.35119 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0.0065081
\end{pmatrix}
\begin{pmatrix}
0 \\
0.0871429
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0.0065081
\end{pmatrix}
\begin{pmatrix}
0 \\
0.0871429
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-0.199994 \\
0
\end{pmatrix}
\begin{pmatrix}
2.29501
\end{pmatrix}
= 
\begin{pmatrix}
2.29501
\end{pmatrix}
\]

4.2.6 Exceedance Amalgamation

The exceedances calculated above are calculated for each potential material in the ply, then summed and multiplied by the appropriate SIMP weight, to take into account the contribution to failure of that particular material. This calculation is shown below:

\[
\lambda_{ij}^l = \sum_{m=1}^{N} [(\lambda_{ij})_m w_m] \quad i, j = 1, 2, 6 \quad (4.12)
\]

where:

- \( l \) is the layer of interest
- \( m \) is the material
- \( w_m \) is the particular SIMP weight corresponding to that material, for that ply, for that element
This step ensures that materials which are nearly ‘gone’ do not contribute greatly to the relative degree of failure of that particular strain in that particular ply. Additionally, materials which are weighted more highly will also contribute more of their constitutive matrices to the ply in question. Note that the result of the above calculation will be that each ply will have a vector of five strain exceedance numbers, each of which give an indication as to how close a given ply is to failure in that particular strain. The reason that the failure loads themselves are not SIMP weighted, and then compared to the local element strains is that the division of the strains by the failure loads would result in divisions by zero when the SIMP weights approach 0. This can be seen mathematically by substituting the weight $w_m$ into the exceedance calculation.

\[
\begin{bmatrix}
\frac{\epsilon_{11}}{X_m(w_m)} \\
\frac{\epsilon_{22}}{Y_m(w_m)} \\
\frac{\gamma_{26}}{S_m(w_m)} \\
\frac{\gamma_{16}}{S_m(w_m)} \\
\frac{\gamma_{12}}{S_m(w_m)}
\end{bmatrix}
= 
\begin{bmatrix}
(\lambda_{11})_m \\
(\lambda_{22})_m \\
(\lambda_{26})_m \\
(\lambda_{16})_m \\
(\lambda_{12})_m
\end{bmatrix}
\]

It is apparent from the above calculation that when the SIMP weight $w_m$ approaches 0, the exceedance would approach infinity, which is an unacceptable result.

**Example: SIMP weight Exceedances**

Given the calculated exceedances from the example above, the SIMP weighted strain exceedance for each strain for the ply can be calculated as below:

\[
\begin{bmatrix}
(\lambda_{11}^0) \\
(\lambda_{22}^0) \\
(\lambda_{26}^0) \\
(\lambda_{16}^0) \\
(\lambda_{12}^0)
\end{bmatrix}
= 
\begin{bmatrix}
(\lambda_{11})_{Al} \\
(\lambda_{22})_{Al} \\
(\lambda_{26})_{Al} \\
(\lambda_{16})_{Al} \\
(\lambda_{12})_{Al}
\end{bmatrix}
+ 
\begin{bmatrix}
(\lambda_{11})_{GrEp\theta(0)} \\
(\lambda_{22})_{GrEp\theta(0)} \\
(\lambda_{26})_{GrEp\theta(0)} \\
(\lambda_{16})_{GrEp\theta(0)} \\
(\lambda_{12})_{GrEp\theta(0)}
\end{bmatrix}
+ 
\begin{bmatrix}
(\lambda_{11})_{GrEp\theta(45)} \\
(\lambda_{22})_{GrEp\theta(45)} \\
(\lambda_{26})_{GrEp\theta(45)} \\
(\lambda_{16})_{GrEp\theta(45)} \\
(\lambda_{12})_{GrEp\theta(45)}
\end{bmatrix}
\]
Chapter 4. Structural Analysis

4.2.7 Highest Exceedance Selection

Each ply in the element will have a vector of five strain exceedances, all of which are bounded by 0 and infinity, with numbers above 1 indicating failure. The largest of these can be selected and then inverted, to determine what multiple of the current load is required to fail the element in question, given the existing strains. This ensures that the strain which exceeds it’s failure limit by the most in all of the plies will be the indicator of failure.

Example: Highest Exceedance

The calculated exceedances for the lower ply, as calculated in the example above, are shown below.

\[
\begin{bmatrix}
10.7396 \\ 10.2319 \\ 0.765003
\end{bmatrix}
= 
\begin{bmatrix}
25.3427 \\ 25.3443 \\ 0
\end{bmatrix}
\times 
\begin{bmatrix}
6.87557 \\ 5.35119 \\ 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\ (0.33) \\ 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\ (0.33) \\ 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\ 0
\end{bmatrix}
= 
\begin{bmatrix}
0.00522346 \\ 0.000167225 \\ 2.29501
\end{bmatrix}
\]
Additionally, exceedances for the upper ply were calculated as well, using identical techniques to those described above, and are listed below:

\[
\begin{bmatrix}
\lambda_{11}^1 \\
\lambda_{22}^1 \\
\lambda_{26}^1 \\
\lambda_{16}^1 \\
\lambda_{12}^1
\end{bmatrix}
= \begin{bmatrix}
10.7408 \\
10.2321 \\
0 \\
0 \\
0.765051
\end{bmatrix}
\]

The highest exceedance, that is, the degree to which the failure strain is exceeded the most, for the lower ply, ply 0, is given by \(\lambda_{11}^0\), which in this case is indicating that the imposed strain in this direction is actually 10.7396 times higher than the failure strain of the ply. The highest exceedance of both plies is actually that of the upper ply which is slightly higher. This value is given as: \(\lambda_{11}^1 = 10.7408\). Therefore, for this particular arbitrary point in the element, the highest value is selected and inverted to give 0.09310, which indicates that the imposed strains would need to be reduced to 0.09310 of their current level to ensure that this point on the element would not fail in its current condition. The fact that the upper ply exceeds its failure criteria slightly more than the lower makes logical sense, considering the initial strain state of the element, in which a positive in-plane strain along the x-direction was coupled with a x-plane curvature. The curvature is enough to relieve the imposed in-plane strain slightly on the lower ply, while increasing it on the upper. It is important to recall at this juncture that the plies currently consist of a SIMP-weighted amalgam of the three materials, and that the stiffness of the plies are directly weighted by the appropriate weights. Further, the failure exceedances, as discussed above, have also been weighted, to appropriately account for the influence of each material’s failure criteria. As the SIMP weight for a specific material approaches one, therefore, both the stiffness and the failure exceedance will approach those of the material in question.

### 4.2.8 Failure Load Aggregation

Once the multiple of the current load to failure is calculated for each Gauss point of a particular element, a Kreisselmeier-Steinhauser (KS) function is then used to establish an appropriate amalgamated failure load number for the whole element. The KS function used in this instance
is given by the equation:
\[
KS(g_j(x)) = c_{max} + \frac{1}{\rho} \ln\left(\sum_{j} e^{\rho(-g_j(x) + c_{max})}\right) \geq 0
\] 
\hspace{1cm} (4.13)

where:

\(g_j(x)\) is the failure calculation function. This corresponds to the failure function calculation described above, and is shown to be a function solely of the design variables \(x\), which are the SIMP weights.

\(N\) is the number of exceedances to consider. If the plate elements in question are bicubic, and are evaluated at each gauss point, this would give 16 exceedance numbers per element to be incorporated into the function.

\(c_{min}\) is the minimum value of all of the exceedance numbers, which is taken to be the absolute lower limit. This particular factor is included to reduce the numerical difficulties which occur when computing the exponential of large numbers.

Therefore, this KS function is applied over each gauss point for all elements, giving a sample aggregated failure number. This number represents the point closest to failure in the entire structure. This number can then be applied as a constraint for the optimizer.

### 4.2.9 Secondary KS requirement

With the assumption that the maximum failure load will be selected from the plies analyzed above, a significant problem is encountered when more than one ply is present. This problem is related to the particular details of the SIMP weights, which tend to change the stiffness as well as the failure properties of the materials in question. The problem can be best illustrated by an example. Consider a single plate element, comprised of two plies, with each ply consisting of Aluminum, or Graphite/Epoxy aligned at 0 degrees, and the material properties as specified in Sections \[4.1.1\] and \[4.2.1\]. Further, consider that the SIMP weights are given as:

\[
\begin{bmatrix}
(w_0^0) \\
(w_0^1) \\
(w_1^0) \\
(w_1^1)
\end{bmatrix}
= \begin{bmatrix}
x \\
1 - x \\
0.5 \\
0.5
\end{bmatrix}
\]
Where $x$ is plotted from $0.1 < x < 0.9$. The superscript on weight indicates the layer number, and the subscript indicates the material number, where 0 indicates the lower ply and aluminum, respectively. If an arbitrary force resultant along the x-axis is imposed on the above element, the expectation is that the primary failure values will be isolated to only the $\epsilon_1$ values for simplicity. If the resulting minimum P to failure value for all layers is plotted, and the minimum P to failure value for $\epsilon_1$ for each separate layer is plotted, the results are as shown below, in figure 4.1. As can be seen in the figure, the minimum failure value experiences a pronounced cusp near the 0.5 weight area, due to the minimum failure strain switching from the upper ply to the lower ply as the SIMP weight of aluminum in the lower ply increases. This assumes that the weights on the upper ply remain constant, which would not be the case during a typical optimization process, but the fact that a cusp exists is enough to indicate that the failure function is not C1 continuous and therefore cannot be used as currently defined in a gradient based optimizer without further modification. A suggested modification would be to apply another KS function to the minimum failure loads, to remove any cusps and remove the non-differentiable results. The resulting function, applied to the above case, would appear as shown in figure 4.2. In order to ensure continuity across multiple plies, at each gauss point all failure numbers are aggregated across the plies using a KS function, and then a further KS function is applied across all gauss points of the structure to aggregate the failure numbers for the entire structure.
4.3 Sensitivity Calculation

The failure number which is supplied by the structural analysis may be used as either an objective or a constraint, dependent upon problem formulation. In either case, it is necessary that the sensitivity of the structure the changes in the design variables be determined as efficiently as possible. Unlike the SIMP or DOC sensitivities discussed in sections 2.7 and 3.8, respectively, the structural sensitivity numbers are not as amenable to analytic differentiation, due to the complexity in the implementation of the finite element method. Furthermore, the applied loading will create displacements which are functions of the design variables through the SIMP weighted effect on the constitutive relationships discussed in section 4.1. The mathematical relationship is given by a chain rule:

$$\frac{df}{dx_n} = \frac{\partial f}{\partial x_n} + \frac{\partial f}{\partial y_i} \frac{dy_i}{dx_n}$$  \hspace{1cm} (4.14)

where:

- $f$ represents the failure number output by the structural solution
- $x$ represents the SIMP weights
- $y(x)$ represents the displacements which are functions of the SIMP weights
\( i \) is an index indicating the number of displacements

\( n \) is an index indicating the number of SIMP weights

In order to satisfy the governing equations, the residual of the problem can also be differentiated:

\[
\frac{dR}{dx_n} = \frac{\partial R}{\partial x_n} + \frac{\partial R}{\partial y_i} \frac{dy_i}{dx_n} = 0
\]

\[
\frac{\partial R}{\partial y_i} \frac{dy_i}{dx_n} = -\frac{\partial R}{\partial x_n}
\]

Where \( R \) is the structural residual, \( R = Ku - f \).

After rearrangement, this allows a solution of the form:

\[
\frac{df}{dx_n} = \frac{A}{\frac{\partial f}{\partial x_n}} + \frac{B}{\frac{\partial f}{\partial y_i} \frac{\partial R}{\partial y_i}} - \frac{D}{\frac{\partial R}{\partial x_n}} \Psi
\]  \hspace{1cm} (4.15)

where \( \Psi \) is the adjoint.

Addressing each of the sub components of this equation, partials \( A = \frac{\partial f}{\partial x_n} \) and \( D = \frac{\partial R}{\partial x_n} \), are calculated simultaneously with a central-difference perturbation of \( x_n \).

Partial \( B = \frac{\partial f}{\partial y_i} \) is solved using a chain rule:

\[
\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial P} \frac{\partial P}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial y_i}
\]

where:

\( P \) is the failure load value, discussed in section 4.2

\( \frac{\partial f}{\partial P} \) and \( \frac{\partial \varepsilon}{\partial y_i} \) are calculated analytically within the finite element code

\( \frac{\partial P}{\partial \varepsilon} \) is calculated using a complex step method

The complex step method is a more accurate method than a finite difference, whose general form is:

\[
\frac{df}{dx} \approx \frac{Im[f(x + ih)]}{h} \]  \hspace{1cm} (4.16)

The complex step method removes the possibility of subtractive error and theoretically is capable of determining the sensitivity of a function with machine precision [46]. The complex step sensitivity is used here rather than the central-difference scheme because it is easy to implement in the computer code for this calculation, whereas the central difference scheme is retained for partials \( A \) and \( D \) due to a greater degree of complexity in complex step implementation for these two equations.
Partial $C = \frac{\partial R}{\partial y_i}$ is equivalent to the stiffness matrix $K$, which is already calculated during each iteration. The resulting sensitivity is a semi-analytic solution, in that some components are arrived analytically, and others using differencing schemes.
Chapter 5

Results and Discussion

5.1 Sample Problem

A sample problem is selected to validate the modelling framework and to explore the implications of accounting for cost on the individual plate design. The sample problem consists of a stiffened square plate composed of orthogonal blade stiffeners at fixed height and spacing, assumed to be manufactured from some combination of metallic and composite lamina. Manufacturing specifics of stiffener attachment are not considered, since they are assumed to be dependent on final material selection, as well as final load distribution, both of which are unknown at the outset of the design.

5.1.1 Geometric Details

Plate geometry was selected to be similar in size to other similar stiffened plate analyses, such as the composite fuselage listed in NASA CR-159302 [17]. Baseline plate geometric essentials are listed in Table 5.1. Ply thicknesses are limited by finite element aspect ratio considerations.

<table>
<thead>
<tr>
<th>Plate Geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffener Spacing (s)</td>
<td>166.6 mm</td>
</tr>
<tr>
<td>Stiffener Blade Height (h)</td>
<td>60.0 mm</td>
</tr>
<tr>
<td>Plate Length (x)</td>
<td>333.3 mm</td>
</tr>
<tr>
<td>Plate Width (y)</td>
<td>333.3 mm</td>
</tr>
</tbody>
</table>

Table 5.1: Plate geometries

For this reason, the lowest practical limit on ply thickness is typically a few times thicker than typical composite laminates. This has no effect on cost or mass estimates, since the effects are
taken into account through the cost and performance calculations for the aircraft. The effect of thicker but fewer plies does have structural implications, which are discussed further in this chapter. The number of plies for both stiffeners and plate are fixed.

5.1.2 Finite Element Mesh Details

The part is meshed with an element spacing of approximately 55.55 mm, all elements on the skin being square and all elements on the stiffeners being approximately 20 x 55.55 mm. The finite element mesh relates to the design variable discretization with a ratio of 3 square finite elements to one design variable patch, such that 9 finite element stiffnesses are controlled by one set of design variables, dependent on the number of layers, as shown in Figures 5.1(a) and 5.1(b). This level of discretization was selected after a h-convergence study showed that this number of elements was sufficient to reduce the variation in calculated displacements by less than 0.11%, as compared to the next level of discretization fineness. The convergence study is included in Appendix B.

5.1.3 Materials

Each element of the plate and stiffeners may be composed of one of several materials, which is dependent upon the SIMP weighting scheme. The materials considered in the sample problem are aluminum and graphite / epoxy unidirectional prepreg laminate at ±60, ±45 ±30, 0 and 90 degrees from the x-axis as indicated in Figures 5.1(a) and 5.1(b). The stringers are also considered to be composed of similar materials. For the stiffeners, 0 degree composites are considered to be aligned with the long axis. Material properties are tabulated in Table 5.2.
5.1.4 Loading and Boundary Conditions

In order to ensure that the plate is capable of withstanding simulated flight loads, five representative load cases were selected from the composite fuselage design study in NASA CR-159302 [47]. These five load cases are imposed in succession, and the results are incorporated into the optimization function simultaneously as inequality constraints for each load case. These load cases are listed in Table 5.3. Note that the above loads are intended for a plate of approximately 500 x 500 mm cross section. Therefore, if the plate area is reduced, the pressure load will need to be incremented by an amount equal to the inverse square of the area reduction, in order to simulate approximately similar loads. This also applies if plate area is increased, except the pressure is then reduced. Since the plate is also simulated with geometry at 333.3 x 333.3, the

<table>
<thead>
<tr>
<th>Load Cases</th>
<th>Axial</th>
<th>Shear</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ultimate Compression with Positive Pressure</td>
<td>525.0</td>
<td>105.0</td>
<td>0.0914</td>
</tr>
<tr>
<td>Ultimate Compression with Negative Pressure</td>
<td>525.0</td>
<td>105.0</td>
<td>-0.0517</td>
</tr>
<tr>
<td>Ultimate Tension with Positive Pressure</td>
<td>262.0</td>
<td>105.0</td>
<td>0.0914</td>
</tr>
<tr>
<td>Ultimate Tension with Negative Pressure</td>
<td>262.0</td>
<td>105.0</td>
<td>-0.0517</td>
</tr>
<tr>
<td>Ultimate Burst Pressure</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1215</td>
</tr>
</tbody>
</table>

Table 5.3: Imposed Load Cases
pressures imposed in that instance would be 2.25 times higher than those listed in Table 5.3. The load cases for the 333.3 x 333.3 plate are imposed in the fashion shown in Figures 5.2(a) through 5.2(d) and 5.3. Pressure loads are imposed only on surfaces perpendicular to the

(a) Load Case 0: Ultimate Compression with Positive Pressure
(b) Load Case 1: Ultimate Compression with Negative Pressure
(c) Load Case 2: Ultimate Tension with Positive Pressure
(d) Load Case 3: Ultimate Tension with Negative Pressure

Figure 5.2: Load Cases 0 through 3

pressure; therefore, blade stiffeners are not acted upon directly by pressure. Loading cases are taken to be ultimate loads, and therefore include a safety factor of 1.5. The boundary conditions imposed in this instance are to constrain the plate in the $w$ displacement direction at all edge nodes, while also preventing a single corner node from moving in the $u$ and $v$ directions and ensuring that a single other corner node is constrained in $v$. This configuration, when the
forces are in equilibrium as illustrated in the figures, ensures that the boundary conditions do not actually exert any reaction forces in in-plane directions, thereby reducing the likelihood of stress concentrations. An illustration of the boundary conditions is shown in Figure 5.4.

5.2 Optimization Function

The final form of the optimization function is:

minimize:

\[ \text{DOC}(c(X), m(X)) \]

w.r.t

\[ X = \begin{bmatrix} x_{e0}^l & x_{e0}^l & \ldots & x_{en}^l \end{bmatrix} \quad l = 1, 2, \ldots, k \]

\[ e = 1, 2, \ldots, m \]

subject to:

\[ g_j(X) \geq 0 \quad j = 1, 2, \ldots, m \]

\[ h_e^l(X) = 0 \]

where:

\[ X \] is an \( n \)-dimensional vector of design variables

\( l \) represents the layer of interest, where there are \( k \) layers
Figure 5.4: Example Plate Boundary Conditions

\(e\) represents the element of interest, where there are \(m\) elements

\(DOC(X)\) is the objective function

\(g_j(X)\) are the inequality constraint functions corresponding to the failure calculation in the structural analysis for each load case \(j\), and are described by the KS function \(g_j(X) = KS(g_j(X))\)

\(h_k(X)\) are the equality constraint functions enforcing that the sum of the weights for each layer \(l\) in each element \(e\) sums to 1. The functional form is \(h^l_e(X) = \sum_{i=1}^{n} x^l_{ei} - 1\)

This optimization function is used with a set penalty function value for the SIMP weight calculations, until the SQP based optimizer meets internal optimality conditions. At that time there is an internal check which is based upon the DMO conditions as discussed in Section 2.6. If the DMO convergence criteria are met, then an optimum is said to be found, and final results are output. If the DMO convergence criteria are not met, then the penalty value is incremented and the optimizer is restarted with the design variables in the current positions. This essentially reformulates the objective function, since the new penalty function value creates new weighting; however, this is acceptable, since the optimizer resets all previous data except the design variable values upon re-initialization. A flowchart of the conceptual data flows between each of the sub-modules during the optimization process is illustrated in Figure 5.5.
5.3 Reference Plates

In order to create relevant comparison data and demonstrate the validity of accounting for DOC during the preliminary design phase, five reference plates of either homogeneous aluminum or composite, following a typical ply stacking sequence were selected. These were analyzed and salient data recorded. Due to the rapid increase of design variables associated with including many plies, and the difficulty in resolving non-failed results with extremely thin plies, an overall thickness of no less than 4 mm was specified for the result runs. Individual ply thickness was
varied to ensure that both stiffener and plate thickness values remained at this thickness regardless of ply numbers. Details with respect to these five reference plates may be found in Table 5.4. In the table, where appropriate, separate identifiers are used to indicate ply parameters for the skin, represented as $sk$, and the stiffeners, represented as $st$. Note that for costing purposes,

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>#DV</th>
<th>DOC ($)</th>
<th>Cost ($)</th>
<th>Weight (kg)</th>
<th>LC0</th>
<th>LC1</th>
<th>LC2</th>
<th>LC3</th>
<th>LC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>(0)        4</td>
<td>576</td>
<td>36,595,689</td>
<td>242.75</td>
<td>2.8394</td>
<td>0.6580</td>
<td>0.8389</td>
<td>1.4489</td>
<td>2.2312</td>
<td>2.3864</td>
</tr>
<tr>
<td>Al</td>
<td>(0)       8</td>
<td>1152</td>
<td>37,638,694</td>
<td>442.92</td>
<td>2.8394</td>
<td>0.5622</td>
<td>0.7967</td>
<td>1.2995</td>
<td>2.0376</td>
<td>1.9219</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0 90)</td>
<td>576</td>
<td>35,059,113</td>
<td>496.86</td>
<td>1.3333</td>
<td>1.3967</td>
<td>2.2860</td>
<td>0.6784</td>
<td>1.3494</td>
<td>0.8652</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(±45 0 90)</td>
<td>1152</td>
<td>35,836,375</td>
<td>667.60</td>
<td>1.3334</td>
<td>0.8286</td>
<td>0.6353</td>
<td>0.9642</td>
<td>1.1955</td>
<td>2.0192</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0 ±60 90) sk; (90 ±60 0) str</td>
<td>1152</td>
<td>35,846,747</td>
<td>669.87</td>
<td>1.3334</td>
<td>1.2200</td>
<td>0.8521</td>
<td>0.7219</td>
<td>1.4050</td>
<td>0.8851</td>
</tr>
</tbody>
</table>

Table 5.4: Initial Cases

since the possibility of laminated composite (GrEp) and Aluminum plates are allowed, a homogeneous aluminum plate as in the first two rows of Table 5.4 has fabrication costs associated with the production of each ‘ply’ of the aluminum making up the laminate. Although this is not entirely accurate, there is no way of knowing a priori whether a homogeneous aluminum part will result during optimization, therefore, since each ply is treated independently, each ply cost must be estimated independently.

It is also important to note that the above table indicates that despite the fact that the fabrication cost of aluminum is lower than that of the composites, the effect of the individual plate mass difference, when extrapolated to some percentage of the total aircraft acquisition cost and overall empty weight actually dominates the DOC. In fact, given the assumed initial conditions, total DOC for the all-composite parts is lower than the aluminum parts. The columns indicating the load case failure numbers provide an indication of which load cases are more critical, where a negative number would indicate failure, and 0 would indicate that a location in the plate lies precisely on the failure strain. Note also that although the 8 ‘ply’ aluminum and 4 ‘ply’ aluminum are essentially identical parts, since the overall thickness is identical, the failure numbers for the 8 ply are actually slightly closer to failure. This makes sense in the context of the Reissner-Mindlin plate model, which averages strains in each ply. Since the outer plies of the 8 ply case are half the thickness of the 4 ply case, the average strains due to bending in the outer fiber plies will be slightly higher. Note, however, that the failure numbers in all cases are above 0, and since these are applied as constraints during the optimization process, in all 5 of the above cases, the design point lies away from any inequality constraint boundary.
5.4 Optimization Results

Several avenues within the framework were explored and any implications of the defined models were examined.

5.4.1 4 Ply Optimization

Given the initial geometries of Table 5.1, and setting the individual ply thickness to 1.0 mm, a 4 ply plate optimization was carried out. The result was an all composite plate, with all plate plies oriented at 90 degrees in the direction of the applied load, and all of the resulting stiffener orientations aligned at 0 degrees, as shown in Figures 5.6(a), 5.6(b), 5.6(c) and 5.6(d). The 4 ply optimal design is compared to the two 4 ply reference plate designs in Figures 5.7(a) and 5.7(b).
Chapter 5. Results and Discussion

Figure 5.7: 4 Ply Optimum vs. Reference Plate Comparisons

(a) 4 Ply DOC Comparison

(b) 4 Ply Individual Plate Cost Comparison

(c) 4 Ply Individual Mass Comparison
individual plate weight and load case failure numbers and their percentage differences when compared to the 4 ply reference plates is presented in Table 5.5. As can be seen, the optimal plate reduced the DOC from the pure aluminum case by 4.34%, while reducing the DOC from the conventional ply stacking sequence by 0.14%. The significant reduction in DOC when switching from aluminum to composite comes from the difference in plate mass, since the mass remains unchanged between the two composite designs, with only the fabrication cost affecting the total DOC. The optimizer has also reduced the failure loads as much as practical and the new ply stacking sequence structure has load case failure numbers much closer to 0, indicating the proximity to failure. In the case of Load Case 4, the failure constraint is actually slightly negative. This is ignored, since the value is 2 orders of magnitude lower than the other values, and some slight conservatism is present in both ply and constraint aggregation KS functions. Breaking down the DOC into the sub-disciplines which are affected by plate mass or cost and then comparing the optimal 4 ply case against the two reference 4 ply cases results in the percentage differences as shown in Table 5.6. The DOC comparison table shows that the fuel costs and landing fees are entirely related to plate mass, which is why they are unchanged with a different ply stacking sequence. The comparison table also shows the significant increases which occur for the loan, insurance and depreciation when comparing the optimal design to a pure aluminum design. Referring to the DOC equations discussed in Section 3.7, it is apparent that these sub-disciplines are only affected by plate cost, and not plate mass. Due to the increased fabrication time and feedstock costs associated with composites, the plate cost is higher for the optimal design than for the homogeneous aluminum design; however, despite this and the fact that some sub-disciplines are more expensive with the optimal design, the overall DOC is still reduced. These relationships may be non-intuitive, and therefore are valuable to consider when

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC ($)</th>
<th>Cost ($)</th>
<th>Weight (kg)</th>
<th>LC0</th>
<th>LC1</th>
<th>LC2</th>
<th>LC3</th>
<th>LC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gr/Ep (90)</td>
<td>4 sk</td>
<td>35,006,202</td>
<td>485.23</td>
<td>1.3333</td>
<td>0.1011</td>
<td>0.2192</td>
<td>0.2609</td>
<td>0.4112</td>
<td>-0.0373</td>
</tr>
<tr>
<td>Al (0)</td>
<td>4</td>
<td>-4.34%</td>
<td>99.89%</td>
<td>-53.04%</td>
<td>-84.64%</td>
<td>-73.87%</td>
<td>-81.99%</td>
<td>-81.57%</td>
<td>-101.56%</td>
</tr>
<tr>
<td>Gr/Ep (0 90)</td>
<td>4 sk</td>
<td>-0.15%</td>
<td>-2.34%</td>
<td>0.0%</td>
<td>-92.76%</td>
<td>-90.41%</td>
<td>-61.54%</td>
<td>-69.53%</td>
<td>-104.31%</td>
</tr>
</tbody>
</table>

Table 5.5: Load Case Percentage Change - 4 Ply Optimum vs. Reference Plates

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC Total</th>
<th>Loan</th>
<th>Insurance</th>
<th>Depreciation</th>
<th>Fuel</th>
<th>AF Maint</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al (0)</td>
<td>4</td>
<td>-4.34%</td>
<td>9.29%</td>
<td>6.49%</td>
<td>6.49%</td>
<td>-12.50%</td>
<td>-17.70%</td>
<td>-13.20%</td>
</tr>
<tr>
<td>Gr/Ep (0 90)</td>
<td>4 sk</td>
<td>-0.15%</td>
<td>-2.40%</td>
<td>-0.29%</td>
<td>-0.29%</td>
<td>0.0%</td>
<td>-0.12%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

Table 5.6: DOC Percentage Change - 4 Ply Optimum vs. Reference Plates
optimizing a part for cost.

### 5.4.2 8 Ply Optimization

When optimizing an 8 ply case, with the ply thicknesses set at 0.5 mm, or one half of the thickness defined for the 4 ply case, the optimizer again converged to a fully composite case, with the skin plies oriented at 90 degrees, and the stiffeners plies all oriented at 0 degrees, or along their length. This 8 ply case was run with a one to one discretization scheme, which means that overall failure prediction accuracy is reduced. A comparison between the optimal 8 ply result to the reference 8 ply cases in terms of load is illustrated in Figures 5.8(a) 5.8(b) and 5.8(c). The percentage differences are listed in Table 5.7. As for the 4 ply case, the improvement against the homogeneous aluminum plate is better than against the composite plates, due to the approximately 50% plate weight reduction associated with switching material. This weight reduction lowers the DOC by affecting fuel cost directly. Conversely, all of the DOC savings which the optimal design achieves when compared to the other composite stacking sequences is

![Figure 5.8: 8 Ply Optimum vs. Reference Plate Comparisons](image-url)
related to the fabrication time savings related to laying down the composite in a single direction throughout. In terms of the individual DOC sub disciplines, the percentage improvements are shown in Table 5.8. Examining the DOC percentage change table for the 8 ply case, it is apparent that similar trends to the 4 ply optimal case are emerging. It is also of interest in both cases to note that the homogeneous aluminum case is primarily at a disadvantage when comparing the effect of fuel burn in the DOC cost, and that this value is directly dependent upon the price of jet fuel, which is a constant in the model. Fuel price included in equation 3.28 and also discussed in terms of the DOC model in Sections 3.7.3.

### 5.4.3 Fuel Cost Effects

A prominent finding from the optimization results shown in Sections 5.4.1 and 5.4.2 is that even if the individual plate fabrication costs are significantly higher, the weight penalty is such that with the existing July 7, 2008 fuel price constant as quoted in Table 3.9 and all other constants as stated, a fully composite, more individually expensive design is preferable. A new optimization run was started with the fuel cost changed to the price of 0.23 $/L, the average price on January 1, 2000 [48]. The resulting optimum plate meets the minimum DOC costs by selecting aluminum for the stiffeners and remaining with composite in the skin section, as shown in Figures 5.9(a), 5.9(b), 5.9(c), 5.9(d) where the grey coloured panels indicate aluminum. In order to properly gauge whether the optimizer has truly chosen a better configuration for the new fuel cost, the fuel price is left at the year 2000 cost and the reference cases are recalculated, as well as the optimal material distribution determined when fuel was at the 2008 level. These
Figure 5.9: 4 Ply Optimum Ply Orientations - Year 2000 Fuel Costs
are then compared to the new material configuration in Tables 5.9 and 5.10. The extreme

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC (§)</th>
<th>Cost ($)</th>
<th>Weight (kg)</th>
<th>LC0</th>
<th>LC1</th>
<th>LC2</th>
<th>LC3</th>
<th>LC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gr/Ep</td>
<td>(90)4 skin</td>
<td>23,889.581</td>
<td>328.81</td>
<td>2.1153</td>
<td>1.4974</td>
<td>1.8965</td>
<td>1.8676</td>
<td>1.8965</td>
<td>1.6441</td>
</tr>
<tr>
<td>Al</td>
<td>(0)4</td>
<td>-0.56%</td>
<td>3.545%</td>
<td>-25.50%</td>
<td>33.01%</td>
<td>88.44%</td>
<td>5.07%</td>
<td>-40.92%</td>
<td>-37.57%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)4</td>
<td>-0.71%</td>
<td>-33.82%</td>
<td>58.65%</td>
<td>7.21%</td>
<td>-17.04%</td>
<td>175.29%</td>
<td>40.54%</td>
<td>90.02</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0)4 skin</td>
<td>-0.49%</td>
<td>-32.24%</td>
<td>58.65%</td>
<td>1381.52%</td>
<td>765%</td>
<td>615.78%</td>
<td>361.18%</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 5.9: Plate Attribute Percentage Change - fuel at year 2000 cost, Optimum vs. Reference Plates

percentage difference Load Case failure number values between the original optimum and the new cheap fuel optimum are a result of the much higher failure numbers associated with the new optimum, as compared to the original. As can be seen in the table, by choosing a combination

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC Total</th>
<th>Loan</th>
<th>Insurance</th>
<th>Depreciation</th>
<th>Fuel</th>
<th>AF Maintain</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>(0)4</td>
<td>-0.56%</td>
<td>3.29%</td>
<td>2.30%</td>
<td>2.30%</td>
<td>-5.80%</td>
<td>-8.61%</td>
<td>-6.34%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0 90)</td>
<td>-0.71%</td>
<td>-5.85%</td>
<td>-4.21%</td>
<td>-4.21%</td>
<td>7.66%</td>
<td>10.91%</td>
<td>7.91%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)4 skin</td>
<td>-0.49%</td>
<td>-5.47%</td>
<td>-3.93%</td>
<td>-3.93%</td>
<td>7.66%</td>
<td>11.05%</td>
<td>7.91%</td>
</tr>
</tbody>
</table>

Table 5.10: DOC Percentage Change - 4 Ply, fuel at year 2000 cost, Optimum vs. Reference Plates

of aluminum and composite the optimizer has minimized DOC by reducing the acquisition cost as opposed to a totally composite design, as shown by the percentage decreases in the loan, insurance and depreciation values when compared to the fully composite designs. Additionally, although the aluminum-composite plate is heavier than the pure composite plates, this plate is still better than a pure aluminum plate, because the weight savings associated with the composite skin reduce the fuel cost by 5.8% per year. The results of this analysis indicate that there must be a ‘cross-over’ point, where the original fully composite design for 2008 fuel prices becomes uneconomic as compared to the 2000 fuel price optimum design. This cross-over point can be determined by varying the fuel price between the 2000 and 2008 prices, and comparing the DOC of the two optimal designs as these prices vary, as shown in Figure 5.10. The actual cross-over point occurs when fuel is priced at approximately $0.345 per litre. Beyond this, the cheaper DOC cost is obtained by using the fully composite plate. This demonstrates the advantage of taking DOC into account when considering preliminary design cost for a part. In terms of the fuel cost, it can be seen that a drastic change in terms of percentage breakdown of
Figure 5.10: DOC Cost Comparison of Year 2000 Fuel Cost optimum plate vs. Year 2008 Fuel Cost optimum plate

DOC has occurred with contemporary fuel prices as shown in the DOC percentage breakdown of the all-composite optimal plate with fuel prices at 2008 levels versus the DOC percentage breakdown of the mixed material optimal plate with fuel prices at 2000 levels, shown in Figures 5.11(a) and 5.11(b). The significant effect of the recent fuel prices on the operating environment is quite evident in the DOC breakdowns shown. The sensitivity of plate DOC to fuel price is independent of fabrication cost and changes dependent upon the individual plate mass. The relationship between individual plate mass and the DOC sensitivity with respect to fuel price change is shown in Figure 5.12. The figure implies a non-linear relationship between plate mass and the effect on DOC due to fuel cost, when all other geometries are fixed. The effect of fixing the geometry in this instance creates a linear relationship between individual plate mass and fuselage mass. Fuselage mass is used to determine fuel burn calculations, but is related to the overall aircraft weight; therefore, the non-linear effect is due to the increasing contribution of the fuselage to overall empty weight as individual plate mass is scaled up. For this reason, the sensitivity to fuel price is overall higher as individual plate mass increases.

5.4.4 Labour Cost Effects

Labour cost effects are more complicated to account for than fuel costs, since they are taken into account explicitly during the DOC calculations through maintenance and service costs, as discussed in Section 3.7. Labour costs are taken into account implicitly in the DOC cost,
Chapter 5. Results and Discussion

Figure 5.11: DOC Percentage Breakdown Comparison

(a) 4 Ply Optimum Plate DOC - Year 2008 Fuel Cost  
(b) 4 Ply Optimum Plate DOC - Year 2000 Fuel Cost

Figure 5.12: DOC Fuel Cost Sensitivity $\frac{\partial (\text{DOC})}{\partial \text{fuel cost}}$ with respect to Individual Plate Mass
because they are used within the fabrication cost estimation to provide an individual plate cost estimate. This plate cost estimate is then used to scale the airframe acquisition cost, which is used to determine all of the financial implications on the DOC model. The effect of varying labour cost is therefore broken down by varying operating labour cost and varying fabrication labour cost separately. Both avenues are explored to determine the effects on DOC, although it should be noted that the underlying assumption of the framework is that the airframe designer is using it as a preliminary design tool. This implies that the airframe designer is the primary stakeholder; therefore, fabrication cost is likely the only labour cost within his direct control. Unlike the fuel cost variation analysis, no alternate optimal plate emerged when scaling these labour costs down; however an identical design to the cheap fuel optimal design emerged when labour costs were scaled sufficiently higher.

**Fabrication Labour Cost Effects**

A fabrication labour cost study was performed by comparing the effect of scaling labour cost on the reference 4 ply plates, and the optimum 4 ply plate discussed in Section 5.4.1. The resulting effect on overall DOC and on individual plate cost are shown in Figures 5.13(a) and 5.13(b). The figures show that when labour cost is sufficiently high, the additional expense associated with composites will eventually effect the DOC through the acquisition cost such that a switch to aluminum is warranted, as would be seen when the aluminum plate DOC values cross those of the composite plates in Figure 5.13(a). Although overall individual plate cost for the aluminum is lower throughout the range of varying labour cost analysis, as in Figure 5.13(b), the overall DOC savings associated with the reduced mass in the composite plate is enough to counteract
this result. However, as labour costs increase, the individual plate cost difference between composite and aluminum becomes more pronounced, until eventually, at a calculated cross-over point of 3.719. This value is a modifier on existing labour costs and therefore corresponds to a 3.719 increase over current labour prices of $35.09/hr as quoted in Table 3.10. Note that this labour cost also includes maintenance overhead factors. The cross-over point does not need to be reached for a mixed material to emerge, since a full optimization with all other values fixed and the fabrication cost modifier set to 3.5 results in an optimal material arrangement identical to that presented in Figures 5.9(a), 5.9(b), 5.9(c) and 5.9(d). DOC percentage differences between the optimal material arrangement, the reference plates, and the optimal ply stacking sequence at a labour modifier of 1.0 are shown in Table 5.11. A listing of the individual plate cost and

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC Total</th>
<th>Loan</th>
<th>Insurance</th>
<th>Depreciation</th>
<th>Fuel</th>
<th>AFMaint</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>(0)₄</td>
<td>-1.36%</td>
<td>5.13%</td>
<td>3.78%</td>
<td>3.78%</td>
<td>-5.80%</td>
<td>-7.64%</td>
<td>-6.34%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0 90)₄</td>
<td>-1.41%</td>
<td>-11.73%</td>
<td>-9.02%</td>
<td>-9.02%</td>
<td>7.66%</td>
<td>7.45%</td>
<td>7.91%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)₄ skin</td>
<td>-0.93%</td>
<td>-10.75%</td>
<td>-8.24%</td>
<td>-8.24%</td>
<td>7.66%</td>
<td>7.87%</td>
<td>7.91%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0)₄ st</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.11: DOC Percentage Change - 4 Ply, labour costs multiplied by 3.5, Optimum (Gr/Ep (90)₄ skin, Aluminum (0)₄ st), vs. Reference Plates

mass is included in Table 5.12. Load case failure numbers are not included, since these are identical to those already presented. The data indicate that when labour cost is high enough, the effect of acquisition cost is enough to overcome the additional weight penalty associated with using aluminum. This is similar to the results of the fuel variation analysis; however, it is an opposite effect, in that if the individual plate cost becomes too high relative to plate mass, the optimal material choice changes to aluminum. When looking at a DOC sub-discipline breakdown, a comparison between the optimal design with a labour modifier of 3.5 and the original optimal plate reveals that despite a higher DOC for the higher labour costs, the overall percentage breakdowns are identical.
Maintenance Labour Cost Effects

The DOC model takes into account labour costs related to airframe, engine and aircraft turnaround servicing. These three values are not functions of individual plate cost whatsoever, as discussed in Section 3.7. However, they are all affected by directly varying the labour cost. The airframe maintenance calculation is related to plate mass, therefore a change in the individual plate mass will result in a change in the sensitivity of the airframe maintenance value with respect to plate mass. The sensitivity of airframe maintenance, engine maintenance and servicing costs with respect to their direct labour costs as plate mass changes is plotted in Figure 5.14. This figure essentially shows the partial derivative of these subdisciplines with respect to mass when they have been differentiated with respect to labour cost. Since individual plate mass has no direct effect on engine maintenance or servicing, the lack of change of these disciplines as plate mass changes is reasonable, whereas for airframe maintenance, the cost sensitivity increases with an increase in plate mass.

5.4.5 Stiffener Height Effects

Changing the stiffener height from the original specification of 60 mm will change the DOC, through both less overall surface area affecting fabrication, and also through reduced mass. The more significant effect of changing stiffener height will be to change the strain distribution.
Throughout the plate. This affects the failure numbers, and allows a determination of the lowest possible stiffener height based upon the reference plates and the standard optimal design. Designs which showed a failure number lower than -0.05 were not considered. Figure 5.15 shows the resulting DOC vs. stiffener height when considering a range of stiffener heights from 3 to 90 mm for the two reference 4 ply plates and the standard optimal plate. The figure shows that when stiffener height is 60 mm, as originally defined, the optimal plate has a lower DOC; however, as the stiffener height is decreased, the optimal plate is no longer available as an option, as it does not meet the failure criteria. Conversely, the fully composite reference plate, with a (0 90)_s stacking sequence is viable for much lower stiffener heights. Since the cross-ply laminate has a far better stiffness value in the off-axis direction, an examination of the magnitude of the moment resultant shows that the plate is significantly more compliant when the original optimal ply sequence of (90)_4 for the plate and (0)_4 for the stiffeners, as expected. When the stiffener height is 60 mm, this reduced stiffness is offset by the fact that the stiffeners are large enough to compensate. If the stiffener height is reduced below 60 mm, this is no longer the case, and some form of cross-ply is required in order to provide the required stiffness. The difference in magnitude between the moment resultant, $\kappa_y$, delineated as $sy_1$ in the plots, is shown in Figure 5.16. The figures show the significant increase in moment resultant in the unidirectional plate as against the cross-ply and the stiffener height difference indicates the degree to which this is significant, since an additional 33 mm of height can be removed when selecting an appropriate ply. Table 5.13 illustrates percentage differences showing the improvement in individual panel
Figure 5.16: Moment resultant $\kappa_y$ values for $(0\ 90)_s$ cross-ply vs. $(90)_4$ skin and $(0)_4$ stiffener optimum at stiffener heights of 60 and 27 mm
cost and weight, as well as load case failure numbers when comparing the 27 mm stiffener cross-ply design to the 33 mm homogeneous aluminum and 60 mm optimum (90),4 for the plate and (0),4 for the stiffeners. Table 5.14 shows the comparison of the cross ply to the other two plates

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>St Ht (mm)</th>
<th>Cost($)</th>
<th>Weight (kg)</th>
<th>LC0</th>
<th>LC1</th>
<th>LC2</th>
<th>LC3</th>
<th>LC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gr/Ep (0 90),s</td>
<td>27</td>
<td>432.90</td>
<td>0.953</td>
<td>0.5090</td>
<td>0.2407</td>
<td>0.2698</td>
<td>0.7284</td>
<td>0.0927</td>
<td></td>
</tr>
<tr>
<td>Al (0),4</td>
<td>33</td>
<td>47.13%</td>
<td>-128.43%</td>
<td>104.63%</td>
<td>23.90%</td>
<td>60.36%</td>
<td>-12.92%</td>
<td>35.81%</td>
<td></td>
</tr>
<tr>
<td>Gr/Ep (90),4 skin</td>
<td>60</td>
<td>-12.09%</td>
<td>-39.97%</td>
<td>80.14%</td>
<td>8.91%</td>
<td>3.30%</td>
<td>43.54%</td>
<td>140.27</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.13: Plate Cost and Mass, Load Case failure number percentage change compared to (0 90),s cross-ply

with respect to DOC and the relevant DOC sub-disciplines. The tables indicate the significant

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>St Ht (mm)</th>
<th>DOC Total</th>
<th>Loan</th>
<th>Insurance</th>
<th>Depreciation</th>
<th>Fuel</th>
<th>AFMaint</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al (0),4</td>
<td>33</td>
<td>-3.97%</td>
<td>7.26%</td>
<td>5.26%</td>
<td>5.20%</td>
<td>5.20%</td>
<td>-12.68%</td>
<td>-19.24%</td>
<td>-12.88%</td>
</tr>
<tr>
<td>Gr/Ep (90),4 skin</td>
<td>60</td>
<td>-2.86%</td>
<td>-1.86%</td>
<td>-1.33%</td>
<td>-1.33%</td>
<td>-1.33%</td>
<td>-4.12%</td>
<td>-7.45%</td>
<td>-4.02%</td>
</tr>
</tbody>
</table>

Table 5.14: DOC Percentage Change with various stiffener heights, compared to (090),s cross-ply with a stiffener height of 27 mm

weight savings which can be realized by reducing the stiffener heights. This indicates a major area for extension of the framework, since the effect of reducing stiffener height has such a pronounced effect.

5.4.6 Plate Size Effects

Since the fabrication cost model is based upon a first-order dynamic assumption, the presence of a steady-state velocity implies that a larger single panel should have a lower fabrication cost for a given linear, areal or volumetric measure as opposed to a smaller plate. Since the total surface area of the fuselage is fixed, the result of increasing individual plate size would be a smaller number of these larger plates which may have lower overall acquisition costs. To explore this advantage in the model, square plate models of 166.5 x 166.5 through 1000.0 x 1000.0 mm were simulated, in increments of 166.5 mm. All other geometric features were kept fixed at the original values given in Table 5.1.4 Ply models with ply thicknesses of 1 mm were run with the two reference plates, the optimum, (90),4 skin, (0),4 st plate and the (90),4 skin, (0),4 Aluminum st plate determined to be optimal for high labour costs and low fuel costs. The results were plotted against one another in terms of DOC and plate size in Figure 5.17(a) cost
per unit area in Figure 5.17(b), mass per unit area in Figure 5.17(c) and cost per unit mass in Figure 5.17(d). In examining the plot of DOC with respect to plate size change Figure 5.17(a).

![Figure 5.17: Size Variation Effects](image)

(a) DOC change with respect to plate size

(b) Cost per unit area, as a function of plate size

(c) Mass per unit area, as a function of plate size

(d) Cost per unit mass, as a function of plate size

It is apparent that the DOC does decrease with respect to larger plate size. Additionally, as expected in the fabrication cost model, the overall reduction in DOC is more pronounced with smaller plates, as the fabrication cost model has evidently not achieved steady-state velocity in material deposition. Once this occurs, any further gains are solely due to the reduction in mass per unit area which occurs due to the increased number of stiffeners. The mass over area value approaches \((\alpha + \beta \frac{2h}{L})\) asymptotically, where \(h\) is the stiffener height, \(L\) an individual plate length, \(\alpha\) the amount of mass per unit area for the plates and \(\beta\) the amount of mass per unit area for the stiffeners. For an elaboration of this equation, consult Appendix C. For the current problem, \(h\) is 60 mm, and \(L\) is 166.5, and the mass per unit area depends upon the material,
volume and density of the respective material. Since the plates and stiffeners are defined as 4
mm thick, the mass per unit area for the two materials considered are:

- Aluminum $10.8 \frac{kg}{m^2}$
- GrEp $4.904 \frac{kg}{m^2}$

The asymptotic mass over area values for the 4 plates considered are listed in Table 5.15. The

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>$\frac{M}{\pi \text{ size} \to \infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>(0)4</td>
<td>21.6</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(090)4</td>
<td>9.808</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)4 skin</td>
<td>9.808</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0)4 st</td>
<td></td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)4 skin</td>
<td>15.704</td>
</tr>
<tr>
<td>Al</td>
<td>(0)4 st</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.15: Limits of mass over area values

limits quoted in Table 5.15 are consistent with the mass over area values shown in Figure 5.17(c).
The resulting analysis reveals why the overall DOC decreases so rapidly with increasing plate
size, similar to the decrease in cost per unit area with respect to plate size in Figure 5.17(b).
When the plate length is between 166.6 and approximately 600 mm, the cost decreases rapidly
as the fabrication processes ramp to full speed, but as plate length continues to increase, there
is no more reduction in fabrication cost due to the attainment of steady state velocity. Rather,
all subsequent gains occur due to the reduction in plate mass per unit area as length increases,
as shown in Figure 5.17(c). The reason that cost per unit area continues to decrease as the plate size increases past 600 mm is due to the fact that the quoted plate cost numbers include
both fabrication costs, whose contribution is negligible at this stage, and feedstock costs, which
are functions of the total plate mass. Since plate mass is monotonically decreasing, both mass
per unit area, cost per unit area, and cost per unit mass will decrease, although it is important
to note that these numbers will all be affected by the asymptotic behaviour of the mass over
area data. In terms of DOC reduction, it is evident that the larger the sample plate, the lower
the DOC, and ideally a larger plate would be considered in the optimization scheme, especially
considering the fact that plates of lengths below 600 mm have significant room for improvement
due to the fabrication cost model parameters. However, the current architecture is only a proof-
of-concept, and the current optimization scheme would require a great deal of time to optimize
a plate with relevant fidelity and the requisite number of design variables. From the current
analyses a percentage comparison of the DOC, cost and weight between the (90)4 skin, (0)4 st
plate and the other 3 plates at the largest size of 1000 mm is shown in Table 5.16. The table
Table 5.16: Plate cost and weight comparison - all plate sizes 1000 x 1000 mm

<table>
<thead>
<tr>
<th>Material</th>
<th>Angles</th>
<th>DOC ($)</th>
<th>Cost ($)</th>
<th>Weight (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gr/Ep</td>
<td>(90)</td>
<td>33,600,834</td>
<td>2170.15</td>
<td>10.6157</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(0)</td>
<td>-0.16%</td>
<td>-4.68%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Gr/Ep</td>
<td>(90)</td>
<td>-2.28%</td>
<td>48.20%</td>
<td>-34.02%</td>
</tr>
<tr>
<td>Al</td>
<td>(0)</td>
<td>-5.00%</td>
<td>150.73%</td>
<td>-53.04%</td>
</tr>
</tbody>
</table>

indicates that the optimal plate manages to reduce the fabrication cost by 4.68% from the all composite cross-ply plate. It is interesting to note that the reduction value is significantly larger than the percentage difference reduction exhibited between these two plates when they were 333.3 x 333.3 mm, as in Table 5.5. The fabrication model only discriminates between 0 and 90 degree layup in terms of time if the surface being considered is not square; otherwise the amount of time is equal. For this reason, the driving factor behind the reduced fabrication cost between the optimum and cross-ply arrangement illustrated in Table 5.16 must be the stiffeners. A significant time advantage is achieved if the layup proceeds along the longitudinal length of the stiffener, rather than the short axis, since this would necessitate much stopping and starting. Since the cross-ply laminate has two 90 degree plies in the stiffeners as well, the increased reduction in time is due to the fact that in the larger plates, an economy of scale is gained for the 0 degree and 90 degree plies in the skin and the 0 degree plies in the stiffeners, but no benefit is gained for any 90 degree plies in the stiffeners. In fact, the inclusion of these plies simply increases the time by directly requiring more repetitions of the same time length as before, since the stiffener height for both the 333.3 x 333.3 mm and 1000 x 1000 mm plates are identical. As in the previous comparisons, the more significant reduction in DOC when compared against a mixed material or all aluminum plate come about entirely due to the reduction in mass associated with a fully composite plate. As is typical for plates with aluminum in this analysis, the fabrication cost for these plates are actually much more favourable than the full composite plate, but given the current assumptions in the model, the reduction in cost associated with the reduced weight of composites is enough to offset this factor.

5.4.7 Stiffener Spacing Effects

Stiffener spacing plays a significant role both in terms of the effect on cost and in terms of the structural effect. By increasing the stiffener spacing, less mass per flat unit area results in a structurally lighter design, but this has effects on the failure loads. Additionally, changes in overall plate mass and stiffener number for a given flat area result in monotonic variations of
the DOC, dependent upon whether the number of stiffeners increases or decreases. In order
to determine the effect of stiffener spacing, the two reference plate designs, as well as the

\[(90)_{4}\text{ skin}, (0)_{4}\text{ st optimal plate}\]

and the low fuel price \[(90)_{4}\text{ skin}, (0)_{4}\text{ Aluminum st plate}\]

were defined with an overall 666.6 mm plate, stiffer height of 60 mm, and varying
stiffener spacings of 333.3, 222.2, 166.6, 133.3 and 111.1 mm. The larger plate size of 666.6
x 666.6 was selected to minimize fabrication cost effects related to insufficient time to reach
steady state velocity, as determined in Section 5.4.6. This plate size is judged to be sufficient
such that any further cost advantage is primarily mass based. In terms of comparisons, it
would be instructive to observe overall structural deflection at a common point among the
different plates, unfortunately, the current architecture only allows stiffeners to be located along
element boundaries; therefore, these numbers also serve as spacing for groups of three plate
finite elements, following a discretization level of 3 finite elements per design variable patch.
This means that the nodes among each of these plate designs will not be coincident; therefore,
the structural results may not be fully consistent. The total deflection in the z direction as
a result of the pure pressure load of Load Case 4 was plotted for each plate at each stiffener
spacing and is included as Figure 5.18(a). The variation in DOC of each type of plate at the
different stiffener distances is plotted in Figure 5.18(b). Both the DOC and the displacement trend

\[\begin{align*}
\text{(a) w displacement change with respect to stiffener spacing} & \\
\text{(b) DOC change with respect to stiffener spacing}
\end{align*}\]

Figure 5.18: Stiffener Spacing Variation Effects

follow intuitive trends, with the DOC decreasing as the spacing increases, and the displacement
increasing in the same direction. It is interesting to note that the least compliant plate is the
pure aluminum, whereas the most compliant design is the \((0,90)_{s}\) cross-ply laminate. Of the
composite plates, the fact that the pure composite cross-ply is the most compliant makes sense,
considering that the stiffeners provide a great deal of out of plane rigidity, and in the cross-
ply design, half of the stiffener plies are oriented orthogonal to the resultant in-plane normal forces generated by plate bending. The low stiffness in the matrix direction for composites results in a structure with essentially half of the ability to resist out of plane pressure than the other two ply arrangements considered. Typically, with closer stiffener spacing this effect is reduced, regardless of structural properties, as shown in the left part of Figure 5.18(a). As the stiffener spacing increases the weakness of the cross-ply stiffener arrangement with respect to pressure loads becomes more pronounced. In terms of cost optimization, the ideal arrangement would be to reduce stiffeners as much as structurally allowable. Of the designs considered and displayed in the figures, all were above the failure margin, and therefore were structurally acceptable within the bounds of the optimization criteria. It is important to consider that no explicit restriction on compliance has been applied, therefore even though the cross-ply plate experiences a displacement of approximately 3.5 mm, this is not enough to fail the part, and therefore remains acceptable. In order to reduce DOC as much as possible among the chosen spacings the widest of all spacings should be used. In order to maintain the same size elements as originally defined and reduce the stiffener spacing to the maximum shown by this study, the discretization level should be set to 6. This will ensure elements that capture the same structural required in the convergence study, while reducing the contribution of the stiffeners.

5.4.8 DOC Model Plate Cost and Mass Effects

Although not requiring a plate optimization, it is instructive to determine the relative effects and sensitivities of the DOC model to changes in the individual plate cost and mass, and to determine the points at which individual plate mass and individual plate cost become more or less influential. An examination of the effect on DOC of the total fuselage cost and mass is performed first. Figure 5.19 shows the effect of varying overall fuselage mass and cost on the DOC. The overall sensitivity of DOC to mass is more pronounced than cost, as is visible in the plot. The carpet plot also implies that there is a DOC level at which it is more effective to reduce cost rather than mass, although the point at which this occurs is dependent upon the particular values of both. Examining the sensitivity of the DOC to changes in the fabrication cost of the plate, the value was found to be invariant, implying a second derivative of 0, and implying that the slope of the carpet plot is linear in the cost direction regardless of mass. The calculated sensitivity value is 0.2795 DOC($) / Cost($). Conversely, the sensitivity of the DOC to plate mass varies with plate mass, and is plotted in Figure 5.20. The results of this analysis indicate that the effect of varying mass on DOC decreases as the plate mass and hence fuselage mass increase overall. The reduction of the effect of mass on DOC as the plate weight increases was also discussed in Section 5.4.3 and this effect was indirectly observed through varying the fuel cost.
Figure 5.19: DOC compared to fuselage mass and cost

Figure 5.20: DOC compared to fuselage mass and cost
5.5 Conclusions

This study considered an integrated fabrication-cost-model for a representative aerospace component, along with traditional weight estimation. This was coupled to an empirical Direct Operating Cost model, based on assumptions surrounding a typical in-service aircraft. The modelling framework considered multiple material options in the context of a laminated, stiffened plate, taking into account material stiffness, fabrication time, feedstock cost and material mass. By using a weighted penalty method, the problem of handling discrete material options was reformulated as continuous and differentiable. A gradient based optimization algorithm was used to provide design variable weights used to determine final material selection. Analytic sensitivities of all objective and most constraints were calculated to reduce optimization time. The structure was subjected to multiple representative load cases to reduce the likelihood of single design point optima. The optimization algorithm had as it’s goal the reduction of Direct Operating Cost of the aircraft, given specific assumptions. By varying certain assumptions while keeping all others fixed, the framework was tested to determine the validity of the optimization scheme with different starting parameters. Parameters which were varied and studied were: fuel costs, labour costs, stiffener heights and stiffener spacing. Several conclusions were obtained from this study:

- By including Direct Operating Cost effects early in the model development, rapid trade-studies balancing current and perceived future aircraft operating conditions can be integrated into primary structural design. This allows greater design flexibility and a more informed preliminary design capability.

- Labour and fuel costs traditionally play an indirect role in the preliminary design; however, by considering them during the design phase, the optimal structural design can be determined which meets both structural and operating cost considerations.

- Optimization algorithms present an opportunity to allow rapid preliminary design synthesis, provided adequate accurate information exists and the optimization problem is intelligently posed.

- The Gutowski fabrication cost model framework is an excellent base upon which to provide a link between material choice and placement, and cost estimation.

- Material mass remains an important metric in determining cost; however, this must be supplemented by fabrication and assembly costs. Further, it must be recognized that both the part cost estimation and mass are part of a larger system, whose overall costs and requirements may provide different weights for these factors.
With respect to composite layup sequences, a great deal of potential exists in terms of structural tailoring as well as cost optimization.

5.6 Future Work

During the research, creation, development and exploration of the framework, several other important areas for further study were identified:

- The time required to optimize the part was highly dependent upon both the fineness of the Finite Element mesh and number of design variables. Up to 90% of the computational time was taken with determining the finite difference values for the failure constraint sensitivities. In order to speed up the framework significantly and develop the capability to solve larger problems, the requirement exists to determine a faster way to calculate the partial derivatives of the failure number with respect to the design variables \( \frac{\partial f}{\partial x_n} \), structural residual with respect to the design variables \( \frac{\partial R}{\partial x_n} \) and failure load value with respect to the structural strains \( \frac{\partial P}{\partial \epsilon} \).

- An SQP gradient-based algorithm was used to optimize the problem in this case. It would be instructive to attempt to optimize the problem with other gradient-based algorithms or non-gradient based algorithms to determine the points at which each of these become more effective.

- Although the framework is only a proof-of-concept, the ability exists to take into account more materials in the design. It would be informative to include a low density, low stiffness spacer for sandwich structures, as well as varied alloys into the framework. This would greatly increase the potential range of designs, all while taking into account the fabrication, mass, stiffness and feedstock implications on the Direct Operating Cost. Additionally, the option for stiffeners to be removed entirely would result in far more targeted designs in terms of structural tailoring.

- An advantage that exists with the Gutowski fabrication cost model is that it explicitly allows refinement based on real-world validation. Although every effort was taken to be as accurate as possible with the researched fabrication time constants, to truly ensure more accurate estimates, some simple physical validation runs of the most dominant fabrication steps could be performed.

- The current structural solution assumes that the structure acts linearly. However, one of the primary failure modes for thin stiffened laminated plates is by either skin or stiffener
buckling. The inclusion of a non-linear buckling structural analysis would provide better confidence and fidelity in terms of the failure calculations.

- During the trade study phase stiffener height, plate size and stiffener spacing were changed manually to more fully explore the design space. Ideally, these geometric features should be incorporated as design variables in the framework, to allow the optimizer to more fully exploit advantages in the design space. Ideally, these geometric design variables would need to have analytic sensitivities calculated as well.

- The ability to vary the number of plies, as well as determining their ‘thickness’, with some discretization scheme, would allow a much more nuanced approach to identifying potential stacking sequences for composites. Again, determining sensitivities in a very time-effective way for these new design variables would be required.

- One of the common results of building a structure out of smaller modular parts is that a learning curve effect may manifest itself. Incorporating the effect of learning curves on repetitive operation fabrication time would provide greater fidelity in cost modelling, as well as providing greater information in terms of determining the optimum plate size for aerospace structural construction.

- Fabrication costs, material mass, final material placement, and structural stiffness are all subject to non-deterministic effects. Ideally, a stochastic model including properly derived variability would provide greater confidence when optimizing a design.

- An area for exploration exists in changing the underlying assumptions of the Direct Operating Cost model. The DOC model framework is based upon empirical correlations of past aircraft. As such, it is limited in utility for considering unconventional aircraft. Further research towards generalized operational costing methods for aerospace is warranted.

- The current optimization framework took the approach of minimizing cost while setting failure as a constraint. It would be instructive to include some structural parameter such as compliance into the objective, to determine whether this has merit.

- The current framework does not allow for stiffeners to be present anywhere else other than along existing element edges. Ideally, the presence of stiffeners should be enabled between any adjacent nodes in the finite element mesh. This would greatly increase the versatility of the model and grant much better flexibility to the optimizer in choosing an ideal plate topography. Coupled with stiffener height selection, this would vastly increase the scope of the design space, and present many more opportunities for cost and structural savings.
References


Appendix A

DOC Sensitivity Calculation Constants

A.1 Discussion

In order to determine the constants associated with the DOC sensitivity calculations, the following procedure was used:

- Assign constant numbers to initial constants appearing in original equation
- Take all governing variables that are not constants, and extract their governing equations
- Assign new constant numbers to any constants appearing in these governing equations
- Combine constants where appropriate
- Repeat as necessary until only fuselage cost or fuselage weight remain as variables

The above procedure was followed for all of the sensitivity calculations; therefore, in listing the constants associated with the final sensitivity calculations, the above procedure is to be taken into account in terms of the relationships between constants. Note that some of the constants between the different sub-disciplines will be identical, since they must all eventually relate back to the underlying governing equations. Note also that not all sensitivities will be addressed, only those with unexplained constants in Section 3.8.

A.2 Fuel

Fuel cost is calculated using the equation:

\[ fuel(m) = fuel_{year}(m)(mis_{year}) \]

The fuel sensitivity equation is:

\[ \frac{\partial fuel}{\partial m} = (-C_{10})(\frac{1}{\sqrt{C_6 + m} - \sqrt{C_7 + m}})^2(0.5)(\frac{1}{\sqrt{C_6 + m} - \sqrt{C_7 + m}}) \]
The related constants are:

\[ C_1 = \text{fuel\_price as listed in Table 3.9} \]

\[ C_2 = \text{mis\_year as calculated in equation 3.27} \]

\[ C_3 = \text{Mission distance } d \text{ as given in Table 3.3} \]

\[ C_4 = \text{Aircraft Fuel Capacity } f_{\text{cap}} \text{ as in Table 3.3} \]

\[ C_5 = \left( \frac{2}{\rho_{\infty}} \right) \left( \frac{2}{\rho_{\infty}} \right)^{\frac{1}{2}} \left( \frac{C_L}{C_D_{\text{max}}} \right), \text{ where } SFC \text{ is the thrust specific fuel consumption found in Table 3.4, } \rho_{\infty} \text{ is the air density at the given flight altitude, as explained in Section 3.7.3, } S \text{ is the wing surface area, as in Table 3.3 and } \frac{C_L}{C_D_{\text{max}}} \text{ is calculated in equation 3.24} \]

\[ C_6 = AC_{EW-LE-LF} + AC_{MF-MP} + ENG_n \cdot \frac{EN_w}{2.20462}, \text{ where } AC_{EW-LE-LF} \text{ is discussed in Section 3.7.1, } AC_{MF-MP} \text{ is provided in Table 3.3 and } ENG_n \text{ and } EN_w \text{ are the number of engines and their individual mass, as in Table 3.4} \]

\[ C_7 = AC_{EW-LE-LF} + (ENG_n \cdot \frac{EN_w}{2.20462}) + AC_{MF-MP} - f_{\text{cap}} \rho_{\text{fuel}}, \text{ where } \rho_{\text{fuel}} \text{ is found in Table 3.3} \]

\[ C_8 = 0.0003048 \]

\[ C_9 = 1.484796 \]

\[ C_{10} = \frac{C_1 C_2 C_3 C_4}{C_5 C_8 C_9} \]

### A.3 Airframe Maintenance

Airframe maintenance is calculated using the equation:

\[ AFM_{\text{year}}(m, c) = AF_{\text{lpyear}}(m) + AF_{\text{mpyear}}(c) \]

and is discussed in more detail in Section 3.7.11. Airframe maintenance is a function of both cost and mass. The mass sensitivity equation is:

\[ \frac{\partial \text{maint}}{\partial m} = C_{22} + C_{26} + C_{24}C_{11}(C_{11}(m) + C_{18})^2 + C_{25}C_{11}(C_{11}(m) + C_{18})^{-2} \]

The related constants are:

\[ C_1 = \text{Material Costs Per Flight, given by the relevant equation in the Airframe Maintenance section of Section 3.7.11} \]

\[ C_2 = \text{mis\_year as calculated in equation 3.27} \]

\[ C_3 = \text{Maintenance labour cost } l, \text{ as given in Table 3.10} \]
$C_4 = \text{Maintenance burden } \theta \text{ as given in Table } 3.10$

$C_5 = \text{Mission Length } t \text{ as calculated in equation } 3.26$

$C_6 = \text{Maintenance labour reduction rate } i_{70}, \text{ to take into account for inflation between the model correlation year and the present date, given in Table } 3.10$

$C_7 = \sqrt{\text{mach}} \text{ where mach is the mission mach number, given in the Cruise Speed discussion Section of Section } 3.7.3$

$C_8 = 1.10231E - 4$

$C_9 = AC_{EW - LE - LF}, \text{ as discussed in Section } 3.7.1$

$C_{10} = 6$

$C_{11} = 2.20462E - 3$

$C_{12} = 630$

$C_{13} = 120$

$C_{14} = 0.59$

$C_{15} = C_8 C_9$

$C_{16} = C_9 C_{11}$

$C_{17} = C_{10} + C_{15}$

$C_{18} = C_{13} + C_{16}$

$C_{19} = C_2 C_3 C_4 C_6 C_7$

$C_{20} = C_5 C_{14} C_{19}$

$C_{21} = C_1 C_2$

$C_{22} = C_8 C_{19}$

$C_{23} = C_{17} [C_{19} + C_{20}] + C_{21}$

$C_{24} = C_{12} C_{19}$

$C_{25} = C_{12} C_{20}$

$C_{26} = C_8 C_{20}$
The cost sensitivity equation is:

$$\frac{\partial \text{maint}}{\partial c} = \left(\frac{\text{mis}_{\text{year}}}{1E + 6}\right)(3.08t + 6.24)$$

### A.4 Landing

The landing fee equation is:

$$\text{LFee}(m) = FR + (AC_{\text{seats}})(SM) + AC_{\text{CURLW}}(LM)$$

The landing sensitivity equation is given as:

$$\frac{\partial \text{landing}}{\partial m} = C_{20} - \frac{(0.5)[C_{22}(\frac{1}{\sqrt{m + C_{11}}}) - (\frac{1}{\sqrt{m + C_{12}}})]}{(\sqrt{m + C_{11}} - \sqrt{m + C_{12}})^2}$$

Note that the landing fee constants are specifically related to the empirical multipliers levied by the particular airports used. In this case, the assumption of landing equally consistently at Ottawa and Regina airports is made, therefore the constants reflect the landing fees at those two airports. The related constants are:

- $C_1 = 5.35E - 3$
- $C_2 = 6.05E - 3$
- $C_3 = 499.44 + 1.44(AC_{\text{seats}})$, where $AC_{\text{seats}}$ is given in Table 3.3
- $C_4 = \frac{\text{mis}_{\text{year}}}{2}$ where $\text{mis}_{\text{year}}$ is as calculated in equation 3.27
- $C_5 = AC_{\text{EW-LE-LF}}$, as discussed in Section 3.7.1
- $C_6 = 102.27(AC_{\text{seats}})(LF)$ where $AC_{\text{seats}}$ and $LF$, the aircraft load factor, are given in Table 3.3
- $C_7 = \rho_{\text{fuel}}$, found in Table 3.3
- $C_8 = 1852$ km, the mission reserve distance, as discussed in Section 3.7.12
- $C_9 = \text{Aircraft Fuel Capacity } f_{\text{cap}}$ as in Table 3.3
- $C_{10} = (\frac{SFC}{\rho_{\infty}})(\frac{2}{\rho_{\infty}}S)^{\frac{1}{2}}(\frac{C_{1}^{2}}{C_{D_{\text{max}}}})$, where $SFC$ is the thrust specific fuel consumption found in Table 3.4, $\rho_{\infty}$ is the air density at the given flight altitude, as explained in Section 3.7.3, $S$ is the wing surface area, as in Table 3.3 and $\frac{C_{1}^{2}}{C_{D_{\text{max}}}}$ is calculated in equation 3.24.
Appendix A. DOC Sensitivity Calculation Constants

\[ C_{11} = AC_{EW-LE-LF} + AC_{MF-MP} + ENG_n + \frac{EN_n}{20462}, \]
where \( AC_{EW-LE-LF} \) is discussed in Section 3.7.1, \( AC_{MF-MP} \) is provided in Table 3.3, and \( ENG_n \) and \( EN_n \) are the number of engines and their individual mass, as in Table 3.4.

\[ C_{12} = AC_{EW-LE-LF} + (ENG_n \frac{EN_n}{20462}) + AC_{MF-MP} - f_{cap} \rho_{fuel}, \]
where \( \rho_{fuel} \) is found in Table 3.3.

\[ C_{13} = 0.0003048 \]
\[ C_{14} = 1.48479 \]
\[ C_{15} = C_5 + C_6 \]
\[ C_{16} = \frac{C_5 C_8 C_9}{C_{10} C_{13} C_{14}} \]
\[ C_{17} = C_1 C_4 \]
\[ C_{18} = C_2 C_4 \]
\[ C_{19} = C_3 C_4 \]
\[ C_{20} = C_17 C_18 \]
\[ C_{21} = C_{15}(C_17 C_18) + C_{19} \]
\[ C_{22} = C_{16}(C_17 + C_18) \]
Appendix B

H Convergence

In order to ensure that an appropriate mesh size was selected, an H convergence study was performed. This study compared three representative stiffened plates under pure pressure loads, identical to those quoted for Load Case number 4 in Section 5.1.4. The three plates were all subject to the boundary conditions outlined in section 5.1.4. Three material combinations were selected for the test: Aluminum, all composite at 0 degrees, and all composite at 90 degrees. Since the finite elements used are 16 node bicubic elements, in order to ensure nodal locations remain unchanged, the mesh refinement proceeded in multiples of 3. The stiffened plate geometry measured 500 x 500 mm, and stiffener heights remained fixed at 60 mm. The three representative meshes are presented below, in Figures B.1(a), B.1(b) and B.1(c). Displacements in the z direction were taken at the exact centre location of each plate, and compared. The resulting stiffener and plate element counts are given in Table B.1. Results of these comparisons

<table>
<thead>
<tr>
<th>Discretization</th>
<th># Plate Els</th>
<th># Stiff Els</th>
<th>Total Els</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>24</td>
<td>33</td>
</tr>
<tr>
<td>3</td>
<td>81</td>
<td>216</td>
<td>297</td>
</tr>
<tr>
<td>9</td>
<td>729</td>
<td>1944</td>
<td>2673</td>
</tr>
</tbody>
</table>

Table B.1: Elements per discretization level

and percentage differences between the displacements of the previous discretization level and the current one are listed in Table B.2. Table B.2 indicates that an order of magnitude increase in the number of elements, which occurs between the discretization levels of 3 and 9 garners approximately a 1% improvement in displacements. Considering the amount of time required for a structural analysis, this may be acceptable. However, considering that finite difference approximations of the displacements with respect to the design variables are required for sensitivity calculation, then the time penalty associated with a mesh this fine is too great. Therefore,
Figure B.1: Finite Element Meshes used for convergence study
discretization level 3, corresponding to an elemental spacing of 55.5 mm and nodal spacing of 18.518 mm for the plate elements is selected as an acceptable trade-off between finite element accuracy and computational speed.
Appendix C

Mass over Area Proof

When comparing Direct Operating Cost data, it was found within the current framework that the ratio of mass to a given flat plate area declined as the plate length and width increased. This decrease is due to the difference between the total surface area growth as compared to the flat plate growth. This value must not reach zero, therefore it must be asymptotically approaching a given number.

Assuming that:

- a stiffened plate is discretized into equally sized square plate and stiffener sections, which are present at all edges of the square plates
- \( n \) represents the number of plate elements
- \( L \) represents a discretized plate edge length
- \( h \) represents the stiffener height
- \( \alpha \) represents the mass per unit area for the plate elements
- \( \beta \) represents the mass per unit area for the stiffener elements

Then the number of stiffeners on a given square plate is given as:

\[
    n_{stiffs} = 2(n + \sqrt{n})
\]

The flat plate total area is:

\[
    A_{plate} = nL^2
\]

and the stiffener total area is:

\[
    A_{stiffs} = 2(n + \sqrt{n})Lh
\]
Appendix C. Mass over Area Proof

The ratio of interest is:

\[ \frac{\text{Mass}}{A_{\text{plate}}} \]

where \( \text{Mass} = \alpha A_{\text{plate}} + \beta A_{\text{stiffs}} \) and \( A_{\text{tot}} = A_{\text{plate}} + A_{\text{stiffs}} \).

To determine the value which the ratio is approaching asymptotically, the limit is taken:

\[
\lim_{n \to \infty} \frac{\alpha A_{\text{plate}} + \beta A_{\text{stiffs}}}{A_{\text{plate}}} = \frac{\alpha nL^2 + \beta 2(n + \sqrt{n})Lh}{nL^2} = \frac{\alpha + \beta 2 Lh [n + \sqrt{n}]}{nL^2}
\]

\[
\lim_{n \to \infty} \frac{\alpha}{a} + \frac{\beta 2h}{L} \left( \frac{L}{\sqrt{n}} + \frac{2h}{\sqrt{n}} \right)
\]

The resulting equation is used to determine the mass over flat area ratio which the given plate design is approaching, which provides some insight into the potential of increasing individual plate size.