3D Finite Element Cosserat Continuum Simulation of Layered Geomaterials

by

Azadeh Riahi Dehkordi

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Abstract

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Azadeh Riahi Dehkordi

Department of Civil Engineering

University of Toronto

The goal of this research is to develop a robust, continuum-based approach for a three-dimensional, Finite Element Method (FEM) simulation of layered geomaterials. There are two main approaches to the numerical modeling of layered geomaterials; discrete or discontinuous techniques and an equivalent continuum concept.

In the discontinuous methodology, joints are explicitly simulated. Naturally, discrete techniques provide a more accurate description of discontinuous materials. However, they are complex and necessitate care in modeling of the interface. Also, in many applications, the definition of the input model becomes impractical as the number of joints becomes large. In order to overcome the difficulties associated with discrete techniques, a continuum-based approach has become popular in some application areas. When using a continuum model, a discrete material is replaced by a homogenized continuous material, also known as an 'equivalent continuum'. This leads to a discretization that is independent of both the orientation and spacing of layer boundaries. However, if the layer thickness (i.e., internal length scale of the problem) is large, the
classical continuum approach which neglects the effect of internal characteristic length can introduce large errors into the solution.

In this research, a full 3D FEM formulation for the elasto-plastic modeling of layered geomaterials is proposed within the framework of Cosserat theory. The effect of the bending stiffness of the layers is incorporated in the matrix of elastic properties. Also, a multi-surface plasticity model, which allows for plastic deformation of both the interfaces between the layers and intact material, is introduced. The model is verified against analytical solutions, discrete numerical models, and experimental data. It is shown that the FEM Cosserat formulation can achieve the same level of accuracy as discontinuous models in predicting the displacements of a layered material with a periodic microstructure. Furthermore, the method is capable of reproducing the strength behaviour of materials with one or more sets of joints. Finally, due to the incorporation of layer thickness into the constitutive model, the FEM Cosserat formulation is capable of capturing complicated failure mechanisms such as the buckling of individual layers of material which occur in stratified media.
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“Strange, is it not? That of the myriads who before us pass’d the door of Darkness through. Not one returned to tell us of the Road, which to discover we must travel too” “Khayyam, Fitzgerald Translation”

Like the waves of the sea, our essence is defined by perpetual motion. This work marks the end to only a part of my journey and hopefully, the beginning of yet another. This journey would not be possible without the help, encouragement, friendship, and guidance of so many people, to all of whom I wish to express my sincere thanks. I am especially grateful to all my teachers, the first of whom were my parents. This thesis is dedicated to them for teaching me the value of education and instilling in me the capacity of reasoning. It is also dedicated to them for their unconditional love, support, and sacrifice over all these years.

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List of Symbols

Roman Symbols

\(a_b\) Half of the length of a block
\(B, B_1, B_2\) Bending stiffness of a plate, a beam or a layer
\(B_{ep}\) Elasto-plastic bending stiffness
\(B_N\) Matrix of derivative of shape functions in spatial coordinates arranged in Voigt notation
\(b\) Body force
\(b_b\) Half of the width of a block
\(C, C_{ijkl}, C_{ab}\) Elasticity tensor, fourth-order elasticity tensor, and Voigt elasticity matrix
\(\hat{C}\) Elasticity tensor or matrix in local coordinates of anisotropy
\(c, c^*\) Cohesion parameter and reduced cohesion parameter in Mohr-Coulomb failure criterion
\(c_i\) Tensile strength of the material
\(D, D_{ijkl}, D_{ab}\) Compliance tensor, fourth-order compliance tensor, and Voigt compliance matrix
\(D^e_{ijkl}, D^{ep}_{ijkl}\) Elastic and elasto-plastic constitutive tensors
\(dV\) Infinitesimal volume
\(E, E_1, E_2\) Young’s modulus
\(e_i, e_2, e_3, e_i\) Base vector of coordinates
\(e_{ijk}\) Permutation tensor
\(F\) Deformation gradient
\(F^{int}\) Internal force of finite element formulation
\(F^s\) Factor of safety in SSR method
\(F(\sigma, \kappa)\) Yield function
\(F^s(\mu)\) Yield function of micromoments in Cosserat plasticity
\(G_N\) Matrix of derivative of shape functions in material coordinates arranged in Voigt notation
\(G\) Lamé constant, shear modulus
\(G_c\) Additional material constant for a Cosserat particulate material
\(H^p\) Plastic modulus
\(h\) Internal length, layer thickness, joint spacing, or diameter of a particle
\(I_1, I_2, I_3\) First, second and third invariants of the stress tensor
\(I^{(2)}, I\) Second-order unit tensor
\(I^{(4s)}\) Symmetric fourth-order unit tensor
\( J \) Jacobian of deformation
\( J_1, J_2, J_3 \) First, second and third invariants of deviatoric part of the stress tensor
\( J_{s1}, J_{s3} \) Second and third invariants of symmetric part of the stress tensor
\( \bar{K} \) Curvature measure in material coordinates
\( K, K_{NM} \) Stiffness Matrix
\( K^{\text{mat}}, K^{\text{geo}} \) Material and geometric stiffness matrices, respectively
\( K_{\text{uu}}, K_{\text{u} \theta}, K_{\theta \theta} \) Sub-matrices for displacement-displacement field and displacement-rotation field in Cosserat buckling stiffness matrix
\( K_{\theta \theta}, K_{\theta u} \) Sub-matrices for rotation-rotation field and rotation-displacement field in Cosserat buckling stiffness matrix
\( k_n \) Normal stiffness of the interface
\( k_s \) Shear stiffness of the interface
\( l \) Length of a beam or column
\( l_a, l_b \) Length and width of a plate
\( \bar{M} \) Couple moment in material coordinates
\( m, s, m_{\text{dil}} \) Constant parameters in Hoek-Brown failure criterion
\( m \) Body couple in Cosserat formulation
\( N \) Number of layers
\( n \) Normal vector to a surface
\( \bar{n}_x, \bar{n}_y, \bar{n}_z \) Distributed normal and shear loads on edges of a plate
\( p_z \) Distributed load acting normal to the surface a plate
\( Q \) Orthogonal second-order rotation matrix
\( Q(\sigma, \kappa) \) Potential function
\( R^C \) Cosserat rotation tensor
\( R_c \) Anisotropy ratio
\( R^R \) Pure rotation part of the deformation gradient
\( R_e \) Matrix arrangement of second-order rotation tensor of strain in Voigt notation
\( R_{\sigma} \) Matrix arrangement of second-order rotation tensor of stress in Voigt notation
\( S, S_{ij} \) Deviatoric part of stress
\( t_{\sigma} \) Stress traction vector
\( t_m \) Couple stress traction vector
\( U \) Right stretch tensor
\( u, u_i \) Displacement filed
\( V \) Left stretch tensor
\( v \) Eigenvector
\( w \) Deflection of a plate
\( W \) Energy function
\( x_1, x_2, x_3 \)  
Global coordinates

\( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \)  
Local coordinates of anisotropy

**Greek Symbols**

- \( \alpha \)  
  Index of the plastic surface in multi-surface plasticity

- \( \beta \)  
  Orientation of local coordinates of anisotropy with respect to the maximum principal stress

- \( \Gamma, \Gamma_{ij} \)  
  Cosserat strain measure in material coordinates

- \( \gamma_b \)  
  Voigt strain vector

- \( \gamma, \gamma_{ij} \)  
  Cosserat strain measure

- \( \delta_{ij} \)  
  Kronecker delta

- \( \varepsilon, \varepsilon_{ijkl}, \dot{\varepsilon}_{ijkl} \)  
  Strain, second-order strain tensor and incremental strain tensor

- \( \varepsilon^e, \varepsilon^p, \dot{\varepsilon}^p \)  
  Elastic and plastic strain, and increment of plastic strain

- \( \ddot{\varepsilon} \)  
  Strain in local coordinates of anisotropy

- \( \theta^L \)  
  Lodé angle

- \( \ddot{\theta} \)  
  Nodal rotation vector in finite element formulation

- \( \theta^R \)  
  Rotation vector

- \( \Theta, \Theta_i \)  
  Cosserat rotation angle, \( i^{th} \) component of Cosserat rotation vector

- \( \kappa_h \)  
  Hardening parameter in plasticity formulation

- \( \kappa, \kappa \)  
  Curvature tensor in global and local coordinates

- \( \kappa^e, \kappa^p \)  
  Elastic and plastic part of curvature measure

- \( \lambda, \lambda \)  
  Lamé constant in theory of elasticity

- \( \bar{\lambda} \)  
  Plastic multiplier in theory of plasticity

- \( \mu \)  
  Cosserat couple stress in current configuration

- \( \nu, \nu_1, \nu_2 \)  
  Poisson’s ratio, Poisson’s ratio in local coordinates of anisotropy

- \( \Sigma \)  
  Cosserat stress tensor in material coordinates in analogy with Biot stress of classical continuum

- \( \sigma, \sigma_{ij} \)  
  Cauchy stress tensor in classical continuum, and stress tensor in analogy with Cauchy stress in Cosserat continuum

- \( \sigma_{ij} \)  
  Incremental stress tensor

- \( \sigma_1, \sigma_2, \sigma_3 \)  
  First, second, and third principal stress

- \( \sigma_c \)  
  Uniaxial compressive strength

- \( \sigma_n \)  
  Normal stress

- \( \bar{\sigma} \)  
  Stress in local coordinates of anisotropy

- \( \tau_a \)  
  Voigt stress vector

- \( \tau \)  
  Shear stress
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$, $\phi^*$</td>
<td>Friction coefficient and reduced friction coefficient in Mohr-Coulomb failure criterion</td>
</tr>
<tr>
<td>$\phi_{dil}$</td>
<td>Dilation angle in Mohr-Coulomb failure criterion</td>
</tr>
<tr>
<td>$\phi_N$</td>
<td>Shape function for $N^{th}$ node of a finite element</td>
</tr>
</tbody>
</table>
Chapter 1
Introduction

1.1 Introduction and outline

Many geomaterials are often intersected by discontinuous surfaces such as: regular bedding planes, foliations, and joints, producing a layered or blocky structure. Jointed media are quite common both in nature and industry. As a result, topics related to this type of media are widely discussed in various fields of science and engineering. Analytical, experimental, and numerical research in this field has attracted considerable attention in different areas of application including geomechanics, composite science, aerospace, and biomechanics. From a numerical perspective, simulation of these classes of materials can be carried out using discrete or discontinuous models in which the interface surfaces are explicitly introduced into the input model.

It is clear that due to the discrete nature of the material, discontinuous models can provide a more accurate description of the physical behaviour of jointed materials. However, they are complex and necessitate care in modeling of the interface planes. For example, in modeling the interface, the Finite Element Method (FEM) which utilizes contact or interface elements can be applied. The literature shows that inappropriate selection of these elements will lead to erroneous results. Also, when the number of interfaces becomes excessively large, in many applications, explicit definition of interfaces becomes impractical. In order to overcome difficulties associated with discontinuous models, the continuum-based approach has been adopted with considerable success. In this case, the interface is homogenized and its features are carefully considered in a continuum model. The technique is associated with three major advantages: First, considerable reduction in the size of the problem and the effort in modeling, which leads to a more computationally efficient solution. Second, avoidance of an inconsistent
formulation associated with the use of contact or interface elements. Third, elimination of explicit definition of the interface planes.

In the classical equivalent continuum approach, the original discontinuous material is replaced by a continuous, anisotropic material. The properties of this idealized, continuous material are then based specifically on the orientation and the properties of the interface surfaces and the intact material. The homogenized anisotropic material exhibits unequal physical properties along different axes and in contrast to an isotropic material, its elastic and elasto-plastic behaviours depend on the orientation of applied load. Anisotropy can be due to planes of weakness associated with the origin of the material, e.g., the crystal structure of the material, or it can be due to the development of bedding planes, foliation, schistocity, joints, fractures, shear planes, or faults. Figure 1-1 shows examples of sedimentary and metamorphic rocks with a layered structure. Due to the existence of joints or planes of weakness these rocks exhibit transversely isotropic characteristics.

![Figure 1-1 Layered rock structure](image)

(a) (b)

Figure 1-1 Layered rock structure (a) sedimentary rock, (b) metamorphic rock (Kolymbas, 2007).

The intrinsic assumption of the classical continuum theory is that the material is homogeneous and the dimensions of the constituent elements of a material such as particles, voids, and gaps are small compared to the dimensions of the characteristic volume. However, the assumption of homogeneity is not necessarily valid in materials with microstructure such as a layered rock. In other words, the classical continuum approach disregards the condition of discontinuity in deformation which exists at the
interface planes. As a result, the internal length scale, which is a parameter defined by the spacing of the discontinuous surfaces, is disregarded in its formulation. Figure 1-2 shows a schematic representation of a layered rock slope and a manufactured layered sample which is subjected to a similar loading condition. It is clear that in these examples, in addition to the anisotropic characteristics, the bending of individual layers of material plays a significant role in the overall response of the structure. The behaviour of these examples is similar to the bending of a deck of cards, where bending stiffness of the layers sustains the applied load. There are numerous cases both in natural and manufactured materials, where the response of a layered material is dominated by the bending stiffness of the layers. In other words, the dimensions of the constituent components of these materials are not negligible compared to the dimensions of the characteristic volume. Therefore, the overall response depends on the dimensions of the constituent components. This internal dimension is often referred to as the internal length scale or the internal characteristic length. For a particulate medium, the internal length scale is the dimension of the particles. For layered and blocky materials, the internal length scale is the layer thickness and dimensions of the blocks, respectively.

In the equivalent continuum concept based on classical continuum theory, effects of weakness planes on the elastic and elasto-plastic properties of the material are introduced through the constitutive equations. However, the influence of layer thickness or block size is disregarded. Thus, it can be a large source of error in the simulation. This limits the validity of the classical continuum approach to the cases where lattice-structure anisotropy is involved or where joint spacing is small compared to the dimensions of the engineering structures. In these cases, the conditions of stress uniformity and continuity of the deformation field are valid on a macro scale. Otherwise, discrete methods should be applied, where the discontinuous nature of deformation and subsequently, the internal length scale of the problem, need to be taken into consideration.

Concerns over the computational aspects of discrete methods and the validity limits for continuous models were the key factors that shifted the goal of this research towards the development of an enhanced continuum-based description of jointed materials. It should be noted that a continuum description of the behaviour of jointed materials is not a new
idea. In fact, many strength criteria such as the well-known Hoek-Brown failure criterion are based on the same concept. Hoek-Brown predicts the strength of an isotropic jointed rock based on the characteristics and population of interface surfaces within the rock. Based on these ideas, the initial intention of this thesis was to develop an anisotropic Hoek-Brown failure criterion for jointed rock masses. Further research into the strength characteristics of anisotropic and jointed rock masses revealed that they differ in certain aspects from materials with microstructural anisotropy. One example is the reduced load bearing capacity of the hanging wall of a stope due to the buckling of individual layers of the host rock. Buckling is not a strength related property and should be considered as a stability problem. In other words, the buckling mechanism should not be marked by constant material properties which define the yield criterion and the stress space in which irreversible plastic deformation of a material occurs. The buckling load of a stratified structure is more a function of thickness and length of the layers, elastic properties of the layers, the elastic and elasto-plastic properties of the interface surfaces, and the geometry and the boundary condition of the structure, and consequently, is affected by the internal length scale of the material.

![Figure 1-2](image)

*Figure 1-2 (a) A schematic representation of the flexural toppling failure of layered rock slope, (b) a manufactured model from ilmenite-gypsum mixtures (Adhikary & Dyskin, 2007).*

It was decided that instead of focusing on developing a strength criterion based on the mapping of experimental data - which could ultimately be applied to a narrow class of materials - it would be more beneficial to provide a continuum-based numerical tool in which the natural mechanism and the effect of the internal length scale evolve as a result
of: i) an enhanced mathematical description of the mechanics of the medium, and ii) an
algorithmic implementation of the physical phenomena involved in the overall behaviour.

This thesis is structured into 9 chapters. In addition to the main chapters, there are a
number of appendices that are referred to in the text. In the author’s judgment, these
supporting materials are crucial for the reader. However, for the sake of briefness and
flow of the text, it was decided to include them as appendices.

Following a self-contained introduction, Chapter 2 focuses on the numerical techniques
for the simulation of jointed materials and their fundamental governing equations,
assumptions, and computational aspects.

Chapter 3 briefly explains the principles of classical continuum theory and the elastic and
inelastic behaviour of anisotropic materials. Also, the concept of an equivalent continuum
and the applied homogenization technique are discussed in this chapter.

In Chapter 4, the formulation of an equivalent continuum is modified based on Cosserat
type which is an enhanced mathematical description of materials with microstructure.
Cosserat theory is one of the models that describe the behaviour of a generalized
micropolar medium. In the micropolar theory of elasticity, the stress and strain tensors are
not symmetric, and the difference between the shear components of the stress is
equilibrated by micromoments.

This dissertation is focused on a 3D finite element Cosserat formulation of layered
geomaterials. In order to incorporate the bending stiffness of the layers into the Cosserat
continuum formulation, the constitutive equations that describe the response of the
layered materials are modified based on the mechanical model of a stacked plate shown
in Figure 1-3. The attractive aspect of the application of Cosserat theory to the analysis of
stratified media is the fact that the link between the kinematic and kinetic variables of
Cosserat theory and the physical behaviour of the bending of a layered beam or plate
structure can be made in a mechanically rigorous fashion. A direct consequence of the
application of Cosserat theory is that the internal length scale of the material is introduced
into the constitutive equations of the system. In the case of layered continua, the couple
stress measures or micromoments are linked to the curvature of a beam or a plate through the bending stiffness of the layers which is introduced into the elasticity matrix. Subsequently, the theory provides a continuum description of discrete materials, such as jointed materials, whose behaviour is governed by an internal length scale (e.g. joint spacing). It should be noted that the application of Cosserat theory or in general the equivalent continuum approach is only valid when a periodic or sequential pattern of discontinuity exists within the material. This chapter demonstrates that the developed 3D FEM Cosserat continuum model can simulate the displacement behaviour of layered materials with periodic microstructure as accurately as discontinuous models.

Chapter 5 focuses on the stability and buckling analysis of layered structures. In this chapter the large deformation theory of a Cosserat continuum is briefly discussed. Also, mathematical assumptions that are compatible with the nature of buckling behaviour are proposed. Subsequently, a full 3D buckling stiffness matrix is developed. It is shown through a number of benchmark problems that by introducing the internal length scale to the problem, the FEM formulation is potentially capable of capturing complicated failure mechanisms such as buckling and foliation of individual layers of material.

Chapter 6 presents the application and verification of the FEM Cosserat method in the strength prediction of anisotropic jointed rock. This chapter also includes a brief review of available experimental data. Examples demonstrate the capabilities and shortcomings of the FEM Cosserat model through the reproduction of experimental strength graphs suggested for jointed rocks.

Chapters 7 and 8 focus on application areas; these chapters attempt to address a number of challenging rock mechanics problems using the proposed FEM Cosserat model. Chapter 7 describes the application of the proposed FEM Cosserat method to slope stability analysis based on the Shear Strength Reduction (SSR) method. In these examples, the SSR method is applied to the FEM Cosserat model and a number of rock slope problems which involve different mechanisms of failure are simulated. The deformation mode, failure mechanism, and factor of safety predicted by the FEM Cosserat model are verified against two discontinuous models. High levels of consistency
indicate that the proposed FEM Cosserat model combined with a multi-surface plasticity algorithm is an efficient and robust numerical approach for analysing the stability of jointed rock slopes.

Chapter 8 investigates the effect of anisotropy and layer thickness on the response of excavations in a layered rock mass. The effects of in-plane anisotropy and out-of-plane anisotropy are shown using two examples. Also, the application and performance of the method is compared with three different techniques, using Phase\textsuperscript{2}, 3DEC, and FLAC3D.

Finally, Chapter 9 includes a general conclusion and suggestions for further improvement of the proposed FEM Cosserat model.

![Figure 1-3 A stacked beam in (a) undeformed position, (b) deformed position (Kolymbas, 2007), b represents the jump in the displacement at the interface between the layers.](image)

**1.2 Major contributions of the thesis**

The intention of this dissertation is to provide a robust and efficient numerical model for 3D analysis of elasto-plastic behaviour of layered geomaterials. The equivalent continuum is not a new concept, nor is the application of Cosserat theory. This research serves as an extension to an existing two-dimensional (2D) Cosserat model for layered and blocky materials.

The first contribution of this thesis was the development of a full 3D FEM formulation based on Cosserat continuum theory and a 3D constitutive matrix for a Cosserat layered material. The major challenge of a 3D formulation of a Cosserat continuum lies in the nature of the rotation measures. In this research, 3D stress, strain, curvature and rotation measures, as well as their transformation matrices are proposed, and the FEM
formulation is modified based on these measures. Also, a new constitutive matrix for a Cosserat material with a plate-like microstructure is proposed.

The developed Cosserat model has two major advantages over the previous Cosserat continuum models. First, in contrast to most of the previous work, in the Cosserat continuum formulation, the possibility of yielding of the joint planes and the intact material was considered, and the plasticity formation was modified based on this notion. Another major difference between the FEM formulation of the equivalent continuum approach developed in this research and previous work (based on the same concept) is the numerical simulation of multiple failure mechanisms. Most of the equivalent continuum models based on classical and Cosserat theories are restricted to one plastic surface expressing the elasto-plastic deformation of the interface or joints. To the author’s knowledge, none of the proposed models have addressed the issue of multiple plasticity surfaces involved in this approach in a mathematically rigorous fashion. As a result, plastic correction may be performed with respect to the surfaces which are not in fact active, and this in turn may lead to an incorrect estimation of yield and post-yield behaviour of layered geomaterials. Clearly, this error becomes more significant when large load steps are used.

Another important contribution is the 3D formulation for bifurcation and buckling analysis of layered materials. Buckling and stability analysis of layered media is an important issue which has attracted considerable attention, particularly, in the composite industry. Buckling is also a cause of failure in many geo-structures such as tunnels and hanging walls of stopes. However, few researchers have addressed this problem in geomechanics. This is due to the dimensions of engineering structures and the excessive number of layers typical of problems in this field. The Cosserat model is, however, an exceptionally good candidate for such analysis. In this research, the FEM Cosserat formulation is modified for buckling analysis and its efficiency is verified against a number of benchmark solutions (Riahi & Curran, 2008).

In addition to the theoretical aspects, the proposed FEM Cosserat model has been applied to the analysis of a number of analytical and application problems concerning layered
media. First, the effect of anisotropy and internal length scale on the deformational response of a layered material was investigated (Riahi & Curran, 2007). Second, the Shear Strength Reduction (SSR) method was used with the FEM Cosserat model to study the stability of jointed rock slopes. The results were compared to the predictions by discrete techniques (Riahi & Curran, 2008). Finally the effect of anisotropy and layering on the response of excavations was analysed.

It can be concluded that through an enhanced mathematical description of the mechanics and elasto-plastic response of materials with a layered structure and through application of a robust and stable algorithmic approach, the FEM Cosserat model is capable of predicting the natural response of a layered geomaterial. Using the FEM Cosserat approach, it is possible to predict the variation of strength with orientation for different classes of anisotropic materials, simulate buckling and toppling failure, and determine the stress distribution, deformation mode, and failure mechanisms of layered geomaterials.

In addition to the aforementioned advantages, it is believed that there are two main reasons why the application of Cosserat theory in the FEM simulation should be viewed differently from the past. Attractive aspects of the FEM Cosserat model include its generality and flexibility for extension. Cosserat theory is essentially a generalized form of continuum theory. Therefore, by applying the same principles with minimal effort, the formulation can be extended to dynamic, thermodynamic, and coupled solid-fluid analysis of layered geomaterials. Performing such analysis by discrete techniques involves extensive modification and reformulation of the method or specially devised interface elements. Also, it is noteworthy to mention that Cosserat theory provides a robust and general platform for the analysis of solid mechanics problems. Ongoing research in Cosserat theory shows that the formulations of finite and discrete element methods reduce to the same fundamental governing questions, and that the differences between these formulations lie merely in their numerical implementation.
1.3 Scope and limitations of the finite element Cosserat continuum model

It will be explained in detail in the following chapters that Cosserat theory provides a mathematically enhanced description of materials with microstructures by considering higher order gradients of displacements and by introducing the internal length scale of the problem into the constitutive equations.

It should be reminded, however, that the constitutive equations are derived based on the assumption of the existence of a periodic microstructure within the material. As a result, application of the method is limited to the problems where there is a sequential pattern in the distribution of interfaces. For example, the model fails to produce reasonable results for the stress distribution around an excavation which is intersected by a single joint plane.

Also, the FEM Cosserat approach falls within the FEM continuum-based approaches, and does not allow for mechanisms such as total detachment and impact at the joint surfaces. Numerical simulation of these mechanisms in FEM requires the application of contact technology and explicit formulation of the subsequent arising issues.
Chapter 2
Numerical Models for Jointed Materials

2.1 Introduction to numerical techniques

The objective of this work is to develop a continuum-based numerical technique for the constitutive modeling of layered geomaterials. In the next chapter, anisotropy and its types will be defined, along with the subsequent effects on the elastic and the elastoplastic behaviour of materials. The discussion in this chapter mainly concerns the numerical simulation of materials in which anisotropy is induced as a result of pre-existing discontinuity surfaces with a preferred orientation. Many composites and rocks fall within this category, and considerable resources have been allocated to the constitutive modeling of this class of materials. In geomechanics, discontinuity surfaces or interfaces are referred to as ‘joints’.

In computational geomechanics, two main approaches have been adopted to simulate jointed materials: i) techniques which explicitly model the discontinuous nature of the material, e.g., the Discontinuous Displacement Analysis (DDA), the Discrete Element Method (DEM), as well as the Finite Element Method (FEM), and the Finite Difference Method (FDM), which utilize an interface element, and ii) the FEM or FDM equivalent continuum model.

Discontinuous methods are founded on the assumption of the discontinuity of deformation. In these techniques, joints are explicitly simulated and the input model consists of an assembly of individual blocks or particles. Due to the fundamental assumption that deformations have a discrete nature, discontinuous models provide a more accurate description of the physical behaviour of jointed materials. However, application of explicit techniques may lead to unreasonably high stress gradients, which is not physically meaningful. Also, in many applications, as the number of interfaces,
blocks, or particles increases, the definition of the problem geometry becomes more difficult and from a numerical point of view, the numerical model becomes computationally more demanding. Alternatively, continuum-based approaches, where joints are smeared to produce a continuous material and the input model is a continuous body with modified properties, are more convenient. Figure 2-1 classifies the available numerical techniques for the simulation of jointed or discrete materials. Basic assumptions and numerical characteristics of each method are briefly discussed in the following sections. For a more detailed review refer to (Jing & Hudson, 2002; Jing 2003)

![Diagram of Numerical Modeling of Discrete Materials](image)

**Figure 2-1** Numerical approaches used in the simulation of jointed materials.

### 2.2 Discrete or discontinuous analysis

#### 2.2.1 The discrete element method (DEM)

The Discrete Element Method (DEM) is based on equilibrium of a number of rigid or deformable discontinuous bodies. The fundamental assumption of the method is that the material consists of separate, discrete particles or blocks. Equilibrium of force and moment are satisfied locally and a finite-difference time integration scheme is applied to velocities in order to compute the displacement of blocks. The most widely used discrete
technique is the Distinct Element Method originally formulated by Cundall (1971; 1979). DEM has been widely applied to impact rock mechanics problems. The theoretical basis of the method was detailed by Williams et al. (1985) who showed that DEM could be viewed as a generalized finite element method.

An advantage of DEM is its efficiency in the absence of appropriate constitutive models. DEM is computationally intensive, and, as a result, its application is limited to small scale problems consisting of a limited number of particles or blocks. Advances in parallel processing capabilities now allow for a scaling up of the number of particles and a decrease in the computational time. For further reading on the subject one can refer to Pande et al. (1990) and the proceedings on the Third Conference on Discrete Element Methods (Cook & Jensen, 2002).

In recent years, hybrid methods such as the Finite Difference-Discrete Element Method (FDM/DEM) and the Finite Element-Discrete Element Method (FEM/DEM) (Munjiza, 2004) have grown in popularity. The combined methods consider a number of discontinuous bodies which are internally meshed as a finite element or finite difference domain. The contact forces that result from interaction of the individual blocks are taken into account in the formulation.

### 2.2.2 The discontinuous deformation analysis (DDA)

Discontinuous Deformation Analysis (DDA), which is sometimes referred to as the implicit DEM, is a work-energy method based on the principle of minimum total energy, originated by Shi (1988). DDA considers a number of independent blocks, separated by joints. The original DDA formulation applies a first-order polynomial displacement function. Consequently, the stresses and strains within a block are constant. In order to allow for stress variation within a block, hybrid methods have been developed wherein a finite element mesh is applied to the domain of the blocks in circumstances where displacement inside the block is highly nonlinear and cannot be ignored (Shyu, 1993; Chang, 1994; Grayeli & Mortazavi, 2006).
DDA is widely applied to slope stability analysis and its implicit time marching allows for quasi-static simulation of problems such as creep. However, like all discontinuous analysis, it is computationally intensive.

### 2.2.3 The finite element or finite difference explicit interface approach

FEM is the most powerful and widely used numerical method for solving solid mechanics problems. FEM is based on discretization of the weak form of the governing equations of the system. In solid mechanics problems, FEM is often based on an energy principle, e.g., the virtual work or minimum total potential energy principle (Bathe, 1996; Zienkiewicz, 2005).

The principles of the Finite Difference Method (FDM) are similar to those of FEM. However, instead of discretizing the domain into a finite element mesh, the boundary value problem is solved through a finite difference scheme, and the derivatives that occur in the governing partial differential equations are replaced by finite difference approximations (Griffiths & Smith, 2006). There are two major disadvantages associated with FDM. First, using FDM, solution of elliptic, parabolic, and hyperbolic systems requires different numerical techniques. The characteristics of the system of equations and the appropriate numerical techniques in each case are explained by Griffiths and Smith (2006). Also, spatial discretization of the boundary value problem into a structured grid in the standard FDM leads to serious problems in the definition of irregular and curved boundaries. However, significant progress has been made so that irregular meshes can be used (Perrone & Kao; 1975, Brighi et al., 1998). The most significant improvement is the development of finite volume or control volume methods (Wheel, 1996; Fryer et al. 1991).

Two and three-dimensional isoparametric conventional elements, beam, shell, and plate elements, which are in general referred to as structural, solid, or continuum elements are based on continuum theory, which considers a continuous and homogeneous body of material. As a result, discontinuous surfaces need to be explicitly simulated using specially formulated elements which simulate cracks, contacts, or other interfaces such as
joints. The application of structural elements together with interface elements is widely popular in geomechanics and in the composite materials industry.

Joint elements are devised to take into account the sliding and separation that may occur along the interface between adjacent blocks. In general, this occurs if the shear resistance at the interface is significantly lower than the limiting shear resistance of the block material. Different formulations for interface elements have been proposed. One approach, using special joint elements and adding their stiffness to the global stiffness of the structure, was first used in rock mechanics by Goodman et al. (1968). In this work, the authors represent the joint element as a simple one-dimensional element with eight degrees of freedom, offering resistance to compressive and shear forces acting normally and parallel to its surface. For a brief literature review on the state-of-the-art in the FEM formulation of discontinuity surfaces one can refer to Tzamtzis (2003)

The original DEM and the DDA both assume that individual bodies are rigid or first-order elastic deformable, while FEM explicit interface modeling allows for a complicated elasto-viscoplastic mechanism of failure within each block. The main disadvantage of FEM explicit interface models is that the size and the orientation of the mesh are dictated by the joint spacing and orientation. Also, as the number of joints increases, the definition of the input models can become impractical particularly in 3D analyses, if it is not performed in an automated fashion.

2.3 Continuum-based analysis

2.3.1 The finite element or finite difference equivalent continuum approach

A continuum is a continuous, deformable body in which all properties are describable through continuous functions. The behaviour of the continuum is often defined through a set of partial differential equations governing the response of the body. In continuum-based numerical techniques, a computational scheme is devised to solve the governing equations of the system. The difference between the FEM and the FDM lies mainly in the numerical scheme utilized to solve the weak form of the differential equations.
In an equivalent continuum approach, through the application of a homogenization technique, the original discontinuous material is replaced by a hypothetical continuous material. The homogenization and numerical technique utilized by the equivalent continuum method is discussed in detail in the next chapter. However, it should be noted that both the classical FEM and FDM equivalent continuum approaches are applicable to situations where joint spacing is small compared to the physical dimensions of the problem. This is due to the intrinsic assumptions of continuum theory, which disregards the internal length scale of the problem. Consequently, the classical equivalent continuum method can introduce large errors into the solution if applied to problems in which the internal length scale is significant.

The equivalent continuum approach was introduced into the FEM analysis by Zienkiewicz and Pande (1977). In their approach, every gauss point of the regular continuum element has a set of modified material properties. An attractive aspect of this method is that joints are not explicitly defined in the model, and the FEM mesh is totally independent of the spacing and orientation of the joints. From a numerical aspect, it is possibly the most computationally efficient scheme. Furthermore, complicated FEM analyses, involving multi-phase media and thermo-dynamics can easily be simulated using the equivalent continuum approach.

2.4 Conclusion

This chapter presents a brief review of the basic assumptions and computational aspects of available numerical techniques for the simulation of discrete materials or materials with microstructure. It is clear that while discrete methods provide a more accurate description of the discontinuous nature of deformation for these classes of materials, they are relatively unpopular. Discrete models are disadvantageous due to several problems including difficulties in defining the input model, high processing time, application to a narrow class of problems, over-simplified assumptions concerning the deformation of individual bodies of material, and the inability to perform more advanced analyses. The FEM, on the other hand, has proven to be the most powerful and efficient tool in problems concerning solid mechanics. Taking this into consideration, it is therefore
advantageous to devise a finite element continuum-based method that provides a more accurate description of the behaviour of materials such as layered geomaterials which exhibit an internal length scale effect.
Chapter 3
Continuum Theory of Anisotropic Materials

3.1 Introduction to continuum theory

Continuum theory is the mathematical description of the mechanics of deformable continuous solid bodies. Continuum theory is based on homogeneity and continuity assumptions, and subsequently any microstructure of the material is disregarded. In plain terms, it assumes that there are no discontinuities or no gaps within the material. As a result, all mathematical functions that describe the behaviour of the body and their derivatives are continuous functions. Theories of elasticity and plasticity are founded in continuum theory and provide a mathematical, quantitative description that matches well with physical observations. These theories, along with specialized theories of beams, plates, and shells are all based on continuum theory, and allow for the analysis of most solid mechanics engineering problems that satisfy the basic assumptions of the theory. For a comprehensive description of these theories, one can refer to Malvern (1969) and Baser and Weichert (2000).

This chapter focuses on the elastic and elasto-plastic behaviour of layered geomaterials. Due to the existence of interfaces with preferred orientations, layered geomaterials exhibit anisotropic behaviour. The mathematical definition of anisotropy within the framework of the theory of elasticity is discussed in the next section. Section 3.3 is an introduction to the classical theory of plasticity and the anisotropy of strength criteria. The equivalent continuum approach and the homogenization technique applied to the method are discussed in Section 3.4, and subsequently, the elasticity tensor and strength properties of the equivalent continuum are presented. Numerical aspects and an integration scheme for the proposed approach are explained in Section 3.4.4. Finally, the
advantages, disadvantages, and the limits of validity of the proposed method are briefly discussed.

3.2 Elastic behaviour of anisotropic materials

3.2.1 Hooke’s law and material symmetry

A constitutive law is a mathematical description of the relation between stress and strain. Regarding linear elasticity, Hooke’s law usually defines the stress-strain relation in the following form:

\[ \sigma = C \varepsilon \quad \text{or} \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]  \hspace{1cm} (3-1)

where stress, \( \sigma \), and strain, \( \varepsilon \), are both second-rank tensors, and the matrix of elastic properties, \( C \), is a fourth-rank tensor. In its most general form, in three-dimensional (3D) space, the elasticity matrix has 81 components. If stress and strain are symmetric, this results in minor symmetry of \( C \), producing the following:

\[ C_{ijkl} = C_{jikl} = C_{jlk} = C_{jlk} \]  \hspace{1cm} (3-2)

As a result, the number of independent components of \( C \) reduces to 36 independent components. Hyperelastic materials admit a strain energy density function, \( W = W(\varepsilon) \), in the following form:

\[ \sigma = \frac{\partial W}{\partial \varepsilon} \quad \text{or} \quad \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \]  \hspace{1cm} (3-3)

Due to the linear nature of the stress-strain relation, the strain energy density function can be written in the following quadratic form:

\[ W = \frac{1}{2} \left( C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \right) \]  \hspace{1cm} (3-4)
From Equation 3-4, it can be concluded that the elasticity tensor is symmetric with respect to the first and second sets of indices, which is referred to as major symmetry. Major symmetry is expressed in indicial notation by:

$$C_{ijkl} = C_{klij}$$  \hspace{1cm} (3-5)

Major symmetry together with minor symmetry results in 21 independent components in the elasticity tensor (Nemat-Nasser, 1998).

### 3.2.2 Linear elasticity in Voigt notation

Stress, strain, and elasticity tensors are usually expressed in Voigt notation (Voigt, 1910), where symmetric second-order stress and strain tensors reduce to 6-component vectors and the fourth-rank elastic tensor reduces to a six by six second-rank matrix. The first two indices $ij$ are represented by a single index, $a$, and the second two indices $kl$ are denoted by a single index, $b$. The matrix of elastic properties, $C_{ab}$, is defined with respect to components of the fourth-rank tensor, $C_{ijkl}$, in the following form:

$$C_{ab} = \begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\
C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212}
\end{pmatrix}$$  \hspace{1cm} (3-6)

The stress strain relation can be expressed as:

$$\tau_a = C_{ab} \gamma_b$$  \hspace{1cm} (3-7)

where repeated indices are summed. In the above formulation, $\tau$ and $\gamma$ are vectors of dimension six and are defined as:

$$\tau_1 = \sigma_{11}, \ \tau_2 = \sigma_{22}, \ \tau_3 = \sigma_{33}$$  \hspace{1cm} (3-8)
\[ \tau_4 = \sigma_{23} = \sigma_{32}, \ \tau_5 = \sigma_{31} = \sigma_{13}, \ \text{and} \ \tau_6 = \sigma_{12} = \sigma_{21} \]

and

\[ \gamma_1 = \varepsilon_{11}, \ \gamma_2 = \varepsilon_{22}, \ \gamma_3 = \varepsilon_{33} \]
\[ \gamma_4 = 2\varepsilon_{23}, \ \gamma_5 = 2\varepsilon_{31}, \ \text{and} \ \gamma_6 = 2\varepsilon_{12} = 2\varepsilon_{21} \] (3-9)

The strain energy density function can be expressed as:

\[ W = \frac{1}{2} \gamma^a \tau_{ab} \gamma^b = \frac{1}{2} C_{ab} \gamma_a \gamma_b \] (3-10)

where

\[ \tau = \frac{\partial W}{\partial \gamma} \quad \text{or} \quad \tau^a = \frac{\partial W}{\partial \gamma_a} \] (3-11)

### 3.2.3 Compliance and stiffness

The compliance matrix and the stiffness matrix, also referred to as the elasticity matrix, are representative of how resistant a material is towards deformation. Compliance relates the increment of strain to the increment of stress through the following relation:

\[ \varepsilon_{ij} = D_{ijkl} \sigma_{kl} \quad \text{or} \quad \tau^a = C_{ab} \gamma^b \] (3-12)

It is clear that the fourth-rank compliance matrix, \( D_{ijkl} \), is the inverse of the elasticity matrix, \( C_{ijkl} \):

\[ D_{ijkl} = (C_{ijkl})^{-1} \quad \text{and} \quad C_{ijkl} = (D_{ijkl})^{-1} \] (3-13)

It can be proven that the six by six matrix, \( C_{ab} \), is positive definite and admits a unique inverse, \( D_{ab} \), as:
However, it should be mentioned that the relationship between components of the second-order elastic modulus, \( C_{ab} \), and the fourth-rank elastic modulus, \( C_{ijkl} \), is different from the relationship between components of \( D_{ab} \) and \( D_{ijkl} \).

### 3.2.4 Elastic symmetry

#### Isotropy

In classical continuum theory, major and minor symmetries result in 21 independent components for the elasticity tensor. Furthermore, depending on the number of symmetry planes that exist in the material, different types of elastic symmetries are achievable.

Isotropy is the most general type of symmetry in which any arbitrary plane is a plane of symmetry. The number of independent elastic constants for isotropic materials reduces to two, and the elastic tensor is defined by:

\[
C = \lambda \mathbf{I}^{(2)} \otimes \mathbf{I}^{(2)} + 2G \mathbf{I}^{(4s)} \quad \text{or} \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)
\]  

where \( \mathbf{I}^{(2)} \) is the second-order unit tensor, \( \mathbf{I}^{(4s)} \) is the symmetric fourth-order unit tensor, and \( \lambda \) and \( G \) are called the Lamé constants. For isotropic materials, the components of the elasticity matrix can be defined with respect to any two independent constants. The elasticity matrix of isotropic materials is expressed in the following form:

\[
C_{ab} = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & (C_{11} - C_{22})/2 & 0 & 0 \\
0 & 0 & 0 & 0 & (C_{11} - C_{22})/2 & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{22})/2 \\
\end{bmatrix}
\]  

where
\[ C_{11} = \frac{E}{1+\nu} \left(1 - \frac{\nu}{1+\nu}\right), \quad C_{12} = \frac{E}{1+\nu} \frac{\nu}{1+2\nu} \] (3-17)

and \( E \) is Young’s modulus and \( \nu \) is the Poisson’s ratio of the material.

**Transverse isotropy**

The next level of symmetry is exhibited by a transversely isotropic material, in which there exists one plane of isotropy. In this case, the number of independent elastic components reduces to five. If the plane of isotropy coincides with the \( x_1 - x_2 \) plane, the elastic compliance matrix is expressed by:

\[
D_{ab} = \begin{bmatrix}
D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\
D_{12} & D_{11} & D_{13} & 0 & 0 & 0 \\
D_{13} & D_{13} & D_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & 2(D_{11} - D_{12})
\end{bmatrix}
\] (3-18)

The components of the compliance matrix are then defined in the following form:

\[
D_{11} = \frac{1}{E}, \quad D_{33} = \frac{1}{E_3},
\]

\[
D_{12} = D_{21} = \frac{-\nu}{E}, \quad D_{13} = D_{31} = \frac{-\nu_3}{E_3}, \quad \text{and}
\]

\[
D_{44} = \frac{1}{G_3}
\]

where \( E_1 = E_2 = E \) and \( \nu_1 = \nu_2 = \nu \) are the Young’s modulus and Poisson’s ratio in the plane of isotropy, respectively, and \( G_{12} = G_{21} = \frac{E}{2(1+\nu)} \) is the shear modulus associated with this plane. \( E_3 \) and \( \nu_{13} = \nu_{31} = \nu_3 \) are the Young’s modulus and Poisson’s ratio in the \( x_3 \) direction, respectively, and \( G_{13} = G_{31} = G_3 \) is the corresponding shear modulus.
The elasticity matrix, which is the inverse of the compliance matrix, can be expressed by:

\[
C_{ab} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2
\end{bmatrix}
\]  

(3-20)

where

\[
C_{11} = D \left\{ \frac{1}{EE_3} - \frac{v_3^2}{E_3^2} \right\}, \quad C_{12} = D \left\{ \frac{v}{EE_3} + \frac{v_3^2}{E_3^2} \right\}, \quad C_{13} = D \left( \frac{1 + v}{EE_3} v_3 \right),
\]

\[
C_{33} = D \frac{1 - v^2}{E^2}, \quad C_{44} = \mu_3 = G_3, \quad \text{and} \quad D = \frac{E_4^2 E_3^2}{(1 + v) ((1 - v) E_3 - 2 v_3^2 E)}.
\]

(3-21)

**Orthotropy**

A material is called orthotropic if two mutually orthogonal planes of symmetry exist within the material. In this case, the material is also symmetric with respect to a plane perpendicular to these two planes of symmetry. For orthotropic materials, the number of independent elastic parameters reduces to nine. For details on the structure of the elasticity tensor, one can refer to Nemat-Nasser (1998).

**Monoclinic**

A material is called monoclinic if only one plane of symmetry exists within the material. In this case, the total number of independent components reduces to thirteen.

The type of elastic symmetry is not restricted to the aforementioned categories; many other elastic symmetric conditions can be considered. For a more general description, one can refer to Figure 3-2 and Nemat-Nasser (1998).
3.2.5 Transformation of stress, strain and elastic properties

The elasticity tensor is usually defined in a coordinate system coincident with the material symmetry directions. In most types of analysis, the stress, strain, and elasticity tensors should be defined with respect to a fixed global coordinate (Figure 3-1).

\[
\sigma = Q \hat{\sigma} Q^T \text{ or } \sigma_{ij} = Q_{im} Q_{jn} \hat{\sigma}_{mn} \quad (3-22)
\]

and

\[
\epsilon = Q \hat{\epsilon} Q^T \text{ or } \epsilon_{ij} = Q_{im} Q_{jn} \hat{\epsilon}_{mn} \quad (3-23)
\]

where \( Q \) is the tensor of cosine directions expressed in Appendix 1.

The fourth-rank elastic modulus then follows the following transformation rule:

\[
C_{ijkl} = Q_{im} Q_{jn} Q_{sk} Q_{rl} \hat{C}_{mnsr} \quad (3-24)
\]

The transformation of \( C_{ab} \) and \( D_{ab} \), however, are more complicated and can be expressed as follows:
\[ C = R_\sigma \tilde{C} R_\sigma^T \]  
(3-25)

and

\[ D = R_\varepsilon \tilde{D} R_\varepsilon^T \]  
(3-26)

where \( R_\sigma \) and \( R_\varepsilon \) are the transformation matrices defined by:

\[ \tau = R_\sigma \tilde{\tau} \quad \text{and} \quad \gamma = R_\varepsilon \tilde{\gamma} \]  
(3-27)

For details on the structure and components of \( R_\sigma \) and \( R_\varepsilon \) refer to Appendix 1.
Figure 3-2 The relationship between the traditional, distinct, elastic symmetries (Cowin & Mehrabadi, 1987).

**Triclinic Symmetry**  
(no plane of symmetry)

**Monoclinic Symmetry**  
(one plane of symmetry)

- **Rhombic or orthotropic symmetry**  
(Three mutually orthotropic planes of symmetry)
- **Tetragonal Symmetry**  
(Five plane of symmetry, the normals to four of the planes all lie in the fifth plane of symmetry and are at an angle of 45° with respect to one another)
- **Hexagonal Symmetry**  
(Three planes of symmetry whose normals all lie in one plane and are at angles of 60° with respect to one another)
- **Tetragonal Symmetry**  
(6)
- **Hexagonal Symmetry**  
(6)

- **Hexagonal Symmetry or Transverse Isotropy**  
(One plane of isotropy)

- **Cubic Symmetry**  
(Nine planes of symmetry all intersecting at 90° or 45°)

**Isotropy Symmetry**  
(Every plane is a plane of symmetry and a plane of isotropy)

**Zero Level Symmetry:**  
Triclinic symmetry is the total absence of elastic symmetry.

**First Level Symmetry:**  
Monoclinic symmetry is developed by assuming one plane of symmetry.

**Second Level Symmetry:**  
These symmetries are developed by assuming two planes of symmetry, in each case it was found that more than two must exist.

These two symmetries are special cases of similar ones above, and involve no additional planes.

**Third Level Symmetry:**  
Symmetries at this level have nine or more planes of symmetry.

**Forth Level Symmetry:**  
Isotropy is total symmetry.
3.3 Inelastic behaviour of anisotropic materials

3.3.1 General plasticity in stress space

Plasticity refers to the irreversible straining of material. The objective of the mathematical theory of plasticity is to define a constitutive model that describes the relation between stress and strain once a certain level of stress is met. In order to formulate plastic behaviour, two basic functions, i.e., “yield surface” and “plastic potential” are required. In this section, the general framework of classical plasticity in strain-space\(^1\) is discussed.

Yield surface

The yield surface defines the stress level at the onset of plastic deformation, and can be written as a function of the stress tensor, \(\sigma\), and the hardening parameter, \(\kappa_h\):

\[
F(\sigma, \kappa_h) = 0
\] (3-28)

In the classical theory of plasticity, the yield criterion defines the admissible states of \((\sigma, \kappa_h)\) in stress-space. For an isotropic material, the yield criterion is independent of the orientation of the coordinate system employed, and therefore, it should be a function of stress invariants only:

\[
F(I_1, I_2, I_3, \kappa_h) = 0
\] (3-29)

where \(I_1\), \(I_2\) and \(I_3\) are the first, second, and third invariants of the stress tensor (Simo & Hughes, 1991).

---

\(^1\) Plasticity formulation can be performed in stress-space. For further details refer to Simo & Hughes (1991).
Plastic potential

The plastic potential function, $Q$, determines the direction of plastic straining after yield is reached, and is defined as follows:

$$Q(\sigma) = \text{const.} \quad (3-30)$$

It is assumed that the plastic strain increment is proportional to the stress gradients of $Q$ through the proportionality constant, $\dot{\lambda}$, also known as the plastic multiplier (Owen & Hinton, 1980), or consistency parameter (Simo & Hughes, 1991). The flow rule is defined by:

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial Q}{\partial \sigma} \quad (3-31)$$

For some materials, such as metals, $F = Q$, which gives rise to the associative theory of plasticity. However, for rocks and soils, $F \neq Q$, which results in a non-associative theory of plasticity. Similar to the yield function, the potential function is expressed in stress-space. For isotropic materials, the potential function is independent of the orientation of applied load.

The Kuhn-Tucker complementary condition and the consistency condition

The plastic multiplier or consistency parameter obeys the following relations:

$$\dot{\lambda} \geq 0, \quad F(\sigma, \kappa_h) \leq 0 \quad \text{and} \quad \dot{\lambda} F(\sigma, \kappa_h) = 0 \quad (3-32)$$

The above relations are called the Kuhn-Tucker complementary condition or loading/unloading condition. In addition, $\dot{\lambda} \geq 0$ satisfies the consistency requirement, expressed as follows:

$$\dot{\lambda} F(\sigma, \kappa_h) = 0 \quad (3-33)$$

For interpretation of these relations one can refer to Simo and Hughes (1991).
Finally, it should be mentioned that some researchers found the concept of plastic potential restrictive. Also, it is suggested that the loading criteria should be expressed explicitly rather than in the form of Kuhn-Tucker condition expressed by Equation 3-32. For further reading on these subjects refer to (Casey, 2002).

**Elasto-plastic tangent moduli**

A constitutive relation is the mathematical description of the relation between the states of stress and strain. This relation in its incremental form can be defined as:

\[ \dot{\sigma}_{ij} = D_{ijkl}^{ep} \dot{e}_{kl} \]  

(3-34)

where \( \dot{\sigma}_{ij} \) is the incremental stress tensor, \( \dot{e}_{kl} \) is the incremental strain tensor, and \( D_{ijkl}^{ep} \) is the elasto-plastic constitutive tensor.

In the derivation of the elasto-plastic constitutive tensor, it is assumed that the material is hyperelastic, in other words, the strain-stress relations can be expressed by Equation 3-3. Also, it is assumed that the strain tensor can be decomposed into an elastic part and a plastic part denoted by \( \varepsilon^e \) and \( \varepsilon^p \), respectively, according to:

\[ \varepsilon^{lp} = \varepsilon^e + \varepsilon^p \]  

(3-35)

Using the above mentioned assumptions along with the Kuhn-Tucker consistency condition and flow rule, the elasto-plastic constitutive matrix can be defined as:

\[ D_{ijkl}^{ep} = D_{ijkl}^e \quad \text{if } \dot{\lambda} = 0 \]  

(3-36)

\[ D_{ijkl}^{ep} = D_{ijkl}^e - \frac{\partial F}{\partial \sigma_{ij}} D_{ijkl}^e \frac{\partial F}{\partial \sigma_{kl}} + H^p \quad \text{if } \dot{\lambda} > 0 \]

where \( D_{ijkl}^e \) is the elastic constitutive matrix, relating the increments of stress to increments of elastic strain, \( \partial F / \partial \sigma_{ij} \) is the normal vector to the yield surface in the
stress-space, $\partial Q / \partial \sigma_y$ is the normal vector to the plastic potential, and $H^p$ is the plastic modulus. It should be noted that the elasto-plastic matrix, in general, is not symmetric, and its symmetry is preserved only for associative flow rules. For the derivation of Equation 3-36 refer to Appendix 3.

### 3.3.2 Strain-space plasticity formulation

Traditionally, the elasto-plastic formulation is performed in stress space, where stress is considered as the independent variable. However, the formulation can alternatively be expressed in strain space. In this approach, the yield surface is described as a function of strain rather than stress, and strain is treated as an independent variable. The possibility of such formulation was hinted by Drucker (1950), while a systematic approach was presented by Naghdi and Trapp (1975), Caulk and Naghdi (1978), and Casey and Naghdi (1981).

It is suggested that using a strain space formulation, the ambiguities associated with perfect plasticity and strain softening models can be avoided. Another advantage was noted by Yoder and Iwan (1981) who suggest that using strain space formulation, the difficulties associated with intersecting yield surfaces and determination of active surfaces can be circumvented.

The problems associated with the formulation of plasticity in stress space, where stress is treated as the independent variable can be illustrated using a simple stress-strain curve for three cases of strain hardening, perfect plasticity, and strain softening materials (Moss, 1984). For simplicity, it is assumed that the material is subjected to infinitesimal deformations. Considering point B as shown in Figure 3-3, in strain hardening materials, for a specified value of $\Delta \sigma$, there is a unique answer to $\Delta \varepsilon$. For perfectly plastic materials, only $\Delta \sigma \leq 0$ is admissible; however, the condition of $\Delta \sigma = 0$ does not result in a unique solution for $\Delta \varepsilon$. Finally, in strain softening materials, the condition of $\Delta \sigma < 0$ will result in two solutions: one corresponding to plastic loading and the other corresponding to elastic unloading at point B.
One controversial aspect of the stress-space and strain space formulation is whether the formulation expressed in the two spaces are equivalent (Casey & Naghdi, 1983, Drucker, 1988, Lu & Vaziri, 1997). Also, Simo (1991) noted that “on physical grounds it is difficult to justify the a priori formulation of the response function in strain space”. In fact, most yield criteria are expressed in stress space, and as a result, in this thesis the traditional plasticity formulation using stress-space is adopted.

Figure 3-3 Stress-strain curve for (a) strain hardening, (b) perfectly plastic, (c) strain softening materials.
3.3.3 Plasticity of anisotropic materials

Introduction to plasticity models for anisotropic materials

In the previous sections, elastic symmetries and the effect of anisotropy on the elasticity tensor is discussed. Similar to elastic properties, the strength of anisotropic materials also depends on the orientation and type of anisotropy. As a result, in contrast to isotropic materials, the yield and potential functions of anisotropic materials are not only a function of stress invariants, but also depend on the angle of applied load with respect to the anisotropy direction. The shape of the curve relating the uniaxial compressive strength, \( \sigma_c \), and the orientation angle, \( \beta \), is designated as the “type of anisotropy”. Figure 3-4 is a schematic graph that demonstrates yield strength variation of highly anisotropic rocks with respect to \( \beta \). According to these graphs, the response is generally classified into three types, namely, “U-shaped”, “shoulder shaped”, and “wavy shaped”.

![Figure 3-4 Variation of \( \sigma_c \) versus \( \beta \) for inherent and induced anisotropy (Nasseri et al., 1996).](image)

The ratio of maximum to minimum strength of an anisotropic material is referred to as the anisotropic ratio (Hudson, 1993), and is written as:

\[
R_c = \frac{\sigma_{c90}}{\sigma_{c\text{min}}} \tag{3-37}
\]

where \( \sigma_{c\text{min}} \) is the minimum strength and \( \sigma_{c90} \) is the strength of the material when joints are perpendicular to the direction of the applied load.
Various failure criteria for anisotropic materials have been proposed. According to their assumptions and techniques, they can be classified into three main categories (Duveau et al., 1998): mathematical continuous approaches, empirical continuous models, and discontinuous plane of weakness. A comprehensive assessment and comparison of some representative criteria for each category is presented by Duveau et al. (1998). In this review, nine representative criteria are selected and the results are compared to experimental data for Angers schist. Quantitative comparisons of predictions obtained from each model strongly suggest that the discontinuous plane of weakness approach demonstrates superior performance for jointed materials. Consequently, in this research, the discontinuous plane of weakness approach was adopted to represent the strength behaviour of the material. Table 3-1 classifies the most widely used available anisotropic criteria. In the following section, the basic assumptions, technique, and characteristics of each group are briefly discussed.

Mathematical continuous approach

In mathematical continuous criteria, a continuum is considered and a continuous variation of strength is assumed. The strength function is generally described by a mathematical technique which considers the type of symmetries existing in the material.

One of the first anisotropic criteria of this kind was proposed by Hill for frictionless materials by extending the Von-Mises isotropic theory (Hill, 1964). A more general approach was proposed by Goldenblat and Kopnov (1965), who suggested the use of strength tensors of different order to take into account the anisotropy. Tsai and Wu (1971) proposed a failure criterion which uses strength tensors of first and second order for fibre reinforced composites. This failure criterion is widely used for different kinds of materials; for transversely isotropic materials it can be expressed by:

\[
F_1 \sigma_{11} + F_2 (\sigma_{22} + \sigma_{33}) + F_3 \sigma_{11}^2 + F_4 (\sigma_{22}^2 + \sigma_{33}^2) + 2F_5 \sigma_{12} \sigma_{21} + 2F_6 \sigma_{23} \sigma_{32} + \frac{1}{2} (F_{22} - F_{23}) \sigma_{23}^2 + F_{55} (\sigma_{11}^2 + \sigma_{33}^2) = 1
\]  

\(3-38\)
where $F_{ij}$ are constant coefficients that are determined experimentally by performing uniaxial, tensile, and pure shear tests.

For geological materials, a widely used criterion was proposed by Pariseau (1972) by modifying the Hill criterion in order to take into account the strength difference in tensile and compressive loading and the strength dependency on the mean stress. Parallel to these works, a more rigorous and general approach was developed by Boehler and Sawczuk (1977) in the framework of the theory of invariant tensorial functions. Specific failure criteria were also proposed for rock materials (Allirot & Boehler, 1979), and for composites (Boehler & Raclin, 1982). More recently, a new invariant failure criterion was developed by Cazacu (1995) by extending the Stassi isotropic criterion.

The main challenge of these models is the determination of material constants which should be determined experimentally. The number of constants depends on the proposed criterion, and can vary for the different approaches. Also, introducing new terms of stress requires deep insight into the physical behaviour of the material so that the introduced terms result in the observed behaviour.

**Empirical continuous models**

In regard to empirical models, an isotropic strength criterion is considered and strength anisotropy is described by determining some empirical laws that define the variation of material parameters with respect to loading orientation. The functions describing these variations are fully empirical in nature and are calibrated from simple fitting of experimental data. Clearly, these models lack any physical or mathematical basis. One of the representative criteria of this kind was proposed by Jaeger (1971). Known as “the variational cohesion theory”, the Mohr-Coulomb failure criterion was modified by using a variable material cohesion as a function of loading orientation and a constant value of friction. A simple modification of this criterion was proposed by McLamore and Gray (1967), who used a variation of the friction coefficient in the same way as the cohesion coefficient is used in the original criterion. Ramamurthy et al. (1988) proposed a modification of the McLamore and Gray criterion by using a non-linear form of the failure envelope in the Mohr plane.
The main disadvantage of these models is that they lack any physical or mathematical basis. It is clear that parameters such as the friction angle and cohesion represent intrinsic characteristics of a material, and do not vary with the orientation of the applied load. Also, determining variational laws requires a large amount of experimental data, as well as an appropriate curve-fitting process. Therefore, the application of the proposed models is restricted to a narrow group of materials exhibiting similar behaviour.

**Discontinuous planes of weakness models**

In contrast to the previously mentioned techniques, these models emphasize the mathematical description of physical mechanisms active in the failure process. The basic assumption is that the failure of an anisotropic body is either the result of failure of one or more weakness planes or the result of failure of the intact matrix. Consequently, a number of different strength criteria should be considered simultaneously, and it is the geometric superposition of these functions that defines the elastic region. In this technique, based on the type and number of failure mechanisms activated in the material, all classes of anisotropy and curves depicted in Figure 3-4 can be obtained.

![Figure 3-5 Schematic presentation of the failure mechanism in the plane of weakness criterion.](image)

The most representative model of this kind, known as the single plane of weakness theory was proposed by Jaeger (1976). By considering the planes of weakness as orientated Griffith cracks, and, based on an extension of the modified Griffith theory (McClintock & Walsh, 1962), other criteria were proposed by Walsh and Brace (1964), Hoek and Brown (1980, 1983), and recently Duveau et al. (1998) proposed to use the Barton
criterion for sliding along schistose planes. The type of strength versus orientation graph obtained from these models is schematically shown in Figure 3-5. It should be noted that since a continuous body of material is considered in these models, the method falls within the category of continuous approaches. However, since discontinuous surfaces are assumed to exist at a specific orientation in the material, the method is referred to as the discontinuous plane of weakness.

Table 3-1 Classification of widely used anisotropic failure criteria (Duveau et al., 1998).

<table>
<thead>
<tr>
<th>Continuous Criteria</th>
<th>Discontinuous Criteria *</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical Approach*</td>
<td>Experimental Approach</td>
</tr>
<tr>
<td>Von Mises</td>
<td>Casagrande and Carillo</td>
</tr>
<tr>
<td>Hill</td>
<td>Jaeger (variable cohesive strength theory)</td>
</tr>
<tr>
<td>Olszak and Urbanowicz</td>
<td>McLamore and Gray</td>
</tr>
<tr>
<td>Goldenblat</td>
<td>Ramamurthy, Rao and Singh</td>
</tr>
<tr>
<td>Goldenblat and Kopnov</td>
<td></td>
</tr>
<tr>
<td>Boehler and Sawczuk</td>
<td></td>
</tr>
<tr>
<td>Tsa and Wu</td>
<td></td>
</tr>
<tr>
<td>Pariseau</td>
<td></td>
</tr>
<tr>
<td>Boehler</td>
<td></td>
</tr>
<tr>
<td>Dafalia</td>
<td></td>
</tr>
<tr>
<td>Allirot and Boehler</td>
<td></td>
</tr>
<tr>
<td>Nova and Sacchi</td>
<td></td>
</tr>
<tr>
<td>Boehler and Raclin</td>
<td></td>
</tr>
<tr>
<td>Raclin</td>
<td></td>
</tr>
<tr>
<td>Kaar et al.</td>
<td></td>
</tr>
<tr>
<td>Cazacu</td>
<td></td>
</tr>
</tbody>
</table>

* It should be noted that in all of these models, the constant parameters are determined based on experimental observations and this classification only concerns the applied technique.

### 3.4 The equivalent continuum approach

The equivalent continuum approach is based on a homogenization technique in which the original discontinuous material is replaced by a continuous material. The hypothetical material exhibits similar elastic and inelastic behaviour to the original discontinuous material.
In the following sections, assumptions and techniques involved in the development of the constitutive relations of the equivalent continuum are discussed. Also, the FEM formulation and the integration technique for the proposed constitutive model are presented.

### 3.4.1 The mechanical model of the equivalent continuum

A jointed material consists of an intact matrix and a number of discontinuous surfaces which are called interfaces or joints. From a mathematical point of view, such a material can be simplified to a number of elasto-viscoplastic modules interacting in series (Sharma & Pande, 1988), as shown in Figure 3-6. Each module represents one component of the material, i.e., an intact matrix or a set of interfaces. In this representation, springs represent the elastic behaviour and dashpot-slide modules represent the viscoplastic behaviour.

![Figure 3-6 A mechanical representation of the behaviour of jointed materials.](image)

### 3.4.2 Elastic properties of the equivalent continuum

Using a simple spring-dashpot mechanical model, as presented in Section 3.4.1, the elasticity matrix of the equivalent continuum can be calculated based on the elastic properties of the different components of a material using the following equations:
Chapter 3                Continuum Theory of Anisotropic Materials

\[ D_{Eq} = D_{\text{matrix}} + D_{j_1} + \ldots + D_{j_n} \quad \text{and} \quad \frac{1}{C_{Eq}} = \frac{1}{C_{\text{matrix}}} + \frac{1}{C_{j_1}} + \ldots + \frac{1}{C_{j_n}} \]  

(3-39)

where \( D \) denotes the compliance tensor, \( C \) denotes the elasticity tensor, and sub-indices \( Eq \) and \( j \) denote equivalent continuum and joints, respectively.

Assuming general anisotropy for each component, and choosing \((x_1, x_2, x_3)\) as the global reference coordinates, the compliance and elasticity matrix of the equivalent continuum can be expressed in the global or reference coordinates as:

\[ D_{Eq} = R^\varepsilon_{\text{matrix}} \hat{D}_{Eq} R^\varepsilon_{\text{matrix}}^T + R^\varepsilon_{j_1} \hat{D}_{j_1} R^\varepsilon_{j_1}^T + \ldots + R^\varepsilon_{j_n} \hat{D}_{j_n} R^\varepsilon_{j_n}^T \]  

(3-40)

and

\[ \left( C_{Eq} \right)^{-1} = \left( R^{\sigma}_{\text{matrix}} \hat{C}_{Eq} R^{\sigma}_{\text{matrix}}^T \right)^{-1} + \left( R^{\sigma}_{j_1} \hat{C}_{j_1} R^{\sigma}_{j_1}^T \right)^{-1} + \ldots + \left( R^{\sigma}_{\mu} \hat{C}_{j_\mu} R^{\sigma}_{\mu}^T \right)^{-1} \]  

(3-41)

where \( R^\sigma \) and \( R^\varepsilon \) are the transformation matrices discussed in Section 3.2.5. From a computational point of view, instead of a direct calculation of the elasticity tensor based on Equation 3-41, it is more convenient to calculate the compliance using Equation 3-40, and to express the elasticity matrix as the inverse of the compliance as explained in Section 3.2.3.

Elastic properties of discontinuous surfaces are usually defined with respect to the normal stiffness of the interface, \( k_n \), the shear stiffness of the interface, \( k_s \), and the joint spacing, \( h \). The elasticity and compliance matrices in the local surface coordinates, are defined as:

\[
\begin{align*}
\hat{D}_j &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/k_s h & 0 & 0 & 0 \\
0 & 0 & 0 & 1/k_s h & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\hat{C}_j &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k_s h \\
0 & 0 & 0 & 0 & k_s h & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]  

(3-42)

Through out this work, it is assumed that the plane of the joints exhibits isotropic symmetry. Usually, a normal vector to this plane is an input parameter that defines the
orientation of the local coordinates of anisotropy with respect to the global coordinates (see Appendix 3).

Using Equation (3-40), and choosing a global coordinate system that coincides with the local coordinates of anisotropy, the compliance matrix of the equivalent continuum with an isotropic intact matrix and one set of joints can be expressed by:

$$\mathbf{D}_{\text{Eq}} = \begin{pmatrix}
\frac{1}{E} & -\nu/E & -\nu/E & 0 & 0 & 0 \\
-\nu/E & \frac{1}{E} & -\nu/E & 0 & 0 & 0 \\
-\nu/E & -\nu/E & 1/E + 1/hk_n & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G + 1/hk_s & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G + 1/hk_s & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G \\
\end{pmatrix}$$

(3-43)

where $E$ and $\nu$ are the Young’s modulus and Poisson’s ratio of the intact material, respectively. For the above material, the elasticity matrix, which is the inverse of $\mathbf{D}_{\text{Eq}}$, in the local coordinates of the joint has the following form:

$$\mathbf{C}_{\text{Eq}} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{12} \\
\end{pmatrix}$$

(3-44)

where

$$A_{11} = A_{22} = \frac{E}{1-\nu^2 - \frac{\nu^2(1+\nu)^2}{1-\nu^2 + E/hk_n}}, \quad A_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu) - \frac{(1-\nu)E}{E/hk_n}}$$

(3-45)

$$A_{32} = A_{23} = A_{13} = A_{31} = \frac{\nu E}{(1+\nu)(1-2\nu) + (1-\nu)E/hk_n}, \quad A_{12} = A_{21} = \frac{\nu A_{11}}{1-\nu},
\quad G_{23} = G_{13} = \frac{Ghk}{G + hk_s}, \text{ and } G_{12} = G$$
Upon comparison of the structure of the matrices in Equations 3-43 and 3-44 with the compliance and elasticity matrices of transversely isotropic materials expressed by Equations 3-18 and 3-20, one finds that an isotropic material embedded with one set of joints exhibits transversely isotropic characteristics. It should be noted that, in this case, isotropy of the intact material results in invariance properties with respect to the rotation of the coordinate system, and therefore, it is possible to add the compliance matrices of the intact material and the joints in the local coordinates of the joints. However, for general anisotropic materials, the approach proposed in Equations 3-40 and 3-41 should be followed.

3.4.3 Inelastic behaviour of the equivalent continuum

The approach adopted by this research is based on the discontinuous planes of weakness theory discussed in Section 3.3.3. As previously explained, in this method it is assumed that a jointed material is composed of an intact material and one or more sets of joints. Failure of the equivalent continuum can be due to the failure of each set of joints or failure of the intact material, which are simultaneously present at all FEM integration points. Each component exhibits a separately specified strength and post-peak behaviour. Once the stress level at a point reaches the strength of any of these components, its corresponding mechanism becomes active, and its potential rule is mobilized.

From a numerical point of view, the existence of multiple yield mechanisms can result in serious complications in the determination of the activated surfaces and the post peak behaviour. The FEM algorithm adopted in this research falls within the multi-surface plasticity schemes discussed in Section 3.4.4. In the proceeding sections, yield and potential surfaces applied for both the joints and the intact material are introduced.

Yield criterion of the intact material

The yield surface of isotropic materials is expressed in the following form:

\[ F(\sigma, \kappa_h) = 0 \]  

(3-46)
In the method adopted in this research, any type of yield criterion can be considered for the intact matrix. In geomechanics, two widely used $J_2$ plasticity models, the Mohr-Coulomb and Hoek-Brown models, are often used to describe the behaviour of isotropic rocks. It should be noted that in this research, effects of hardening or softening have been excluded in all application examples (i.e., $F(\sigma, \kappa_n) = F(\sigma) = 0$).

The Mohr-Coulomb criterion expresses shear failure in the following form:

$$F_1 = \sqrt{r} + \sigma_n \tan \phi - c = 0 \quad (3-47)$$

where $\phi$ is the friction coefficient and $c$ is the cohesion parameter. The above criterion can be expressed in terms of principal stresses by:

$$F_1 = \frac{1}{2}(\sigma_1 - \sigma_3) = -\frac{1}{2}(\sigma_1 + \sigma_3) \sin(\phi) + c \cos(\phi) \quad (3-48)$$

Also, by expressing principal stresses in terms of stress invariants, the Mohr-Coulomb criterion for isotropic materials can be written in the following form:

$$F_1 = \frac{I_1}{3} \sin(\phi) + \sqrt{J_2} \left[ \cos(\phi) - \frac{1}{\sqrt{3}} \sin(\theta^L) \sin(\phi) \right] - c \cos(\phi) \quad (3-49)$$

where $I_1$ is the first invariant of the stress tensor, $J_2$ is the second invariant of deviatoric part of the stress, and $\theta^L$ is the Lodé angle, defined in Appendix 2.

It is important to note that for isotropic materials with Mohr-Coulomb behaviour, infinite planes of weakness oriented in all directions exist within the material. For this type of material, failure happens on the critical plane, which is the plane on which the yield criterion is violated first. Analytical calculation reveals that two conjugate planes with an orientation of $(\pi/4 - \phi/2)$ with respect to maximum principal stress are the critical planes of weakness for isotropic Mohr-Coulomb materials.
In addition to shear failure, many materials, such as rock and concrete, exhibit low resistance to tensile forces. Such behaviour can be expressed as the tensile strength of the material, written in the following form:

\[
F_2 = \sigma_1 - c_t = \frac{2}{\sqrt{3}} \sqrt{J_2} \sin(\theta^L) + \frac{2\pi}{3} + \frac{I_1}{3} - c_t = 0 \tag{3-50}
\]

where \( \sigma_1 \) is the maximum principal stress and \( c_t \) is the tensile strength of the material.

Another yield criterion that can be considered for intact rock is the Hoek-Brown criterion, expressed as follows:

\[
F_1 = \frac{m\sigma_c}{3} I_1 + m\sigma_c \sqrt{J_2} \left[ \cos(\theta^L) - \frac{\sin(\theta^L)}{\sqrt{3}} \right] - s\sigma_c^2 + 4J_2 \cos^2(\theta^L) \tag{3-51}
\]

where \( m \) and \( s \) are two constants determined by the type of rock, and \( \sigma_c \) is the uniaxial compressive strength of the intact rock.

**Potential surface of the intact material**

For Mohr-Coulomb type materials, the potential surface is defined by:

\[
Q_1 = \frac{I_1}{3} \sin(\varphi_{dil}) + \left[ \sqrt{J_1} \cos(\varphi_{dil}) - \frac{1}{\sqrt{3}} \sin(\theta^L) \sin(\varphi_{dil}) \right] - c \cos(\varphi_{dil}) \tag{3-52}
\]

where \( \varphi_{dil} \) is the dilation angle of the material. Similar to Equation 3-52, the potential function for the tension-cut-off surface can be expressed by Equation 3-50, in which the friction angle, \( \varphi \), should be replaced by the dilation angle, \( \varphi_{dil} \).

For Hoek-Brown type materials, the potential surface is defined in the following form, with recommended values of \( m_{dil} < m / 4 \):

\[
Q_1 = \frac{m_{dil}\sigma_c}{3} I_1 + m_{dil}\sigma_c \sqrt{J_2} \left[ \cos(\theta^L) - \frac{\sin(\theta^L)}{\sqrt{3}} \right] - s\sigma_c^2 + 4J_2 \cos^2(\theta^L) \tag{3-53}
\]
Flow and potential vector of the intact material

The yield and potential flow vectors are calculated using the approach discussed in Owen and Hinton (1980). The flow vector for the yield surface is obtained by:

\[
\frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial I_1} \frac{\partial I_1}{\partial \sigma} + \frac{\partial F}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial \sigma} \tag{3-54}
\]

And the flow of the potential surface is obtained as follows:

\[
\frac{\partial Q}{\partial \sigma} = \frac{\partial Q}{\partial I_1} \frac{\partial I_1}{\partial \sigma} + \frac{\partial Q}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma} + \frac{\partial Q}{\partial \theta} \frac{\partial \theta}{\partial \sigma} \tag{3-55}
\]

The yield strength of the interface

The yield surface of an anisotropic material is not only a function of stress and the hardening parameter, but also depends on the orientation of applied load with respect to state of stress, as follows:

\[
F(\sigma, \kappa_h, \beta) = 0 \tag{3-56}
\]

Different strength criteria have been proposed for the interface and joint surfaces. In this research, the Mohr-Coulomb failure criterion presented in Equation 3-47 is considered for the joints. However, in contrast to isotropic materials that assume an infinite number of planes of weaknesses in every direction, in jointed materials there is only one plane where slip can happen, oriented at a fixed angle, \( \beta \).

In order to take into account the orientation dependence properties of the anisotropic material, the stress tensor is projected onto the joint surface. The violation of yield functions is checked in the local coordinates of the joints as follows:

\[
F_1 = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} + \sigma_{33} \tan \varphi - c = 0 \tag{3-57}
\]
where \( \bar{x}_3 \) is the normal vector to the interface and \( \tilde{\sigma} \) is the stress expressed in the local coordinates of the joint and can be obtained by:

\[
\tilde{\sigma} = R_\beta^T \sigma
\]  

(3-58)

For plane-strain problems, the state of stress reduces to:

\[
\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{33} & \sigma_{13} & \sigma_{22} \end{bmatrix}
\]

(3-59)

and the rotation matrix reduces to:

\[
R = \begin{bmatrix} S^2 & C^2 & -2CS & 0 \\ CS & -CS & S^2 - C^2 & 0 \end{bmatrix}
\]

(3-60)

where \( S = \sin(\beta) \) and \( C = \cos(\beta) \). If the global axes are chosen to be the principal stress directions, the strength criterion for each set of joints with orientation \( \beta \) is expressed as:

\[
\sigma_{11} - \sigma_{33} = \frac{2(c + \sigma_{33} \tan \phi)}{(1 - \tan \phi \cdot \tan \beta) \sin 2\beta}
\]

(3-61)

The above criterion is equivalent to what is expressed by Jaeger in the “single plane of weakness theory”.

Expressing the above criterion in terms of stress invariants, the Mohr-Coulomb criterion reduces to:

\[
\sqrt{J_2} \cos(\theta^c) = (c + (2\sqrt{J_2}/\sqrt{3} \sin(\theta^i + 4\pi/3) + 4/3) \tan \phi)/((1 - \tan \phi \cdot \tan \beta) \sin 2\beta)
\]

(3-62)

Potential function of the interface

The potential function indicates the direction of plastic flow once yield occurs, and can be expressed by:
\[ Q_a(\sigma) = \text{const.} \]  (3-63)

The plastic potential function of an isotropic Mohr-Coulomb type material with a dilatancy angle, \( \varphi_{\text{dil}} \), is as follows:

\[ Q_i = |\tau| + \sigma_n \tan \varphi_{\text{dil}} + \text{const} = 0 \]  (3-64)

It is suggested that the dilatancy angle be less than the friction angle (Sharma & Pande, 1988).

The potential flow of the joints can also be expressed in the local coordinates of anisotropy by:

\[ Q_i = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} + \sigma_{33} \tan \varphi_{\text{dil}} - c = 0 \]  (3-65)

**Flow vector and potential flow vector of the interface**

In the theory of plasticity, the gradient of the yield surface and the potential surface in stress-space represent the flow vector and the potential flow vector, respectively. The yield surface and plastic potential for an anisotropic material can be expressed in the local coordinates of anisotropy. Gradients of these two functions with respect to stress components in the local coordinates, i.e., \( \partial F / \partial \tilde{\sigma} \) and \( \partial Q / \partial \tilde{\sigma} \), express the flow vector and the potential flow vector in the local coordinates of anisotropy. The yield and potential flow vectors are second-order tensors, and, therefore, they follow the rules of transformation of second-order tensors (see Appendix 5). Similar to stress and strain, these second-order tensors are usually arranged in Voigt notation, and subsequently, they follow the rules of transformation of Voigt stress vectors as follows:

\[ \frac{\partial F}{\partial \sigma} = R^\sigma \frac{\partial F}{\partial \tilde{\sigma}} \quad \text{and} \quad \frac{\partial Q}{\partial \sigma} = R^\sigma \frac{\partial Q}{\partial \tilde{\sigma}} \]  (3-66)
where $R_\sigma$ is the matrix used for the transformation of coordinate system of Voigt stress (see Appendix 1), and the potential vector is defined as:

$$
\frac{\partial Q}{\partial \sigma} = \left[ \frac{\partial Q}{\partial \sigma_{11}}, \frac{\partial Q}{\partial \sigma_{22}}, \frac{\partial Q}{\partial \sigma_{33}}, \frac{\partial Q}{\partial \sigma_{23}}, \frac{\partial Q}{\partial \sigma_{13}}, \frac{\partial Q}{\partial \sigma_{12}} \right]
$$

(4-67)

### 3.4.4 Multi-surface plasticity

In Section 3.3.1, the numerical algorithm for the integration of the elasto-plastic constitutive relations of single surface plasticity models is presented. This section is concerned with the cases where the elastic region is defined by multiple convex yield surfaces intersecting in a non-smooth fashion (see Figure 3-7). In general, each yield surface may exhibit a hardening or softening mechanism. The multi-surface plasticity model is sometimes referred to as a multi-mechanism model.

**Elasto-plastic constitutive relations for multi-surface plasticity**

This section discusses an algorithmic approach to the numerical implementation of plasticity models of materials with multiple convex yield surfaces intersecting in a non-smooth fashion. This situation is of considerable interest to many fields, such as soil mechanics, where a Cam-Clay combined with a cap model is used. In this research, this method is applied to rock mechanics problems where a matrix of intact material is intersected with one or multiple sets of joints.

![Schematic interaction of two yield surfaces in a non-smooth fashion](image)

Figure 3-7 Schematic interaction of two yield surfaces in a non-smooth fashion: the failure surface of the Lade model (Pasternack & Timmerman, 1986).
It is assumed that \( n \) different yield surfaces with their associated or non-associated potential surfaces are intersecting in a non-smooth manner. The mechanical model presented in Section 3.4.1 demonstrates that the essence of the elasto-plastic relation in the presence of multiple mechanisms is similar to that presented for single yield surface cases, i.e.:

\[
\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \tag{3-68}
\]

and

\[
\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{ij}^e \tag{3-69}
\]

In materials with multi-mechanisms, the total increment of plastic strain is obtained by the summation of the plastic straining of all active plasticity surfaces. This can be expressed by:

\[
\dot{\varepsilon}_{ij}^p = \sum_{\alpha=1}^{n} \left( \dot{\varepsilon}_{ij}^p \right)_{\alpha} = \sum_{\alpha} \lambda_{\alpha} \frac{\partial Q_{\alpha}}{\partial \sigma_{ij}} \tag{3-70}
\]

where \( \alpha \) denotes the index of the activated surfaces.

Also, the hardening parameter for each plasticity surface can be expressed by:

\[
\left( \mathbf{K}_h \right)_{\alpha} = \dot{\lambda}_{\alpha} \left( \frac{\partial \mathbf{K}_h}{\partial \varepsilon_{ij}^p} \right)_{\alpha} \tag{3-71}
\]

Finally, the Kuhn-Tucker and consistency conditions must be satisfied for any active yield surface, as follows:

\[
F_{\alpha}(\sigma, \mathbf{K}_h) \leq 0, \quad \dot{\lambda}_{\alpha} \geq 0, \quad \text{and} \quad \dot{\lambda}_{\alpha} F_{\alpha} = 0 \tag{3-72}
\]

\[
\dot{\lambda}_{\alpha} \dot{F}_{\alpha} = 0
\]

The Kuhn-Tucker and consistency conditions in Equation 3-72 result in \( \alpha \) number of equations and \( \alpha \) unknowns \( \dot{\lambda}_{\alpha} \). The procedure for the calculation of \( \dot{\lambda}_{\alpha} \) is discussed in
Finally, the elasto-plastic constitutive tensor of the material can be expressed by:

\[ D_{ijkl}^{e} = D_{ijkl}^{e} - \sum_{a=1}^{n} \left( \frac{\partial F}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} D_{ijkl}^{a} - \frac{\partial F}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} \right) + H^{p} \] (3-73)

**Determination of active surfaces**

In the presence of a single yield surface, the condition for a smooth plasticity surface to be active is the violation of yield criterion, as follows:

\[ F_{n+1}^{trial} (\sigma, \kappa_{h}) > 0 \] (3-74)

However, in the context of multi-surface plasticity, the condition expressed by Equation 3-74 does not guarantee that all surfaces for which the yield condition is violated are necessarily active (Simo, 1988). In other words, there may be cases for which:

\[ F_{n+1}^{trial} (\sigma, \kappa_{h}) > 0 \text{ and } F_{n+1}^{trial} (\sigma, \kappa_{h}) < 0 \] (3-75)

The condition expressed by Equation 3-75 can arise due to the intersection of plasticity surfaces in a non-smooth manner, or, in other words, due to the existence of singular points. In the context of multi-surface plasticity, for any surface to be active, it should satisfy the following condition (Simo, 1988):

\[ F_{n+1}^{trial} (\sigma, \kappa) > 0 \text{ and } \dot{\lambda}_{n+1} (\sigma, \kappa) > 0 \] (3-76)

The condition expressed by Equation 3-74 can be used as an indicator of potential active surfaces by applying the following formula:

\[ J_{active}^{trial} = \{ \alpha \in \{1, 2, ..n\} | f_{\alpha,n+1}^{trial} > 0 \} \] (3-77)
Figure 3-8 represents the geometric interpretation of the possible conditions arising from intersection of two yield surfaces. These conditions can be classified into three different situations:

1- If only one yield condition is violated, then the condition \( F_{\alpha,n+1}^{\text{trial}} > 0 \) necessitates that \( \dot{\lambda}_{\alpha,n+1} > 0 \). Subsequently, all plasticity calculations can be performed with respect to \( F_{\alpha} \) and \( Q_{\alpha} \).

2- If more than one yield condition is violated, then the condition \( F_{\alpha,n+1}^{\text{trial}} > 0 \) for any surface does not guarantee that \( \dot{\lambda}_{\alpha,n+1} > 0 \). This situation has been depicted in Figure 3-8 for the case where two yield surfaces \( \alpha = 1 \) and \( \alpha = 2 \) are interacting, and can be categorized as follows:

   a) If \( F_{\alpha,n+1}^{\text{trial}} > 0 \) and \( \dot{\lambda}_{\alpha,n+1} > 0 \) for both \( \alpha = 1 \) and \( \alpha = 2 \), the situation reflects a corner region where both the surfaces are active, i.e., \( F_{\alpha,n+1} = 0 \). This region is depicted in Figure 3-8 as \( r_{12} \).

   b) If \( F_{\alpha,n+1}^{\text{trial}} > 0 \) and \( \dot{\lambda}_{\alpha,n+1} \leq 0 \) for \( \alpha = 1 \) or \( \alpha = 2 \) the situation reflects that only one of the surfaces are active. This region is depicted in Figure 3-8 as \( r_{1} \) and \( r_{2} \).

In order to determine the active surfaces, \( J_{\text{active}} \), it is necessary to systematically enforce the Kuhn-Tucker conditions for all the surfaces expressed by Equation 3-77. Two different return mapping algorithms can be followed:

**Procedure 1:**

(I-I) Solve the closest-point-projection iteration with \( J_{\text{active}} = J_{\text{active}}^{\text{trial}} \), until a converged solution is obtained.
(I-2) Check the sign of $\hat{\lambda}_{\alpha,n+1}$ for all $\alpha \in J_{\text{trial}}^{\text{active}}$. If for any $\alpha$, $\hat{\lambda}_{\alpha,n+1} < 0$, drop this index from $J_{\text{trial}}^{\text{active}}$, and go back to (I-1), otherwise exit.

**Procedure 2:**

(II-1) Solve one iteration with $J_{\text{active}}^{\text{trial}} = J_{\text{active}}^{\text{trial}}$, where $k$ is the index of the current iteration.

(II-2) Check sign of $\Delta \hat{\lambda}_{\alpha,n+1}$ for all $\alpha \in J_{\text{trial}}^{\text{active}}$. If for any $\alpha$, $\Delta \hat{\lambda}_{\alpha,n+1} < 0$, drop this index from $J_{\text{trial}}^{\text{active}}$, and restart the iteration, otherwise proceed to the next iteration.

---

Figure 3-8 Geometric illustration of the geometry at a corner point, $\sigma \in \partial E$ intersection of two yield surfaces $\{J_{\text{act}} = \{1, 2\}\}$: (a) definition of regions $r_1$, $r_2$, and $r_{12}$, (b) region $r_1$ is characterized by $\hat{\lambda}_{n+1}^1 > 0$ and $\hat{\lambda}_{n+1}^2 < 0$, and (c) region $r_{12}$ is characterized by $\hat{\lambda}_{n+1}^1 > 0$ and $\hat{\lambda}_{n+1}^2 > 0$ (Simo, 1988).
Algorithm for the integration of elasto-plastic, multi-surface constitutive relations

The algorithm begins with the trial elastic prediction:

$$\sigma_{n+1}^\text{trial} = \sigma_n + D\dot{\varepsilon}_{n+1}$$

(3-78)

Violation of yield is checked for all plasticity surfaces using the following conditions:

If $$F_a(\sigma_{n+1}^\text{Trail}, \kappa_{hn}) \leq 0$$, for all values of $$\alpha \in \{1, 2, \ldots, m\}$$: Elastic step

(3-79)

If $$F_a(\sigma_{n+1}^\text{Trail}, \kappa_{hn}) > 0$$, for at least one value of $$\alpha \in \{1, 2, \ldots, m\}$$: Plastic step

(3-80)

If the condition for yield is not violated, the elastic solution is admissible and the FEM solution can go to the next step.

If the trial stress violates one of the surfaces, only that surface is active, and the plastic corrections are performed according to the mechanism of the active yield surface.

If the trial stress violates several yield surfaces, this does not imply that all those yield surfaces are active. The active surfaces are determined using the Kuhn-Tucker conditions, which imposes positive values of plastic correction. This situation is discussed in detail in Section 3.4.4. The iteration starts with the assumption that all the yield surfaces that are violated by the trial stress are active. After finding the updated state, in each iteration, the sign of the plastic multipliers is checked. Yield surfaces with a negative plastic multiplier are removed from the list of active mechanisms and the process is repeated.

3.5 Conclusion

In this chapter, the theoretical aspects of a continuum-based approach to the simulation of anisotropic jointed materials are discussed. Based on the number and orientation of the joints, different types of elastic and elasto-plastic asymmetries can be simulated using the proposed model. The type of anisotropy can be general: from a transversely isotropic
material to any other type of anisotropy. However, the emphasis is on anisotropies which are induced as a result of existing discontinuity surfaces with preferred orientations.

The adopted procedure is an equivalent continuum approach, where the elastic and elasto-plastic properties of a continuous homogenized material are specified based on the properties and orientation of discontinuity surfaces and the intact material. An elasticity tensor for the equivalent continuum is derived based on the elastic properties of the interfaces and the intact matrix. There are different techniques available for the calculation of the yield and the potential surface of an equivalent material. These techniques are classified as mathematical, experimental, and discontinuous plane of weakness approaches. In mathematical models, stress terms of different orders with unknown coefficients are added to an isotropic yield criterion in order to reflect the anisotropic characteristics. This technique demands insight into the behaviour of the material and the nature of the stress terms that may be involved. Also, the unknown constants should be determined by performing a number of experiments at different anisotropy orientations. The second class of strength criteria are experimental models, in which an isotropic yield function is considered and the material parameters involved in the formulation are defined as functions with respect to the anisotropy angle to reflect the dependence of the strength on the orientation of applied load. However, these laws are purely experimental and are not recommended for general use.

In this research, the elasto-plastic behaviour of the equivalent continuum was simulated using the discontinuous planes of weakness approach, in which, without explicit simulation of interfaces, planes of weakness with specified orientations are assumed to pass through each point of the material.

The numerical technique adopted in this research is the Finite Element Method (FEM). The elastic and inelastic properties of the equivalent continuum are specified at each integration point. It is assumed that a number of plasticity surfaces representing the behaviour of different sets of joints and intact material are present at each gauss point, and the condition of yield is checked separately for all plastic surfaces. The main advantage of the method is that, based on the number, orientation, and characteristics of
the plastic surfaces, different elastic asymmetries and elasto-plastic anisotropies can be simulated. Also, the method does not require the determination of additional material parameters or unknown coefficients.

There are two main disadvantages associated with the equivalent continuum approach presented in this chapter. The first problem is associated with the basic assumption of continuum theory, which is that the deformable body is continuous. As a result of this assumption, the effect of internal length scale, i.e., joint spacing or layer thickness, is disregarded in the formulation. This approximation can introduce large errors into the solution, and will be discussed in detail in the following chapter. The second problem is related to the numerical efficiency of the method. As the number of plastic surfaces increases, simultaneous evaluation of the yield criteria and the return mapping of stress, based on a number of different functions, can become computationally intensive. Also, stability and accuracy of the numerical solution will decrease as the number of yield surfaces increase at each integration point. In these cases it may be appropriate to develop a single anisotropic constitutive model based on the average properties of the material.
Chapter 4
Cosserat Continuum

4.1 Introduction to micropolar and Cosserat theory

In the previous chapter, continuum theory, including its basic assumptions and limitations, has been briefly discussed. In classical continuum theory the effect of micromoments is neglected when deriving the governing equations of the system. Classical continuum theory postulates that the stress tensor is symmetric, a postulation which is sometimes referred to as Boltzman’s Axiom. Micropolar theory, or the theory of asymmetrical elasticity, disregards the symmetry of Cauchy stress and the resulting minor symmetry of elasticity. There are a number of different mathematical models describing the mechanics of micropolar media such as Cosserat and gradient continuum theories (Dyszlewicz, 2004; Forest, 2005).

Cosserat theory was first proposed by the Cosserat brothers at the beginning of the past century (Cosserat, 1909). Later, prominent researchers such as Mindlin (1965), Kroner (1967) and Shaefer (1967) contributed to the theoretical aspects of gradient continuum theories. Micropolar theory is based on the assumption that micromoments exist at each point of the continuum. A direct consequence of this assumption is that the stress tensor is generally not symmetric, and the difference in the shear components of stress is equilibrated by micromoments. In Cosserat theory, one of the mathematical models describing the mechanics of general micropolar continua, each point of the continuum is associated with independent rotational degrees of freedom in addition to translational degrees of freedom. The basic kinematic variables of Cosserat theory are the displacements, the first-order displacement gradients, the microstructural rotations, and the rotation gradients. Higher-order displacement gradients are not considered.
The first numerical application of Cosserat theory was on the localization of shear bands in granular materials (Mühlhaus & Vardoulakis, 1987), (de Borst, 1993), (de Borst & Mühlhaus, 1992) and (Dietsche et al., 1993). Generalized homogenization procedures for granular materials and their comparison with discrete equations have been discussed by Pasternak and Mühlhaus (2005). In materials with a strain softening regime or a non-associated flow rule, the differential equations of the system lose ellipticity in the post-peak range. In FEM, this ill-posedness manifests itself in pathological mesh dependence. Different techniques have been developed to improve the performance of the FEM solution in localization problems, including the application of Cosserat theory. In the Cosserat continuum approach, as a result of the existence of micromoments, a length scale parameter is introduced into the governing equations of the system. Subsequently, in the localization of shear bands, the bandwidth remains narrow. However, if one adheres to classical continuum theory, the use of discrete crack elements or discrete shear band elements are necessary in order to properly model localization in strain softening materials. Recently, Khoei et al. (2007) have worked on the adaptive aspect of the Cosserat solution in localization problems.

The application of micropolar theory to the FEM formulation of layered and blocky materials was proposed by Mühlhaus (1993) and subsequently, was continued by Adhikary and Dyskin (1997). Rock masses are often intersected by discontinuities, and as a result they do not satisfy the assumptions of classical continuum theory. Due to their discrete nature, it is logical that simulations of layered and blocky rock masses should be carried out using discontinuous models. Alternatively, such simulations could be carried out with the Cosserat continuum theory which provides a large-scale (average) description of such a material by taking into account the effect of system micromoments. The Cosserat model presented in this chapter falls within the equivalent continuum approach discussed in the previous chapter. Inter-layer interfaces or joints are considered to be smeared across the mass, and the effects of joints are implicit in the choice of the stress-strain formulation. However, in contrast to the formulation described in the previous chapter, Cosserat continuum theory introduces the internal length scale of the problem into the governing equations of the system. For example, in the case of layered
media, the bending stiffness of individual layers is introduced into the constitutive equations.

Considering the FEM discretization aspect, a distinct advantage of the Cosserat formulation over the discrete methods is that the simulation can be performed using regular continuum elements with dimensions greater than the layer thickness. This is in contrast to explicit interface models, where the size of the finite elements cannot exceed the layer thickness. In addition, the orientation of elements is independent of the orientation of the joints or interfaces. These advantages are especially attractive where three-dimensional (3D) simulation of geometries with one or multiple arbitrarily-oriented discontinuity surfaces are involved.

There are two main reasons that the applicability of Cosserat theory, or, in general, micropolar continuum theories, should be viewed differently than in the past. First, progress in numerical methods and computational tools makes the processing time of problems with the inherent complexity of micropolar solids only marginally greater than processing time of conventional continuum models. Second, there are clear cases where continuum models cease to represent physical reality in a meaningful manner, namely when the field equations lose ellipticity. Upon the introduction of strain softening in the constitutive equations, the FEM solution becomes mesh dependent and in order to get meaningful results, it is inevitable that higher-order continuum models must be employed (de Borst, 1991).

In this chapter, a full 3D finite element formulation based on Cosserat theory is presented. The 2D finite element formulation of Cosserat layered materials was proposed by Mühlhaus (1993; 1995), and Adhikary and Dyskin (1997). In this research, the formulation is extended to 3D analysis of layered media with a plate-like microstructure. In Section 4.2, fundamentals of the Cosserat continuum, Cosserat rotations, and measures of strain and stress are discussed. Subsequently, the governing equations of the Cosserat continuum and work conjugacy of stress and strain measures are discussed in Section 4.3. Section 4.4 contains a detailed FEM formulation of the 3D Cosserat continuum. Section 4.5 explores the mechanics and the constitutive equations of the layered continuum based
on the theory of plates. It should be emphasized that although the application problems address the issue of the internal length scale of stratified continua, the 3D FEM formulation proposed in this chapter is general and upon introduction of physically representative constitutive equations, can be applied to other problems such as 3D analysis of the localization of shear bands in granular materials. In order to capture the elasto-plastic behaviour of the layered material, the plasticity formulation described in the previous chapter should be modified based on micropolar theory. Section 4.6 develops the FEM formulation for the elasto-plastic behaviour of layered materials in the Cosserat framework. Finally, in Section 4.7 the superior performance of the FEM Cosserat model is demonstrated through a number of numerical examples.

As a closing note to this introduction, it should be mentioned that Cosserat theory provides a continuum description of materials with microstructures, which is the focus of this research. From a numerical point of view, this theory has been applied to the FEM and the FDM analyses of particulate, layered, and blocky materials. Also, Cosserat theory has been applied to the formulation of discrete methods such as DEM analysis. From an analytical point of view, micropolar theories of elasticity and plasticity are broad, ongoing subjects of research in applied mathematics. For a comprehensive review on the subject of micropolar elasticity refer to Dyszlewicz (2004) and references therein.

In addition to the aforementioned line of research, a more general form of Cosserat theory using the concept of directed media has been the focus of much research in this field. In the following section, a brief review of the theory is presented.

### 4.1.1 Theory of directed media

Another line of research within the Cosserat continuum approach is referred to as the theory of directed media where the motion of a 3D continuum is described through a position vector as well a number of director vectors which represent the deformation of each material point. There is no restriction on the number of director vectors and they represent both the rotation and the stretch of material points. The formulation is often expressed in a convected coordinate system, where directors do not necessarily remain orthogonal. In the theory of directed continua, in addition to the laws of conservation of
mass and linear and angular momentum, balance of director momentum should also be satisfied (Rubin, 2000).

In most cases, the physical interpretation of director vectors is not immediately apparent. However, the approach has been applied with much success to the formulation of 3D structural components such as shells, rods, and points. A review of Cosserat theory of shells can be found in Naghdi (1972; 1982). A shell structure is modelled as a surface of zero thickness with a director vector that describes the deformation of a material fibre through the physical thickness of the shell structure.

Theory of Cosserat rods can be attributed to Ericksen and Truesdell (1958), Suhubi (1968), and Green et al., (1974 a, b). Rod structures are assumed to be curved lines which are thin in 2 directions, and director vectors are applied to describe the deformation of cross-sections of a rod.

The first application of a Cosserat point to the continuum problems may be traced to the work of Rubin (1985a, b; 1995). Theory of Cosserat point is also related to Cohen and Muncaster (1988)'s theory of pseudo-rigid bodies and Slawianowski (1982). The generalized form of the Cosserat point theory was developed by Green and Naghdi (1991) to include any number of directors. In the Cosserat theory of points, a three-dimensional element, which is thin in all three directions, is represented by a zero-dimensional point and director vectors that describe the deformation of a 3D small volume.

Cosserat theories of shells, rods, and points have been applied with much success to the thermomechanical, dynamic, and nonlinear numerical analyses of problems concerning the aforementioned structures (Rubin, 1995; 2005). Nadler and Rubin (2003a, b) extended the Cosserat theory of points by considering inhomogeneous strains and thus developed a 3D brick element, characterized by eight directors. Using the Cosserat approach, it is argued that the unphysical shear locking and hour glassing problems associated with classical elements can be avoided (Loehnert et al., 2005). More recent works on this subject include Liu et al. (2007) and Jabareen and Rubin (2008).
A major difference in the implementation of a Cosserat point with the standard procedure in nonlinear elasticity is in the formulation of constitutive equations. For example, in the Cosserat point formulation, the constitutive parameters in the inhomogeneous part of the strain energy functional are determined based on the known analytical and experimental solutions corresponding to different deformation modes such as bending and torsion of the element geometry. Therefore, the balance equations are obtained by a direct approach and the method eliminates the use of numerical integration over the element domain.

4.2 Kinematics of Cosserat continuum

4.2.1 Cosserat rotations

In this section, a full 3D representation of Cosserat continuum theory is presented in the small deformation framework. Compared to a classical continuum (Malvern, 1969), an enhanced or Cosserat continuum, is obtained by adding a rotation, \( R^c \) to each point of the continuum. Cosserat rotation is defined as the independent rotation of a rigid triad attached to each material point which rotates independently with respect to the material triad. The representation of micropolar rotation in its most general form follows (Steinmann, 1994):

\[
R^c = \exp(spn(\theta^c))
\]

(4-1)

where \( \theta^c \) is the axial vector of rotation, or the independent rotation vector, and defines the axis of rotation with rotation angle \( \theta^c \). \( \theta^c \) can be expressed by:

\[
\theta^c = \theta e_i
\]

(4-2)

where \( e_i \) is the \( i^{th} \) component of the base vector, and rotation angle \( \theta^c \) is defined as:

\[
\theta^c = ||\theta^c||
\]

(4-3)

Note: Within this text bold notation or single sub index represent vectors, while double indices denote tensors of second-order or matrices.
The skew symmetric tensor associated with the axial vector is expressed by:

\[ spn(\theta^c) = e \cdot \theta^c \]  \hspace{1cm} (4-4)

and can be expressed in the following matrix form:

\[
spn(\theta^c) = \begin{pmatrix}
0 & -\theta_3 & \theta_2 \\
\theta_3 & 0 & -\theta_1 \\
-\theta_2 & \theta_1 & 0
\end{pmatrix}
\]  \hspace{1cm} (4-5)

The mathematical definition of the rotation tensor, \( R^c \), is:

\[
R^c = \exp(sp\n(\theta^c)) = \cos(\theta^c) I + \frac{\sin(\theta^c)}{\theta^c} sp\n(\theta^c) + \frac{1 - \cos(\theta^c)}{(\theta^c)^2} \theta^c \otimes \theta^c
\]  \hspace{1cm} (4-6)

The above formula is the exact representation of the finite rotation of a generalized continuum, and, generally, its exact terms cannot be obtained. One exception is where the rotation vector \( \theta^c \) coincides with one of the base vectors, for example, \( e_3 \). In this case the exact terms of Equation 4-6 can be expressed as the following:

\[
R^c = \begin{pmatrix}
\cos(\theta_3) & -\sin(\theta_3) & 0 \\
\sin(\theta_3) & \cos(\theta_3) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (4-7)

The above formula has been used by Adhikary et al. (1999) in the buckling analysis of stratified media.

In the case of finite rotations, \( R^c \) can be approximated by the series expansion of Equation 4-6.

\[
\exp(f(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} ((f(x))^n
\]  \hspace{1cm} (4-8)
Simo et al. (1990), and Büchter and Ramm (1994) have applied various orders of the series expansion of the rotation in an updated formulation of shell elements.

In a small rotation framework, the rotation matrix, $R^c$, is approximated by the following:

$$
R^c \cong I + spn(\theta^c) = \begin{pmatrix}
1 & -\theta_3 & \theta_2 \\
\theta_3 & 1 & -\theta_1 \\
-\theta_2 & \theta_1 & 1
\end{pmatrix}
$$

The geometric interpretation of the Cosserat rotation in a 2D representation, where only $\theta_3$ is relevant, is depicted in Figure 4-1. In 2D, a local rigid cross is associated with each point of the continuum (Mühlhaus, 1995). During deformation, the crosses are displaced by $(u_1, u_2)$ and rotated by $\theta_3$. This figure demonstrates the interaction between the layers of material. If no slip occurs at the interface between layers, then the Cosserat continuum reduces to the classical continuum. However, if slip does occur at the interface between layers, the rigid crosses will have some independent rotation with respect to the material axes.

In Cosserat theory $\theta_3^{rel}$ represents the relative rotation between the material element and the corresponding rigid coordinate cross, as expressed in the following:

$$
\theta_3^{rel} = \omega_3 - \theta_3 \quad \text{and} \quad \omega_3 = \frac{1}{2}(u_{2,1} - u_{1,2})
$$

where $\omega_3$ is the rotation of an infinitesimal element $(dx_1, dx_2)$ of the continuum.
4.2.2 Micropolar strains

In the 3D Cosserat continuum, the small deformation strain measure, \( \gamma \), is defined as follows:

\[
\gamma = F^T + \left( R^e \right) - I
\]  

(4-11)

where \( L \) is the deformation gradient.

Using the above formula, the strain components can be expressed by:

\[
\begin{align*}
\gamma_{11} &= \frac{\partial u_1}{\partial x_1}, \quad \gamma_{22} = \frac{\partial u_2}{\partial x_2}, \quad \gamma_{33} = \frac{\partial u_3}{\partial x_3}, \\
\gamma_{12} &= \frac{\partial u_3}{\partial x_1} + \theta_1, \quad \gamma_{23} = \frac{\partial u_3}{\partial x_2} - \theta_1, \\
\gamma_{13} &= \frac{\partial u_2}{\partial x_1} - \theta_2, \quad \gamma_{13} = \frac{\partial u_3}{\partial x_1} + \theta_2, \\
\gamma_{21} &= \frac{\partial u_1}{\partial x_2} + \theta_1, \text{ and } \gamma_{12} = \frac{\partial u_2}{\partial x_1} - \theta_3
\end{align*}
\]  

(4-12)

The strain measure, \( \gamma \), is expressed in indicial notation as:
\[ \gamma_{ij} = u_{j,i} - \epsilon_{ijk} \dot{\theta}_k \quad (4-13) \]

where \( \epsilon_{ijk} \) is the permutation symbol. It should be recalled that the strain measures in conventional continuum mechanics are expressed by:

\[ \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \text{ where } u_{ij} = \frac{\partial u_i}{\partial x_j} \quad (4-14) \]

Figure 4-2 represents the geometric interpretation of the effect of the Cosserat rotation on the small deformation shear strain components in 2D, where only \( \theta_3 \) is relevant.

### 4.2.3 Micropolar curvatures

In a continuum with microstructure, in addition to the rotation of the rigid triad with respect to the material axes, which is defined as a Cosserat rotation, the variation of rotations of the adjacent triads is a second measure of deformation, referred to as curvature. Curvature is a third-order tensor and has the following mathematical definition (Steinmann, 1994):

\[ \kappa = (R^c)^T (R^c \otimes \nabla) \]

or in the indicial notation:

\[ \kappa_{ij,s} = R^c_{ki,s} R^c_{kj,s} \quad (4-16) \]
However, the third-order curvature measure is anti-symmetric with respect to the interchange of the first two indices, i.e., $\kappa_{ij} = -\kappa_{ji}$; thus it can be reduced to a second-order tensor using the following notation:

$$\kappa_{is} = \frac{1}{2} (e_{ij} R^c_{k,i} R^c_{j,s})$$  \hspace{1cm} (4-17)

By replacing $R^c$ according to Equation 4-9 into Equation 4-17, and by disregarding any higher order terms of rotation, the expression for the second-order curvature tensor becomes:

$$\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix} = \begin{pmatrix} -\theta_{1,1} & -\theta_{1,2} & -\theta_{1,3} \\ -\theta_{2,1} & -\theta_{2,2} & -\theta_{2,3} \\ -\theta_{3,1} & -\theta_{3,2} & -\theta_{3,3} \end{pmatrix}$$  \hspace{1cm} (4-18)

The second-order curvature measure can be expressed in indicial notation as:

$$\kappa_{ij} = -\theta^e_{i,j}$$  \hspace{1cm} (4-19)

The expression in Equation 4-19 is in accordance with the curvature measure derived based on work conjugacy in Appendix 8.

Under a superposed rigid body rotation, $Q$, the second-order strain, $\gamma$, and curvature matrices, $\kappa$, are transformed into the following form:

$$\gamma = Q \tilde{\gamma} Q^T \text{ and } \kappa = Q \tilde{\kappa} Q^T$$  \hspace{1cm} (4-20)

where $Q$ is the second-order orthogonal rotation tensor.
4.3 Governing equations, micropolar stress, and micropolar couple stress

Micropolar or Cosserat theory assumes that micromoments exist at each point of the continuum. In Cosserat theory, equilibrium of forces and equilibrium of moments are expressed in the following form (Truesdell & Tupin, 1960):

\[
\sigma_{ij,i} + b_j = 0 \tag{4-21}
\]

\[
m_k + \mu_{kj,j} + e_{kij}\sigma_{ij} = 0 \tag{4-22}
\]

where \( b \) is the body force, \( m \) is the body couple moment, and \( \sigma \) and \( \mu \) are Cosserat stress and Cosserat couple stress, or moment stress, respectively. The stress tensor \( \sigma \) is analogous with the Cauchy stress of the classical continuum. Figure 4-3 and Figure 4-4 show the representation of stress and couple stress measures in 3D and 2D, respectively.

The first subscript of the stress refers to the direction of the surface normal pertinent to the surface on which the stress acts. The second subscript of the stress refers to the direction that the stress acts. The first subscript of the couple stress (or moment stress) refers to the axis around which it rotates, while the second subscript denotes the surface on which the moment stress acts. The notation adopted for stress tensor components is similar to the standard notation used in classical continuum theory; however, it is different from the notation of some of the previous works on Cosserat theory referred in this dissertation (Mühlhaus, 1993; 1995; Dawson & Cundall, 1993; Dai et. al, 1996). The notation adopted for couple stresses is compatible with most literature on Cosserat theory, however, it differs from the standard notation used in plate theory, where the moment subscript refers to the stress components by which the moments are produced (see Appendix 9).

In the absence of body couple moments, and when couple stress terms are self equilibrated (Dai et al., 1996), Equation 4-22 reduces to:

\[
e_{kij}\sigma_{ij} = 0 \text{ or } \sigma_{ij} = \sigma_{ji} \tag{4-23}
\]
The condition expressed by Equation 4-23 implies that the symmetry of the Cauchy stress and its work conjugate strain measure is preserved, and the Cosserat continuum reduces to the classical continuum. Symmetry of stress and strain tensors also leads to minor symmetry of elasticity. The micropolar theory of elasticity, or theory of generalized continua, disregards this assumption.

The stress vector or stress traction and the couple stress vector or moment traction are defined according to:

\[ t_\sigma = \sigma \cdot \mathbf{n} \quad \text{and} \quad t_\mu = \mu \cdot \mathbf{n} \]  

(4-24)

where \( \mathbf{n} \) is the normal vector to the surface.

Using the principle of virtual work, it can be concluded that \( \sigma \) and \( \mu \) are the work conjugate measures of the strain measure \( \gamma \) and curvature measure \( \kappa \) defined in the previous sections. The strain measures can be obtained directly from the equilibrium equations; details are provided in Appendix 8.

Under the superimposed rigid body rotation, \( Q \), the second-order stress and couple stress matrices are transformed by the following:

\[ \sigma = Q\hat{\sigma}Q^T \quad \text{and} \quad \mu = Q\hat{\mu}Q^T \]  

(4-25)

where \( Q \) is the second-order orthogonal transformation matrix.

Figure 4-3 3D representation of stress and couple stress measures.
4.4 Finite element formulation

In the FEM formulation of a Cosserat continuum, each node is associated with three displacement and three rotational degrees of freedom. The vector of nodal degrees of freedom is defined as:

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{\theta}
\end{bmatrix} =
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix}
\]  

Using a notation similar to Voigt notation, the second-order strain and curvature tensors can be expressed in the following vectorial form:

\[
\begin{bmatrix}
\mathbf{\gamma} \\
\mathbf{\kappa}
\end{bmatrix} =
\begin{bmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{21} \\
\gamma_{22} & \gamma_{23} & \gamma_{31} & \gamma_{32} \\
\gamma_{33} & \gamma_{32} & \gamma_{31} & \gamma_{21}
\end{bmatrix}
\]  

Finally, using FEM discretization techniques and the interpolation function, \( \phi \), the strain and curvature field can be interpolated with respect to the vector of nodal degrees of freedom, \( \mathbf{u} \) and \( \mathbf{\theta} \) through:

\[
\begin{bmatrix}
\mathbf{\gamma} \\
\mathbf{\kappa}
\end{bmatrix} =
B_N
\begin{bmatrix}
\mathbf{\bar{u}}_N \\
\mathbf{\bar{\theta}}_N
\end{bmatrix}
\]  

Figure 4-4 2D representation of stress and couple stress measures.
The operator $B_N$ has a block structure and is expressed in the following form:

$$B_N = \begin{pmatrix} B_{N1} & B_{N2} \\ 0 & B_{N3} \end{pmatrix}$$

(4-29)

with

$$B_{N1} = \begin{pmatrix} \phi_{N,1} & 0 & 0 \\ 0 & \phi_{N,2} & 0 \\ 0 & 0 & \phi_{N,3} \\ 0 & 0 & \phi_{N,2} \\ \phi_{N,3} & 0 & 0 \\ \phi_{N,1} & 0 & 0 \end{pmatrix}, \quad B_{N2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\phi_N & 0 & 0 \\ +\phi_N & 0 & 0 \\ 0 & -\phi_N & 0 \\ 0 & 0 & -\phi_N \end{pmatrix}, \quad \text{and} \quad B_{N3} = \begin{pmatrix} -\phi_{N,1} & 0 & 0 \\ 0 & -\phi_{N,2} & 0 \\ 0 & 0 & -\phi_{N,3} \\ -\phi_{N,3} & 0 & 0 \\ -\phi_{N,2} & 0 & 0 \\ 0 & -\phi_{N,1} & 0 \end{pmatrix}$$

(4-30)

where $\phi_N$, the shape function for the $N^{th}$ node, is used for interpolation of both the displacement field and the rotation field.

In small deformation analysis, the internal force is defined as:

$$F^{\text{int}} = \int_V B_N^T [\sigma, \mu] dV = \int_V (B_{N1}^T \sigma + B_{N2}^T \sigma + B_{N3}^T \mu) dV$$

(4-31)

Finally, the material stiffness matrix is defined in the following form:

$$K_{NM}^{\text{mat}} = B_N^T DB_M$$

(4-32)

where $D$ is a block diagonal matrix which relates the stress and couple stress measures to their work conjugate measures: strains and curvatures, respectively, through the constitutive laws $D_1$ and $D_2$. In the local coordinate system, the constitutive laws are defined as:
Using Equations 4-32 and 4-33, the material stiffness matrix can be expressed in the following form:

\[
K_{NM} = \begin{bmatrix}
B^T_{1N}D_1B_{1M} & B^T_{1N}D_1B_{2M} \\
B^T_{2N}D_1B_{1M} & B^T_{2N}D_1B_{2M} + B^T_{3N}D_2B_{3M}
\end{bmatrix}
\]  
(4-34)

From Equation 4-34, it is clear that the stiffness matrix remains symmetric, provided that \( D_1 \) and \( D_2 \) are symmetric.

### 4.5 Constitutive equations

For the derivation of the components of the elasticity tensor, the equivalent continuum concept, discussed in Section 3.4.2, is applied. It is assumed that the intact material and all sets of joints interact similar to a number of springs in series. Subsequently, the elastic compliance of these components can be added in order to express the elastic compliance of the equivalent continuum.

In the Cosserat formulation, however, the constitutive equations of the equivalent continuum should be modified based on the mechanics of a Cosserat material. Cosserat theory and classical theory differ in the way that shear stresses are distributed and in the way that micromoments are related to their work conjugate curvature measures. The additional Cosserat parameters should be determined based on the mechanical response of a material with a particular microstructure. The successful application of Cosserat theory is due to the fact that the link can be made between the kinetic and kinematic variables of a Cosserat continuum and the physical behaviour of materials with microstructure such as particulate, blocky, and layered material. In the case of a layered material, it was discussed in the introduction (see Figure 1-2), that the bending stiffness of the individual layers plays a significant role in the overall response of the material. In the case of blocky materials, the rotation and bending of the blocks can be significant in
the system. However, in the latter, due to the complicated mechanisms that may arise, derivation of the constitutive equations requires considerable simplifying assumptions. Therefore, applications of Cosserat theory to blocky materials has been less encouraging compared to layered materials in which each layer of the material can be represented by a beam (2D) or a plate (3D) component. In the following sections, the 3D elasticity tensor for a Cosserat layered material is derived. Also, a general discussion on the mechanics of materials with a blocky structure is provided.

### 4.5.1 Layered plates

When deriving the constitutive equations for a layered continuum, it is assumed that the microstructure follows a sequential pattern, with a constant thickness for all layers. Using the concept of the equivalent continuum, and the mechanical model that was discussed in Section 3.4, the elasticity matrix of a layered material, arranged in Voigt notation, was expressed by Equation 3-44. Two aspects of the derived elasticity tensor should be noted here. First, in the elasticity matrix expressed by Equation 3-44, effects of bending stiffness of the layers were disregarded, which is due to an intrinsic assumption of classical continuum theory that the dimensions of the microstructures are small compared to the dimensions of a representative volume. As a result, the elasticity matrix derived in Equation 3-44 represents the behaviour of a classical transversely isotropic material. Second, due to the symmetry of shear stresses, only one out of each pair of conjugate shear components is stored in the elasticity tensor. If symmetry of the Cauchy stress and its conjugate strain measure is not applicable, the full 9-by-9 elasticity matrix can be expressed in the following form:
Chapter 4  Cosserat Continuum

\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_{11} & G_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_{11} & G_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(4-35)

with:

\[
A_{11} = A_{22} = \frac{E}{1-\nu^2} - \frac{\nu^2(1+\nu)^2}{1-\nu^2 + E/hk_n}, \quad A_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu) - \frac{\nu}{E/hk_n}},
\]

(4-36)

\[
A_{12} = A_{21} = \frac{\nu E}{(1+\nu)(1-2\nu) + \frac{\nu}{hk_n}}, \quad A_{31} = A_{31} = \frac{E h k_n}{(1-\nu)(2hk_n\nu^2 - (1+\nu)(E + h k_n))},
\]

\]

\[
G_{11} = 1/\left(1 + 1/G + \frac{1}{hk_s}\right) = \frac{G h k_s}{G + h k_s}
\]

where \(E\) and \(\nu\) correspond to Young’s modulus and Poisson’s ratio, respectively of the intact material comprising the layers. \(G\) is the shear modulus of the intact material, \(h\) is the layer thickness, and \(k_n\) and \(k_s\) are the normal and shear stiffness of the interface. In the above equations, all components are similar to those expressed by Equation 3-43. The only difference is in the structure of the matrix in which all 4 shear coefficients are stored.

In the Cosserat continuum, shear stresses are not symmetric. Also, in addition to the true stress tensor, a couple stress tensor is assumed to exist at each point of the material. In order to derive the additional material parameters that are involved in the elasticity tensor of a layered material, the mechanical model of a stack of interacting plates is considered. Figure 4-5 shows a representative element of a Cosserat layered material and the non-
zero stress and couple stress measures acting on it. The intention of this section is to derive a set of constitutive parameters that describe the behaviour of this element of material. It is clear that the elastic interaction between the layers is determined by the elastic properties of interface, i.e., \( h k_n \) and \( h k_s \), and is accounted for in the coefficients of the elasticity matrix expressed by Equation 4-36.

The mechanical response of each layer of the material is similar to a plate. A single thin plate, with normal vector \( e_3 \), is shown in Figure 4-6.

![Figure 4-5 3D representative volume of a Cosserat layered continuum and the nonzero stress and couple stress components acting on it.](image1)

![Figure 4-6 3D representation of mechanics of a single plate.](image2)

The mechanical behaviour of a plate can be represented by the following assumptions:
\[ \hat{\theta}_1 \neq 0, \hat{\theta}_2 \neq 0, \text{ and } \hat{\theta}_3 = 0 \]  

(4-37)

A direct consequence of the assumptions expressed by Equation 4-37 is that all curvature measures corresponding to \( \hat{\theta}_i \) can be neglected, i.e.:

\[ \hat{\kappa}_{31} = \hat{\kappa}_{32} = \hat{\kappa}_{33} = 0 \]  

(4-38)

In plate theory, in addition to the bending moments, twisting moments need to be accounted for. The curvature changes to the deflected middle surface are expressed by (refer to Appendix 9):

\[ k_x = -\frac{\partial^2 w}{\partial x^2}, \quad k_y = -\frac{\partial^2 w}{\partial y^2}, \quad \chi = -\frac{\partial^2 w}{\partial x \partial y} \]  

(4-39)

where \( \chi \) represents the warping of the plate, and is equal to zero if the twisting mechanism is neglected. For a brief review of the theory of thin plates refer to Appendix 9. In the Cosserat formulation of materials with plate-like microstructures, the curvature measures due to the twisting of the section are represented by \( \hat{\theta}_{ij} \), while the curvature measures due to the bending are represented by \( \hat{\theta}_{ij} \). Finally, in the following derivation of the constitutive equations, the membrane behaviour is disregarded.

Considering the mechanics of a plate depicted in Figure 4-6, it can be interpreted that part of each shear stress component across the thickness of the layer, e.g., \( \tilde{\sigma}_{13} \), is in equilibrium with its conjugate shear component, i.e., \( \tilde{\sigma}_{31} \), and is related to the corresponding strain components through the shear coefficient of the equivalent continuum, \( G_{11} \), expressed by:

\[ \tilde{\sigma}_{31} = G_{11}(\hat{\gamma}_{13}+\hat{\gamma}_{31}), \quad \tilde{\sigma}_{32} = G_{11}(\hat{\gamma}_{23}+\hat{\gamma}_{32}) \]  

(4-40)

where \( G_{11} = 1/(G + \frac{1}{h_k s}) = \frac{G h_k s}{G + h k_s} \)
However, the stress occurring across the thickness of the layer has a contribution from the bending of the layers, which is related to $\tilde{\gamma}_{13}$ through the shear modulus of the individual layers, $G$. Thus, the stress component in the direction of the layers then can be expressed as:

$$\tilde{\sigma}_{13} = G_{11}(\tilde{\gamma}_{13} + \tilde{\gamma}_{31}) + G\tilde{\gamma}_{13}, \quad \tilde{\sigma}_{23} = G_{11}(\tilde{\gamma}_{23} + \tilde{\gamma}_{32}) + G\tilde{\gamma}_{23}$$

(4-41)

In Cosserat theory, the couple stress measures or micromoments also need to be defined with respect to their work conjugate curvature measures. Using the mechanics of a single plate element (see Appendix 9), it can be concluded that the nonzero couple stress measures in a layered medium represented in Figure 4-5 are $\tilde{\mu}_{21}$ and $\tilde{\mu}_{12}$ which are due to the bending mechanism and $\tilde{\mu}_{11}$ and $\tilde{\mu}_{22}$, which are due to the twisting mechanism.

The nonzero bending couple stresses, $\tilde{\mu}_{21}$ and $\tilde{\mu}_{12}$ can be related to the curvature of the system through the bending stiffness of the interacting layers, as follows:

$$\tilde{\mu}_{12} = B(\tilde{\kappa}_{12} + \nu\tilde{\kappa}_{21}) \quad \text{and} \quad \tilde{\mu}_{21} = B(\tilde{\kappa}_{21} + \nu\tilde{\kappa}_{12})$$

(4-42)

where

$$B = \frac{Eh^2}{12(1-\nu^2)} \left( \frac{G - G_{11}}{G + G_{11}} \right)$$

In the above formulation, the Poisson effect on the bending moments has been taken into account (Timoshenko & Goodier, 1970; Szilard, 2004).

The curvature measures $\tilde{\kappa}_{11}$ and $\tilde{\kappa}_{22}$ are analogous with the curvature measure $\chi$ in plate theory. Thus, the twisting couple stresses are related to their corresponding curvature measures through:

$$\tilde{\mu}_{11} = (1-\nu)B(\tilde{\kappa}_{11}) \quad \text{and} \quad \tilde{\mu}_{22} = (1-\nu)B(\tilde{\kappa}_{22})$$

(4-43)

By assuming zero values for these components, the effect of twisting moments in a plate structure is disregarded in the formulation, which can lead to considerable errors in the Cosserat solution.
Using the above arguments and assuming isotropic behaviour for the individual plate, \( \mathbf{D}_1 \) and \( \mathbf{D}_2 \) matrices, defined by Equation 4-33, are expressed in the following forms:

\[
\mathbf{D}_1 = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G_{22} & G_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G_{11} & G_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{22} & G_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{11} & G_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G & G \\
0 & 0 & 0 & 0 & 0 & 0 & G & G \\
\end{bmatrix}
\]  

(4-44)

and

\[
\mathbf{D}_2 = \begin{bmatrix}
(1-\nu)B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1-\nu)B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
[\mathbf{0}]_{2\times7} & [\mathbf{0}]_{7\times2} & \begin{bmatrix}
B & \nu B \\
\nu B & B \\
\end{bmatrix}
\end{bmatrix}
\]  

(4-45)

where coefficients of \( A_{ij} \) and \( G_{11} \) are similar to those in a classical material and are expressed by Equation 4-36, and additional Cosserat parameters are:

\[
G_{22} = G + G_{11} \quad \text{and} \quad B = \frac{Eh^2}{12(1-\nu^2)} \left( \frac{G - G_{11}}{G + G_{11}} \right)
\]  

(4-46)

Finally, it should be noted that the rotation measure \( \tilde{\theta}_i \) and the corresponding rotation measure in plate theory, are opposite in signs. So, in order to use the above proposed
constitutive equations, it is necessary to change the sign of the first column of $B_{x3}$ in Equation 4-30.

In the proposed constitutive formulation, two limit cases are of special interest. If plates are non-interacting then $k_n \to \infty$ and $k_s \to 0$, and the bending stiffness of the system is equal to the sum of the bending stiffnesses of the individual layers. In the case where $k_s \to \infty$, the second parenthesis of $B$ expressed by Equation 4-46 will approach zero, and the effect of bending stiffness can be neglected. In this case, the Cosserat formulation reduces to the classical continuum formulation represented in the previous chapter.

For a comprehensive study on the derivation of Cosserat parameters in a layered beam, and the 2D elasticity tensor one can refer to Zvolinski and Shkhinek (1984) and Mühlhaus (1995). In this thesis, the technique proposed in these references is extended to 3D.

The major difference between 2D and 3D formulations of the Cosserat continuum lies in the complications created by the 3D nature of the rotation vector. In a 2D analysis, the local coordinate of the Cosserat rotation always coincides with the out of plane axis of the global coordinate system. However, in a 3D analysis of an arbitrarily-orientated plate or beam, in spite of the simplifications involved in the 3D formulation of Cosserat rotations, generally, the projection of the rotation vector in global coordinates results in 3 components. Consequently, for an arbitrary oriented plate or beam, all 9 components of the curvature tensor should be preserved in the FEM formulation. Therefore, under a rigid body rotation, stress, couple stress, strain, and curvature tensors follow the transformation rule of second-order tensors, and transformation of $\hat{D}_1$ and $\hat{D}_2$ follows the transformation of fourth-rank tensors. This is in contrast to 2D formulation where curvature transformation follows the rule of transformation of a first-order tensor or a vector quantity and therefore, transformation of $\hat{D}_2$ follows the transformation rule of a second-order tensor.
4.5.2 Layered beams

A beam is a simplified version of a plate, in which the effect of rotation around one of the axes is neglected. Figure 4-7 shows the geometry of a beam aligned in the $\vec{e}_1$ direction in 3D space. It can be inferred from the figure that the only relevant curvature measures are $\kappa_{21}$ and $\kappa_{23}$. For a beam structure, the curvature measure, $\kappa_{23}$, is zero, and consequently, the bending coefficient corresponding to that direction is zero. Finally, the couple stress measure, $\mu_{21}$, can be related to the curvature, $\kappa_{21}$, through the bending stiffness of the beam, $B$.

![Figure 4-7 2D representation of a layered beam.]

4.5.3 Blocky materials

In deriving the constitutive relations of blocky materials, a sequential pattern with a brickwork structure is considered. Cosserat continuum theory in this case accounts for relative rotation between the adjacent blocks.

The mechanical behaviour of materials with such a structure is rather complicated; however, simplifying assumptions are adopted so that the most salient aspects are considered in the formulation. It is assumed that the side length of block, $a_b$, is at least 2 or 3 times larger than the side length, $b_b$, (see Figure 4-8.) Therefore, it can be assumed that moments are exclusively transferred across the block interface parallel to the $x_i$ direction (Mühlhaus, 1993).

The vertical displacement and rotation rates of the centre of the block (2) with respect to displacement and rotation of block (1) can be expressed as (see Figure 4-9):
\[ \dot{u}_2' = \dot{u}_2 + \dot{u}_{z_i} n_i \quad \text{and} \quad \dot{\theta}_3' = \dot{\theta}_3 + \ddot{\kappa}_3 n_i \quad (4-47) \]

where the vector \( \mathbf{n} = (a_b, 2b_b) \) joins the centroids of the two blocks. In the case where the relative rotation between the material element and the corresponding rigid coordinate cross is equal to zero, i.e., \( \theta_z = \omega \), the Cosserat continuum reduces to the isotropic classical continuum.

In 2D analysis, the constitutive matrix of a material with blocky microstructure is proposed to be:

![Figure 4-8 Forces and moments acting on a block (Mühlhaus, 1993).](image)

![Figure 4-9 Relative displacement and relative rotation between rigid blocks (Mühlhaus, 1993).](image)
with

$$A_{11} = K + \frac{4}{3}G, \quad A_{12} = A_{21} = K - \frac{4}{3}G, \quad A_{22} = A_{11}$$

$$G_{11} = G + G^e, \quad G_{12} = G - G^e, \quad G_{22} = G_{11}$$

where $K$ is the bulk modulus and $G$ is the shear modulus of the block material. For identification of the bending coefficients, the relative elastic rotation of blocks of a characteristic volume has to be considered. The bending coefficients are then defined as:

$$B_1 = \frac{E2(a_b)^2}{12} \left( \frac{a_n}{2b_n} \right)^2 \quad \text{and} \quad B_2 = \frac{E2(a_b)^2}{12}$$

It should be mentioned that all the parameters used in the constitutive matrix can be adjusted in a practical manner similar to the way they are used in the distinct element method (Dai et al., 1996). Recently, Sulem and Muhlhaus (2007) have closely compared the elastic and elasto-plastic Cosserat continuum formulation of 2D blocky structures with discrete equations.

### 4.6 Cosserat Plasticity

#### 4.6.1 Introduction to micropolar theory of plasticity

By the 1970s, the mechanics of a Cosserat continuum were clearly understood. It was accepted at the time that a Cosserat continuum is mostly suitable for describing the kinematics of granular media. However, applications of Cosserat theory were less
encouraging due to concerns over the meaning and significance of couple stresses in granular media (Brown & Evans, 1972; Bogdanova & Lippmann, 1975).

New interest in the application of Cosserat theory initiated in the mid 1980s because the link was made by Mühlhaus (1986) and Mühlhaus and Vardoulakis (1987) between Cosserat continuum description and localization analysis. One of the most comprehensive works focusing on this subject was published by Steinman (1991), who addressed the mechanics of elasto-plasticity in a general large deformation framework. Later, other researches focused on various aspects of micropolar plasticity (Mühlhaus, 1995), (Grammenoudis et al., 2007). A comprehensive review on elasto-viscoplastic constitutive modelling of generalized continua is given in Forest and Sievert (2003). Part of this work is concerned with plasticity models in Cosserat media, which will be discussed subsequently.

Cosserat elasto-plasticity theory was first applied to the FEM analysis of strain localization problems by de Borst (1991; 1993). In the post-peak behaviour of materials exhibiting a strain softening regime, the FEM governing equations loose their ellipticity. As a result, pathological mesh dependence occurs. In order to overcome the problems associated with the FEM formulation of localization problems, de Borst developed a pressure-dependent $J_2$-flow theory within the Cosserat continuum formulation. The kinematics of Cosserat continua for granular materials are similar to those explained in Section 4.2; however the elasticity matrix, or constitutive equations, are developed to take into account the internal length scale of a particulate media which governs the shear band size. The Cosserat continuum model for granular materials is a rather mature subject (Vardoulakis, 1995). A 2D version of the elasticity matrix for Cosserat particulate materials is provided in Appendix 7. In the plasticity formulation, it is proposed that the definition of $J_2$ be modified to take into consideration the existing micromoments of the system (Vardoulakis, 1995; Manzari, 2004).

Appendix 10 briefly explains the Cosserat plasticity formulation developed for $J_2$ pressure sensitive materials. In the next section the theoretical aspects of the Cosserat theory of plasticity are discussed.
4.6.2 Mathematical models for the micropolar theory of plasticity

Similar to the classical theory of plasticity, the strain measures can be decomposed into an elastic and an elasto-plastic part:

\[ \varepsilon = \varepsilon^e + \varepsilon^p \quad \text{and} \quad \kappa = \kappa^e + \kappa^p \]  

(4-51)

The classical theory of plasticity can be extended to Cosserat media by choosing a potential function, \( Q \), such that:

\[ \dot{\varepsilon}^p = \frac{\partial Q}{\partial \sigma} \quad \text{and} \quad \dot{\kappa}^p = \frac{\partial Q}{\partial \mu} \]  

(4-52)

Two main classes of potentials have been proposed in the past. In the first class, the potential function is expressed as a coupled function of stress and couple stresses. These models involve a single yield function, \( F(\sigma, \mu) \), and a single plastic multiplier, \( \lambda \):

\[ \dot{\varepsilon}^p = \lambda \frac{\partial F}{\partial \sigma} \quad \text{and} \quad \dot{\kappa}^p = \lambda \frac{\partial F}{\partial \mu} \]  

(4-53)

The method described in Appendix 10 and the first extension of von Mises failure plasticity to Cosserat theory (de Borst & Mühlhaus, 1992; de Borst, 1993; Mühlhaus & Vardoulakis, 1987) belong to the first class of methods. In these models, the second invariant of stress is modified to take into account the effect of the micromoments of the system.

In the second class of models, the potential is described as the sum of two independent functions of force stress and couple stress. In these models, two yield surfaces, \( F(\sigma) \) and \( F_c(\mu) \), and two plastic multipliers are considered:

\[ \dot{\varepsilon}^p = \lambda \frac{\partial F}{\partial \sigma} \quad \text{and} \quad \dot{\kappa}^p = \lambda_c \frac{\partial F_c}{\partial \mu} \]  

(4-54)

The method described in Forest et al. (2000) falls within the second group of theories.
4.6.3 Cosserat elasto-plasticity of layered materials

Elasto-plastic formulation for Cosserat layered materials is not clearly addressed in the mathematical framework of plasticity theory. Previous works (Adhikary et al., 1996; Adhikary & Dyskin, 1998; Dawson & Cundall, 1993; Mühlhaus, 1995) have addressed the elasto-plastic behaviour of the interface in 2D analyses. It is suggested that a method similar to the one adopted for classical continua should be applied; however, the details and assumptions of such a method have not been discussed. The possibility of yield in the intact rock has only been considered by Adhikary and Guo (2002) and Adhikary and Dyskin (2007).

In this research, the formulation of the elasto-plastic behaviour of a Cosserat layered material is based on the concept of an equivalent continuum with multiple potential plasticity surfaces, as discussed in the previous chapter. The formulation, however, is modified to consider the effect of asymmetrical stress and micromoments.

**Yield and potential criteria of the interface**

In this section, the elasto-plastic formulation of the classical equivalent continuum proposed in Section 3.4.3 is modified based on Cosserat theory.

Similar to Section 3.4.3, a simple Mohr-Coulomb criterion with a tension cut-off surface has been considered for the interface. The stress tensor is projected to the local coordinates of anisotropy using the appropriate rotation matrix. The shear and tensile failure of joints can be expressed by:

\[
F_1 = \sqrt{\tilde{\sigma}_{31}^2 + \tilde{\sigma}_{32}^2} + \tilde{\sigma}_{33} \tan \varphi - c = 0 \tag{4-55}
\]

and

\[
F_2 = \tilde{\sigma}_{33} - c_i = 0 \tag{4-56}
\]

Similarly, the potential surface with a dilatancy angle \( \varphi_{\text{dil}} \) can be expressed by:
\[ Q_1 = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} + \hat{\sigma}_{33} \tan \varphi_{dil} - c = 0 \]  

(4-57)

and

\[ Q_2 = \hat{\sigma}_{33} - c = 0 \]  

(4-58)

It is clear that both the yield and potential functions of the interface depend only on the normal and shear components of stress occurring on the surface. The only difference between the Cosserat and the classical formulation of the yield and potential functions for the interface is that in the Cosserat formulation these functions depend only on shear terms that are acting on the plane, i.e., \( \hat{\sigma}_{31} \) and \( \hat{\sigma}_{32} \) (see Figure 4-3) and are totally independent of conjugate shear terms acting perpendicular to the layers. Equations 4-55 to 4-58 also suggest that yield strength and potential surfaces of the interface are not direct functions of micromoments. However, the existence of the bending mechanism and micromoments affects the evolution of the stress and shear terms of the system.

**Yield flow and potential flow of the interface**

It is suggested that the plasticity formulation of the Cosserat layered material be carried out in a fashion similar to classical theory. In Cosserat continuum theory, in order to obtain the plastic strain, Equation 4-53 should be applied. Direct differentiation of the flow function with respect to the components of stress and micromoments will result in a plastic flow vector defined by:

\[
\frac{\partial Q}{\partial \sigma} = \begin{bmatrix} 0, 0, \tan \varphi_{dil}, 0, \frac{\hat{\sigma}_{32}}{\sqrt{\sigma_{31}^2 + \sigma_{32}^2}}, 0, \frac{\hat{\sigma}_{31}}{\sqrt{\sigma_{31}^2 + \sigma_{32}^2}}, 0, 0 \end{bmatrix}
\]

\[
\frac{\partial Q}{\partial \mu} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
\]

(4-59)

The suggested flow rule is in accordance with the 2D strain rate suggested by Dawson and Cundall (1993). Equation 4-59 suggests that once slip happens at the interface, the flow rule and the plastic straining on the plane of the joint, e.g., the \( \bar{x}_j, \bar{x}_l \) plane, would
cause no plastic straining on the conjugate plane, $\bar{x}_i\bar{x}_j$. The structure of the flow vector also guarantees that the stress components, $\bar{\sigma}_{31}$ and $\bar{\sigma}_{13}$, would be returned to the yield surface in a symmetric fashion. In other words, the correction that is applied to the shear stress component acting on the surface should be equally applied to the conjugate stress component acting through the thickness. This is due to the fact that part of the stress component acting along the thickness of a layer is in equilibrium with its conjugate shear stress occurring on the surface and the other part, which is due to the bending, is equilibrated by micromoments. Therefore, once the stress tensor is updated, at any integration point, the equilibrium of moments expressed by Equation 4-22 still holds, since the part of stress which is due to slip at the interface is equally reduced in both conjugate shear components. In other words, the difference in shear stress components remains in equilibrium with the corresponding micromoments.

The onset of plastic straining at the interface indicates that the shear force on the plane of the interface has reached the shear resistance of the interface. Subsequently, the elastic shear modulus associated with that direction, represented by $G_{11}$, reduces to an elastoplastic level. This is very similar to the formulation of the Goodman explicit joint model in which, once slip occurs at the joints, the shear modulus, $k_s$, is removed from the system. Considering the elastic matrix, $C$, the structure of the flow rule presented in Equation 4-59 is such that values of all four conjugate coefficients of $G_{11}$ in Equation 4-44 will be reduced to their elastoplastic level.

In Cosserat layered materials, once slip happens at the interface, the corresponding shear resistance reduces, and accordingly, the influence of the bending coefficient becomes more significant in the system. As a result, in addition to the constitutive parameters that relate the increment of stress to the increment of strain, i.e., $D_1$, the elastic parameters that relate couple stresses to curvature measures, i.e., components of the $D_2$ matrix, should be updated to their elastoplastic level. The bending coefficient should be updated based on the elastoplastic shear coefficient, $G_{11}^{ep}$, which can be expressed by:
\[
B_{ep} = \frac{E h^2}{12(1-\nu^2)} \left( \frac{G - G_{11}^{ep}}{G + G_{11}^{ep}} \right)
\]  \hspace{1cm} (4-60)

It is suggested that \( B_{ep} \) be replaced by the bending stiffness of the individual blocks where slip happens at the interface of layers, and \( B_{ep} \) be set to zero when slip happens within the intact material (Adhikary & Guo, 2002).

**Yield and potential criteria of intact material**

In this research, for the intact material, an isotropic \( J_2 \) plasticity model was considered. The strength criteria applied to the intact material, are discussed in Section 3.4.3. The yield criteria for the Cosserat formulation are the same as discussed in that section. However, the formulation should be modified based on Cosserat theory.

In classical plasticity theory, the yield and the potential surfaces are expressed as scalar functions of stress. In Cosserat theory, the contribution of micromoments must be taken into consideration. Only a few investigations (Adhikary & Guo, 2002), (Adhikary & Dyskin, 2007), have considered the possibility of yield in the intact material. In order to capture tensile failure of rocks, Adhikary and Guo (2002) suggested that the effect of micromoments be added to the axial stresses of the layers based on mechanical considerations in beam theory:

\[
\tilde{\sigma}_{11}^{\text{modified}} = \tilde{\sigma}_{11} + \frac{6|\tilde{\mu}_{51}|}{h}
\]  \hspace{1cm} (4-61)

In addition, it is suggested that \( B \) should be set to zero if yielding occurs within the rock layer. There are a number of concerns over the proposed method by Adhikary and Guo (2002).

First, the method suggests incorporating micromoments into the formulation by introducing additional axial forces within the rock layer. The formulation is based on the maximum value of axial stress which occurs at the outer fibre of a layer, and results in overestimation of the tensile axial forces and subsequent underestimation of the tensile
strength of rock. In addition, it is not clear how the return mapping of stress should be performed once the yield criterion is violated.

Second, the method assumes that because the tensile strength of rock is relatively low compared to its compressive strength, in presence of high micromoments the dominant mode of failure is tensile cracking, and the effect of micromoments in shear failure is disregarded. It is apparent that the latter assumption is not necessarily valid, and any procedure for the design of beam components requires simultaneous evaluation of both criteria.

Third, in the proposed method, the direction of tensile failure is assumed to be in the direction parallel to the axis of the layers, which is not valid in many cases. The numerical verifications conducted by Adhikary and Guo (2000, 2002) are all concerned with cases that the tensile failure within the intact rock is known to be in this direction.

Finally, the proposed procedure by Adhikary and Guo (2002) requires that the elasto-plastic calculation be carried out in the local coordinates of the beam. Following their procedure in 3D analyses becomes considerably more complicated due to the nature of the differences in the nature of the Cosserat rotation in 2D and 3D. In 2D analysis, the local out-of-plane axis of the beam, $\bar{x}_3$, coincides with the global $x_3$ axis, as depicted in Figure 4-10. This results in the frame indifference characteristic of the Cosserat rotation, $\theta_3$, in 2D analysis. Consequently, the second-order curvature and micromoment measures are indifferent with respect to the first coordinate system in which they transform rules apply only to the second set of variables. In other words, only the second component of these quantities is sensitive to a change of coordinate system, and subsequently, they transform as first-order tensors. This is also reflected in the transformation of the elasticity tensor of 2D Cosserat materials proposed by Mühlhaus (1995).

In a 2D analysis of Cosserat layered materials, values of the micromoments are zero at the faces aligned along the axis of the beam; as a result, the value of $\mu_{31}$ used in Equation 4-61 can easily be obtained by:
\( \tilde{\mu}_{31} = \sqrt{\mu_{31}^2 + \mu_{32}^2} \)  

(4-62)

In 3D analyses, a second-rank rotation matrix should be applied to transform micromoments from global to the local coordinates of the plate in order to obtain 2 non-zero values of micromoments in the local coordinates. Subsequently, two axial stress components should be modified, and finally, a consistent approach should be applied to modify the bending stiffness in two perpendicular directions in the plate.

Figure 4-10 Frame indifference property of Cosserat rotation in 2D analysis.

Considering the aforementioned shortcomings, in this research, the formulation of plasticity for the intact material is based on the effect of the symmetric part of the stress tensor, and the effect of micromoments is disregarded. Clearly, the applied method is an approximation to the exact solution, since the values of the micromoments directly influence the yield and potential functions, as suggested by beam theory.

The adopted method, however, allows the plasticity calculation to be carried out in global coordinates and with respect to stress invariants in a fashion similar to classical plasticity models. It is shown through a number of application examples that the results were satisfactory. Nevertheless, an attempt was made to devise a similar approach that has been adopted for granular materials (see Appendix 10), in order to take into account the
effect of micromoments in the stress invariants. It is proved that the first stress invariant remains insensitive to micromoments, while values of $J_2$ and $J_3$ are affected by them (see Appendix 11).

**Yield and potential flow**

Similar to classical theory, the yield and plastic flow for $J_2$ materials in micropolar theory can be expressed by:

\[
\frac{\partial F}{\partial (\sigma, \mu)} = \frac{\partial F}{\partial I_1} + \frac{\partial F}{\partial J_2} + \frac{\partial F}{\partial \theta^L} \quad (4-63)
\]

and

\[
\frac{\partial Q}{\partial (\sigma, \mu)} = \frac{\partial Q}{\partial I_1} + \frac{\partial Q}{\partial J_2} + \frac{\partial Q}{\partial \theta^L} \quad (4-64)
\]

Similar to the stress and couple stress vectors, yield flow and plastic potential flow both have eighteen components, nine representing stress terms and nine representing couple stress terms of the system. In the method adopted for the plastic formulation of the intact material, the effect of the couple stress components is disregarded. It should be recalled that the method applied in this research is only an approximation and for obtaining an exact solution, the equilibrium of moment necessitates that along with the true stress tensor, the couple stress tensor must be returned to the yield surface in an elasto-plastic stress update scheme.

**4.6.4 Mohr circle of non-symmetric stress tensor**

In order to appreciate the effect of non-symmetric stresses in micropolar theory, the normal and shear stresses on an arbitrary plane are considered. The 2D geometric representation of the case is depicted in Figure 4-11. The non-symmetric stress results in a Mohr circle which can be expressed by (Vardoulakis, 1995):
\[(\sigma - \sigma_e)^2 + (\tau - \tau_e)^2 = r^2\]  \hspace{1cm} (4-65)

where

\[\sigma_e = \frac{\sigma_{11} + \sigma_{22}}{2}, \quad \tau_e = \frac{\sigma_{12} - \sigma_{21}}{2}, \quad \text{and} \quad r^2 = \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \left(\frac{\sigma_{12} + \sigma_{21}}{2}\right)^2\]  \hspace{1cm} (4-66)

The geometric representation of the Mohr circle is depicted in Figure 4-12. The shift of the centre of the circle along the \(\tau\) axis is a measure of the loss of symmetry. If the circle does not intersect with the \(\sigma\) axis, the principal stresses are no longer real-valued, and are complex conjugates. The condition for the eigenvalues and principal stresses to remain real-valued is:

\[\tau_e^2 \geq r^2 \quad \text{or} \quad (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}\sigma_{21} \geq 0\]  \hspace{1cm} (4-67)

Figure 4-11 Stress and couple stress on an arbitrary plane in a micropolar continuum.

Figure 4-12 Mohr circle representation in a 2D micropolar continuum.


4.7 Examples

4.7.1 Effect of internal length scale and interaction of layers on magnitude of elastic deformation of layered structures

This section demonstrates the capability of the FEM Cosserat model in predicting the deformation of a layered structure. The chosen examples mostly concern layered plate structures for which an analytical solution, or an approximation to the analytical solution, is available. The performance of Cosserat theory is of particular importance when bending of the layers plays a significant role in the deformation. Three examples with different geometry and boundary conditions were chosen. The Young’s modulus and the Poisson’s ratio of the layers are chosen to be 20 GPa and 0.3, respectively. The effect of layer thickness and the interaction of layers were investigated using these examples.

Cantilever layered strip plate subjected to a transverse shear force

This example concerns a layered strip plate subjected to a 1 MPa transverse shear traction applied at the end of the plate. The geometry, boundary conditions, and FEM discretization are shown in Figure 4-13. The plate has a length of 2 m, a height of 0.25 m, and is divided into \( N \) horizontal layers independent of the number of elements used along the thickness. Due to the boundary conditions in the \( x_2 \) direction, the plate behaves as an extruded beam.

\[
\begin{align*}
x_2 &= 0 \\
u_1 &= u_2 = u_3 = 0 \\
\theta_1 &= \theta_2 = \theta_3 = 0 \\
u_2 &= 0 \\
\theta_1 &= 0 \\
x_1 &= 2 \text{ m}
\end{align*}
\]

Figure 4-13 Geometry and boundary conditions for the layered strip plate.

In order to study the effect of slip at the interface, the bending stiffness of the layers, and mechanics of a Cosserat continuum, it is assumed that the beam shown in Figure 4-13 is
divided into 4 layers, each with a thickness of 0.0625 m. The shear stiffness of the interface, \( k_s \), varies from zero to a very large value (1e15 MPa/m).

Figure 4-14 to Figure 4-16 show the distribution of shear stresses and micromoments occurring in the layered strip plate. Figure 4-17 shows the vertical displacement at the tip of the beam vs. \( \log(ks) \). Over the entire range of \( k_s \), the results were compared to the results predicted by the FEM explicit joint model using the Phase² FEM package (Rocscience, 2005). For the case where the shear stiffness of the joints is equal to zero, the extruded beam reduces to 4 non-interacting beams, while for very large values of \( k_s \), the model reduces to a composite beam with a depth of 0.25 m. For these two limit cases, where \( k_s \to 0 \) and \( k_s \to \infty \), the solution is compared to the analytical solution for the deflection of a beam predicted by Timoshenko and Goodier (1970):

\[
\frac{u_s(l)}{2} = \frac{4\tau_l l^3}{Eh^2} \left(1 - \nu^2\right)
\]

Also, for the case where \( k_s = 0 \), the displacements at the tip of the layered beam are compared to the analytical solution of Equation 4-68, for various values of layer thickness. The results are presented in Figure 4-18 and Table 4-1. The values of displacement predicted by the Cosserat solution are normalized by the analytical values predicted by Equation 4-68. In order to better interpret the results, it should be noted that some of the chosen values for the layer thickness, result in partial layers in the given structure, which may not be physically meaningful. It is recalled that in the Cosserat solution, individual layers are not explicitly present, and the layer thickness is an independent input parameter. Considering the specified boundary conditions and the condition of \( k_s = 0 \) on the interface between the layers, in the following examples the Cosserat solution reduces to the solution of a single layer with an equivalent load of \( p_z/N \) and 0.25/N, where \( N \) is not necessarily an integer number.

This example is one of the benchmarks that demonstrate the inability of the classical continuum model to represent the behaviour of stratified media. A similar 2D version of this example was previously solved by Adhikary and Dyskin (1997). It is clear that as the
ratio of $k_s/k_n$ decreases, the behaviour of the layered continuum becomes similar to the bending of a deck of cards.

![Figure 4-14 Distribution of shear stress $\sigma_{13}$, $\sigma_{31}$ and micromoment $\mu_{21}$ in a layered strip plate ($k_s=0$, $\tau = 10$ KPa).](image)

![Figure 4-15 Distribution of shear stress $\sigma_{13}$, $\sigma_{31}$ and micromoment $\mu_{21}$ in a layered strip plate ($k_s=100$ MPa/m, $\tau = 10$ KPa).](image)

![Figure 4-16 Distribution of shear stress components $\sigma_{13}$, $\sigma_{31}$, and micromoment $\mu_{21}$ in a layered strip plate ($k_s=\infty$, $\tau = 10$ KPa).](image)
From a numerical point of view, in this case, using classical anisotropic continuum theory will result in very small values of the shear modulus in the elasticity tensor and subsequent ill-conditioning of the stiffness matrix of the system. In the context of the theory of elasticity, the direct consequence of prescribing zero values of \( k_s \) at a specified
direction is that no shear stress can develop in that direction. Classical theory of elasticity postulates that the stress tensor is symmetric; as a result no shear stress can develop on the conjugate direction, which is the direction of applied shear force.

In the Cosserat formulation, the internal length parameter of the material (i.e., the layer thickness) is considered in the formulation. In this example, the layer bending mechanism sustains the shear load applied to the system. In the case where $k_s$ is zero, the shear stress along the axis of the beam, i.e., $\sigma_{31}$, remains zero, while the shear stress along the thickness of the beam, i.e., $\sigma_{13}$, develops according to the magnitude of the applied force. As $k_s$ increases, the difference in components of conjugate shear stresses reduces, and in the case where $k_s \rightarrow \infty$, the shear stresses become symmetric. In each case, the difference between the conjugate shear stress components is equilibrated by gradient of micromoment along the axis of the beam.

### Table 4-1 Displacements at the tip of the cantilever layered plate subjected to a shear force ($k_s=0$, $\tau = 10 \text{ KPa}$).

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Timoshenko solution (mm)</th>
<th>FEM Cosserat solution (mm)</th>
<th>Normalized displacement of Cosserat solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>145.6</td>
<td>145.4</td>
<td>0.9988</td>
</tr>
<tr>
<td>0.02</td>
<td>36.40</td>
<td>36.37</td>
<td>0.9990</td>
</tr>
<tr>
<td>0.04</td>
<td>9.100</td>
<td>9.097</td>
<td>0.9996</td>
</tr>
<tr>
<td>0.06</td>
<td>4.044</td>
<td>4.046</td>
<td>1.000</td>
</tr>
<tr>
<td>0.08</td>
<td>2.275</td>
<td>2.277</td>
<td>1.001</td>
</tr>
<tr>
<td>0.1</td>
<td>1.456</td>
<td>1.458</td>
<td>1.002</td>
</tr>
<tr>
<td>0.12</td>
<td>1.011</td>
<td>1.014</td>
<td>1.002</td>
</tr>
<tr>
<td>0.14</td>
<td>0.7428</td>
<td>0.7453</td>
<td>1.003</td>
</tr>
<tr>
<td>0.25</td>
<td>0.2329</td>
<td>0.2349</td>
<td>1.011</td>
</tr>
</tbody>
</table>
Convergence test 1:

In this section, the effect of the element size on the accuracy of the FEM Cosserat solution and the relevance of the element size to the layer thickness is investigated. The cantilever strip plate is discretized into 2, 7, 14, and 28 elements along the length and into 1 and 8 elements along the thickness. In each example, the layer thickness varies from 0.01 m to 0.25 m and consequently, the number of layers varies from 15 to 1. It is verified that the values of vertical displacement is totally independent of the number of elements used along the thickness of the beam, however, it depends on the number of elements used along the axis of the beam. The FEM mesh and the corresponding vertical displacements are presented in Figure 4-19 and Table 4-2, respectively.

![Finite element mesh discretization](image)

**Figure 4-19** Finite element mesh discretization applied in the convergence test for a cantilever strip plate.

Convergence test 2:

In the second convergence test, value of $k_s$ is set to 10 MPa/m, and the strip plate consists of 4 layers with a thickness of 0.0625 m. In this case, the beam is divided into 14 elements along the length and 1, 2, 4, 8, and 16 elements along the thickness. Values of total displacement obtained from the Cosserat model are shown in Table 3-1.

In contrast to the values of total displacement in the first convergence test, values of total displacements in Table 4-3 indicate that in this case, the FEM Cosserat solution is sensible to the refineness of the discretization in the direction of the depth of the beam.
Table 4-2 Vertical displacement at the tip of the layered beam for various discretization with $k_s=0$ (convergence test 1).

<table>
<thead>
<tr>
<th>Layer Thickness (m)</th>
<th>Displacement (mm)</th>
<th>B2-1*</th>
<th>B2-8</th>
<th>B7-1</th>
<th>B7-8</th>
<th>B14-1</th>
<th>B14-8</th>
<th>B28-1</th>
<th>B28-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td></td>
<td>1365</td>
<td>1365</td>
<td>1449</td>
<td>1449</td>
<td>1454</td>
<td>1454</td>
<td>1456</td>
<td>1456</td>
</tr>
<tr>
<td>0.02</td>
<td></td>
<td>341.4</td>
<td>341.4</td>
<td>362.3</td>
<td>362.3</td>
<td>363.7</td>
<td>363.7</td>
<td>364.0</td>
<td>364.0</td>
</tr>
<tr>
<td>0.04</td>
<td></td>
<td>85.47</td>
<td>85.47</td>
<td>90.66</td>
<td>90.66</td>
<td>90.97</td>
<td>90.97</td>
<td>91.02</td>
<td>91.02</td>
</tr>
<tr>
<td>0.06</td>
<td></td>
<td>38.07</td>
<td>38.07</td>
<td>40.34</td>
<td>40.34</td>
<td>40.46</td>
<td>40.46</td>
<td>40.47</td>
<td>40.47</td>
</tr>
<tr>
<td>0.08</td>
<td></td>
<td>21.47</td>
<td>21.47</td>
<td>22.72</td>
<td>22.72</td>
<td>22.77</td>
<td>22.77</td>
<td>22.78</td>
<td>22.78</td>
</tr>
<tr>
<td>0.14</td>
<td></td>
<td>7.091</td>
<td>7.091</td>
<td>7.446</td>
<td>7.446</td>
<td>7.453</td>
<td>7.453</td>
<td>7.455</td>
<td>7.455</td>
</tr>
<tr>
<td>0.25</td>
<td></td>
<td>2.279</td>
<td>2.279</td>
<td>2.355</td>
<td>2.355</td>
<td>2.356</td>
<td>2.356</td>
<td>2.356</td>
<td>2.356</td>
</tr>
</tbody>
</table>

* The first number indicates total number of elements used along the axis of the plate, while the second index indicates the total number of elements used along the depth of the plate.

Table 4-3 Vertical displacement at the tip of the layered beam for various discretization with $k_s=10$ MPa/m (convergence test 2).

<table>
<thead>
<tr>
<th>Displacement (mm)</th>
<th>B14-1*</th>
<th>B14-2</th>
<th>B14-4</th>
<th>B14-8</th>
<th>B14-16</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM Cosserat solution</td>
<td>32.76</td>
<td>32.81</td>
<td>32.87</td>
<td>32.88</td>
<td>32.88</td>
</tr>
<tr>
<td>FEM explicit interface solution</td>
<td>N/A</td>
<td>N/A</td>
<td>33.2</td>
<td>33.4</td>
<td>33.7**</td>
</tr>
</tbody>
</table>

* The first number indicates total number of elements used along the axis of the plate, while the second index indicates the total number of elements used along the depth of the plate.
** Values of displacement of the table are not the exact nodal values of FEM model but an average value obtained by the contouring technique.

In this example, both the FEM Cosserat solution and the FEM explicit interface element indicate that as number of elements used along the thickness increases, the values of total displacements increase. The Cosserat solution, independent of the ratio of the element size to layer thickness, exhibits a similar pattern to the FEM explicit joint model. The result can be interpreted based on the fact that in the first case, due to prescription of zero values of $k_s$, the solution becomes totally independent of $x_3$ direction, and the distribution of all nodal and internal variables is uniform for any cross-sectional plane at a fixed distance in $x_1$. As a result, the number of integration points along the thickness of the
plane becomes unimportant. In the second convergence test, variation of displacements and stress is not uniform for each cross-sectional plane. Therefore, both the FEM Cosserat solution and the FEM classical solution with the explicit interface elements become dependent on the number of elements used along this direction.

Based on the two aforementioned tests, two aspects of the Cosserat solution become evident. First, no additional error is introduced into the solution as a result of a relation between the element size and the layer thickness. It can be concluded that the element size is totally independent of the layer thickness and the error which is introduced into the FEM Cosserat solution due to the mesh discretization is the same error that is introduced to the FEM classical solution due to the spatial discretization. Second, the displacement values predicted by the Cosserat solution are as accurate as the displacement values predicted by the FEM classical model using the explicit joint elements.

**Rectangular layered plate fixed at all four edges subjected to a uniform transverse load**

The layered plate in this case has a span of 1 m in both directions, a depth of 0.15 m, and is fixed on all sides. The geometry, boundary conditions, and mesh discretization used in this example are shown in Figure 4-20.

In order to simulate the non-interacting behaviour of the layers, \( k_s \) was set to zero, while \( k_n \) was chosen to have a relatively large value (\( k_n / E = 1 \times 10^{-10} \)). The plate is divided into \( N \) layers, where the number of layers, \( N \), varies from 1 to 15. The Cosserat solution is compared to the analytical solution for a single rectangular plate fixed at all sides obtained from Roark (2002) and Szilard (2004):

\[
Max(w) = \frac{\alpha (p_z / N) l_b^4}{E h^3} \quad \text{with} \quad \alpha = 0.0138 \tag{4-69}
\]

where \( l_b \) is the width of the plate in the shorter direction and \( \alpha \) is a coefficient which depends on the boundary conditions, the ratio of length to width, and the loading of the plate.
Due to the geometry and loading, as well as the non-interacting behaviour of the plates, it is expected that the deformation at the center of the stacked plate will be similar to the deformation of each of the individual layers of the plate with an equivalent thickness of \( h/N \) and a transverse traction load of \( p_z / N \).

As a result, the Cosserat solution was also verified against the FEM solution of a single plate with an equivalent 0.15\( /N \) thickness subjected to \( p_z / N \), where \( p_z \) is the magnitude of the transverse load. It should be noted that the 3D Cosserat formulation proposed in this research does not represent the behaviour of a single plate, but is representative of the
behaviour of a 3D continuum with plate-like microstructure. Similarly, the FEM formulation based on Cosserat theory, is not a replacement for plate or shell elements, but can be used as an alternative to a combination of structural and interface elements. In general, the FEM simulation of layered plates can be performed by discretizing the problem into a combination of solid continuum, beam, plate, or shell elements representing individual layers, and by using interface elements along discontinuity surfaces.

Figure 4-21 and Figure 4-22 show the vertical displacements at the centre of the plate discretized into a coarse mesh and a fine mesh, respectively. Also, the exact values of displacements and the normalized values are presented in Table 4-2 and Table 4-3. All graphs presented in Figure 4-21 and Figure 4-22 indicate the dependence of vertical displacement on $h^2$, where $h$ is the thickness of the individual layers, or the internal characteristic length of this problem. Using a coarse discretization, the results of both the Cosserat FEM solution and the classical FEM solution of a single plate are considerably different from the predictions by the plate theory and cannot be used for interpretation. Using a fine mesh, however, the results show a high level of consistency with the analytical solution based on plate theory presented by Equation 4-69, except for the range of $h > l/10$, where $l$ is the span of the plate. It is discussed in Appendix 9 that if the layer thickness is relatively large compared to the span of the plate, then the theory of thin plates reaches its limits of validity, and application of theory of moderately thick plates, or a full 3D analysis is required. Also, the results obtained from the Cosserat solution using both fine and coarse discretizations indicate that if the twisting moments are disregarded in the constitutive equations, the Cosserat solution considerably overestimates the displacements.

In all cases, the results predicted by the FEM Cosserat solution are very close to the results of the FEM solution for a single plate. Close correlation between the results of FEM Cosserat solution and the results of the FEM solution of a single plate indicates that the spatial discretization error of FEM is carried over to the Cosserat solution. In the following section, the effect of spatial discretization on the error of the Cosserat solution is established through a convergence test.
Figure 4-21 Maximum deflection versus layer thickness for a layered rectangular plate subjected to a uniformly distributed load with $p_z = 100 \text{ kPa}$ and $k_z=0$ (mesh P4-3).

Figure 4-22 Maximum deflection versus layer thickness for a layered rectangular plate subjected to a uniformly distributed load with $p_z = 100 \text{ kPa}$ and $k_z=0$ (mesh P16-3).
Table 4-4 Displacements at the center of a layered plate fixed at all edges with $p_z = 100 \text{ KPa}$ and $k_z=0$ (mesh P4-3).

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Plate theory (mm)</th>
<th>FEM Cosserat solution (mm)</th>
<th>FEM solution of a single plate (mm)</th>
<th>Normalized displacement of Cosserat solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>4.600</td>
<td>1.156</td>
<td>1.400</td>
<td>0.2512</td>
</tr>
<tr>
<td>0.015</td>
<td>2.044</td>
<td>0.8420</td>
<td>0.8284</td>
<td>0.4119</td>
</tr>
<tr>
<td>0.02</td>
<td>1.150</td>
<td>0.6135</td>
<td>0.6012</td>
<td>0.5334</td>
</tr>
<tr>
<td>0.025</td>
<td>0.736</td>
<td>0.4576</td>
<td>0.4474</td>
<td>0.6217</td>
</tr>
<tr>
<td>0.03</td>
<td>0.5111</td>
<td>0.3514</td>
<td>0.3431</td>
<td>0.6874</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1840</td>
<td>0.1561</td>
<td>0.1520</td>
<td>0.8484</td>
</tr>
<tr>
<td>0.06</td>
<td>0.1278</td>
<td>0.1155</td>
<td>0.1123</td>
<td>0.9038</td>
</tr>
<tr>
<td>0.075</td>
<td>0.0817</td>
<td>0.0798</td>
<td>0.0775</td>
<td>0.9762</td>
</tr>
<tr>
<td>0.1**</td>
<td>0.0460</td>
<td>0.0500</td>
<td>0.0485</td>
<td>1.086**</td>
</tr>
<tr>
<td>0.15**</td>
<td>0.0204</td>
<td>0.0269</td>
<td>0.0263</td>
<td>1.317**</td>
</tr>
</tbody>
</table>

** Assumptions of the applied plate theory are not valid in this range.

Table 4-5 Displacements at the center of a layered plate fixed at all edges with $p_z = 100 \text{ KPa}$ and $k_z=0$ (mesh P16-3).

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Plate theory (mm)</th>
<th>FEM Cosserat solution (mm)</th>
<th>FEM solution of a single plate (mm)</th>
<th>Normalized displacement of Cosserat solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>4.600</td>
<td>4.566</td>
<td>4.515</td>
<td>0.9925</td>
</tr>
<tr>
<td>0.015</td>
<td>2.044</td>
<td>2.039</td>
<td>2.016</td>
<td>0.9972</td>
</tr>
<tr>
<td>0.02</td>
<td>1.150</td>
<td>1.152</td>
<td>1.139</td>
<td>1.002</td>
</tr>
<tr>
<td>0.025</td>
<td>0.7360</td>
<td>0.0741</td>
<td>0.7328</td>
<td>1.010</td>
</tr>
<tr>
<td>0.03</td>
<td>0.5111</td>
<td>0.5176</td>
<td>0.5117</td>
<td>1.013</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1840</td>
<td>0.1914</td>
<td>0.1895</td>
<td>1.040</td>
</tr>
<tr>
<td>0.06</td>
<td>0.1278</td>
<td>0.1352</td>
<td>0.1340</td>
<td>1.058</td>
</tr>
<tr>
<td>0.075</td>
<td>0.0817</td>
<td>0.0891</td>
<td>0.0885</td>
<td>1.090</td>
</tr>
<tr>
<td>0.1**</td>
<td>0.0460</td>
<td>0.0532</td>
<td>0.0530</td>
<td>1.157**</td>
</tr>
<tr>
<td>0.15**</td>
<td>0.0204</td>
<td>0.0274</td>
<td>0.0275</td>
<td>1.343**</td>
</tr>
</tbody>
</table>

** Assumptions of the applied plate theory are not valid in this range.
Convergence test

Convergence test 1:

In order to investigate the effect of the mesh discretization on the error of the Cosserat solution and any possible relation between the element size and the layer thickness, the layered plate is discretized into 4, 16, 32 elements along its span and 1 and 3 elements along the thickness. For a number of different layer thickness values, it is verified that the FEM Cosserat solution is independent of the number of elements used along the thickness, however, depends on the refinement of the mesh used in the span of the plate.

Table 4-6 Displacements at the center of a layered plate fixed at all edges with $p_z = 100 \, KPa$ and $k_s=0$ (convergence test1).

<table>
<thead>
<tr>
<th>Layer thickness (mm)</th>
<th>Maximum deflection (mm)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P4-1*</td>
<td>P4-3</td>
<td>P16-1</td>
<td>P16-3</td>
<td>P32-1</td>
<td>P32-3</td>
</tr>
<tr>
<td>150.</td>
<td>0.027</td>
<td>0.027</td>
<td>0.027</td>
<td>0.027</td>
<td>0.027</td>
<td>0.027</td>
</tr>
<tr>
<td>75.</td>
<td>0.079</td>
<td>0.079</td>
<td>0.089</td>
<td>0.089</td>
<td>0.089</td>
<td>0.089</td>
</tr>
<tr>
<td>50.</td>
<td>0.156</td>
<td>0.156</td>
<td>0.191</td>
<td>0.191</td>
<td>0.192</td>
<td>0.192</td>
</tr>
<tr>
<td>25</td>
<td>0.457</td>
<td>0.457</td>
<td>0.741</td>
<td>0.741</td>
<td>0.744</td>
<td>0.744</td>
</tr>
</tbody>
</table>

* The first number indicates total number of elements used along the axis of the plate, while the second index indicates the total number of elements used along the depth of the plate.

Similar to the first convergence test of the previous example, due to prescription of zero values of $k_s$, the FEM Cosserat solution becomes independent of the number of elements used along the thickness. However, similar to the classical FEM solution, the error depends on the number of elements used along the span. It is expected that as the value of $k_s$ increases, the Cosserat solution becomes dependent on the number of elements used along the thickness of the plate. However, it is verified in the second convergence test of the previous example that this error is due to the spatial discretization of the FEM solution, and no special relation exists between the size of the elements used and the layer thickness.
Layered circular plate subjected to a uniform pressure on the top

The structure in this example has a radius, $r_0$, of 1 m and a depth of 0.25 m, and is divided into $N$ non-interacting layers (see Figure 4-23).

![Figure 4-23 Geometry, boundary conditions, and mesh for a layered circular plate subjected to a uniformly distributed load.](image)

The Cosserat solution of this example is verified against the FEM solution of a single plate subjected to $p_z/N$ traction load and to the plate theory prediction for an equivalent traction load of $p_z/N$. The analytical solution for a single plate is obtained from Szilard (2004):

$$\text{Max}(w) = \frac{(p_z/N)r_0^4}{64D} \quad \text{where} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

(4-70)

Graphs of maximum deflection at the center of the circular plate versus layer thickness are depicted in Figure 4-24. Values of maximum deflection predicted by the Cosserat solution, and the normalized values with respect to the analytical solution are presented in Table 4-4.

Similar to the previous examples, the FEM Cosserat solution for a circular layered plate shows a high level of accuracy compared to the analytical solution and the classical FEM solution.
Figure 4-24 Maximum deflection versus layer thickness for a layered circular plate subjected to a uniformly distributed load with $p_z = 100$ KPa and $k_s=0$.

Table 4-7 Displacements at the center of a layered circular plate fixed at all edges with $p_z = 100$ KPa and $k_s=0$.

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Plate theory (mm)</th>
<th>FEM Cosserat solution (mm)</th>
<th>FEM solution of a single plate (mm)</th>
<th>Normalized displacement of Cosserat solution (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>8.531</td>
<td>8.271</td>
<td>8.072</td>
<td>0.9695</td>
</tr>
<tr>
<td>0.04</td>
<td>2.133</td>
<td>2.094</td>
<td>2.045</td>
<td>0.9817</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.8736</td>
<td>0.8668</td>
<td>0.8476</td>
<td>0.9922</td>
</tr>
<tr>
<td>0.08</td>
<td>0.5332</td>
<td>0.5343</td>
<td>0.5231</td>
<td>1.002</td>
</tr>
<tr>
<td>0.12</td>
<td>0.2370</td>
<td>0.2446</td>
<td>0.2400</td>
<td>1.032</td>
</tr>
<tr>
<td>0.25 **</td>
<td>0.0546</td>
<td>0.0661</td>
<td>0.0660</td>
<td>1.211</td>
</tr>
</tbody>
</table>

** Assumptions of the applied plate theory are not valid in this range.

4.7.2 Effect of reduced shear modulus and layer thickness on the pattern of elastic deformation of a jointed material in a uniaxial compressive test

The purpose of this section is to demonstrate how reduced shear stiffness and internal length scale affect the deformation pattern of transversely isotropic materials. The first
part of this section investigates the effect of reduced shear modulus and the subsequent anisotropy in the elastic response of a classical anisotropic material, while the second part focuses on the effect of layer thickness on the response of the problem.

The chosen example is a layered column with a width of 1 m and a height of 2 m. Both horizontal and vertical displacements are fixed at the bottom of the sample. In the Cosserat model, the Cosserat rotational degrees of freedom are also constrained at the bottom. A constant uniform traction of 5 MPa is applied at the top of the column.

The intact rock is assumed to be isotropic with a Young’s modulus of 10 GPa and a Poisson’s ratio of 0.25. In all of the samples, the shear stiffness, \( h_k_s \), of the joints was equal to 194 MPa. In this example of a layered rock, the shear modulus of the joints, \( h_k_s \), is taken to be approximately 20 times smaller than that of the intact rock, while a relatively large value is chosen for \( h_k_n \), causing the elastic properties to be equal in both directions, i.e., \( \tilde{E}_{11} = \tilde{E}_{22} \).

**Elastic response of a transversely isotropic column subjected to uniform pressure on top**

In order to study the effect of interface orientation and bending stiffness independently, the displacements of a classical transversely isotropic material subjected to uniform pressure on top for three different orientations were studied (Figure 4-25). All properties of the models are similar, except for the interface orientation. In these models, anisotropy is introduced only as a result of reduced shear resistance.

Figure 4-25 indicates that when the plane of isotropy is oriented at 30° with respect to a horizontal plane total displacement is aligned parallel to the direction of anisotropy. However, when joints are oriented at 45°, contours of total displacements do not indicate any trend in orientation, and when joints are oriented at 60°, displacements are perpendicular to the orientation of the plane of weakness. The analytical solution for this problem is presented in Appendix 12. It is verified that in the case where \( \tilde{E}_{11} = \tilde{E}_{22} \), the horizontal and total displacements are a function of \( \tan(2\alpha) \).
Elastic response of a layered column subjected to a uniform pressure on top

This section investigates the effect of internal length scale on the deformation of a column of layered rock with a constant shear modulus but varying layer thickness subjected to uniaxial compression (Riahi & Curran, 2007). The results indicate that the layer thickness or the internal length scale not only influences the magnitude of displacements, as reported previously, but it can also change the pattern of predicted deformation.

The examples were solved using three different models: the classical anisotropic model, the Cosserat model and the explicit joint model. For the formulation of the explicit joint model, refer to the Phase² theory manual (Rocscience, 2005).

Figure 4-25 Deformation pattern, based on contours of total displacement, of a transversely isotropic material with plane of isotropy oriented at (a) 30°, (b) 45°, and (c) 60°.
In order to show the effect of layer thickness on the deformational behaviour of the sample, four models with constant shear resistance, $hk_s$, but different values of $h$ and $k_s$, as expressed in Figure 4-26, are considered. The joints are oriented at an angle of 45° with respect to the horizontal axis, and the only varying parameter in the system is the bending stiffness of the material or joint spacing. Figure 4-26(b-d) show the geometry and loading for models with different values of joint spacing used for the explicit joint analysis.

In the equivalent Cosserat formulation joints are not explicitly defined, but values of $h$ were introduced into the formulation through the bending stiffness of the layers. It should be noted that in the FEM Cosserat model, the Cosserat rotations should be constrained at the bottom of the sample.

![Figure 4-26 Boundary conditions and joint spacing of the column of layered rock subjected to a uniform axial pressure.](image)

<table>
<thead>
<tr>
<th>Model</th>
<th>$hk_s$ (MPa)</th>
<th>$h$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>194.3</td>
<td>0.0</td>
</tr>
<tr>
<td>(b)</td>
<td>194.3</td>
<td>0.088388</td>
</tr>
<tr>
<td>(c)</td>
<td>194.3</td>
<td>0.176777</td>
</tr>
<tr>
<td>(d)</td>
<td>194.3</td>
<td>0.353553</td>
</tr>
</tbody>
</table>
Figure 4-26(a) shows the conventional anisotropic model, in which joint spacing is not directly specified. This model was solved using the classical formulation in which only the value of the shear modulus was reduced to the specified value. This case was also solved with the Cosserat formulation, with $B$ set to zero.

Figure 4-27 shows values of maximum displacement of the model predicted by the FEM Cosserat solution and the FEM explicit joint model. Both models show a high level of consistency in the values of maximum deformation of the sample. Also, it is clear that for a constant value of shear modulus, as the layer thickness increases, maximum deformation of the jointed column increases.

Figure 4-28 to Figure 4-31 demonstrate the effect of the internal length scale on the pattern of deformation of the layered sample predicted by the Cosserat solution and the FEM explicit joint solution. Figure 4-28 and Figure 4-29 correspond to the deformed shape of the jointed column, while Figure 4-30 and Figure 4-31 show the contours of horizontal displacement.

![Figure 4-27 Maximum displacements of the jointed column vs. layer thickness.](image)
It is clear that both models predict that as the layer thickness increases, displacements become more aligned toward the direction of the joints. Figure 4-30(a) and Figure 4-31(a) correspond to the case where the effect of the internal length scale is disregarded using the classical anisotropic model and the Cosserat model with coefficient $B$ equal to zero, respectively. Comparison of these figures confirms that as $h$ approaches zero values, the Cosserat model reduces to the conventional anisotropic model. In the case where $h$ approaches zero, the Cosserat model prediction is similar to the model shown in Figure 4-25(b). However, as $h$ increases, the discrepancy between the classical anisotropic model and the Cosserat model increases.
Chapter 4  Cosserat Continuum

The classical equivalent continuum approach, or ubiquitous joint model, based on classical continuum theory is a limit case of the Cosserat approach in which the bending stiffness of the layers is disregarded. Therefore, it is not sensitive to values of $h$, and yields similar results when $h$ varies but $hk_s$ is held constant.

A comparison of the results suggests that the joint spacing affects both the pattern and the magnitude of deformation. It is interesting to note that as layer thickness increases, horizontal displacements of the layers increase, and total displacements become more aligned toward the direction of the joints.

Figure 4-29 Total displacement predicted by the Cosserat model for a column of layered rock.
In Cosserat models, the effect of joint spacing is incorporated into the constitutive equation, while in the explicit joint models, the internal length scale is explicitly modelled by utilizing joint elements which allow for discontinuity of deformation at the interface of the layers.
4.8 Conclusion

In this chapter, the formulation of the FEM equivalent continuum method was modified based on Cosserat theory to take into account the effect of the internal length scale in layered materials. The main contribution of this dissertation is the development of a 3D FEM Cosserat formulation for layered media. Constitutive equations were modified based on the theory of plates.

For the elasto-plastic formulation, the concept of an equivalent plane of weakness was again adopted. Failure of the interface and the intact rock was considered in the plasticity formulation. In the past, few investigations had addressed the issue of the failure of the intact material in the Cosserat simulation of layered continua. In this research a simple
approach to the formulation of plasticity for the cases where failure within the intact matrix occurs is presented. Also, the integration algorithm was modified based on the multi-surface plasticity algorithm explained in the previous chapter.

A number of benchmark examples demonstrate the effect of layer thickness in the elastic analysis of layered media. Application of the proposed FEM formulation to a number of problems suggests that the method can accurately be applied to the analysis of layered structures, such as layered plates and shells. Using the classical FEM formulation, simulation of this class of problems requires explicit definition of interface elements between the layers. The results also emphasize that the internal length scale of layered materials can significantly impact the response of the problem and cannot be disregarded.

Two main aspects of the proposed method need further research. First, the effect of micromoments or couple stresses on yield within the intact material was not addressed in this work. It is suggested that the significance of the moment stresses in the yield of the intact material be investigated. This requires experimental tests to determine how values of micromoments affect the yield point of the intact matrix. It is advantageous to incorporate these effects in terms of stress invariants. Also, this thesis is focused on the development of a FEM formulation for numerical simulation of layered materials. Although, the 3D FEM formulation of the Cosserat continuum proposed in this chapter is general, due to the restrictions on the constitutive equations of blocky materials, the application examples only investigate the effects of internal length scale in layered continua.
Chapter 5
Buckling Analysis of Layered Continua

5.1 Introduction to buckling analysis

One of the major motivations of this dissertation is application of Cosserat theory to simulations of buckling mechanisms that occur in layered materials. This chapter offers an introduction to the FEM analysis of buckling and instability problems. Subsequently, a new FEM formulation for the simulation of the buckling of layers in Cosserat media is presented.

Buckling is a bifurcation point, at which the solution becomes unstable. From a mathematical point of view, buckling occurs when the stiffness matrix becomes singular. There are different methods for carrying out a buckling analysis of structures. In FEM, eigenvalue analysis and geometric nonlinear analysis are applied to buckling and stability analysis of structures. A brief review of each method is presented in the following sections.

5.1.1 Eigenvalue analysis

The objective of an eigenvalue analysis is to examine the stability of the problem under investigation. In an eigenvalue analysis, the solution to the generalized eigenvalue problem expressed by the following equation is sought:

\[ Av = \lambda Bv \]  

where \( A \) and \( B \) are symmetric matrices, and \( \lambda \) and \( v \) are the eigenvalue and eigenvector, respectively. An eigenvalue analysis seeks the answer to the following question:
“Assuming that the steady state solution of the system is known, is there another solution into which the system could bifurcate if it were perturbed from its equilibrium position” (Bathe, 1996).

In FEM eigenvalue analysis, initially, the static solution of the system is obtained, and subsequently, an eigenvalue analysis is performed to find the lowest buckling load of the system. However, an eigenvalue analysis only predicts the onset of bifurcation of an ideal elastic structure, and is not valid beyond this point.

From a computational point of view, an eigenvalue analysis determines the unknown coefficient of the stress stiffness matrix that causes the determinant of the total stiffness matrix to become zero. As a result, an eigenvalue analysis is appropriate where the structure does not sustain large deformation before the bifurcation point, such as the buckling of columns subjected to uniaxial compression loads.

### 5.1.2 Geometric nonlinear analysis

In geometric nonlinear analyses, the FEM formulation is modified based on the large deformation continuum theory. Large deformation analysis is applied where the deformation gradient of the problem, $F$, is not close to identity matrix, $I$. The deformation gradient is expressed by:

$$ F = u \nabla x = x_{i,j} $$

The deformation gradient can be decomposed into a pure rotation and a pure stretch (Malvern, 1969):

$$ F = R^R U = VR^R $$

where $R^R$ is the pure rotation, $U$ is the right stretch tensor, and $V$ is the left stretch tensor.
Equation 5-3 indicates that large deformation can result from finite strain, finite rotation or a combination of two. In the FEM analysis, the large deformation formulation is usually classified into four categories:

**Large strain**: In large strain analysis it is assumed that strains are no longer infinitesimal. Shape changes due to stretches, deflections, and rotations may be arbitrarily large.

**Large rotation**: In large rotation analysis, stretch terms of the deformation gradient are infinitesimal; however, rotations may be arbitrary large. In this case, the structure does not experience a noticeable shape change, and strains can be evaluated using linearized functions. The effect of the rigid body motion of the structure should be taken into account in the calculation.

**Stress stiffening**: In stress stiffening analysis, it is assumed that both strains and rotations are small. A first-order approximation to rotations is used to capture some nonlinear rotation effects (ANSYS Inc., 2006).

**Spin softening**: In spin softening, it is assumed that both strains and rotations are small. This option accounts for the radial motion of a body's structural mass as it is subjected to an angular velocity. Hence, it is an approximation with a large deflection but a small rotation (ANSYS Inc., 2006).

For a comprehensive understanding of the underlying assumptions of each case, one can refer to the following references (Belytschko, 2000), (Bathe, 1996), and (ANSYS Inc., 2006).

In a full, large deformation analysis, all the above effects are intrinsic in the formulation. Usually, a material (initial) or spatial (current) frame of reference is chosen, and calculations are performed with respect to the chosen frame of reference. Stress and strain measures are only meaningful with respect to the coordinate system they are defined in, and the objectivity of the stress measures and related issues should be correctly addressed. In the FEM large deformation analysis, the geometric nonlinear terms are computed iteratively and nonlinear geometric stiffness matrices are added to the material stiffness matrix at each step or iteration of the solution. If a bifurcation point exists, the
geometric stiffness matrices will cause the determinant of the total stiffness matrix to become zero at that point. For FEM analysis to predict instability, a small perturbation is applied to the system. In addition, increments of the applied load should be small enough so that buckling can be captured numerically.

The advantage of large deformation analysis over eigenvalue analysis is that accounts for initial imperfections, plastic behaviour, and any large-deflection response before the bifurcation point. This type of analysis is also valid in a post-buckling regime. However, a full nonlinear geometric analysis is theoretically complicated and computationally intensive. Generally, depending on the physical nature of the problem, some simplifying assumptions are applied to eliminate insignificant terms. In buckling or collapse analysis, if the elastic structure sustains little deformation before the bifurcation point, both strains and rotations remain small. In such cases, only stress stiffening terms contribute to the total stiffness matrix. In this case, the buckling load predicted by the FEM nonlinear analysis should be similar to the prediction derived from an eigenvalue analysis. It should be noted that, beyond the buckling point, deformations are arbitrarily large, and in the post-buckling regime, application of a full geometric nonlinear formulation is necessary.

This research is primarily concerned with the buckling analysis of layered materials where deformations are small prior to collapse. Also, the objective is to find the onset of buckling, without considering the post-buckling behaviour. Therefore, a number of simplifying assumptions are adopted that lead to the buckling stiffness matrix, proposed in the following section. The method adopted here is inspired by Adhikary et al. (1999). However, the aforementioned work does not clearly indicate the assumptions and limitations of its proposed approach. Also, the proposed stiffness matrix is not objective with respect to rotation. One of the major contributions of this chapter is the development of a full 3D buckling matrix for layered plates based on the Cosserat formulation, which takes into account the effect of coupled rotation terms.
5.2 Large deformation strain and rotation measures

Small deformation Cosserat measures of strain are defined in indicial notation as:

\[
\gamma_{ij} = u_{j,i} - e_{ijk} \theta_k
\]  
(5-4)

In a general large deformation framework, the applied strain measure can be expressed by:

\[
\Gamma = \left( R^C \right)^T F = \left( R^C \right)^T R^R U
\]  
(5-5)

in which \( R^C \) is the Cosserat rotation defined by Equation 4-1, and \( R^R \) is the pure rotation part of the deformation gradient. It is interesting to note that the term \( \left( R^C \right)^T R^R \) represents the relative rotation, in other words it is the rotation of the relative angle \((\theta^C - \theta^R)\). When the Cosserat rotation is equal to rotation \( R^R \), then the measure of strain reduces to the right Cauchy-Green tensor.

It is important to notice that if deformations are small, the large deformation measure of strain, \( \Gamma - I \), reduces to the transpose of the small deformation measure of strain, \( \gamma \).

\[
\Gamma_{ij} - \delta_{ij} \rightarrow \gamma_{ji}
\]  
(5-6)

For clarification on the change of indices of the two measures refer to Appendix 8.

In general, the exact solution to the expression of the rotation tensor presented by Equation 4-1 cannot be obtained. In 2D analysis where the only relevant rotation component is \( \theta_3 \), the large rotation matrix \( R^C \) reduces to:

\[
R^C = \begin{pmatrix}
\cos(\theta_3) & -\sin(\theta_3) & 0 \\
\sin(\theta_3) & \cos(\theta_3) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  
(5-7)
In this research, in order to obtain the 3D large rotation tensor, a second-order approximation to the series expansion expressed by Equation 4-8, is applied. It is proposed to use the following expression for the Cosserat rotation matrix:

\[ R^c \approx I + spn(\theta) + \frac{1}{2} spn^2(\theta) \]  

(5-8)

Consequently, for finite rotations, the rotation matrix can be expressed by Equation A-XIII-20 in Appendix 13.

### 5.3 Buckling stiffness matrix

As previously explained, in the derivation of the buckling stiffness matrix, it is assumed that both strains and rotations are infinitesimal prior to reaching the bifurcation point. Subsequently, only the effect of stress stiffness matrix needs to be added to the system.

In addition, it is assumed that curvature measures are infinitesimal, and therefore the geometric nonlinearities associated with curvatures and micromoments are disregarded. This is in accordance with most beam and plate large deformation analyses. Using the aforementioned simplifications, and the second-order approximation to the rotation tensor expressed by Equation 5-8, the 3D buckling matrix is proposed in the following form:

\[
K_{NM} = \begin{bmatrix}
K_{uu} & K_{u\theta} \\
K_{\theta u} & K_{\theta\theta}
\end{bmatrix}
\]  

(5-9)

where

\[
K_{uu} = \int \left[ \begin{array}{ccc}
0 & \sigma_{13} \phi_{N,1} + \sigma_{23} \phi_{N,2} + \sigma_{33} \phi_{N,3} & \sigma_{31} \phi_{M,1} \\
-\sigma_{13} \phi_{N,1} - \sigma_{23} \phi_{N,2} - \sigma_{33} \phi_{N,3} & 0 & \sigma_{21} \phi_{M,2} + \sigma_{31} \phi_{M,3} \\
+\sigma_{12} \phi_{N,1} + \sigma_{22} \phi_{N,2} + \sigma_{32} \phi_{N,3} & -\sigma_{21} \phi_{N,2} - \sigma_{31} \phi_{N,3} & 0
\end{array} \right] dV
\]

\[
K_{u\theta} = \int \left[ \begin{array}{ccc}
0 & \sigma_{13} \phi_{N,1} + \sigma_{23} \phi_{N,2} + \sigma_{33} \phi_{N,3} & \sigma_{31} \phi_{M,1} \\
-\sigma_{13} \phi_{N,1} - \sigma_{23} \phi_{N,2} - \sigma_{33} \phi_{N,3} & 0 & \sigma_{21} \phi_{M,2} + \sigma_{31} \phi_{M,3} \\
+\sigma_{12} \phi_{N,1} + \sigma_{22} \phi_{N,2} + \sigma_{32} \phi_{N,3} & -\sigma_{21} \phi_{N,2} - \sigma_{31} \phi_{N,3} & 0
\end{array} \right] dV
\]
Chapter 5  Buckling Analysis of Layered Continua

\[
K_{\theta \theta} = \int_{V} \begin{bmatrix}
-2(\sigma_{22} + \sigma_{33}) & (\sigma_{12} + \sigma_{31}) & (\sigma_{13} + \sigma_{21}) \\
(\sigma_{12} + \sigma_{21}) & -2(\sigma_{11} + \sigma_{33}) & (\sigma_{23} + \sigma_{32}) \\
(\sigma_{13} + \sigma_{21}) & (\sigma_{23} + \sigma_{32}) & -2(\sigma_{11} + \sigma_{22})
\end{bmatrix} \phi_{M} \phi_{M} dV
\]

and in 2D analyses, the buckling matrix reduces to:

\[
\begin{bmatrix}
0 & \phi_{M} \\
\phi_{M} & 0
\end{bmatrix}
\]

For a detailed explanation of the assumptions, impact of individual terms, and derivation of the above matrices, see Appendix 13. It is clear that by neglecting the stress terms related to the out-of-plane direction, the 3D buckling matrix reduces to the 2D buckling matrix expressed by Equation 5-10. One characteristic of these matrices is their objectivity with respect to rotation. The buckling stiffness matrix expressed by Equation 5-10 differs in the sign of \( \sigma_{22} \) from the 2D matrix proposed by Adhikary et al. (1999). Sulem and Cerrolaza (2002) also suggest that the sign of \( \sigma_{22} \) should be as expressed in Equation 5-10.

5.4 Internal force

In this research in order to find the bifurcation point, a linear buckling analysis is performed. Therefore, it is assumed that the internal force varies linearly in each step of the solution. However, if the buckling stiffness matrix proposed in this research is to be applied in a large deformation FEM analysis, the internal force should be modified based on the nonlinear terms involved in the system, and the definition of the strain measure \( \Gamma \).
In accordance with the assumption that curvature measures are infinitesimal, the internal force in large deformation analysis can be replaced by:

\[ F^{\text{int}} = \int_V \left( G_N^T \Sigma dV + B_{N3}^T \mu \right) dV \]  

(5-11)

where \( \Sigma \) is the work conjugate measure to \( \Gamma \). In the above expression, \( G_N \) is the operator which defines the strain field with respect to nodal displacements, as follows:

\[ \delta \Sigma = G_N \left[ \delta \bar{U}_N \quad \delta \bar{\theta}_N \right] \]  

(5-12)

5.5 Examples

5.5.1 Elastic buckling of stratified columns and plates

The objective of these examples is to evaluate the accuracy and efficiency of the buckling stiffness matrix derived in this research for the stability analysis of stratified structures. As a result, a linear buckling analysis or an eigenvalue analysis is sufficient. However, the proposed matrix can be applied to a nonlinear geometric analysis, provided that other aspects of geometric nonlinearities such as reference configuration, objectivity of stress rate, and nonlinearity in internal force are accounted for. In contrast to most buckling analyses of plates and shells, here, regular continuum solid elements are applied. The elements are 20-noded brick elements with a full order of integration; however, the formulation is based on Cosserat theory. In these examples, a number of layered structures with different geometry and boundary conditions are simulated. The layers have a Young’s modulus of 100 MPa and a Poisson’s ratio of 0.3. In all cases, except for the last example, values of the shear stiffness of the interfaces are set to zero, while a relatively large value of normal stiffness \( \left( k_s / E = 1 \times 10 \right) \) was chosen. In the cases where \( k_s \) is zero, the layers are totally non-interacting, and the solution can be compared to the analytical solution of individual layers. In this case, due to the adopted assumptions, it is expected that the buckling load obtained for each structure will be similar to the analytical value of the buckling load of the individual layers.
In order to find the buckling load, two different methods were applied. First, an eigenvalue analysis was performed, and the stress level at which the full stiffness matrix became singular was determined. In the second approach, a full Newton-Raphson iterative scheme was adopted, and the buckling stiffness expressed by Equation 5-9 was added to the material stiffness matrix in each iteration of the calculation, while a linear variation of internal force was assumed. By adding a small perturbation to the system, the critical load at which the structure became unstable and the deformed shape corresponding to that load was calculated. For verification purposes, two different methods, a direct Gauss Elimination (GE) solver, and an iterative Bi-Conjugate Gradient (BCG) solver were applied. The results obtained from the aforementioned procedures and the analytical solutions, presented below.

**Buckling of layered columns**

This example concerns the buckling analysis of stratified columns. A square column with a width of 1.0 m and a height of 5.0 m is subjected to a uniform pressure at the top. The column consists of 10 individual layers each with a thickness of 0.1 m. The geometry and boundary conditions for each example are shown in Figure 5-1. The column is discretized into 30, 20-noded brick elements.

The analytical solution is obtained from the Euler buckling formulation:

\[
\sigma_{cr} = \frac{E}{12(1-\nu^2)} \left( \frac{\pi h}{l} \right)^2
\]

where \(\alpha\), a parameter which depends on the boundary conditions, is 0.25, 1, and 4 for case (a), (b), and (c), respectively.

Values of the analytical and the FEM buckling load for columns with boundary conditions shown in Figure 5-1(a), (b), and (c) are presented in Table 5-1. It is verified that when \(k_s = 0\), the column’s bifurcation point occurs at the buckling load of the individual layers. As \(k_s\) increases, the buckling load of the column increases. In the limit case, where \(k_s = \infty\), the 10 layers become rigidly attached and the layered structure
behaves as a single 1.0 m thick column. It is clear that due to the ratio of width to length of the thick column, the Euler analysis is not valid for this structure. Similarly, since the thick column undergoes large deformations prior to buckling, the proposed FEM formulation, which is based on the small deformation assumptions, is valid for buckling analysis of a thick column.

Deformed shapes corresponding to Mode I buckling of the columns predicted by the FEM Cosserat solution are depicted in Figure 5-2. The analytical solution for the Mode I buckling shape for boundary conditions specified in Figure 5-1 is expressed by:

\[ u_x = W_1 \left( 1 - \cos \left( \frac{\pi}{2l} z \right) \right) \]  
\[ u_y = W_1 \left( \sin \left( \frac{\pi}{l} z \right) \right) \]  
\[ u_z = \frac{W_1}{2} \left( 1 - \cos \left( \frac{2\pi}{l} z \right) \right) \]  

where \( W_1 \) is an undetermined coefficient.

Figure 5-1 Geometry, mesh, and boundary conditions for the Euler buckling load of a layered column.

(a) (fixed-fee)  (b) (hinged-hinged)  (c) (fixed-fixed)
2D deformed shape and buckling mode for Euler buckling of a layered column predicted by the FEM Cosserat solution and the analytical solution (Equation 5-14).

Table 5-1 Buckling load of a layered column predicted by the FEM Cosserat and analytical solutions.

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Load increment (KPa)</th>
<th>FEM Cosserat solution (KPa)</th>
<th>Analytical prediction (KPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 0.1</td>
<td>0.1</td>
<td>9.00</td>
<td>9.038</td>
</tr>
<tr>
<td>(b) 0.1</td>
<td>1</td>
<td>35.98</td>
<td>36.15</td>
</tr>
<tr>
<td>(c) 0.1</td>
<td>2</td>
<td>136.0</td>
<td>144.6</td>
</tr>
</tbody>
</table>

Buckling of layered plates

From a theoretical point of view, if bifurcation points exist for a plate structure, in the absence of any transverse loading, at the buckling load the plate deflects laterally. The differential equation of plate buckling is rather complicated and can be expressed as (Szilard, 2004):

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left( \bar{n}_x \frac{\partial^2 w}{\partial x^2} + 2\bar{n}_y \frac{\partial^2 w}{\partial x \partial y} + \bar{n}_y \frac{\partial^2 w}{\partial y^2} \right) \tag{5-15}
\]

where \( w \) is the deflection of the plate and \( \bar{n}_x, \bar{n}_y, \) and \( \bar{n}_{xy} \) are the distributed edge loads. Except for some specific boundary conditions, analytical solutions are difficult to obtain. In this section, the FEM Cosserat solutions for buckling of layered plate structures are
verified against the analytical solution for a number of benchmark problems: (a) layered plate, (b) simply supported layered plate, and (c) a simply supported layered rectangular plate subjected to in-plane axial compression.

**Hinged strip plate subjected to in-plane axial compression**

A layered plate structure with a length of 2 m and a width of 1 m was considered. The structure has a depth of 0.25 m and consists of a number of layers of equal thickness. The geometry and boundary conditions of the layered plate are shown in Figure 5-3. Due to the boundary conditions at the sides (i.e., \( u_y = 0 \) and \( \theta_x = 0 \)), the plate behaves as an extruded column similar to the one in Figure 5-1 (b).

Four different layer thicknesses were considered: \( h = 0.01 \text{ m} \), \( h = 0.025 \text{ m} \), \( h = 0.05 \text{ m} \), and \( h = 0.1 \text{ m} \). Table 3-1 shows the buckling load predicted by the FEM Cosserat solution and the analytical solution (Equation 5-13).

The deformed shape corresponding to Mode I buckling, identified by the FEM Cosserat solution for the specified geometry and boundary conditions are depicted in Figure 5-4. The predicted results are verified against the analytical solution given by 5-14-b.

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Load increment (KPa)</th>
<th>Cosserat buckling load (KPa)</th>
<th>Analytical buckling load (KPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.01</td>
<td>2.310</td>
<td>2.259</td>
</tr>
<tr>
<td>0.025</td>
<td>0.01</td>
<td>14.40</td>
<td>14.12</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1</td>
<td>57.60</td>
<td>56.49</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>221.0</td>
<td>225.9</td>
</tr>
</tbody>
</table>
Layered square and rectangular plates supported at all edges subjected to in-plane axial compression

A square and a rectangular plate with uniformly distributed compressive edge loads acting in the $x$ direction were considered. Both plates have a width, $l_b$, of 1 m and a depth of 0.25 m, with an aspect ratio, $l_a/l_b$, of 1 and 2, respectively. The plates are simply supported at all edges. Once again, by setting $k_s = 0$, each structure is divided into a number of non-interacting plates with different thicknesses. In each example, the buckling load and mode were verified against the analytical solution predicted by Szilard (2004). Since the edge load is applied only in one direction, the edge load, $n_s$, can be expressed by:
\[(n_x = \lambda \bar{n}_{x0} = \lambda \frac{\pi^2 D}{l_a^2})\] (5-16)

where

\[D = \frac{Eh^3}{12(1-\nu^2)}\] (5-17)

For a simply supported plate, the deflected surface of the buckled plate can be expressed by (Szilard, 2004):

\[w = \sum_m \sum_n W_{mn} \sin\left(\frac{m\pi x}{l_a}\right)\sin\left(\frac{n\pi y}{l_b}\right), \text{ } m = 1, 2, \ldots \text{ and } n = 1, 2, \ldots \] (5-18)

which satisfies the boundary condition \((w = 0, \; m = 0, \; n = 0)\) along all edges.

Values of \(\lambda\) can be expressed by:

\[\lambda = \left(m + \frac{n^2 l_a^2}{m l_b^2}\right)^2\] (5-19)

For a given aspect ratio, \(l_a/l_b\), the critical load is obtained by selecting \(m\) and \(n\) so that \(\lambda\) in Equation 4-84 is minimized. By substituting values of \(m\) and \(n\) into Equations 5-16 and 5-18, the buckling load and buckling mode associated with \(\lambda\) can be predicted.

\[\begin{array}{c}
\text{Figure 5-5 Simply supported rectangular plate subjected to a compressive axial force.} \\
\text{The geometry, boundary conditions, and the FEM mesh of these examples are shown in Figure 5-6. Table 5-3 shows the buckling load predicted by the FEM Cosserat solution and the analytical solution for the layered plates with various values of layer thickness.}
\end{array}\]
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Figure 5-6 Geometry, boundary conditions, and the FEM mesh for a rectangular layered plate simply supported at all edges, (a) $l_a/l_b = 1$, and (b) $l_a/l_b = 2$.

At all edges:
\[ u_z = 0 \]
\[ \theta_x, \theta_y, \theta_z : \text{free} \]
\[ u_x, u_y : \text{free} \]

Figure 5-7 Deformed shape and mode I buckling displacements for a rectangular layered plate simply supported at all edges, (a) $l_a/l_b = 1$, and (b) $l_a/l_b = 2$. 
Table 5-3 Buckling load of a simply supported layered plate predicted by the FEM Cosserat and analytical solutions.

<table>
<thead>
<tr>
<th>Layer thickness (m)</th>
<th>Load increment (KPa)</th>
<th>FEM Cosserat solution (KPa)</th>
<th>Analytical solution** (KPa)</th>
<th>FEM Cosserat solution (KPa)</th>
<th>Analytical solution** (KPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>37.04</td>
<td>36.15</td>
<td>37.13</td>
<td>36.15</td>
</tr>
<tr>
<td>0.02</td>
<td>1</td>
<td>147.8</td>
<td>144.6</td>
<td>147.7</td>
<td>144.6</td>
</tr>
<tr>
<td>0.04</td>
<td>1</td>
<td>578.9</td>
<td>578.4</td>
<td>578.7</td>
<td>578.4</td>
</tr>
</tbody>
</table>

** Analytical solutions are based on \( m = 1 \) and \( n = 1 \) for the square plate and \( m = 2 \) and \( n = 1 \) for the rectangular plate.

Layered square plate supported at all edges subjected to in-plane axial compression with interacting layers

The geometry, boundary conditions, and FEM mesh of this example are similar to the stacked plate shown in example 5-6(a), except that the height of the layered plate is chosen to be 0.0625 \( m \). In this example, the stacked plate structure consists of 10 layers each with a thickness of 0.00625 \( m \). The layers are interacting with values of \( k_s \) varying from zero to a relatively large value (\( k_s / E = 1 \times 10 \)).

The FEM Cosserat solution for the buckling load of the layered structure for various values of \( k_s \) is presented in Table 5-4 and Figure 5-8. For two limit cases where \( k_s = 0 \) and where \( k_s \rightarrow \infty \), the results can be compared to the analytical solution predicted by Equation 5-16. In the limit case where \( k_s = 0 \), the Cosserat load predicts that the critical load of the layered structure is equal to the critical load of the individual plates with a thickness of 0.00625 \( m \). As values of \( k_s \) increases, the critical load of the structure increases and in the limit case where \( k_s \rightarrow \infty \), the buckling load of the layered structure is equal to the critical load of a single plate with a thickness of 0.0625 \( m \).
Table 5-4 Buckling load of a simply supported layered plate with interacting layers predicted by the Cosserat solution.

<table>
<thead>
<tr>
<th>( K_s ) (Pa/m)</th>
<th>0.0</th>
<th>1e3</th>
<th>1e5</th>
<th>1e6</th>
<th>1e7</th>
<th>1e8</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM Cosserat solution (KPa)</td>
<td>14.48</td>
<td>14.48</td>
<td>15.98</td>
<td>26.97</td>
<td>95.89</td>
<td>529.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( K_s ) (Pa/m)</th>
<th>1e9</th>
<th>1e10</th>
<th>1e15</th>
<th>1e20</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM Cosserat solution (KPa)</td>
<td>1088</td>
<td>1288</td>
<td>1288</td>
<td>1288</td>
</tr>
</tbody>
</table>

Figure 5-8 Buckling load of a layered square plate with interacting layers (h=0.00625m).

5.6 Conclusion

This chapter focused on the stability analysis of layered structures. In this research, a simplified buckling formulation was proposed, in which it was assumed that the layered structure sustains little deformation prior to bifurcation. Therefore, consistent assumptions on the physical nature of the buckling behaviour were adopted, and their effect on the mathematical terms involved in the formulation was discussed. As a result, a full 3D buckling stiffness matrix was proposed which could predict the critical load of a layered structure based on linear analysis. Since only the stress stiffening effect was considered in the formulation, the buckling load of the structure could be obtained using
two different methods. First, an eigenvalue analysis on the total stiffness matrix was performed to obtain the stress state at which the stiffness matrix became singular. Second, a linear buckling analysis was performed, in which the proposed matrix was added to the material stiffness matrix in each iteration of the solution. In this analysis, in order to be consistent with the adopted assumptions, it was assumed that the variation of internal force with respect to the rotation terms is linear. A small perturbation was applied to the system in order to obtain the deformed shape or buckling mode. The formulation was applied to the buckling analysis of a number of 2D and 3D layered plate structures. The mode of buckling and the corresponding buckling load were compared to the analytical solutions. The results predicted by both methods demonstrated a high level of consistency with the available analytical solutions.

The theoretical aspects of this formulation are complete, and it can be applied to the stability analysis of elastic and elasto-plastic problems. However, if the buckling stiffness matrix is to be used in a nonlinear analysis, the corresponding issues should clearly be addressed.

The application examples of this chapter mainly involved elastic structural components for which analytical solutions were available. However, the proposed method can be equally useful in the stability analysis of geostructures, where buckling instability is a dominant mode of failure. Numerical methods such as DEM, which are based on a large deformation concept, are intrinsically capable of stability analysis and capturing the buckling mechanism. Also, by taking into account the effects of dislocation, internal rotation, and local or global instabilities into the formulation, numerical models such as DEM can use simple constitutive models to reproduce the complex elasto-plastic response of natural materials such as rock. Most numerical models based on continuum theory, are not formulated to capture the aforementioned effects. Therefore, the effect of internal rotations, dislocations, and buckling instabilities on the load bearing capacity of geomaterials is often reflected in purely empirical constitutive laws. Also, from a computational aspect, buckling and large deformation analysis is computationally intensive. Clearly, the degree of complexity of the solution depends on the total degrees of freedom of the problem. If layers are to be modelled explicitly, stability analysis
becomes impractical except for small scale problems. The Cosserat formulation provides a suitable platform for such analysis, since the internal length scale which dominates the critical load of a structure is implicit in the governing equations of the system. As a result, stability analysis can be performed with a relatively coarse discretization and considerably fewer degrees of freedom.

The formulation proposed in this research, eliminates the need for empirical laws for capturing the reduced load bearing capacity of layered materials which results from buckling instabilities, and makes the FEM continuum simulation similar to the DEM formulation, in which the natural response of the problem can be captured by an advanced formulation of physical behaviour. In this research, however, the applications were restricted to a number of benchmark elastic problems. The natural extension of this work is the application of the proposed FEM Cosserat formulation to elastic and elastoplastic analysis of practical problems associated with layered geomaterials, including rock mass strength, slope stability, and design of underground excavations.
Chapter 6
Application to Strength Prediction

6.1 Strength prediction of anisotropic jointed rocks

Anisotropy is a characteristic of metamorphic and sedimentary rocks that is due to the existence of schistocity, foliation, cleavage, bedding planes and laminations. These planes of weakness affect the strength and deformational behaviour of rocks depending on the orientation of the applied stresses.

The effects of anisotropy and internal length scale on deformational behaviour of materials were discussed in the previous chapter. This chapter focuses on the application and verification of the FEM Cosserat method adopted in this research to the strength prediction of jointed rocks. Jointed rock masses comprise interlocking angular particles or blocks of hard brittle material separated by discontinuity surfaces which may be coated with weaker materials. The strength of jointed rock masses depends on the strength of both the intact pieces and the discontinuity surfaces. Depending on the number, orientation, and nature of the discontinuities, the intact rock pieces will translate, rotate, and crush in the presence of stresses imposed on the rock.

A complete understanding of this problem presents formidable theoretical and experimental problems. In order to understand the strength behaviour of this class of materials, it is necessary to start with the components of jointed rock masses which control the overall behaviour of the continuum. A comprehensive review of available strength criteria for jointed rocks was presented in Chapter 3. In this chapter, the FEM Cosserat formulation, based on the discontinuous plane of weakness theory, is verified against analytical methods and experimental data.
6.1.1 Uniaxial compressive strength of jointed rocks and effect of anisotropy

This test simulates a uniaxial compressive test on a column of rock with two sets of pre-existing joints. The column has a width of 1 m and a height of 2 m and is subjected to uniaxial compression on the top surface.

The Young’s modulus and Poisson’s ratio of the intact material were chosen to be 20 GPa and 0.3, respectively. The elastic properties of the joints, as well as layer thickness does not have a significant influence on the compressive strength of the sample; the results presented below are obtained using values of $k_n=1000 \text{ GPa/m}$, $k_s=10 \text{ GPa/m}$, and $h=0.1$. It is assumed that the isotropic intact material fails according to Mohr-Coulomb yield criterion with a cohesion of 25 MPa, a friction angle of 40°, and a dilation angle of 40°. Also, it is assumed that a primary set of joints with a cohesion of 10 MPa, a friction coefficient of 20°, and a dilation angle of 20°; and a secondary set of joints with a cohesion of 15 MPa, a friction coefficient of 30°, and a dilation angle of 30° can potentially become activated in the material.

The uniaxial strength of a sample of rock with one set of joints is verified against the analytical solution, also known as Jagear’s single plane of weakness criterion:

$$\sigma_1 - \sigma_3 = \frac{2(c + \sigma_3 \tan \phi)}{(1 - \tan \phi \tan \beta) \sin 2\beta}$$  \hspace{1cm} (6-1)

The definition of the anisotropy angle $\beta$, is shown in Figure 6-1.

Figure 6-1 Definition of anisotropy angle $\beta$. 
Figure 6-2 and Figure 6-3 show a comparison between the FEM solution and the analytical solution for this test, when plastic failure is allowed only for one set of joints.

The uniaxial compressive strength of a material with two sets of joints is presented in Figure 6-4. It is assumed that the secondary set of joints has an angle of 35° with respect to the primary set of joints (e.g., where the first set of joints is aligned vertically, the second set of joints is at 35° with respect to the applied load). Since the intersection of two sets of joints results in a blocky structure, and subsequently, requires reformulation of the constitutive equations of a Cosserat material, the effect of weakness in the elastic properties were disregarded. In this case, it is only verified that the strength of the equivalent material is the minimum strength obtained from the geometric superposition of the graphs of each set of joints. Finally, Figure 6-4 shows the strength versus orientation of the equivalent material, where plastic failure of the intact rock is also considered. The shape of these graphs depends on the strength properties of the intact material and the joints. A comparison of numerical predictions and available experimental data for the strength of anisotropic jointed materials suggest that the U shape graphs are observed when only one orientation of weakness exists within the material.

![Figure 6-2 Uniaxial compressive strength versus anisotropy orientation for joints with a friction angle of 20° and a cohesion of 10 MPa.](image-url)
Figure 6-3 Uniaxial compressive strength versus anisotropy orientation for a jointed material for joints with a friction angle of 30° and a cohesion of 15 MPa.

Figure 6-4 Uniaxial compressive strength versus orientation for equivalent material with two sets of joints.

The numerical simulation also suggests that the shoulder effect is associated with the presence of a relatively weak intact material, which is insensitive to the orientation of the applied load. The shoulder effect cannot be observed when the intact material is strong.
and when its strength does not fall within the range of the maximum-minimum strength of the joints, or when no bedding planes are present in the material. Also, it can be concluded that the wavy shape graphs are usually associated with materials in which there is a secondary plane of weakness present. The conclusions based on the numerical model are consistent with the classification suggested by Hoek (1983), and observations reported by Nasseri (1992).

![Figure 6-5 Uniaxial compressive strength versus orientation for equivalent material with two sets of joints embedded in weak isotropic rock.](image)

**Effect of joint properties on the degree of anisotropy**

The degree of anisotropy is the ratio of the maximum to the minimum unconfined compression strength (ucs) of anisotropic materials. Often, the maximum ucs of an anisotropic material is observed when the load is perpendicular to the joint plane. If a Mohr-Coulomb type behaviour is chosen for the joints, using the proposed method, the degree of anisotropy can obtained from Equation 6-2. From an analytical point of view, the minimum strength always occurs when the weakness plane is oriented at an angle of \((\pi / 4 - \varphi / 2)\), with respect to the maximum principal stress. This plane is the critical plane or plane of failure of an isotropic Mohr-Coulomb material with identical properties. Thus, the degree of anisotropy can be expressed as:
where $\sigma_c$ is the uniaxial compressive strength of the intact rock.

Equation 6-2 clearly indicates that as the cohesion, $c$, of the joints decreases the degree of anisotropy increases. In other words, as the joints become weaker the effect of anisotropy becomes more pronounced in the material. Also, it can be proven that the coefficient $(1 - \tan \varphi \cdot \tan(\frac{\pi}{4} - \frac{\varphi}{2})) \cos \varphi$ is a strictly decreasing function in the range $[0, \pi / 2]$. The maximum of this function is obtained when $\varphi = 0$, while the minimum is obtained at $\varphi = \pi / 2$. Therefore, it can be concluded that as the strength of the joints increases the degree of anisotropy decreases, a result which is consistent with the experimental observation for rocks. It should be noted that the parameters involved in Equation 6-2 are not independent of each other, and usually a decrease or increase in one parameter affects other parameters as well.

**Effect of confining pressure**

The expression for the degree of anisotropy in the presence of a confining pressure, $\sigma_3$, is:

$$R_c = \frac{\sigma_c \left(1 - \tan \varphi \cdot \tan(\frac{\pi}{4} - \frac{\varphi}{2})\right) \cos \varphi}{\sigma_3 + 2(c + \sigma_3 \tan \varphi)}$$  \hspace{1cm} (6-3)

which clearly indicates a decrease in the degree of anisotropy with increasing confining pressure.

**Effect of internal length scale**

FEM Cosserat analysis based on various values of bending coefficient suggest that when sliding on the joint planes is the mode of failure, the internal length scale does not have a
significant effect on the compressive strength of the material. Also, in post peak regime, progressive failure occurs independent of the values of bending coefficient.

### 6.1.2 Comparison with experimental observations

The purpose of this section is to conduct a comparison between the strength predictions of the FEM Cosserat solution against experimental data for natural materials such as rock. As explained in the previous section, the developed model indicates that the effect of internal length on the uniaxial strength of materials is insignificant if sliding on the plane of joints is the dominant mode of failure. A comprehensive study on the effect of anisotropy on the strength and deformation of a number of schistose rocks has been conducted by Nasseri (1992). From an experimental point of view, prediction of the anisotropic responses of the strength and deformation of rocks involves testing the specimens at different joint orientation angles, $\beta$. A review of these observations indicates that maximum failure strength is either at $\beta = 0$ or $\beta = 90^\circ$, with the minimum value usually around $\beta = 30^\circ$.

In a comparison of the results with experimental data, determination of the input parameters for the rock mass is crucial. In this section, the results are compared to the results presented by Saroglou and Tsiambaos (2008). Two different types of gneiss which exhibit strong anisotropy were chosen. The parameters presented in this reference are in terms of Hoek-Brown parameters, and the equivalent Mohr-Coulomb parameters were estimated based on the proposed values (Table 6-1).

| Table 6-1 Hoek-Brown and equivalent Mohr-Coulomb properties of rocks used for the uniaxial compression test. |
|-------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Gneiss A                                        | $66.5$          | $23.2$          | $18$            | $33$            | $9$             | $28$            |
| Gneiss B                                        | $85.7$          | $24.6$          | $24$            | $31$            | $5.5$           | $23$            |

<table>
<thead>
<tr>
<th>$\sigma_c$ (MPa)</th>
<th>$m_b$ (MPa)</th>
<th>$c$ (MPa)</th>
<th>$\phi$</th>
<th>$c$ (MPa)</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_c$</td>
<td>$m_b$</td>
<td>$c$</td>
<td>$\phi$</td>
<td>$c$</td>
<td>$\phi$</td>
</tr>
</tbody>
</table>
The prediction of the FEM Cosserat solution is compared with the experimental data reported for gneiss A and gneiss B in Figure 6-6 and Figure 6-7, respectively.

![Figure 6-6 Uniaxial compressive strength versus anisotropy angle for gneiss A.](image)

![Figure 6-7 Uniaxial compressive strength versus anisotropy angle for gneiss B.](image)

Experimental observations indicate that for a wide range of anisotropic rocks, including the rocks presented in this example, there is a significant reduction in the strength of the
material when joints are oriented parallel to the orientation of the applied load. This reduction is mainly due to the tensile failure on the plane of joints which is accompanied by local instabilities within the material. However, using the FEM method, such effects can not be captured. The discrepancy in the results is due to the fact that, at anisotropy angle of 0, the FEM method predicts the strength of the material based on the shear strength of the intact rock. In order to be able to capture this effect, experimental strength criteria can be used, in which the strength reduction at anisotropy orientations close to 0, is experimentally determined. Also, numerical techniques based on the DEM formulation have proven to be efficient since their formulation accounts for instabilities that occur within the material due to tensile failure. Preliminary studies by the author indicate that, using a full large deformation formulation, the Cosserat solution will be able to capture these effects.

6.1.3 Shear test and effect of layer thickness

In this section the effect of layer thickness in the shear strength of stratified materials is investigated. A sample of layered rock subjected to a uniform shear load on top is considered. The Young’s modulus and Poisson’s ratio of the layer material are chosen to be $20 \text{ GPa}$ and 0.25, respectively. The elastic properties of the joints, as well as the layer thickness vary depending on the tests. In order to study the effect of layer thickness, it is assumed that the joints fail according to a Mohr-Coulomb law in a pure shear mode with a cohesion of 1 MPa, a very small friction angle (0.23°), and a dilation angle of 0°.

The effect of joint spacing on the shear strength of the sample is studied for two cases where joints are oriented horizontally and vertically, using varying values of elastic shear resistance on the plane of joints, ($hk_s=1000 \text{ MPa}$, $hk_s=100 \text{ MPa}$, and $hk_s=10 \text{ MPa}$). Figures 6-8 to 6-10 show the numerical predictions of the shear strength by the FEM Cosserat model for the sample with vertical layers. The graphs suggest that for a constant value of shear modulus, as values of layer thickness increase, the shear strength of the material with vertical layers increases. However, as discussed in the previous chapters, the effect of internal length scale is less pronounced for higher values of $hk_s$. The graphs also suggest that the internal length scale or layer thickness has a significant effect on the
post-peak behaviour. In the case of vertical layers, once failure on the joint planes occurs, the bending mechanism sustains the applied load. This mechanism prevents progressive failure of the material. In the limit case where the cohesion and friction parameters are set to zero, the shear stress on the joint planes reduces to zero, while shear stress across the thickness of the beam equilibrates the applied load. If the bending stiffness of the layers is zero, the shear stresses remain symmetric and progressive failure occurs.

Figure 6-8 Shear strength of layered rock vs. layer thickness for joints oriented vertically ($hk_s=1000\text{MPa}$).

Figure 6-9 Shear strength of layered rock vs. layer thickness for joints oriented vertically ($hk_s=100\text{MPa}$).
Figure 6-11 shows the strength of the sample for the case where the layers are aligned horizontally. The results suggest that when layers are aligned in the direction of the applied shear load, layer thickness has no effect on the strength properties of the material. Also, in the post-peak behaviour, progressive failure occurs independent of values of layer thickness.

Figure 6-10  Shear strength of layered rock vs. layer thickness for joints oriented vertically ($h_{ks}=10$ MPa).

Figure 6-11  Shear strength of layered rock for joints parallel to the direction of the shear load.
From a theoretical point of view, in the limit case with the bending coefficient equal to zero, a Cosserat layered material reduces to a perfectly homogenized transversely isotropic material. In this case, for a material with the aforementioned elasto-plastic properties, the shear strength, whether the joints are parallel or perpendicular to the orientation of the applied load, is similar provided that the joints are frictionless. The conclusion comes from the proposition of symmetry of stress in the absence of any internal length scale within the material.

### 6.2 Conclusion

The strength versus orientation relationship obtained from the FEM Cosserat solution is similar to the graphs presented in Figure 6-2 to Figure 6-5. If slip on the plane of the joints is the mode of failure, the method applied in this research yields similar results to Jaeger’s criterion. Hoek and Brown (1983) state that the “single plane of weakness” theory, as proposed by Jaeger and Cook (1976) and as presented in Figure 6-12a, is sufficient for the prediction of strength when the rock behaves anisotropically due to the presence of a single plane of weakness (e.g. discontinuity plane), but does not describe adequately the strength behaviour of intact rock possessing inherent anisotropy, due to the presence of bedding or foliation. In the author’s view, numerical application of the single plane of weakness theory provides a physically justifiable and practical method, yet comparison of the experimental data with this method indicates that certain mechanisms in jointed rock lead to complicated behaviour that cannot be captured by the proposed method (see Figure 6-12b).

In addition to the numerical complications discussed in Section 3.4, the general shortcomings of the method are briefly summarized below. The most significant shortcoming associated with this method is the range of the anisotropy angle over which shear failure on the plane of joints can occur. This shear failure can happen for \( \beta < 90 - \phi \), while for \( \beta > 90 - \phi \), the strength of the rock mass is determined by the strength of the intact rock, or, in other words, shoulder effects are observed. However, most experimental data suggests a U shape graph with a smooth transition for the range...
of $\beta > 90 - \phi$. Consequently, this method can significantly overestimate the strength of the rock mass in this range. In addition, the friction coefficient does not have a significant effect on the degree of anisotropy, yet significantly influences the shape of the failure curve.

![Figure 6-12 Schematic strength versus orientation graph obtained from: (a) the single plane of weakness theory, (b) experimental observations.](image)

It has been observed that as confining pressure increases, the minimum strength shifts to higher orientations (Nasseri et al., 2003). A similar observation was reported by McLamore and Gray (1967). Using the method adopted in this research, this phenomenon cannot be captured, and minimum strength always happens at an anisotropy angle of $\beta = 45^\circ - \phi/2$. This phenomenon is due to the fact that the Mohr-Coulomb friction angle of rocks is not constant with respect to confining pressure. This effect has also been considered in the equivalent Mohr surface of the Hoek-Brown criterion, which utilizes variable Mohr-Coulomb parameters (Hoek & Brown, 1997) in order to reflect a decreasing friction parameter with increasing values of confining pressure.

The results of this chapter also suggest that the developed FEM Cosserat model does not indicate any correlation between the internal length scale of a layered material and the uniaxial compressive strength. However, the developed model is capable of capturing the effect of joint spacing in the shear strength and the post-peak behaviour of stratified media, which is due to the bending mechanism of the layers. The results predicted for the
simulated shear test explain the difference in shear strength of anisotropic materials at
different orientations of anisotropy. It also suggests that by adding a new physical
parameter, which is bending stiffness of the layers, to the governing equations, the model
becomes capable of predicting a more complex constitutive behaviour. In the shear test,
the significance and the effect of bending of the layers and the subsequent developed
micromoments in the yield strength of the intact material was not investigated, which was
due to two main reasons: First, from physical and experimental points of view, the effect
of micromoments in the yield behaviour of the intact material is not clear. Second, the
plasticity formulation of yield within the intact matrix is based on the true stress tensor,
and the effect of micromoments is disregarded.

To conclude, it should be mentioned that although considerable resources have been
allocated to experimental and analytical methods for strength prediction of anisotropic
rocks, due to the shortcomings explained in 3.3.3, most of them are inapplicable. A brief
review of the most widely used numerical packages presented in Section 8.2 suggests that
most numerical implementations of anisotropic strength criteria are based on either the
single plane of weakness concept or Hill’s modified Mises criterion, discussed in Chapter
3 (Hill, 1964).
Chapter 7
Application to Slope Stability Analysis

7.1 Slope stability analysis using the Shear Strength Reduction (SSR) method

This chapter focuses on the application of the FEM Cosserat method to the Shear Strength Reduction (SSR) analysis of the stability of jointed rock slopes. It is assumed that failure of the equivalent material can result from failure of the joints, failure of intact material, or a combinatory mechanism. A consistent elasto-plastic algorithm which takes into account the failure of multiple plastic surfaces at FEM integration points is adopted. Deformation mode, failure mechanism, and the factor of safety predicted by this approach are verified against discontinuous analyses including the Discrete Element Method (DEM) and the FEM explicit joint model. The results indicate that the Cosserat continuum approach is a reliable tool for identifying various modes of failure for jointed slope problems. In addition, application of a robust elasto-plastic algorithm based on the multi-surface plasticity concept enables the method to predict the safety factor accurately.

7.1.1 Overview of the Shear Strength Reduction Method

SSR has proven to be an efficient tool in the analysis of slope stability problems for materials exhibiting both a discrete and a continuous nature. In this method, a systematic search is performed to find the reduction factor, \( F_s \), that brings the slope to the limit of failure. The reduced shear strength of a Mohr-Coulomb material is described by the following equation:

\[
\frac{\tau}{F_s} = \frac{c}{F_s} + \frac{\sigma_n}{F_s} \tan(\varphi) \quad \text{or} \quad \frac{\tau}{F_s} = c^* + \sigma_n \tan(\varphi^*)
\]  

(7-1)
where \( c^* = c / F_s \) and \( \varphi = \arctan(\tan(\varphi^*) / F_s) \) are factored Mohr-Coulomb shear strength parameters. The factor of safety is the maximum value of \( F_s \) for which the numerical solution remains stable. The problem should be solved for different values of \( F_s \). A sudden increase in values of displacements in the graph of \( F_s \) versus maximum displacement indicates the onset of instability. Also, non-convergence of the solution is suggested to be an indicator of failure in the FEM solution (Hammah et al., 2007).

### 7.2 Examples

Three different examples were chosen to demonstrate the performance of the proposed FEM Cosserat continuum approach to the slope stability analysis of layered rock slopes using the SSR method. The geometry of the slope and the mesh of the FEM Cosserat model are depicted in Figure 7-1. The slope is simulated in 3D, however, the boundary conditions are set so that the response can be compared to the 2D analyses predicted by Phase2 and UDEC. As a result, application of 1 element in the out-of-plane direction is sufficient. In this example, a total of 1113 brick elements (20-noded) are applied, and the total number of degrees of freedom is 23223.

For the intact rock, an isotropic material with Young’s modulus of 9.072 GPa and a Poisson’s ratio of 0.26 was considered. The joints have a normal stiffness, \( k_n \), and a shear stiffness, \( k_s \), of 100 MPa/m and 10 MPa/m, respectively. For the elasto-plastic behaviour of the intact rock, a non-associative Mohr-Coulomb failure criterion with a
friction coefficient of 43°, dilation angle of 0°, and cohesion of 675 KPa, with zero tensile strength, was considered. In examples 7.2.1 and 7.2.3, the joints exhibit a Mohr-Coulomb failure behaviour with a friction coefficient of 40°, dilation angle of 0°, cohesion of 100 KPa, and tensile strength of 100 KPa. In example 7.2.2, the cohesion and tensile strength of the joints were set to 0. The load was applied incrementally in 30 steps and the tolerance of the solution was set to 0.001.

The results were verified against the results predicted by the DEM, using the UDEC numerical package reported by Duncan and Mah (2004), and the FEM explicit joint model using Phase² reported by Hammah et al. (2006).

In order to investigate the effect of layer thickness, the elastic and elasto-plastic response of all examples were compared to the results predicted by the ubiquitous joint model, which is a limit case of the Cosserat model with zero bending stiffness.

### 7.2.1 Slope with daylighting joints

The joints in this example are approximately 21 m apart and dipping at 35°. The factor of safety calculated by the FEM Cosserat model was 1.26 (see Figure 7-2), while UDEC predicted 1.28, and the FEM explicit joint model predicted 1.32.

The failure mechanism was determined based on the deformation mode predicted by the Cosserat solution depicted in Figure 7-4 and the effective plastic strain from the FEM Cosserat model depicted in Figure 7-4. Similar to the failure mechanism identified by UDEC and Phase², the FEM Cosserat solution indicates that the mechanism of failure is a combination of sliding at the toe and a curved tension crack at the top of the slope. The failure mechanism for the FEM Cosserat solution was based on the contours of total displacement and the deformed shape of the slope, and contours of the effective plastic straining of the activated plastic surfaces, depicted in Figure 7-3 and Figure 7-4, respectively. The deformed shape and contours of total displacement of the UDEC and Phase² solution are presented in Figure 7-5(a) and 7-5(b), respectively.
Figure 7-2 Shear strength reduction factor vs. maximum displacement for slope with daylighting joints predicted by the FEM Cosserat model.

Effect of layer thickness on the elastic response of the problem for varying values of shear stiffness of the joints was investigated using the FEM Cosserat model, and the ubiquitous joint model (Cosserat model with zero bending stiffness). Figure 7-6 shows that over the whole range of $k_s$, both models predict similar results. In particular, when the shear stiffness of the joints becomes very small; both models become unstable. This indicates that the bending rigidity of the layers has no significant effect on the load bearing capacity of this slope, since progressive slip on the joint plane is the mode of deformation.

Figure 7-3 Failure mode predicted by the FEM Cosserat model: (a) contours of total displacement and (b) deformed shape for slope with daylighting joints.
Figure 7-4 Failure mechanism predicted by the FEM Cosserat model based on contours of effective plastic straining of: (a) slip on the plane of joints and (b) tensile failure of the intact rock for slope with daylighting joints.

Figure 7-5 Failure mechanism and contours of total displacement for slope with daylighting joints predicted by: (a) UDEC and (b) Phase² FE-SSR.

Figure 7-6 Maximum elastic displacement vs. $k_s$ for slope with daylighting joints predicted by the FEM Cosserat model and the classical ubiquitous joint model.
Similarly, for the elasto-plastic response, since the dominant mode of failure is sliding on the joint planes, it is expected that both the Cosserat and ubiquitous joint models would predict similar results. The ubiquitous joint model predicts a safety factor of 1.25, and a similar mechanism of failure.

7.2.2 Slope with non-daylighting joints

The joints in this example are spaced at approximately 20 m and dip at 70°. The factor of safety calculated by the FEM Cosserat solution is 1.45 (see Figure 7-7), while UDEC predicted 1.5, and the FEM explicit joint model predicted 1.53.

The failure mechanism predicted by the Cosserat solution is depicted in Figure 7-8 and Figure 7-9. This mechanism is a combination of shearing through the intact rock in the lower part, near the toe of the slope, and sliding along joints in the upper part of the slope. The failure mechanism predicted by UDEC and the FEM explicit joint model is depicted in Figure 7-10.

![Figure 7-7 Shear strength reduction factor vs. maximum displacement for slope with non-daylighting joints calculated by the FEM Cosserat model.](image)
limit case of the Cosserat model, in which the bending stiffness of the layers is disregarded, Figure 7-11 indicates the significance of the layer thickness in the load bearing capacity of the slope with non-daylighting joints. It is clear that the geometry of the problem and boundary conditions prevent unstable sliding from occurring on the joint planes. In this example, part of the load is carried by bending of the layers, while the rest is carried by the elastic resistance of the intact material.

Figure 7-8 Failure mode predicted by the FEM Cosserat model: (a) contours of total displacement, (b) deformed shape for slope with non-daylighting joints.

Figure 7-9 Failure mechanism predicted by the FEM Cosserat model based on contours of effective plastic straining of: (a) shear within the intact material and (b) sliding along the joints for slope with non-daylighting joints.

Figure 7-10 Failure mechanism and contours of total displacement for slope with non-daylighting joints predicted by: (a) UDEC and (b) Phase² FE-SSR.
In the elasto-plastic response, if the bending stiffness is set to zero, the Cosserat model reduces to the classical ubiquitous joint model. In this case, the ubiquitous joint model predicts a factor of safety of 1.39, yet a similar mechanism of failure to the Cosserat model. In addition to the difference in the value of the SSR factor, the ubiquitous joint model showed poorer convergence properties and higher values of displacements in the post failure regime, which indicates that in the Cosserat model, the bending mechanism sustains part of the load applied to the system.

![Figure 7-11 Maximum elastic displacement vs. $k_s$ for slope with non-daylighting joints predicted by the FEM Cosserat model and the classical ubiquitous joint model.](image)

### 7.2.3 Failure involving flexural toppling

The slope in this example has a single set of joints, approximately 20 $m$ apart, which dip 70° into the slope face. The FEM Cosserat model predicts a safety factor of 1.45 (see Figure 7-12), while UDEC analysis gives 1.3, and the FEM explicit joint model predicts 1.40.

Similar to the FEM explicit joint model and UDEC analysis, the FEM Cosserat joint model identifies the mode of deformation to be flexural bending of the rock columns (see Figures 7-13 and 7-14). However, interpretation of the mechanism of failure is rather
complicated. By referring to the deformed shape, both Hammah et al. (2007) and Duncan and Mah (2004) concluded that the mode of failure is flexural bending of the layers. Investigation of the response of this problem suggests that the layers behave similarly to a number of cantilever beams and flexural bending cannot lead to total collapse in this example, unless it is accompanied by failure within the intact rock. Contours of effective plastic straining at the interface predicted by the FEM Cosserat model are depicted in 7-16(a). Shear failure of the joints predicted by the FEM explicit joint model is depicted in Figure 7-17, where in the region above the black line joint elements have failed in shear. Comparison of the results suggests that both models are highly consistent in predicting the shear failure at the interface. Using the Phase$^2$ explicit joint model, it is not clear what mechanism of failure within the intact rock leads to the total collapse of this slope. It is clear that as slip occurs at the interface of the joints tensile failure within the intact rock propagates in that region. Investigation of the results by the author and formation of a clear shear strain band in the critical stage (Figure 7-17) suggest that a combination of shear and tensile failure within the intact rock along with flexural bending of the layers leads to the failure of this slope. Unfortunately, the mechanism of failure within the intact rock is not reported by Duncan and Mah (2004). Yet, they do mention that due to high stress gradients that occur within the layers, UDEC requires a very fine discretization of blocks. The discrepancy between the values of SSR factor for Phase$^2$ and UDEC may be explained by the fact that the FEM model is more accurate for predicting yielding within the intact rock.

Finally, it should be noted that the high value of SSR factor predicted by the FEM Cosserat model compared to Phase$^2$ may be due to underestimating the tensile forces within the intact rock. Similar to Phase$^2$, the FEM Cosserat model identifies shear failure of the intact rock at the toe of the slope. However, tensile cracking is limited to a region near the wall and at the toe of the slope. The FEM Cosserat model fails to capture the tensile cracking which is due to the bending of layers or high values of curvature near the top. The tensile forces can be explained using beam theory, which states that the beam bending moments result in axial stress along the layers.
In order to demonstrate the effect of flexural layer bending on the stability of the slope, the deformational behaviour of the problem for different values of $s_k$, has been investigated. Figure 7-18 shows that as values of $s_k$ decreases, the bending mechanism of the layers sustains the force. In the limit case, where $s_k$ is equal to zero, the elastic displacements converge to a value of 3.38 m. However, in the ubiquitous joint model, as values of $s_k$ decrease, the problem becomes totally unstable, and total collapse of the slope occurs due to the near-zero shear resistance of the material.

![Figure 7-12 Shear strength reduction factor vs. maximum displacement for slope with flexural toppling predicted by the FEM Cosserat model.](image)

![Figure 7-13 Failure mode predicted by the FEM Cosserat model: (a) contours of total displacement and (b) deformed shape for slope involving flexural toppling.](image)
In the elasto-plastic response, the ubiquitous joint model (the Cosserat model with zero bending stiffness) predicts a safety factor of 1.3; however, it fails to predict the correct mechanism of failure. The considerable reduction in the SSR factor for the ubiquitous joint model is consistent with Duncan and Mah (2004)’s conclusion that in the flexural bending problems, the SSR factor reduces as the layer thickness decreases. The contours of total displacement and the deformed shape of the model, predicted by the ubiquitous
joint model, are presented in Figure 7-15. Also, contours of plastic straining due to sliding on the joint planes predicted by the Cosserat model and the ubiquitous joint model are depicted in Figure 7-16.

Comparison of the results indicates that since layer bending is the dominant mode of deformation, classical continuum models fail to predict the correct mechanism of failure for this example. The fundamental difference of the FEM Cosserat model and ubiquitous joint model in this type of failure is also shown in Section 6.1.3, where effect of layer thickness in the strength and the post-peak behaviour of stratified materials is investigated.

Figure 7-17 Contours of shear straining and slip on the plane of joints for slope involving flexural toppling predicted by Phase² SSR.

Figure 7-18 Maximum elastic displacement vs. $k_s$ for slope involving flexural toppling predicted by the FEM Cosserat model and the classical ubiquitous joint model.
7.3 Conclusion

The objective of this chapter was to investigate the performance of the proposed FEM Cosserat technique with regard to the stability analysis of discrete or jointed materials. The results predicted by the proposed FEM Cosserat model were verified against two different discontinuous methods. This comparison was based on the deformation mode, failure mechanism, and the SSR factor.

Cosserat theory provides an enhanced continuum description of materials with microstructures. In the FEM Cosserat approach, the bending stiffness of layers is introduced into the governing equations of the system; subsequently, the method becomes capable of capturing modes of failure such as flexural layer bending that could previously be identified only by discontinuous models. Experimental observation of the failure of jointed rock slopes clearly indicates that the ultimate collapse often involves the failure of different components of the material. Numerical prediction of the failure mechanism depends on capturing the deformation mode and the correct formulation of the strength and post-peak behaviour of the material. The FEM Cosserat continuum approach was further improved by applying a multi-surface plasticity algorithm which allows for simultaneous failure of different components of the material. In total, four plastic surfaces could be potentially active at each point of the continuum. Consistency between SSR factor obtained by the FEM Cosserat method and predictions by discontinuous models clearly indicate the accuracy of the integration algorithm in the presence of multiple plastic surfaces.

In general, it was found that the mode of failure and deformation can be influenced by the effect of bending stiffness or layer thickness. It should be noted that in addition to the material parameters, the SSR factor predicted by the FEM Cosserat solution can be affected by a number of numerical aspects such as the stability and accuracy of the stress update integration scheme.

The results clearly suggest that the FEM Cosserat model can be an efficient and reliable tool in the stability analysis of jointed rock slopes. An attractive aspect of the FEM Cosserat approach is the fact that discontinuity surfaces are not explicitly modeled, and
that the solution can be performed by an FEM discretization which is totally independent of joint spacing and orientation. In conclusion, the FEM Cosserat solution can be an efficient tool in slope stability analysis of 3D geomaterials with arbitrarily-orientated layers.
Chapter 8
Analysis of Excavations in Layered Rock

8.1 Analysis of 2D and 3D excavations

It is widely appreciated that joints can have a significant effect on the stress distribution and deformational behaviour of excavations. In addition to experimental observations, their effect has been investigated using numerical techniques (Fishman et al., 1991; Jia & Tang, 2008). This section focuses on the application of the FEM Cosserat model to 2D and 3D analyses of excavations. The engineering dimensions of this class of problems are such that the explicit definition of joints in most cases is impractical. In the next section, a brief review of the alternative numerical options for analysis of this class of problems is presented. This review is intended to compare the applicability of the proposed FEM Cosserat method against some of the more widely used numerical packages through two examples: an extruded tunnel excavated in layered rock, and a full 3D analysis of a circular hole with out-of-plane layers.

8.2 Review of alternative numerical options

During the course of this research, the author has regularly reviewed the alternative numerical techniques for analyzing jointed or stratified materials. This section is by no means a comprehensive review; however, it contains a brief description of alternative techniques available in some of the widely used commercial numerical packages for analysis of layered and stratified continua (see Table 3-1).

ANSYS (ANSYS Inc., 2006) is a powerful, widely used, and general numerical package for the analysis of multiphysics problems. ANSYS provides options for linear and nonlinear elastic analysis of anisotropic materials as well as elasto-plastic analysis based
on Hill’s modified von-Misses criterion (Hill 1969). In addition, ANSYS provides a layered element which allows up to 250 different material layers within a solid or shell element. However, this option only accounts for changes in the material properties and does not allow for slip at the interface of the layers. Despite its generality, ANSYS is more suitable for analysis of structural problems, as opposed to geomechanics problems. Consequently, a material model such as the ubiquitous joint model is not directly provided. Alternatively, a generalized plastic material model that allows up to six user-defined compressive, tensile, and shear strength parameters to obtain anisotropic strengths in different directions may be used. ANSYS allows for two types of interface simulations using contact and interface elements. Contact technology (Cescotto & Charilier, R., 1992) allows for changes in the normal and the shear stiffness, isotropic and orthotropic frictional slip based on Coulomb’s law, and the contact traction and penetration. Contact analysis involves identifying contact pairs, and designating and defining contact and target surfaces. The process of initiating contact analysis is rather complicated and can only be applied to a limited number of bodies in contact. Interface elements define a cohesive zone material that characterizes the separation behaviour at the interface. Interface elements do not take into account the change in elastic properties but are suitable for simulation of delaminating and failure of the interface zone (Xu & Needleman, 1994). Similar to contact problems, the interface zone should be defined geometrically and be meshed separately.

| Table 8-1 Available methods for 3D modelling of jointed and layered material by the most widely used numerical packages. |
|---|---|---|---|---|---|---|
| Type of analysis | Transversely isotropic material | Orthotropic material | Layered material | Ubiquitous joint model | Explicit interface definition |
| ANSYS-3D 3D | Elasto-plastic | Elasto-plastic | Yes | No | Yes |
| FLAC3D 3D | Elasto-plastic | Elastic | No | Yes | Yes |
| 3DEC 3D | Elasto-plastic | Elastic | NO | Yes | N/A |
| PLAXIS3D 2D | Elasto-plastic | Elastic | NO | Yes | Yes |

FLAC3D (Itasca Inc., 2004) is an explicit finite difference numerical package that provides both the ubiquitous joint model and the explicit interface technique. There are major shortcomings associated with the application of the ubiquitous joint technique in FLAC3D. First, this constitutive model is restricted to one set of joints in a Mohr-
Coulomb material. Second, the application does not consider any anisotropy in elastic properties due to the existence of joints. In other words, the ubiquitous joint model only accounts for plastic deformation in a specified orientation. In addition to the aforementioned restrictions, the stress update algorithm adopted by FLAC3D is not theoretically justifiable. In this multi-surface plasticity model, the plastic correction is first performed based on the properties of the intact matrix. If the updated stress state violates the yield surface of the joints, a subsequent correction will be performed based on the joint properties. In the author’s judgement, this approach does not have any physical basis, and may lead to an incorrect plastic path. Explicit simulation of interfaces is rather complicated in FLAC3D. The implemented technique requires that individual bodies of material separated by interface surfaces be geometrically defined so that common nodes between neighbouring blocks initially do not occupy the same coordinates. The interface is then defined on one block, and subsequently the individual blocks need to be moved to their true spatial position. Clearly, the required procedure restricts the application of the FLAC3D interface technique to cases that involve a limited number of interfaces which form blocks of regular geometries.

3DEC (Itasca Inc., 2003) is a discrete element method based on an explicit integration scheme. 3DEC requires discrete definition of individual bodies; however it supports the ubiquitous joint model as an option. The ubiquitous joint model of 3DEC is similar to the FLAC3D bilinear strain-hardening/softening ubiquitous joint model, and exhibits similar shortcomings. In addition, accurate solutions to plasticity problems can only be achieved by putting special restrictions on the discretization of the blocks. In general, it is recommended not to use 3DEC unless the problem has a discrete nature, and to use alternative continuum methods if joints are to be modelled with a ubiquitous joint technique. 3DEC is especially suitable for problems that involve a large number of arbitrarily-oriented joints. In contrast to FLAC3D and ANSYS, 3DEC does not require the definition of individual blocks separated by interfaces. In this method, an original body can be divided into any number of sub-blocks or layers simply by defining the orientation, spacing, and number of joints that are passing through the body. In the author’s judgement, 3DEC is the only practical option for users when more than a few 3D orientated joints are involved in the problem. The main disadvantage of 3DEC, however,
results from the fact that a reasonable discretization of the rigid sub-blocks into deformable blocks adds considerably to the amount of memory required by the program, making the solution computationally intensive.

PLAXIS3D (2004) Tunnel is an extension of the PLAXIS 2D finite element code that is designed for the analysis of tunnels. The 3D model consists of equal parallel cross-sectional planes that are copied in the out-of-plane direction, and subsequently, does not allow for any geometry variation along the tunnel axis. PLAXIS 3D Tunnel provides explicit interface elements. However, due to the aforementioned restriction, simulation of 3D oriented layers is not possible. PLAXIS3D also provides a ubiquitous joint model referred to as “jointed rock”. The jointed rock model accepts an elastic transversely isotopic intact material, and up to three sliding directions. The first sliding plane corresponds to the direction of elastic anisotropy. A maximum of two other sliding directions may be defined. On each plane, a local Coulomb condition, along with a tension cut-off criterion, is applied. The ubiquitous model of PLAXIS, does not allow for yield in the intact material. Also, the theoretical aspects of the numerical integration algorithm applied by the application are not presented in the manual.

8.3 Examples

This section focuses on the application of the FEM Cosserat model to the analysis of the effects of the in-plane and out-of-plane joints on the response of excavations. If joints are in-plane, then the normal vector to the joint plane falls within the 2D cross-sectional plane of the problem. Under certain loading and boundary conditions, analysis of 3D problems with in-plane joints can be performed in 2D. If joints are out-of-plane, then the normal to the joint plane makes a non-zero angle with respect to the 2D cross sectional plane of the problem, and in most cases a full 3D analysis is required.

**Effect of in-plane layers on the elastic and elasto-plastic response of excavations**

The primary purpose of this example is to investigate how layers and the subsequent induced anisotropy affect the elastic and elasto-plastic response of excavations. For a
number of different anisotropy orientations, the elastic and elasto-plastic response of the problem is studied using the FEM Cosserat model; the results are also compared the predictions by the ubiquitous joint model and Phase² explicit joint model (Rocscience, 2005).

The geometry, boundary conditions and mesh discretization of the FEM Cosserat solution are shown in Figure 8-1. The joint orientations and information regarding the mesh discretization of Phase² explicit joint model are presented in Figure 8-2. The top boundary is subjected to a uniform pressure with a magnitude of 100 MPa, while all other boundaries are fixed. In the plastic solution the load was applied in 10 load steps. In order to compare the results to Phase², in the 3D Cosserat model the displacements associated with the out-of-plane direction (i.e., \( u_y \)) and the Cosserat rotation around the x axis (i.e., \( \theta_x \)) are constrained. As a result, the full 3D model behaves similarly to a 2D plane strain problem and therefore, it is justified to use one element in the out-of-plane direction.

Figure 8-1 Geometry, mesh discretization, and boundary conditions for the FEM Cosserat solution of a 2D-extruded tunnel excavated in layered rock.
Chapter 8  Analysis of Excavations in Layered Rock

Figure 8-2 Geometry, mesh specifications, and joint orientations for the FEM explicit joint model of a 2D-extruded tunnel excavated in layered rock.

The intact rock is modelled as an isotropic Mohr-Coulomb material with a Young’s modulus of 17.8 GPa, Poisson’s ratio of 0.25, friction coefficient of 33.5°, dilation angle of 16.72°, cohesion of 10.28 MPa, and tensile strength of 0.5 MPa. The joints are spaced 2.5 m apart and exhibit a Mohr-Coulomb failure behaviour with a friction coefficient of 30°, dilation angle of 30°, cohesion of 0.5 MPa, with no tensile strength. The elastic response of the model is investigated for two cases: i) $k_n = 20 \text{ GPa/m}$ and $k_s = 200 \text{ MPa/m}$, and ii) $k_s / E = 1 \times 10^5$ and $k_s = 200 \text{ MPa/m}$. The Cosserat solution is compared to the Phase^2 explicit joint model and the classical transversely isotropic FEM ubiquitous joint model. The ubiquitous joint model is a limiting case of the Cosserat model in which the effect of bending stiffness of the layers is disregarded ($B=0$).
Figure 8-3 shows the values of the total elastic displacement at the center of the tunnel roof for case (i). In this case, all models predict similar values for the total displacement at the top of the tunnel. In case (ii), \( k_n \) is relatively large. Therefore, the effect of the bending stiffness of the layers can be observed in the results. Figure 8-4 shows the values of the total elastic displacement at the center of the tunnel roof for case (ii). Due to the bending mechanism that resists the load, the displacements of the model with horizontal layers are approximately 20% less than the case where joints are oriented vertically. However, the ubiquitous joint model yields the exact results for both cases (provided that the ratio of \( k_n / E \) is relatively large), since the shear modulus of the equivalent material is equal in both directions. It is clear that by disregarding the bending rigidity of the layers the ubiquitous joint model can overestimate the elastic displacements. Figure 8-5 shows the values of total displacement for the elasto-plastic response of the case (ii). A comparison of the results indicates that as yielding occurs, the effect of bending stiffness becomes more pronounced.

![Graph](image)

**Figure 8-3** Total elastic displacement at the center of the tunnel roof for joints with \( k_n = 20000 \text{ MPa/m} \) and \( k_s = 200 \text{ MPa/m} \).

Figure 8-6 to 8-9 show the contours of total displacement predicted by the FEM Cosserat model and the explicit joint model. Figures 8-10 and 8-11 show the contours of
maximum principal stress predicted by the FEM Cosserat model, and the ubiquitous joint model for case (i).

![Diagram](image)

Figure 8-4 Total elastic displacement at the center of the tunnel roof for joints with $k_n/E = 1e10$ and $k_s = 200 \text{ MPa/m}$.

![Diagram](image)

Figure 8-5 Total elasto-plastic displacement at the center of the tunnel roof for joints with $k_n = 20000 \text{ MPa/m}$ and with $k_s = 200 \text{ MPa/m}$.
Figure 8-6 Contours of total elasto-plastic displacement predicted by (a) FEM Cosserat model, and (b) FEM explicit joint model, for joints oriented horizontally.

Figure 8-7 Contours of total elasto-plastic displacement predicted by (a) FEM Cosserat model, and (b) FEM explicit joint model, for joints oriented at 30° with respect to a horizontal plane.

Figure 8-8 Contours of total elasto-plastic displacement predicted by (a) FEM Cosserat model, and (b) FEM explicit joint model, for joints oriented at 60° with respect to a horizontal plane.
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Figure 8-9 Contours of total elasto-plastic displacement predicted by (a) FEM Cosserat model, and (b) FEM explicit joint model, for joints oriented vertically.

Figure 8-10 Distribution of maximum principal stress around the excavation predicted by the FEM Cosserat model for joints dipping (a) 0°, (b) 30°, (c) 60°, and (d) 90°.
It is clear that the induced anisotropy influenced the magnitude and location of the maximum displacement on the boundary of the excavation. Similar to displacements, the magnitude and distribution of stresses are also affected by anisotropy orientation and the bending mechanism. Figure 8-10 and 8-11 show the contours of maximum principal stress predicted by the Cosserat model, and the ubiquitous joint model for case (i). It is clear that both models predict a similar pattern, however, principal stress predictions differ as the layers become more horizontal, i.e., joints dipping at 0° and 30°. This is in accordance with Figures 8-2 and 8-3, which also suggest that the effect of layer thickness on the displacement of the model becomes more significant as the layers approach horizontal orientations.

Figure 8-11 Distribution of maximum principal stress around the excavation predicted by the FEM ubiquitous joint model for joints dipping (a) 0°, (b) 30°, (c) 60°, and (d) 90°.
Effect of out-of-plane layering on the elastic and elasto-plastic response of excavations

This example is concerned with the effect of out-of-plane layers on the deformation and stability of excavations. The elastic and elasto-plastic response of a circular hole excavated in a layered rock with layers oriented in an out-of-plane direction is studied. Figure 8-12 shows the geometry, boundary conditions, and the definition of the anisotropy orientation of this problem. The length of the extrusion is 60 m and a distributed load with a magnitude of 100 MPa/m is applied to a width of 9 m on the top surface. Due to symmetry, in the FEM Cosserat model, half of the problem is simulated using 2430 brick elements (20-noded). The intact material is an isotropic rock with Young’s modulus of 20 GPa and Poisson’s ratio of 0.3. The intact material and the joints exhibit a Mohr-Coulomb failure behaviour along with a tension-cut-off. The elasto-plastic material parameters for the intact rock and the joints are presented in Table 8-2.

**FEM Cosserat model:**

- **Element type:** 20-noded brick
- **Number of elements:** 2420
- **Number of elements in out-of-plane direction:** 20
- **Number of FEM nodes:** 48696
- **Number of DOF in 3D:** 11565

*Figure 8-12 Geometry, boundary conditions, and mesh discretization of the FEM Cosserat solution of a 3D excavation in jointed rock.*
Table 8-2 Strength properties of the material with out-of-plane layers.

<table>
<thead>
<tr>
<th></th>
<th>Intact rock parameters</th>
<th>Joint parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c$ MPa</td>
<td>$\varphi$ deg.</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>33.5</td>
</tr>
</tbody>
</table>

The effect of layer thickness on the elastic response is studied for three different models with varying values of layer thickness: (a) $h_k = 30 \text{ GPa}$, $h_s = 300 \text{ MPa}$, $h = 0 \text{ m}$, (b) $h_k = 30 \text{ GPa}$, $h_s = 300 \text{ MPa}$, $h = 1.5 \text{ m}$, and (c) $h_k = 60 \text{ GPa}$, $h_s = 300 \text{ MPa}$, $h = 3 \text{ m}$. Once again, it is recalled that in case (a), by setting the layer thickness to zero, the bending stiffness of the layers becomes zero, and the Cosserat model reduces to a classical transversely isotropic model, i.e., ubiquitous joint model. For the case where $h = 1.5 \text{ m}$, the geometry of the 3DEC models are shown in Figure 8-13. In the 3DEC model, in order to simulate the deformability and elasto-plastic response of the intact material comprising the layers, the blocks need to be discretized into zones. The number of blocks and number of zones in the 3DEC examples vary depending on the joint spacing and orientation.

Figure 8-13 3DEC model for joints dipping (a) $0^\circ$, (b) $30^\circ$, (c) $60^\circ$, and (d) $90^\circ$. 
Figure 8-14 shows the values of maximum vertical displacement of the model for varying values of layer thickness. The results indicate that, independent of the bending stiffness of the layers, for all of the aforementioned layer thicknesses, the most critical orientation occurs when joints are oriented horizontally. However, comparison of the results with the FEM Cosserat model with zero bending stiffness, which is equivalent to a classical transversely isotropic elastic material or a ubiquitous joint model, indicates that as the layers approach a horizontal orientation, the effect of layer thickness becomes more significant in the system.

Figure 8-14 Effect of layer thickness on the elastic response based on graphs of maximum displacement versus joint orientation.

The elasto-plastic response of the problem with the elastic properties of $h = 1.5 \ m$, $hk_n = 30 \ GPa/m$, and $hk_y = 300 \ MPa$ has also been studied using the FEM Cosserat and 3DEC solutions. Contours of total elasto-plastic displacement predicted by the FEM Cosserat model and the 3DEC model are presented in Figure 8-15 and Figure 8-16, respectively. Both models are highly consistent in their predicted pattern of displacement, and indicate that the deformation response varies significantly with respect to the joint orientation.
Contours of effective plastic straining for models with different joint orientations are depicted in Figure 8-17 to Figure 8-20, and indicate the active mechanisms of failure. It is clear that in all cases, the dominant mechanism of failure is shearing through the intact rock at the walls of the tunnel. In this example, due to the geometry of the problem and the orientation of layers with respect to the excavation, shear failure on the joint planes cannot be the dominant mode of failure. Progressive slip on the joints cannot occur unless it is accompanied by failure within the intact rock. Figure 8-17 shows when layers are horizontal, slip on the interface is not significant. As layers become more vertical the effect of slip on the interface can be observed in the contours of effective plastic straining.

![Contours of total elasto-plastic displacement predicted by the FEM Cosserat solution joints dipping](image)

Figure 8-15 Contours of total elasto-plastic displacement predicted by the FEM Cosserat solution joints dipping (a) 0°, (b) 30°, (c) 60°, and (d) 90°.

Finally, vertical displacement versus joint orientation at the center of the tunnel roof along the excavation predicted by the Cosserat FEM model and the 3DEC is presented in Figure 8-21. Results predicted by both models show a high level of consistency in how joint orientation and layer thickness influence the response. It should be noted that in the 3DEC models the solution was obtained using both the large and the small deformation formulation. Also, the results were studied using both a coarse and a fine block zoning. In
this example, 3DEC solution exhibited a high sensitivity to the refinement of the block zoning. As expected, the FEM Cosserat model is most consistent with the 3DEC results using the small deformation formulation with a fine zoning. Total elasto-plastic displacement of the tunnel roof predicted by the FEM Cosserat model and the 3DEC model is shown in Figure 8-21.

Figure 8-16 Contours of total elasto-plastic displacement for strong rock predicted by 3DEC joints dipping (a) 0°, (b) 30°, (c) 60°, and (d) 90°.
Figure 8-17 Contours of effective plastic straining for joints dipping $0^\circ$: (a) slip on the plane of joints (b) tensile failure of the joints, (c) shearing through the intact rock, and (d) tensile failure of the intact rock.

Figure 8-18 Contours of effective plastic straining for joints dipping $30^\circ$: (a) slip on the plane of joints (b) tensile failure of the joints, (c) shearing through the intact rock, and (d) tensile failure of the intact rock.
Figure 8-19 Contours of effective plastic straining for joints dipping 60°: (a) slip on the plane of joints (b) tensile failure of the joints, (c) shearing through the intact rock, and (d) tensile failure of the intact rock.

Figure 8-20 Contours of effective plastic straining for joints dipping 90°: (a) slip on the plane of joints (b) tensile failure of the joints, (c) shearing through the intact rock, and (d) tensile failure of the intact rock.
Finally, in order to verify the accuracy of the elasto-plastic stress integration algorithm, the FEM Cosserat solution is verified against FLAC3D ubiquitous joint model. In this case, in the FEM Cosserat model the ratio of $h k_s / E$ and $h k_n / E$ is set to a relatively large value (i.e., 1e10), and effects of bending stiffness are disregarded. Values of maximum vertical displacement at the center of the tunnel roof, predicted by the FEM Cosserat model and the FLAC3D ubiquitous joint model are presented in Figure 8-22. Also, contours of total displacement predicted by the FEM Cosserat model and the FLAC3D ubiquitous joint model are presented in Figure 8-23 and Figure 8-24, respectively.

Close consistency between values of displacements predicted by the FEM Cosserat model and FLAC3D ubiquitous joint model suggests that the stress integration algorithm developed in this research is accurate and reliable for analysis of elasto-plastic response of the material when more than one failure mechanism is involved. Also, comparison between the pattern and the values of total displacement presented in Figure 8-23 and Figure 8-24, and the results presented in Figure 8-15 and Figure 8-16, indicates that
elastic anisotropy due to existence of weak layer interfaces and the bending stiffness of the layers both have a significant impact on the response of the problem, and disregarding their effects leads to large errors in the solution.

![Graph showing total elasto-plastic displacement at the middle of the tunnel roof versus joint orientation predicted by the FEM Cosserat model with $h=0$ and FLAC3D ubiquitous joint model.](image)

Figure 8-22 Total elasto-plastic displacement at the middle of the tunnel roof versus joint orientation predicted by the FEM Cosserat model with $h=0$ and FLAC3D ubiquitous joint model

![Contour plots showing displacement predicted by the FEM Cosserat model for joints dipping (a) $0^\circ$, (b) $30^\circ$, (c) $60^\circ$, and (d) $90^\circ$ ($k_h/E = 1e10$, and $h = 0$).](image)

Figure 8-23 Contours of total displacement predicted by the FEM Cosserat model for joints dipping (a) $0^\circ$, (b) $30^\circ$, (c) $60^\circ$, and (d) $90^\circ$ ($k_h/E = 1e10$, and $h = 0$).
Figure 8-24 Contours of total displacement predicted by the FLAC3D ubiquitous joint model joints dipping (a) 0°, (b) 30°, (c) 60°, and (d) 90°.

8.4 Conclusion

This chapter considers the effect of anisotropy and bending rigidity on the elastic and elasto-plastic response of excavations in layered rock. The results are important from two perspectives:

First, the effect of in-plane and out-of-plane layers in the deformation and the stress distribution around excavations in stratified media are closely studied. The relationship between the magnitude of displacements and the anisotropy orientation was established,
and the effect of bending rigidity was studied in each case. Also, the effect of anisotropy on the pattern of response was investigated. Interesting results, such as the difference in pattern of deformation with in-plane layers when the joints are oriented at 30° and 60° were observed.

From a numerical perspective, this chapter provides a review on the application and performance of the proposed 3D FEM Cosserat model against alternative methods such as the FEM explicit interface elements and the Distinct Element Method (DEM). Reviews of the limitations and the assumptions of other 3D numerical packages that can be used as an alternative to the FEM Cosserat model are provided in this chapter. The results are compared to the predictions of two widely used numerical packages: Phase² (Rocscience, 2005) and 3DEC (Itasca Inc., 2003). In the case where a problem can be represented as a 2D extrusion, the FEM Cosserat model is verified against the FEM explicit joint model using the Phase², and the results show close consistency with the Phase² predictions. For full 3D analyses, the results are compared to 3DEC numerical package. Comparison between solutions produced by the FEM Cosserat model and 3DEC show good agreement in both elastic and elasto-plastic analysis results. It is interesting to note that while the FEM Cosserat model and 3DEC are fundamentally different in terms of their governing equations, contact formulations, and numerical integration, the results exhibit a good level of consistency. Also, for this model, the stress integration algorithm was verified against FLAC3D ubiquitous joint model, and the results exhibited a high level of consistency.

In general, it is found that the proposed FEM model can be safely applied to problems with a discrete nature, provided that the microstructures exhibit a sequential pattern. Reviews of the techniques and pre-processing aspects of different methods suggest that if the generation of individual bodies or interfaces is not performed in an automated fashion, continuum-based methods formulated in Cosserat theory remain the only practical method for the 3D analysis of problems with a discrete nature. In the author’s judgement, 3DEC provides a unique tool for the 3D analysis of discrete problems. However, there are a number of disadvantages associated with its implementation. First, the restrictions on the discretization of the rigid blocks into deformable blocks may
reduce the accuracy of the solution, especially in plasticity analyses. The most significant
problem, however, remains in the extension of the method to the analysis of problems
with a more complicated nature, such as coupled solid-fluid analyses.

From a pre-processing perspective, the FEM Cosserat model is, without question, the
most convenient tool for the analysis of problems with a discrete nature. Also, the
solution’s accuracy and stability makes the method a promising and reliable tool. The
major disadvantages of the method, however, remain in the lack of development of more
generalized constitutive equations and the difficulty in extending the method to cases
where more than one set of joints are involved.
Chapter 9
Conclusion and Final Remarks

9.1 Concluding remarks

The primary goal of this research was to develop a robust and efficient 3D continuum-based method for the analysis of layered materials. In geomechanics, practical concerns, limit the application of explicit or discrete methods to cases where only a limited number of discontinuity surfaces are involved, and push users to use classical anisotropic models, such as the ubiquitous joint model, beyond the limits of their validity. It was argued that when the number of surfaces exceeds a limited number, the explicit definition of interfaces is practically impossible, even for a relatively simplified geometry in a layered medium. Existence of more than one set of joints adds considerably to the level of complexity involved in the user definition of interfaces. Furthermore, if the interface definition is performed automatically in a uniform or stochastic fashion, further discretization of the blocks into a 3D FEM mesh or the FDM grid may lead to a computationally large system of equations. Finally, the existence of intersecting arbitrarily-oriented surfaces may lead to a poor quality FEM mesh or FDM grid, which affects the accuracy, convergence, and stability of the solution, especially in elasto-plastic analyses.

Cosserat theory provides a mathematically enhanced continuum description of materials with microstructure. Consequently, the numerical techniques based on this theory can be exceptionally useful in geomechanics and the composite industry which usually involve an excessive number of layers.

In conclusion, there are a number of general aspects of the FEM Cosserat solution that deserve to be emphasized:
• Through a series of problems that concern layered plate structures, it is shown that the FEM Cosserat solution as formulated is as accurate as the FEM explicit interface method in predicting the elastic displacements of media with a periodic layered microstructure.

• In elasto-plastic response, the method showed good agreement with alternative discrete techniques. It was pointed out that the accuracy of the solution in this range was affected by the accuracy of the elasto-plastic stress integration algorithm. Disregarding the effect of the micromoments in the yield criteria of the intact material could also reduce the solution accuracy.

• Through a number of benchmark verification tests and application examples it is concluded that the developed FEM Cosserat model is robust. In other words, under a wide range of values for each of the input parameters, the FEM Cosserat model exhibits consistent behaviour and predicts a reasonable response.

• Considering the computational efficiency of the method, it can be argued that the intrinsic characteristic of the model is that the input geometry and the FEM mesh are independent of the layer thickness and orientation, which allows for the solution of the problem with fewer DOFs and clearly leads to a more computationally efficient system of equations.

• Considering the characteristics of the solvers, the stiffness matrix remains sparse and symmetric. However, in general, as elastic unsymmetries increase within the material, the nodal stiffness matrix becomes more populated compared to the nodal stiffness matrix of an isotropic material. Also, both the material and the buckling stiffness matrices presented in this research are symmetric. However, similar to a classical material, the symmetry of the material stiffness matrix is violated if the plastic flow of the material follows a non-associative flow rule.

• The rigorous mathematical basis that underlies the formulation provides an extendable platform for the analysis of different classes of problems of concern in continuum theory. For example, analysis of visco-plastic, coupled solid-fluid media,
and dynamic problems can be extended to discrete materials using the Cosserat continuum approach (Birsan, 2008).

In conclusion, the FEM Cosserat model demonstrated a high level of consistency with discrete models and it can be safely applied to the simulation of materials with a sequential microstructure.

9.2 Major contributions of the thesis

The contributions of this thesis can be classified into two categories. First, derivation of a series of new formulations for the analysis of 3D Cosserat layered media. The new aspects of the formulation can be summarized as follows:

- The finite element formulation of a Cosserat continuum is extended to 3D. In particular, the nature of the rotation tensor and the curvature measure varies between 2D and 3D analyses. In this work, an infinitesimal rotation tensor is applied and the strain and curvature tensors are proposed based on the rotation matrix. Finally, the issues arising from the existence of anisotropy within the material are addressed.

- Based on the mechanical consideration of a single plate component, the relevant rotation, curvature, and micromoment measures within a 3D Cosserat layered material were determined and subsequently, constitutive matrices were proposed.

- The FEM elasto-plastic analysis was reformulated in the Cosserat framework. In particular, the definition of the flow vector and the plastic flow vector, their nature, and the evolving issues due to the non-symmetrical nature of the stress tensor were addressed.

- In this research, the possibility of yield of the interface and intact material was considered. Since violation of multiple plastic surfaces can occur at the FEM integration points, a consistent elasto-plastic algorithm which automatically identified the activated surfaces was applied. Application of the model is new in rock mechanics analysis of layered materials and within the Cosserat framework.
• A finite rotation tensor was proposed and based on micropolar theory of finite rotation and finite deformation, a new FEM buckling stiffness matrix was presented.

• The proposed formulation was implemented in the Keck Geomechanics Code (KGC), a 3D object-oriented implicit FEM application, developed at the University of Toronto. This implementation required developing Cosserat nodes, stress, strain, elements, and material objects, as well as developing a new plasticity model for the numerical integration of stress, based on the idea of the existence and potential violation of multiple plastic surfaces at each gauss point. In addition, the structure of the KGC was made compatible with the Cosserat data structure.

Also, from an application point of view, the FEM Cosserat model was applied to a series of practical rock mechanics problems:

• A series of slope stability problems using the shear strength reduction method.

• 3D analysis of an excavation with out-of-plane joint orientation.

### 9.3 Further development of the model

#### 9.3.1 Developments in the application of Cosserat continuum theory

Application of Cosserat theory to the FEM analysis of materials can be extended in the following aspects:

• This research is mainly focused on the effect of anisotropy and internal length scale on the deformability and the strength of layered geomaterials from a numerical perspective. Other aspects, such as the effect of large deformation due to the interface slip, and the effect of the rotation of the anisotropy direction, remain to be addressed if a more accurate analysis is required.

• The development of constitutive equations that provide a mathematical, quantitative description of the physical behaviour of materials remains as a major challenge. The 2D and 3D constitutive equation for layered and blocky materials is developed.
However, the mechanical response of a material with blocky structure, is rather complicated and in general involves more than one internal length parameter. Therefore, developing a mathematical, quantitative description of the physics of blocky materials is not straightforward. The assumptions applied in this case are simplified and the current model can only be applied to the 2D simulation of a limited class of materials in which the dimension of the block in one direction is considerably larger than in the other direction. The natural extension of this dissertation is to further develop the constitutive equations for 3D blocky materials with more general shapes.

- In the elasto-plastic response, the FEM Cosserat model was formulated based on the possibility of yield on the interface or within the intact material. The Cosserat model is accurate in predicting the elasto-plastic response of the interface. However, the elasto-plastic formulation of yielding within the intact material is based only on the true stress tensor, and the effect of micromoments or the couple stress tensor is disregarded in the formulation. It can be argued that in a Cosserat layered material, micromoments have an effect similar to the true moments of a beam, and therefore, their effects need to be incorporated in the yielding of the intact material. The significance and exact effect of micromoments on yield within the intact rock requires extensive theoretical and experimental research, and brings new dimensions to the theory and application of the Cosserat continuum.

- Another subject relevant to this research is the comparison of the governing equations of continuum methods based on Cosserat theory, or on general micropolar theory, with the governing equations of discrete element methods such as the DDA and DEM, which intrinsically account for the internal length scale associated with a particular the problem. Preliminary work on this subject has been conducted by Doolin (2003), and shows that under a Cosserat formulation, both the DEM and DDA reduce to the same fundamental equations, with differences only in their numerical techniques.
9.3.2 Developments in the strength prediction of anisotropic materials

In the author’s judgement, the strength prediction method adopted in this research is the only practical yet physically justifiable method for a continuum-based analysis of anisotropic materials. The only acceptable alternative approach is to perform extensive experimental tests and to determine the strength and flow rule on an individual basis for each material. The advantages of the method are discussed in Chapters 3 and 6. However, comparison with experimental observation indicates that, despite its numerous advantages, the discrepancy between the predictions of the method and the experimental observations are considerable for natural materials such as jointed rock. One suggestion in this area is the application of a large deformation formulation, which accounts for the rotation of anisotropy orientation within the material. Preliminary work by the author based on the large deformation Cosserat theory, yielded promising results. The modified formulation predicts the strength behaviour in a fashion similar to the Distinct Element Method, which can be applied to reproduce the strength behaviour of materials.

9.3.3 Developments in the numerical integration of the stress update algorithm for multiple plastic surfaces

The integration scheme is based on a multi-surface plasticity algorithm, which allows for simultaneous violation of different yield criteria and provides a return mapping algorithm based on the particular situation. The method is numerically stable provided that the intersection of the plastic surfaces forms a convex shape. Although the analytical solution imposes no restrictions on the number of surfaces, to the author’s knowledge, the numerical application of this method was restricted to plasticity models like the Cam-Clay model along with a cap surface. In this research, the method was applied to the simulation of layered materials in which there is a plane of weakness with a specified orientation within a matrix of isotropic material. All verification examples demonstrated a good consistency with other numerical methods. However, two main aspects of the method deserve further research:

- First, it is suggested that the accuracy, efficiency, and stability of the method be investigated when more that two surfaces are involved. Also, the numerical damping
that this algorithm may introduce into the solution should be closely studied. In addition, the difference between the proposed algorithm and the algorithms applied by other numerical packages, such as FLAC3D and ANSYS, should be closely monitored.

- Finally, it is suggested that an adaptive load stepping algorithm be implemented in order to restrict the number of violated surfaces to a reasonable minimum number. This would boost the accuracy and efficiency of the return-mapping algorithm considerably.
Appendices

Appendix 1: Rotation matrices of Voigt stress, Voigt strain, elasticity matrix, and compliance matrix

Direction cosines for the rotation of the coordinate system are expressed as follows:

\[
\begin{bmatrix}
\hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\
x_1 & l_1 & m_1 & n_1 \\
x_2 & l_2 & m_2 & n_2 \\
x_3 & l_3 & m_3 & n_3 \\
\end{bmatrix}
\]

where

\[
a_i^j = \cos(\hat{x}_i, \hat{x}_j)
\]

and \(a_i^j\) is the cosine of the angle between the new axis, \(\hat{x}_i\), denoted by hat, and the original axis, \(x_j\).

Due to symmetry of the Cauchy stress tensor, one component of each shear pair is stored in the Voigt stress vector. In order to reflect this fact in the calculation of the stress vector in the new coordinate system, the rotation coefficients of any two conjugate shear stress components are summed. The Voigt stress vector is transformed using the following relations:

\[
\tau = R_{\sigma} \tilde{\tau} \quad \text{and} \quad \tilde{\tau} = R_{\sigma}^{-1} \tau
\]

where
Accordingly, in order to get the correct value of the strain energy, the stored shear strain terms (multiplied by two) are stored in the Voigt strain vector. The transformation of the Voigt strain vector then follows:

\[ \gamma = R_\varepsilon \tilde{\gamma} \quad \text{and} \quad \tilde{\gamma} = R_\varepsilon^{-1} \gamma \]  

where

\[ R_\varepsilon = \begin{pmatrix}
  l_1^2 & m_1^2 & n_1^2 & 2n_1m_1 & 2l_1n_1 & 2l_1m_1 \\
  l_2^2 & m_2^2 & n_2^2 & 2n_2m_2 & 2l_2n_2 & 2l_2m_2 \\
  l_3^2 & m_3^2 & n_3^2 & 2n_3m_3 & 2l_3n_3 & 2l_3m_3 \\
  l_1l_2 & m_1m_2 & n_1n_2 & m_1n_2 + n_2m_1 & l_1n_2 + l_2n_1 & l_1m_2 + l_2m_1 \\
  l_1l_3 & m_1m_3 & n_1n_3 & m_1n_3 + n_3m_1 & l_1n_3 + l_3n_1 & l_1m_3 + l_3m_1 \\
  l_2l_3 & m_2m_3 & n_2n_3 & m_2n_3 + n_3m_2 & l_2n_3 + l_3n_2 & l_2m_3 + l_3m_2
\end{pmatrix} \quad (A-I-6)

From Equation A-I-6, it is clear that normal components of strain in the new coordinate system are obtained by storing half of the rotation terms corresponding to the shear component in the matrix, while shear components in new coordinate system are obtained by having twice the coefficients of all terms present in the \( R_\varepsilon \) matrix.

It is important to note that due to the arrangement of the Voigt stress and strain vectors, \( R_\sigma \) and \( R_\varepsilon \) are not orthogonal, in other words:

\[ R_\sigma^{-1} \neq R_\sigma^T, \quad \text{and} \quad R_\varepsilon^{-1} \neq R_\varepsilon^T \]  

(A-I-7)

It can be proven that:
The transformation of the elasticity matrix, $C$, follows:

$$\tilde{\tau} = \tilde{C} \tilde{\gamma}$$  \hspace{1cm} (A-I-10)

$$R^{-1}_\sigma \tau = \tilde{C} R^{-1}_\sigma \gamma$$

$$\tau = R^{-1}_\sigma \tilde{C} R^{-1}_\sigma \gamma \Rightarrow C = R^{-1}_\sigma \tilde{C} R^T_\sigma$$

Also, the transformation of compliance matrix, $D$, follows:

$$\tilde{\gamma} = \tilde{D} \tilde{\tau}$$  \hspace{1cm} (A-I-11)

$$R^{-1}_e \gamma = \tilde{D} R^{-1}_\sigma \tau$$

$$\gamma = R^{-1}_e \tilde{D} R^T_\sigma \sigma \Rightarrow D = R^{-1}_e \tilde{D} R^T_\sigma$$
Appendices

Appendix 2: Principal stresses and stress invariants

Stress invariants are defined by:

\[ I_1 = \text{tr}(\sigma) = \sigma_{ii} \]  
\[ I_2 = \frac{1}{2}(\sigma : \sigma - I_1^2) = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) \]  
\[ I_3 = \det \sigma = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqrs} \sigma_{ip} \sigma_{jq} \sigma_{kr} \]

where \( \epsilon_{ijk} \) is the permutation symbol.

Principal stresses can be expressed in terms of stress invariants as (Owen & Hinton; 1980):

\[ \sigma_1 = \frac{2}{\sqrt{3}} \sqrt{J_2} \sin(\theta^L \pm \frac{2\pi}{3}) + \frac{I_1}{3} \]  
\[ \sigma_2 = \frac{2}{\sqrt{3}} \sqrt{J_2} \sin(\theta^L) + \frac{I_1}{3} \]  
\[ \sigma_3 = \frac{2}{\sqrt{3}} \sqrt{J_2} \sin(\theta^L + \frac{4\pi}{3}) + \frac{I_1}{3} \]

where \( \theta^L \) is the Lodé angle defined by:

\[ \sin 3\theta^L = -\frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^{3/2}}} \quad \theta^L = \frac{1}{3} \sin^{-1} \left[ -\frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^{3/2}}} \right] \]  

where \( \sigma_1 > \sigma_2 > \sigma_3 \) and \(-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \).
Appendix 3: Derivation of the elasto-plastic constitutive matrix for a single plasticity surface

The derivation of the constitutive matrix starts with the following relations:

\[ F(\sigma, \kappa) = 0, \quad dF = \frac{\partial F}{\partial \sigma} \dot{\sigma} + \frac{\partial F}{\partial \kappa} \dot{\kappa} = 0 \]  \hspace{1cm} (A-III-1)

\[ \dot{\sigma}_{ij} = D_{ijkl}^{e} \dot{\epsilon}_{kl} = D_{ijkl}^{e} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) \]  \hspace{1cm} (A-III-2)

\[ \dot{\epsilon}_{kl}^p = \lambda \frac{\partial Q}{\partial \sigma_{kl}} \]  \hspace{1cm} (A-III-3)

Substituting \( \dot{\sigma}_{ij} \) from Equation A-III-2 into Equation A-III-1 produces:

\[ dF = \frac{\partial F}{\partial \sigma} D_{ijkl}^{e} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) + \frac{\partial F}{\partial \kappa} \dot{\kappa} = 0 \]  \hspace{1cm} (A-III-4)

Substituting \( \dot{\epsilon}_{kl}^p \) form Equation A-III-3 into the above expression produces:

\[ dF = \frac{\partial F}{\partial \sigma} D_{ijkl}^{e} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) + \lambda \frac{\partial Q}{\partial \sigma} D_{ijkl}^{e} \frac{\partial Q}{\partial \sigma_{kl}} + \frac{\partial F}{\partial \kappa} \dot{\kappa} = 0 \]  \hspace{1cm} (A-III-5)

Using the definition of the plastic modulus:

\[ H^p = -\frac{\partial F}{\partial \kappa} \frac{\dot{\kappa}}{\dot{\lambda}} \]  \hspace{1cm} (A-III-6)

Equation A-III-1 can be expressed as:

\[ dF = \frac{\partial F}{\partial \sigma} D_{ijkl}^{e} \dot{\epsilon}_{kl} - \lambda \frac{\partial F}{\partial \sigma} D_{ijkl}^{e} \frac{\partial Q}{\partial \sigma_{kl}} + \frac{\partial F}{\partial \kappa} \dot{\kappa} - H^p \dot{\lambda} = 0 \]  \hspace{1cm} (A-III-7)

Now the plastic multiplier can be determined by:
\[
\dot{\lambda} = \frac{\partial F}{\partial \sigma_{ijkl}} D^e_{ijkl} \dot{\varepsilon}_{ijkl} +\frac{\partial Q}{\partial \sigma_{ijkl}} + H^p
\]  
(A-III-8)

Substituting the expression of Equations A-III-8 and A-III-3 in Equation A-III-2 produces:

\[
\dot{\sigma}_{ijkl} = D^e_{ijkl} \varepsilon^p_{ijkl} = D^e_{ijkl} (\dot{\varepsilon}_{ijkl} - \dot{\varepsilon}_{ijkl}^p) = D^e_{ijkl} \left( \dot{\varepsilon}_{ijkl} - \dot{\lambda} \frac{\partial Q}{\partial \sigma_{ijkl}} \right)
\]  
(A-III-9)

\[
= D^e_{ijkl} \left( \dot{\varepsilon}_{ijkl} - \frac{\partial F}{\partial \sigma_{ijkl}} \frac{\partial Q}{\partial \sigma_{ijkl}} \right) = \frac{\partial F}{\partial \sigma_{ijkl}} \frac{\partial Q}{\partial \sigma_{ijkl}} + H^p
\]  

and, finally, the elasto-plastic matrix can be expressed by:

\[
D^e_{ijkl} = D^e_{ijkl} - \frac{\partial F}{\partial \sigma_{ijkl}} \frac{\partial Q}{\partial \sigma_{ijkl}} D^e_{ijkl} + H^p
\]  
(A-III-10)
Appendix 4: Change of the orthogonal basis for a transversely isotropic material

Direction cosines for the rotation of the coordinate system are expressed by Equation A-I-1 in Appendix A-1. Figure A-1 shows the local and global coordinate system for a transversely isotropic material.

![Diagram of coordinate systems](image)

Figure A-1 Change of orthogonal basis for transversely isotropic materials.

In the FEM three-dimensional (3D) analysis, the normal vector to the plane of isotropy is an input parameter:

\[ \tilde{x}_3 = \mathbf{n} = (n_1, n_2, n_3) \]  
(A-IV-1)

The angle between \( \tilde{x}_3 \) and its projection on the \( x_1x_2 \) plane is defined as:

\[ \alpha = \sin^{-1}(n_3) \quad \text{and} \quad n_3 = \sin \alpha \]  
(A-IV-2)

Also, the angle between the projection of the normal vector on the \( x_1x_2 \) plane and the \( x_i \) axis is defined as \( \beta \) :

\[ \cos(\alpha) \cos \beta = n_i \quad \text{and} \quad \beta = \cos^{-1}(n_i / \cos(\alpha)) \]  
(A-IV-3)
Since the material is isotropic in the $\tilde{x}_1\tilde{x}_2$ plane, any two perpendicular axes on this plane can be considered as the local axes. Thus, the local axis $\tilde{x}_2$ is proposed to be an intersection of the two planes of $\tilde{x}_1\tilde{x}_2$ and $x_1x_2$, and is expressed by:

$$\tilde{x}_2 = x_3 \times \tilde{x}_3 \quad (A-IV-4)$$

Finally, in the orthogonal coordinate system $\tilde{x}_1$ is defined as:

$$\tilde{x}_1 = \tilde{x}_2 \times \tilde{x}_3 = -\tilde{x}_3 \times \tilde{x}_2 \quad (A-IV-5)$$

and the cosine directions can be represented by:

$$a_i^j = \begin{bmatrix} 
    \sin(\alpha) \cos(\beta) & -\sin(\beta) & n_1 \\
    \sin(\alpha) \cos(\beta) & \cos(\beta) & n_2 \\
    -\cos(\alpha) & 0 & n_3
\end{bmatrix} \quad (A-IV-6)$$
Appendix 5: Transformation of the yield and plastic flow vectors

The yield and plastic flow vector are gradients of the yield and the potential surfaces in stress space. The yield vector and the potential surface are both scalar quantities, while stress is a second-order tensor. Therefore, similar to the stress measure, the yield and plastic flow vectors are second-order tensors:

\[
\frac{\partial F}{\partial \sigma_{ij}} e_i \otimes e_j \quad \text{and} \quad \frac{\partial Q}{\partial \sigma_{ij}} e_i \otimes e_j
\]

Thus, the transformation of the yield and plastic flow vectors follows the transformation of a second-order stress tensor:

\[
\hat{\sigma} = R^T \sigma R, \text{ or } \hat{\sigma}_{mn} = R_{im} \sigma_{ij} R_{jn}
\]

and

\[
\sigma = R \hat{\sigma} R^T, \text{ or } \sigma_{ij} = R_{im} \hat{\sigma}_{mn} R_{jn}
\]

Using the chain rule it can be proven that:

\[
\frac{\partial F}{\partial \sigma_{ij}} e_i \otimes e_j = \frac{\partial F}{\partial \hat{\sigma}_{mn}} \hat{e}_m \otimes \hat{e}_n \frac{\partial \hat{\sigma}_{mn}}{\partial \sigma_{ij}} e_i \otimes e_j
\]

\[
= \frac{\partial F}{\partial \hat{\sigma}_{mn}} \hat{e}_m \otimes \hat{e}_n \left( R_{mi} \sigma_{sk} R_{kn}^T \right) e_s \otimes e_t \frac{\partial F}{\partial \sigma_{ij}} e_t \otimes e_j
\]

\[
= \frac{\partial F}{\partial \hat{\sigma}_{mn}} \hat{e}_m \otimes \hat{e}_n \frac{R_{mi} \hat{\sigma}_{sk} R_{kn}^T}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{ij}} e_t \otimes e_j
\]

\[
= \frac{\partial F}{\partial \hat{\sigma}_{mn}} \hat{e}_m \otimes \hat{e}_n \frac{R_{mi} \hat{\sigma}_{sk} R_{kn}^T}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{mn}} e_t \otimes e_j
\]

\[
= R_{im} \frac{\partial F}{\partial \hat{\sigma}_{mn}} \hat{e}_m \otimes \hat{e}_n
\]
Appendix 6: Derivation of the constitutive tensor in multi-surface models

The essence of multi-mechanism plasticity models is similar to single-surface plasticity models. The incremental stress-strain relationship is defined by:

\[
\sigma_{ij} = D_{ijkl}^p \epsilon_{kl} \quad (A-VI-1)
\]

\[
\dot{\sigma}_{ij} = D_{ijkl}^e \dot{\epsilon}_{kl} = D_{ijkl}^e (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p)
\]

where \(\sigma_{ij}\) is the incremental stress tensor, \(\epsilon_{kl}\) is the incremental strain tensor and \(D_{ijkl}^p\) is the elasto-plastic constitutive tensor.

Similar to single-mechanism models, the consistency condition, Hooke’s law, and the flow rule are used in the derivation of the constitutive matrix. However, in the presence of multiple mechanisms, the consistency condition should be satisfied for all yield surfaces. Furthermore, the increment of plastic strain is expressed as the sum of plastic straining of all active mechanisms:

\[
F'_\beta (\sigma, \kappa) = 0, \quad dF'_\beta = \frac{\partial F'_\beta}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial F'_\beta}{\partial \kappa_h} \dot{\kappa}_h = 0 \quad (A-VI-2)
\]

\[
\dot{\epsilon}_{kl}^p = \sum_{\beta} \lambda_{\beta} \frac{\partial Q_{\beta}}{\partial \sigma_{kl}} \quad (A-VI-3)
\]

For simplicity, it is assumed that only two mechanisms are active and the hardening parameters of these mechanisms are uncoupled scalars. Substituting \(\dot{\sigma}_{ij}\) from Equation A-VI-1 into Equation A-VI-2 produces:

\[
dF_1 = \frac{\partial F_1}{\partial \sigma_{ij}} D_{ijkl}^e (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) + \frac{\partial F_1}{\partial \kappa_{h1}} \dot{\kappa}_{h1} = 0 \quad (A-VI-4)
\]

\[
dF_2 = \frac{\partial F_2}{\partial \sigma_{ij}} D_{ijkl}^e (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) + \frac{\partial F_2}{\partial \kappa_{h2}} \dot{\kappa}_{h2} = 0
\]

Using Equation A-VI-3 to replace \(d\epsilon^p\), the above equations can be rewritten as:
By using the definition of plastic modulus:

\[ H_1^p = -\frac{\partial F_1}{\partial \kappa_{hl}} \dot{\kappa}_{hl}, \quad H_2^p = -\frac{\partial F_2}{\partial \kappa_{h2}} \dot{\kappa}_{h2} \] (A-VI-7)

the expression in Equation A-VI-6 reduces to:

\[ dF_1 = \frac{\partial F_1}{\partial \sigma_{ij}} D_{ijkl}^e \dot{\epsilon}_{kl} - \dot{\lambda}_1 \frac{\partial Q_1}{\partial \sigma_{kl}} - \dot{\lambda}_2 \frac{\partial Q_2}{\partial \sigma_{kl}} + \frac{\partial F_1}{\partial \kappa_{hl}} \dot{\kappa}_{hl} = 0 \] (A-VI-8)

\[ dF_2 = \frac{\partial F_2}{\partial \sigma_{ij}} D_{ijkl}^e \dot{\epsilon}_{kl} - \dot{\lambda}_1 \frac{\partial Q_1}{\partial \sigma_{kl}} - \dot{\lambda}_2 \frac{\partial Q_2}{\partial \sigma_{kl}} + \frac{\partial F_2}{\partial \kappa_{h2}} \dot{\kappa}_{h2} = 0 \]

By defining new parameters as follows:

\[ H_{mn}^e = \frac{\partial F_m}{\partial \sigma_{ij}} D_{ijkl}^e \frac{\partial Q_o}{\partial \sigma_{kl}} \] (A-VI-9)

Equation A-VI-8 can be expressed by:

\[ dF_1 = \frac{\partial F_1}{\partial \sigma_{ij}} D_{ijkl}^e \dot{\epsilon}_{kl} - \dot{\lambda}_1 \left( H_{11}^e + H_{12}^p \right) - \dot{\lambda}_2 H_{12}^p = 0 \] (A-VI-10)

\[ dF_2 = \frac{\partial F_2}{\partial \sigma_{ij}} D_{ijkl}^e \dot{\epsilon}_{kl} - \dot{\lambda}_1 H_{21}^e - \dot{\lambda}_2 \left( H_{22}^e + H_{22}^p \right) = 0 \]

Also, the plastic multiplier can be determined as:
Replacing Equation A-VI-11 and Equation A-VI-3 in Equation A-VI-2 leads to:

\[
\dot{\sigma}_{ij} = D_{ijkl}^e \varepsilon_{kl} = D_{ijkl}^e (\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^p) = D_{ijkl}^e (\dot{\varepsilon}_{kl} - \dot{\lambda}_1 \frac{\partial Q_1}{\partial \sigma_{kl}} - \dot{\lambda}_2 \frac{\partial Q_2}{\partial \sigma_{kl}})
\]

\[
= D_{ijkl}^e \left( \frac{\partial Q_1}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e \dot{\varepsilon}_{kl}}{H^{ep}} - \frac{\partial Q_2}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e \dot{\varepsilon}_{kl}}{H^{ep}} \right)
\]

\[
= D_{ijkl}^e \left( \frac{\partial Q_1}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e \dot{\varepsilon}_{kl}}{H^{ep}} - \frac{\partial Q_2}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e \dot{\varepsilon}_{kl}}{H^{ep}} \right) \dot{\varepsilon}_{kl}
\]

Finally, the elasto-plastic constitutive matrix can be formulated as:

\[
D_{ijkl}^{ep} = D_{ijkl}^e \left( \frac{\partial Q_1}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e}{H^{ep}} - \frac{\partial Q_2}{\partial \sigma_{kl}} \frac{N_{lj} D_{ijkl}^e}{H^{ep}} \right)
\]

It should be noted that in this research the effects of hardening or softening has been disregarded in all application examples.
Appendix 7: The elasticity tensor of a Cosserat particulate continuum

The 2D elasticity matrix for Cosserat particulate materials is defined by (de Borst, 1993):

\[
\mathbf{D}^e = \begin{bmatrix}
2Gc_1 & 2Gc_2 & 2Gc_2 & 0 & 0 & 0 & 0 \\
2Gc_2 & 2Gc_1 & 2Gc_2 & 0 & 0 & 0 & 0 \\
2Gc_2 & 2Gc_2 & 2Gc_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G + G_c & G - G_c & 0 & 0 \\
0 & 0 & 0 & G - G_c & G + G_c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2Gh^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2Gh^2
\end{bmatrix}
\]  

(A-VII-1)

where

\[c_1 = \frac{(1 - \nu)(1 - 2\nu)}{1 - 2\nu}\]  

and \(c_2 = \nu / (1 - 2\nu)\)  

(A-VII-2)

and \(G\) and \(\nu\) are the classical shear modulus and Poisson’s ratio, respectively, \(G_c\) is an additional material constant, and \(h\) is the internal length scale of the problem which can be physically evaluated in a granular material (Arslan & Sture, 2008)
Appendices

Appendix 8: Work conjugacy in a Cosserat continuum

The starting point of this derivation is the strong form of the equilibrium equations, expressed by the equilibrium of force and moment in the Cosserat continuum as:

\[ \sigma_{ij,i} + b_j = 0 \]  \hspace{1cm} (A-VIII-1)

\[ m_k + \mu_k(j_i - e_{kij}\sigma_j) = 0 \]  \hspace{1cm} (A-VIII-2)

It is important to note that the equilibrium equations are expressed in a notation compatible with classical continuum theory in which the first index of the stress term, \( \sigma_{ij} \), denotes the normal to the surface, and the second index denotes the direction of the stress.

By applying an admissible virtual displacement, the expression for virtual work can be obtained by:

\[ \delta W = \int_V \delta u_j (\sigma_{ij,j} + b_j) + \int_b \delta \theta_k (m_k + \mu_k(j_i - e_{kij}\sigma_j)) dV = 0 \]  \hspace{1cm} (A-VIII-3)

\[ \delta W = \int_V (\delta u_j \sigma_{ij,j} + \delta u_j b_j) dV + \int_b (\delta \theta_k m_k + \delta \theta_k \mu_k j_i - \delta \theta_k e_{kij}\sigma_j) dV = 0 \]  \hspace{1cm} (A-VIII-4)

Using integration by parts, the above expression leads to:

\[ \delta W = \int_V (\delta u_j \sigma_{ij,j}) dV - \int_b \delta u_j \sigma_{ij} dV + \int_V \delta u_j b_j dV + \int_b \delta \theta_k m_k dV + \int_V (\delta \theta_k \mu_k) dV - \int_b \delta \theta_k \mu_k dV - \int_V \delta \theta_k e_{kij}\sigma_j dV = 0 \]  \hspace{1cm} (A-VIII-5)

The integral \( \int_V \delta u_j b_j dV \) accounts for the body forces of the system, and the integral \( \int_V \delta \theta_k m_k dV \) accounts for the body moment of the system. Also, using Green’s theorem,
the two integrals \( \int_V (\delta u_j \sigma_{ij})_j dV \) and \( \int_V (\delta \theta_k \mu_{kj})_j dV \), reduce to integrals over the surface. Substituting these integrals in Equation A-VIII-5 leads to:

\[
\delta W = + \int_S (\delta u_j \sigma_{ij}) n_j dS - \int_V \delta u_{j,i} \sigma_{ij} dV \\
+ F_{\text{body force}} + F_{\text{body moment}} \\
+ \int_S (\delta \theta_k \mu_{kj}) n_j dS - \int_V \delta \theta_{k,j} \mu_{kj} dV \\
- \int_V \delta \theta_{k} e_{kij} \sigma_{ij} dV = 0
\]  

(A-VIII-6)

In the above expression, the integral \( \int_S (\delta u_j \sigma_{ij}) n_j dS \) expresses the force traction, while \( \int_S (\delta \theta_k \mu_{kj}) n_j dS \) expresses the moment-traction force. Thus, the expression in Equation A-VIII-6 can be rewritten in the following form:

\[
\delta W = + F_{\text{force traction}} + F_{\text{moment traction}} \\
+ F_{\text{body force}} + F_{\text{body moment}} \\
\int_V (\delta u_{j,i} \delta \theta_k e_{kij}) \sigma_{ij} dV \\
- \int_S \delta \theta_{k,j} \mu_{kj} dV = 0
\]  

(A-VIII-7)

From the above equation, measures of strain conjugate with the defined measures of stress and couple stress are obtained. It is clear that the measure of strain conjugate to \( \sigma_{ij} \) is \( (u_{j,i} - e_{kij} \theta_k) \) and the measure of strain conjugate to \( \mu_{kj} \) is \( \theta_{k,j} \). In order to be compatible with conventional notation, the small deformation measure of strain is defined so that \( \gamma_{ij} \) is conjugate to \( \sigma_{ij} \). Therefore, \( \gamma_{ij} \) can be expressed as:

\[
\gamma_{ij} = u_{j,i} - e_{kij} \theta_k
\]  

(A-VIII-8)

Here, a switch in the order of the indices of \( \gamma_{ij} \) and \( u_{j,i} \) is made. The large deformation measure of strain, \( \Gamma_{ij} \), is defined as \( R^T F \). Subsequently, it should be noted that \( \Gamma_{ji} \) is conjugate to \( \sigma_{ij} \), also, in the limit case where deformations are small, \( \Gamma_{ij} \) reduces to \( \gamma_{ji} \).
Appendix 9: Classical small deflection theory of thin plates

The classical theory of thin plates is based on an approach by Kirchhoff (1876), which provides a reasonably good approximation to the solution of the differential equations of three-dimensional elasticity. In order to obtain an exact solution, a plate should be considered as a three-dimensional continuum. Such an approach results in insurmountable mathematical difficulties and consequently is impractical. Therefore, based on their structural differences, plates are divided into four categories (Szilard, 2004):

Stiff plates \((1/50 < h/L < 1/10)\), which are usually referred to as thin plates, have flexural rigidity and carry the load by bending and twisting moments as well as transverse shear.

Membranes \((h/L < 1/50)\) are very thin plates in which any flexural rigidity can be neglected. Membranes carry loads by axial forces and central shear.

Moderately thick plates \((1/10 < h/L < 1/5)\) are similar to stiff plates, except that the effect of transverse shear forces on the normal stress components are taken into account.

Thick plates \((1/5 < h/L)\) have an internal stress condition similar to a 3D continuum.

Below the formulation of stiff plates, or thin plates, is presented. For the simplifications and assumptions used in the derivation of the plate equations one can refer to Szilard (2004).

It is assumed that the plate is subjected to a transverse uniform load on the surface with a magnitude of \(p_z\). The external and internal forces acting on the element of the middle surface are shown in Figure A-2, and the resultant stress distribution is shown in Figure A-3. The formulations are in terms of transverse deflection, \(w(x,y)\). The internal and external forces, stresses, and deflection components are considered positive when they point toward the positive direction of the coordinate axis. Also, moments are assumed positive when they produce tension at the lower face of the pertinent section. In the following formulation the subscripts of the bending and twisting moments refer to the
stresses by which they are produced. For example, $M_x$ is the moment caused by $\sigma_x$, and does not refer to a moment rotating around the $X$ axis.

Figure A-2 Schematic illustration of external and internal forces on an element of the middle surface (Szilard, 2004).

Figure A-3 Stress components on a plate element (Szilard, 2004).
By assuming that the plate is only subjected to lateral forces, the following relations can be applied:

\[ \Sigma M_x = 0, \quad \Sigma M_y = 0, \quad \Sigma p_z = 0 \] (A-IX-1)

Considering the stress and force distribution on an element of the material, the above equations lead to:

\[ \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} = q_x \] (A-IX-2)

\[ \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_y}{\partial y} = q_y \] (A-IX-3)

\[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p_z \] (A-IX-4)

Substituting Equations A-IX-2 and A-IX-3 into Equation A-IX-4 and observing that \( m_{yx} = m_{xy} \) leads to:

\[ \frac{\partial^2 m_x}{\partial x^2} + 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = -p_z(x,y) \] (A-IX-5)

In the above equation the bending and twisting moments depend on strains and, subsequently, displacements \((u,v,w)\).

Using the geometry of a deflected plate as depicted in Figure A-4, the angle of rotation of lines I-I and II-II can be expressed as:

\[ \theta = -\frac{\partial w}{\partial x} \quad \text{and} \quad \theta + \ldots = \theta + \frac{\partial \theta}{\partial x} dx \] (A-IX-6)

The strain measure can be expressed by:

\[ \varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \quad \text{and} \quad \gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \] (A-IX-7)
and the curvature changes of the deflected middle surface are defined by:

\[ \kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \text{and} \quad \chi = -\frac{\partial^2 w}{\partial x \partial y} \]  

(A-IIX-8)

where \( \chi \) is the warping of the plate.

Finally the moments can be expressed as:

\[ m_x = \int_{-h/2}^{+h/2} \sigma_x z dz = D(\kappa_x + v\kappa_y) \]  

(A-IIX-9)

\[ m_y = \int_{-h/2}^{+h/2} \sigma_y z dz = D(\kappa_y + v\kappa_x) \]

\[ m_{xy} = m_{yx} = \int_{-h/2}^{+h/2} \tau_{xy} z dz = \int_{-h/2}^{+h/2} \tau_{yx} z dz = D(1-v)\chi \]
where $m_x$ and $m_y$ are the bending moments and $m_{xy} = m_{yx}$ are the twisting moments that produce the in-plane shear stresses $\tau_{xy}$ and $\tau_{yx}$. $D$ is the flexural rigidity of the plate expressed by:

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

(A-IX-10)
Appendix 10: Cosserat plasticity for granular materials

A comprehensive review on the kinematics and mechanics of the Cosserat continuum model for granular materials has been carried out by Vardoulakis (1995). The link between Cosserat theory and localization analysis was made by Mühlhaus and Vardoulakis (1987) and Steinmann and Willam (1991). In recent years, discussions on the applicability of the Cosserat continuum approach to granular materials and the nature of stress and couple stress tensors have received considerable attention. For a review of these discussions refer to Friio et al. (2006). Different aspects of the Cosserat continuum approach in representation of an assembly of particles, and especially, comparison of the Cosserat formulation with discrete formulation have been the subject of more recent research (Ehler et al.; 2003; Pasternak & Mühlhaus, 2005).

In micropolar theories, the second invariant of the deviatoric stress, $J_2$, is modified in order to incorporate the effect of coupled moment stresses (de Borst, 1993):

$$ J_2 = a_1 s_{ij} s_{ij} + a_2 s_{ij} s_{ji} + a_3 \frac{m_{ij}m_{ji}}{\ell^2} $$  \hspace{1cm} (A-X-1)

or (Manzari, 2004)

$$ J_2 = a_1 s_{ij} s_{ij} + a_2 s_{ij} s_{ji} + a_3 \frac{m_{ij}m_{ji}}{\ell^2} + a_4 \frac{m_{ij}m_{ji}}{\ell^2} \quad \text{with} \quad a_4 = 0 \hspace{1cm} (A-X-2) $$

In Equation A-X-1 the summation convention with respect to repeated indices has been adopted. Here, $s_{ij}$ is the deviatoric stress tensor, $m_{ij}$ is the coupled stress tensor, and $a_1$, $a_2$, and $a_3$ are material parameters. In the absence of couple stresses, i.e., $m_{ij} = 0$ and $s_{ij} = s_{ji}$, Equation A-X-1 reduces to:

$$ J_2 = (a_1 + a_2) s_{ij} s_{ij} \hspace{1cm} (A-X-3) $$

As the Cosserat continuum reduces to the classical continuum in the absence of micromoments, the following constraint should be satisfied:
Based on micromechanical considerations, Vardoulakis (1995) and Mühlhaus and Vardoulakis (1987) proposed the values $a_1 = \frac{3}{4}$, $a_2 = -\frac{1}{4}$, and $a_3 = -1$ for the static solution, and values of $a_1 = \frac{3}{8}$, $a_2 = \frac{1}{8}$, and $a_3 = \frac{1}{4}$ for the kinematic solution. The other values proposed are $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{4}$, and $a_3 = \frac{1}{2}$, which give rise to a particularly simple numerical algorithm. Nevertheless, the basic characteristics are not affected by the precise choice of this set of material parameters. Using the latter set of parameters, $J_2$ can be expressed by:

$$J_2 = \frac{1}{2} \sigma^T P \sigma$$

(A-X-5)

where $P$ for plane strain problems is expressed by:

$$P = \begin{bmatrix}
\frac{2}{3} & -1/3 & -1/3 & 0 & 0 & 0 & 0 \\
-1/3 & \frac{2}{3} & -1/3 & 0 & 0 & 0 & 0 \\
-1/3 & -1/3 & \frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

(A-X-6)

Similar to a classical, non-polar continuum, an associated flow rule can be expressed by:

$$\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}}$$

(A-X-7)

Also, the effective plastic strain can be expressed by:

$$\bar{\varepsilon}^p = \left[ \frac{1}{3} e_{ij}^p e_{ij}^p + \frac{1}{3} e_{ij}^p e_{ji}^p + \frac{2}{3} \kappa_{ij}^p \kappa_{ij}^p \ell^2 \right]^{1/2}$$

(A-X-8)

The Lode angle is defined as follows (Manzari, 2004):
\[
\theta^i = \frac{1}{3} \sin^{-1} \left( \frac{-3\sqrt{3}}{2} \left( \frac{J_{3s}}{J_{2s}^{1/2}} \right) \right)
\]

(A-X-9)

where \( J_{3s} \) and \( J_{3s} \) are the second and third invariants of the symmetric part of the (force) stress tensor.
Appendices

Appendix 11: Modification of stress invariants for Cosserat layered materials

In this section, based on micromechnical considerations for layered materials, it is proposed that stress invariants be modified so that they take into account the effect of the micromoments of the system. This work can serve as a suggested guideline, but further analysis and verifications are required to assess the validity and further extension of the formulation.

In the Cosserat formulation, the effects of discontinuities are implicit in the system. The stress measures represent an average value of the stresses due to the existence of microstructures within the material. It is proposed that the stress invariants of Cosserat layered materials can also be modified in an average sense over the characteristic length or thickness of each layer by considering the mechanics of a single beam as depicted in Figure A-5.

![Figure A-5 Mechanics of a single beam under bending.](image)

This derivation is based on the mathematical definition of invariants of second-rank tensors, as quantities whose values do not depend on the coordinate system they are defined in. The stress measure is treated as a second-rank tensor.

In order to take into account the effect of micromoments, it is proposed to use a modified stress tensor:

\[
\sigma_{\text{mod}} = \begin{bmatrix}
\sigma_{11}^{\text{mod}} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22}^{\text{mod}} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}^{\text{mod}}
\end{bmatrix}
\]

\( (A-XI-1) \)
Using the mechanics of a single beam, the axial stress in the beam can be expressed based on mechanical moments by:

\[ \sigma_{11}^m = \frac{M_{21}x_3}{I} \quad \text{(A-XI-2)} \]

where \( I \) is the moment of inertia expressed by:

\[ I = \frac{bh^3}{12} \quad \text{(A-XI-3)} \]

In the Cosserat continuum, the state of stress at each point of the material is expressed by a force stress tensor, \( \sigma \), and a moment stress tensor, \( \mu \). For a layer with a thickness of \( h \), the mechanical moment, \( M \), can be calculated based on micromoments by:

\[ M_{ij} = \mu_{ij}h \quad \text{(A-XI-4)} \]

Using Equations A-XI-1 and A-XI-2, the axial stress in the beam can be modified based on values of micromoments by:

\[ \sigma_{11}^{\text{mod}} = \sigma_{11}^\sigma + \sigma_{11}^m \quad \text{and} \quad \sigma_{22}^{\text{mod}} = \sigma_{22}^\sigma + \sigma_{22}^m \quad \text{(A-XI-5)} \]

where \( \sigma_{11}^m \) is the axial force which results from the micromoments of the system:

\[ \sigma_{11}^m = \frac{M_{21}x_3}{I} = \frac{m_{12}hx_3}{I} \quad \text{(A-XI-6)} \]

\[ \sigma_{22}^m = \frac{M_{12}x_3}{I} = \frac{m_{12}hx_3}{I} \]

In the above formulation, the notation for the moment subscripts is in accordance with the convention followed in the Cosserat formulation, which is different from the conventions of plate theory.

It is noteworthy that, similar to stress and couple stress measures, stress invariants are point-wise functions. However, in the Cosserat continuum model proposed in this
research, the definition of stress invariants are modified based on micromoments and average over the thickness of the layer. The first invariant of stress, $I_1$, can be expressed by:

$$I_1 = \sigma_{ii}$$

(A-XI-7)

$$I_1' = \frac{1}{h} \int_{-h/2}^{+h/2} I_1 dx_3 = \frac{1}{h} \int_{-h/2}^{+h/2} \sigma_{ii}^{\text{mod}} dx_3 = \left[ \left( \int_{-h/2}^{h/2} (\sigma_{11} + \sigma_{11}^m) dx_3 \right) + \left( \int_{-h/2}^{h/2} (\sigma_{22} + \sigma_{22}^m) dx_3 + \sigma_{33}^\sigma \right) \right]$$

Due to dependence of first invariant to even terms of power of $x_3$:

$$\int_{-h/2}^{h/2} \sigma_{11}^m dx_3 = \left( \int_{-h/2}^{h/2} \mu_2 h x_3^2 \right)_{h/2} = 0$$

and it can be concluded that:

$$I_1' = \sigma_{ii} = \left[ \sigma_{11}^\sigma + \sigma_{22}^\sigma + \sigma_{33}^\sigma \right]$$

Therefore, the first invariant of stress is indifferent with respect to values of the micromoments of the system.

The second invariant of stress is expressed by:

$$I_2 = \frac{1}{2} (\sigma_{ij} \sigma_{ij} - \sigma_{ii} \sigma_{jj})$$

(A-XI-8)

In order to calculate the second invariant of the deviatoric part of the stress, the modified deviatoric stress, $S^{\text{mod}}$, at each point along the layer thickness, $h$, is defined by:

$$S^{\text{mod}} = \begin{bmatrix} \sigma_{11}^{\text{mod}} - \frac{I^{\text{mod}}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}^{\text{mod}} - \frac{I^{\text{mod}}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}^{\text{mod}} - \frac{I^{\text{mod}}}{3} \end{bmatrix}$$

(A-XI-9)

where

$$I^{\text{mod}} = (\sigma_{11}^\sigma + \sigma_{22}^\sigma + \sigma_{33}^\sigma) + (\sigma_{11}^m + \sigma_{22}^m)$$
It should be noted that the above stress tensor is a point-wise function; therefore, the value of $I^{\text{mod}}$ is based on the point-wise values of the matrix trace. Using the above definition, and replacing values of $\sigma_{11}^{\text{mod}}$ and $\sigma_{22}^{\text{mod}}$ by Equation A-XI-5, the diagonal terms of the above expression can be replaced by:

$$S_{11}^{\text{mod}} = S_{11}^{\sigma} + \frac{2}{3} \sigma_{11}^{m} - \frac{1}{3} \sigma_{22}^{m}, \quad S_{22}^{\text{mod}} = S_{22}^{\sigma} + \frac{2}{3} \sigma_{22}^{m} - \frac{1}{3} \sigma_{11}^{m}, \quad \text{and}$$

$$S_{33}^{\text{mod}} = S_{33}^{\sigma} - \left( \frac{\sigma_{11}^{m} + \sigma_{22}^{m}}{3} \right)$$  \hspace{1cm} (A-XI-10)

The effect of micromoments on the definition of $J_2$ is investigated considering the terms that are affected:

$$J_2^* = \frac{1}{2h} \left[ \int_{-h/2}^{h/2} S_{11}^{\text{mod}} S_{22}^{\text{mod}} - S_{12} S_{21} + S_{22}^{\text{mod}} S_{33}^{\text{mod}} - S_{23} S_{32} + S_{11} S_{33}^{\text{mod}} - S_{13} S_{31} - S_{23} S_{32} \right] \, dx_3 \hspace{1cm} (A-XI-11)$$

Using values of the modified stress expressed by Equation A-XI-10, the above expression reduces to:

$$J_2^* = \frac{1}{2} \left( S_{11}^{\sigma} S_{22}^{\sigma} + S_{11}^{\sigma} S_{33}^{\sigma} + S_{22}^{\sigma} S_{33}^{\sigma} + S_{11}^{\sigma} S_{22}^{\sigma} + S_{11}^{\sigma} S_{22}^{\sigma} - S_{12} S_{21} - S_{13} S_{31} - S_{23} S_{32} \right)$$

$$+ \frac{1}{2h} \int_{-h/2}^{h/2} \left( \frac{1}{9} \sigma_{11}^{m} \sigma_{22}^{m} - \frac{1}{3} \left( \sigma_{11}^{m} \right)^2 - \frac{1}{3} \left( \sigma_{22}^{m} \right)^2 \right) \, dx_3 \hspace{1cm} (A-XI-12)$$

Substituting $\sigma_{11}^{m}$ and $\sigma_{22}^{m}$ according to Equation A-XI-6 into Equation A-XI-12 leads to:

$$J_2^* = \frac{1}{2} \left( S_{11}^{\sigma} S_{22}^{\sigma} + S_{11}^{\sigma} S_{33}^{\sigma} + S_{22}^{\sigma} S_{33}^{\sigma} + S_{11}^{\sigma} S_{22}^{\sigma} + S_{11}^{\sigma} S_{22}^{\sigma} - S_{12} S_{21} - S_{13} S_{31} - S_{23} S_{32} \right)$$

$$+ \frac{1}{h^2} \left( -2 \mu_{12}^2 - 2 \mu_{21}^2 + \frac{2}{3} \mu_{12} \mu_{21} \right) \hspace{1cm} (A-XI-13)$$

The third invariant of stress, which is the determinant of the stress matrix, can be modified in a similar way. Nevertheless, the definitions of the second and third invariants are rather complication and make it difficult to apply the proposed method.
Appendix 12: Elasticity solution for the deformation of a column of transversely isotropic material subjected to uniaxial compression

This appendix seeks the analytical solution to the deformation of a transversely isotropic material under uniaxial compression. The example is solved numerically in Section 0. In order to simplify the expressions, the Poisson’s ratio is assumed to be zero. Also, it is assumed that $E_{11} = E_{22}$.

Figure A-6 Geometry and boundary conditions for elasticity solution of a column of transversely isotropic materials.

Using the appropriate transformation matrices and choosing the stress state as $\sigma = \{0 \quad \sigma \quad 0 \quad 0 \quad 0 \}$, the components of strain can be expressed as:

\[
\varepsilon_{11} = 2 \left( -\frac{2E_{11} + G_{12}}{E_{11}G_{12}} \right) \sin^2(\alpha) \cos^2(\alpha) \sigma \\
\varepsilon_{22} = \left( G_{12} \cos^4(\alpha) + 4E_{11} \cos^2(\alpha) \sin^2(\alpha) + G_{12} \sin^4(\alpha) \right) \sigma \\
\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} \left( \frac{E_{11} - 2G_{12}}{E_{11}G_{12}} \right) \frac{\sin(4\alpha)}{4} \sigma
\]

Using the following relations:
\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \] (A-XII-2)

\[ \varepsilon_{11} = \frac{\partial u_2}{\partial x_2} \]

\[ \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \]

the displacements can be obtained by:

\[ u_1 = \int \varepsilon_{11}dx_1 + f(y) \] (A-XII-3)

\[ u_2 = \int \varepsilon_{22}dx_2 + f(x) \]

\[ f'(x) + f'(y) = 2\varepsilon_{12} = \text{const} \]

The above relations can be expressed as:

\[ u_1 = \left[ 2 \left( \frac{-2E_{11} + G_{11}}{E_{11}G_{12}} \right) \sin^2(\alpha) \cos^2(\alpha) \sigma \right] x + ay + b \] (A-XII-4)

\[ u_2 = \left[ \left( \frac{G_{12}}{E_{11}G_{12}} \cos^4(\alpha) + \frac{4E_{11}}{E_{11}G_{12}} \cos^2(\sigma) + G_{12} \sin^4(\alpha) \right) \sigma \right] y + cx + d \]

\[ a + c = 2\varepsilon_{12} = \left( \frac{2E_{11} - G_{11}}{E_{11}G_{12}} \right) \frac{\sin(4\alpha)}{4} \sigma \]

Using the boundary condition at \( u_2(x,0) = 0 \):

\[ u_2 = cx + d = 0 \Rightarrow c = 0; \quad d = 0 \] (A-XII-5)

and \[ a = \left( \frac{E_{11} - 2G_{11}}{E_{11}G_{12}} \right) \frac{\sin(4\alpha)}{4} \sigma \]

In order to determine the coefficient \( b \), the rigid body motion of the problem should be constrained. The solution depends on the location at which the horizontal displacement is constrained. However, the slope of the contours of the horizontal displacement is constant and can be expressed as a tangent of the points with constant horizontal displacement:
\[
\tan(\beta) = -\frac{(2 \sin(\alpha) \cos(\alpha))}{\cos(2\alpha)} = -\tan(2\alpha)
\] (A-XII-6)

This explains the pattern of the contours of the horizontal displacement obtained in Figure 4-28. The above equation is independent of the values of the shear modulus and Young’s modulus and is valid for transversely isotropic materials where Young’s modulus \( E_{22} = E_{11} \). Similarly the slope of the contours of total displacement can be obtained as a function of \( \tan(2\alpha) \).
Appendices

Appendix 13: Derivation of the geometric stiffness matrix

A comprehensive study of the micropolar theory of finite rotations and finite strains is conducted by Steinmann (1994). The starting point of this derivation is the large deformation Cosserat measure of strain, expressed as:

\[ \Gamma = (R^c)^T F = (R^c)^T R^c U \quad \text{(A-XIII-1)} \]

The above measure of strain is used by Adhikary et al. (1999) in large deformation analysis of 2D layered continua. Using the above measure of strain, the expression of virtual work leads to:

\[ \delta W_{\text{int}} = \int_{V'} \sum_{ij} \delta \Gamma_{ij} dV + \int_{V'} M_{ij} \delta K_{ij} dV \quad \text{(A-XIII-2)} \]

where \( V \) is the domain of integration in the material or initial configuration. It is noteworthy that the work conjugate stress measure to \( \Gamma_{ij} \) is analogous to Biot's stress tensor of a classical continuum, and \( M \) and \( K \) are conjugate couple stress and curvature measures defined in the initial configuration:

\[ \Sigma = JF^{-1} \sigma R^c \quad \text{(A-XIII-3)} \]
\[ M = JF^{-1} \mu R^c \]

In the above, \( \sigma_{ij} \) is analogous to the classical Cauchy stress and is the work conjugate stress measure to the small deformation measure of strain, \( \gamma \) (see Appendix 8), and \( m \) is the couple stress measure defined in the current configuration. For more information on this subject one can refer to Steinmann (1994).

It is clear that if the deformations are small then, it can be safely assumed that \( \Sigma = \sigma \).

Using the above assumptions, the variation of virtual work can be expressed by:

\[ \Delta \delta W_{\text{int}} = \Delta \left( \int_{V'} \sum_{ij} \delta \Gamma_{ij} dV + \int_{V'} M_{ij} \delta K_{ij} dV \right) = \int_{V'} \Delta \sum_{ij} \delta \Gamma_{ij} dV + \int_{V'} \sum_{ij} \Delta \delta \Gamma_{ij} dV \]
\[ + \int_{V'} \Delta M_{ij} \delta K_{ij} dV + \int_{V'} M_{ij} \Delta \delta K_{ij} \quad \text{(A-XIII-4)} \]
In the above, the integrations are over the initial configuration. This research deals with a relatively simple version of the large deformation theory; similar to what is assumed in engineering beam and plate buckling theories, the geometric nonlinearity associated with curvature and coupled stresses are neglected. Consequently, the effect of the last term in Equation A-XIII-4 can be disregarded.

Since, it is assumed that strains and rotations are small prior to buckling, the deformation measure, $\delta \Gamma_{ji}$, reduces to $\delta \gamma_{ij}$. In contrast to the assumption of linear variation of curvature, which is valid in most large deformation analyses, this assumption restricts the formulation to cases where deformations are small ($F \cong I$, and $J \cong 1$), yet allows for formulation of initial stress stiffening effects.

Using these simplifying assumptions, the expression for the variation of virtual work reduces to:

$$\Delta \delta W_{\text{int}} = \int_V \Delta \sigma_{ij} \delta \gamma_{ij} dV + \int_V \sigma_{ij} \Delta \delta \Gamma_{ji} dV + \int_V \Delta \mu_{ij} \delta \kappa_{ij} dV$$

(A-XIII-5)

Since the derived matrix is applied in a linear buckling analysis, it can be assumed that $F \cong I$, and that the initial and current domains of integration coincide. Also, based on the aforementioned assumptions, it can be argued that the stress measure, $\sigma_{ij}$, is the work conjugate to the strain measure, $\gamma_{ij}$, and in the limit case, where the Cosserat continuum reduces to classical continuum, $\sigma_{ij}$ reduces to the Cauchy stress measure. In the above expression, the effect of the first and last terms appear in the material stiffness matrix, expressed by Equation 4-34 in Section 4.4, while the following formulation accounts for the buckling geometric stiffness matrix:

$$\int_V \sigma_{ij} \Delta \delta \Gamma_{ji} dV = (\delta \vec{U}_N)^T K_{\text{geom}} \Delta \vec{U}_M$$

(A-XIII-6)

Replacing Equation A-XIII-1 into Equation A-XIII-6, leads to:
\[ \int_V \sigma_y \Delta \delta (R_c^T F)_{ji} dV = \int_V \sigma_y \Delta (\delta R_c^T F + R_c^T \delta F)_{ji} dV = \int_V \sigma_{ij} (\Delta \delta R_c^T F + \delta R_c^T \Delta F + \Delta R_c^T \delta F + \Delta R_c^T \Delta \delta F)_{ji} dV \quad (A-XIII-7) \]

It is clear that the following statement is always valid:

\[ \Delta \delta F = 0 \quad (A-XIII-8) \]

Therefore, the expression in Equation A-XIII-7 reduces to:

\[ (\delta \bar{U}_N)^T K_{\text{geom}} \Delta \bar{U}_M = \int_V \sigma_{ij} (\Delta \delta R_c^T F + \delta R_c^T \Delta F + \Delta R_c^T \delta F)_{ji} dV \quad (A-XIII-9) \]

Using the appropriate rotation tensor, the geometric stiffness matrix for buckling analysis can be evaluated in both 2D and 3D frameworks.

**2D geometric stiffness matrix**

In 2D analysis, the rotation matrix can be expressed by:

\[ R^C = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) \end{bmatrix} \quad (A-XIII-10) \]

Substituting \( R^C \) into the expression for \( \Delta \delta \Gamma \) leads to:
Derivation of the stress stiffness matrix is the goal of this appendix. Because a linear buckling analysis is performed for verification of the proposed matrix, as explained in the previous section, further simplifying assumptions are made. It is assumed that both strains and rotations are infinitesimal prior to the onset of buckling, and that the post-
buckling behaviour is not considered. Based on these assumptions, it can be concluded that:

\[
F = \frac{\partial x_i}{\partial X_j} \cong I
\]  

Also, the assumption concerning small rotations leads to:

\[
\sin(\theta_i)\delta\theta_i = 0 \quad \text{(A-XIII-14)}
\]
\[
\cos(\theta_i)\delta\theta_i = \delta\theta_i
\]

Based on the relations of A- XIII-14, \( \delta(R_c)^T \) can be expressed by:

\[
\delta(R_c)^T = \begin{pmatrix} 0 & \delta\theta_3 \\ -\delta\theta_3 & 0 \end{pmatrix}
\]  

and \( \Delta\delta(R_c)^T \) reduces to:

\[
\Delta\delta(R_c)^T = \begin{pmatrix} -\delta\theta_3\Delta\theta_3 & 0 \\ 0 & -\delta\theta_3\Delta\theta_3 \end{pmatrix}
\]  

Using the following relations:

\[
x_i = X_i + u_i \quad \text{(A-XIII-17)}
\]
\[
\delta x_i = \delta X_i + \delta u_i = \delta u_i
\]

and using the interpolation functions, \( \phi \), for the displacement and the Cosserat rotation field, the geometric stiffness matrix can be expressed by:

\[
\begin{pmatrix}
0 & 0 & \left[\left(-\sigma_{22}\phi_{N,2} - \sigma_{12}\phi_{X,1}\right)\right]\phi_M \\
0 & 0 & \left[\left(\sigma_{11}\phi_{X,1} + \sigma_{21}\phi_{N,2}\right)\right]\phi_M \\
\left[\left(-\sigma_{22}\phi_{M,2} - \sigma_{12}\phi_{M,1}\right)\right]\phi_N & \left[\left(\sigma_{11}\phi_{M,1} + \sigma_{21}\phi_{M,2}\right)\right]\phi_N & \left[\left(-\sigma_{11} - \sigma_{22}\right)\phi_M \phi_N \right]
\end{pmatrix}
\]  

(A-XIII-18)
3D geometric stiffness matrix

The intention of this section is to evaluate the geometric stiffness matrix expressed by Equation A-XIII-9. For the derivation of the 3D geometric stiffness or buckling matrix, the same rules and assumptions explained in the 2D analysis are applied. The main complication in performing 3D analysis is the evaluation of the 3D rotation matrix and the coupling terms that can arise as a result of a rotation vector with more than one component.

In this research, it is proposed to use a second-order approximation to the rotation matrix:

\[ R^c \approx I + \text{span}(\theta) + \frac{1}{2} \text{span}^2(\theta) \]  
(A-XIII-19)

The chosen order of approximation depends on the highest order of differentiation that is present in the formulation. Here, the term \( \Delta \delta (R_c)^T \) requires application of at least a second-order approximation so that the effect of the cosine terms in Equation 4-6 is taken into account in the formulation. Using Equation A-XIII-19, the rotation matrix, \( R^c \), reduces to:

\[
R^c \approx \begin{bmatrix}
1 & -\theta_3 & \theta_2 \\
\theta_3 & 1 & -\theta_1 \\
-\theta_2 & \theta_1 & 1
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
-\theta_2^2 - \theta_3^2 & \theta_1 \theta_2 & \theta_1 \theta_3 \\
-\theta_1 \theta_2 & (\theta_1)^2 - (\theta_3)^2 & \theta_2 \theta_3 \\
-\theta_1 \theta_3 & \theta_2 \theta_3 & (\theta_1)^2 - (\theta_2)^2
\end{bmatrix}
\]  
(A-XIII-20)

Based on the small rotations assumption, and similar to the assumption expressed by Equation A-XIII-14, it can be concluded that:

\[ \theta_i \delta \theta_j \approx 0 \]  
(A-XIII-21)

Based on Equation A-XIII-21, the effects of the second-order terms in \( \delta R^c \) are negligible, and the expression for the virtual rotation reduces to:
\[ \delta R^c \cong \begin{bmatrix} 0 & -\delta \theta_3 & \delta \theta_2 \\ \delta \theta_3 & 0 & -\delta \theta_1 \\ -\delta \theta_2 & \delta \theta_1 & 0 \end{bmatrix} \]  

(A- XIII-22)

which is in accordance with Equation A- XIII-14.

Variation of the rotation matrix, \( \Delta \delta R^c \), is obtained by applying Equation A- XIII-20 and the following relation to Equation A- XIII-19:

\[ \Delta \delta \theta_i = 0 \]  

(A- XIII-23)

As a result of the relations expressed by Equation A-XIII-20 and A-XIII-22, any contribution from the first matrix in Equation A-XIII-19 will be neglected, and the expression for \( \Delta \delta R^c \) reduces to:

\[ \Delta \delta R^c \cong \frac{1}{2} \begin{bmatrix} -2(\Delta \theta_i \delta \theta_2 + \Delta \theta_2 \delta \theta_i) & \Delta \theta_i \delta \theta_2 + \delta \theta_i \Delta \theta_2 & \Delta \theta_i \delta \theta_3 + \delta \theta_i \Delta \theta_3 \\ \Delta \theta_1 \delta \theta_3 + \Delta \theta_3 \delta \theta_1 & -2(\Delta \theta_i \delta \theta_1 + \Delta \theta_1 \delta \theta_i) & \Delta \theta_2 \delta \theta_3 + \delta \theta_2 \Delta \theta_3 \\ \Delta \theta_2 \delta \theta_3 + \Delta \theta_3 \delta \theta_2 & \Delta \theta_2 \delta \theta_3 + \delta \theta_2 \Delta \theta_3 & -2(\Delta \theta_i \delta \theta_1 + \delta \theta_i \Delta \theta_2) \end{bmatrix} \]  

(A- XIII-24)

In the above expression, the off-diagonal terms account for the coupling of rotation components.

By applying the assumption concerning small deformation expressed by Equation A-XIII-13, and substituting Equations A- XIII-24 and A- XIII-23 in Equation A- XIII-9, the relation for the geometric stiffness matrix can be expressed as:

\[ \left( \delta \tilde{U}_N \right)^T K^{geom} \Delta \tilde{U}_M = \int_V \begin{bmatrix} \sigma_{11}(\delta \theta_3 \Delta F_{11} - \delta \theta_2 \Delta F_{12}) & \sigma_{21}(\delta \theta_3 \Delta F_{21} - \delta \theta_2 \Delta F_{22}) & \sigma_{31}(\delta \theta_3 \Delta F_{31} - \delta \theta_2 \Delta F_{32}) \\ \sigma_{12}(\delta \theta_3 \Delta F_{11} + \delta \theta_2 \Delta F_{22}) & \sigma_{22}(\delta \theta_3 \Delta F_{21} + \delta \theta_2 \Delta F_{22}) & \sigma_{32}(\delta \theta_3 \Delta F_{31} + \delta \theta_2 \Delta F_{32}) \\ \sigma_{13}(\delta \theta_2 \Delta F_{11} - \delta \theta_1 \Delta F_{21}) & \sigma_{23}(\delta \theta_2 \Delta F_{21} - \delta \theta_1 \Delta F_{22}) & \sigma_{33}(\delta \theta_2 \Delta F_{31} - \delta \theta_1 \Delta F_{32}) \end{bmatrix} dV \]  

(A- XIII-25)
Finally, using Equation A-XIII-13 and the interpolation function, \( \phi \), for both the displacement and the Cosserat rotation field, the geometric stiffness matrix can be expressed by Equation 5-9.
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